Investigating Student's Systems of Thinking

Regarding Graphs of Continuous Functions in Coordinate Planes

by

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ABSTRACT

Authors of calculus texts often include graphs in the text with the intent that the graph depicts relationships described in theorems and formulas. Similarly, graphs are often utilized in classroom lectures and discussions for the same purpose. The author or instructor includes function graphs to represent quantitative relationships and how a pair of quantities vary. Previous research has shown that different students interpret calculus statements differently depending on their meanings of points in the coordinate plane. As a result, students' widely differing interpretations of graphs presented to them.

Researchers studying how students understand graphs of continuous functions and coordinate planes have developed many constructs to explain potential aspects of students' thinking about coordinate points, coordinate planes, variation, covariation, and continuous functions. No current research investigates how the different ways of thinking about graphs correlate. In other words, are there some ways of thinking that tend to either occur together or not occur together? In this research, I investigated student's system of meanings to describe how the different ways of understanding coordinate planes, coordinate points, and graphs of functions in the coordinate planes are related in students' thinking. I determine a relationship between students' understanding of number lines or coordinate planes containing an infinite collection of numbers and their ability to identify a graph representing a dynamic situation. Additionally, I determined a relationship between students reasoning with values (instead of shapes) and their ability to create a graph to represent a dynamic situation.

DEDICATION

I would like to dedicate this dissertation to my parents and my sons for their unwavering love and support throughout my doctoral program, especially to my dad, who spent his last years of life supporting and encouraging me through the program and for his incredible help in caring for my sons so that I had the opportunity to pursue this degree. Without him, I would not have been able to pursue my dream, and his support was greatly appreciated and deeply missed.

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my advisor for his insightful comments and suggestions in guiding me through determining a research direction, narrowing the scope (repeatedly), and carrying out my research. I am additionally thankful for the expertise and advice offered by the rest of my committee and for the assistance throughout. I would like to extend my sincere thanks to the graduate advisors who helped me stay on track with deadlines. Finally, I would like to thank my mother, who has been my constant support and encouragement through a very challenging time.

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CHAPTER 1

INTRODUCTION

From a constructivist perspective, meanings that a learner evokes depend on how they interpret the situation they are in. A student's evoked system of meanings would be the collection of meanings triggered in the student's thinking while reasoning. Current research has focused on single aspects of student thinking (for example, how the student thinks about a coordinate plane or how the student thinks about variation). However, a student's interpretation of a graph of a function depends on their evoked system of meanings at the time they interpret the graph. Research on students' system of meanings related to graphs informs researchers and instructors on how to support students in constructing a productive and cohesive system of meanings for calculus and identify meanings that are essential or prohibitive for constructing a productive system of meanings.

Research Questions:

- In what ways do students understand graphs, and how are the different ways that students understand graphs and coordinate planes related in the student's understanding?
- How are productive meanings for graphs related to the different ways that students understand graphs and coordinate planes?

Based on current research, I consider a productive meaning for graphs of continuous functions to be one in which the student understands a graph of continuous functions as consisting of an infinite collection of coordinate pairs representing coordinated values of two covarying quantities which emerge through imagining a graph as emerging or having emerged, by coordinating values of two quantities as their values vary continuously through intervals.

To answer the research questions, I created items so that student responses give insight into their meanings. Developing items involves a process of designing an item, testing the item, then analyzing and redesigning the item ((Thompson, 2016; Carlson, Oehrtman, & Engelke, 2010).). Since meanings are in the student's mind, we cannot directly access them. However, researchers can build models of what they believe is a student's system of meanings. Thompson (2016) stated that meanings are invoked in interpretation. To design an item to assess a student’s meaning, the item must be designed to reveal something about how the person interpreted the item. The constructs used to describe student thinking are a tool of researchers to classify and explicate their model of what they believe is the student’s meaning by describing a collection of meanings that explain what a student said or did and why those meanings explain the student’s work. In saying two different students think of variation discretely, it is not to say that the student’s meanings about variation are identical – but that their meanings seem consistent with thinking described by a theoretical construct. The researcher would expect that the students would interpret situations similarly and that the construct would be useful in explaining the student's actions. Theoretical constructs are tools researchers use to explain aspects of students’ meanings, develop and test hypothesis about the boundaries of their meanings, and develop hypothetical learning trajectories to support students in constructing productive meanings. In the case of my study, theoretical constructs such as *emergent shape thinking* and *value thinking* are employed to provide insight into how to design items that differentiate between different meanings of graphs students might have.

David et al. (2019) showed that students interpreted calculus statements differently based on their understanding of points in the coordinate plane. As a result, students' meanings of graphs influenced how they understood the mathematical ideas the graph was intended to support.

Intermediate Value Theorem
Let f be a continuous real-valued function on [a,b]. Then for all real numbers N between f(a) and f(b), there exists c in (a,b) such that $f(c) = \mathrm{N}$.

For example, David et al. (2019) described *location thinking* as when students thought about the spatial location of a coordinate point in the plane and tended to label points on the graph as values of the function). A student thinking about a coordinate point on a function this way might interpret the statement $f(0) < y < f(5)$ as the piece of the graph between $(0, f(0))$ and $(5, f(5))$ (see Figure 1). A student whose meaning of $f(0) < y < f(5)$ is the curve between $(0, f(0))$ and $(5, f(5))$ would likely interpret "for all real numbers N between f(a) and f(b)" as "for any location on the curve between $(0,f(0))$ and $(5, f(5))$". In the example in Figure 1, the student understood the inequality by referencing the curve, including the part of the curve between x = 0 and x = 5 and above $y = f(0)$. Note that I am referring to $(0, f(0))$ from my perspective to clarify the coordinate pair I am referencing. The student might be thinking about the coordinate point $(0, f(0))$ as $f(0)$.

The ways students might think about the piece of the graph between the two marked locations can also differ. A student could be imaging a curve or path that directs them from one point $(0,f(0))$ to the next point $(5,f(5))$, a discrete collection of points (and a path between them), or as an infinite collection of ordered pairs being represented by the curve. The student in Figure 1 highlighted the curve, but when asked how many coordinate pairs were on the curve, the student stated that there were 6 (one for each whole number from 0 to 5 inclusively).

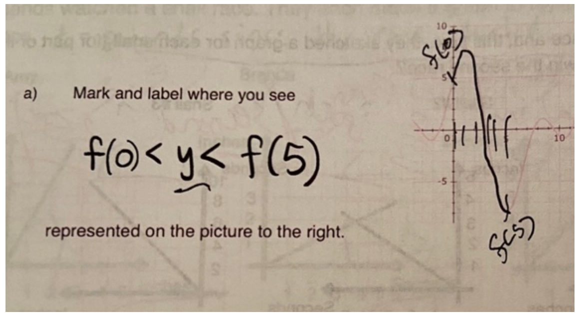


Figure 1. Example of Thinking Discretely

Interview Excerpt:

S6: f of 0 to f of 5, that is here, from here to here [highlights curve]
Me: The inequality includes all the curve that is between the 2 marks that you made?
Me: How many different coordinate pairs are on the curve?
S6: 1, 2, 3, 4, 5, 6 [student counts while drawing marks on x-axis.]

According to Bass (2019), there are, in principle, two ways to interpret a number line: (1) as an empty line to which numbers are added and (2) as a line full of points, each of which corresponds to a magnitude whether labeled or not. In the first case, the only numbers represented on the number line are the numbers marked on the line. Additional numbers can be added to the line by adding additional marks to the picture. To a person with this conception, the number line In Figure 2 represents only the labeled or stated numbers. Such a student might see the only numbers currently represented on the number line as 1, 2, 3, 4, 5, …, 13, and 14. The construct of empty number lines could account for limitations in students' conceptions of a number line imposed by their current numerical schemes. In the second conception, wherein the line is full of points and each point is some distance from a reference point, the student imagines numbers as being represented on the number line even if they are not marked.

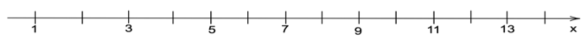


Figure 2. A Blank Number Line

Imagine that an instructor draws a number line as a tool in a discussion involving the variation of a quantity whose value varies. The instructor might draw a picture like in Figure 3 while describing the variable's value as changing from 2 to 7. The teacher might intend for the student to see the picture and imagine the variable's value as changing by varying through each number on the real line, as represented by the line to the teacher. The instructor might discuss the variable as varying through the numbers on the number line between 2 and 7 as part of their class discussion. The instructor's intention in this discussion might be for the student to imagine the variable as varying through all the real numbers between 2 and 7.

A Number Line Indicating Variation From Two To Seven

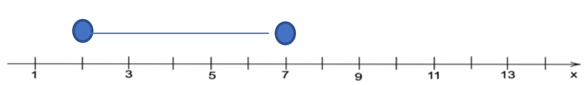


Figure 3. A Number Line Indicating Variation From Two To Seven

A student who understands the number line as representing only the listed numbers, however, might interpret the discussion as meaning the variable started at 1, then was 2 (the next number), then 3 (the next number), and so forth without imagining the variable as taking any value between the marked values. For example, they might imagine counting on the number line, as shown in Figure 4.

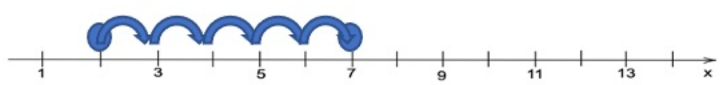


*Figure 4***.** A Number Line Demonstrating Imagining Counting 2, 3, 4, 5, 6, 7

The cartesian coordinate plane, Figure 5 below, is the standard coordinate system on which students learn to graph continuous functions of two variables. The cartesian coordinate plane is two number lines superimposed perpendicularly and intersecting at 0 on each number line. As with the number line, two people looking at the same picture might have different understandings of what is being represented by the picture. Those understandings influence how the person thinks about functions graphed in the coordinate plane and how they might imagine concepts such as covariation of the values of the quantities related by the function graphed in the coordinate plane.

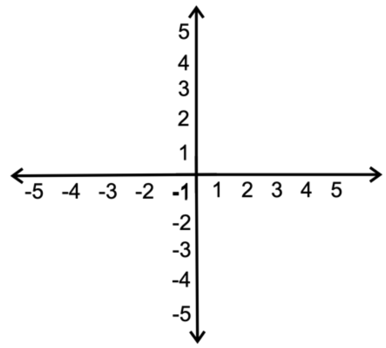


Figure 5. Cartesian Coordinate Plane

Imagine someone looking at the plane graphed in Figure 5 whose meaning for number line is that it includes only the marked values or locations. The person might look at the picture and sees two number lines arranged perpendicularly and mentally imagine creating a grid that spans the plane. To this person any location in the plane describes information about two quantities whose values are represented on each number line simultaneously by referencing values or locations on each number line. In other words, when the first person thinks about a coordinate pair (for example (1,2)) in the coordinate plane, they imagine that the coordinate pair graphed in the plane is a way of representing a coordination between two quantities when the value of the quantity represented on the horizontal number line has a value of 1. The value of the quantity represented on the vertical axis is 2. The student imagines the coordinate point as being represented even if no point is physically plotted in the plane.

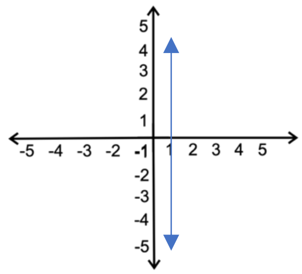
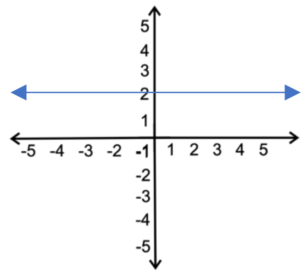
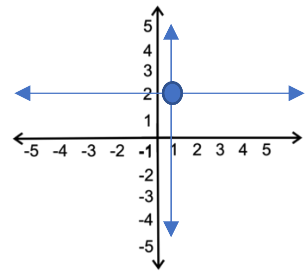
		
Imagining everywhere in the coordinate plane where x is 1.	Imagining everywhere in the coordinate plane where y is 2	The coordinate point (1,2) imagined by coordinating the values of both x and y.

Figure 6. Imagining A Coordinate Point As A Coordination Of Two Values

Suppose the student understands the number line as only representing a discrete collection of values. In that case, they might understand the coordinate plane as representing only coordinated values for the values represented on the number lines. A student understanding the coordinate plane as only including marked values might extend that thinking to functions graphed in the coordinate plane. For example, to the student looking at the function graphed in Figure 6 there were only six coordinate pairs of the function represented by the function's graph. The student explains that more can be added, but in the student’s current thinking, they are not represented until additional dashes are added, such as the ones the student made between 1 and 2 at the bottom of the graph.

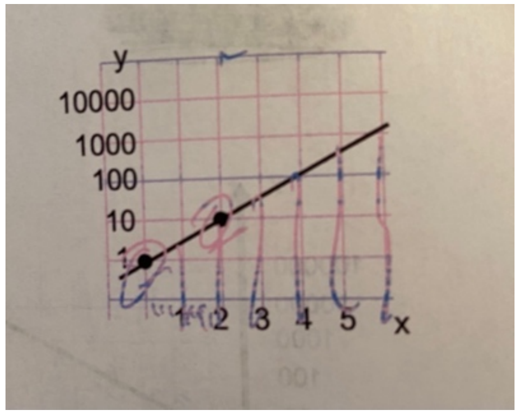


Figure 7. Imagining An Empty Coordinate Plane

Interview Except:

Me: How many coordinate pairs are being represented by that function?
S6: 1, 2, 3, 4, 5, 6
Me: If we wanted it to include for example, decimals, what would we need to do?
S6: You would just need to like small dashes to include that. This graph is not that zoomed out. It is zoomed in. You can see that there are decimal points as 1.1, 1.2

Researchers studying how students understand graphs of continuous functions and coordinate planes have developed many constructs to explain potential aspects of students' thinking about coordinate points, coordinate planes, variation, covariation, and continuous functions. Existing research studies focus on one aspect of the students thinking. There is no current research investigating how the different ways of thinking about graphs, coordinate planes, and covariation of functions are related in a student's system of thinking. Student's individually understand the classroom instruction involving graphs based on their collection of evoked meanings. The collection of meanings in the student's mind, which may be evoked in situations involving graphs, forms a related system of (in the moment) meanings.

Researchers have introduced constructs to characterize students' meanings for coordinate points, coordinate planes, variation, covariation, and graphs of functions in coordinate planes. In this dissertation, I build upon current constructs such as emergent shape thinking, static shape thinking, value thinking, and chunky variational reasoning to investigate relationships among student meanings for coordinate points, coordinate planes, variation, covariation, and graphs of continuous functions. Research into these relationships can provide insight to researchers and educators on how to support students in building productive meanings critical to constructing a productive system of meanings for calculus. In the following chapters, I review the existing constructs regarding different ways of thinking about coordinate points, coordinate planes, and graphs of continuous functions in a coordinate plane. I then discuss a collection of questions that give insight into a student's thinking about graphs and collectively give information about the student's system of thinking. Trends between individual student's systems of thinking can give important insight into ways of thinking that tend to occur in the thinking of students whose system of meanings is productive or unproductive for success in calculus concepts. This knowledge can give researchers and educators information on unproductive meanings needing to be addressed in order to support the students in constructing a stronger system of meanings and what meanings the students need to be supported in constructing.

CHAPTER 2

LITERATURE REVIEW

In this chapter, I discuss the current research literature which informed my research. The literature review contains three sections.

- Existing constructs regarding different meanings that might exist in student thinking regarding graphs.
- Literature which informed creation of items which give information on student meanings.
- Existing literature and a conceptual analysis of what is meant by a productive meaning for graphs of continuous functions and coordinate planes.

Literature Review of Graphing Constructs

In this section I discuss constructs already existing in literature regarding different meanings students might utilize when interpreting graphs that informed my item creation. I start by reviewing conceptions involving a single quantity or variable (number lines and variation) and then conceptions involved in coordinating two quantities or variables (coordinate points, coordinate planes, covariation, and graphs of functions).

Conceptions of a Number Line and Measurements

Hyman Bass (2019) argued that two conceptions of a number line are likely based on two different classroom ways students are introduced to number lines. The first narrative starts with an empty line and places numbers as students learn about them. Students first build an image where only counts (1,2, 3,...) are represented on the number line (Clements & Stephens, 2001). As the student learns about integers, rational, and irrational numbers, they are expected to fill in the gaps in their conception of the number line, eventually understanding the number line as representing all real numbers. Bass, however, argued that students constructing their understanding of the number line this way are likely left conceiving of a number line that still has gaps between visible marks or labels on the number line. To the student, a number is not represented on the number line until somehow depicted as on the line. It is also important to note that in this construction, numbers on the number line are not connected to any unit of measurement in the student's understanding.

In contrast, Bass talks about another potential classroom narrative of the number line based on work by Davydov (1975), where students are supported in constructing a concept of the number line through instruction on measurement. From an early age, students have a sense of measurements of quantities such as length, area, volume, and weight. Children can understand ideas such as the difference of two areas by overlapping two shapes, cutting the larger shape into two pieces with one matching the smaller shape and the other representing the difference between the two areas (Davydov (1975); Clements & Stephen (2001)). Students can be supported in constructing a conception of a number line in which numbers are related to iterating a unit of measurement (an interval with a direction). Bass argued that the number line conceived of as constructed by imagining the distances of locations on the line from a reference point is inherently continuous (without gaps). Additionally, numbers learned later are understood as naming magnitudes that were always on the line but were unnamed. Sirotic and Zazkis (2007) discussed that student's numeric schemes might be limited based on believing there are only a finite number of rational numbers. Student's numeric schemes likely play a role in how they understand graphs and variation, but it is too big of a topic to address in this dissertation.

In this study I attempt to distinguish between students whose numeric scheme constrains them to thinking about a discrete collection of numbers and those whose numeric schemes imply to them that a number line contains an infinite number of numbers (points). I do not address the matter of their understanding the density of numbers on a number line. Instead, I focus on the extent their variation scheme is limited to thinking about varying discretely.

Variation of a single variable or quantity

To conceive of a quantity in a situation, someone would need to conceive of an object with a measurable attribute, a unit of measurement, and a method of quantifying the measurement (Thompson, 1993). In Thompson's theory of quantitative reasoning, the object is not a physical object, but a student's conception, and thus is idiosyncratic to the individual. Similarly, the student's quantitative structure (the relationships the student conceives of between quantities) is idiosyncratic. It is possible that a student does not conceive of a quantity in a situation.

A student's conception of variation of a variable's value is how the student imagines the value of the variable varies (I am using the word variable instead of quantity because the student might not conceive of a quantity). Thompson and Carlson's variation framework (2017) describes six potential conceptions of variation of a variable, extending Castillo-Garsow's (2010) constructs of chunky and smooth variational reasoning. At the lowest two levels, the student does not conceive of variation (understanding the variable as a symbol that has nothing to do with variation or imagining the variable has only one value in the situation). Discrete variation describes a student who understands variation as the variable's value varies by replacing the value with a next value. Discrete variation differs from no variation because the student imagines the variable having different values within the same situation. If the student envisions that the value of the variable increases or decreases (a comparison of the values of the variable) but still doesn't envision the variable as taking values between the specific values, then the student's thinking would be classified as gross variation of the quantities' values. The framework's highest two levels of variation are chunky continuous variation and smooth continuous variation. In both, the student imagines that the variable's value varies through intervals. The distinction is how the student imagines the value varying through values within each interval. Smooth continuous variation describes thinking where the student anticipates that the value varies smoothly and continuously through every sub-interval of the interval. Chunky continuous variation describes thinking where the student conceives of the variable's value as increasing (or decreasing) through the entire chunk at once, as if laying a ruler repeatedly. They focus their thinking on the endpoints of each interval, without the immediate realization that the variable takes on every value within the interval.

A student's conception of the number line has implications on how the student conceives of variation represented on the number line. A student who imagines a number line as representing only the marked values and gaps between the marked values is limited to conceiving of variation as at most gross variation (discretely with direction). There are no numbers between the marked values in the student's conception. Their variational reasoning is limited to replacing the number with the next number they perceive.

Coordinate Planes, Coordinate points, and graphs of functions

Lee and colleagues (Lee, 2016; Lee 2018; Lee et al., 2020; Paoletti et al., 2022) discussed two possible conceptions students could have when conceiving of a coordinate plane. Imagining a situational coordinate system describes imagining a grid of two perpendicular number lines overlaying the objects. The grid establishes a frame of reference in which objects' attributes can be measured, for example, measuring distances on a map. Coordinate pairs would be interpreted as locations within that frame of reference. Lee discussed that situational coordinate systems could be a useful conception for the student in constructing relationships between quantities for specific values of the quantities. Varying the quantities' values would require imagining the frame of reference as background of a movie.

Lee's definition for a quantitative coordinate plane is two perpendicular number lines that are understood as representing possible measurements (in the quantities' units) for two covarying quantities (respectively). Lee discussed that for someone to conceive of a quantitative coordinate system, they must conceive of two covarying quantities within a situation. The student then needs to image each axis as representing measurements in the units of the respective quantity as the quantity's value varies. To plot a point, the student must then extract the coordinated values of quantities from the situation and represent the values in the quantitative coordinate system. This way of thinking about coordinate points is consistent with what David, Roh & Sellers (2019) described as *value thinking*. Lee's quantitative coordinate plane is consistent with a conception of a coordinate plane needed for a student to imagine the graph of a function as emerging through coordinating the values of the quantities as their values vary (described as *emergent shape thinking* in Moore and Thompson, 2015)).

Lee hypothesized that a student's interpreting a graph differently than intended might be due to the student interpreting the coordinate plane as a frame of reference to directly measure within the plane (overlaying a ruler over a picture to measure a length in the picture) instead of a quantitative coordinate plane where the coordinate values represent the simultaneous values of two quantities being related. Lee suggested that this could explain why a student interprets a time-speed graph as if it was the biker's path.

Both coordinate plane conceptions described by Lee assume that the student conceives of measurements related to the coordinate system. Students might not be conceiving of measurements. For example, with *location thinking* (David, Roh, & Sellers, 2019), the student is thinking about the spatial location of the coordinate point in the coordinate plane. Students thinking about coordinate points as locations would compare the points by comparing spatial locations (higher, lower, left, right). Location thinking is consistent with the conception of a point as *over-and-up* (Frank, 2016), where the student conceives of the coordinate pairs as directions to find the spatial location of the point in the plane. Frank discussed that conceiving of the coordinate point as over-and-up would hinder students from conceiving of a graph of a function as entailing covariation of two quantities. Instead, their reasoning would focus on appropriate shapes for graphs (consistent with *static shape thinking* (Moore and Thompson, 2015)). *Static shape thinking* describes conceiving of a graph of a function as if a piece of wire and a point on a graph like a bead that moves along the wire. Moore and Thompson said that a student exhibiting *static shape thinking* is focused on their perceived physical features of the graph (such as increasing means that the graph goes up). They might infer information about the variables (such as determining slope) by associating memorized facts with their perceived physical features of the graph. The student's thinking is focused on visual features of the graph and not on the variable's values or variations in the variable's values. David, Roh & Sellers described that students who were thinking about points as spatial locations tended to label the graph with just the value of the second coordinate of a coordinate pair and were likely to interpret statements such as $f(\mathrm{a}) < \mathrm{y} < f(\mathrm{b})$ as the part of the graphed curve between $(a,f(a))$ and $(b,f(b))$.

Physical features of the graph (such as increasing/ decreasing, steepness, …) depend on orientation, scale, and which quantity is represented on which axis. Students whose conceptions are limited to static shape thinking will not recognize that two graphs represent the same relationship if graphed on coordinate planes that switch the axis, change the scale, or reverse orientation (Moore et al., 2019). For example, Moore described the work of a preservice teacher who associated slope with the perceived slantiness of the line. To this teacher, a line slanting "up as you go over" meant a positive slope, and a line heading down meant a negative slope. In a task where the preservice teacher was writing an equation for a line presented on a coordinate plane with x on the vertical axis and y on the horizontal axis rotated the graph so that x was on the horizontal axis with the positive direction heading left. The line in this orientation visually heads down. The preservice teacher concluded that the slope was negative and insisted that the slope was negative even after correctly identifying a coordinate on the graph and identifying that the equation she wrote was inconsistent with the coordinate pair she identified. In contrast, students whose thinking is consistent with *emergent shape* *thinking* can leverage their thinking to accommodate different graphing conventions and conclude that graphs that look different can represent the same relationship between two quantities (Moore, 2016).

Value thinking describes understanding coordinate pairs as representing values of the two variables. The student might be thinking about the value of one quantity and separately the value of the other quantity and coordinating the values to plot a coordinate point. To think about another coordinate pair, the student needs to think about each value individually and coordinate to plot another point. Thinking about coordinate points this way would be inherently discrete (one value of one quantity at a time). In thinking about coordinate points discretely, it is possible that the student only conceives of the graph of a function as containing the coordinate points that are physically marked, and the curve drawn through the points as indicating a direction of movement from one point to the next (Tasova et al., 2021).

It is also possible that the student, conceiving of two covarying quantities in the situation and uniting the coordination of the value of the two quantities as a mental object, conceives of the point as a *multiplicative object* (Saldahana & Thompson, 1998; Frank 2016). Conceiving of a point as a multiplicative object means that in addition to conceiving of the coordinates as values of the two quantities, the student also has an image of a new quantity, such as "distance at a moment in time") that as the value of one quantity varies, the other quantity always has a value associated with it. If the student then imagines the graph of a function as emerging by maintaining a record of the coordinated values as they continuously covary for some duration, then the student's conception of a graph is consistent with *emergent shape thinking* (Moore and Thompson, 2015). Tasova et al. (2020) and Thompson et al. (2017) discussed examples of students whom they claimed conceived of a point as a multiplicative object but did not maintain the conception as the values varied. This indicates that for a student to conceive of the graph of a function emergently, the student needs to conceive of the variation of the quantities as happening simultaneously and maintain their conception of the multiplicative object as the values of the quantities covary.

Covariation

Thompson (1994), Carlson et al. (2002) and Thompson and Carlson (2019) defined frameworks for understanding a student's covariational reasoning based on how the student coordinates the quantities' values or variations of the quantities' values. Carlson et al.'s framework describe levels of variational reasoning based on how the student envisions variables' values varying and describes levels of covariational reasoning according to the ways they coordinate quantities' values. Higher-level variational reasoning involves envisioning a variable's value varying continuously. Higher level covariational reasoning entails the coordination of variables that vary continuously. Lower-level variational reasoning involves envisioning quantities values one at a time, or discretely. Lower-level covariational reasoning entails coordinating values of quantities envisioned as varying discretely or not varying by way of subsitituting one value at a time.

The lowest levels in both covariational frameworks describe thinking in which students coordinate values of the quantities but do not have an image of the quantities values as varying simultaneously. At the lowest level (coordination) in Carlson's framework, the student conceives that the value of one quantity is coordinated with the value of the other. At this level, the student coordinates point by point (discretely) by identifying for a single value of x the coordinated value of y. For students thinking discretely, a new value for x is a new situation. The lowest two levels in Thompson and Carlson's framework are that the student has no image of the values varying together (no coordination) or the student conceives of variations asynchronously (pre-coordination).

The next level in both frameworks (directional and gross coordination of values) is the first level where the student envisions that the quantities values vary together. Directional covariation describes envisioning one quantity's value having a directional change (increases or decreases) as the other quantity's value increases. They might not be able to quantify the amount of change. If the student's image of variation is discrete, then it is likely that the student understands the graph as only including only the plotted points, and any curve or line drawn through might represent only a direction of travel from one point to the next (as seen in Tasova et al., 2021).

In the highest three levels in Carlson's framework, the student quantify the “next” value of a quantity by determining an amount of change for at least one variable. In level 3 of Carlson et al.'s framework (quantitative coordination), the student coordinates successive values of a dependent quantity by imagining an amount of change from the previous value of one quantity and coordinates the new value with the new value of the independent quantity as determined by a fixed amount of change in its value. The focus of the student’s thinking at this level is on the amount of change in the second quantity, and the student might conceive of the rate of change as being the amount of change of the second quantity for one fixed change in the other. A typical behavior at this level is portioning the horizontal axis into intervals of a fixed length, plotting points for each interval, and connecting the points. In level 4 (Average rate), the student's thinking focuses on constructing secant lines or estimating the slope of a graph over a small interval). In level 5 (Instantaneous rate), the student additionally has an image the rate is (continuously) changing (increasing or decreasing).

The highest two levels of covariation on Thompson and Carlson's framework (chunky continuous and smooth continuous) both require that the student envisions that the changes in both quantities' values happen simultaneously and smoothly. In other words, the student must conceive of a multiplicative object that unites the values of the quantities and a point on the graph to represent the multiplicative object graphically. The difference between chunky continuous and smooth continuous reasoning is in how the student imagines quantities' values varying. In chunky continuous, the student imagines the variation as happening chunk by chunk. The student imagines the values of the variables varied through all the numbers on the interval as part of the chunk. However, their focus is on coordinating the values of the variables at each endpoint. With smooth continuous covariation, the student has an image of the variables as simultaneously covarying through all the values on the respective intervals and by "all values" they have conceptualized the interval as a continuum of values.

Castillo-Garsow et al. (2013) argued that both continuous and chunky variational reasoning are essential for developing a concept of rate of change of one quantity with respect to another. Chunky variational reasoning is needed to quantify amounts of variation. Research on supporting students in developing productive conceptions of functions like exponential functions (Confrey and Smith (1994); Ellis et al. (2016)) designed activities that support chunky continuous thinking. However, extending from thinking about covariation on chunks to the function defined on all real numbers relies on the student having a smooth continuous variation conception. Castillo-Garsow (2014) stated that smooth variational reasoning could lead to the constructions of chunks by mentally pausing the perceived variation when quantification is needed.

Additionally, Castillo-Garsow argued that chunky reasoning, even when refined to smaller chunks, is not likely to lead to continuous variational reasoning because the student is thinking of the chunks (no matter how small) as a completed variation. For a student with chunky thinking to quantify variation within a perceived chunk, the student needs to break the chunks into smaller chunks so that the value of the quantity they wish to consider resides at an endpoint. Castillo-Garsow (2014) called for more research on how to support students in moving from chunky to smooth variation.

Paoletti and colleagues (Paoletti and Moore, 2017; Paoletti and Vishnubhotla, 2022; Paoletti et al, 2021) have reported on teaching experiments involving students emergently constructing graphs. Paoletti and Moore (2017) argued that students became aware of their parametric reasoning through representing the covariational relationship between two quantities. Paoletti and Vishnubhotla (2022) presented the results of a teaching experiment with middle school students which supported the students in reasoning about different types of covariational relationships through activities which involved graphically representing the relationship between two quantities. Paoletti et al (2021) reported on a teaching experiment where the students represented a system of covarying quantities. The emphasized the importance and interplay of both chunky and continuous thinking in order for the students to represent the relationships between the quantities. The activities in all three studies involved the students representing relationship by constructing graphs which emerged through comparing values or changes in values of the covarying quantities.

Literature Review on Creating Tasks to Assess Meanings

Literature on designing tasks aimed to reveal the mathematical meanings persons use to understand a task describes a cyclic process involving utilizing previous research, item design/ redesign, clinical interviews, and piloting the items. (Thompson, 2016; Carlson, Oehrtman, & Engelke, 2010). In both articles, it is suggested that existing research on mathematical understandings can inspire items. Thompson also suggested that interactions with students in the classroom can inspire items. Item design involved designing an item, evaluating the item by administering it (either in interviews or by administering a collection of open-ended items), and analyzing the results to determine the extent to which the item was open to being understood by students in ways that reflect the meanings they used to interpret the item, and then redesigning the item. The cycle of revise, evaluate, and revise items should be repeated until analysis indicates that the questions are being interpreted as intended and elicit the types of responses that are desired (for example, not interpreted as asking for a single numeric answer).

In designing items, Thompson (2016) stated that individual items must have a focus for what meanings the question is meant to reveal, the item needs to be designed to reveal something about the student's interpretation of the item, and the collection meanings assessed by the items need to create a coherent picture of the person's meanings. Thompson additionally suggested starting by focusing on developing items for meanings that might be productive or unproductive for future learning and asking follow-up questions designed to understand what their response implies for them in how they understood the item. Thompson also suggested requesting symbolic responses sparingly, deciding early on the item's focus, and piloting items early and often.

Carlson, Oehrtman, & Engelke’s approach to initially designing items was to start by running a series of studies to understand the reasoning abilities and meanings they believed are foundational to the central ideas of Calculus and Precalculus, resulting in a taxonomy of the reasoning abilities and mathematical concepts that guided their item design. They designed and administered a set of open-ended questions based on their taxonomy. Then they conducted clinical interviews to determine the reasoning abilities and meanings needed to provide a correct response. From the grounded analysis of students’ responses, they refined the wording in the questions, developed potential distractors, and inferred meanings the students had that led to that response. Additional items for the PCA were added to the collection of questions and modified through feedback from colleagues and clinical interviews until all questions met the following three criteria.

- Interviews revealed that the only students whose meaning was consistent with the targeted meaning would select the correct answer.
- The items could be adapted to multiple-choice form.
- The item was appropriate for students in precalculus.

Thompson's item design went through a design, analyze, and redesign process, but the initial approach to developing the items differed. In the PCA study, the item was developed starting with a mathematical concept, and student’s meanings emerged through grounded analysis and redesigning items and answers until students had consistent meanings when selecting particular responses. Thompson's item design focused on what meanings he wanted to reveal with the item and how to design the item so that students with different meanings would respond differently.

Both the PCI authors and Thompson analyzed qualitative data based on grounded theory (Strauss and Corbin, 1997). With multiple choice items, grounded analysis was utilized to develop the answers. This involved a cycle of administering the questions, using grounded theory to analyze student responses to identify patterns in student thinking, then readministering the questions and conducting interviews to verify the student interpreted the questions as intended, and that student's selected particular answers only when their meanings were similar. Thompson described coding open-ended questions as involving a process of analyzing responses (which sometimes led to a modification of items, modifying the theory, or discarding the item), grouping responses by levels of productive reasoning, and creating a rubric that could be used to code the item. Since more than one person would be coding the items, each rubric went through a revision process until consensus was reached and inter-scoring agreement was assessed to ensure consistency in the items' scoring. Thompson's study included both open-ended and multiple-choice items. Thompson mentioned a potential issue of multiple-choice questions overestimating the number of students with a particular meaning. This would happen if a specific answer sounds correct to the student, but they would not have come up with that answer themselves.

Items in the Force Concept Inventory (FCA) were developed through designing an inventory of concepts they wanted to explore and common misconceptions (alternate ways of thinking that are incorrect from the perspective of the authors) (Hestenes, Wells, & Swackhamer, 1992). Multiple-choice items were designed based on the authors' understanding from previous experience and research.

There are two significant differences in how the collection of questions for the FCI were designed. The first is that the questions and answers were designed based on the authors' understanding of the Newtonian concept of force and common misconceptions. The lack of redesigning the items through analysis of student responses explains why some questions in the first version had to be discarded because they were misread (meaning that students commonly interpreted the question was not what was intended), some questions which were weak discriminators due to the "Newtonian" response being arrived at by more than one way of reasoning, and some items having commonly chosen answers for which the authors did not know the reasoning students used to arrive at those answers.

Incorporating interviews, analysis, and redesigning the questions prior to implementing the test would have identified the items that were not interpreted as intended, had multiple meanings that led to the same answer, and the meaning that led to each of the answers included for the items. The items could then have been redesigned or replaced prior to administering the assessment and would have improved the interpretability of the results.

Inferring an individual student's meanings through an assessment requires items to be developed so that students selecting the same answer have consistent meanings. Students with productive meanings might employ less productive meanings in tasks where the less productive meaning is sufficient. For example, when analyzing how students interpret expressions, Parr (2021) noted that students who sometimes interpreted an expression as a measured length or distance from a reference point might, in a different question, interpret an expression using spatial arrangements. Since students might use less productive meanings in tasks where the less productive meaning is sufficient, items meant to identify particular meanings need to be carefully designed so that other meanings will not lead the students to the same responses.

A second distinction is that the authors of FCI saw the entire test as a measure of detecting if a student's thinking could be classified as "Newtonian" or not. They argued that for individual questions, there could be false positives (students whose thinking is not consistent with being Newtonian but choose the answer a Newtonian would) and false negatives (Newtonians who select an answer that was designed based on a misconception), but that only those whose thinking was consistent with Newtonian thinking would have a consistent pattern of Newtonian answers, and emphasized analyzing the assessment as a whole. Thompson also noted that a single item is insufficient to give insight into the boundaries and connections within the student's meanings. Thompson discussed the need to aggregate the data from many questions, but how to aggregate the data remained an open qauestion.

Literature Review on Expert Graph Understanding

Kop et al. (2015) discussed two frameworks for levels of recognition for graphing (one for graphing from a formula and one for creating a formula from a graph), which they used to compare the graphing activities of experts and novices. For experts, they interviewed three mathematicians, a teacher educator, and a textbook author who all have a master's or higher in mathematics and more than ten years of teaching experience. For novices, Kop et al. invited three teachers who had been, respectively, teaching for 2, 6, and 30 years. Kop et al. did not discuss their reasoning for determining the teachers were novices, and the mathematicians, teacher educators, and textbook authors were experts. The classification is questionable based on the observations by Kop et al. that the teacher with 30 years of experience was as accurate as of the experts in providing standard interpretations of the graphs. Additionally, Roth and Bowen (2003) also noted that scientists who regularly teach courses with textbooks with similar graphs were much more likely to provide standard interpretations (the interpretations expected of students in the classes from which the graphs appear in their textbooks) than scientists in the private sector.

Table 1 shows Kop et al.'s framework for recognition of graphs when presented with a formula, and Table 2 shows their framework for recognition of the formula when presented with a graph.

Table 1

Kop et al.'s(2015) framework for recognition of graphs when presented with a formula

		Heuristic search (strong → weak)				
Levels of recognition (high → low)	A	A1. Graph is instantly recognized as a whole				
	B	B1. Recognition of family (with characteristics); possible graphs are known	B2. Search for 'parameters' of the graph	B3. Investigate the family characteristics, for instance via zeroes, derivative		
	C	C1. Split formula in subformulas, graphs of subformulas being known	C2. Compose the graphs by qualitative reasoning	C3. Compose the graphs by making a table		
	D	D1. Characteristic aspect of graph is recognized; rest of graph is unknown				
	E	E1. Graph is not recognized, algebraic formula is starting point for strategic exploration	E2. Qualitative reasoning for instance about domain, or vertical asymptote, or symmetry, or infinity behavior, or increase/decrease	E3. Algebraic manipulation	E4. Strategic search, for instance for zeroes or extreme values (via derivative)	E5. Calculate strategically chosen point(s)
	F	F1. No recognition at all	F2. Standard repertoire of research	F3. Make table with random x values		

Table 2

Kop et al.'s (2015) framework for recognition of the formula when presented with a graph.

	Heuristic search (strong → weak)			
Levels of recognition(high → low)	A	A1. Formula is instantly recognized		
	B	B1. Recognition of family of formulas ('is something like…')	B2. Searching for 'parameters' of the formula (e.g., translation or via zeroes)	
	C	C1. Graph is decomposed into subgraphs	C2. Finding formulas for subgraphs and composing the formula	
	D	D1. 'Parts of the graph give parts of formula' (part of formula is recognized)	D2. Adjusting a formula to characteristics via qualitative reasoning	
	E	E1. Mentioning algebraic formulas ('can it be something like this?') without direct link to graph	E2. Checking a formula via qualitative reasoning	E3. Checking a formula by general methods (zeroes, extreme values, table)
	F	F1. No recognition at all		

Kop et al. built their framework to describe levels of recognition required to construct graphs from formulas or to construct formulas from graphs but did not address how the experts reasoned with the graphs. Nor did Kop et al. analyze how the participants conceived of the quantities and the covariation of those quantities. Kop et al. instead prioritized efficiency in graphing and static shape thinking in their frameworks. The designation of higher and lower levels fits with their conception of an expert as one with a high level of recognition. However, it is not clear if their levels relate to being an expert in interpreting novel graphs and reasoning with graphs. Kop et al.'s frameworks are also a reflection of the emphasis on shape recognition in the current curriculum on the teaching of graphs (Moore & Thompson, 2016).

In their analysis of the interviews, Kop et al. coded instances in the participants' work where they displayed aspects of the framework to create paths that documented the graphing activity of each participant throughout the problems. Kop et al. expected that the codes in the paths of the novices would have consisted primarily of codes in the lower levels of their recognition framework and that codes in the paths of the experts would have primarily consisted of codes in the higher levels of the recognition framework.

Kop et al. noticed that the teachers with less experience did primarily utilize the lower levels of their framework but were surprised by the paths of the experts. Two of the experts' paths consisted primarily of codes in the lower levels of the framework, and the paths of two of the other experts also started in the lower levels of the framework. Kop et al.'s experts' ways of thinking about graphs involve more than merely recognizing graphs. Kop et al.'s experts participated in the process of interacting with the graphs and formulas to determine additional information about the other representation. The experts' process of interacting with graphs suggests a need for explicit curricular goals for students' interpretations of graphs, and the relationship between formulas and graphs in math education needs to involve more than just shape recognition. Kop et al.'s article supports defining productive meanings about graphs as understanding graphs as a coordination of quantities' values of two covarying quantities as they continuously covary. In the next section I explicate a high-level understanding of graphs of continuous functions and coordinate planes.

A conceptual analysis of a high-level understanding of graphs of continuous functions and coordinate planes.

Prior to calculus, most graphing instruction involves graphs on a rectangular coordinate plane, where two number lines are placed perpendicularly (intersecting at 0). The positive values for the number lines are conventionally oriented right and up (respectively). The marks on each number line are conventionally spaced equal distance with the same increase in value between each mark. These choices are a convention. The coordinate plane could be designed by orienting the axis in a different direction or intersecting the number lines at a place other than 0. Semi-log and logarithmic coordinate planes have equally spaced marks on one or both axis which have a constant ratio instead of a constant additive increase. Polar coordinates can also be used to describe a coordinate plane. Conceptions of coordinate planes and graphs of functions that enable the student to envision the relationship of the values of two quantities on any coordinate plane would be considered a high-level understanding of graphs.

Understanding the graph of a function to represent the relationship between the values of two covarying quantities requires that the student conceives of two quantities. To conceive of a quantity, the student must conceive of an object with a measurable attribute, a unit of measurement, and a method to quantify the measurement. In addition to conceiving of quantities, there are three foundational meanings that are necessary for a high-level of productive meanings of graphs (how they conceive of the coordinate plane, how they conceive of points in the coordinate plane, how they conceive of covariation).

The first foundational meaning is that the student conceives of each axis as a measurement tool, representing measurements in the respective units of the two quantities. Changing conventions, like reversing orientations, would be like reversing the direction of the ruler. If the student does not conceive of measurements as related to the coordinate plane and graph, their thinking about graphs would be entirely based on spatial locations or viaual appearance. Their analysis of graphs cannot be based on values (since they are not conceiving of values). Thus, they reason about graphs by reasoning about perceived aspects of the graph (steepness, increasing as up, decreasing as down, …). A change in conventions would, in their thinking, therefore, represent a different relationship.

The second foundational meaning is to conceive of a multiplicative object that unites the values of those quantities and a coordinate point in a plane as representing the multiplicative object. Conceiving of a multiplicative object means that the student has an image that the values of one quantity are linked to the values of the other quantity and that as one varies, the other necessarily varies with it. If coordinate points represent, to the student, the multiplicative object which unites the values of the two variables, then inherently thinking about a coordinate point would require thinking about values of both quantities. If they do not conceive of a multiplicative object, they might associate a location on the curve as giving information about a single quantity (or variable). David, Roh, & Sellers (2019) noticed that students who are *location thinking* (thinking about coordinate points as locations) tended to label a location on the curve as 'y' or '$f(x)$' (not as an ordered pair), and interpreted statements such as $f(a) < y < f(b)$ as being on the graph of the curve between $f(a)$ and $f(b)$. $f(a)$ is understood to be the location on the curve above a. a is understood as identifying the location for which y (like an index, y_a).

The third foundational meaning is that they need to maintain their conception of a multiplicative object as they envision quantities' values varying smoothly and continuously, meaning they imagine the values varying simultaneously and both values varying through a continuum of values. Chunky continuous covariation is imagining that each quantity's has values within a chunk of variation but coordinates values of each quantity only at the end of each chunk. They are not coordinating the values of the quantities within each chunk. For these students, to reason about what is happening within a chunk, they would need to break the chunks into smaller chunks so that the value they need to reason about is at an endpoint of a chunk. Though they conceive each quantity's value as having all values, they think about values at the end of the chunks differently from those within a chunk because they are only mentally coordinating values at the end of each chunk. A chunky-continuous conception might be sufficient to recognize that the same relationship graphed in different coordinate systems is the same, even though they look different. However, they might have difficulty reasoning about the relationship between the quantities within a chunk. They would need to find an appropriate way to partition the values so that the value they want to reason about is at the end of the chunk. For smooth continuous covariation, the student imagines both values varying simultaneously and coordinates the values of the two quantities as they vary. They can imagine any amount of variation from any starting coordination of values.

An unproductive way of thinking about covariation of the values of two quantities (or spatial locations) is thinking about the value (or location) of each variable as varying asynchronously and discretely. They first think of the value (or location) of one variable being replaced by its next value (or location). Once that is completed, they think about the other variable's value (or location) as being replaced by its next value or location (first over to the next 'x' and then up to the next 'y'). The student might connect the plotted points, but the lines or curves they use to connect them indicate a path or direction to the next point. The only coordinate points (representing a relationship between the two quantities or a location in the plane) in their image are the plotted ones. I hypothesize that students whose conception of a number line represents counts and not measurements or whose conception of the coordinate plane is of spatial locations are likely limited to coordinating the value/ location of one variable at a time. In other words, to support a student in constructing a conception of variation that is either chunky continuous or smooth continuous, the student would first need to be supported in constructing an image of coordinate planes as representing measurements for two quantities whose values are being coordinated and that a point in the coordinate plane represents the coordination of the values of the two quantities. Another unproductive way of thinking about graphs is that a student does not imagine variation at all. They think about one coordination at a time, and a new value or location would be a different situation.

CHAPTER 3

THEORETICAL PERSPECTIVE

The types of questions a researcher asks and how a researcher analyzes and interprets the data depend on the researcher's underlying theoretical perspective. Theoretical constructs do work for research in two ways. First, the theoretical constructs serve as a lens that informs our research. The constructs frame how we design studies, what questions we ask, how we collect data, and how we analyze the results. Secondly, the underlying theoretical constructs are important to make explicit in sharing our research with others so that others can look at the research through a similar lens. For the purpose of understanding an individual student's system of meanings, I have adopted a cognitive perspective. Cobb (2017) explained that theories from cognitive psychology could be used to explain student's mathematical activities and the differences in their individual reasoning. The cognitive perspective focuses on a single individual's reasoning, making it useful to investigate an individual student's system of meanings. At the same time, Thompson (2000) makes clear that a cognitive perspective can also be used to accont for social interactions when viewing individiuals in social settings as interpreting others' actions and thereupon basing their own actions on those interpretations.

In this chapter, I explain the theoretical perspective that has informed my research design and discuss how the perspective informs the research design and analysis. I start by discussing Von Glasersfeld's (1995) theory of radical constructivism and central principles of Piaget's genetic epistemology (1971, 1977), which Von Glassersfeld built upon. I then elaborate on what I mean by an individual student's system of meanings and how the cognitive perspective informs this study.

Radical Constructivism

Thompson (2000) explained that the constructivist perspective operates as a foundation upon which theories in mathematics education can be built. Different theories in mathematics education serve different purposes and frame the type of research that is done, what questions are asked, and the role the researcher plays in the process of data collection and analysis (Thompson, 2000). Radical constructivism frames that research with a goal of understanding individual student's actions.. Radical constructivism is based on two principles. The first is that knowledge is not passively received, but instead constructed within the person's mind. The second is that cognition serves to organize the individual's experiential world (as they understand it in the moment) and adapts over time in response to new experiences (Glasserfeld, 1995). Additionally, how a researcher interprets the observed actions and utterances is based on the researcher's own cognitive structures. Researchers can, at best, offer models of other's mathematical reasoning and must be careful to distinguish between their own meanings and their student's hypothesized meanings, being careful not to impute their meanings to the students.

I am interested in understanding and building models of individual student's system of thinking related to how they understand and interpret graphs and coordinate planes. For this purpose, I adopt a perspective of radical constructivism. Based on Piaget's genetic epistemology, Von Glasersfeld (1995) explained that from a constructivist perspective, knowledge is constructed in an individual's mind as they encounter new experiences, form actions (both mentally and physically), and reflect on the outcome of those actions. Von Glasersfeld further explained that these encounters occur in the person's experiential world, based on how they perceive the situation and not on "things in the world" that have an independent existence. Von Glaserfeld explained that to Piaget, interaction is the cognitive subject interacting with their understanding of the objects and situation (both physical objects and mental). The cognitive subject does so based on their existing (previously constructed) cognitive structures. That is, they are not interacting with objects "as they really are" in the world, but instead based on their concetions and perceptions. Since their cognitive structures are individual and internal to the subject, research cannot aim to nor achieve to state exactly what the subject's cognitive structure consists of. Rather the goal is to build a model of the potential system of thinking which explains observable actions and utterances.

Radical constructivism makes sense as a foundational construct for this study since the goal of this study is to describe how individual students understand and interpret graphs. The student's system of thinking would be understood as the result of the reflections on the outcomes of their actions in previous problems involving graphs, individually constructed through the student's interactions, and idiosyncratic to the student. While it is impossible to know exactly what is in a student's system of thinking, the goal is to create a second-order model of student thinking which explains the student's mathematical activity (Steffe & Thomson, 2000).

In a first-order model, a person interprets a conversation based upon their own meanings without attempting to understand the conversation from the perspective of the other person in the conversation. In a second-order model, the person attempts to understand the conversation through the lens of their tentative understanding of the other person’s meanings (and not their own meanings). Researchers aim to explain the mental operations of someone which explains how they interpret the problem and their work in resolving the problem. Researchers who adopt radical constructivism as a base for their theoretical perspective attempt to create second-order models of student thinking. The models often make predictions on how the student might respond when presented with a new situation (Steffe et al, 1983). The researcher can then gather more data, test the viability of the model, and adapt the model based on how well the current model explains the new data.

Piaget’s Genetic Epistemology and the definition of an individual’s meanings

Three constructs from Piaget’s genetic epistemology (assimilation, accommodation, and schemes) are foundational to defining what is meant by individual student meanings. To Piaget, these constructs serve as tools to describe how an individual (consciously or unconsciously) constructs new meanings, modifies existing meanings, and integrates those meanings within their current meanings.

A person's meaning is how the person understands a situation, combined with immediate implications of that understanding in their thinking. A person has an initial assimilation of a situation (an understanding), and that understanding implies actions or images within the person’s cognitive system. Thompson et al. (2014) described stable meanings as the implications that result from having assimilated to a scheme. In other words, the student makes sense of a situation by making connections with (or triggering) a scheme. Actions result from the triggered scheme (possibly both mental and physical actions), along with expectations regarding the result of the action. If the expectation is met, the new situation is assimilated into and becomes a part of the preson’s understanding. Thompson et al. (2014) described a *stable understanding* as the result of assimilation to a scheme. However, if the expectation of result is not met, the student experiences cognitive dissonance which might result in an accommodation to the scheme or the creation of a new scheme. A functional accommodation is an accommodation to a scheme that occurs in the context of using it (Steffe, 1991). With a functional accommodation the student has coordinated their existing schemes in a new way or made a new distinction within the existing scheme in the moment of using it. This type of coordination often is not permanent and is what Thompson and Harrel call an *understanding in the moment* (Thompson et al., 2013). *Understandings in the moment* explain how a student might accommodate without learning. A *stable understanding*, in contrast is the result of a metamorphic accommodation—a permanent accommodation to their schemes that the student can use to assimilate future situations. In this proposed research it is not possible to know if students' work and answers are a result of in-the-moment meanings or stable meanings. It is possible that different questions trigger different schemes, resulting in the student engaging in different actions. It is also possible that the understanding the student used to make sense of one problem was not stable and would not be accessible to assimilate the new situation. For the purpose of this study, when I attend to a student's system of meanings I am referring to their in-the-moment meanings that explain the observable actions, work, and utterances.

A scheme is a complex structured organization of triggers to the scheme, images, operations, and expectations of outcomes generated by the triggered actions. Meanings are understood as the collection of actions and expectations of results from the actions. Meanings, as such, are built over time in the student's mind through the student's repeated interpretations and actions. One goal of cognitive research is to be able to explicate a model of potential student thinking. Student thinking is idiosyncratic, and as such we can only develop models which explicate the way the student might be thinking which accounts for the observed student work and utterances on a task. There will be aspects of the student's system of thinking which are not observable in the student's work or utterances. The goal is to create a model which fits what is observed in the student's work and utterances and explains the students' behavior.

It is important to investigate how a student interprets a task in order to create a model of their meanings. How the student understands the task might not be what the researcher intended. To create a model of the student's thinking and to draw inferences from the analysis of the student's work, it is important to base it on how the student understood the task and not on what was intended by the creator of a task (if they are not the same). A good task is one that is open-ended so that the student's work gives evidence of the student's interpretation of the task and provides insight into the student's meanings regarding the task through their work and conclusions.

A student, for example, could have assimilated the meaning "$f(x)$ is just y" into their scheme for functions and "the curve is $f(x)$" into their scheme for graphs of functions. Since the curve represents the function, expressions such as $f(0)$ and $f(5)$ would likely be interpreted as locations on the curve. In interpreting a statement such as "$f(0) < y < f(5)$", a student with those meanings would likely interpret it as meaning the curve between the locations $(0,f(0))$ and $(5,f(5))$. In contrast, the numbers 0 and 5 in the expression $0 < y < 5$ are not represented in function notation. Since 0 and 5 are not expressed in function notation, the student interprets the second statement as the curve (since y is f(x)) between the heights of 0 and 5. The student's meaning for f(0) and f(5) as locations on the graph can be used to potentially explain the reason why the student's interpreted the statement involving function notation as referring to part of the curve between two locations and the other as referring to part of the curve between specific heights.

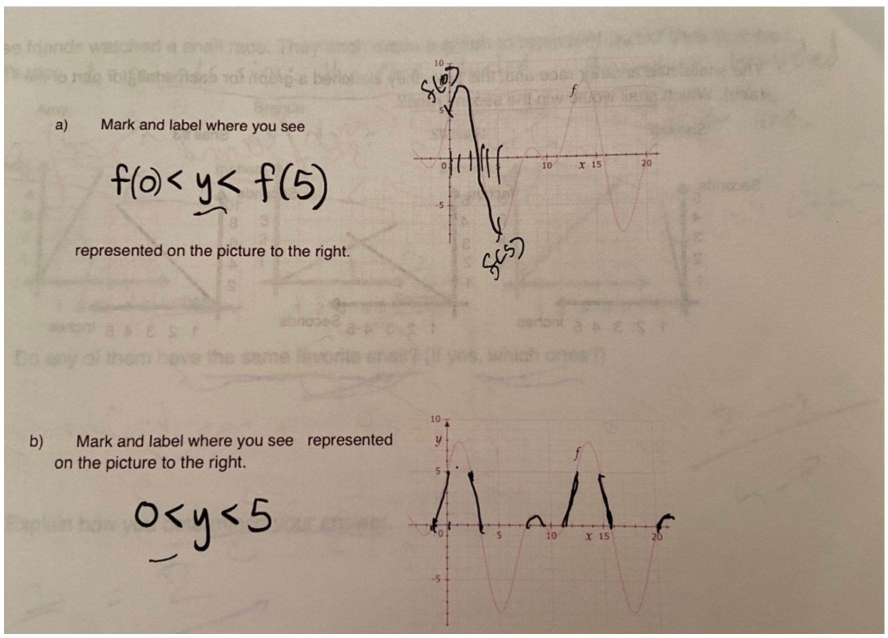


Figure 8. A student indicating on the graph where they see the statements $f(0) < y < f(5)$ *and* $0 < y < 5$.

Domain-specific theories

The role of background theories, such as radical constructivism, is to frame the type of research questions we ask and the role of the researcher in collecting and analyzing the data to answer the questions. Radical constructivism as an epistemological stance does not offer insight into what are productive meanings for the student to have regarding graphs of functions and coordinate planes. Instead, domain-specific theories serve that role and help the researcher to describe student thinking and distinguish between ways of thinking that are productive or prohibitive for the student's future mathematical learning (Thompson, 1991, 2002). For this research study, I am building upon many existing constructs which explain different aspects of student thinking in order to investigate the student's system of meanings. I leverage existing constructs for variational and covariation reasoning and coordinate points.

Quantitative and Covariational Reasoning

A *quantity* exists in the mind of the person perceiving of the *quantity*. A quantity is a measurable attribute of some object in the individual's reality, combined with a unit of measurement and a method to quantify the attribute (Thompson, 1993). *Quantitative reasoning* is the individual's quantitative structure (in their reality) that organizes the quantities and relationships between quantities that they perceive in the situation. The quantitative structure exists in the mind of the person conceiving the quantities.

Covariational reasoning refers to the mental actions an individual engages in when they coordinate the variation of the values of two quantities they conceived of. Like quantitative reasoning, covariation reasoning is about how the individual conceives of the situations, quantities, and how the quantities' values vary in relation to one another (Carlson, Jacobs, Coe, Larson, & HSU, 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). How the student conceives of quantities and the relationship between their values and how their values vary has implications on how they understand functions and graphs of functions.

Two people could consider the same problem and conceive of different quantities (or none at all) and different relationships. Covariational reasoning constructs provide tools for researchers to describe the student's mental actions regarding the quantities in their models of the student's thinking. Carlson et al. (2002) described a framework for describing covariational reasoning for students engaging in dynamic function tasks. Thompson & Carlson (2017) described a framework that expanded upon variational reasoning about quantities' values. Both frameworks describe the reasoning for students who conceive of quantities. Since conceiving of a quantity in a situation is idiosyncratic, it is possible that a student does not conceive of a quantity when reasoning about the graph of a function. The student might not be coordinating any values (or variation in values) but instead reasoning about locations within the plane when reasoning about the coordinate points (David et al, 2019).

Castillo-Garsow (2013) discussed two constructs for understanding students variational reasoning (*chunky variational reasoning* and *continuous variational reasoning*). The difference between chunky and continuous variational reasoning is in how the student perceives of the value of the variable as varying. A student conceiving of *continuous variational* reasoning conceives of a quantity whose value is varying by varying through all the numbers on the interval. I understand Castillo-Garsow means all the real numbers on the interval when describing continuous variational reasoning. A student whose image of the coordinate plane or number line includes only marked values would then be unable to reason continuously about the value of one of the varying quantities. Chunky variational reasoning is imaging the value of the quantity as varying one chunk at a time, like measuring a length by laying a ruler multiple times against it. They are aware that any time they lay the ruler, the ruler has numbers between the endpoints, but they do not envision the object having intermediate lengths associated with intermediate values on the ruler. Thompson and Carlson (2017) also described discrete variational reasoning (where the student thinks about the value of the variable as varying by taking on discrete values), and situations where the student does not imagine the value as varying at all. Thompson and Carlson's covariational framework extends their variational framework by describing how the student coordinates the variation of each quantity. Their framework includes thinking about variation one quantity at a time (without coordinating the variations), coordinating completed changes in one quantity's value with completed changes in the other quantity's values (one corresponding change at a time), and students imaging the sustained continuous covariation.

Both Carlson et al. and Thompson and Carlson's frameworks are based on how the student's coordinate values or changes in values, which would require that the students conceive of values. Not all students reason with values. As noted by David et. al (2019), some students are reasoning based on spatial location. The covariational frameworks could be extended to describe thinking in which the students are coordinating spatial locations and not values. If the student conceives spatial locations, then "change" could be understood as going to the next location. The curve could be seen as indicating movement to another location. A directional change could be understood as getting higher or lower in the plane while moving from point to point, and reasoning about variation would likely be limited to perceived visual aspects of the curve.

Table 3

Proposed Co-variational Framework

No variation	**The student does not imagine the values of the variables as varying. The student might coordinate values of variables but does not imagine the values of the variables as varying between the coordinated values.**
Single variable variation	The student imagines the values of one or more of the variables as varying but does so one variable at a time and does not coordinate the variations.
Location Coordination	Covariational reasoning based upon spatial location and directional movements in the plane (up, over, left, right)
Discrete Coordination of values	Covariational reasoning based on comparing values of discrete coordinate pairs.
Unit Coordination of changes (additive amounts)	Covariational reasoning based on coordinating directional changes in one variable's value with unit changes in the other variable's value. To reason about values within a unit the student would require re-unitizing to smaller units.
Continuous/ Chunky Continuous Covariation	Covariational reasoning based on coordinating simultaneous continuous (or chunky continuous) changes in one variable's value with corresponding continuous (or chunky continuous) changes in the other variable's value.

Research has shown that covariational reasoning is vital for productive meanings for understanding ideas in calculus (Kaput, 1994; Thompson 1994a, Thompson & Carlson, 2017). For this reason, it is foundational for my conceptual analysis of a productive system of meanings for coordinate planes and graphs of functions. It is also important that it is not assumed that the student has conceived of quantities or constructed a particular quantitative structure in analyzing student responses. One way to investigate the student's reasoning is to ask questions which have follow-up questions within the same situation to reveal information about how the student understood the situation. Additionally, item validation is needed to refine the tasks so that student answers reveal information about the student's thinking.

CHAPTER 4

METHODOLOGY

To investigate a student's system of meanings I developed a collection of questions for an assessment given to a large group of students. Student answers to each question individually provided insights about the student's meanings and collectively they can be analyzed to give insight into the student's system of meanings. Comparing the collection of answers for each student allows me to investigate patterns and relationships in different student's system of meanings.

For the students' answers to provide insight into their thinking, the questions needed to be validated through clinical interviews. The goal of the clinical interviews is to determine how students interpret the questions, and what meanings led students to their interpretations.

In this chapter I discuss

- Methodology for the study
- Item design principles
- Data Collection Protocols
- Summary of Participant Information

Combining qualitative and quantitative data

Two of the possible purposes for mixing qualitative and quantitative research are having the results complement each other, and using the results from one method to develop the other method (Schoonenboom & Johnson, 2017). Developmental mixing refers to mixing methods at the design stage (as opposed to complementary mixing which mix the methods at the analysis stage). Development refers to using the results from one method to help develop or inform the other method. For the purpose of this study, the qualitative analysis serves to both compliment the quantitative analysis and in the development of the questions and coding of the answers for each question. Complementarily mixing the methods refers to using the results of one method to elaborate, enhance, illustrate, or clarify the results of the other method. In this study the results of the qualitative analysis of the items that is a result of the item validation process is used to clarify the meanings of the relationships between students answers in the quantitative analysis.

Qualitative analysis in the development of the items serves to develop and inform the results of the quantitative analysis. Analyzing the percent of students who chose an answer gives insight into student meanings collectively because the items were developed through a design, analyze, redesign cycle. Questions developed in the assessment were designed to reveal students' meanings, with the collection revealing a student's scheme of meanings for graphs. The results from analyzing the clinical interviews produced constructs for classifying students' responses on the quantitative assessment. In particular, the results of the interview analysis produced constructs for classifying student's meanings for a graph (for example, the student's image of covariation, static or emergent shape thinking, is the student conceiving of measurements or locations, …). The analysis of my data happened in two phases. During the item analysis phase student interview data and written work was analyzed to look at if students arrived at the same answers based on similar reasoning . From the results of this analysis, a summary of indicated meanings for each item was developed. From the summary of meanings an indicator variable for each item was developed. The indicator variable scored each item as a 0 or a 1. A score of 1 meant that the student answer indicated they reasoned with the more productive meaning the item was written to investigate. A score of 0 meant that the students reasoning was likely not the productive meaning. These variables were used in the quantitative analysis in two ways. One way of investigating reltionships between meanings in students systems was to identify students whose answers comnsistenly indicted that the student reasoned with a particular meaning. The percent of the students with constent reasoning on other items were then compared with the other students answers on those items. Evidence of a relationship was inferred when the students with consistent reasoning were more likely to also have scores of 1 then the other students. The other way of investigating relationships was through confirmatory factor analysis.

Initial Item Design

In order to investigate student thinking the items were designed within three distinct question sets. The purpose of the question set is to gain insights into students in-the-moment thinking about graphs of functions and coordinate planes. The order of the sets of questions is progressive. Initially, questions do not suggest any method of representing the quantities graphically. Questions which include a blank coordinate plane the student is asked to graph on are included next. Finally, questions which include graphs of functions are included at the end of the collection of items. The order of the items was to make sure the questions themselves did not prompt students to draw graphs when they would not have thought ot on their own.

Set #1 – Tasks without any coordinate plane

These tasks were designed are to understand students thinking about a quantity's value and variation of a quantity's value separate from graphing and to determine if the student thinks about points in a coordinate plane as a way to represent a relationship between the values of two quantities. These questions need to precede questions where a graph or coordinate plane are in the question in order to isolate thinking related to the number line/ graph from the student's thinking separate from a graph.

Set #2 – Tasks with a coordinate plane/ number line, but not graphs of any functions and animation questions (Emergent Graphing GC-File, views 3,8,9).

These tasks were designed to investigate how students think about graphs related to coordinating values or locations of covarying quantities. In these tasks the student is asked to graph on a provided coordinate plane. Questions using coordinate planes other than the standard cartesian coordinate planes are needed to model if the student is reasoning based on expected shapes of functions or through coordinating values of covarying quantities.

Animation questions were included to investigate how the students construct a graph to represent two covarying quantities. In the large data collection the students were first given a static picture that showed the starting moment of the animation and asked to represent the information they see in the static picture on the provided graph using the red pen. Once the students had recorded the information they saw in the static picture on the graph they were asked to switch to the blue pen to record the information they saw in the animation.

Set #3– Questions with static graphs.

These tasks were designed to investigate how a student thinks about a function graph in a coordinate plane. ~~These tasks were designed to reveal how students reason~~

~~about a graph.~~ As in set #2, questions using coordinate planes other than the standard cartesian coordinate planes are needed to determine whether the student is reasoning based on shapes of functions or through coordinating values of covarying quantities. Additionally, one question displays an animation and asks the student to select the graph that represents how two quantities in the animation are varying together.

Clinical Interviews

Clinical interviews (Clement, 2000) often consist of a series of questions or tasks that the interviewer asks the student to work through. The goal of clinical interviews is to characterize the student's (in the moment) meanings. The tasks in clinical interviews are usually open-ended and students in clinical interviews are asked to think aloud as they work through the tasks. Follow-up questions are asked by the researcher to help clarify the student's meaning or for the researcher to test their current hypothesis about how the student is thinking. Clinical interviews are ideal for item validation since the goal is to characterize student thinking, not to influence it.

Item Validation and Evidence of Student Thinking

This section describes the item validation process and provide evidence from student work in support of the coding scheme for the items that is used in the quantitative part of the study. In a pilot study, initial items were given in a test format to inform edits to the items and to inform potential follow-up questions during the clinical interviews on items where student responses were unclear about the student's thinking. Clinical interviews were then conducted on items in the design, redesign cycle to revise the items, provide evidence of student meanings that the item can give insight into, and to develop an initial coding scheme for the quantitative part of the study.

Group Administration of items in Validation phase.

Items were administered to the students in a test format in intermediate algebra classes summer 2022 and fall 2022. There were 7 students from summer 2022 and 14 students from Fall 2022. Students answered the items independently. Written instructions asked the students to show work/ explain their answers. The goal of the group administration during the validation phase was to determine which items to include in the first round of clinical interviews. Student written responses where the students meaning were unclear were used to adjust items and to inform follow-up questions during the clinical interviews.

Validation with Clinical Interviews.

The 7 students interviewed were volunteers from calculus 1 class in fall 2022. The clinical interviews aimed to investigate how the student understood the questions and the meaning they employed in determining their answers. The clinical interview data was analyzed to look for consistency in student interpretation and the student's meanings which which were indicated by specific answers, develop a summary of meanings for each item, and adjust items which student answers did not reliably indicate student meanings. The items were written to investigate particular meanings, so the focus of the analysis was on if the item reliably indicated if the student was (or was not) reasoning with the particular meaning.

Group Administration for Quantitative Data Collection Phase

Once the instrument was complete and a summary of meanings was created, the collection of questions were administered to a large group of students. Invitations to participate in the study were sent via email to students in precalculus and calculus courses at ASU in Spring 2023. A total of 41 students participated in the quantitative data collection. There were four collection times, each which lasted 90 minutes.

Protocols for Validating and Administering Items

Group implementation of items in the validation phase.

During the initial development of questions, a total of 21 students in intermediate algebra classes were given a collection of questions on paper which they completed independently in a classroom setting. Student responses and explanations were compared to look for patterns in their responses and meanings. Questions where students arrived at the same answers but displayed different reasoning were updated. Questions where student reasoning was unclear were used to inform follow-up questions for the subsequent clinical interviews.

Clinical interviews.

Clinical Interviews (Piaget, 1975; Clement, 2000) are interviews with the goal of investigating an individual's thinking. The questions and tasks utilized in a clinical interview are meant to probe the student's (in-the-moment) thinking and the limitations of that thinking. Tasks in clinical interviews usually include open-ended or think-aloud problem-solving tasks. Clement stated that one benefit of clinical interviews (over other types of interviews) is that the open-ended format of clinical interviews allows the researcher to gather data on the student's meanings and "hidden mental structures and processes".

Protocol for clinical interviews

The goal of the clinical interviews for item validation is to collect data on how the student understood the question, the meanings they utilized while working on the task, and to look for consistency in student meanings amongst students who concluded the same answer. Students were asked to think aloud about what they understood the question is asking. Follow-up questions included asking the student to explain how they arrived at their answer, if they noticed something non-conventional about the graph (and if it would change their answer), or an extension question to aid the researcher in understanding the student's thinking that led to their response.

The interviews were recorded via an app on a tablet which links speech to what the student wrote on the tablet. Both the audio recording and the student work were analyzed to look for consistency in how the students' understood the question, and consistency in the student's answers/meanings and to develop a summary of student meanings indicated by student answers to the questions.

The instrument was considered complete when a summary of meanings for each item was developed for which different answers to the same item indicated different meanings and there were several items which would give insight into the same meanings to look for students who consistently reasoned with a particular meaning in the quantitative analysis.

Collecting the Quantitative Data

A total of 41 students from a large public southwestern university completed the written survey. These subjects were recruited by an invitation from their classroom instructors and were given the option to attend one of four 90-minute collection times. ~~The data was collected in a classroom setting with four collection times each consisting of different students for 90 minutes. All students who volunteered to participate were invited to one of the data collection times.~~

The students completed the cover page after signing the consent form and were asked to wait until instructed before turning the page. Asking the students to wait to turn the page was to make sure that all students were given the same instructions before answering an item. The cover page prompted them to provide their name, current class, if they primarily attended school prior to college in the US or outside the US, and number of years they had attended the university, and the first two items. Twenty-four additional assessment questions followed, with instructions for each question, including four items involving animations, appearing on a power point which was presented as the students were instructed to answer the next question. Students were asked to show their work and/or provide a written explanation of the thinking they engaged in when answering the question.

Almost all of the participants were in their first or second year at the college. 30 of the students were international students and the remaining students had most of their mathematics education within the US.

Table 45
Class Levels of Participants

Class Level	Number of Students
Pre-Calculus	9
Calculus 1	17
Calculus 2+	15

Coding the Quantitative Data

Items were designed by utilizing existing constructs from current literature on ways of understanding graphs, coordinate points, and coordinate planes. The goal was to determine what type of item students with a particular way of thinking would necessarily answer differently than a student with a different way of thinking. For example, on a nonstandard coordinate plane with one orientation switched a line with positive slope would appear to go downwards from left to right. Students whose reasoning was based on the steepness of a line would likely answer differently when asked about increasing or decreasing of the output variable then a student whose reasoning is based on the values of the variable.

To investigate whether a student's reasoning about the graph of a function is based on the shape of the function's graph, the question to answer was "what type of item would a student who reasons based on shape necessarily answer differently than a student who is reasoning with values or variations in values?". The items that were designed for this purpose either used a coordinate plane that is not the standard cartesian coordinate plane or a non-standard orientation. In order to investigate the student's reasoning from different perspectives, some items involve the student graphing on the non-standard coordinate plane and some items involve the student answering questions based on a graph of a function in a non-standard coordinate plane. One limitation of these tasks is that I cannot distinguish between students who are reasoning with the quantities' values from those who are reasoning by comparing variations in the quantities' values.

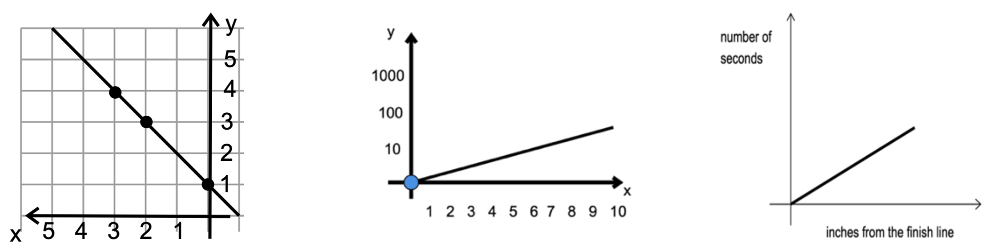


Figure 11. Examples of Graphs of functions on coordinate planes which are not the standard cartesian coordinate plane.

The items which involve the students graphing in a cartesian coordinate plane are utilized to investigate both if the student conceives of a coordinate pair as a way to represent a multiplicative object as well as to investigate how (and whether) they coordinate two covarying quantities. Some items involve students creating or identifying a graph that represents the coordination of the values of two covarying quantities in a dynamic situation shown in an animation and some involve the students graphing a given function in a non-standard coordinate plane. If the student imagines the graph as emerging through the coordination of values they should select a graph which is not a function of the variable represented on the horizontal axis. If the students graphing is limited to recognizing shapes or the expectation of starting on the y-axis and graphing to the right then they should select the second graph.

To investigate the student's variational and covariational reasoning, the question to answer was “what type of item would a student who reasons discretely necessarily answer differently than a student who is reasoning continuously?”. The items that were designed for this purpose investigate if the student is thinking about the curves as a collection of ordered pairs or if they think about a line differently. For example, asking a student which graph contains more coordinate pairs where one graph consists of a collection of dots and the other graph consists of a line. Additional items ask the student how many numbers or coordinate pairs are represented in a graph. One limitation is I am not able to distinguish if students are thinking about numbers discretely because they are unaware there are additional numbers or if their reasoning is just focused on a particular subset. Additionally, students who state they are thinking about an infinite collection of numbers might not be thinking about all real numbers. It can only be indicated that they are thinking about numbers as being in between the marked values on each axis. There are additionally several questions which can indicate the student is thinking about variation as replacing the value of the variable with a new value (without thinking about the values in between).

CHAPTER 5

QUALITATIVE ANALYSIS

This chapter overviews the analysis of the qualitative data, which includes

- Examples of the process of validating the items
- Summary of meanings for each item
- Table of indicator variables used in the quantitative analysis

Examples of Item Validation

This section overviews four examples of the item validation process. The process involved developing an item, administering the item to students, analyzing the student's work or utterances, editing the item, readministering the item to new students, and analyzing again; for this stud,y items were written to see if students were reasoning with a particular meaning or a different meaning. The item was considered ready when students who appeared to be reasoning in a particular manner gave the same answer, and students who reasoned differently gave a different answer.

Example 1

Item 1
How many numbers are between 4 and 9?

Item 2
x changed from 3 to 8; how many different numbers was x?

In the initial version of item #1, students who were thinking of the whole numbers between 4 and 7 and those who were thinking about replacing 4 with 7 would state two numbers on the interval. For the clinical interviews, I adjusted the question (interval from 4 to 9) so that the two ways of understanding would come up with different responses. In the interviews, there were two different reasons students would get 5 (if they think about an operation or if they think about chunks). Table 4 outlines students' four distinct ways of thinking in the collected written work and clinical interviews. For this study, the items were used to indicate whether a student is thinking about an infinite or discrete collection of numbers.

Table 4
The four distinct ways of reasoning for items 1 and item 2

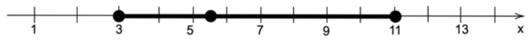
Response	Student meaning
2	The student only thinks about the numbers stated (4 and 9). Possible indication the student is thinking about variation as replacement.
5	Possible indication the student is thinking about the interval as a collection of chunks (pieces of a stick). Also, possible the student is thinking about the result of an operation.
4 or 6	An indication that the student is thinking of whole numbers (might or might not include the endpoints). A possible indication that the student is thinking about variation discretely.
Infinite	Indication the student understands an interval as containing more than whole numbers. Potential indication of continuous thinking.

Table 5
Indicator Code #2 for item 1 and item 2

Response	Code
The response includes a mention of infinite numbers.	1
The response is a finite value	0

Example 2

How many values of x are-indicated in the picture below?



This item was designed to investigate whether students think about the marked values only (where the dots are), the labeled values, the integer values (at the tick marks), or all real numbers. Some students interpreted the illustration as representing two intervals, some as the three values marked by the dots, some as representing the whole numbers between 3 and 11, and some as representing all real numbers between 3 and 11.

One complication was students writing several answers. Students who mentioned both infinite and a finite answers were coded as a 1. Students who only gave a finite answer were coded as a 0.

Table 6

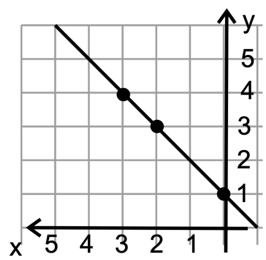
Meanings of different responses for item 3.1

Student response	Meaning
2	Students were thinking about 2 "intervals." In the interviews, students who were thinking about subintervals were sometimes thinking about whole numbers only but also sometimes thinking about numbers between the whole numbers.
3	Students were thinking about x =3, x = 5.5, and x = 11
9	Students were counting the whole numbers from 3 to 11.
14	Students counted the whole numbers on the number line (not looking at the graphed line or dots).
infinite	Students were thinking about numbers other than whole numbers

	(possibly decimals, possibly real numbers)

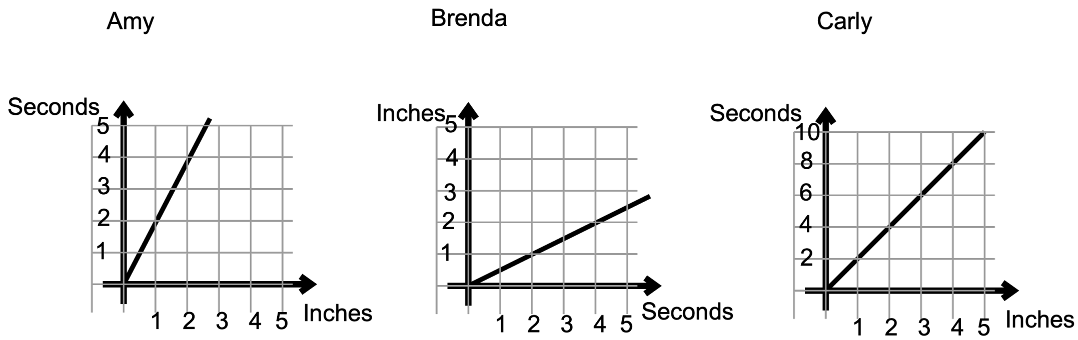
Example 3

Is y increasing or decreasing as x increases? Explain how you determined your answer.



Students answered that y was decreasing for a few reasons. One was that the downward shape meant that y was decreasing. A second was a comparison of y values but with x decreasing as if restricted to moving in the rightward direction and comparing only the y values (despite writing ordered pairs for both). Students answering x increases either referenced ordered pairs (written in an order where the values of x were increasing) or described that x increasing was a leftward direction and that y increased when they moved in that direction. When I asked for clarification in interviews, they explained y's value increased (many drawing triangles along the curve). Some of the students in the interview described imaging what the ggraph would be in a standard coordinate plane. An answer of increasing requires that the student makes the determination by a reason other than the shape of the graph.

Example 4



3 friends watched a snail race. Each friend chose their favorite snail and made a graph of part of that snail's race. Is it possible that any of them had the same favorite snail? Explain how you determined your answer.

Students limited to reasoning based on the shape of the graph would determine that no two friends had the same favorite snail. All three graphs represent the same relationship between inches and seconds. Students who compared coordinate pairs determined that Amy and Carly had the same favorite snails. Some students determined that Amy and Brenda had the same favorite snail by stating it is the same graph with the axis switched. Most seemed to have determined that by comparing two or more coordinate pairs. In the interviews, all students compared the graphs by calculating either inch per second or seconds per inch and comparing those values. Some did not notice Brenda's axis was different and stated that only Amy and Carly had the same favorite snail. When I asked if the axis made a difference, all of them stated it would and said all three had the same favorite snail. The question was simplified to include only the graphs of Amy and Brenda. This was done so that the question would not take as much time and because the third graph did not give more information about student thinking than when the question was administered with only two graphs.

Summary of Meanings

Table 7 summarizes the different meanings that were identified (in either the clinical interviews or written work) and the questions in which those meanings appeared to be utilized by students.

Table 7
Summary of Meanings for Items.

Indicated Meaning	Description or reasoning.	Questions
Reasoning with Discrete Collection of Numbers or locations.	Student thinking is focused on a discrete collection of values or locations. The student might be aware that there are more numbers, but those numbers are not considered in their reasoning in the same way as the discrete collection of numbers or locations. Examples of this thinking are reasoning focused on marked values or dots in the plane. Students thinking this way do not consider the line an infinite collection of coordinated values.	1, 2, 3a, 4, 6b, 7, 10b, 11, 14a, 14b, 23b, 26b
Reasoning with numbers in-between marked values.	Student's reasoning includes thinking about numbers in between marked values. The student may be thinking about all real numbers or a subset of the real numbers, for example, thinking about 'decimals' in between.	1, 2. 3a, 4 , 6b, 7, 10b, 11, 14a, 14b, 23b, 26b
Reasoning about variation as replacement.	Reasoning about variation as x was one number, which was replaced with another number without having been any other number.	2, 3a, 4, 14b,
Reasoning about variation as the amount of change.	Reasoning about variation by how much you have to add to get to the second number. The student's reasoning might focus on computation or how far you move to get to the second number.	1, 2
Representing coordinated values as a coordinate point.	Reasoning about coordinate points as the uniting of values of two quantities. Examples include drawing a coordinate point to represent a given coordination of values.	3b, 5, 7, 8a, 8b, 9, 18b, 21a, 23a, 24

Reasoning based on the shape of the graph.	Reasoning about the graph of a function based on the shape of the graph, for example, how steep the line appears, or up means y is increasing.	6a, 10a, 12, 13, 14, 15, 16, 17, 18a, 19, 20, 21b, 22a, 22b, 26a, 26c
Reasoning with values or variation values.	Reasoning about the graph of a function involves reasoning about coordinated values or coordinating changes in values. Examples of students reasoning with values or changes in values would be students reasoning a function's graph based on coordinate points instead of the shape of the graph.	6a, 8a, 8b, 9, 10a, 12, 13, 15, 16, 17, 18a, 19, 20, 21b, 22a, 22b, 24, 25, 26a, 26c
Reasoning about locations	The student's reasoning is based on location or relative location in the coordinate plane. Any thinking about values is second in the reasoning, with their reasoning focused on spatial locations.	21a, 23a, 24, 25

Summary of Meanings for each item

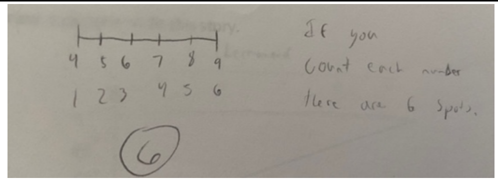
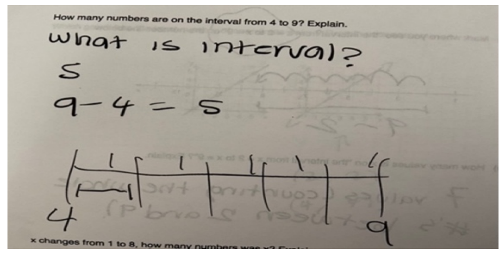
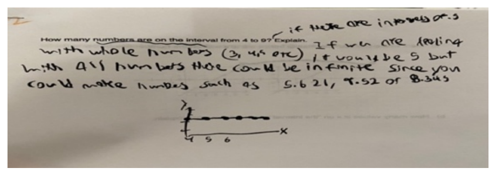
This section goes through each item and details the meanings observed in student work or utterances during the item validation phase. The items are in the order they were in the quantitative analysis.

Set #1 Tasks without any given number lines or coordinate planes.

Question 1
How many numbers are between 4 and 9?

Table 8
Summary of Meanings From the Qualitative Analysis for Question 1

Student Answer	Indicated Meaning	Example

Finite number other than 5 (Example 4 or 6)	Discrete Thinking. The student is thinking about a discrete collection of numbers. They might (or might not) be aware of other numbers, but they are reasoning with the discrete collection of numbers.	
5	Thinking of addition or variation. In this case, the student is not thinking about the numbers in between; their reasoning is based on computation. For example: "From here to here, I moved 5".	
Infinite or	The student is aware there are infinite numbers, but they might only reason with the discrete values.	
Infinite.	The student's reasoning includes thinking about an infinite collection of numbers. It is important to note that students thinking about an infinite collection of numbers might not be thinking about all real numbers.	

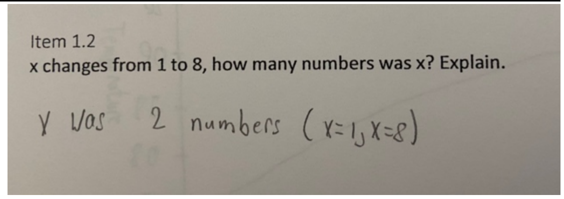
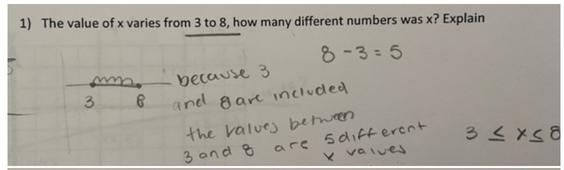
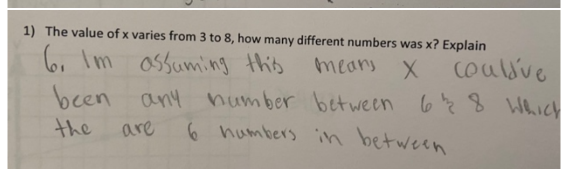
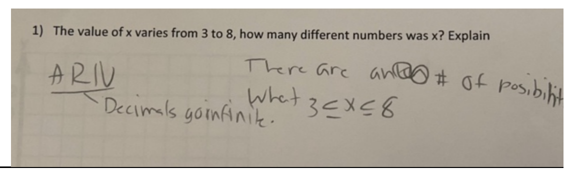
Question 2

x varied from 3 to 8; how many different numbers was x?

Table 9

Summary of Meanings From the Qualitative Analysis for Question

Student Answer	Indicated Meaning	Examples

2	Replacement. The student's image is that x was 3, then it was replaced by 8 without thinking about x's value as taking any number in between.	 Item 1.2 x changes from 1 to 8, how many numbers was x? Explain. X was 2 numbers (x=1, x=8)
5	The student is thinking about the amount of change or a computation.	 1) The value of x varies from 3 to 8, how many different numbers was x? Explain 8 − 3 = 5 3 8 because 3 and 8 are included the values between 3 and 8 are 5 different x values 3 ≤ x ≤ 8
4 or 6 (depending on if they include the endpoints)	Discrete Thinking. The student is thinking about a discrete collection of numbers. They might (or might not) be aware of other numbers, but they are reasoning with the discrete collection of numbers.	 1) The value of x varies from 3 to 8, how many different numbers was x? Explain 6, Im assuming this means x could've been any number between 6 & 8 which the are 6 numbers in between
Infinite	Potentially continuous thinking.	 1) The value of x varies from 3 to 8, how many different numbers was x? Explain ARIV Decimals go infinite. There are an ∞ # of posibilit what 3 ≤ x ≤ 8

Question 3

I woke up at 8 am and went to bed 14 hours later (at 10 pm). The temperature displayed on my thermometer in my backyard was 75 degrees when I woke up and 95 degrees when I went to bed.

3a

a) How many different temperatures was it in my backyard between when I woke up and when I went to bed?

Table 10

Summary of Meanings From the Qualitative Analysis for Question 3a

Student Answer	Indicated Meaning
2	Replacement
20 or a Finite number	Amount of change or discrete Thinking. The student is thinking about a discrete collection of numbers. They might (or might not) be aware of

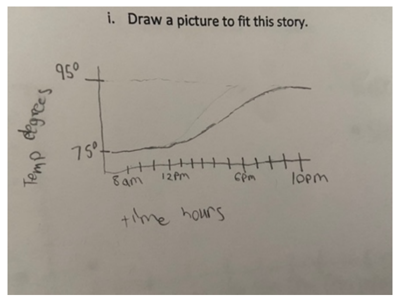
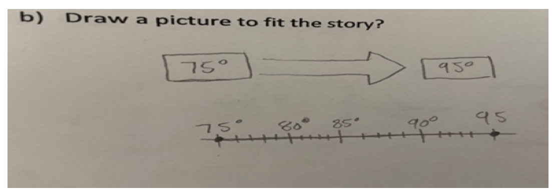
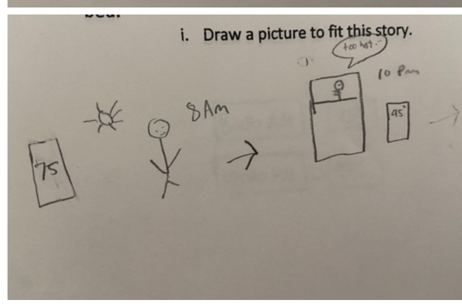
	other numbers, but they are reasoning with the discrete collection of numbers.
Infinite	Potentially continuous thinking.

3b

b) Draw a picture to illustrate the story.

Table 11

Summary of Meanings From the Qualitative Analysis for Question 3b

Student Answer	Indicated Meaning	Examples
Draws a graph in the coordinate plane.	The student thinks about coordinate points in the coordinate plane to represent a relationship between two quantities. A potential indicator that the student conceives of a multiplicative object.	
Draws a number line.	The student is thinking about the variation of one quantity and thinks about the number line as a way to represent that variation.	
Draws any other type of picture.	The student does not think of a coordinate plane as representing a relationship between two quantities.	

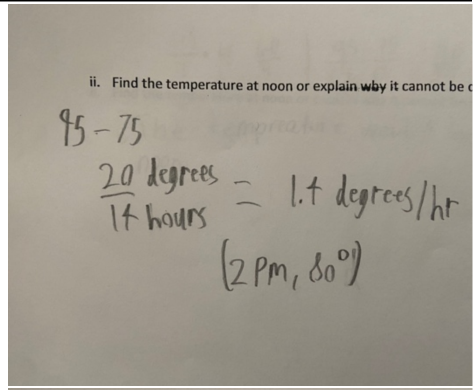
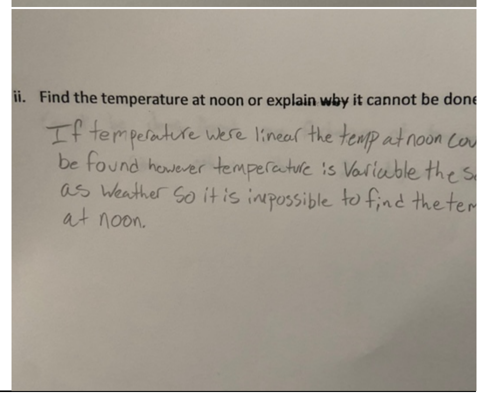
3c

c) If possible, find the temperature at noon. If not possible, explain why.

Table 12

Summary of Meanings From the Qualitative Analysis for Question 3c

Student Answer	Indicated Meaning	Examples

Computes using an assumed CROC.	Student's reasoning focuses on an expected computation.	
States not possible to know/ compute.	Student reasoning involves reasoning about how the values of the quantities covary.	

3d

d) Circle the correct statement:
   i. The temperature only increased while I was awake.
   ii. The temperature might have decreased at some point while I was awake.

Table 13

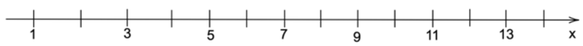
Summary of Meanings From the Qualitative Analysis for Question 3d

Student Answer	Indicated Meaning
i	The student understood the relationship as strictly increasing.
ii	The student's conception of the quantities involves thinking about how the values covary over some period of time and that temperature's value fluctuates over time.

Set #2 Tasks that ask the students to graph on a given coordinate plane.

Question 4

a) Mark where you see "the interval from x = 2 to x = 9" on the number line below.



b) How many values of x are represented on the interval you drew? Explain.

Table 14

Summary of Meanings From the Qualitative Analysis for Question 4

Student Answer	Indicated Meaning
2	The student is thinking about a replacement. x was 2, and then that was replaced with x = 9.
Other Finite number	Discrete Thinking. The student is thinking about a discrete collection of numbers. They might (or might not) be aware of other numbers, but they are reasoning with the discrete collection of numbers.
Infinite	Potentially continuous thinking.

Question 5

On a cold night at the backyard bonfire, John noticed that when he was 2 feet from the bonfire, the temperature was 8 degrees Celsius. Use a single mark to represent the information below.

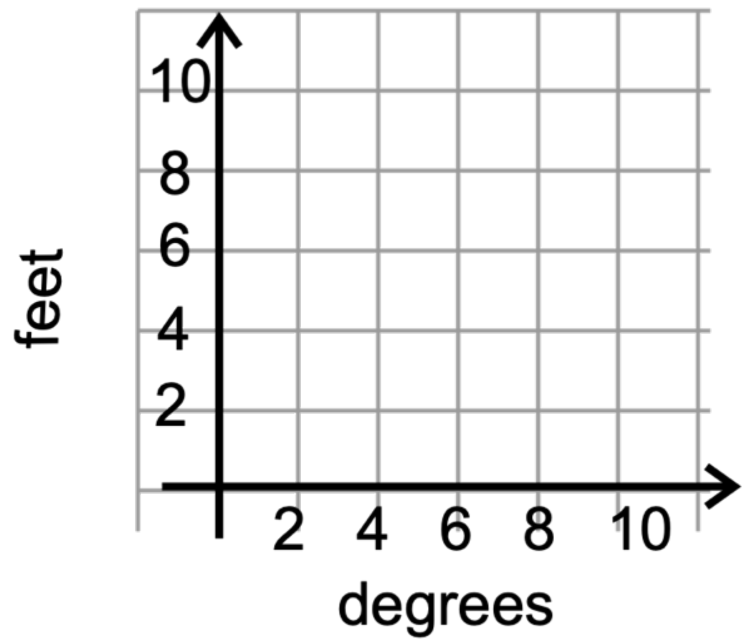
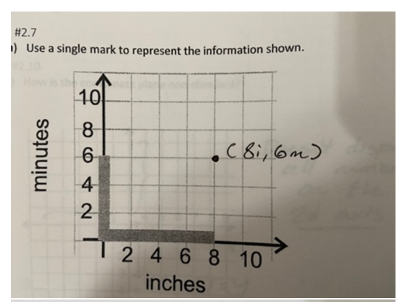
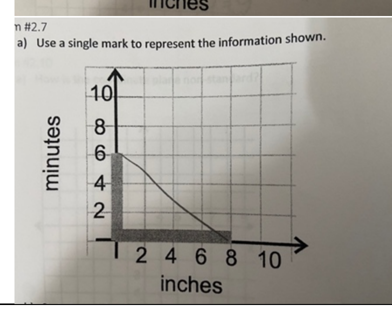


Table 15

Summary of Meanings From the Qualitative Analysis for Question 5

Student Answer	Indicated Meaning	Example
Marks a coordinate point.	The student conceives of a multiplicate object and conceives of a coordinate point as a way to represent the coordination of values.	
Does not mark a single coordinate point.	The student does not conceive of a coordinate point as a way to coordinate the values. It is possible the student does not conceive of a multiplicate object or does not conceive of a coordinate point to represent a multiplicative object.	

Question 6

6a

a) Graph y = 4x on the coordinate plane below.

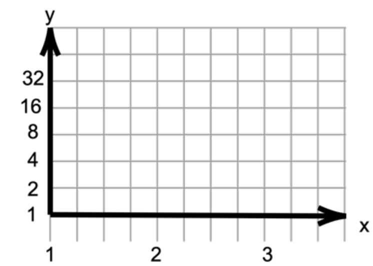
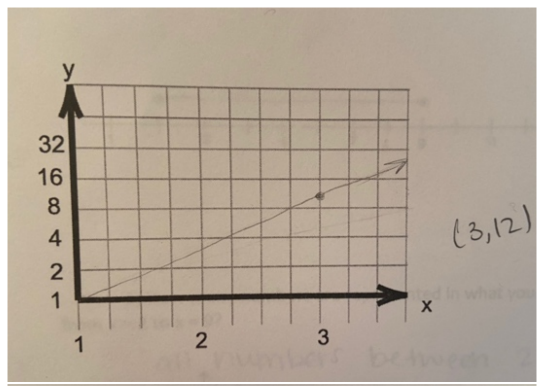
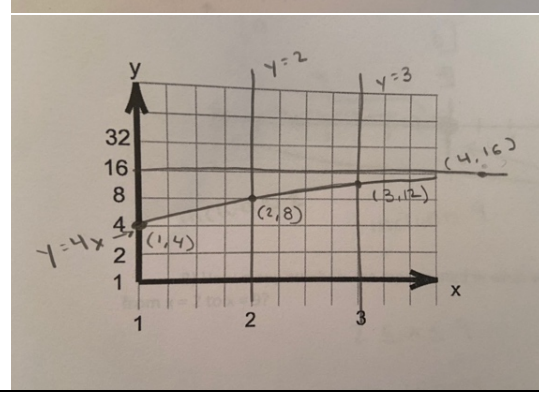


Table 16

Summary of Meanings From the Qualitative Analysis for Question 6a

Student Answer	Indicated Meaning	Example
Draws a line (or points that lie on a line)	Indicator of shape thinking. Linear functions have the shape of a line.	
Draws a curve (or points that lie on the curve)	The reasoning is based on coordinating values relative to the values on the axis.	

6b

How many coordinate pairs are on the graph of the function you drew?

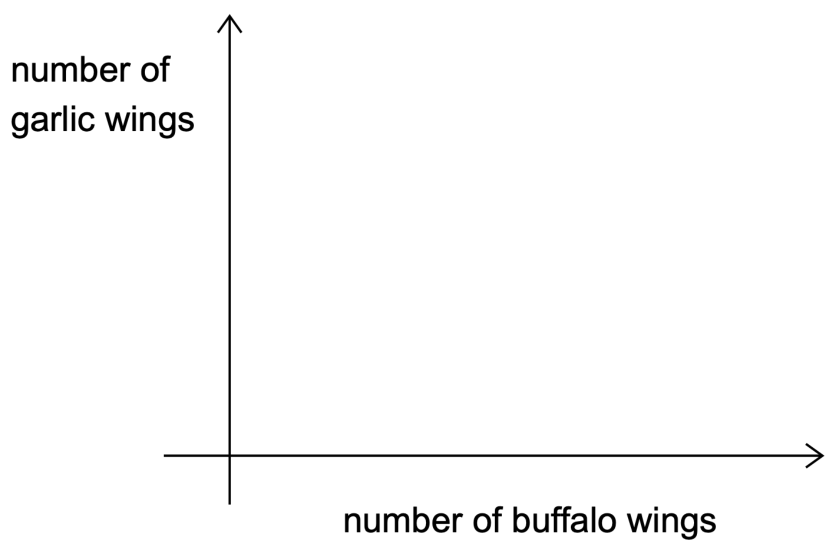
Table 17

Summary of Meanings From the Qualitative Analysis for Question 6b

Student Answer	Indicated Meaning
Finite number (Example 4 or 6)	Discrete Thinking. The student is thinking about a discrete collection of numbers. They might (or might not) be aware of other numbers, but they are reasoning with the discrete collection of numbers.
Infinite	The student thinks of the graph of the line as consisting of an infinite collection of ordered pairs.
Students graph consists of a discrete collection of points	Discrete covariation or discrete coordination of values. The student is coordinating individual coordinate values.

Question 7

There need to be 3 buffalo sauce chicken wings for every 1 garlic chicken wing ordered. Sketch a graph to represent the possible orders of chicken wings.



y

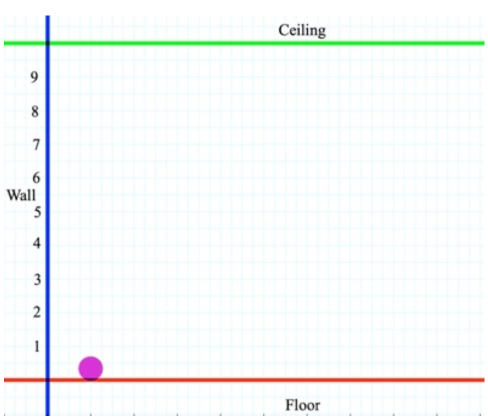
Table 18

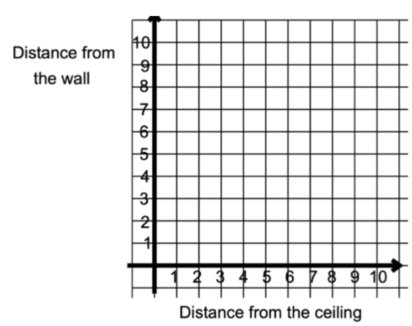
Summary of Meanings From the Qualitative Analysis for Question 7

Student Answer	Indicated Meaning
Draws the line y = 1/3 x	An indicator that the students meaning for the line are not an infinite collection of coordinate pairs. Since there cannot be fractional chicken wings, the students meaning for the line is not an infinite collection of ordered pairs representing the possible orders. The student's Meaning of the line cannot be inferred from this question.
Draws a discrete collection of coordinate points on the line y = 1/3x	An indicator that a graphed line includes an infinite collection of coordinated values to the student. If the student understands the line as containing an infinite collection of ordered pairs, then the student would not draw a line through the points because you cannot have fractional chicken wings in the order.
Draws a line from one axis to the other.	The student does not think of a coordinate point as a way of representing coordinated values. The line does not represent a coordination of values of two quantities.

Question 8a (Animated item)

a. You want to sketch a graph of the relationship between the distance the ball is from the ceiling and the distance the ball is from the wall.
    a. On the graph below, use the red pen mark and label the information you see in the moment of the ball's path seen below.



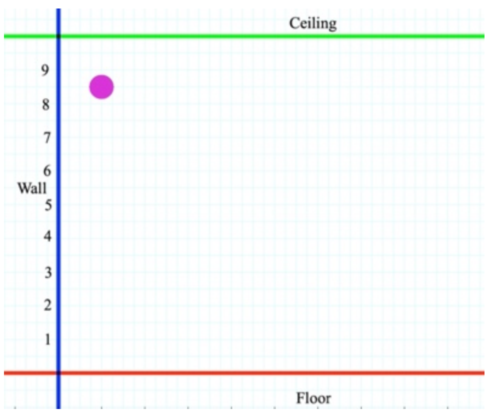


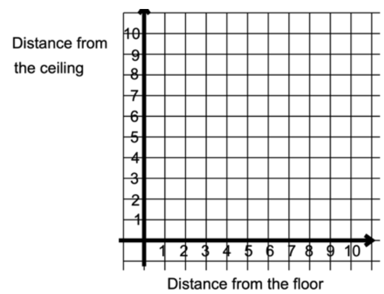
b. On the same graph, using the blue pen, as the ball moves sketch the relationship between the distance the ball is from the ceiling and the distance the ~~snail~~ ball is from the wall.

Question 8b (Animated item)

You want to sketch a graph of the relationship between the distance the ball is from the floor and the distance the ball is from the ceiling.

c. On the graph below using the red pen mark and label the information you see in the moment of the balls path seen below.





d. On the same graph, using the blue pen, as the ball moves sketch the relationship between the distance the ball is from the floor and the distance the ball is from the ceiling.

Table 19

Summary of Meanings From the Qualitative Analysis for Question 8a/b

Student Answer	Indicated Meaning
The initial mark is a coordinate point in the middle of the plane.	A possible indicator that the student conceives of a multiplicative object and a coordinate point in the plane as a way to represents the multiplicative object.
The initial mark is a coordinate point on the y-axis	A possible indicator of graphing is a sensorimotor experience that involves starring on the y-axis and graphing to the right.
The initial mark is not a coordinate point.	The student does not conceive of a multiplicate object or does not conceive of a

	coordinate point as a way to represent the multiplicative object.

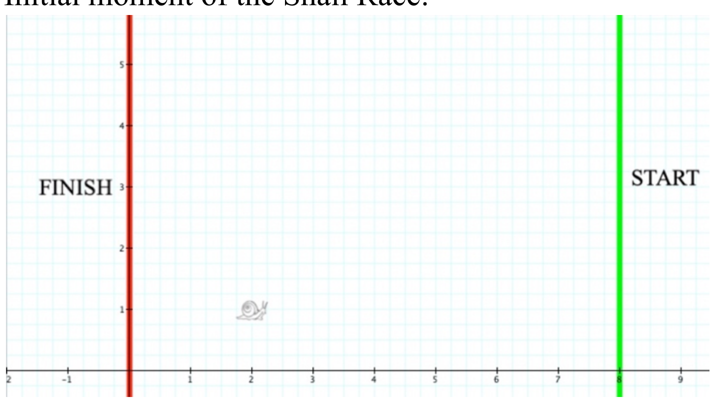
Student Answer	Indicated Meaning
Graphing activity involves graphing to the right only.	A possible indicator of graphing is a sensorimotor experience that involves starring on the y-axis and graphing to the right.
Graphing activity involves graphing to the left or up.	A possible indicator of the student thinking of the graph emerging through the coordination of values.
Other Graphing Activity?	

Question 9 (Animated Item)

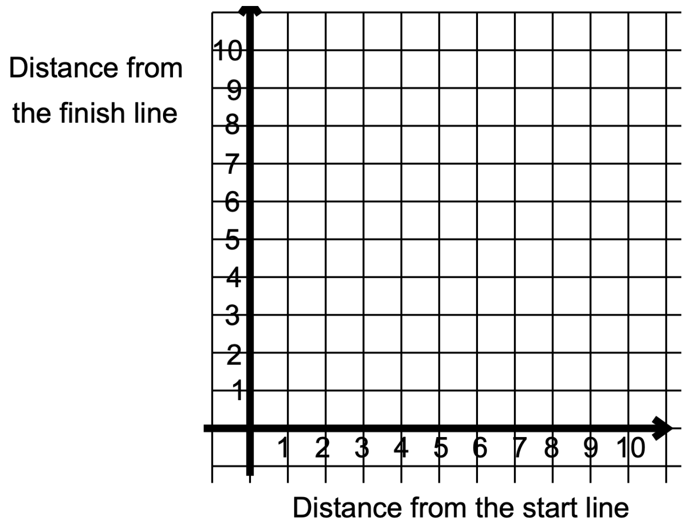
You want to sketch a graph of a relationship between the distance the snail is from the start line and the distance the snail is from the finish line.

a) On the graph below, label the information you see in the moment of the snail race shown in the picture using the red pen mark.

Initial moment of the Snail Race:



Graph:



b) On the same graph using the blue pen, as the snail race plays, sketch the relationship between the distance the snail is from the finish line and the distance the snail is from the start line.

Table 20

Summary of Meanings From the Qualitative Analysis for Question 9

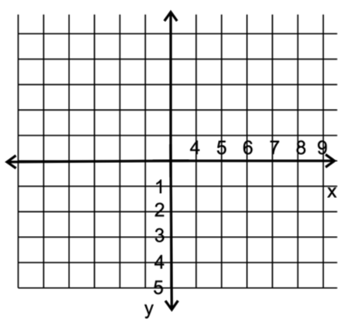
Student Answer	Indicated Meaning
The initial mark is a coordinate point at (6,2)	A possible indicator that the student conceives of a multiplicative object and a coordinate point in the plane to represent the multiplicative object.
The initial mark is a coordinate point on the y-axis	A possible indicator of graphing is a sensorimotor experience that involves starring on the y-axis and graphing to the right.
The initial mark is not a coordinate point, or the student marks the snail's location.	The student does not conceive of a multiplicate object or does not conceive of a coordinate point as a way to represent the multiplicative object.

Student Answer	Indicated Meaning
Graphing activity involves graphing to the right only.	A possible indicator of graphing is a sensorimotor experience that involves starring on the y-axis and graphing to the right.
Graphing activity involves graphing to the left or up.	A possible indicator of the graph emerging through the coordination of values.

Student sketches a line showing the snail's path or other graphing activity.	The student does not think of a coordinate point as a way to coordinate the values of two quantities.

Question 10

a) Graph y = 2x - 4 on the non-standard coordinate plane below.



b) How many coordinate pairs did you draw in your graph?

Table 21

Summary of Meanings From the Qualitative Analysis for Question 10

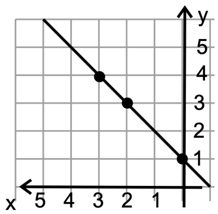
Student Answer	Indicated Meaning (Part a)
The graph is drawn upwards.	The student's Meaning of positive slope is that its graph moves up.
The graph is drawn down.	The student coordinate values relative to the axis to graph the function.

Student Answer	Indicated Meaning (Part b)
Finite Number	The students are thinking about a discrete collection of numbers.
Infinity	A potential indicator that the student thinks about covariation continuously.

Set #3: Tasks with static number lines or coordinate planes

Question 11

Which graph of which function has the least number of coordinate pairs (x, y)?



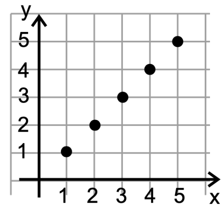
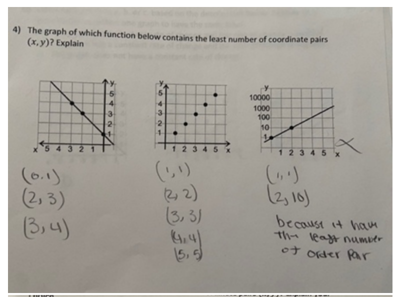
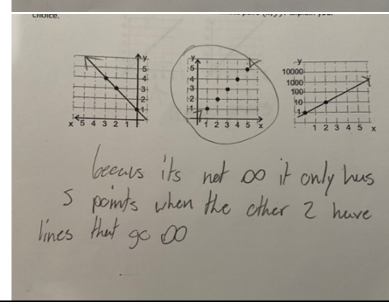


Table 22

Summary of Meanings From the Qualitative Analysis for Question 11

Student Answer	Indicated Meaning	
Right Graph.	Discrete Thinking. The student is thinking about a discrete collection of numbers. They might (or might not) be aware of other numbers, but they are reasoning with the discrete collection of numbers.	
Left Graph	The student is thinking about the line as consisting of coordinate pairs.	

Question 12

Is y increasing or decreasing as x increases? Explain how you determined your answer.

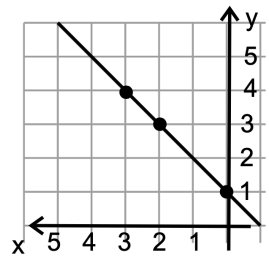
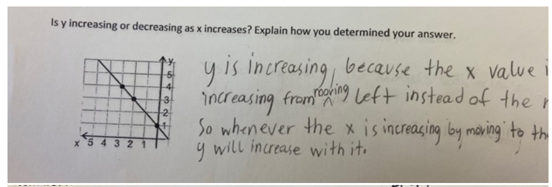
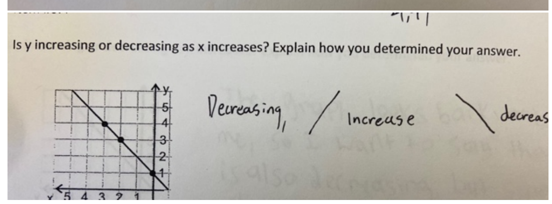
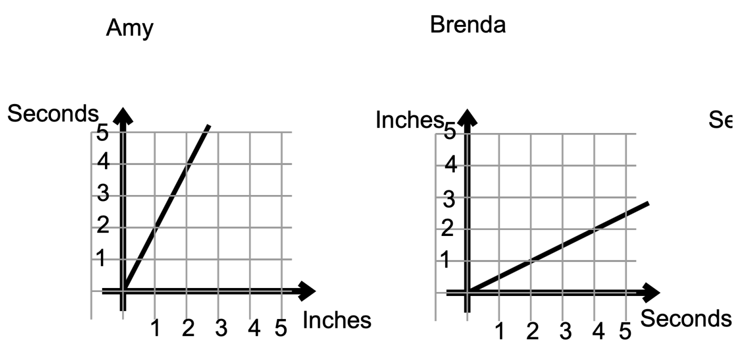


Table 23
Summary of Meanings From the Qualitative Analysis for Question 12

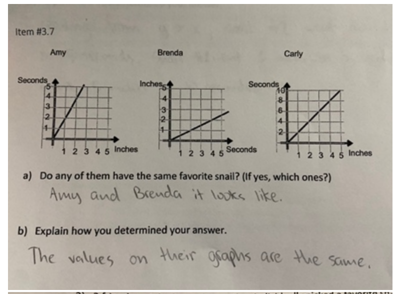
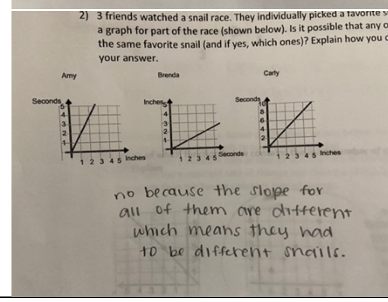
Student Answer	Indicated Meaning	Examples
Increasing	The student is reasoning about values or changes in values.	
Decreasing	A potential indicator of shape thinking. The student is reasoning about the shape of the graph; since the graph slopes down, they respond decreasing.	

Question 13



Two friends watched a snail race. Each friend chose their favorite snail and made a graph of part of that snail's race. Do they have the same favorite snail? Explain how you determined your answer.

Table 24
Summary of Meanings From the Qualitative Analysis for Question 13

Student Answer	Indicated Meaning	Examples
Yes	The student is coordinating amounts of change or values.	
No	A possible indicator of shape thinking. The student is reasoning based on the shape of the graph. To this student, the steeper incline means that the snail is moving faster.	

Question 14

a) How many values of x are being indicated in the picture below?

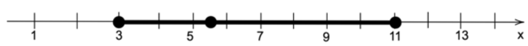
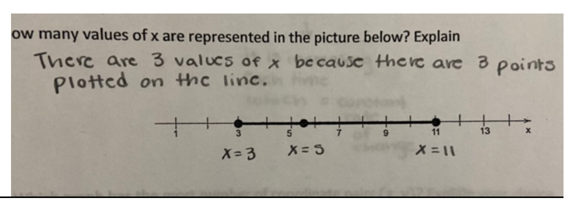
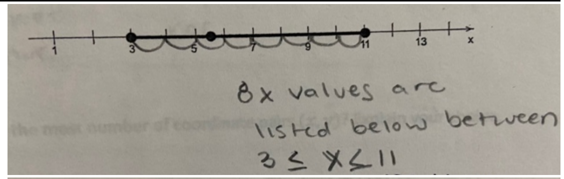
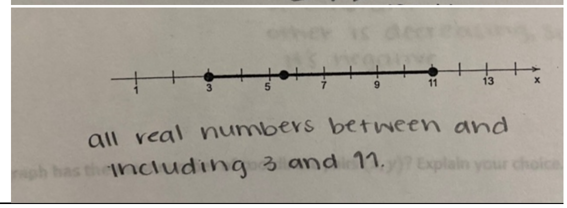


Table 25
Summary of Meanings From the Qualitative Analysis for Question 14a

Student Answer	Indicated Meaning	Examples
3	The student is thinking about the three marked dots. Indicator of discrete thinking.	

<table>
<tr><td>Other finite number.</td><td>The student is thinking about a discrete collection of values.</td><td>
</td></tr>
<tr><td>Infinite</td><td>The student is thinking about values other than the marked ones on the number line.</td><td>
</td></tr>
</table>

b) The value of x is represented on the number line. x varied from 3 to 11; how many values was x?

Table 26

Summary of Meanings From the Qualitative Analysis for Question 14b

Student Answer	Indicated Meaning
2	The student is thinking about a replacement. x was 3, and then that is replaced by 11.
Other finite number.	An indicator shows that the student is thinking discretely.
Infinite	A possible indicator is that the student thinks about variation continuously.

Question 15

The snails readied for another race, and, this time, one friend sketched a graph predicting how each snail would run. Which snail would win the second race, according to these graphs? Explain how you decided.

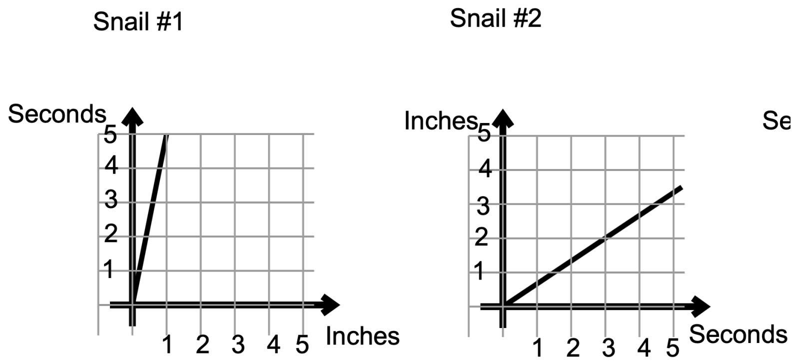


Table 27
Summary of Meanings From the Qualitative Analysis for Question 15

Student Answer	Indicated Meaning
Snail 1	The student's reasoning is based on the steepness of the graph. Since it is steeper, the snail is faster and will win.
Snail 2	The student is reasoning about the values of variation in values to determine that the snail in graph 2 either travels faster or goes farther in the same time.

Question 16

The graph of the function below shows the function f, which gives the number of seconds in the race as a function of the number of inches the snail is from the start line. Did the snail speed up or slow down? Explain.

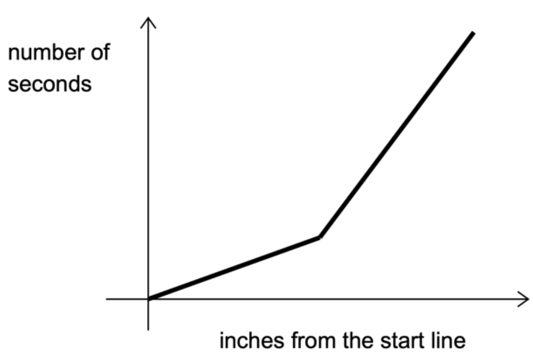
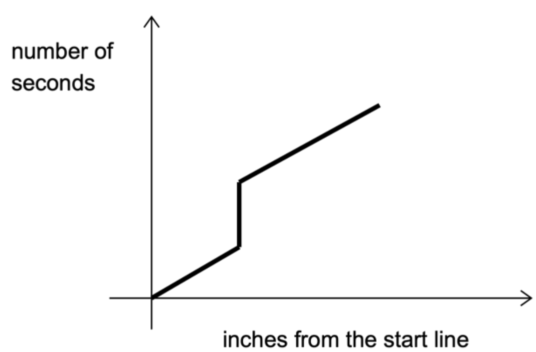


Table 28
Summary of Meanings From the Qualitative Analysis for Question 16

Student Answer	Indicated Meaning
Speed up	The student's focus is on the steepness of the two line segments. Since the second is steeper, they interpret it as moving faster, and the snail sped up.
Slow down	The student is reasoning about variation in values. In the first line segment, the snail goes more inches for the same number of seconds, and in the second. Because the graph does not have any numbers for the student to reason that the snail slowed down, they have to reason about the variation of one quantity for a fixed amount of variation for the other quantity.

Question 17

A snail starts a race, pauses for a bit, and continues the race.
Which graph shows the snail's race?



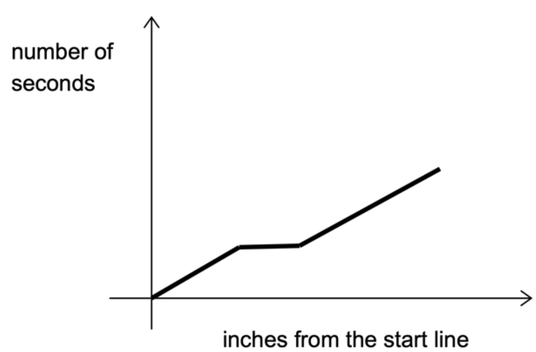


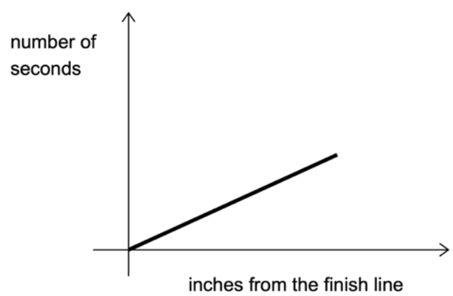
Table 29

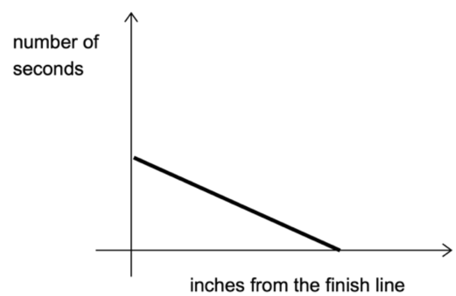
Summary of Meanings From the Qualitative Analysis for Question 17

Student Answer	Indicated Meaning
Graph 1	To select Graph 1, the student would need to think about the snail pausing, implying that while time increased, the number of inches stayed the same, leading to the vertical line segment in the graph.
Graph 2	The student might select graph 2 because the 1st graph is not a function or because they interpret the horizontal line segment as the expected shape for when the snail stopped.

Question 18

a) You want to graph function f, which gives the number of seconds in the race as a function of the number of inches the snail is from the finish line. The graph of which function below could be the graph of the race?





b) On the graph you choose for part a, mark and label where the start of the race is indicated.

Table 30

Summary of Meanings From the Qualitative Analysis for Question 18

Student Answer	Indicated Meaning (part a)
Graph 1	An indicator that the student selects a graph that looks like they expect it to. A possible indicator of shape thinking.
Graph 2	A possible indicator that the student is coordinating values or the two quantities.

Student Answer	Indicated Meaning (part b)

The marked point on the y-axis.	A possible indicator of location thinking. The student expects the start of the race to be indicated on the y-axis.
The marked point with time = 0.	An indicator that the student is coordinating values of two quantities and thinks of a coordinate point to represent the coordination.
Does not mark a coordinate point.	An indicator that the student does not conceive of a multiplicative object.

Question 19

Which graph could represent the snails race in the Animation?

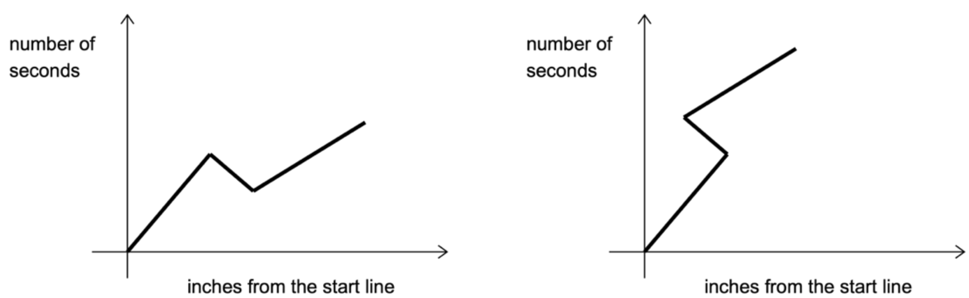


Table 31

Summary of Meanings From the Qualitative Analysis for Question 19

Student Answer	Indicated Meaning (part a)
Graph 1	An indicator that the student selects a graph that looks like they expect it to. A possible indicator of shape thinking.
Graph 2	A possible indicator that the student is coordinating values or the two quantities.

Question 20

Assume the graphs for the two snail races are drawn on the same scale; which snail won the race? Explain.

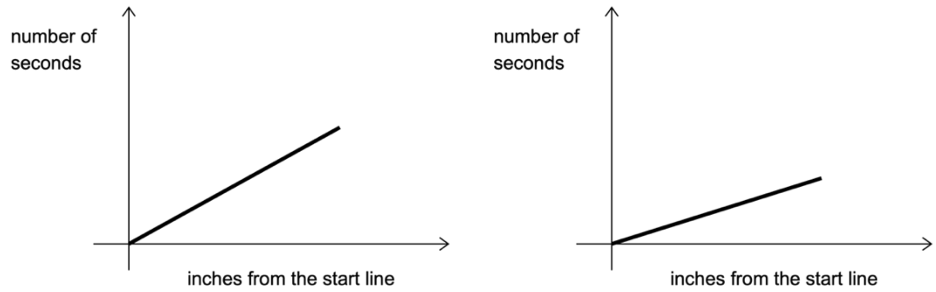
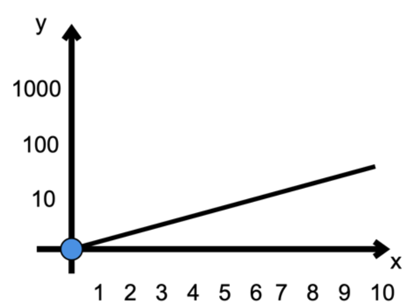


Table 32
Summary of Meanings From the Qualitative Analysis for Question 20

Student Answer	Indicated Meaning (part a)
Graph 1	A possible indicator of shape thinking. The student's reasoning is likely that the snail would win because the graph is steeper.
Graph 2	A possible indicator that the student is coordinating values or the two quantities. The student's reasoning is likely that for the same inches, the snail took less time, so they would win the race.

Question 21
Consider the graph of the function below.



a) Label the blue dot.

Table 33
Summary of Meanings From the Qualitative Analysis for Question 21a

Student Answer	Indicated Meaning
(0,0)	The student understands the intersection of the axis as (0,0) regardless of the labels on the axis. A possible indication of location thinking. The student's reasonings about the coordinate point are based on the spatial location at the intersection of the two axes.
(0,1)	The student is reasoning about the exponential scale on the y-axis to determine the y-coordinate. The student is reasoning about the values on the axis.
Not a coordinate point.	A possible indicator that the student does not conceive of a multiplicative object or does not think of coordinate points as a way to represent a coordination of values.

b) Which best describes this function?

   i) Liner function with an equation of $f(x) = mx + b$ with fixed values m and b.
   ii) Exponential function with equation $f(x) = a(b)^x$ with fixed values a and b.

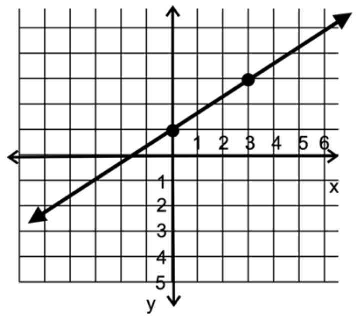
Table 34

Summary of Meanings From the Qualitative Analysis for Question 21b

Student Answer	Indicated Meaning
i	Indicator of shape thinking. Since the graph looks like a line, it must be a linear function.
ii	The student is coordinating values or changes in values.

Question 22

Consider the graph of the function below.



a) Which is the correct form of the equation for the line in the graph?

   a. $y = \frac{2}{3}x + b$

   b. $y = -\frac{2}{3}x + b$

Table 35

Summary of Meanings From the Qualitative Analysis for Question 22a

Student Answer	Indicated Meaning
Positive slope	The reasoning is based on over and up. The student's reasoning is based on spatial changes and not coordination of values or changes in values.
Negative slope	Computed by coordinating values or changes in values. The student's reasoning is based on the values relative to the axis.

b) Does y increase or decrease as x increases? Explain.

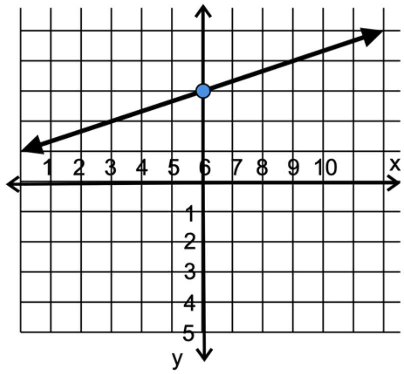
Table 36

Summary of Meanings From the Qualitative Analysis for Question 22b

Student Answer	Indicated Meaning
Increase	Student reasoning is focused on shape.
Decrease	Student reasoning involves coordinating values or changes in values.

Question 23

Consider the graph of the function f below.



a) Label the blue dot on the graph of function f using function notation.

Table 37

Summary of Meanings From the Qualitative Analysis for Question 23a

Student Answer	Indicated Meaning
Labels with an ordered pair.	Possible indicator the student thinks about coordinate
The label is not an ordered pair.	A possible indicator of location thinking.

b) How many coordinate points are on the graph of function f?

Table 38

Summary of Meanings From the Qualitative Analysis for Question 23b

Student Answer	Indicated Meaning
Finite number	The student is thinking of a discrete collection of points, for example, the labeled gridlines or points on the line at the intersection of gridlines.
Infinite	A possible indicator that the student is thinking about values in between those on the gridline. The student may also think about discrete values that continue infinitely because of the arrows.

Question 24

On the provided graphs, mark and label where you think is indicated by the given mathematical statement.

$$f(-5) < y < f(2)$$

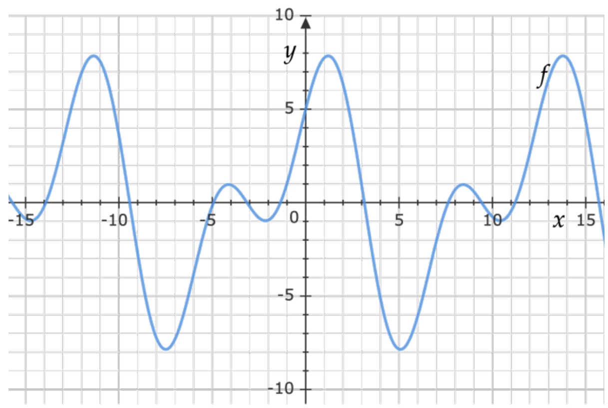


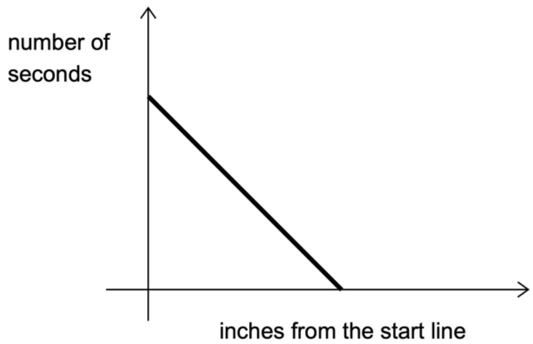
Table 39

Summary of Meanings From the Qualitative Analysis for Question 24

Student Answer	Indicated Meaning
Highlights curve between (-5, ***f***(-5)) and (2, ***f***(2))	The student understands *f*(a) as a location on the curve and y as the curve itself between the two points on the curve.
Indicates a region or piece of the y-axis.	The student understands *f*(a) as a value of y.
Marks some locations or points or other graphing activity.	I am not sure much can be drawn from this answer other than the student doesn't have a way of thinking about the inequality relative to a coordinate plane.

Question 25

The graph of the function below shows the function f, which gives the number of seconds in the race as a function of the number of inches the snail is from the start line.



Mark and label where you see the start of the race indicated on the graph.

Table 40
Summary of Meanings From the Qualitative Analysis for Question 25

Student Answer	Indicated Meaning
Marks where function intersects the y-axis	An indication that to the student, the graph starts on the y-axis.
Marks where the function intersects the x-axis.	Reasoning based on the quantity's values (race starts when the number of seconds is 0).

Question 26

Circle the best response for each graph.

a) Does the function vary at a constant rate of change with respect to x?
   a. Yes, the rate of change is constant and positive
   b. Yes, the rate of change is constant and negative
   c. No, the rate of change is not constant

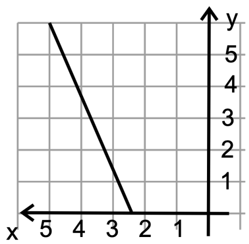


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Table 41

Summary of Meanings From the Qualitative Analysis for Question 26a

Student Answer	Indicated Meaning
a	The reasoning is based on coordinating values or changes in values.
b	The reasoning is based on the shape of the graph.
c	The student likely doesn't have a meaning for a constant rate of change/ guessed at an answer.

b) Does the function vary at a <u>constant rate of change</u> with respect to x?
   a. Yes, the rate of change is constant and positive
   b. Yes, the rate of change is constant and negative
   c. No, the rate of change is not constant

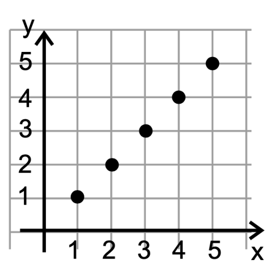

Table 42
Summary of Meanings From the Qualitative Analysis for Question 26b

Student Answer	Indicated Meaning
a	A possible indicator of discrete thinking.
b	The student likely doesn't have a meaning for a constant rate of change/ guessed at an answer.
c	A possible indicator of continuous thinking.

c) Does the function vary at a <u>constant rate of change</u> with respect to x?
   a. Yes, the rate of change is constant and positive
   b. Yes, the rate of change is constant and negative
   c. No, the rate of change is not constant

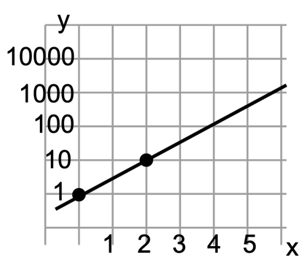

Table 43
Summary of Meanings From the Qualitative Analysis for Question 26c

Student Answer	Indicated Meaning
a	The reasoning is based on the shape of the graph.
b	The student likely doesn't have a meaning for a constant rate of change/ guessed at an answer.
c	Reasoning includes coordinating values or changes in values.

d) Does the function vary at a <u>constant rate of change</u> with respect to x?
   a. Yes, the rate of change is constant and positive
   b. Yes, the rate of change is constant and negative

c. No, the rate of change is not constant

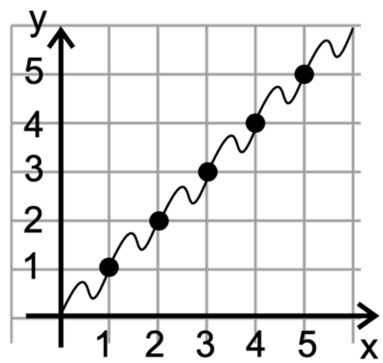

Table 44

Summary of Meanings From the Qualitative Analysis for Question 26d

Student Answer	Indicated Meaning
a	Students focus on the dots which lie on a line. The students meaning for the curve in between the line is not as an infinite collection of coordinate pairs. A possible indicator of discrete thinking. It is also possible the student's understanding of the constant rate of change is that there is a pattern.
b	The student likely doesn't have a meaning for a constant rate of change/ guessed at an answer.
c	It is possible the student's reasoning is that it is not a line, therefore, not a constant rate of change. It is also possible the student is thinking about covariation continuously. The student is reasoning about more than just the collection of dots on a line. However, they might be reasoning that the shape is not a line or reasoning about values.

CHAPTER 6

QUANTITATIVE ANALYSIS

In this chapter, I discuss two main results of the quantitative data collection and what the results seem to imply about the relationship between different ways of thinking about graphs. Additional relationships indicated by the data are included in Appendix __.

The chapter includes

- Data that indicates a relationship between understanding the graph of a function as containing an infinite collection of coordinate points and the student's meanings for a number line.
- Data that indicates a relationship between the student's ability to construct a graph to represent a dynamic situation and if a student reasons with values of a quantity when reasoning about a graph of a continuous function.

Analyzing the Quantitative Data

Quantitative data analysis focused on looking for relationships between the different meanings that were expressed. Items were written to identify a particular productive meaning. An indicator variable was used to score each item. A score of 1 meant that the student's answer was consistent with reasoning with the productive meaning. A score of 0 meant that the student's reasoning was based on a different way of thinking.

Groups of Consistent Reasoning

To investigate relationships, groups of students whose answers indicated they consistently reasoned with a particular meaning were identified. How those students answered other items was then investigated by comparing how students in that subgroup answered other items with how students not in the subgroup answered that item. Table 45 outlines the different subgroups identified and the criteria used to identify the students in the subgroup.

Table 45

Groups of Consistent Reasoning

Code	Description:	Number of Students	Criteria
CRD	Consistently Reasoning Discretely	31 (76%)	Discrete Responses on at least two-thirds of items 1, 2, 3a, 4, 14a, 14b
CRINCN2	Const. Reason Infinite Collection of Numbers	7 (17%)	Infinite Responses on both 14a and 14b
CRICCP	Const. Reason Infinite Collection of Cood. Points.	17 (41%)	Response indicates infinite CP on items 11 and 23
CRV	Const. Reason with Values	18 (44%)	Reasoning with values on items 13, 15, 16, and 20.
CRS	Const. Reason with Steepness	8 (20%)	Reasoning with steepness on both items 16 and 20.
CRVNSP	Const. Reason with values in non-standard CP	8 (20%)	Reasoning with values on at least all items 12, 21b, 26a, 26c.

CRSh	Const. Reason with Shape	10 (24%)	Reasoning with shape on at least three items 12, 21b, 26a, 26c.
CYR	Correct y-region	5 (12%)	Correctly identified the inequality as a range of y-values

Understanding a graph as an infinite collection of Coordinate Pairs (CRICCP)

Items 11 and 23b were written to investigate if student reasoning about graphs of functions focused on a discrete collection of coordinate pairs or if their reasoning consisted of thinking about an infinite collection of coordinate pairs. Students' responses to these items were used to investigate a relationship between the student's reasoning on the number line and the student's reasoning about the graph of a continuous function in a coordinate plane. Table __ summarizes the student responses.

Table 46

CRICN2 responses to items for indicating Infinite Collection of Coordinate Points

	Item 11	Item 23b
CRICN2	100%	100%
CRD	32%	35%

All seven students classified as CRICN2 also reasoned about graphs of functions as containing an infinite collection of coordinate points; in contrast, only a third of the students who consistently reasoned discretely reasoned about graphs as containing an infinite collection of coordinate pairs. In the sample (41 students), 20 of the student's responses to items 11 and 23b indicated they were thinking of an infinite collection of coordinate pairs. 17 of the 41 student's answers indicated they were thinking of an infinite collection of coordinate pairs for both items. If seven students were randomly selected from the 41 students, the probability that all seven students selected answered indicating that they were thinking about an infinite collection of coordinate pairs is 0.004. The probability of randomly selecting seven whose responses indicated they thought of an infinite collection of coordinate pairs for both items is 0.001. This is evidence that students who reason about an infinite collection of numbers on a number line are also likely to reason about the graph of a continuous function as containing an infinite collection of coordinate pairs. Additionally, students who reason about the number line as containing only whole numbers or marked locations on the number line are much less likely to think about the graph of a continuous function as consisting of an infinite collection of coordinate pairs.

Additionally, the data indicates relationships between how they interpret the inequality $f(-5) < y < f(2)$ in a coordinate plane, and if the student is thinking about the function's graph as containing an infinite collection of coordinate pairs or a finite set. Of the forty-one students, only five interpreted $f(-5) < y < f(2)$ as representing an interval of y values (CYR students). 80% of the CYR students were also CRICCP students, and the remaining CYR student answered one of the two items, indicating they were thinking about an infinite collection of coordinate pairs. Table ___ shows that students who understood the inequality as representing an interval of y-values are nearly twice as likely to have answers which indicate they are thinking about an infinite collection of coordinate pairs than other students. The probability of randomly selecting five answers from the answers for item 11 and all five indicated the student was reasoning about an infinite collection of coordinate pairs is 0.021.

Table 47

CYR Responses on CRICCP Items

	Item 11	Item 23b
CYR	100%	80%
Non-CYR	42%	44%
Part of curve	33%	44%

Consistently Reasoning with Values or Variations in Values

These items (8a,8b,9) asked students to construct a graph to represent the information they saw represented in an animation that was displayed on the projector at the front of the classroom. Table ___ shows the percentage of the sample who correctly drew the graph for each item. The relationship between reasoning with values and the ability to draw the graph of a function that represents the animation students was investigated by looking at the student's answers to items 8a, 8b, and 9 for students in the CRV and CRVNSP subgroups.

Table 48

Correctly Draws Graph for Animations

Item	**8a**	**8b**	**9**
Correct Graph and Coordinate Point	27%	29%	44%

Correct coordinate point only	15%	22%	15%
Incorrect	59%	49%	41%

Table 49 below shows the percentage of students who could draw a correct graph for each item for students in each subgroup. CRV and CRVNSP refer to students who consistently reason with values. CRV involves reasoning about values on graphs of continuous functions on a standard coordinate plane with time represented on the vertical axis. CRVNSP involves reasoning about values on graphs of continuous functions on non-standard coordinate planes (semi-log coordinate planes or coordinate planes with one orientation reversed). In both cases, we see that students whose answers indicated they were reasoning with values instead of the shape of the graph were more likely to be able to draw a graph that correctly represented the quantities in the animation. The students who consistently reasoned about values in the non-standard coordinate plane were the most likely to be able to draw the graph, with 50% or more of the students correctly drawing the graph. This provides evidence of a relationship between reasoning about values in the coordinate plane and the ability to sketch a graph to represent the animation.

Table 49

Creating Graphs to Represent Animation

Item	**8a**	**8b**	**9**
All Students	27%	29%	44%

CRV (Values)	44%	44%	56%

CHAPTER 7

CONCLUSION

Understanding the relationship among different graphing constructs in students' systems of meanings gives researchers and educators insights into potential ways to support students in building a productive system of meanings for calculus. The work in this study resulted in identifying relationships among several meanings students may have when reasoning about graphs.

This study investigated relationships among whether students thought about a discrete or infinite collection of values (on a number line and in a coordinate plane), whether students reasoned about the shape of the graph or values of the two variables and the connection between those meanings and the student's ability to recognize or construct a graph which represents a dynamic situation. Future research is needed to extend this work into other meanings important to students' graphical thinking, such as the student's covariational reasoning.

The ability to recognize a graph that represents a dynamic situation and the ability to construct a graph to represent the dynamic situation appears to be related to different underlying meanings. A strong relationship existed between students' ability to recognize which graph represented the dynamic situation and whether students imagined a graph representing an infinite collection of coordinate pairs. The ability to construct a graph that represented a dynamic relationship was closely related to students' reasoning about coordinated values or changes in coordinated values of the two quantities (as opposed to reasoning about the shape of the graph).

Reasoning about a discrete or infinite collection of locations, values, or coordinate points.

Students' understanding of the graph of a function as consisting of a finite collection of coordinate pairs or an infinite collection of coordinate pairs is related to whether they think about intervals on a number lines as representing an infinite collection of numbers and is also related to how the student is thinking about the variation of a single quantity. Students who consistently reasoned about intervals containing an infinite collection of coordinate points were more successful in correctly identifying the graph of a function that represents the dynamic situation. Supporting students in constructing a productive meaning for functions may start with supporting the students in conceiving intervals as containing an infinite number of values and thinking about a variable's value varying through all of them. Bass' (2019) discussion of two potential conceptions of number lines (the empty number line in which numbers are placed starting with counts and a conception of a number line based on measurements) is particularly relevant. Bass stated that the empty number line conception was likely to lead to the student understanding of number lines as one with gaps between numbers while the measurement line (based on Davydov's (1975) work) had the potential to support the students in constructing an understanding of a continuous number line. The results of this study suggest the importance of first supporting students in constructing a continuous number line and have implications for student instruction as early as their elementary education as a foundation for supporting the student in developing a productive understanding of graphs in functions.

Reasoning about values or changes in values

The ability to construct a graph to represent the dynamic situation in the animation seems to be related to the students reasoning about coordinated values (or changes in values) about the quantities. This research suggests a relationship between *emergent shape thinking* and the student's level of covariational reasoning in Thompson and Carlson's 2017 framework (where levels are differentiated based on how students coordinate values or variations in values). Supporting students in reasoning about the values of covarying quantities instead of the shape of the graph is important for supporting students in constructing a productive meaning for graphs. This result is consistent with the results of Paoletti and Vishnubhotia's (2022) small group teaching experiment, where they supported middle school students in emergent shape thinking through activities that encouraged reasoning about directional change in one quantity's value corresponding to change in a second quantity's value. Activities involving reasoning about graphs in coordinate planes other than the cartesian coordinate plane, where students cannot rely on the shape of the graph, may be useful in supporting students in learning to reason with values or variation in values of quantities. This recommendation is consistent with previous research, which suggested that students who only have the opportunity to reason about graphs in conventional coordinate planes are likely to develop an understanding of graphs that are only viable when the graphing conventions are met (Paoletti et al., 2022) and with calls from previous researchers to include graphs with nonconventional coordinate systems (Thompson, 1995; Moore et al., 2019b; Paoletti, 2020).

Reasoning involving formulas and graphs of functions

Students appear to reason differently about two seemingly related items. In one item, students were asked if the value of y increases or decreases as x increases. In the other item, students were first asked to identify the equation of the line and then asked if the value of y increases or decreases as x increases. When students were not asked about the equation, 86% correctly identified that the value of y decreased even though the line slanted upwards in the non-standard plane. When students were first asked to identify the equation, only 46% of the students correctly identified that y increased.

Additionally, 67% of students who consistently reasoned with values correctly identified that y increased compared to only 20% who consistently reasoned with shape. This is additional evidence of the relationship between reasoning with values and having a productive system of meanings for graphs of functions. Student responses to y increasing or decreasing after being asked to select the correct equation were positively correlated with the equation they selected, indicating that students may prefer to reason with formulas. This is consistent with Knuth's (2000) research that indicated that students have an over-reliance on algebraic solution methods and often do not perceive the graphical representations as neither related to symbolic methods nor useful in their reasoning about relationships. This may explain why only four students considered a graph to represent the relationship between temperature and time over a day. This research shows that the students who thought about creating a graph to represent the relationship were more successful in creating graphs to represent the animation. These students also consistently reasoned with the quantities' values (or variations in values). This suggests that supporting students in conceiving graphs as a way to represent information presented non-graphically and in reasoning with graphs to make conclusions about the relationship between two quantities values is an important part of supporting the students in constructing a productive meaning for graphs. This is further supported by the relationship between *emergent shape thinking* and the students correctly understanding the inequality $f(-5) < y < f(2)$ as a range of values on the y-axis.

Future Directions

The relationships identified in this study can be leveraged to create hypothetical learning trajectories and activities to support students in developing *emergent-shape thinking.* Reasoning about the covariation of quantities utilizing a graph of a function may start by supporting students in thinking about variation through an infinite collection of numbers in an interval, about the nature of number lines, and the structure of coordinate planes as representing both marked and unmarked values on intervals. Additionally, students need to be supported in reasoning about coordinated values of quantities as the values covary and in imagining graphs as a way of reasoning about the values and changes in values. Activities utilizing non-standard coordinate planes may be valuable tools in supporting students in reasoning about quantities values instead of the graph's shape.

Due to time limitations for developing and validating items and limitations in the number of items that can be included in a 90-minute survey, only some ways of thinking about graphs were included in this analysis. Additional research is needed to investigate the relationships between student's level of covariational reasoning, student's conception of a coordinate point and to understand the limitations in student's graphical thinking caused by unproductive meanings such as thinking about a graph as only consisting of a finite collection of coordinate pairs.

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APPENDIX A

INTERVIEW CONSENT FORM

Title of research study: Investigating Student's Systems of Thinking Regarding Graphs of Continuous Functions in Coordinate Planes

Investigators: Barbara Villatoro and Dr. Pat Thompson

Why am I being invited to take part in a research study?

We invite you to take part in a research study because you are enrolled in a calculus or pre-calculus course and are willing to participate. [Note that you must be 18 years of age or older.]

Why is this research being done?

We would like to study ways in which students understand certain mathematical concepts which are foundational to success in Calculus.

How long will the research last?

Your participation will entail one 90-minute interviews that will focus on your understanding of certain key mathematical ideas and explanations of how you determined your answer to questions which are being designed to help me understand your thinking on the key mathematical idea.

What happens if I consent to participate?

If you consent to participate, then an interview will be scheduled to take place on campus during April 2023. These interviews will be audio recorded and your scratch work will be collected. Analysis of your responses will aid the researchers in improving the questions to be used later and provide qualitative evidence of potential student thinking for students in the quantitative study who give similar answers. Transcriptions of your explanations may be used to explicate student thinking in any published reports based on this study.

What happens if I do not consent to participate?

You are not required to allow researchers to interview you. There is no penalty for choosing not to participate.

What happens if I consent to participate but I change my mind later?

You can change your mind at any time with no penalty to you. If you change your mind, please contact one of the members of the research team (contact information is at the end of this form). Any data already collected will be deleted or destroyed and no further data will be collected.

Is there any way being in this study could be bad for me?

Audio or transcripts of your interview may be shared with other researchers. If so, then your name will be replaced with a pseudonym. It is still possible that someone may recognize your voice, but we believe the likelihood is small. If data from your interview is used in publications we will transcribe the audio, and replace your name with a pseudonym. In other words, we will make every effort to protect your identity. Otherwise, the only potential negative effect of participating is the normal frustration or discomfort that may come from working through challenging mathematics problems and explaining your reasoning.

Will being in this study help me in any way?

We cannot promise any benefits to you or others from your taking part in this research. However, possible benefits include improved mathematical learning and shifts in your understandings.

Will I be paid for participating in this study?

You will be compensated for your time at a rate of $15 Amazon Card.

If you are paid a total of $600 or more as a research subject in a calendar year, the University is required to report the payment to the Internal Revenue Service as miscellaneous income. ASU will send you a form (IRS form 1099-MISC) in January documenting the payment total. This form is also sent to the IRS to report any money paid to you. You can use the form with your income tax return, as appropriate. Collecting this information allows ASU to meet government reporting obligations and that precautions are in place to assure confidentiality and data security.

What happens to the information collected for the research?

All of the audios will be stored in a password protected folder in a google drive (a cloud storage service). Any physical data will be stored in a locked filing cabinet/desk drawer in a locked office. We will analyze the data to try to describe how your mathematical meanings changed over the duration of the interviews. If the data is used in reports or publications we will make every effort to hide your identity.

Who can I talk to?

If you have questions at any time about the study or the procedures, you may contact either researcher by email: Barbara Villatoro (bvillat2@asu.edu) or Dr. Pat Thompson (pat@pat-thompson.net).

This research has been reviewed and approved by the Social Behavioral IRB. You may talk to them at (480) 965-6788 or by email at research.integrity@asu.edu if:

Your questions, concerns, or complaints are not being answered by the research team.
You cannot reach the research team.
You want to talk to someone besides the research team.
You have questions about your rights as a research participant.
You want to get information or provide input about this research.

Consent To Participate

"I have read and understand the above information. I have received a copy of this form. I agree to participate in this study with the understanding that I may withdraw at any time."

Subject's signature __________________________________Date _________________

Investigator's signature ______________________________Date _________________

APPENDIX B

CLINICAL INTERVIEW PROTOCOL

Review consent forms with student and make sure they have a signed copy saved.

Int: Thank you for taking the time to meet with me today. The purpose of this interview is to gain insight into student thinking. I ask that you try as best as you can to think out loud, explain everything as well as you can. I may ask you to clarify things you say or write down or ask you how you arrived at particular conclusions. This does not mean that you are wrong, I just want to gain insight as to what is going on inside your head. For example, if I asked you what 2 times 3 was and you said 6, I may ask you how you got that answer, why you multiplied, or what the value of 6 represents to you and to see if you could explain it as if I know nothing about the course you are taking. Do you have any questions before we begin?

Answers questions, if any arise.

Int: As we work through each question, I will read the question. Then I ask that you think aloud as you work through the question. Remember I may ask questions about something you said or wrote in order to better understand how you arrived at your conclusion. Before we get started with the first question, do you have any questions?

Answers questions, if any arise.

Protocol for Questions which do not ask the student to graph.

Read the question allowed. If there are graphs, read the labels on the horizontal and vertical axis.
Int: Please remember to think out loud as you work.
Ask clarification follow-up questions as needed.

Protocol for Questions that ask students to graph:

Read the question allowed. Read the labels on the horizontal and vertical axis.
Int: Please remember to think out loud as you work.
Ask clarification follow-up questions as needed.

Protocol for Animation Questions:
Read the question allowed. Read the labels on the horizontal and vertical axis.
Int: In a moment, I will press play to start the animation. First, please start by recording the information you see in the current view of the animation on the provided plane using blue. Please explain how you decided what to do.
Let the student work through and explain what they draw.
Int: Now I will press play for the animation. The animation will play on repeat until we move to the next question. Please use black to record the information you see in the animation on the same plane. As you do so, discuss out loud what you are thinking.
Ask clarification follow-up questions as needed.

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QUANTITATIVE DATA COLLECTION CONSENT FORM

Title of research study: Investigating Student's Systems of Thinking Regarding Graphs of Continuous Functions in Coordinate Planes

Investigators: Barbara Villatoro and Dr. Pat Thompson

Why am I being invited to take part in a research study?

We invite you to take part in a research study because you are enrolled in a calculus or pre-calculus course and are willing to participate. [Note that you must be 18 years of age or older.]

Why is this research being done?

We would like to study ways in which students understand certain mathematical concepts which are foundational to success in Calculus.

How long will the research last?

Your participation will entail one 90-minute survey that will focus on your understanding of certain key mathematical ideas and explanations/ work of how you determined your answer to questions that are being designed to help me understand your thinking on the key mathematical idea. The surveys will be conducted in a classroom survey in a test-like format. Due to the nature of group assessments confidentiality cannot be guaranteed.

What happens if I consent to participate?

If you consent to participate then you will be asked to register for one of the 90 minute survey collection times/ dates in April. During the 90 minutes questions will be presented and you will be asked to show work and provide an answer for each question.

What happens if I do not consent to participate?

You are not required to participate. There is no penalty for choosing not to participate.

What happens if I consent to participate but I change my mind later?

You can change your mind at any time with no penalty to you. If you change your mind, please contact one of the members of the research team (contact information is at the end of this form). Any data already collected will be deleted or destroyed and no further data will be collected.

Is there any way being in this study could be bad for me?

Your written work may be shared with other researchers. If so, your name will be replaced with a pseudonym. It is still possible that someone may recognize your work, but we believe the likelihood is small. In other words, we will make every effort to protect your identity. Otherwise, the only potential negative effect of participating is the normal frustration or discomfort that may come from working through challenging mathematics problems and writing/ explaining your reasoning.

Will being in this study help me in any way?

We cannot promise any benefits to you or others from your taking part in this research. However, possible benefits include improved mathematical learning and shifts in your understandings.

Will I be paid for participating in this study?

You will be compensated for your time at a rate of $15 for your participation at the completion of the completed survey.

If you are paid a total of $600 or more as a research subject in a calendar year, the University is required to report the payment to the Internal Revenue Service as miscellaneous income. ASU will send you a form (IRS form 1099-MISC) in January documenting the payment total. This form is also sent to the IRS to report any money paid to you. You can use the form with your income tax return, as appropriate. Collecting this information allows ASU to meet government reporting obligations and that precautions are in place to assure confidentiality and data security.

What happens to the information collected for the research?

Any physical data will be stored in a locked filing cabinet/desk drawer in a locked office. We will analyze the data to try to describe how your mathematical meanings changed over the duration of the interviews. If the data is used in reports or publications we will make every effort to hide your identity.

Who can I talk to?

If you have questions at any time about the study or the procedures, you may contact either researcher by email: Barbara Villatoro (bvillat2@asu.edu) or Dr. Pat Thompson (pat@pat-thompson.net).

This research has been reviewed and approved by the Social Behavioral IRB. You may talk to them at (480) 965-6788 or by email at research.integrity@asu.edu if:

Your questions, concerns, or complaints are not being answered by the research team.
You cannot reach the research team.
You want to talk to someone besides the research team.
You have questions about your rights as a research participant.
You want to get information or provide input about this research.

Consent To Participate

"I have read and understand the above information. I have received a copy of this form. I agree to participate in this study with the understanding that I may withdraw at any time."

Subject's signature ________________________________Date ________________

Investigator's signature ____________________________Date ________________

APPENDIX D

GROUP SURVEY PROTOCOL

Review consent forms with student and make sure they have a signed copy saved. Then collect the consent forms.

Int: Thank you for taking the time to meet with me today. The purpose of this survey is to gain insight into student thinking. I ask that you try as best as you can to show your work and explain everything as well as you can. Please do not erase or cross out any work. If you change your mind, circle your final work but leave the other work visible. You should have received two different colored pens. In some instances, I will ask you to use a specific color for a question. If you need a new pen of either color please let me know.

Read each question and give students time to work through their answers. Remind them to show work. On questions that include a coordinate plane, read the labels on each axis. Do not give any information not already written for the question.

Protocol for Animation Questions
Read the question allowed. Read the labels on the horizontal and vertical axis.
Int: In a moment, I will press play to start the animation. First, please start by recording the information you see in the current view of the animation on the provided plane using blue.
Give the students time to record the information they see.
Int: Now I will press play for the animation. The animation will play on repeat until we move to the next question. Please use black to record the information you see in the animation on the same plane.

Upon submitting the completed survey students receive their payment for participation.

APPENDIX E

IRB PROTOCOL

		Page: 1 of 7
	PREPARED BY: IRB Staff	**APPROVED BY**: Heather Clark
DOCUMENT TITLE: HRP 503 A Social Behavioral Protocol	**DEPARTMENT:** Office of Research Integrity and Assurance (ORIA)	**EFFECTIVE DATE**: [9/8/2021]

INSTRUCTIONS
Complete each section of the application. Based on the nature of the research being proposed some sections may not apply. Those sections can be marked as N/A. Remember that the IRB is concerned with risks and benefits to the research participant and your responses should clearly reflect these issues. You (the PI) need to retain the most recent protocol document for future revisions. Questions can be addressed to research.integrity@asu.edu. **PIs are strongly encouraged to complete this application with words and terms used to describe the protocol is geared towards someone not specialized in the PI's area of expertise.**

IRB: 1. Protocol Title: Investigating Graphical Thinking

IRB: 2. Background and Objectives

2.1 List the specific aims or research questions in 300 words or less.
2.2 Refer to findings relevant to the risks and benefits to participants in the proposed research.
2.3 Identify any past studies by ID number that are related to this study. If the work was done elsewhere, indicate the location.

TIPS for streamlining the review time:
- ✓ Two paragraphs or less is recommended.
- ✓ Do not submit sections of funded grants or similar. The IRB will request additional information, if needed.

Response: The goal of the study is to investigate the relationship between different ways of thinking about graphs in student's systems of thinking by developing a collection of questions which individually give information about part of the student system of thinking and collectively can be used to investigate their system of thinking. The potential risk to the student is in feeling uncomfortable showing work or explaining their mathematical thinking. No past studies.

IRB: 3. Data Use - What are the intended uses of the data generated from this project?

Examples include: Dissertation, thesis, undergraduate project, publication/journal article, conferences/presentations, results released to agency, organization, employer, or school. If other, then describe.

Response: Dissertation
IRB: 4. Inclusion and Exclusion Criteria 4.1 List criteria that define who will be included or excluded in your final sample. Indicate if each of the following special (vulnerable/protected) populations is included or excluded: ▪ Minors (under 18) ▪ Adults who are unable to consent (impaired decision-making capacity) ▪ Prisoners ▪ Economically or educationally disadvantaged individuals 4.2 If not obvious, what is the rationale for the exclusion of special populations? 4.3 What procedures will be used to determine inclusion/exclusion of special populations? **TIPS for streamlining the review time.** ✓ Research involving only data analyses should only describe variables included in the dataset that will be used. ✓ For any research which includes or may likely include children/minors or adults unable to consent, review content [here] ✓ For research targeting Native Americans or populations with a high Native American demographic, or on or near tribal lands, review content [here] For research involving minors on campus, review content [here]
Response: Subjects will be recruited from college precalculus and calculus courses. The recruitment email will specify that they need to be at least 18 to participate. The study will not target any vulnerable populations.
IRB: 5. Number of Participants Indicate the total number of individuals you expect to recruit and enroll. For secondary data analyses, the response should reflect the number of cases in the dataset.
Response: 100-150 students. Up to 10 for CI and the remainder for participation
IRB: 6. Recruitment Methods 6.1 Identify who will be doing the recruitment and consenting of participants. 6.2 Identify when, where, and how potential participants will be identified, recruited, and consented. 6.3 Name materials that will be used (e.g., recruitment materials such as emails, flyers, advertisements, etc.) Please upload each recruitment material as a separate document, Name the document: recruitment_methods_email/flyer/advertisement_dd-mm-yyyy 6.4 Describe the procedures relevant to using materials (e.g., consent form). ✓

Response:
6.1. Barbara Villatoro will be doing the recruitment and consenting of participants.
6.2. Emails will be sent to teachers of Calculus and Precalculus classes asking the instructors to post the recruitment email to their class. The recruitment email will include a link to select one of the available times to participate in the large group assessment.
6.3 Recruitment Email.
6.4. Students will be sent a copy of the consent form and asked to bring it to the large group assessment. Extra copies will be available for students who forget to bring a signed copy. The consent forms will be collected when the students enter the room for the assessment and any questions the students have about the consent form will be answered.

IRB: **7. Study Procedures**

7.1 List research procedure step by step (e.g., interventions, surveys, focus groups, observations, lab procedures, secondary data collection, accessing student or other records for research purposes, and follow-ups). Upload one attachment, dated, with all the materials relevant to this section. Name the document: supporting documents dd-mm-yyyy

7.2 For each procedure listed, describe **who** will be conducting it, **where** it will be performed, **how long** is participation in each procedure, and **how/what data** will be collected in each procedure.

7.3 Report the total period and span of time for the procedures (if applicable the timeline for follow ups).

7.4 For secondary data analyses, identify if it is a public dataset (please include a weblink where the data will be accessed from, if applicable). If not, describe the contents of the dataset, how it will be accessed, and attach data use agreement(s) if relevant.

TIPS for streamlining the review time.
- ✓ Ensure that research materials and procedures are explicitly connected to the articulated aims or research questions (from section 2 above).
- ✓ In some cases, a table enumerating the name of the measures, corresponding citation (if any), number of items, sources of data, time/wave if a repeated measures design can help the IRB streamline the review time.

Response:
7.1. Clinical interviews to validate items to be used for the assessment. Analysis of the clinical interviews and audio recordings of the clinical interviews are used to improve items and inform the later quantitative analysis. Once 100-150 students will be recruited to answer the items in a classroom setting. The Reponses will then be coded for analysis. Clinical interviews will be audio recorded. (Up to 10 for CI and the remainder for the survey).
File 1: Quant. Data Collection.
File 2: Student Answer Sheet
File 3: Dissertation.AnalyzeDataPlan
File 4: Dissertation.PotentialCodes2
7.2 Clinical Interviews will be conducted by Barbara Villatoro in ECA 342 (Office of Barbara Villatoro). Analysis of clinical interviews will be conducted by Barbara Villatoro. Large group assessment will be conducted in classrooms at the ASU Tempe campus. Barbara Villatoro will collect and code the items, as well as collect the consent form as and answer questions regarding consent.
7.3 Clinical Interviews by April/May 2023, Large group assessment in April/May 2023. Analysis of data May/ June 2023.
Students participate in one clinical interview of 90 minutes. The only identifiable data collected is the students name from when they volunteered to participate. Their name is separated from the collected written work/ recording. The large group assessment is conducted similar to an exam, students work independnely recording their answers on their own pages. Questions are presented either on the paper they are working on, or through the coordinated power point . Students will work on the same question at the same time so that they receive the same instructions. If a student wishes to skip a question they would leave the question blank. The clinical interviews are on a subset of the questions (or potential questions) for the large group assessment. The clinical interviews are to collected qualitative data about what student's meanings are for questions that will later be given in the large group assessment in order to aid in understanding student meanings. Clinical interviews are audio recorded to be reviewed if needed. No video recordings. The large group assessment will last 90 minutes. Students selected for interview/survey based on expressed availability when they email Barbara Villatoro to participate. Same recruitment procedure for both.

IRB: 8. Compensation

8.1 Report the amount and timing of any compensation or credit to participants.

8.2 Identify the source of the funds to compensate participants.

8.3 Justify that the compensation to participants to indicate it is reasonable and/or how the compensation amount was determined.

8.4 Describe the procedures for distributing the compensation or assigning the credit to participants.

TIPS for streamlining the review time.

- ✓ If partial compensation or credit will be given or if completion of all elements is required, explain the rationale or a plan to avoid coercion
- ✓ For extra or course credit guidance, see "Research on educational programs or in classrooms" on the following page: https://researchintegrity.asu.edu/human-subjects/special-considerations.
- ✓ For compensation over $100.00 and other institutional financial policies, review "Research Subject Compensation" at: https://researchintegrity.asu.edu/human-subjects/special-considerations for more information.

Response: Students will be paid $15 Amazon Gift Card for their time and participation of an hour to 90 minutes (in either the large group assessment or the clinical interviews).

Source of funds: Self Funded by Barbara Villatoro unless grant funding can be found.

Compensation is to encourage students to participate. Amount was selected to compensate for up to 90 minutes. Amazon Gift Cards will be given to students when they turn in their answer sheet for the large group assessment and before they leave from the clinical interviews.

IRB: 9. Risk to Participants

List the reasonably foreseeable risks, discomforts, or inconveniences related to participation in the research.

TIPS for streamlining the review time.

- ✓ Consider the broad definition of "minimal risk" as the probability and magnitude of harm or discomfort anticipated in the research that are not greater in and of themselves than those ordinarily encountered in daily life or during the performance of routine physical or psychological examinations or tests.
- ✓ Consider physical, psychological, social, legal, and economic risks.
- ✓ If there are risks, clearly describe the plan for mitigating the identified risks.

Response:

Potential discomfort in showing/ explaining their mathematical thinking in an interview or in an assessment. There is a small risk that student handwriting could be recognized if their written work is used. Due to the nature of group assessments confidentiality cannot be guaranteed.

IRB: 10. Potential Direct Benefits to Participants

List the potential direct benefits to research participants. If there are risks noted in 9 (above), articulated benefits should outweigh such risks. These benefits are not to society or others not considered participants in the proposed research. Indicate if there is no direct benefit. A direct benefit comes as a direct result of the subject's participation in the research. An indirect benefit may be incidental to the subject's participation. Do not include compensation as a benefit.

Response: There is no direct benefit to participants.

IRB: **11. Privacy and Confidentiality**

Indicate the steps that will be taken to protect the participant's privacy.

11.1 Identify who will have **access to the data**.
11.2 Identify where, how, and how long data will be **stored** (e.g. ASU secure server, ASU cloud storage, filing cabinets).
11.3 Describe the procedures for **sharing, managing and destroying data**.
11.4 Describe any special measures to **protect** any extremely sensitive data (e.g. password protection, encryption, certificates of confidentiality, separation of identifiers and data, secured storage, etc.).
11.5 Describe how any **audio or video recordings** will be managed, secured, and/or de-identified.
11.6 Describe how will any signed consent, assent, and/or parental permission forms be secured and how long they will be maintained. These forms should separate from the rest of the study data.
11.7 Describe how any data will be **de-identified**, linked or tracked (e.g. master-list, contact list, reproducible participant ID, randomized ID, etc.). Outline the specific procedures and processes that will be followed.
11.8 Describe any and all identifying or contact information that will be collected for any reason during the course of the study and how it will be secured or protected. This includes contact information collected for follow-up, compensation, linking data, or recruitment.
11.9 For studies accessing existing data sets, clearly describe whether or not the data requires a Data Use Agreement or any other contracts/agreements to access it for research purposes.
11.10 For any data that may be covered under FERPA (student grades, etc.) additional information and requirements is available at https://researchintegrity.asu.edu/human-subjects/special-considerations.

Response:
11. 1 PI's will have access to the data.
11.2 Electronic files will be stored in an encrypted folder in ASU cloud storage. Paper responses will be stored in the office of Barbara Villatoro at ASU until they are digitized (without identifiable data) and the hard copies destroyed. Electronic files will be stored for at least 3 years.
11. 3 Paper copies will be destroyed using ASU Shredding services. Digitial copies (without identifying information) may be shared by granting access to the file in ASU cloud storage.
11.4 There isn't any extremely sensitive data.
11.5 Students names will not be used during any audio recordings and an alias will be used if the students words are used in writing the analysis or in the presentation of the analysis of the data.
11. 6 Signed consent will be collected prior to recording the students voice (at the start of the clinical interview) and at the start of the large group assessment. The consent forms will be stored in the office of Barbara Villatoro until digitized and stored in the ASU secured cloud storage. Hard copies will be destroyed via document shredding. There will be audio recording only.
11. 7 The written work will be stored using identifiers that are not linked back to the students name. Students name is not used while audio recording during the clinical interviews. Pseudonyms will be used for any reported data about a specific set of responses. None of the responses are stored with connection to the student name. Students only participate once and are not tracked.
11.8 Contact information will be collected when the student signs-up to participate (name and email). After the data collection the information will not be stored. Compensation will happen at the time of data collection, follow-up is not planed.
11.9 Not accessing existing data sets.
11.10 Not collecting any data covered under FERPA.

IRB: 12. Consent
Describe the procedures that will be used to obtain consent or assent (and/or parental permission).

12.1 Who will be responsible for consenting participants?
12.2 Where will the consent process take place?
12.3 How will the consent be obtained (e.g., verbal, digital signature)?

TIPS for streamlining the review time.

- ✓ If participants who do not speak English will be enrolled, describe the process to ensure that the oral and/or written information provided to those participants will be in their preferred language. Indicate the language that will be used by those obtaining consent. For translation requirements, see Translating documents and materials under https://researchintegrity.asu.edu/human-subjects/protocol-submission
- ✓ Translated consent forms should be submitted after the English is version of all relevant materials are approved. Alternatively, submit translation certification letter.
- ✓ **If a waiver for the informed consent process is requested, justify the waiver in terms of each of the following: (a) The research involves no more than minimal risk to the subjects; (b) The waiver or alteration will not adversely affect the rights and welfare of the subjects; (c) The research could not practicably be carried out without the waiver or alteration; and (d) Whenever appropriate, the subjects will be provided with additional pertinent information after participation.** Studies involving confidential, one time, or anonymous data need not justify a waiver. A verbal consent or implied consent after reading a cover letter is sufficient.
- ✓ ASU consent templates are [here].
- ✓ Consents and related materials need to be congruent with the content of the application.

Response:
12.1 Barbara Villatoro will be responsible for collecting and storing consent forms for consenting participants.
12.2/3 Consent forms will be collected at the start of the data collection in person. Consent forms will be stored for at least 3 years after completion of research.

IRB: 13. Site(s) or locations where research will be conducted.
List the sites or locations where interactions with participants will occur-

- Identify where research procedures will be performed.
- For research conducted outside of the ASU describe:
  - Site-specific regulations or customs affecting the research.
  - Local scientific and ethical review structures in place.
- For research conducted outside of the United States/United States Territories describe:
  - Safeguards to ensure participants are protected.
- For information on international research, review the content [here].

For research conducted with secondary data (archived data):

- List what data will be collected and from where.
- Describe whether or not the data requires a Data Use Agreement or any other contracts/agreements to access it for research purposes.
- For any data that may be covered under FERPA (student grades, etc.) additional information and requirements is available [here].
- For any data that may be covered under FERPA (student grades, homework assignments, student ID numbers etc.), additional information and requirements is available [here].

Response:
ECA 342 for clinical interviews (ASU office of Barbara Villatoro) and classrooms on the ASU Tempe Campus for the large group assessment collection. All data will be collected on ASU Tempe campus.

IRB: **14. Human Subjects Certification from Training.**

Provide the names of the members of the research team.

ASU affiliated individuals do not need attach Certificates. Non-ASU investigators and research team members anticipated to manage data and/or interact with participants, need to provide the most recent CITI training for human participants available at www.citiprogram.org. Certificates are valid for 4 years.

TIPS for streamlining the review time.

- ✓ If any of the study team members have not completed training through ASU's CITI training (i.e. they completed training at another university), copies of their completion reports will need to be uploaded when you submit.
- ✓ For any team members who are affiliated with another institution, please see "Collaborating with other institutions" [here]
- ✓ The IRB will verify that team members have completed IRB training. Details on how to complete IRB CITI training through ASU are [here]

Response:
PI: Patrick Thompson
Additional Investigator: Barbara Villatoro.

General Tips:

- Have all members of the research team complete IRB training before submitting.
- Ensure that all your instruments, recruitment materials, study instruments, and consent forms are submitted via ERA when you submit your protocol document. Templates are [here]
- Submit a complete protocol. Don't ask questions in the protocol – submit with your best option and, if not appropriate, revisions will be requested.
- If your study has undeveloped phases, clearly indicate in the protocol document that the details and materials for those phases will be submitted via a modification when ready.
- Review all materials for consistency. Ensure that the procedures, lengths of participation, dates, etc., are consistent across all the materials you submit for review.
- Only ASU faculty, full time staff may serve as the PI. Students may prepare the submission by listing the faculty member as the PI. The submit button will only be visible to the PI.
- Information on how and what to submit with your study in ERA is [here]. Note that if you are a student, you will need to have your Principal Investigator submit.
- For details on how to submit this document as part of a study for review and approval by the ASU IRB, visit https://researchintegrity.asu.edu/human-subjects/protocol-submission.

APPENDIX F

IRB APPROVAL

ASU Knowledge Enterprise Development

EXEMPTION GRANTED

Patrick Thompson
CLAS-NS: Mathematical and Statistical Sciences, School of (SoMSS)
480/965-2891
Pat.Thompson@asu.edu

Dear Patrick Thompson:

On 3/31/2023 the ASU IRB reviewed the following protocol:

Type of Review:	Initial Study
Title:	Investigating Student's Systems of Thinking Regarding Graphs of Continuous Functions in Coordinate Planes
Investigator:	Patrick Thompson
IRB ID:	STUDY00017664
Funding:	None
Grant Title:	None
Grant ID:	None
Documents Reviewed:	• ConsentFormCI.pdf, Category: Consent Form; • ConsentFormQDC.pdf, Category: Consent Form; • Revised Recruitment Document, Category: Recruitment Materials;

The IRB determined that the protocol is considered exempt pursuant to Federal Regulations 45CFR46 (2)(ii) Tests, surveys, interviews, or observation (low risk) on 3/28/2023.

In conducting this protocol you are required to follow the requirements listed in the INVESTIGATOR MANUAL (HRP-103).

If any changes are made to the study, the IRB must be notified at research.integrity@asu.edu to determine if additional reviews/approvals are required. Changes may include but not limited to revisions to data collection, survey and/or interview questions, and vulnerable populations, etc.

APPENDIX G

ADDITIONAL QUANTITATIVE RESULTS

Research Question 1: In what ways do students understand graphs and how are the different ways that students understand graphs and coordinate planes related in the student's understanding?

To investigate the first research question, I overviewed and compared student responses for items that were written to give insight into student thinking for the same meanings. Subgroups of students who consistently answered indicating the same reasoning were identified and their answers on other items were investigated to look for relationships between meanings. Evidence of a relationship is found when students with consistent meanings on one collection of questions also tended to reason similarly to each other on other items.

Reasoning with a discrete collection of numbers or locations.

Six questions were written to investigate whether a student is reasoning about a discrete collection of numbers (for example whole numbers) or is reasoning with an infinite collection of values. Note that the student's meaning for infinite was not investigated. Students may mean an infinite collection of terminating decimals or they may be thinking about all real numbers. What these questions investigated are if the student's reasoning was restricted to a discrete collection of numbers/locations or if the student's reasonings included thinking about values in between marked points or whole numbers.

Amongst the students who did not reason about an infinite collection of numbers discretely, there were some who counted the number of whole numbers, some who were imagining the number of unit increases from the first number to the next or the result of an operation. For questions involving variation there were a couple whose reasoning was consistent with thinking about replacing one value with the other. Reasoning about the number of whole numbers between two numbers was the most common response. For Item 1, 97% of the students who answered discretely were imagining the number of whole numbers between the two given numbers. For item 2, 83% of the students who answered discretely were thinking about the whole numbers between the two values, 10% either calculated the difference in the two numbers or counted the number of unit increases, and 7% had answers consistent with thinking about replacing the first value with the second. For the purposes of this analysis, all of these meanings are grouped together as reasoning discretely and may include both students who are thinking about a discrete collection of numbers, a discrete collection of locations, or a discrete number of unit increases. Future research is needed to investigate further the relationships between the different meanings included as discrete thinking in this study. Students are considered as consistently reasoning discretely (CRD students) if their answers to at least four of the six items indicated they were thinking discretely.

Table G1

Reasoning about an infinite collection of values

Item	Percent Reasoning with an infinite collection of values or locations.
1	27%

2	22%
3a	29%
4	29%
14a	22%
14b	32%

Table G1 summarizes how often student's responses were consistent with thinking about an infinite collection of numbers on each of the items individually. 76% of the students had responses consistent with thinking discretely for two-thirds or more of the questions (CRD students). 12% of the students had responses consistent with thinking about an infinite collection of numbers for at least two-thirds of the questions (CRICN students).

Items 14a and 14b were both based on the same number line. Item 14a asked students how many numbers were represented on the line graph on the number line. The line had three solid dots (one at each endpoint and one in between that was not on a whole number). Item 14b then stated that x was represented on the number line and that x varied from the first number to the last number and asked the students how many values was x. Seven students (17%) responded infinite to both. Of those seven students, no fewer then six of the students also responded infinite to items 1, 2, 3a, and 4 (respectively). Additionally, all five CRICN students are included in the seven who answered infinite to both item 14a and item 14b (CRICN2). These students' reasoning was robust in that they consistently reasoned with an infinite collection of numbers. I use the student responses for the seven CRICN2 to investigate relationships between students thinking about an infinite collection of numbers and other meanings and will refer to them as the students who consistently reasoned with an infinite collection of numbers (CRICN2 students).

Table G2
Defining CRD, CRICN, CRICN2, CRICCP *students*

	Number of Students	Criteria
CRD	31 (76%)	Discrete Responses on at least two-thirds of items 1, 2, 3a, 4, 14a, 14b
CRICN	5 (12%)	Infinite Responses on at least two-thirds of items 1, 2, 3a, 4, 14a, 14b
CRICN2	7 (17%)	Infinite Responses on both 14a and 14b
CRICCP	17(41%)	Response indicates infinite CP on item 11 and 23

Reasoning with a finite or infinite collection of coordinate pairs or locations in a plane.

Item 11 and Item 23b are two items written to investigate whether student's reasoning about a finite collection of coordinate pairs or an infinite collection of coordinate pairs. Item 11 asked students to determine which function's graph had more coordinate points where one graph was a linear function with three highlighted coordinate points and the other graph had five plotted points. The Pearson correlation coefficient between the two items is 0.707, which is evidence that the student responses to these two items are positively correlated ($p < 0.00001$). In both items, students tended to either think about an infinite collection of coordinate pairs in both questions or in neither question. 52% of the student's answers for both questions were consistent with thinking about an infinite collection of ordered pairs and 48% of the student's answers were consistent with thinking about a discrete collection of ordered pairs.

Table G3

Reasoning About an Infinite Collection of Coordinate Pairs

	Item 11	Item 23b
CRICN2	100%	100%
CRD	32%	35%

All seven of the CRICN2 students also answered that the graph with a line has more coordinate points in Item 11 and stated that the line had infinite coordinate points in Item 23b. Roughly a third of the CRD students reasoned about an infinite collection of ordered pairs. This suggests that there is a relationship between a student reasoning with an infinite collection of numbers for a single quantity (CRICN2) and the student reasoning with an infinite collection of coordinate pairs (CRICCP) when reasoning about the graph of a function in a coordinate plane.

Reasoning with values or variation in values.

Items 13, 15, 16, and 20 investigate whether the student is reasoning based on coordinating values or reasoning based on the steepness of the line. These items had at least one graph with time on the vertical axis. Items 13 and 15 asked the students to compare graphs with the axes switched. In items 13 and 15 students were able to compute rates of change to compare (if they thought to do so). For Items 13 and 15, most students reasoned based on comparing rates they calculated or comparing values on the graphs relative to the axis labels (83% and 95% respectively). For Items 16 and 20 the graphs did not include unit tick marks on the axis. The question instead stated that the graphs for Item 20 were on the same scale. Item 16 included a single piecewise linear graph on the same axis. All the graphs in Items 16 and 20 had time on the vertical axis and distance on the horizontal axis. If students' reasoning was based on the steepness of the graph then they would reach a different conclusion than if their reasoning was on increases in the quantities values relative to the axis labels. For these items, 63% and 76% percent reasoning were consistent with comparing changes in the quantities values. There appears to be an increase in the percentage of students who reasoning was based on the steepness of the graph when they could not calculate the rates. Students were considered to have consistently reasoned with values (CRV) if their answers to all four items (13, 15, 16, 20) indicated they reasoned with values. Students were considered to have consistently reasoned with steepness (CRS) if their answers to both items 16 and 20 indicated they reasoned based on the steepness of the line.

Table G4

Defining CRV, CRS students

	Number of Students	Criteria
CRV	18 (44%)	Reasoning with values on all four of items 13, 15, 16, and 20.
CRS	8 (20%)	Reasoning with steepness on both items 16 and 20.

Equations and Reasoning about shape or values.

Items 12 and 22b asked the students if *y* was increasing or decreasing as *x* increased for a line graphed on a non-standard coordinate plane. If the student was reasoning about if the line went up or down the student would answer differently than a student whose reasoning was focused on comparing the quantities' values or variations in the quantities' values. 86% of the students reasoned about values or variation in values for Item 12, but only 46% did so for Item 22b. Item 22b followed a question that asked the student to select the correct equation for the line (where the answers had opposite slope values). Student answers to Item 22b were consistent with the sign of the slope for the answer the student selected for Item 22a. 81% of the students who said that the line had an equation with a positive slope stated that y increased and 90% of the students who selected the equation with a negative slope stated that y decreased. It appears that the students' reasoning for their answer in Item 22b was based primarily on the equation they selected. In the absence of an equation, a much larger percentage of students reasoned about values. This seems to suggest that when students have both a graph and an equation to reason about, they tend to reason about the equation.

Reasoning about values or shape in non-standard coordinate planes

Items 12, 21b, 26a, and 26c were all written to investigate whether the students' reasoning was focused on the shape of the line or was based on the quantities' values or variations in the values. Items 12 and 26a both involved graphs of a line in a coordinate plane with non-standard orientation. Items 21b and 26c both involved graphs in a semi-log coordinate system. For students who appeared to reason with values in at least three of these Items (CRVNSP students), 73% of them reasoned about values or variation in items in Items 22a and 67% on Items 22b (compared to 38% and 35% of non-CRVNSP students respectively). Of the students who reasoned with values at most once in Items

12, 21b, 26a, and 26c (CRSh students) 20% reasoned with values on Item 22a and 30% reasoned with values on item 22b. This indicates that students who reasoned about values in non-standard coordinate planes (CRVNSP students) were more likely to reason about values even when there was an equation than students who reasoned about shape (CRSh students).

Table G5

Defining CRVNSP, CRSh students

	Number of Students	Criteria
CRVNSP	15 (37%)	Reasoning with values on at least 3 of items 12, 21b, 26a, 26c.
CRSh	10 (24%)	Reasoning with shape on at least 3 of items 12, 21b, 26a, 26c.

CRVNSP students reasoned with values and an average of 3.3 of 4 on the items for CRV. CRV students reasoned with values on an average of 2.2 of the 4 items for CRVNSP. This indicates that CRVNSP students are likely to also consistently reason with values in the cartesian coordinate system, but the reverse is not true. This result suggests that activities that include reasoning with graphs of functions in non-standard coordinate planes may be useful in supporting students in reasoning with values instead of shape.

CRV, CRVNSP, and CRSh students did similarly on items 14a and 14b (Table 51), with most answers indicating the students were thinking about a discrete collection of numbers. This indicates there may not be a relationship between if the student's reason with an infinite collection of values and whether they reason with values and variations in values. The confirmatory factor analysis (discussed later) also shows there does not seem to be a relationship between these meanings.

Table G6

Percent of students who answered infinite on items 14a and 14b

	Item 14a	Item 14b
CRS	0%	0%
CRV	28%	39%
CRVNSP	20%	30%
CRSh	20%	20%

Research Question 2: How are productive meanings for graphs related to the different ways that students understand graphs and coordinate planes?

Based on existing research, I consider a productive meaning for graphs of continuous functions to be one in which the student understands a graph of continuous functions as consisting of an infinite collection of coordinate pairs representing coordinated values of two covarying quantities which emerge through imagining a graph as emerging, or having emerged, by coordinating values of two quantities as their values vary continuously through intervals. Two types of items were developed to investigate student's productivity. The first type is items in which students construct a graph for a situation with two covarying quantities represented in an animation (Items 8a, 8b, 9). CGD refers to students who consistently created graphs that represented the covarying quantities in the animations. The second type were items where the student identifies the graph which represents a dynamic situation in an animation or a description. CIG refers to students who consistently identified graphs or coordinate points on the graph which represented the dynamic situation. In order to answer research question 2, answers for the items of each type were compared for students who consistently reasoned according to a specific meaning. Evidence of a relationship between meanings is inferred when students who consistently answered items consistent with a particular meaning were more successful in answering the items written to investigate the productivity of the student's meanings. Lastly, confirmatory factor analysis provides additional evidence of the relationships identified by comparing student answers to the items.

Coordinating Values to create a graph to represent a dynamic situation.

Items 8a, 8b, and 9 were all written to see how student's reasoning in creating a graph of a function for a dynamic situation in a coordinate plane. In the data collection these items involved two steps. First the student was asked to look at a still picture which was on their answer sheet. The still picture represents the first moment of the animation. The student was asked to first represent the information they see in the still picture on the coordinate plane using the red pen. Then once all the students were ready the animation played (on repeat) and the students were asked to record the information they see in the animation using the blue pen. Students were scored 1 if both their red and blue marks were (approximately) consistent with the animation, 0.5 if only their red mark was, and 0 otherwise.

Table G7

Results for Items 8a, 8b, and 9

Item	8a	8b	8c
Correct Graph and Coordinate Point	27%	29%	44%

Correct coordinate point only	15%	22%	15%
Incorrect	59%	49%	41%

Item 5 stated that someone noticed that when they were a certain distance from the fire the temperature was a certain temperature and asked the students to represent that information on the provided graph. 98% of the students drew a coordinate point at the correct location on the provided graph for item 5. In comparison, when students had to approximate the values themselves from the still picture on Items 8a, 8b, and 9 an average of about 50% of the students drew a coordinate point that represented the two distances in the still picture. The reasoning involved in graphing a coordinate point to represent a single given relationship between two quantities does not seem sufficient to support the students in graphing the coordinate point to represent the relationship between the two distances in the still picture.

Item 3 gives the student the temperature in a backyard at two times during the day. Part b asked the students to draw a picture to represent the information. Only four students drew a graph in a coordinate plane when asked to draw a picture to represent the given information (SDG students). Of the SDG students, 100% of them graphed the coordinate point correctly on all three graphs for Items 8a, 8b, and 9 and graphed the function correctly on at least two of the three Items (only one incorrect for Item 8b). There appears to be a strong relationship between the reasoning for which the students spontaneously drew a graph to represent a situation involving two quantities and the students constructing a graph to represent the dynamic situation in the animation. Of the SDG students, all four regularly reasoned with values and not the shape of the graph (CRV students). This indicates there may be a relationship between reasoning by coordinating quantities values or variations in values (CRV) and the students being able to sketch a graph to represent the dynamic situation in the animation (CDG students). The confirmatory factor analysis (discussed later) also indicates a relationship between the students being able to sketch a graph of the dynamic situation (CDG) in the animation and reasoning about the values (or variation in the values) of two quantities (CRV).

Identifying the graph based on an animation or description.

Items 17, 18a, and 19 all ask the student to identify the graph which represents a dynamic situation. Item 17 and 18a describe the situation and item 19 asks the students to identify which one represents what they see in the animation. All three items involved graphs with time on the vertical axis and a distance on the horizontal axis. Items 18b and 25 asked the students to identify where they see the start of the race as being represented in the graph. Both graphs had time represented on the vertical axis. The start of the race would be represented by a coordinate point when time equals 0, which for both graphs was a coordinate point that was not on the vertical axis. Students whose meaning for the y-intercept is as a starting point would likely indicate where the function crossed the vertical axis as being the start of the race. The student's answers to the collection of items gives insight into how the student relates the dynamic situation to the graph of the function.

Table G8

Students Meaning for Graphs of Functions for Dynamic Situations

Item	17	18a	18b	19	25
CRV students	89%	61%	56%	94%	78%
Non-CRV students	87%	52%	30%	74%	35%

For all five items which give insight into how the student relates the graph of the function to the dynamic situation, CRV students identified the correct graph or coordinate point (CIG students) at a similar or higher percentage than non-CRV students. This indicates a relationship between how the student relates the graph of a function to a dynamic situation and whether the student is reasoning with the quantities' values or variations in the quantities' values. This relationship is further supported by the confirmatory factor analysis (discussed later).

Table G9
CYR reasoning about an infinite collection of coordinate pairs

	11	23b
CYR	100%	80%
Non-CYR	42%	44%

Item 24 asked the student to identify where they see $f(-5) < y < f(2)$ on the graph on their answer sheet. Five students (12%) correctly identified a range of y-values as representing the inequality (CYR students). CYR students understood a line as containing an infinite collection of coordinate pairs much more frequently than non-CYR students (Table 55). This indicates there may be a relationship between the students meaning for the inequality in the plane (CYR) and the student's thinking about the graph of a function as consisting of an infinite collection of coordinate pairs (CRICCP).

Table G10

CYR students on identifying graphs for dynamic situation

	17	18a	18b	19	25
CYR	80%	80%	80%	80%	80%
Non-CYR	89%	53%	36%	83%	50%

Additionally, CYR students either correctly identified the graph similarly to non-CYR students (Items 17, 18a, 19) or at a higher percentage. In identifying where on the graph would represent the start of the race (Items 18b and 25), CYR students were significantly more likely to correctly identify the coordinate point where time equaled 0. This indicates a relationship between students reasoning about the inequality in the plane (CYR students) and the student’s meaning for the graph of the dynamic situation in the coordinate plane (CGI students).

The above discussion indicated that correctly identifying the inequality as an interval of y-values (CYR) is related to both the student’s ability to identify the graph which represents the dynamic situation in the coordinate plane (CGI students) and whether students were thinking about the graph of the function as representing an infinite collection of coordinate pairs (CRICCP students). This is further supported by the students’ responses to Item 26d. Item 26d involved a graph with dots that lie on a line and a curve with a repetitive pattern that goes through the dots. Students were asked if the graph of the function had a constant rate of change. 43% of the non-CRV students said that the graph was of a function that had a constant rate of change. Of the CRV students only 17% said that the graph in Item 26d had a constant rate of change. None of the five CYR students stated that the graph in Item 26d had a constant rate of change.

Confirmatory Factor Analysis

For the confirmatory factor analysis, items were grouped into six variables based on the qualitative analysis to group items which give information about the same construct. Each item received a score of between 0 and 1, and variables were constructed by summing the scores for each question in the group.

Table G11
List of items for each variable.

Variable	Items
1. Reasoning about numbers not marked on the axis. (one Variable).	3a, 4, 14a
2. Variational Reasoning (one variable)	2, 14b
3. Reasoning about numbers not marked on the axis. (two Variable).	11, 23
4. Coordinating values or changes in values	12, 21b, 22b, 26a, 26c
5. Emergent Shape Thinking (create graph)	8a, 8b, 9
6. Emergent Shape Thinking (select graph)	16, 19, 20

Variable 1: Reasoning about a discrete collection of numbers or an infinite collection of numbers (one variable).

The three items in this group were written to investigate whether students thought about a discrete collection of numbers or locations (integers or marks on the number line) or their reasoning included thinking about an infinite collection of numbers. What students understand as an infinite collection of numbers was not investigated. The students might or might have been reasoning about all real numbers. Individual items were scored with a 1 if their answer stated that there were an infinite number of numbers and 0 otherwise. The score for the variable is calculated by summing the score on the three items.

Variable 2: Variational Reasoning (one variable)

The two items in this group were written to investigate if the student is thinking about a discrete collection of numbers or locations when thinking about the variation of a value or if the student is imaging the variable as varying through an infinite collection of numbers. Similar to variable 1, student thinking about an infinite collection of numbers might be imagining only decimal numbers.

Individual items were scored with a 1 if their answer included that there were an infinite number of numbers and 0 otherwise. The score for the variable is calculated by summing the score on the three items.

Variable 3: Reasoning about numbers not marked on the axis. (two Variable).

The three items in this group were written to investigate whether students thought about a discrete collection of ordered pairs or locations in the coordinate plane when reasoning about the graph of a function. Individual items were scored with a 1 if their answer indicated they were thinking about an infinite collection of ordered pairs and 0 otherwise. The score for the variable is calculated by summing the score on the three items.

Variable 4: Coordinating values or changes in values

The five items in this group were written to investigate whether students reasoned about the shape of a graph of a function or whether students reasoned about either coordinating values or changes in values of the quantities. With the questions used I was not able to distinguish between whether the student's image was a comparison of values or of changes in values. Individual items were scored 1 if the students answer indicated the student was reasoning with values or variation in values and 0 otherwise. The score for the variable is calculated by summing the score on the three items.

Variable 5: Emergent Shape Thinking (create graph)

The three items in this group were written to investigate the students' thinking when asked to create a graph to represent information they see in a still picture and then an animation which starts at the moment of the still picture. I had the students use a red pen to first record the information the student saw in the still picture and then change their pen to a blue pen to record the information they saw in the animation. Individual items received a score of 1 if the graph they drew included a coordinate point at the correct location in red and a blue line showing the correct linear relationship (or blue dots that would lie along the line). The item was scored a 0.5 if they had the red mark correct but not the blue line and a 0 if neither was correct. Individual items were scored 1 if the students answer indicated the student was reasoning with values or variation in values and 0 otherwise. The score for the variable is calculated by summing the score on the three items.

Variable 6: Emergent Shape Thinking (select graph)

The three items in this group were written to investigate the student's thinking about the graph of a function by selecting the graph which represents the animation or written description. Individual items were scored 1 if the student selected the correct graph and a 0 otherwise.

Confirmatory Factor Analysis

In this section, I discuss the results of the confirmatory factor analysis that was conducted to investigate relationships among the different ways of thinking about graphs discussed in the previous section. First, I discuss the one-factor model which leads to looking at a two-factor model which appears to describe the relationship in the variables better. Confirmatory factor analysis is an *accept support test*, meaning that a model is understood to be a good fit if we are unable to reject the null hypothesis. The null hypothesis is that the model's covariance matrix reproduces the population covariance matrix. A large p-value provides support that the model appears to fit the data. A rejection of the model based on a small p-value would mean that the model is not a good fit to the data.

A one-factor model

A one-factor model is a structural relationship where all of the variables are informed by a single (unobserved) meaning. Factor analysis was conducted using the Lavaan Package in R using the variance standardization method and standardizing the loadings. The loadings describe the increase in standard deviations for each variable relative to a 1 unit increase in the factor score. Standardized loadings are between -1 and 1. The closer the loading is to 1 the more of the variation in the observed data is explained by the factor. In the case of the one-factor model, the p-value for the chi-squared test is 0.549 which provides evidence of a fit of the model to the data. When investigating the factor loadings (Table 46), two of the variables are not highly rated as evidenced by the factor scores that are much smaller than the other variables. This led to an investigation of a two-factor model.

Table G12
Factor loadings for the 1-factor model

Variable	Standardized Factor Loading
One-Variable Continuous	0.858
One-Variable Variation	0.649
Two-Variable Continuous	0.679
Coordinating Values or Changes in Values	0.227
Emergent Shape Thinking, Constructing their own graph	0.230
Emergent Shape Thinking, Identifying the correct graph	0.517

Table G12 shows the standardized factor loadings for the one factor model. The factor loadings indicate that the coordinating values of changes in values and the thinking involved in creating a graph to represent a dynamic situation are not strongly related to the other variables.

A two-factor model

In the two-factor model, four of the variables are informed by one factor and the other two variables are informed by a second factor. The p-value for the chi-squared test is 0.648, which again indicates that the model is a good fit for the data. The factor loadings using the two-factor model (Table G13) indicate that this model gives a better picture of the structural relationships between the variables then the 1-factor model.

Table G13
Factor Loadings for the 2-factor model

Variable (Factor 1)	Standardized Factor Loading
One-Variable Continuous	0.865
One-Variable Variation	0.646
Two-Variable Continuous	0.676
Emergent Shape Thinking, Identifying the correct graph	0.516

Variable (Factor 2)	Standardized Factor Loading
Coordinating Values or Changes in Values	0.500
Emergent Shape Thinking, Constructing their own graph	0.485

This model shows the same relationships that were seen in the previous analysis. Factor 1 indicates that students reasoning about the graph of function representing a dynamic situation is related to whether they reasoned about an infinite collection of values or coordinate points. Factor 2 indicates that the reasoning involved in creating a graph to represent a dynamic situation is related to whether students reasoned with the values of quantities (or variations in the quantities values).

APPENDIX H

ITEMS

QUESTION 1:
How many numbers are between 4 and 9? Explain

QUESTION 2:
x varied from 3 to 8; how many different numbers was x? Explain

QUESTION 3

I woke up at 8am and went to bed 14 hours later (at 10pm). The temperature displayed on my thermometer in my backyard was 75 degrees when I woke up and 95 degrees when I went to bed.

a) How many different temperatures was it in my backyard between when I woke up and when I went to bed?

b) Draw a picture to illustrate the story.

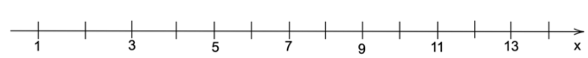
c) If possible, find the temperature at noon. If not possible, explain why.

d) Circle the correct statement. Explain your choice.
   i. The temperature only increased while I was awake.
   ii. The temperature might have decreased at some point while I was awake.

QUESTION 4

The instructions for the question will be displayed on the powerpoint.
Mark where you see "the interval from x = 2 to x = 9" on the number line.

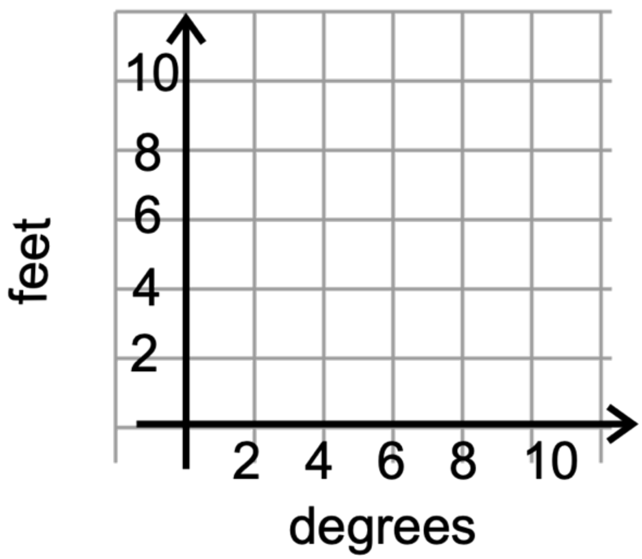
a)



b) How many values of x are represented on the interval you drew? Explain.

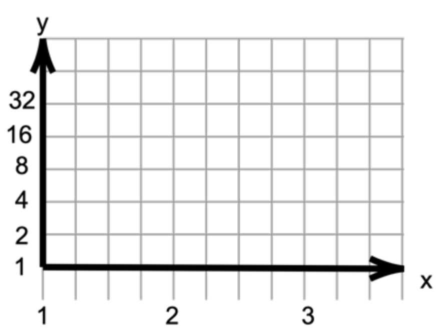
QUESTION 5

On a cold night at the backyard bonfire John noticed that when he was 2 feet from the bonfire, the temperature was 8 degrees Celsius. Use a single mark to represent the information on the provided coordinate plane.



QUESTION 6

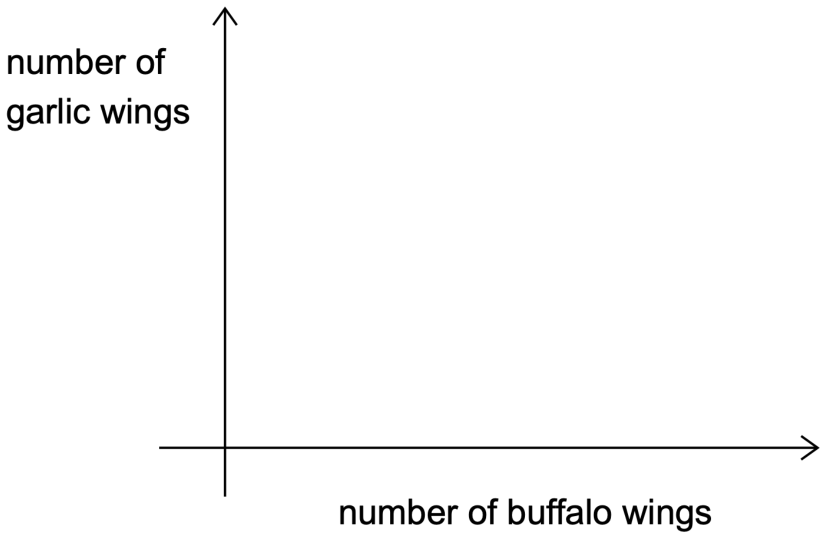
a) Graph y = 4x on the coordinate plane.



b) How many coordinate pairs are on the graph of the function that you drew?

QUESTION 7

a) There needs to be 3 buffalo sauce chicken wings for every 1 garlic chicken wing ordered. Sketch a graph to represent the possible orders of chicken wings.

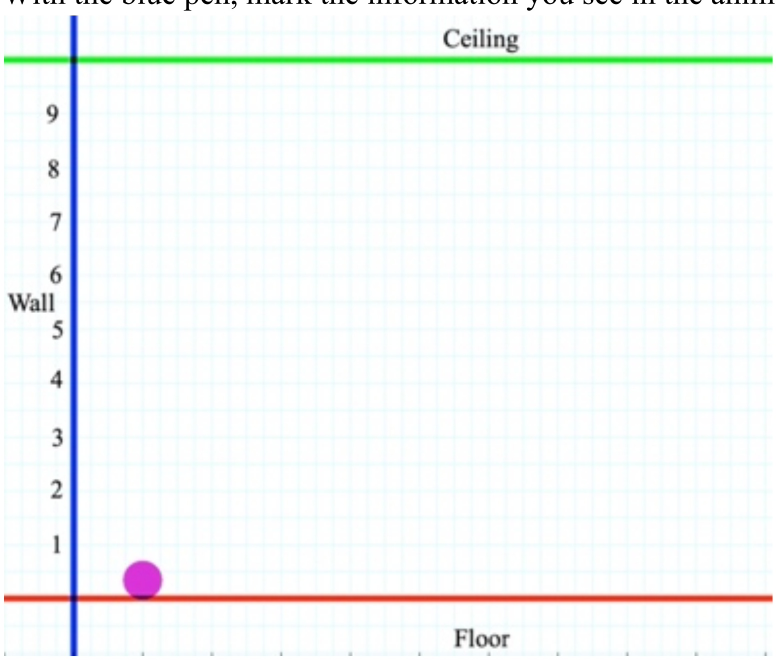


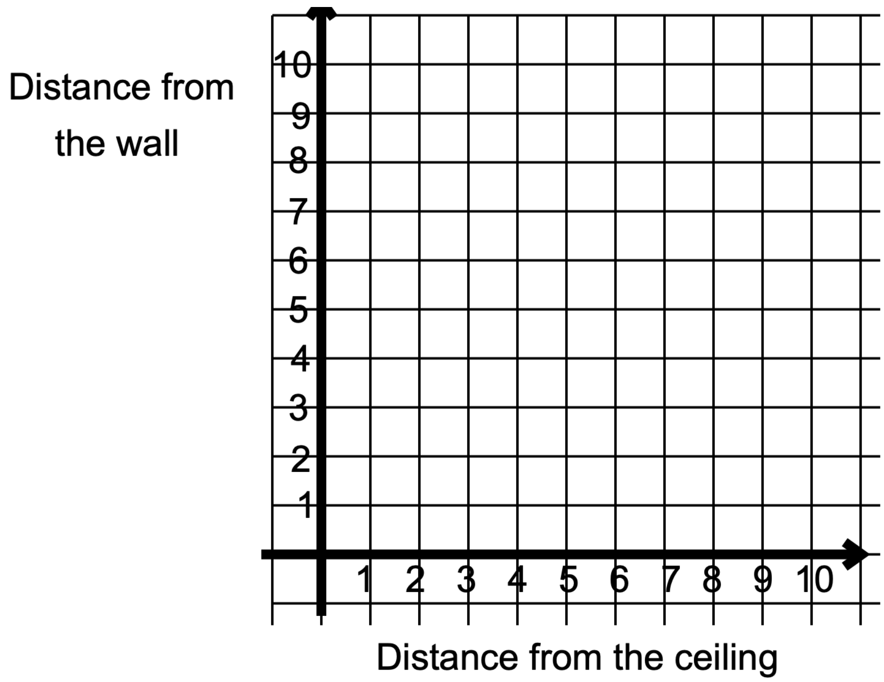
b) How many coordinate pairs are on the graph of the function that you drew?

QUESTION 8 a

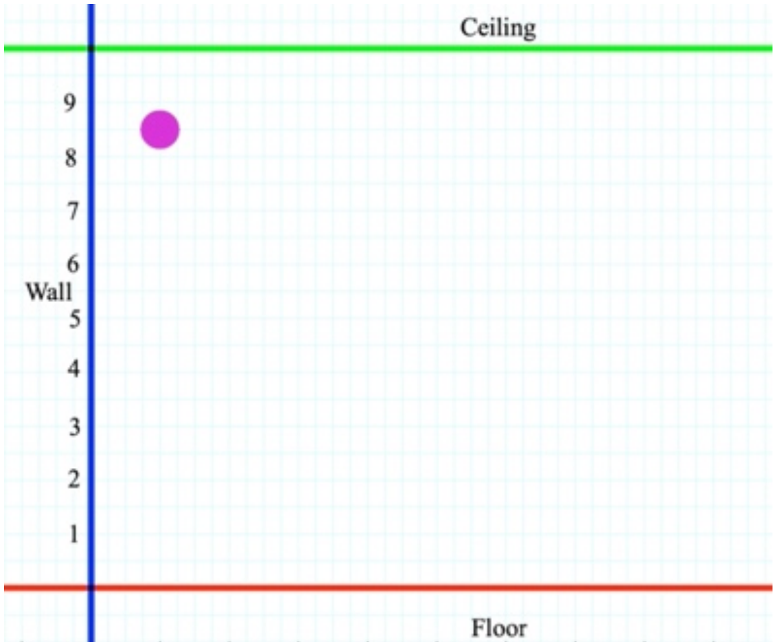
With the red pen mark on the graph the information about the ball that you see in the top image.

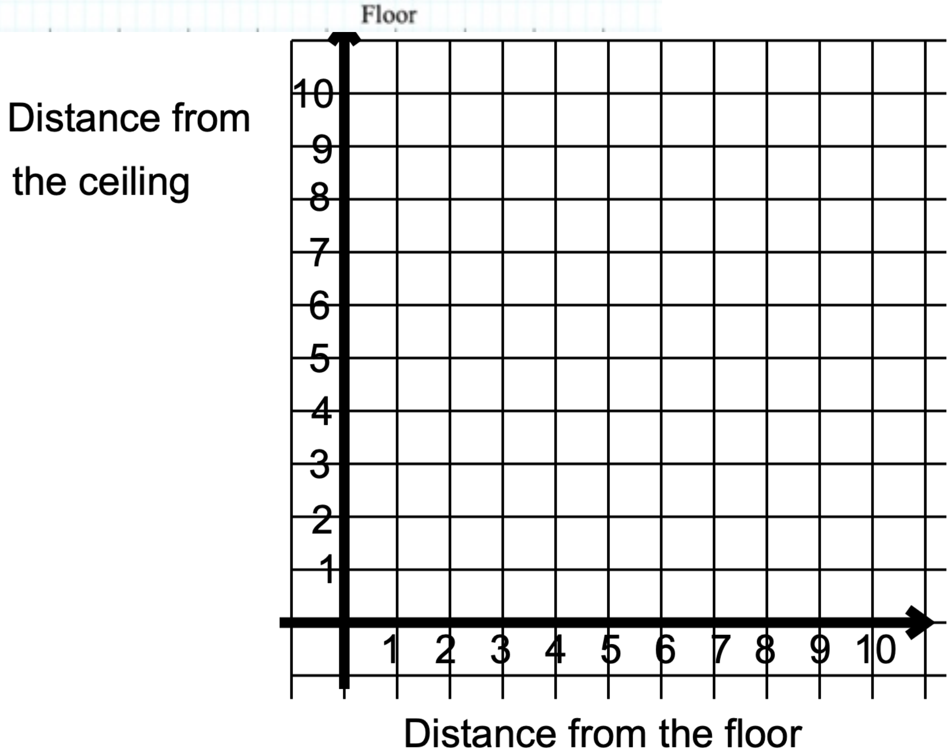
With the blue pen, mark the information you see in the animation which will play shortly.





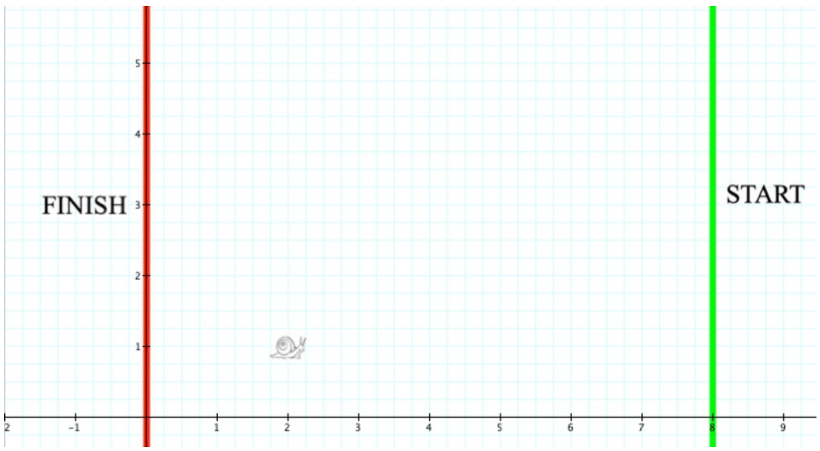
QUESTION 8 b
With the red pen mark on the graph the information about the ball that you see in the top image.
With the blue pen, mark the information you see in the animation which will play shortly.

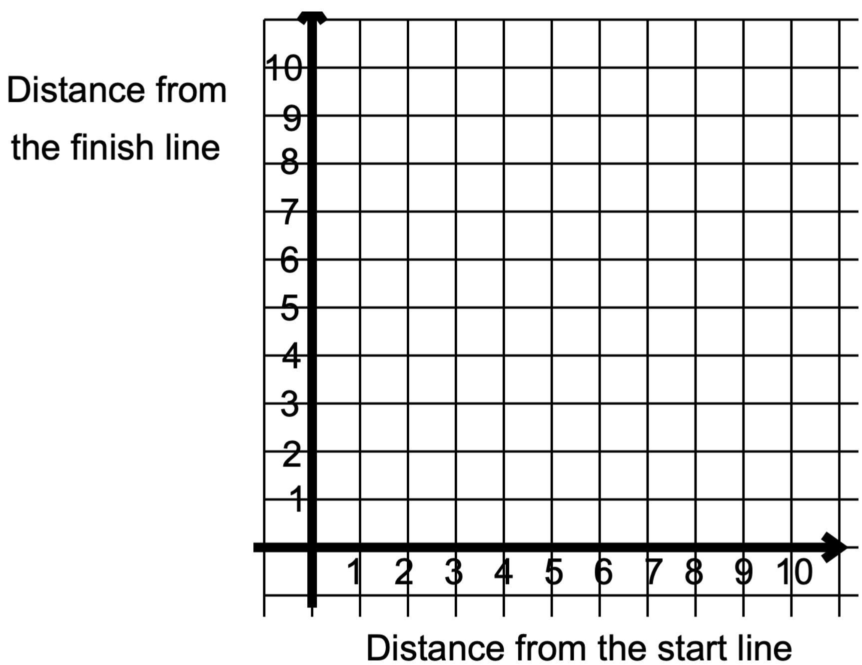

Ceiling
9
8
7
6
Wall
5
4
3
2
1
Floor


Distance from the ceiling
10
9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9 10
Distance from the floor

QUESTION 9

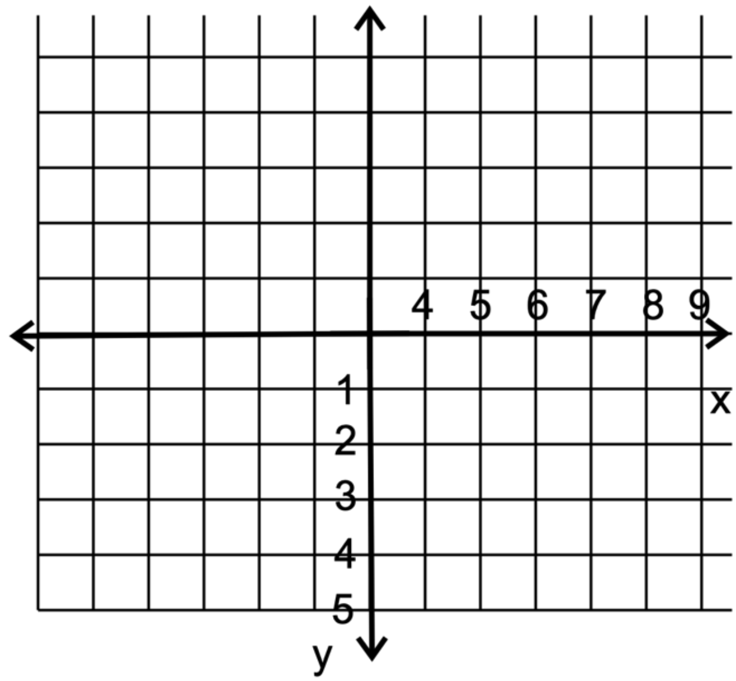
With the red pen mark on the graph the information about the ball that you see in the top image.
With the blue pen, mark the information you see in the animation which will play shortly.





QUESTION 10

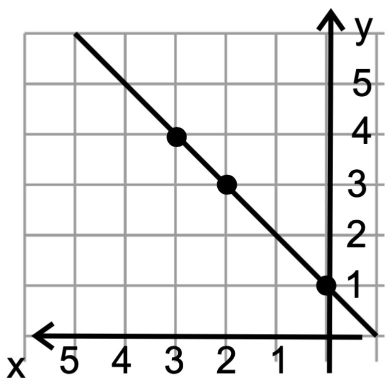
a) Graph y = 2x - 4 on the provided non-standard coordinate plane.

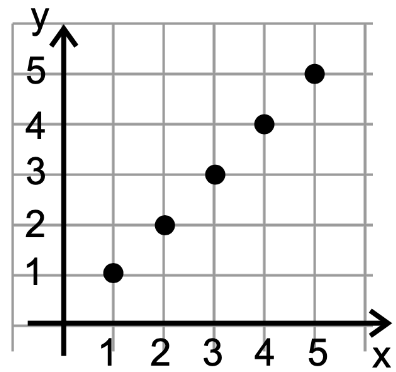


b) How many coordinate pairs did you draw in your graph?

QUESTION 11

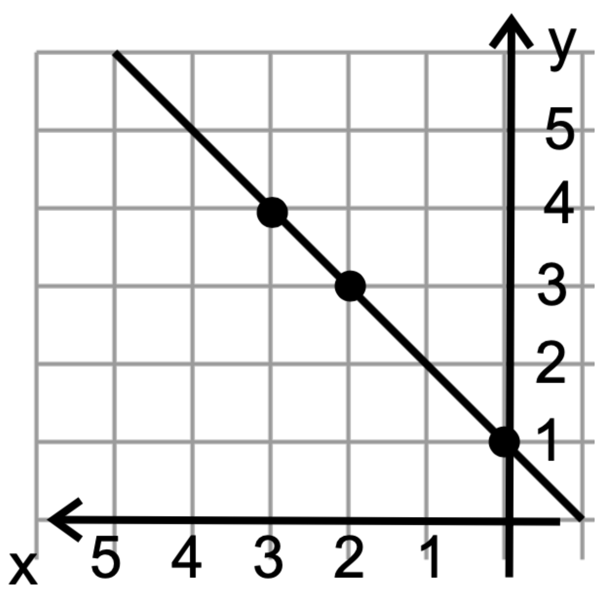
Which graph of which function has the least number of coordinate pairs (x, y)?
Explain how you determined your answer.





Explain:

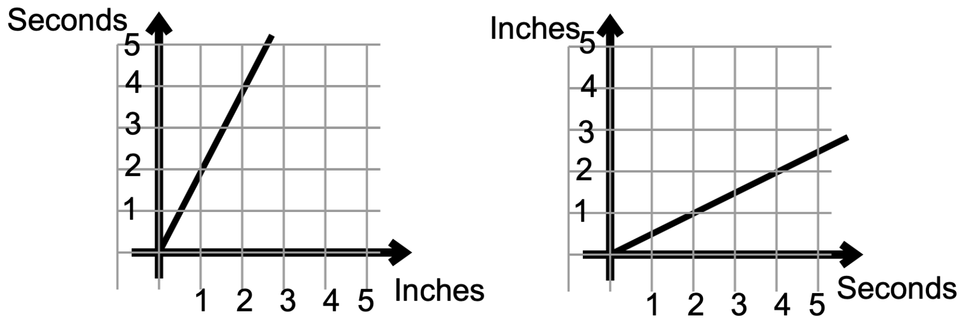
QUESTION 12
Is y increasing or decreasing as x increases? Explain how you determined your answer.



Explain:

QUESTION 13
Two friends watched a snail race. Each friend chose their favorite snail and made a graph of part of that snail's race. Do they have the same favorite snail? Explain how you determined your answer.

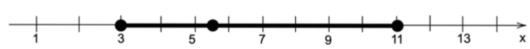
Amy Brenda



Explain:

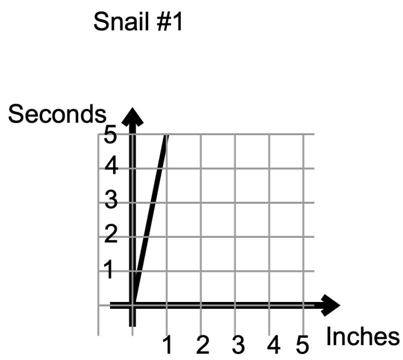
QUESTION 14:
How many values of x are being indicated in the picture below? Explain.

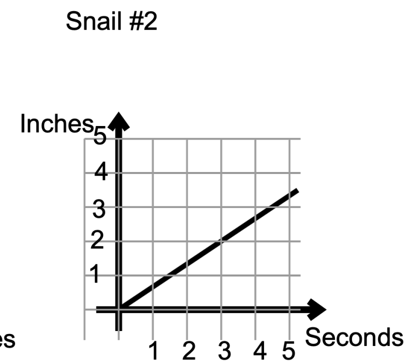
a)



b) The value of x is represented on the number line. x varied from 3 to 11, how many values was x?

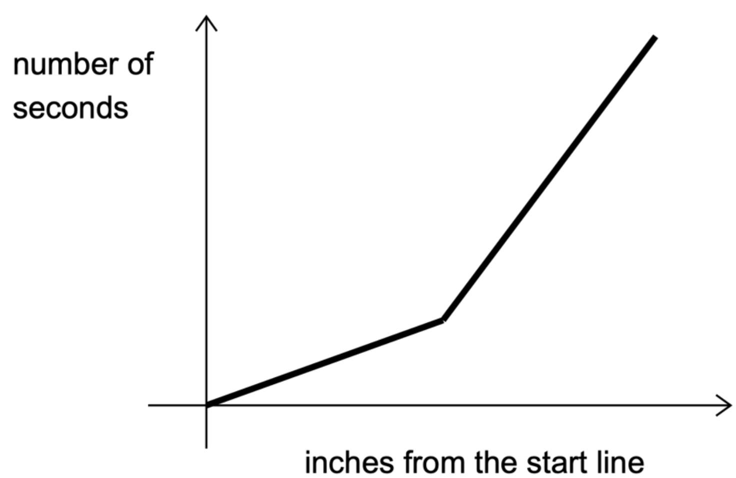
QUESTION 15
The snails readied for another race and, this time, one friend sketched a graph predicting how each snail would run. Which snail would win the second race according to these graphs? Explain how you decided.





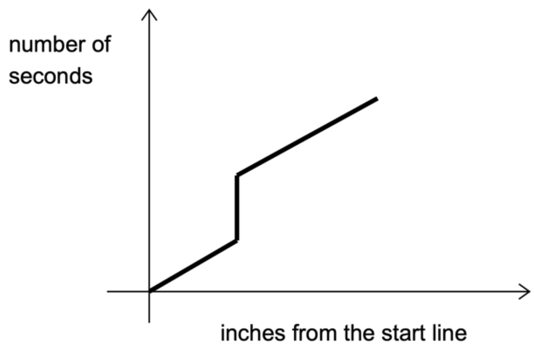
Explain:

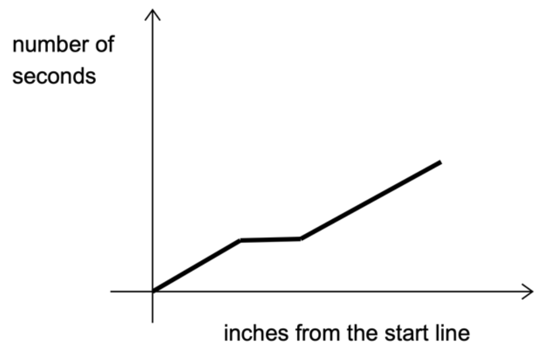
QUESTION 16:
The graph of the function below shows the function f, which gives the number of seconds in the race as a function of the number of inches the snail is from the start line. Did the snail speed up or slow down? Explain.



QUESTION 17
A snail starts a race, then pauses for a bit, then continues the race.

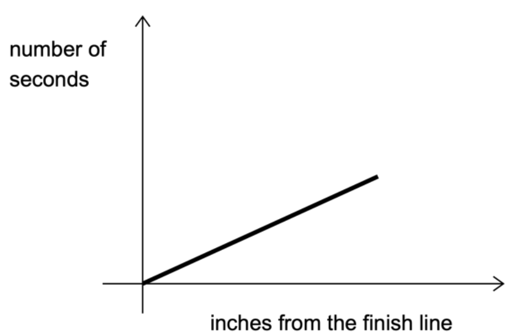
Which graph show the snail's race?

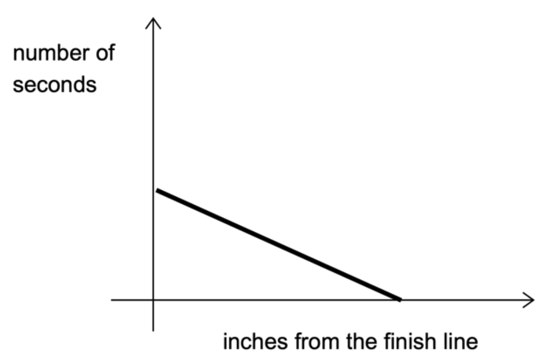




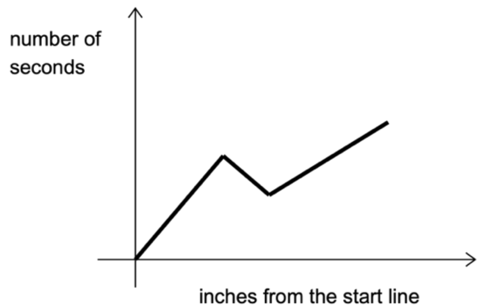
QUESTION 18
The instructions for the question will be displayed on the powerpoint.

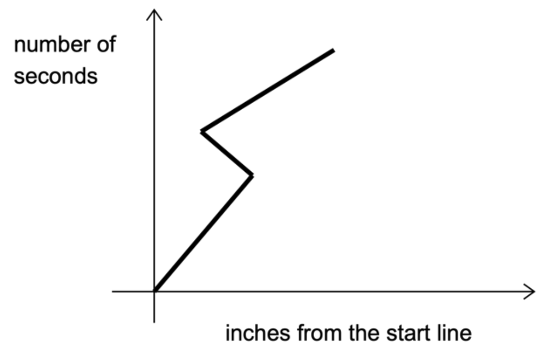
a) You want to graph function f, which gives the number of seconds in the race as a function of the number of inches the snail is from the finish line. The graph of which function below could be the graph of the race?
b) On the graph you choose for part a, mark and label where the start of the race is indicated.



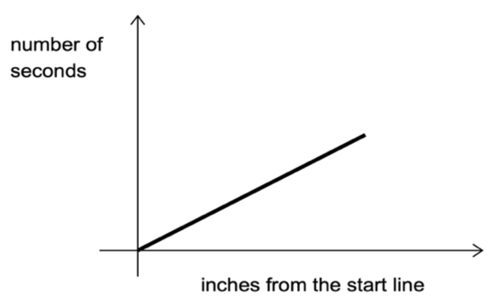


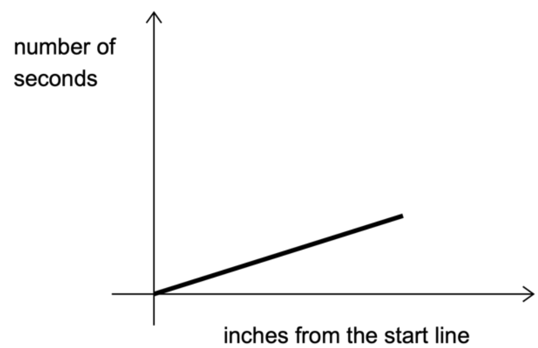
QUESTION 19
Which graph could represent the snail's race in the Animation?





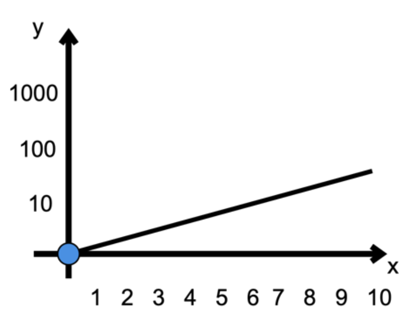
QUESTION 20
Assume the graphs for the two snail's races are drawn on the same scale, which snail won the race? Explain.





QUESTION 21

Consider the graph of the function below.



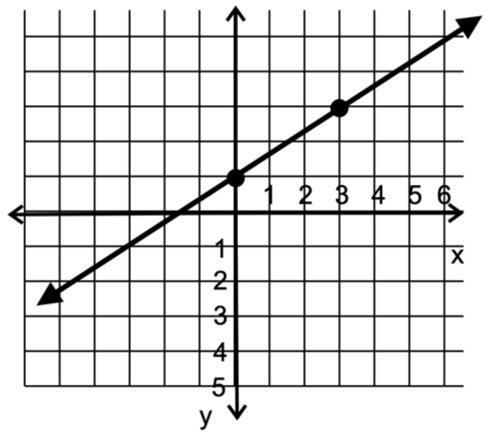
a) Label the blue dot.

b) Which best describes this function?

i) Liner function with an equation of F(x) = mx + b with fixed values m and b.
ii) Exponential function with equation F(x) = a(b)^x with fixed values a and b.

Explain:

QUESTION 22



a) Which is the correct form of the equation for the line in the graph?

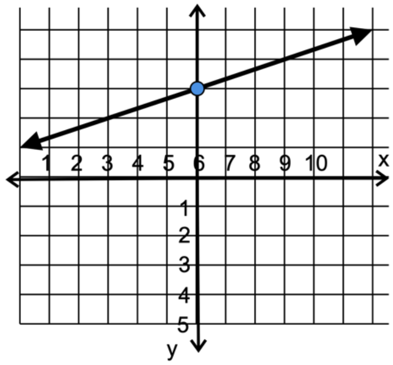
a. $y = \frac{2}{3}x + b$

b. $y = -\frac{2}{3}x + b$

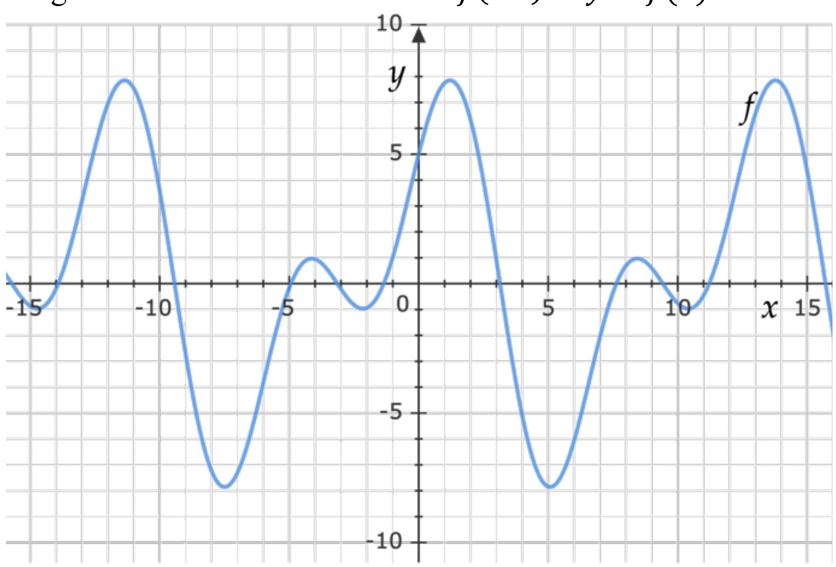
b) Does y increase or decrease as x increases? Explain.

QUESTION 23

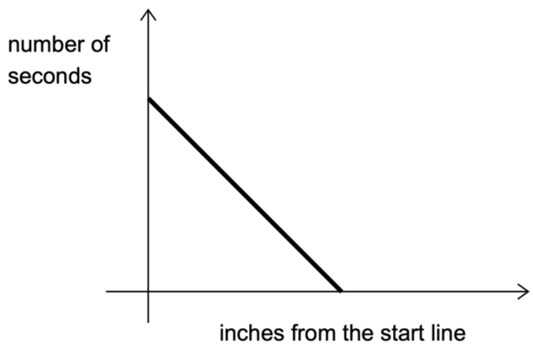
a) Label the blue dot on the graph of function f using function notation.
b) How many coordinate points are on the graph of function f?



QUESTION 24: Use the red pen.
On the provided graph, mark and label with the red pen where you think is indicated by the given mathematical statement. $f(-5) < y < f(2)$

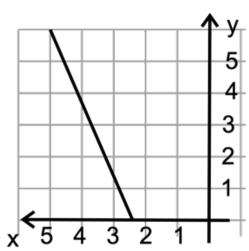


QUESTION 25
The graph of the function below shows the function f, which gives the number of seconds in the race as a function of the number of inches the snail is from the start line.

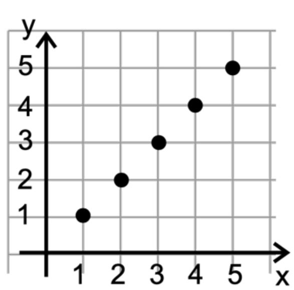


a) Mark and label where you see the start of the race on the graph.
b) Is the snail headed toward the start line or towards the finish line? Explain.

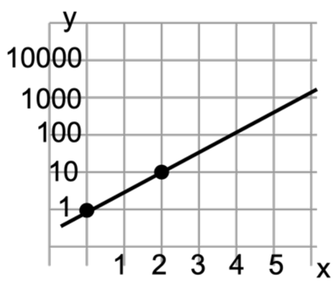
QUESTION 26
Circle the best response for each graph.



a) Does the function vary at a <u>constant rate of change</u> with respect to x?
d. Yes, the rate of change is constant and positive
e. Yes, the rate of change is constant and negative
f. No, the rate of change is not constant

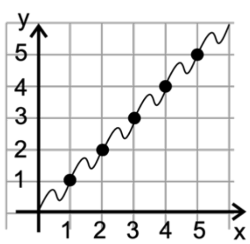


b) Does the function vary at a <u>constant rate of change</u> with respect to x?
g. Yes, the rate of change is constant and positive
h. Yes, the rate of change is constant and negative
i. No, the rate of change is not constant



c) Does the function vary at a <u>constant rate of change</u> with respect to x?

a. Yes, the rate of change is constant and positive
b. Yes, the rate of change is constant and negative
c. No, the rate of change is not constant



d) Does the function vary at a <u>constant rate of change</u> with respect to x?

j. Yes, the rate of change is constant and positive
k. Yes, the rate of change is constant and negative
l. No, the rate of change is not constant