Students' Personal Algebraic Expressions as a Reflection of their Meanings:

The Case of Infinite Series

by

Derek Chance Eckman

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Graduate Supervisory Committee:

Kyeong Hah Roh, Chair
Marilyn Carlson
Jason Martin
John Spielberg
Dov Zazkis

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ABSTRACT

Over the last several centuries, mathematicians have developed sophisticated symbol systems to represent ideas often imperceptible to their five senses. Although conventional definitions exist for these notations, individuals attribute their personalized meanings to these symbols during their mathematical activities. In some instances, students might (1) attribute a non-normative meaning to a conventional symbol or (2) attribute viable meanings for a mathematical topic to a novel symbol. This dissertation aims to investigate the relationships between students' meanings and personal algebraic expressions in the context of one topic: infinite series convergence. To this end, I report the results of two individual constructivist teaching experiments in which first-time second-semester university calculus students constructed symbols (called *personal expressions*) to organize their thinking about various topics related to infinite series. My results comprise three distinct sections. First, I describe the intuitive meanings that the two students, Monica and Sylvia, exhibited for infinite series convergence before experiencing formal instruction on the topic. Second, I categorize the meanings these students attributed to their personal expressions for series topics and propose symbol categories corresponding to various instantiations of each meaning. Finally, I describe two situations in which students modified their personal expressions throughout several interviews to either (1) distinguish between examples they initially perceived as similar or (2) modify a previous personal expression to symbolize two ideas they initially perceived as distinct. To conclude, I discuss the research and teaching implications of my explanatory frameworks for students' symbolization. I also provide an initial theoretical

framing of the cognitive mechanisms by which students create, maintain, and modify their personal algebraic representations.

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This dissertation is the culmination of a five-year foray into academia, a world I knew nothing about when I started this program. As a high school mathematics department head with five years of teaching experience and little upward mobility, graduate school seemed like a natural next step in my career. I looked into the program at Arizona State University and decided to apply. If I were accepted, I would move. If I were not accepted, I would take it as a sign that God wanted me to teach high school for the rest of my career.

To apply, I had to complete an Introduction to Real Analysis course that I was not required to take during my undergraduate schooling. After emailing multiple professors, I persuaded Jack Spielberg to let me attend classes over Skype through a desk-mounted tripod camera. I couldn't always see the board well and had to travel 180 miles (one-way) to take the unit exams in person, but I completed the course with a high grade. After the class, I asked Jack to write me a letter of recommendation for the Ph.D. program. Thank you, Jack, for all you did to make my coming to ASU possible. I wanted you to be on my committee so you could see me complete the journey I started six years ago in your class.

I began my doctoral program under the tutelage of Patrick Thompson and Marilyn Carlson, who taught my introductory mathematics education research courses. Under their guidance, I learned that mathematics education research could focus on epistemology from a psychological perspective. Although Pat retired at the end of my second year, his lessons and discussions on Piaget's Genetic Epistemology paved the way for my decision to study individual student cognition. My interactions with Marilyn have also oriented me toward effective teaching practices and strategies, and her assignments focused on creating descriptive models of student thinking informed my choice to propose a theoretical framework in this dissertation. Pat and Marilyn, you have been central to the ASU mathematics education program for years, and your influence will live for decades to come in current and future mathematics education researchers.

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TABLE OF CONTENTS

CHAPTER Page

CHAPTER Page

CHAPTER Page

CHAPTER Page

LIST OF TABLES

Table Page

LIST OF FIGURES

Figure Page

Figure Page

Figure Page

CHAPTER 1

INTRODUCTION

Unlike most scientific disciplines, mathematics takes objects of analysis that are not perceivable by the five senses (Duval, 2006). For example, it is impossible to see a function, hear a number, or smell a probability. Instead, individuals construct mathematical concepts in their minds (Steffe & Thompson, 2000) and use representations to convey their ideas through language, diagrams, graphs, and algebraic notations (Duval, 2006). These representations play a critical role in individuals organizing, synthesizing, and conveying their thinking; without them, communication between individuals cannot occur (Vergnaud, 2009). Mathematics education researchers have recognized the importance of representations in the learning and teaching of mathematics for decades (e.g., Austin & Howson, 1979; Halliday, 1975; Pimm, 2019; Presmeg et al., 2016) and have adopted or proposed various theories or perspectives by which to study this topic (e.g., Duval, 2006; Glasersfeld, 1995; Godino & Font, 2010; Goldin, 2008; Iori, 2017; Radford, 2013; Vergnaud, 1998)

In this dissertation study, I focus on students' construction of algebraic representations. Over the last 500 years, mathematicians have transitioned from presenting many arguments verbally or in written language to symbolic representations of mathematics (Mazur, 2014). In contemporary mathematics classrooms, algebraic notations are often privileged in textbooks and teacher instruction for infinite series (González-Martín et al., 2011). However, students' acquisition of appropriate meanings for conventional algebraic symbols is often difficult. For example, some students spend a disproportionate amount of time interpreting symbolic notation when reading mathematical arguments (Inglis & Alcock, 2012; Shepherd & van de Sande, 2014). Additionally, Gray and Tall (1994) stated that conventional mathematicians often attribute multiple meanings to a single expression (e.g., considering $\Sigma_{n=1}^{10} \frac{1}{n}$ to denote an additive process and a sum) and posited that the ability to ascribe multiple ideas fluidly to a symbol was a necessary condition to succeed in mathematics coursework. In some cases, students who cannot fluidly attribute multiple meanings to a single expression create novel notation to convey distinct ideas (Eckman & Roh, 2022a, in revision).

Researchers have often chosen to study algebraic representations within targeted grains of analysis, such as students' interpretation of conventional notation (Akgün et al., 2012; Barahmand, 2021; Eckman et al., 2023; Shepherd & van de Sande, 2014), students' collective construction of community inscriptions for instructional topics (e.g., Thompson, 2002; Zandieh et al., 2017), and students' construction of individualized notation for organizing their thinking (Eckman et al., 2023; Eckman & Roh, 2022a, in revision). In this dissertation, I have chosen to investigate individual students' creation and use of algebraic representations to organize their thinking. I have two reasons for this choice. First, I was fascinated by the instances I found in my previous work (e.g., Eckman & Roh, 2022a, in revision) where students (1) attributed non-normative meanings to conventional notation or (2) constructed novel symbols to denote ideas for which conventional symbols exist. Consequently, I hoped to document and categorize more instances where students' individual symbolization did not match convention but seemed capable (in my mind) of functioning as a legitimate representation. Second, I believed that individual interviews would provide a more comfortable setting for students to (potentially) introduce and reason about non-normative symbolization than group or classroom settings, where the pressures of interpersonal communication with peers might inhibit (in my mind) students' creativity.

Although theories of representation (e.g., Duval, 2006; Glasersfeld, 1995; Godino & Font, 2010; Radford, 2013; Vergnaud, 1998) are generally broad enough to apply to multiple contexts, I chose to narrow my focus to students' algebraic representations for one particular topic: infinite series convergence. I focused my study on students' symbolization of series for two reasons. First, González-Martín et al. (2011) have stated that algebraic notations are privileged in calculus textbook prose and teacher instruction. Second, previous researchers have found evidence that students' development of both meaning and symbolization for infinite series is difficult. For example, some researchers have claimed that summation notation, the conventional representation for series, is difficult for students to utilize in their work (Eckman & Roh, 2022a, under review; Katz, 1986; Strand et al., 2012; Strand & Larsen, 2013). Martin (2013) has also positioned series as an intersection of several topics to which researchers have stated that students struggle to provide viable meaning toward, such as limit (e.g., Cottrill et al., 1996; Sierpińska, 1987; Swinyard & Larsen, 2012; Williams, 1991), sequence and sequence of partial sums (e.g., Martin et al., 2011; Martínez-Planell et al., 2012; Oehrtman et al., 2014; Roh, 2008, 2010), and infinity (e.g., Kidron, 2002; Kidron & Tall, 2015; Kidron & Vinner, 1983; Lakoff & Núñez, 2000).

I now summarize my motivation for my dissertation study and propose my research questions. I chose to study student symbolization because representations constitute the only method by which they can externally organize or convey their ideas about mathematical topics. I narrowed my focus to students' algebraic representations because (1) algebraic notations have become ubiquitous as a form of mathematical communication and (2) mathematicians fluidly attribute various meanings to similar symbols according to their needs (implying that students who do not develop this ability will struggle to succeed in a mathematics classroom; Dawkins & Zazkis, 2021; Gray & Tall, 1994). I chose to investigate individual students' attribution of meaning to symbols to remove their feeling of necessity to symbolize according to convention and promote more creative uses of symbols as instantiations of their meanings. Finally, I chose the topic of infinite series because (1) there is a substantial amount of research categorizing students' meanings for topics related to series, (2) algebraic notation is privileged in instruction for series, and (3) the traditional method to symbolize series is problematic for some students. To study individual student attribution of meaning to algebraic representations in the context of infinite series, I propose the following research questions:

- RQ1: *What meanings for series convergence do first-time university calculus students conceive before receiving formal instruction on infinite series?*
- RQ2: *How do students symbolize their meanings for mathematical topics in the context of infinite series?*
- RQ3: *How do students' symbols and attribution of meaning to these symbols change as their thinking about infinite series evolves over time?*

Two grains of analysis are inherent in my research questions: (1) individual student meanings and symbolization at a particular moment (RQ1, RQ2) and (2) the coevolution of the relationship between students' meanings and representations over time (RQ3). To address both grains of analysis, I conducted two individual constructivist teaching experiments (Steffe & Thompson, 2000) comprising an intake interview, seven teaching episodes, and an exit interview with first-time second-semester calculus students. In the following section, I summarize the structure of this dissertation, in which I describe my preparation for this study, how I collected and analyzed my research data, the results of my analysis, and the implications of the study for future research and teacher instruction.

Structure of the Dissertation

This dissertation is comprised of eight chapters. The current chapter, Chapter 1, has three purposes. These purposes include (1) providing a motivation and rationale for my dissertation study, (2) presenting the research questions that guided my study design and analysis, and (3) sharing an overall summary of the dissertation document.

In the next chapter, Chapter 2, I provide an overview and synthesis of the literature that influenced my dissertation study. In this chapter, I divide my discussion into two portions. In the first portion, I address (1) research on the conventional presentation of infinite series in textbooks and teacher instruction and (2) empirical research on students' meanings for various concepts related to infinite series convergence. In the second section, I address students' symbolization of infinite series. Specifically, I summarize (1) research related to summation notation (the conventional algebraic representation for series) and (2) empirical studies of students' attempts to symbolize infinite series.

In the third chapter, Chapter 3, I discuss theories related to the role of symbolization and representations in mathematics education. This chapter contains three major sections. In the first section, I provide the history of representations in mathematics education research and a basic description of theories of symbolization used by contemporary researchers. In the second section, I provide a more detailed comparison of three of these theories to justify my choice of radical constructivism as my perspective for this dissertation study. In the final section, I describe the radical constructivist interpretations of Piagetian constructs related to individual student cognition. This chapter aims to identify my chosen theoretical perspective and introduce some of the initial constructs that I leveraged to create the methodology for this study.

In the fourth chapter, Chapter 4, I describe the structure of my dissertation study, the data collection methodology, and the analysis methods that I used to prepare the results for this study. This chapter contains four major sections. In the first section, I summarize Steffe and Thompson's (2000) five components of a constructivist teaching experiment and how these components were reflected in my data collection methods. In the second section, I summarize the timeline of this dissertation study and describe how the pilot studies I conducted influenced my choice of interview tasks. In the third section, I summarize my data collection methods, including (1) my use of a screening survey and intake interview to identify my two study participants (i.e., Monica and Sylvia), (2) background information about each student, and (3) a summary of the interview tasks across the nine interviews. For more detailed information about my interview tasks, please refer to Appendix B (screening survey) and Appendix C (interview protocols). In the final section, I describe my data analysis, which I conducted in the spirit of grounded theory (Strauss & Corbin, 1998).

I have separated the results of this dissertation study into three major sections, which I will address in individual chapters. In Chapter 5, I describe Monica's and

Sylvia's intuitive meanings for infinite series convergence that emerged during the intake interview. Specifically, I describe one overarching meaning for convergence the students' exhibited and three implications of this meaning that influenced their actions while reasoning about individual series. The material in Chapter 5 is primarily related to my first research question, *what meanings for series convergence do first-time university calculus students conceive before receiving formal instruction on infinite series?*

In Chapter 6, I propose an explanatory framework for contextualizing the number and types of meanings Monica and Sylvia attributed to their various inscriptions and expressions during the teaching experiment. Specifically, I describe three meanings that the students attributed to their symbols and six inscription types they created to re-present these meanings to themselves. The material in Chapter 6 is related to my second research question, *how do students symbolize their meanings for mathematical topics in the context of infinite series?*

In Chapter 7, I discuss two situations in which students' meanings and symbolization coevolved over more than one interview. First, I describe Monica's construction of two distinct mathematical expressions to denote ideas she considered similar. Second, I present Sylvia's modification of the inscriptions comprising one mathematical expression to indicate several examples that she considered to be distinct but share certain properties. The material in Chapter 7 is related to my third research question, *how do students' symbols and attribution of meaning to these symbols change as their thinking about infinite series evolves over time?*

In the final chapter, Chapter 8, I discuss the implications of each results chapter with regard to future research and teacher instruction of infinite series. I also present an initial iteration of a theoretical framework to describe the cognitive mechanisms by which students construct and modify their representations during the course of their symbolizing activity.

CHAPTER 2

LITERATURE REVIEW

The purposes of this chapter are to ground this dissertation study within the current literature regarding (1) students' meanings for and (2) students' symbolization of infinite series convergence. There are two main sections within this chapter. In the first (and largest) section, I address literature related to students' meanings for infinite series convergence (purpose 1). My goals concerning purpose (1) are to provide insight into how my participants might reason about series and further justify the relevance of my research topic. In the second section, I describe research related to how students might symbolize infinite series convergence (purpose 2). My goal regarding the second section is to highlight difficulties students might experience with conventional notation and the explanatory power students might obtain by creating personalized expressions to symbolize series components. The second section also serves to motivate my theoretical perspective chapter, Chapter 3, in which I summarize various general theories of symbolization in mathematics and present the theoretical framing of this dissertation study.

Literature Related to Students' Meanings for Infinite Series Convergence

In this section of the literature review chapter, I address students' meanings and symbolization within a single context: infinite series convergence. There are three components to this section. First, I discuss the current state of infinite series instruction in calculus classrooms. In particular, I will describe a normative meaning for infinite series convergence and discuss how textbooks and instructors present content related to series. Second, I summarize the findings from several empirical studies related to students' meanings for topics pertaining to infinite series, such as sequence, partial sums, limit, and infinity. This section aims to provide insight into how students think about each topic and the difficulties they might encounter while reasoning about series convergence. Finally, I offer an overall summary of the section and contextualize the relevance of this literature to my dissertation study.

Conventional Meanings for Series in Textbooks and Instruction

The purpose of this section is to overview the current state of infinite series coursework in calculus classrooms. I focus on three major areas: (1) the normative meaning and symbolization of infinite series in calculus classrooms, (2) how infinite series are portrayed in mathematics textbooks, and (3) how instructors present infinite series in their instructional sequences. Through this literature review, I highlight the prevalence of algebraic representations of series in coursework and the potentially productive nature of graphical representations of series in instruction.

The Normative Meaning and Symbolization of Infinite Series

The topic of infinite series convergence is central to many key findings in advanced mathematics and informs approximation techniques in physics, engineering, and other physical sciences (Azevedo, 2021). Infinite series convergence is not a simplistic concept but is rather an intersection of many complex mathematical ideas, including sequence, limit, and infinity (Martin, 2013). Much research has been done on the individual topics of sequence (e.g., McDonald et al., 2000; Oehrtman et al., 2014; Przenioslo, 2006; Roh, 2008, 2010b, 2010a), limit (e.g., Cornu, 1991; Cottrill et al., 1996; Roh, 2008; Sierpińska, 1987; Swinyard & Larsen, 2012; Tall & Vinner, 1981; Williams, 1991), and infinity (e.g., Kidron & Tall, 2015; Lakoff & Núñez, 2000; Sierpińska, 1987).

Less work has been done in the area of infinite series, although some researchers have highlighted the importance of the sequence of partial sums (Martínez-Planell et al., 2012), the usefulness of graphs in helping students reinvent the definition of series convergence (Martin et al., 2011), students' struggles to conceive of series convergence appropriately (e.g., Akgün et al., 2012; Barahmand, 2017, 2021; Eckman & Roh, 2022b; Kidron, 2002; Kidron & Vinner, 1983), and students' struggles with creating or interpreting representations of series (Alcock & Simpson, 2004, 2005; Eckman & Roh, 2022a; Strand et al., 2012; Strand & Larsen, 2013).

Broadly speaking, the aforementioned studies have revealed three major obstacles with regard to students' learning of series. First, students struggle to conceive of a sequence as a function (McDonald et al., 2000), which renders the sequence of partial sums useless for reasoning about infinite series (Martínez-Planell et al., 2012). Second, students' conceptions of limit and infinity often preclude them from developing a conventional meaning for series convergence (Barahmand, 2017, 2021; Kidron, 2002; Kidron & Vinner, 1983). Third, many students struggles to create or interpret algebraic representations of series (Alcock & Simpson, 2004, 2005; Eckman & Roh, 2022a; Strand et al., 2012; Strand & Larsen, 2013), which is the privileged medium by which many textbooks portray this topic (González-Martín et al., 2011). Additionally, students' notions of sequence or series behavior and convergence vary from instance to instance (Alcock & Simpson, 2002; Martínez-Planell et al., 2012; Roh, 2008), while the notational conventions for representing these ideas (e.g., summation notation, function notation for general summands) remain consistent.

The normative notation employed by many calculus textbooks (e.g., Larson & Edwards, 2015; Stewart, 2012) and instructors to denote an infinite series is summation (Σ) notation. The general form of an infinite series, the expression $\sum_{n=1}^{\infty} a_n$, is comprised of four distinct inscriptions: (1) the inscription Σ, which represents an additive process; (2) the lower index $n = 1$, which denotes the indexing variable of a series and the position of the first summand of interest in the series calculation; (3) the inscription ∞, which denotes that there is no final summand of interest in the calculation (implying that the additive process never terminates); and (4) the argument a_n, which denotes the general summand of the series in terms of the indexing variable. The entire expression $\sum_{n=1}^{\infty} a_n$ can have one of two distinct meanings. First, the expression can denote the process of summing the infinite terms of the sequence of summands $\{a_n\}_{n=1}^{\infty}$. Second, the expression can indicate this infinite additive process's metaphorical "result" (c.f. Lakoff & Núñez, 2000). I further discuss students' meanings for infinity later in this chapter.

Mathematicians employ the sequence of partial sums, which provides successively more accurate approximations of the value of the infinite series, to bridge the process and metaphorical result meanings mathematicians attribute to the expression $\sum_{n=1}^{\infty} a_n$. To determine the true value of the infinite series, mathematicians determine the limit of the sequence of partial sums, which they might algebraically denote as (1) $\lim_{n\to\infty} S_n$, where $S_n = \sum_{i=1}^{n} a_i$, or (2) $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$. Similar to the expressions for infinite series, mathematicians can also attribute to their algebraic expression $\sum_{i=1}^{n} a_i$ the process of constructing (e.g., writing the terms of) the sequence of partial sums or the resulting sequence as a holistic cognitive entity.

How Textbooks Portray and Symbolize Infinite Series

The purpose of this section is to (a) describe predominant features of textbook instruction on infinite series in North American calculus textbooks, (b) compare the representations predominantly used in textbooks with those shown as productive in the research literature, and (c) further justify the relevance of studying students' symbolization of infinite series as a research topic. I chose to review North American textbooks to further comprehend the conventional tasks that my students were likely to encounter in their coursework. I describe two textbook analyses: (1) González-Martín et al.'s (2011) analysis of infinite series-related content in a selection of Canadian calculus textbooks, and (2) my analysis of infinite series exposition from four American textbooks: Callahan et al. (1995), Larson and Edwards (2015), Stewart (2012), and Thompson et al. (2019).

González-Martín et al. (2011) analyzed the material on infinite series in 17 calculus textbooks used in at least one university course. They concluded that, with rare exceptions, each text presented series in a largely decontextualized manner with few visual or graphical examples. For example, most homework exercises were algebraically-based and included visuals were typically portraits, decorative photographs, and reminders of ancillary mathematical concepts engrained within an example (such as an arbitrary example of the graph of a function). Consequently, González-Martín et al. (2011) stated that algebraic representations of series are privileged within the mathematics community, despite research showing positive impacts on student thinking through reasoning about sequence and series convergence graphically (e.g., Martin et al., 2011; Roh, 2010b).

In my analysis of American calculus textbooks, I also determined a strong preference for algebraic examples of series and few graphical examples (which were generally confined to the introductory sections of the unit and Taylor Series). However, the instructional methods for series varied across texts. For example, two prominent American calculus textbooks (i.e., Larson & Edwards, 2015; Stewart, 2012) order their content by presenting sequences, series, a myriad of convergence tests, Taylor polynomials, Taylor series, and then power series. In contrast, other textbooks (Callahan et al., 1995; see also Thompson et al., 2019) initially present series as an approximation process to determine values of a function within a certain degree of accuracy. These textbooks then present infinite series convergence as a method to produce "exact answers" for the sum of a series. Still, all four textbooks primarily engage students in constructing or reasoning about algebraic representations of series to determine a series' convergence or value.

In summary, many North American calculus textbooks primarily present their infinite series content algebraically. As a result, students must learn to meaningfully express their thinking about series through summation notation (the normative convention for representing infinite series convergence) to reason and communicate about series in their coursework. Consequently, studying how students map their thinking to the summation notation or other representations they create while reasoning about series can provide insight into how to improve students' access to the privileged algebraic representations of infinite series.

I make one final comment about the authors' use of the sequence of partial sums in the textbooks that I reviewed. Although Larson and Edwards (2015) claimed that determining the value of a series is an important question, many of their homework exercises merely required students to assess convergence using a convergence test. One potential disadvantage of limiting students' work to determining convergence is that students rarely utilize the sequence of partial sums. I acknowledge that the exact value of convergence cannot be determined for most series. However, providing students with few opportunities to reason with the sequence of partial sums in their work may lead them to dismiss (or forget) the relevance of this idea (Martínez-Planell et al., 2012). Limiting examples that explicitly address the sequence of partial sums may also be problematic for students' future coursework in higher division mathematics classes, where they must utilize or prove properties of infinite series (Martínez-Planell et al., 2012). One method to increase students' use of the sequence of partial sums in their coursework might be to provide more examples where students must approximate the value of a series within a particular error bound using graphs (see tasks in Martin et al., 2011; Roh, 2010b) or tables (see sample problems in Callahan et al., 1995; Thompson et al., 2019).

Research on Instruction for Infinite Series

This section briefly addresses how instructors present infinite series in the classroom. I offer two examples: the first focuses on primarily algebraic instruction (González-Martín et al., 2011), and the second focuses on the potential positive effects of introducing visual representations during instruction on series (Lindaman & Gay, 2012).

First, González-Martín et al. (2011) interviewed five Canadian calculus instructors to determine common instructional practices regarding infinite series. Most instructors described instructional sequences primarily utilizing formal language and algebraic examples (which reflected the presentational approaches in their textbooks).

When the interviewers asked how the instructors might deepen students' conceptual understanding of series, many proposed algorithmic tasks that lacked visual, graphical, or figural representations. The few instructors who did discuss conceptual tasks related to convergence stated that creating such examples for series was extremely difficult.

Second, Lindaman and Gay (2012) reported a classroom experiment comparing students' exam performance on series-related problems for a control and reform classes. In the reform class, the authors presented visual and graphical examples of series convergence, asked the students to complete written reflections about series, discuss thought-provoking questions during each class session, and participate in collaborative activities (e.g., jigsaw activity). The reform class students scored better on the series-related questions on the chapter exam, midterm, and final than the lecture-oriented control class students.

While the goal of this dissertation is not to study instructional practices about infinite series, the two studies I have summarized in this section informed the design of my teaching experiments tasks. For example, Lindaman and Gay (2012) utilized visual and graphical examples of series convergence. Other researchers (e.g., Martin et al., 2011; Roh, 2008, 2010a, 2010b) have also shown the positive effects that graphical representations of sequences can have on students' meanings for convergence. Additionally, González-Martín et al.'s (2011) report that instructors' commonly use formal algebraic examples of series in their instruction heightens the necessity to study how students construct and utilize these representations in their work. In Chapter 4, I describe my incorporation of graphs and other visualizations into my study design.

Summary of the Conventional Portrayal of Infinite Series

In this portion of the literature review chapter, I have (1) summarized a normative meaning and symbolization for infinite series convergence, (2) presented an analysis of how textbooks conventionally portray the topic of infinite series, and (3) described various instructional practices for teaching the topic of infinite series. In all three sections, the data that I presented confirm that investigating students' creation of an algebraic system by which they can meaningfully describe infinite series is a relevant research topic. I also presented data indicating that presenting visual and graphical examples of sequences (e.g., the sequence of partial sums) can be helpful for students to construct productive meanings for convergence. Including graphical tasks related to series convergence became an important part of my task design for this study, which I discuss in more detail in Chapter 4.

Students' Acquisition of Meaning for Infinite Series

In the following sections, I describe research related to students' meanings for general sequences, the sequence of partial sums, limit, and infinity. I address each topic in an individual subsection and summarize the major points of each topic at the end of this section. My discussion aims to present infinite series as a complex topic comprising various concepts for which students must construct viable meanings to reason productively about convergence. For example, a student who reasons productively about infinite series convergence must leverage their meanings for two topics: sequences (e.g., sequence of partial sums) and limit (or convergence). Additionally, the student's image of limit will be influenced by her meaning for infinity.

Research on Students' Meanings for Sequence

Most empirical studies investigating student thinking about sequences have focused on properties of sequences, such as the limit of a sequence (e.g., Cornu, 1991; Oehrtman et al., 2014; Roh, 2008; Williams, 1991) or sequence as a particular case of function (e.g., Breidenbach et al., 1992; Sierpinska, 1992). Additionally, some researchers have categorized students' general meanings for sequences. For example, McDonald, Mathews, and Strobel (2000) proposed two independent student conceptions for sequence. In the first case, some students considered a sequence as a list of numbers, separated by commas, that portrayed a pattern. Przenioslo (2006) reported that such students might believe a sequence needs to have a discernable pattern (e.g., explicit, recursive, graphical). Sierpińska (1987) stated that such students conceive of a sequence as a well-ordered set. In the second case, McDonald et al. (2000) reported students who conceived of a sequence as a type of function defined by a correspondence between the index and terms of the sequence. Przenioslo (2006) conjectured that students' ability to identify various representations of sequences (e.g., numeric, diagrammatic, graphical) related to their perception of a sequence as an ordering of terms or a correspondence between an index and a set of terms. The second case (i.e., sequence as a function) reflects the conventional method in which mathematicians consider a sequence. However, some scholars (e.g., Martínez-Planell et al., 2012; McDonald et al., 2000) have reported (1) fewer students consider a sequence as a function than as a list, and (2) that there are different levels of operational ability for students within each conception of a sequence.[1]

[1] Both McDonald et al., (2000) and Martínez-Planell et al. (2012) employed APOS theory (e.g., Dubinsky, 1991) to describe the various levels of actions that students with each conception for sequence were capable of performing.

Research on Students' Meanings for Partial Sums

Research on student thinking about the sequence of partial sums is almost nonexistent. The research that does exist addresses students' coordination of visual and symbolic representations for partial sums (Kar et al., 2011), students' construction of novel symbolic representations for partial sums (Eckman & Roh, 2022a, in revision), students' difficulties in constructing the general term of the sequence of partial sums (Eckman & Roh, under review, in revision; Kar et al., 2011), and students' use of the sequence of partial sums to reason about series convergence (Martin, 2013; Martínez-Planell et al., 2012).

This small body of literature makes five major claims about students' reasoning about partial sums. First, many students experience little difficulty reasoning about or symbolizing individual partial sums (Eckman & Roh, 2022a, 2022b). Second, students often struggle to construct symbolic representations of a sequence of partial sums from a visual depiction and vice versa (Kar et al., 2011). Third, students who can verbally describe the process of computing an arbitrary partial sum may not convey their thinking through conventional algebraic symbols (Eckman & Roh, 2022b). Fourth, many students struggle to construct an explicit rule for the general term of the sequence of partial sums (Eckman & Roh, in revision; Kar et al., 2011). Instead, students might employ recursive rules or representational variables to reason about terms in the sequence of partial sums (Eckman & Roh, in revision). Finally, students often struggle to apply the sequence of partial sums to reason about convergence (Martínez-Planell et al., 2012), with many students initially reasoning about series convergence by considering a single, dynamic partial sum (Eckman & Roh, 2022b; Martin, 2013).

In summary, students often comprehend the nature of partial sums but struggle to (1) construct a general term by which to define the sequence of partial sums and (2) utilize the sequence of partial sums to reason about the value of an infinite series. Since the limit of the sequence of partial sums is the value of a series, improving students' comprehension of the sequence of partial sums is a worthwhile research pursuit. Increasing students' exposure to the sequence of partial sums is particularly relevant because some calculus textbooks (e.g., Larson & Edwards, 2015; Stewart, 2012) only sporadically acknowledge this topic, which may prompt students to lose sight of the relevance of partial sums in series convergence (Martínez-Planell et al., 2012).

Research on Students' Meanings for Convergence, Limit, and Infinity

This section aims to provide an overview of the research on convergence, limit, and infinity and apply this research to students' meanings for infinite series. There are two parts to this section. In the first part, I summarize a naturalistic meanings students might possess for series convergence and three approaches students might consider while determining whether a series converges with this meaning. In the second part, I review literature related to various ways in which students at all levels have considered the limit concept. In particular, I summarize research related to (1) two meanings students might possess for convergence of a sequence or function and (2) four meanings students might possess for infinity. For each meaning of infinity, I also contextualize this meaning in research related to student thinking about series convergence.

Research on the limit concept is vast and covers several major mathematical topics. For example, some limit-oriented research has focused on students' meanings for the limit of a function (Cornu, 1991; Cottrill et al., 1996; Swinyard & Larsen, 2012; Tall

& Vinner, 1981; Williams, 1991). Other research has primarily investigated students' meanings for the limit of a sequence (e.g., Barahmand, 2017; Oehrtman et al., 2014; Roh, 2008, 2010; Sierpińska, 1987). Additional studies have focused on the limit concept in relation to infinite or power series (e.g., Barahmand, 2021; Kidron, 2002; Martin, 2013; Martínez-Planell et al., 2012). Other studies have covered miscellaneous topics related to infinite series convergence, such as using technology to think about actual infinity while approximating an infinite series (Kidron, 2002), students' attempts to construct definitions for series convergence (Martin et al., 2011), and comparing students' attempts to determine the value of a series (1) algebraically and (2) within word problems (Akgün et al., 2012).

The research that I cited in the previous paragraph implies that most students either (1) conceive of limit as a dynamic process consistent with approaching a value of the independent variable while tracking values of the function or (2) construct a rigorous definition of limit logically equivalent with the formal definition. The dynamic process meaning for limit is common (Cornu, 1991; Cottrill et al., 1996) and may be a prerequisite for developing more formalized thinking about the limit concept (Swinyard & Larsen, 2012). Developing a rigorous meaning for limit is not intuitive or easy to construct for students (Roh, 2008; Swinyard & Larsen, 2012), although there are instructional recommendations for helping students to make this transition (e.g., Martin et al., 2011; Roh, 2010b; Swinyard & Larsen, 2012).

In the following two sections, I discuss specific student meanings for the limit concept in the context of infinite series (i.e., series convergence). First, I will address students' intuitive approaches to series convergence before receiving formal instruction on infinite series. Second, I provide a general description, grounded in empirical studies, of the evolution of student thinking with regard to limit and certain activities that can facilitate productive student meanings for series convergence.

Students' Intuitive Meanings for Series Convergence. Eckman and Roh (2022b) reported three distinct approaches two students with no exposure to the sequences and series unit in second-semester calculus employed to determine (1) whether a series converged and (2) the value to which a series converged. The students were named Monica and Sylvia.[2] The findings that I report in this section are an abbreviated version of the dissertation findings that I report in Chapter 5 and serve as a preview of the topics that I will discuss in that chapter.

The overarching image of convergence the students leveraged to reason about convergence was a dynamic running total approaching an asymptotic value or an *asymptotic running total meaning*. The students employed three approaches in different interview moments to consider the running total's behavior while reasoning about the convergence of various series. In the first approach, *decreasing summands convergence*, Monica and Sylvia reasoned that if the values of the summands consistently decreased for each successive term in a series, then the running total must approach an (asymptotic) value and converge. In the second approach, *monotone running total divergence,* the students reasoned that (some) monotone series would diverge because they envisioned the value of the running total perpetually increasing (or decreasing), eventually surpassing any potential upper (or lower) bound (i.e., asymptotic value). In the final

[2] Monica and Sylvia are the same students that I report in the results section of this dissertation. The data that I report in this paragraph came from their intake interview and was reported during the 2022 PME-NA conference.

approach, *running total recreation through grouping*, the students combined successive terms in an alternating series to create an almost monotone series. Then they reasoned about the new series to determine the convergence of the original series. Eckman and Roh (2022b) considered the three approaches to constitute meanings in the moment (Thompson et al., 2014). They also reported that students' meanings for series convergence changed across the series they presented in their study. I provide a more detailed analysis of Monica and Sylvia's initial reasoning about series convergence in Chapter 5.

Each method I described in the previous paragraph shows a necessary (but insufficient) property of convergent or divergent series. For example, it is a necessary (but insufficient) property of convergent series that the magnitude of the summands approaches zero as the index increases without bound. It is a necessary (but insufficient) property of a monotone divergent series that the sequence of partial sums perpetually increases (or decreases). Finally, grouping terms to reason about convergence in an alternating series is a property of an absolutely convergent (but not conditionally convergent) series. To differentiate between instances where the intuitive notions of *decreasing summands convergence*, *monotone running total divergence*, and *running total recreation through grouping* are appropriate, a student must develop the ability to reason about series through the sequence of partial sums (Martínez-Planell et al., 2012, make a similar claim about series convergence in general). Consequently, studying how students develop a more rigorous sense of series convergence from their intuitive methods, particularly through constructing and reasoning with the sequence of partial sums, is a relevant line of research work.

Students' Meanings for Limit of a Sequence and Function. The purpose of this subsection is to present research on the various types of meaning that students exhibit for the limit concept, including their ideas about infinity and series convergence. In *particular*, I review two ways in which Swinyard and Larsen (2012) and Oehrtman et al. (2014) reported students' reasoning about limit points. The authors called these two methods for conjecturing and justifying limit points domain-first and range-first perspectives. I devote one paragraph to defining and contextualizing each term.

A student with a *domain-first perspective* might, by initially focusing on the inputs of the function and then the corresponding outputs, determine a limit candidate but be unable to justify his chosen value. Such students will often consider the limit concept to denote a dynamic process of function values tending to an asymptotic value (Cornu, 1991; Roh, 2008; Tall & Vinner, 1981; Williams, 1991) or a cluster point (Cornu, 1991; Roh, 2008). Researchers have also reported that students with a *domain-first perspective* often use terms such as "approaching," "close enough," or "infinitely close" to justify their choice of a limit point (Swinyard & Larsen, 2012; Williams, 1991).

Formal definitions of convergence[3] at infinity rely on a *range-first perspective* or focus on universally quantified error bounds (e.g., "for all $\epsilon > 0$") for the dependent variable of the function (as opposed to a dynamic process of the independent variable increasing without bound). Students' transition from domain to range-first perspectives is often tricky. For example, researchers have reported that students struggle to quantify error bounds defined linguistically (e.g., "infinitely close," "close enough;" Swinyard &

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[3]One such formal definition is "a sequence (a_n) *converges* to a real number a if, for every positive number ϵ, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$ (Abbott, 2015, p. 43).

Larsen, 2012) and in terms of ϵ (Roh 2008, 2010a). Still, research has shown that certain tasks, such as explicitly asking students to quantify error bounds (Swinyard & Larsen, 2012), engaging in graphing activities (Martin et al., 2011; Roh, 2008, 2010a, 2010b), and using manipulatives (Roh, 2010b) might help students to adopt formal definitions of limit and convergence more easily.

Students' Meanings for Infinity and Convergence. The notion of infinity is present in both the limit concept and the nature of infinite sequences and series. Two general conceptions of infinity have existed since the time of Aristotle: potential infinity and actual infinity (Lakoff & Núñez, 2000). Potential infinity refers to dynamic processes that continue without end, and actual infinity is "infinity conceptualized as a real 'thing'" (Lakoff & Núñez, 2000, p. 158). Sierpińska (1987) leveraged the notions of potential and actual infinity to describe four ways a student's conception of infinity might impact his meaning for the limit of a sequence: a finitist attitude, a potentialist model, a potentialist actualist model, and an actualist model. In the following sections, I describe Sierpińska's (1987) findings and relate her constructs to the work that has been conducted on students' meanings for infinite and Taylor series.

The finitist *attitude and its relationship to series.* A student might believe that infinity does not exist or only exists theoretically and cannot be rigorously applied to a limit problem. Sierpińska (1987) called this meaning a *finitist attitude* towards infinity. To such a student, infinity is an abstract idea that cannot be applied to specific situations, including those involving the limit concept. Students with a finitist attitude generally believe that every sequence has a last term and that this term either approximates the limit or is the limit value.

Students who apply a finitist attitude to series might assume that properties of finite addition apply to infinite addition. For example, Barahmand (2021) reported the results of a questionnaire on infinite series given to a group of Iranian high school students asked to compare two divergent series, $A = \sum_{n=1}^{\infty} n$ and $B = \sum_{n=1}^{\infty} 2n$. Some of the students stated that $B > A$ because they considered the ith partial sum of B to be $B_i = 2 + 4 + \cdots 2i = 2(1 + 2 + \cdots i) = 2A_i$, or twice the value of the corresponding partial sum of series A. This generalized process of comparing partial sums is a critical element of the comparison test for series convergence. However, these students' decision to make general statements about the values of two divergent series by comparing partial sums indicates that the students likely considered the series to terminate at a large, finite number.

The potentialist *model and its relationship to series.* Some students might apply a meaning consistent with *potential infinity* to describe the limit of a sequence as a dynamic process of approaching a value. Sierpińska (1987) called this meaning a *potentialist model* for limit. The potentialist model is consistent with a domain-first *perspective*. Sierpińska (1987) stated that students adopting the potentialist model envision the limit of a sequence as a function of time. In this case, a student might say that a sequence can approach (but never reach) a limit value because the process of generating sequence terms will never end. This way of thinking is consistent with reports of students considering sequence convergence (e.g., Roh, 2008) and series convergence (e.g., Eckman & Roh, 2022b; Martin, 2013) as a dynamic value approaching an asymptotic value.

The potentialist actualist *model and its relationship to series***.** A student with a domain-first *perspective* who envisions the limit concept as a never-ending process of generating and evaluating sequence values could consider (hypothetically, of course) what might occur if the process were to end after an infinite amount of time. Sierpińska (1987) stated that such a student possesses a *potentialist actualist* model for limit. As with the finitist attitude, a student whose meaning for the limit concept contains a potentialist actualist model will consider the limit of a sequence to be the ultimate term. However, a student with a potentialist actualist model for limit considers the limit process as an infinite, dynamic process that has (theoretically) ended. For example, Barahmand (2021) stated that most students comparing $A = \sum_{n=1}^{\infty} n$ and $B = \sum_{n=1}^{\infty} 2n$ selected that $A > B$ because, to these students, when the infinite series was completed, series A (the sum of the natural numbers) would have all the elements of series B (the sum of the even numbers) as well as all of the odd numbers. These students likely envisioned both the individual terms of the series and the result of adding these terms. Students utilizing a potentialist actualist model of infinity to reason about convergence are likely in the process of reconciling potential and actual infinity.

The actualist *model and its relationship to series.* Although Sierpińska (1987) did not report an optimal student meaning for limit, she did provide a fourth set of meanings related to students who considered *actual infinity*. Sierpińska (1987) called these meanings the *boundist* and infinitesimalist models of the limit concept. Students with a boundist model for limit distinguished sequences as bounded or boundless and identified the values that bound the sequence. Students who exhibited an *infinitesimalist* model for limit stated that the limit of a sequence is a value such that the difference between the limit and the tail of the sequence is infinitely small.

Martin (2013) reported two methods of determining Taylor Series convergence analogous to Sierpińska's (1987) *boundist* model. First, some individuals created closer approximations of a desired function by appending terms to a single Taylor polynomial until a pre-determined condition was reached. This image of convergence, called a *dynamic partial sum* image, was the most common image for series displayed by the participants in the study. Second, some mathematicians and students focused on (1) evaluating a Taylor series at a desired value of the independent variable of the target function and (2) appending terms to a corresponding Taylor polynomial until they achieved their desired approximation accuracy. This image of convergence, called a *particular x* image, was used by participants who wanted to determine convergence at a single point and not over an interval. These two meanings are similar to Sierpińska's (1987) *boundist model* since the mathematicians and students focused on creating a Taylor polynomial approximating the series within a certain bound.

Martin (2013) reported two other methods of determining Taylor Series convergence that I consider analogous to Sierpińska's (1987) *infinitessimalist* model. First, some mathematicians focused on the distance between the function and the approximation (i.e., error or remainder) going to zero in the expanded Taylor series. In other words, these mathematicians envisioned a dynamic process of simultaneously appending additional terms to a Taylor polynomial and imagining the distance between the function and the approximating Taylor polynomial shrinking. Second, most mathematicians stated at some point during their interviews that the sequence of Taylor polynomials (i.e., sequence of partial sums) converged to the Taylor series. In each case, the mathematicians envisioned either directly (i.e., focusing on the distance between the Taylor polynomial and series shrinking) or indirectly (i.e., focusing on “convergence”) that the difference between the Taylor polynomial and the target function becomes negligible as the number of terms increases without bound.

Summary of Research on Students’ Meanings for Infinite Series

The literature on students’ conceptions for sequence, partial sums, limit, and infinity provides substantial insight into students’ meanings for these topics. For example, students often consider sequences as ordered lists or as a function mapped between an index and the values in the list. Students have little trouble reasoning about individual partial sums but struggle to construct or symbolize the sequence of partial sums. Students’ intuitive sense of series convergence is often that of a running total (i.e., dynamic sum) approaching an asymptote. Students attempting to develop a more formal notion of the limit concept often struggle to (1) transition their thinking to consider error bounds, (2) quantify these error bounds, and (3) reconcile the competing notions of *potential* and *actual* infinity to contextualize a limit value. Researchers have identified explicit teaching interventions to help students overcome their obstacles for many of these topics, including explicitly asking students to quantify error bounds (Swinyard & Larsen, 2012), providing opportunities to reason about convergence graphically (Martin et al., 2011; Roh, 2010b), and utilizing manipulatives to address issues of quantification (Roh, 2010b). The complexity of student thinking related to convergence (as reported in the literature) justifies further research in students’ meanings for infinite series. Additionally, the obstacles posited by several researchers (e.g., Kidron, 2002) regarding comprehending and symbolizing infinite series highlight the need for additional research on how students represent their meanings through symbolic expressions.

Students' Symbolization of Infinite Series

The goal of this study is to compare students' meanings for infinite series convergence with their methods of symbolizing their meanings. Symbolization research is vital for infinite series because most instruction and examples students experience for this topic are grounded in algebraic symbols (González-Martín et al., 2011). Additionally, many researchers (e.g., Eckman & Roh, 2022a; González-Martín et al., 2011; Kidron, 2002; Martínez-Planell et al., 2012) have described the dual meaning of notations related to addition (e.g., summation notation), which mathematicians use to describe infinite series as a process or an infinite sum according to their needs. Thus, students wishing to participate in discussions about infinite series in their calculus coursework must develop an ability to normatively utilize summation (or some other) notation to convey their thinking about series.

In this section, I present two major ideas. First, I summarize the research on students' and mathematicians use and interpretation of summation notation as a representational system. Second, I summarize research related to students' development of representations to denote their thinking about components of infinite series. This section aims to (1) showcase how students might experience difficulty conveying their thinking about infinite series through conventional notations and (2) highlight the advantages that encouraging students to create novel representations affords with regard to modeling student thinking.

Research on Summation Notation as a Representational System

Little research explicitly targets the usefulness of summation notation as a symbolic system and students' use of summation notation as a convention. The research that exists primarily critiques or studies students' conceptions of the indices of summation notation. For example, Katz (1986) suggested that the separation of the upper and lower indices in the summation notation (i.e., m, n in $\sum_{i=m}^{n} a_i$) might encourage students to disassociate these bounds and overstate the relationship between the variable i and the lower index m. Katz's (1986) remark is syntactic, and his recommended change to the notation, $\sum_{i}^{m,n} a_i$ serves to (1) remove the equal sign from the expression and (2) make the relationship between the indices more explicit.

Strand et al. (2012) reported two ways in which second-semester calculus students struggled to interpret the indices in summation notation. First, students were often unsure how to increment the summation index when approximating the area under a curve.[4] For example, Strand and Larsen (2013) hypothesized that a student who wrote the expanded notation $\sum_{i=2}^{4}(i-1) = (2-1) + (4-1) + (6-1) + (8-1)$ considered the lower index $i = 2$ to imply that the space between each value of the index was two units. Second, students struggled to determine whether the upper index value referred to the number of summands in the sum, the final index value by which to evaluate the general term, or both.[5] Although Strand and Larsen (2013) did not provide this interpretation in their report, it is possible that the student who wrote $\sum_{i=2}^{4}(i-1) = (2-1) + (4-1) +$

[4] Conventionally, the increment of the index is one because the domain associated with summation notation is the natural numbers. However, the increment of the index is not given explicitly in the notation.
[5] Conventionally, the upper index refers to the final index value. The upper index only refers to the number of summands in a sum if the lower index value is 1.

$(6-1)+(8-1)$ interpreted the upper index 4 to represent the number of summands that he needed to generate. If this were the case, then the students' work can be explained in the following manner:

1. The lower index $i = 2$ simultaneously refers to the initial value of the index and the distance between each subsequent value of the index. So, the domain of the index is the set $\{2, 4, 6, 8, \ldots\}$
2. The upper index 4 refers to the number of summands in the finite sum, meaning there will be four summands. These summands are calculated using the index values $i = 2, 4, 6, 8$
3. Thus, the expression $\sum_{i=2}^{4}(i-1)$ can be expanded as $\sum_{i=2}^{4}(i-1) = (2-1)+(4-1)+(6-1)+(8-1)$.

While the purpose of this dissertation is not to provide productive instructional practices for teaching summation notation as a convention, the publications I reported in the previous paragraphs provide insights into necessary areas of focus while introducing this notation. In particular, instructors should (1) focus their discussion on the properties of the indices when discussing summation notation for denoting additive processes and (2) distinguish between the starting value of the index, the domain of the index, the final value of the index, and the number of summands in a particular sum. One possible approach might be to present the students with a finite sum written in expanded form, such as $3+7+11+15$, and an expression in summation notation with a starting value other than 1 (e.g., $\sum_{i=5}^{8} 4(i-5)+3$). The instructor might then provide students with two *peer interpretations* (Halani et al., 2013) in which hypothetical students (1) expand the summation notation using the method I described above based on Strand and Larsen's

(2013) data and (2) expand the notation in a normative way. Such an activity would likely provide students insight into the role and rules for indices in summation notation.

Research on Students' Symbolization of Infinite Series

There is very little research on students' construction of representations to describe infinite series convergence. In the following paragraphs, I summarize the results from my work on individual student symbolization for two students: Emily and Cedric. I have previously reported the results of Emily's notation construction in Eckman and Roh (2022b; under review) and Cedric's symbolizing practices in Eckman and Roh (in revision).

Both students participated in two exploratory interviews to reason about and create personalized representations to symbolize arbitrary partial sums. I devote one section to describing Emily's symbolization, another to describing Cedric's symbolization, and a final section to summarizing the contributions the two studies provide to understanding students' symbolization of infinite series.

Emily's Symbolization of Infinite Series

During her first interview, Emily developed a novel set of inscriptions by which she described the process of computing the value of an arbitrary partial sum. These inscriptions functioned (to her) as a literal translation of a written rule she constructed to describe how she computed partial sums. In the second interview, Emily introduced a modified version of summation notation to denote a "holistic" image of a partial sum or series. She then began to use her novel notation to describe series for which she could not discern a general summand and summation notation to denote series whose general term she could describe. After the interviewer encouraged Emily to reflect on how she might represent a series whose general term she could not discern using summation notation, she constructed an inscription for an unknown general summand that she could use in her modified summation notation. After this moment, Emily used her newly constructed notation purposefully to describe individual partial sums, arbitrary partial sums, and infinite series.

Cedric's Symbolization of Infinite Series

Cedric introduced summation notation on the first task but had several non-normative meanings and inscriptions he included in his summation notation. For example, Cedric initially believed that the upper index could only be equal to infinity and openly wondered whether writing a finite value in the upper index constituted an act of mathematical heresy. Cedric also compared summation notation to a computer loop. He used this metaphor to propose a novel symbol as the placeholder for the series' closed-form general summand rule. Like Emily, Cedric struggled to symbolize and give meaning to series for which he could not construct a rule for the general term of summation.

During Cedric's second interview, he introduced a new inscription, ☺, to describe three potential scenarios he envisioned regarding the general term of a series: (1) he knows a general term exists and he can describe this term, (2) he knows a general term exists, but he cannot yet describe the term, and (3) the general term does not exist because the summands have been generated randomly. After this moment, Cedric purposefully used his modified version of summation notation to represent specific partial sums, arbitrary partial sums, and infinite series.

Summary of Emily's and Cedric's Symbolization

Emily and Cedric successfully created representation systems to symbolize arbitrary partial sums and infinite series. Eckman and Roh's (under review) report of Emily's symbolization focused on her thinking and how the meanings she attributed to her inscriptions changed over time. In particular, they noted Emily's conception of the general term of a series changed from the result of a carried-out process to an arbitrary placeholder for an unknown (but imagined) rule. Emily's attribution of her new meaning to a pre-existing inscription (to which she had previously attributed her "carried-out process" meaning) allowed her to purposefully use her representations to describe any partial sum or series which we presented. Eckman and Roh's (in revision) report of Cedric's symbolization focused on instructional interventions that helped orient Cedric towards productive reflection on his meanings and symbolization of infinite series. Specifically, they noted that engaging Cedric in Radford's (2000) three-phase approach to symbolizing (i.e., verbally reason, construct a written rule, create a symbol) and asking him to apply his symbols to model phenomena in new situations culminated in Cedric's construction of a flexible notational system by which he could describe any series which we presented. Additionally, Eckman and Roh (in revision) noted that providing Cedric the flexibility to construct his own notation and asking him to compare instantiations of his notation for consistency promoted opportunities for reflection, resulting in new inscriptions or attributing additional meanings to existing inscriptions.

The results from Eckman and Roh's (under review, in revision) reports on Cedric and Emily justify the relevance of this dissertation and other studies investigating students' construction of representations to organize or convey their thinking in the context of infinite series. In the next section of the literature review chapter, I broaden the scope of the discussion on students' symbolization to a description of general theories of representation in mathematics education.

Summary of Literature Review Chapter

This chapter had two purposes: (1) to summarize literature related to students' meanings for infinite series and (2) to summarize literature related to students' symbolization of series. In the first section of this chapter, I described how textbooks and instructors portray infinite series and summarized empirical studies related to students' meanings this topic (purpose 1). I concluded that the research literature provides substantial insight into how students think about sequences, partial sums, convergence, and infinity. Additionally, I stated that my work has the potential to provide further insight into the intuitive ways students might consider series convergence and how their thinking changes over time. In the second section of this chapter, I described research related to the convention of summation notation and students' attempts to construct personal representations for arbitrary partial sums (purpose 2). I concluded that studying students' symbolization of infinite series is a productive line of research. It can provide novel insights into how students think about series and how they use conventional and novel symbols to communicate their meanings for this topic. In the next chapter, Chapter 3, I introduce the theoretical framing I will utilize to report my dissertation data. Specifically, I leverage the radical constructivist perspective on Piaget's (2001) theory of reflective abstraction as a lens to study students' symbolization practices and the nature of the algebraic representations they create to convey their thinking.

CHAPTER 3

THEORETICAL PERSPECTIVE

In this chapter, I discuss theories related to the role of symbolization and representational in mathematics education. This chapter contains three major sections. In the first section, I provide the history of representations in mathematics education research and a basic description of theories of symbolization used by contemporary researchers. In the second section, I provide a more detailed comparison of three of these theories to justify my choice of radical constructivism as my perspective for this dissertation study. In the final section, I describe the radical constructivist interpretations of Piagetian constructs related to individual student cognition. I also present various constructs related to students' symbolization that I have described in other research reports. Overall, the goals of this chapter are to (1) describe the importance of representations in mathematics and mathematics education research, (2) justify my chosen theoretical perspective for this study, and (3) provide an initial description of the theoretical constructs I leveraged to construct my tasks, guide my interactions with students, and inform my analysis.

Theories of Representation in Mathematics Education Research

Representations (e.g., words, graphs, symbols, gestures) constitute a fundamental component of communication (Vergnaud, 2009). They are also the primary mechanism by which individuals access mathematical ideas not discernable by their five senses (Duval, 2006). "Without words and symbols, representation and experience cannot be communicated" (Vergnaud, 2009, p. 92). Over the last 500 years, mathematicians have gradually moved from describing mathematics verbally to symbolically (Cajori, 1993;

Mazur, 2014). In modern mathematics classrooms, instructors expect students to comprehend mathematical ideas and adopt the corresponding normative forms of representations mathematicians employ to communicate about these topics. For these reasons, students' development of the ability to construct and employ appropriate representations in their mathematics coursework is an essential topic for research study and instructional focus.

A Brief History of Mathematics Education Research and Representations

Mathematics education researchers have studied the role of representations in mathematical discourse for decades. In the 1970s, early mathematics education researchers studied representations by adopting constructs from the field of linguistics. For example, these researchers hosted conferences to compare linguistics to mathematics (see Austin & Howson, 1979, for a summary). They also adopted linguistic constructs such as *register* (Halliday, 1975) to describe students' symbolization (see also Duval, 1999, 2006; Pimm, 2019).

Researchers later attempted to describe representations by leveraging linguistics or semiotics theories. For example, Glasersfeld (1995) adopted the linguistics theory proposed by Ferdinand de Saussure (1857-1913; Switzerland) to provide a cognitively oriented description of students' symbolization. Alternately, other researchers have used the principles of semiotics proposed by Charles Sanders Peirce (1839-1914; United States) to propose sociocultural-oriented theories of symbolization (Presmeg et al., 2016; Radford, 2006, 2008, 2013) or cross-cutting theories encompassing both individual cognition and community activities (Font et al., 2007, 2013; Godino et al., 2007; Godino & Font, 2010).

Other mathematics education researchers have attempted to construct original theories of representation. For example, Goldin (2008) described students' construction of representational systems to convey their neurological activity to others. Additionally, Vergnaud (1998, 2009) presented representations and Piagetian schemes as two distinct but related concepts through which individuals come to communicate their experiences. Mathematics education researchers have also studied various forms of representation, including visual imagery (e.g., Alcock & Simpson, 2004; Arcavi, 2003), graphs (e.g., David et al., 2019; Moore & Thompson, 2015), and algebraic notation (e.g., Lannin et al., 2006; Radford, 2000; Tillema, 2007; Zandieh et al., 2017).

The Relationship between Representations and Cognition

The role of representations in mathematics is to coordinate three facets of semantic meaning: (1) the representation, (2) the thing being represented, and (3) the entity (e.g., individual, community) which maintains the meaning. The various theories of representation I described in the previous section attribute different degrees of variability to individual facets and the nature of the relationships between the three facets. In the following paragraphs, I will use the inscriptions (1), (2), and (3) to refer to the individual facets during my description.

Most theories of representation, including those based in part on linguistics (Duval, 2006; Glasersfeld, 1995), semiotics (Presmeg et al., 2016; Radford, 2008), or mathematics education research (Goldin, 2008; Vergnaud, 2009), claim that mathematical concepts (2) and symbolic notations (1) are distinct entities. Additionally, many of these same researchers claim that representations (1) play a crucial role in thinking and reasoning mathematically (2). For example, students use representations to think about mathematics (Duval, 2006; Vergnaud, 2009), carry out operations in contexts that enable them to modify their thinking (Radford, 2013), develop connections between components of their experience (Font et al., 2007) and construct complex syntactical structures to model these connections (Goldin, 2008). Inherent in students' development of representation systems is their acquisition of the ability to reason about the underlying semantic content of the representation (Glasersfeld, 1995; Radford, 2013).

Within the various theories for representation, there are several characterizations of the ontological nature of what is being represented (2). For example, some theories characterize social interactions as the impetus by which society (3) passes historical forms of representation (and subsequent culturally-preserved meanings) from generation to generation (e.g., Radford, 2008). In this case, the meaning (2) is a series of actions and reflections embedded into students' minds through participating in cultural events (Radford, 2013). In contrast to the sociocultural interpretation of meaning, other theories posit that cognition (3) is the motor by which individuals construct forms of representation (1) to which they attribute their model for a particular experience (2) (e.g., Glasersfeld, 1995; Vergnaud, 2009). In these cognitive-psychology-based theories, the thing being represented (2) is an idiosyncratic, cobbled-together set of components of previous experience, which the student imputes to a semiotic representation (1).

There are also theories of representation that attempt to address both individual and social aspects of representation and knowledge. One example is the Onto-Semiotic Approach (Font et al., 2007; Godino & Font, 2010). While theories of representation from a sociocultural and cognitive psychology perspective describe the entity that maintains the meaning (3) from a single perspective (e.g., the individual or the community), the Onto-Semiotic Approach takes facet (3) as a parameter that can be assigned according to the entity (e.g., individual, department, school, country) which appears (to the researcher) to maintain the coordination of the thing being represented (2) and the representation (1).

Comparing Three Approaches to Describing Students' Symbolization

In the previous section, I provided a global description of theories of representation within the field of mathematics education. In this section, I share examples of how a researcher might employ particular representation theories in their data collection and analysis. I also state the types of research questions each perspective affords and how these questions align (or do not) with my research goals for this dissertation study. This section aims to (1) contrast major theoretical perspectives for describing students' symbolizing practices and (2) justify my choice of theoretical perspective.

The three theories that I will summarize include the following:

1. Radford's (2006, 2013) theory of knowledge objectification, which focuses on symbolization as a social process oriented towards gaining access to institutionalized knowledge or systems;
2. Duval's (1999, 2006) semio-cognitive (semiotic and cognitive) approach, which describes symbolization as the ability to represent a concept using various forms of semiotic representations, including the ability to fluidly construct (normatively) equivalent representations both within and across these forms of representation; and

3. Glasersfeld's (1995) interpretation of Piaget's (2001) theory of reflective abstraction focuses on symbolization as an individual process oriented toward constructing an individualized symbol with a corresponding meaning. Glasersfeld's (1995) views are commonly called a *radical constructivist* perspective.

I devote one subsection to each perspective. I conclude this section by comparing each theory to my research questions to justify my choice of theoretical perspective, Piaget's theory of reflective abstraction (as described by Glasersfeld).

Theory of Knowledge Objectification: Symbolizing Activity as a Social Process

Radford's (2006, 2013) theory of *knowledge objectification* focuses on the social activities by which students become exposed to representations, such as students' acquisition of (conventional) algebraic notations during instructional sequences in the classroom. In the theory of knowledge objectification, "knowledge is an ensemble of culturally and historically constituted embodied processes of reflection and action" (Radford, 2013, p. 10), and "objectification is a social process of progressively becoming critically aware of encoded forms of thinking" (Radford, 2013, p. 27). According to Radford's (2013) theory, the learning process consists of students engaging in social activity to become aware of social and cultural practices. The individual imbues these activities with meaning—thus constructing a knowledge object that can mediate the individual's participation in social activity in the future. Consequently, an individual creates expressions to represent what she perceives as potential invariants involved in the social activity (Iori, 2017). A researcher who adopted the theory of knowledge objectification would likely agree with Bagni (2005a, 2005b), who claims that the historical processes through which conventional mathematical topics emerged can provide critical insights into how students might learn about particular concepts.

A researcher might propose this question while adopting the theory of knowledge objectification to examine students' symbolizing activity related to infinite series: *what social activities might a student need to engage in to generalize the notion of the limit of the sequence of partial sums as the value of an infinite series?* To answer this question, the researcher would likely adopt a similar methodology to Radford (2000), who had groups of students examine a sequence portrayed in a diagram and determine (1) the value of several terms in the sequence, (2) a written rule to determine the value of any term in the sequence, and (3) a symbolic rule—based upon the written rule—by which the group could calculate the value for any term in the sequence. These three steps could be modified as follows to investigate students' meanings for the sequence of partial sums: (1) have students approximate the value of the limit within several specific error bounds, (2) determine a written rule (i.e., definition) to determine the value of the limit within any given error bound, and (3) develop a symbolic rule—based upon the written rule—for representing the process of finding the limit and its subsequent value (if it exists). Martin et al. (2011) adopted methodological steps (1) and (2) of the approach that I described to study students' conception of series pointwise convergence by conjunction with the principle of *guided reinvention* from the Realistic Mathematics Education approach (Freudenthal, 1973).

Semio-Cognitive Approach: Symbolizing Activity as Corresponding Representations

Some theories on students' acquisition and use of algebraic expressions focus on students' development of systems of semiotic representation, including students' ability to translate an expression into an equivalent form in a different system of representation (e.g., algebraic expression, graph, written language rule, drawing). Duval's (1999, 2006) *semio-cognitive approach* describes mathematical activity as an interaction between mathematical objects and forms of semiotic representation or signs. In the semio-cognitive approach, a mathematical object is "the invariant of a set of phenomena or the invariant of some multiplicity of possible representations" (Duval, 2006, p. 129) and the purpose of semiotic sign systems is to "*provide the capacity of substituting some signs for others"* (Duval, 2006, p. 106; italics in original). Additionally, students' development of signs and sign systems is essential to their development of mathematics because they cannot observe mathematical objects empirically—their only access to these objects is through semiotic representations.

Duval (2006) uses the term *representational register* to describe a sign system where representations in the sign system can be transformed. He describes these transformations in two ways: a *treatment*, or transformation that stays within the same register (e.g., $3x + 2 = 2x + 5$ is equivalent to $x = 3$), and a *conversion*, where a transformation results in an equivalent semiotic representation in another register (e.g., constructing the graph of $f(x) = 3x$). In the semio-cognitive approach, researchers only posit students' mathematical comprehension of a concept when they can coordinate at least two registers to discuss the topic.

A researcher who adopts Duval's (1999, 2006) semio-cognitive approach to examine students' symbolizing activities about infinite series might propose this question: *what cognitive difficulties do students experience as they attempt to create equivalent algebraic representations for series (e.g.,* $\lim_{n\to\infty} \sum_{i=1}^{n} a_i$ *,* $\sum_{i=1}^{\infty} a_i$ *,* $\lim_{n\to\infty} S_n$*) and corresponding graphical representations?* To answer this question, the researcher would likely create tasks that prompt subjects to construct or identify an equivalent (to the researcher) representation for a given representation. The researcher would then analyze the data to (1) describe the different methods in which the subjects produced the researcher's desired answer, (2) partition these desirable responses into treatments and conversions between semiotic representations to describe the subjects' mathematical activity, and (3) investigate the difficulty of each treatment or conversion by comparing across subjects to see which transformations were accomplished by the largest or fewest number of subjects (for an example of this kind of study, see Sipes, 2019).

(Glasersfeldian) Reflective Abstraction: Symbolizing Activity as Individual Meaning

Some theories on students' acquisition and use of algebraic expressions focus on students' reflection and abstraction of their experiences and subsequent imputation of the results of these abstractions to individually meaningful signs. Glaserfeld's (1995) interpretation of Piaget's (2001) theory of reflective abstraction (often called a *radical constructivist* perspective) examines students' acquisition of knowledge as a psychological endeavor by which individuals organize neurological stimuli to construct a model of their experience or *experiential reality*. Piaget defined the cognitive entities comprising an individual's experiential reality as *schemes*. In the simplest sense, schemes consist of an individual's organization of the neurological stimuli relayed to the brain, the actions the individual enacted based on the stimuli, and the consequences of those actions. Schemes also induce what Thompson et al. (2014) called a *space of implications*, or the set of possible actions that an individual might make and the outcomes he anticipates from each action. Schemes constitute a dynamic system of meanings that continuously evolve as the individual persists through experiential reality (Thompson et al., 2014).

From a radical constructivist perspective, the purpose of a symbol is to activate a scheme, which evokes a sophisticated re-presentation of a previous experience in the individual's mind. In other words, the purpose of a symbol is to stand as a proxy for some component of previous experience (e.g., relationship, operation, concept). In this way, symbols serve as a recollection tool and the mechanism by which inferences occur (i.e., space of implications leading to action).

A researcher who adopts a radical constructivist perspective to examine students' symbolizing activity concerning infinite series might ask this question: *what components of students' experience do they leverage while reasoning about infinite series convergence, and in what ways do they attribute these schemes to representations?* To answer this question, the researcher would likely (1) construct a set of tasks related (in the researcher's mind) to series convergence, (2) have the students work through these problems, (3) repeatedly ask each student to describe his or her meaning for the different components of each task, and (4) attempt to elucidate instances of potential cognitive conflicts inherent (to the researcher) in the students' reasoning to induce changes in the students' schemes.

In this sort of study, the researcher would have three goals. First, the researcher would determine the types of tasks a student could readily reason through, the representations they appear to employ comfortably, and the corresponding meanings the student's reasoning and symbolizing activity about series convergence affords (a process Piaget called *assimilation*). Second, the researcher would seek to determine tasks that evoke cognitive conflict within students' symbolizing practices and how they resolve these conflicts (a process Piaget called *accommodation*). The researcher's report would summarize her models for (1) how students discerned commonalities among the various examples related to infinite series and attributed these commonalities to representations, (2) the nature of the representations students created to convey their meanings, (3) the instances where the student experienced cognitive conflict and their resolution of this conflict, and (4) propose the meanings by which students operated in each of these situations.[6] I discuss the processes of assimilation and accommodation in more detail later in this chapter.

Comparing the Three Theories to my Research Questions

In Chapter 1, I proposed the following three research questions to guide my study:

- Research Question 1 (RQ1): W*hat meanings for series convergence do first-time university calculus students conceive before receiving formal instruction on infinite series?*

[6] These three points fall in line with the three results chapters that I present in this dissertation. In Chapter 5, I address a common meaning for series convergence possessed by both students in my study. In Chapter 6, I describe the various meanings students' attributed to their symbols. In Chapter 7, I address how students resolved their cognitive conflicts about how to symbolize certain situations about series.

- Research Question 2 (RQ2): *How do students symbolize their meanings for mathematical topics in the context of infinite series?*
- Research Question 3 (RQ3): *How do students' symbols and attribution of meaning to these symbols change as their thinking about infinite series evolves over time?*

The theories I outlined in the previous section can provide insight into these questions for various objects of analysis. For example, adopting the theory of knowledge objectification (Radford, 2013) would allow me to investigate groups of students' meanings for symbols to represent infinite series and how the evolution of the thinking within the group correlates with the group's symbolic expressions. Adopting Duval's (2006) semio-cognitive approach would enable me to describe various representational transformations that individual students perform within and across types of semiotic representations as they attempt to reason about infinite series. Although both Duval's (2006) semio-cognitive approach and Radford's (2013) theory of knowledge objectification can provide insight into my research questions, the foci of the theories do not align with my goals for this study. For example, I wish to study individuals' symbolizing activities, not the activity of a collaborative group. Additionally, I hope to study students' creation of individual symbolic expressions (and corresponding meanings), not their correlations between various forms of representations.

Adopting Glasersfeld's (1995) interpretation of Piaget's theory of reflected abstraction allows me to focus on individual student attribution of meaning to individual symbolic expressions. Additionally, the radical constructivist approach enables me to describe students' symbolization as a process of reflections and abstractions whose purpose is to modify cognitive structures. Consequently, I choose to adopt radical constructivism for my theoretical perspective in this study. In the following section, I provide a more in-depth explanation of (1) the essential constructs in the Glaserfeldian interpretation of Piaget concerning student cognition and (2) the constructs that I have previously introduced (e.g., Eckman & Roh, 2022a, in revision) to describe students' symbolization practices.

The Radical Constructivist Perspective on Individual Student Cognition

This section consists of two major components. First, I describe several Piagetian constructs (e.g., *re-presentation*, *assimilation*, *accommodation*) from a radical constructivist perspective. These constructs form the overarching ontological perspective that I adopted in this study to describe individual student learning and cognition. In the second section, I share several of my own theoretical constructs (e.g., *symbolizing* activity; *personal*, *communicative*, and *conventional expressions*) that I have previously reported in other studies (e.g., Eckman et al., accepted; Eckman & Roh, 2022) to motivate the methodology for this study that I describe in Chapter 4.

Piaget's Theory of Reflected Abstraction (According to Radical Constructivists)

Glaserfeld's (1995) interpretation of Piaget's (2001) theory of reflective abstraction (often called a *radical constructivist* perspective) frames acquisition of knowledge as a psychological endeavor by which an individual organizes neurological stimuli to construct a model of her experience, or *experiential reality*. A consequence of maintaining models of experience is that humans can metaphorically re-experience components of their previous experience as if they were occurring in the moment, a process called a *re-presentation* (Glasersfeld, 1995). Such re-presentations can occur consciously (e.g., recalling an event to tell a story) or subconsciously (e.g., an unexpected smell elicits a memory or emotion). In other words, to re-present is to consider the past in the present. Individuals develop cognitive structures called schemes by reflecting on and coordinating various re-presentations of situations, actions, and consequences,

Many scholars use Piaget's notion of scheme as a theoretical construct (Dubinsky, 1991; Glasersfeld, 1995; Thompson et al., 2014; Vergnaud, 2009), but definitions for scheme vary amongst researchers. I adopt the definition proposed by Thompson et al. (2014) that a scheme is "an organization of actions[7], operations, images, or schemes—which can have many entry points that trigger action—and anticipations of outcomes of the organization's activity" (p. 11). In the simplest sense, schemes consist of (1) an individual's organization of the neurological stimuli relayed to the brain, (2) the actions the individual enacted based on the organization of the stimuli, and (3) the consequences of those actions. The construction of schemes allows an individual to consider components of his current experience as re-presentations of what he has experienced before. The activation of a scheme also induces what Thompson et al. (2014) called a *space of implications*, or set of possible actions, that an individual might make, and the corresponding outcomes he anticipates from each action.

The primary mechanisms by which individuals utilize or modify their schemes are *assimilation* and *accommodation*. An individual *assimilates* an experience when he associates his current experience with a scheme constructed through his previous experience. In other words, an individual imbues his current experience with meaning by

[7] I use the term *action* throughout this section in the sense of Piaget, who said that actions consisted of "all movement, all thought, or all emotions that respond to a need" (Piaget, 1968, p. 6).

(1) associating it with previous experiences and (2) making inferences regarding how to act based on their re-presentation of his previous experiences (Johnckheere et al., 1958; Thompson, 2013). An *accommodation* occurs when an individual experiences cognitive conflict and must modify their schemes to permit assimilation (Tallman, 2015). Thus, schemes constitute a dynamic system of meanings that continuously evolve as the individual persists through experiential reality (Thompson et al., 2014). I discuss how the notions of *assimilation* and *accommodation* relate to students' symbolizing activity in the discussion section of Chapter 7 of this dissertation.

I introduced the constructs *scheme, re-presentation, assimilation,* and *accommodation* to contextualize Glasersfeld's (1995) definition for a (linguistic) symbol: "a word will be considered a symbol, only when it brings forth in the user an abstracted re-presentation" (p. 99). In other words, the cognitive purpose of a symbol (in the Glasersfeldian sense) is to activate a scheme which evokes a sophisticated re-presentation of a previous experience in the individual's mind. An individual might re-present several kinds of experience through a symbol, such as a process, concept, or relationship, and their corresponding spaces of implications. The actions, properties, and implications an individual re-presents through a symbol constitute the *meaning* he attributes to that symbol. My description of *meaning* is similar to that of Piaget, who likely considered meaning and understanding synonymous with *assimilation* to a scheme (Thompson, 2013; Thompson et al., 2014). Thompson et al. (2014) further characterize *meanings at the moment* as the space of implications existing when an individual assimilates a particular moment of their experience to a scheme. Throughout the remaining sections of this chapter, I will refer to the material a student re-presents through a representation they create as the meaning (in the moment) that they attribute to that symbol.

Preliminary Constructs: Symbolizing Activity and Types of Symbols

In this section, I describe the construct *symbolizing activity*, which I have previously used (e.g., Eckman et al., accepted; Eckman & Roh, 2022) to describe students' use of representations in mathematics. Additionally, I present three types of symbols to contextualize how students and mathematicians utilize representations. The three types of symbols I describe include *personal, communicative*, and *conventional expressions*. I have separated the constructs into subsections to facilitate easier reading of this section.

Symbolizing Activity

I begin by describing the construct *symbolizing activity*. Eckman and Roh (under review) provided the following definitions for symbolizing activity, symbolization, and symbol:

> We define *symbolizing activity* as a process of mental activities that entails students' creation or interpretation of a perceptible artifact (writing, drawing, gesture, verbalization) to organize, synthesize, or communicate their thinking. We refer to *symbolization* as the status of completing the symbolizing activity and the perceptible artifacts as *symbols* (p. 3).

Our use of the term symbolizing activity is similar to and different from other researchers. For example, Tillema (2007, 2010) employed a radical constructivist perspective in his studies but primarily used symbolizing activity to describe interpersonal communications between students about their symbolization. Additionally, Zandieh et al. (2017) used the term symbolizing activity to refer to groups of students who collectively created, debated, and modified symbolic expressions for linear algebra concepts. In this study, I define symbolization to broadly encompass a range of activities, including individuals' creation of symbols to organize their thinking, convey their ideas to others, create shared symbolizing norms, or interpret representations presented as conventions by an authority figure.

Personal Expression

I now describe three types of symbols that students might use in their symbolizing activity. The first of these symbol types is a *personal expression*. Eckman and Roh (under review, a) defined a personal expression in the following way:

> We use the term *personal expression* to describe students' investment of meaning to a self-generated form of representation. In other words, a personal expression is created by an individual to re-present (to himself) or convey (to others) his meaning for a particular topic. There are two components to a personal expression: a meaning and a perceptible artifact to which the student attributes their meaning. A *perceptible artifact* includes any action or product a student produces to convey their meanings (writing, drawing, gesture, verbalization), which another individual might observe with his or her five senses (pp. 3-4).

The two portions of a personal expression, the meaning and perceptible artifact, comprise a student's attempt to re-present their thinking for a particular situation. In this dissertation, I primarily focus on students' creation of algebraic personal expressions. A student's algebraic personal expression may (or may not) have an equal sign and may (or may not) reflect traditional mathematical notation (Eckman & Roh, 2022a). Additionally, students might attribute novel meanings to their expressions whose perceptible artifact appears to mirror convention (Eckman et al., 2023; Eckman & Roh, in revision).

Algebraic personal expressions often contain more than one written component to which an individual ascribes meaning. I use the term *inscription* to refer to each of these components. Eckman and Roh (under review) define an inscription as "a written mark utilized by an author to succinctly describe a property, action, or relationship that the author has envisioned (p. 4)." I use the term *inscription* in this context to refer to instances when a student creates a personal expression for a single situation and it is not clear (to the researcher) whether the student might apply the same personal expression structure to re-present other situations which the individual perceives as similar.

When a student creates a general expression which she uses across examples she perceives as similar, I use the term *inscription* to refer to the syntactic positions of the expression and the term *mark* to refer to the perceivable artifact that a student writes for an inscription in a particular personal expression. For example, suppose a student chooses to construct the personal expression $f(x) = 2x + 3$ to re-present a linear relationship between two covarying quantities. In this case, the inscriptions within the expression would be f, ( ), x, $=$,2, x, $+$, and 3, and the student would attribute a particular meaning (e.g., function name, rate of change) to each inscription. However, if the researcher asked the student to compare two linear relationships and the student created the expressions $f(x) = 2x + 3$ and $g(x) = -3x - 5$, the researcher would need a method to differentiate between the symbols that the student used in each expression for her inscriptions. In this case, I use the term *inscription* to refer to common symbols in both expressions (e.g., function name) and the term *mark* to refer to what the student writes for the "function name" inscription in each expression (e.g., the student wrote the mark g for the function name inscription in personal expression 2). I provide more insight into students' uses of marks, inscriptions, and personal expressions in Chapters 6 and 7 of this dissertation.

The constructs of *personal expression*, *inscription*, and *mark* provide several theoretical advantages to researchers. First, the definition of *personal expression* allows a researcher to clearly distinguish between a student's meaning and the perceptible artifact he constructs to re-present his meaning. For example, describing students' symbolization in terms of personal expressions allows a researcher to clearly describe situations in which a student attributes a normative meaning to novel symbols or a non-normative meaning to conventional notation. Second, my distinction between *inscription* and *mark* accounts for situations in which students might write different things across instantiations of personal expressions through which (in the researcher's mind) the student is re-presenting similar experiences. For example, I report an instance in Chapter 7 where a student successfully modified her personal expression by introducing a new mark for one of her inscriptions, allowing her to re-present an additional class of situations through her altered expression. The purpose of my constructs is not to discuss students' meanings or representations separately. Instead, I employ *personal expression*, *inscription*, and *mark* to describe students' coordination of meanings and perceptible artifacts and the coevolution of these concepts as students' thinking evolves.

Communicative Expression

The second type of symbol that a student might utilize within their symbolizing activity is a *communicative expression*. Eckman and Roh (under review) defined a communicative expression in the following way:

> We propose the term *communicative expressions* to describe expressions that students use within communicative discourse with others. In other words, a communicative expression is a perceptible artifact whose meaning is negotiated within a group of individuals through communicative discourse. For such expressions the users of the expressions in the moment of communication may or may not be the creator of the expression (p. 4).

Interpersonal communication is at the heart of communicative expressions, which adds a broad range of dimensionality to this construct. For example, the evoked meaning by the perceiver of a communicative expression (in the moment of interaction) may not match the intended meaning of the expression creator. An individual might propose their *personal expression* as a medium through which a community might discuss a particular idea. However, the *personal expression* becomes a *communicative expression* at the moment the group of individuals begins to negotiate the use and meaning of the expression. This negotiation can happen informally within groups of students (Zandieh et al., 2017) or in formal situations where an instructor asks students to adopt a mathematical convention in their symbolizing activity (Eckman et al., 2023). Finally, individuals can maintain individualized meanings for a communicative expression (and its inscriptions) even if they have verbally agreed on a shared definition for the expression (Thompson, 2002).

Conventional Expressions

The third type of symbol that a student might utilize within their symbolizing activity is a *conventional expression*. Eckman and Roh (under review) defined a conventional expression as “the lexicon of normative representations that mathematicians uphold as conventional forms of communication within the mathematical community (p. 5). Students are often exposed to conventional expressions while reading mathematics textbooks and attending mathematics classes. In contrast with personal and communicative expressions, the student has little to no negotiating power regarding the meaning that they are supposed to (in the eyes of some authority) re-present through conventional expressions. In other words, personal expressions are created by individuals, communicative expressions are negotiated within communities, and conventional expressions are (expected to be) passively received and adopted by authorities.

An individual may employ the syntax of a conventional expression as a component of a personal expression she creates during her symbolizing activity. However, in the moment of creation, the student ascribes her individualized meaning to the conventional syntax, rendering the conventional expression personal. For instance, a student might choose to write re-present the series $1 + 2 + 3 + 4 + \cdots$ using the conventional expression $\sum_{n=1}^{\infty} n$ but consider both n inscriptions to refer to the position of the summands in the series (which is conventional for the lower index n but unconventional for the general summand n). Still, even if the student had re-presented a meaning analogous to the conventional interpretation of the expression $\sum_{n=1}^{\infty} n$, her attributed meaning (and corresponding perceptible artifact) would have been her own creation, or personalized expression of her meaning.

Summary of Theoretical Perspective Chapter

The purposes of this chapter were to (1) describe the importance of representations in mathematics and mathematics education research, (2) justify my chosen theoretical perspective for this study, and (3) provide an initial description of the theoretical constructs I leveraged to construct my tasks, guide my interactions with students, and inform my analysis. For goal (1), I presented two distinct arguments. First, I provided an overview of several theories related to representations in mathematics education to justify symbolization as a relevant research topic and summarize the longevity of this representation research in mathematics education. Second, I juxtaposed how theories of mathematics education define representations, what is being represented, and the entity maintaining the representation differently.

With regard to goal (2), I showed three specific examples of how a researcher conducting a study on student symbolization might leverage different theories of representation to collect, analyze, and report data. I compared each of these examples to my research questions to justify my choice of a radical constructivist perspective as a guiding theoretical lens for this study.

For goal (3), I described several Piagetian constructs from a radical constructivist perspective to highlight how I construed student learning and cognition in relation to this study. I also presented several theoretical constructs I have proposed in my previous work to describe students' symbolization, including *symbolizing activity*, *personal expression*, *communicative expression*, and *conventional expression* as (1) processes of mental actions students engage in to create or interpret representations and (2) the purposes of these representations and the power the student possesses to negotiate their meaning.

In the next chapter, Chapter 4, I present the methodological information related to this study, including data collection and data analysis methods. In Chapters 6 and 7, I provide empirical data to verify and extend the constructs I presented in this chapter.

CHAPTER 4

METHODOLOGY

The purpose of this chapter is to review the methodological principles that I employed to collect and analyze the data for this dissertation study. My data comprised two individual nine-session constructivist teaching experiments (Steffe & Thompson, 2000) with first-time second-semester university calculus students. The interviews consisted of intake and exit clinical interviews (Clement, 2000) and seven exploratory teaching interviews (Castillo-Garsow, 2010; Moore, 2010; Sellers, 2020) focused on students' construction of appropriate meanings and personal expressions to reason about infinite series convergence. For my analysis, I employed principles of grounded theory (Strauss & Corbin, 1998), including open and axial coding, to interpret my data and prepare the results section.

This chapter comprises four major sections. In the first section, I describe the five components of a constructivist teaching experiment presented by Steffe and Thompson (2000) and how I incorporated these components into my study. This section contains a large portion of logistical information related to the roles of the researchers, the technology I utilized to create tasks and host interviews, and how I recorded and maintained the data. In the second section, I summarize the timeline of my dissertation study and a general schedule of my data collection interviews. In the third section, I describe my data collection methods in detail. Specifically, I address (1) the screening survey and intake interview process and analysis by which I selected my two participants, (2) background information for each participant, and (3) a description of the tasks that I provided for each participant for each interview in the study. In the final section, I present my data analysis methods for interpreting, contextualizing, and reporting the results in Chapters 5, 6, and 7.

Constructivist Teaching Experiment Methodology

In this study, I utilized the constructivist teaching experiment methodology (Steffe & Thompson, 2000) to inform the overarching design of my tasks. The primary purpose of a teaching experiment is for a researcher to experience students' reasoning and learning over time. Such reasoning and learning might include students' language, actions, and thinking patterns, including mistakes that a student consistently makes (Steffe & Thompson, 2000). A researcher conducting a teaching experiment models students' mathematical meanings and posits hypotheses related to the conceptual boundaries of these meanings (Steffe & Thompson, 2000). For example, in my study, I carefully modeled students' meanings for infinite series convergence, how they attributed these meanings to their personal expressions, and how the relationship between their meanings and symbols changed over time.

A constructivist teaching experiment consists of several methodological components. These components include (1) a teacher-researcher, (2) a witness to the teaching sessions, (3) one or more students, (4) a series of teaching episodes, and (5) a method to record the teaching experiment (Steffe & Thompson, 2000). I address these components in the subsections below and delineate how I incorporated each element into my study.

Component 1: The Teacher-Researcher

In each interview, I served as the teacher-researcher. The role of a teacher-researcher is to direct the teaching experiment, manage the design and modification of tasks and task sequences, interview participants during teaching episodes, and supervise both ongoing analysis (between each teaching episode) and retrospective analysis (after the conclusion of the teaching experiment). During the teaching sessions, the teacher-researcher must attempt to perform two distinct tasks: (1) develop viable models for the depth and breadth of meanings students employ during the tasks (what Steffe & Thompson, 2000, called *students' mathematics*) and (2) assist the students in constructing viable meanings to aid them during the tasks (Castillo-Garsow, 2010). Performing these two tasks requires the teacher-researcher to coordinate both an in-the-moment perspective of the interactions between teacher and student and an introspective perspective of these interactions reminiscent of a researcher and his subject (Castillo-Garsow, 2010).

Component 2: The Witness

My advisor, Dr. Kyeong Hah Roh, served as the witness during the teaching experiment sessions. The witness of a constructivist teaching experiment has two primary responsibilities. First, they maintain an observer's perspective on the interactions between the teacher-researcher and the student(s) during each interview. Second, the witness corroborates or challenges the conclusions of the teacher-researcher during each stage of analysis (Castillo-Garsow, 2010). For example, Dr. Roh intermittently provided in-the-moment questions to me through the Zoom chat feature to consider during interviews. Additionally, she participated in various debriefing and preparation interviews in which we discussed our current models for each student's thinking and how to prepare tasks to inspire productive reflections in the upcoming interviews.

Component 3: The Students

A teaching experiment can be conducted with an individual or a group of students (Steffe & Thompson, 2000). For this study, I chose to investigate individual students' creation and modification of personal expressions for series convergence. To this end, I chose two first-time second-semester university calculus students, Monica and Sylvia, to participate in individual teaching experiments. I selected each student after they completed a screening survey and an intake interview. I provide additional information regarding why I selected Monica and Sylvia (as opposed to other students) during the data collection section of the methodology chapter.

Component 4: A Series of Teaching Episodes

Steffe and Thompson (2000) provided little detail about the overarching structure of the teaching episodes in a teaching experiment. In the following subsections, I describe two reasons for this lack of specificity regarding the sessions of a teaching experiment. First, I address how their meanings for "experiment" and "teaching" allow a flexible interview and task structure that can be modified at any time during the experiment. Second, I address how various types of interviews, such as clinical and exploratory teaching, can occur during a teaching experiment according to the interviewer's needs.

The Meanings of **Experiment** *and* **Teaching**

Steffe and Thompson (2000) stated that the term "experiment" in constructivist *teaching experiment* relates to generating and testing hypotheses regarding students' mathematics. For example, as the teacher-researcher, I began a teaching episode with a model for each student's thinking and actions based on her reasoning during the previous teaching session. However, since no model is perfect (Box, 1976), it was inevitable that at some point during the episode, the student acted in a manner that appeared (to me) to be spontaneous, novel, and surprising. In these moments, I constructed a new hypothesis for the student's actions and immediately tested my new hypothesis. If the new hypothesis failed, I either (1) adopted a new hypothesis and proceeded with the current task or (2) presented a spontaneous task that appeared to be a natural and intuitive evolution of the conversation (even if I didn't have a well-formulated hypothesis for why the new task might be useful). For example, I describe an instance in Chapter 7 where I allowed Monica to reason about partial sums graphically and presented spontaneous tasks and questions within this discussion that helped her to coalesce her thinking about this topic.

The term "teaching" in a constructivist teaching experiment refers to the interactions the teacher-researcher initiates with students based on their image of their actions (Steffe & Thompson, 2000). As the teacher-researcher, I sometimes introduced a specific task or dialogue to compare the students' consequent actions against a hypothesis and corresponding model for the students' mathematics that I was considering. I present two instances of specific tasks that I prepared between the Day 1 and Day 2 interviews to address my model of students' thinking about partial sums (i.e., Monica) or symbolization (i.e., Sylvia) in Chapter 7. However, there were other times during the interviews when I was unsure how to model a student's actions. In these cases, I tried to move forward in what appeared (to me) to be a naturalistic way to further the discussion on the topic at hand.

A methodological consequence of these meanings of "teaching" and "experiment" is that the structure and order of tasks in a teaching experiment are fluid, evolving with the hypotheses the teacher-researcher generates during the episode. Consequently, Steffe and Thompson (2000) state that the results of a teaching experiment are unique to the students involved in the teaching experiment (due to their idiosyncratic meanings). Additionally, the results of a teaching experiment cannot be easily generalized or directly replicated.

Types of Interviews Comprising the Teaching Episodes

One goal of a teaching experiment is to help a student achieve a particular learning goal (Steffe & Thompson, 2000). Since teaching experiments often comprise several teaching episodes, the nature of individual interviews can differ. For example, some interviews (or tasks) may be more formal, allowing the interviewer to model student thinking in preparation for a teaching task or assess student thinking after an instructional sequence. Alternately, some interviews may primarily consist of tasks designed to perturb student thinking toward reflection and construction of new ideas. In the paragraphs below, I address the role of each type of interview in my study: formal interviews (i.e., clinical interviews) and teaching interviews (i.e., exploratory teaching interviews).

A *clinical interview* constitutes a formalized setting where the interviewer presents a task to a student and then observes the student's actions (Clement, 2000). A clinical interview aims to monitor and model student thinking without providing teaching or other interventions that might directly impact students' thinking about the interview topic. In my study, I used the clinical interview format to conduct an intake and exit interview with each student. The purpose of the intake interview was to observe students' naturalistic approaches to determining series convergence. I have previously reported the results of my analysis of the intake interview data in Eckman and Roh (2022a). The exit interview aimed to discern the efficacy of the instructional interventions I presented during the teaching interviews regarding students' meanings and symbolization for series convergence. In this way, the clinical interview format fits the intake and exit interview purposes: to evaluate and model student thinking without interventions designed to elicit reflection and changes in thinking. In other words, the intake and exit interviews served as diagnostic and assessment tools rather than teaching mechanisms or interventions.

The *exploratory teaching interview*, first proposed by Moore (2010) and Castillo-Garsow (2010), was adopted as a method for completing the exploratory teaching phase of Steffe and Thompson's (2000) constructivist teaching experiment methodology (Sellers, 2020). The purpose of an exploratory teaching interview is to propose or refine models for student thinking, similar to a clinical interview (Sellers, 2020). However, exploratory teaching interviews differ from clinical interviews in that the teacher-researcher in the exploratory teaching interviews actively seeks to perturb and change student thinking rather than merely describing the interviewee's mental actions (Castillo-Garsow, 2010; Moore, 2010; Sellers, 2020). Exploratory teaching interviews can occur independently, as a prerequisite for a teaching experiment (Sellers, 2020), or during a teaching experiment (Moore, 2010). The exploratory teaching interview format aligns with the purposes of the teaching sessions for this study: to actively perturb and attempt to resolve students' thinking related to series and the students' corresponding personal expressions to represent these topics. In this way, the exploratory teaching interviews constituted an interactive experience designed to promote reflection and learning, rather than a diagnostic or assessment tool.

Component 5: Method to Record the Teaching Experiment

I conducted the teaching experiment virtually through the Zoom platform due to the continued prevalence of the COVID pandemic at the time of my data collection. I recorded each interview simultaneously through the Zoom recording feature and the Camtasia screen recording application. The Zoom recording feature captured (1) the audio, (2) the webcam video of the teacher-researcher and the student, and (3) any screen sharing that occurred throughout the teaching experiment. The Camtasia recording captured the teacher-researcher's screen, including the interview protocol, the Zoom application window, and any chat messages shared between the teacher-researcher and the witness.

I constructed and embedded most of the interview tasks for this study into the Microsoft OneNote application. The only exception was the ϵ-strip applet, which functioned much better on the Geogebra website than as an embedded file in OneNote. I selected the OneNote application because (1) it allowed for collaborative work between simultaneous users and (2) it synced annotations made between devices while preserving the absolute position of the annotations on the OneNote file. I also saved each OneNote file for use in figures, tables, and data analysis.

The largest disadvantages to using Microsoft OneNote were that (1) the application did not sync video playback (or GeoGebra applet manipulation) between devices and (2) there was often a lag between when the student wrote something on their screen and when it appeared on my screen. To mitigate these issues, I asked the student to share their screen through the Zoom application while working on the tasks. Viewing the student's shared screen allowed me to see their annotations in real time (removing disadvantage 2) and their video playback or GeoGebra applet behavior (removing disadvantage 1). I also shared my screen during the times when I was playing a video for the student or modifying an applet to allow the student to see my work.

Timeline of Dissertation Study and Overview of Interview Sessions

The following table, Table 1, shows an abbreviated schedule for the research activities involved in this study (a more detailed plan for the interviews will appear later in the next subsection).

Table 1

Data Collection, Analysis, and Defense Schedule for Dissertation Study

Timeline	**Participants**	**Research Activity**	**Method**
Fall 2019-Spring 2020	Four volunteers	Pilot Study 1: Refining Tasks	Clinical interview
Fall 2020-Spring 2021	Four volunteers	Pilot Study 2: Refining Tasks	Clinical interview
Summer 2021	Three volunteers	Pilot Study 3: Refining interview protocol and tasks	Exploratory teaching interviews
Fall 2021	Second-semester calculus student volunteers	Recruit participants from second-semester calculus	Screening survey Theoretical sampling
	Two students	-Intake interview -Seven exploratory teaching interviews -Exit interview	Teaching experiment (clinical interview, exploratory teaching interview)
Spring 2022-Spring 2023	Teacher-researcher Witness	Ongoing analysis Retrospective Analysis	Grounded theory
Fall 2022-Spring 2023	Teacher-researcher	Write the results of the study	N/A
Summer 2023	Teacher-researcher	Dissertation defense	Presentation

I completed three formal pilot studies in preparation for my dissertation study during the 2019-2020 and 2020-2021 academic years. These pilot studies aimed to test potential dissertation tasks and types of participants. While I was unable to construct rigorous research artifacts from the data I collected during Pilot Study 1 and Pilot Study 2, I have submitted or published various reports from Pilot Study 3 (i.e., Eckman & Roh, 2022a, in revision, under review). I collected my dissertation data during the Fall 2021 semester and have published a research report detailing students' intuitive meanings for series convergence during the intake interview (i.e., Eckman & Roh, 2022b).

I engaged in ongoing analysis of the data from each teaching experiment session to prepare for future interviews. I conducted a retrospective analysis of the data from the Spring 2022 semester through the conclusion of writing this dissertation in the Spring 2023 semester. Although I did some writing during my initial analysis, I wrote most of the dissertation during the Spring 2023 semester. I completed the dissertation defense during the Summer 2023 session to fulfill the requirements to graduate with a Ph.D. in mathematics education from Arizona State University.

In the following sections, I describe (1) my pilot studies and their influence on my methodology for my dissertation study and (2) a schedule of interviews and tasks that I presented during the interview sessions of the dissertation study.

Pilot Study Data and its Influence on my Dissertation Study

In this section, I address the nature of my three pilot studies and how they influenced my task design and study participant selection for my dissertation study. I will not address Pilot Study 1 or Pilot Study 2 in any other place in this dissertation study. I have previously reported several results from Pilot Study 3 (Eckman & Roh, 2022a, under review, in revision) and have commented on this study in several places throughout this dissertation. I address each pilot study in an individual subsection.

Pilot Study 1

I conducted Pilot Study 1 during the 2019-2020 academic year to investigate four second-semester calculus students' meanings for the limit of the sequence of partial sums and infinite series using conventional notation. For Pilot Study 1, I designed tasks that required students to construct symbolic representations of the sequence of partial sums of a sequence of real numbers in the context of physical situations (e.g., comparing the number of seats in subsets of rows in an auditorium).

There were two outcomes from Pilot Study 1 that contributed to my task design for this study. First, I conceived a preliminary version of my *personal expression* construct during my analysis of Pilot Study 1. Second, my reflection on students' difficulties reasoning about the sequence of partial sums influenced my decision to prepare separate tasks addressing individual partial sums, the sequence of partial sums, and the limit of the sequence of partial sums in my dissertation study. I also determined from this study that I wanted to select dissertation study participants who had not yet experienced formal instruction in sequences and series in a second-semester calculus course (at least at the beginning of the teaching period).

Pilot Study 2

I conducted Pilot Study 2 during the 2020-2021 academic year to examine four first-semester calculus students' meanings for series convergence and the limit value of a convergent series. I chose to investigate series convergence in Pilot Study 2 because I recognized from my analysis of Pilot Study 1 that students' thinking about series convergence was likely to be different from their thinking about partial sums and the limit of the sequence of partial sums.

During my analysis of Pilot Study 2, I identified three unconventional student meanings related to series convergence. First, several students stated an infinite series could not converge because it has no end, a phenomenon reported by previous researchers (e.g., Sierpińska, 1987; Akgün et al., 2012). Second, some students suggested monotonic series can't converge because the magnitude of the partial sums perpetually increases (Eckman & Roh (2022b) called meaning *monotone running total divergence*). Finally, several students stated that if the values of the summands in a series converge to zero, the series will converge (Eckman & Roh (2022b) called this meaning *decreasing summands convergence*). My reflection on these three unconventional meanings for series convergence influenced the types of series I included in my dissertation study tasks.

Pilot Study 3

The goal of Pilot Study 3 was to examine second-semester calculus students' ability to construct and utilize personal expressions to re-present partial sums, series convergence, and the value to which a series might converge. I purposefully selected students who had not (at the beginning of the interviews) received formal instruction on sequences and series. Pilot study 3 was broadly successful and provided substantial insights into students' symbolizing activity. For example, I have presented various reports about two students, Emily and Cedric, and their symbolizing activity related to arbitrary partial sums and series (e.g., Eckman & Roh, 2022a, under review, in revision).

In preparation for Pilot Study 3, I modified my tasks and interview protocol based on my findings from Pilot Studies 1 and 2. For example, I leveraged my findings from Pilot Study 1 to construct different tasks related to specific and arbitrary partial sums. I also revised the series I presented during the tasks based on my findings from Pilot Study 2. Finally, I created a video defining inscriptions and personal expressions to introduce these ideas to the students and provide an example of creating a novel personal expression (I describe this video later in the chapter). I also aligned my task sequence with methodological tools reported in the literature, such as Radford's (2000) three-stage process to create symbolic sequence rules and some of Zazkis and Hazzan's (1998) questioning techniques.

My analysis of the Pilot Study 3 data prompted me to include a formal opportunity for students to reflect upon their meanings for their personal expressions at the beginning of each teaching session. After completing the three pilot studies and successfully defending my dissertation prospectus, I determined that my interview protocols and study design were sufficient to begin my formal dissertation study.

Overview of the Teaching Experiment Sessions and Tasks

The purpose of this section is to overview the structure of the study, learning and research goals, and interview tasks comprising each teaching episode. For this study, I conducted two individual constructivist teaching experiments (Steffe & Thompson, 2000) with second-semester calculus students. The teaching episodes were comprised of an intake interview, seven teaching interviews, and an exit interview. The intake interview aimed to determine students' intuitive meanings for series convergence; the purpose of the exit interview was to discern the changes in students' thinking and symbolization of series convergence that took place throughout the teaching experiment. The seven teaching experiment days can be divided into three major sections.

The first section comprised Days 1-4. In this section of the experiment, the students constructed personal expressions to reason about specific partial sums, arbitrary partial sums, sequences, sequences of partial sums, and series. The overarching learning goal for the first section was for students to construct and confidently utilize a set of personal inscriptions by which they could organize their thinking about topics related to series convergence. The overarching research goals for the first section were to (1) investigate the meanings that students exhibited for the sequence of partial sums, (2) determine which ideas related to sequences and series the students believed merited symbolizing, and (3) model how students' meanings and symbolization coevolved over time.

The second section comprised Days 5-6. This section of the teaching experiment was modeled mainly after Roh's (2010) ϵ-strip activity, which I modified to focus on the sequence of partial sums. The learning goals for this section of the teaching experiment were for the students to (1) develop a conventional meaning for sequence convergence, (2) adopt a corresponding written rule through which to re-present their meaning, and (3) symbolize the various graphical components of the ϵ-strip activity by assimilating them to their existing inscriptions or creating new inscriptions. The research goals for this second section were (1) to continually monitor the coevolution of students' meanings and symbolization and (2) to examine the relationship between students' symbolization of series scenarios presented numerically and graphically.

The final section comprised the Day 7 interview. The purpose of the Day 7 tasks was to determine whether students' reasoning about the sequence of partial sums during Days 1-4 and the convergence of the sequence of partial sums on Days 5-6 would provide insights (to them) about infinite series convergence. The learning goals for Day 7 were that the students would (1) symbolize their written rule for sequence convergence; (2) posit relationships between sequences, sequences of partial sums, and infinite series; and (3) construct (and symbolize) a written rule for series convergence based (in part) on their written rule for sequence convergence. The research goals for Day 7 included (1) examining how students coordinate various inscriptions and expressions to symbolize mathematical statements, (2) investigating the relationships students conceive between sequences, sequences of partial sums, and series, and (3) determining how students' thinking about sequence of partial sums convergence influences their meanings for series convergence.

In summary, the teaching experiment focused on (a) students' creation of personal expressions to re-present the sequence of partial sums (Days 1-4), (b) students' development of written rules and symbols to re-present sequence convergence (Days 5-6), and (c) how students' meanings for the sequence of partial sums influenced their thinking and symbolization of infinite series convergence (Day 7). The intake and exit interviews served as benchmark assessments to determine students' initial meanings for series convergence (intake interview) and the effects of the teaching experiment on their thinking and symbolization of infinite series (exit interview). The following table, Table 2, summarizes the basic structure of the teaching experiment.

Table 2

Structure and Overarching Themes of Teaching Experiment

Section	Purpose	Research Goals
Intake	Pre-assessment	• Model students' intuitive meanings for series convergence
Days 1-4	Develop personal expressions for sequence of partial sums	• Investigate the meanings that students exhibited for the sequence of partial sums • Investigate students' development of personal expressions to re-present their thinking about the sequence of partial sums • Monitor the coevolution of students' meanings and symbolization over time
Days 5-6	Develop written rule and symbols for sequence of partial sums convergence	• Investigate students' development of written rules and personal expressions to re-present their thinking about the convergence of the sequence of partial sums • Examine the relationship between students' symbolization of series presented numerically and graphically • Monitor the coevolution of students' meanings and symbolization over time
Day 7	Compare convergence of sequence of partial sums and infinite series	• Examine how students coordinate various inscriptions and expressions to symbolize mathematical statements • Investigate the relationships students conceive between sequences, sequences of partial sums, and series • Determine how students' thinking about sequence of partial sums convergence influences their meanings for series convergence
Exit	Post-assessment	• Evaluate changes in students' meanings and symbolization for infinite series convergence throughout interviews

The following table, Table 3, provides an even more detailed description of the topics of each interview, including the dates and task names for each participant's interviews. Each student participated in an intake interview near the beginning of October 2021. After I selected Monica and Sylvia as participants, I scheduled a recurring weekly interview to conduct the remainder of the teaching episodes. Monica generally attended her interviews at the regularly scheduled times, apart from the week of Thanksgiving (Day 6) and finals week (Day 7). Sylvia experienced a medical issue between Day 2 and

Day 3, requiring her to reschedule the Day 3 interview. Additionally, she rescheduled the Day 6 interview due to Thanksgiving. Both students scheduled their exit interviews at a convenient time (for them) during finals week, which happened to be on the same day. In the following section, I describe each teaching episode's tasks, research, and learning goals.

Table 3

Exploratory Teaching Interview Dates and Topics for Each Student

Interview	**Monica**		**Sylvia**	
	Tasks	Date	Tasks	Date
Intake	**Task 1:** Analyze six series for convergence values **Task 2:** Provide a general definition for series convergence	10/08	**Task 1:** Analyze six series for convergence values **Task 2:** Provide a general definition for series convergence	10/12
Day 1	**Task 1:** Determine specific summand and partial sums in various series **Task 2:** Create a written rule (in English) for describing an arbitrary partial sum **Task 3:** Transcribe a series **Task 4:** View personal expressions video and create personal expression for arbitrary partial sum	10/20	**Task 1:** Determine specific summand and partial sums in various series **Task 2:** Create a written rule (in English) for describing an arbitrary partial sum **Task 3:** Transcribe a series **Task 4:** View personal expressions video and create personal expression for arbitrary partial sum	10/21
Day 2	**Task 1:** Compare integral notation to summation notation for representing partial sums[8] **Task 2:** Use personal expressions to describe specific partial sums, arbitrary partial sums, and infinite series	10/27	**Task 1:** Creating personal expressions to describe series with alternating patterns of + and – signs **Task 2:** Using personal expressions to describe specific partial sums, arbitrary partial sums, and infinite series	10/28
Day 3	**Task 1:** Compare contrasting graphs for sequences (e.g., continuous, dots) **Task 2:** Instruction on sequence, series, and sequence of partial sums **Task 3:** Create a personal expression for sequence of partial sums	11/03	**Task 1:** Using personal expressions to describe series with random summands **Task 2:** Instruction on sequence, series, and sequence of partial sums **Task 3:** Create a personal expression for sequence of partial sums	11/08
Day 4	**Task 1:** Categorize inscriptions in glossary **Task 2:** Compare inscriptions for sequence and sequence of partial sums in table **Task 3:** Compare graphs of sequences and initial attempt to symbolize components of sequence graphs **Task 4:** Initial task about convergence in the context of graphs	11/10	**Task 1:** Categorize inscriptions in glossary **Task 2:** Compare inscriptions for sequence and sequence of partial sums in table **Task 3:** Compare graphs of sequences and initial attempt to symbolize components of sequence graphs **Task 4:** Initial task about convergence in the context of graphs	11/11
Day 5	**Task 1:** Construct inscription for general term of a sequence **Task 2:** Introduction of GeoGebra applet for ϵ-strip activity **Task 3:** Symbolize graphical components of ϵ-strip activity	11/17	**Task 1:** Construct inscription for general term of a sequence **Task 2:** Introduction of GeoGebra applet for ϵ-strip activity **Task 3:** Symbolize graphical components of ϵ-strip activity	11/18
Day 6	**Task 1:** Compare inscriptions in glossary to screenshot of ϵ-strip activity **Task 2:** ϵ-strip activity: comparing two definitions for convergence	11/26	**Task 1:** Compare inscriptions in glossary to screenshot of ϵ-strip activity **Task 2:** ϵ-strip activity: comparing two definitions for convergence	11/23
Day 7	**Task 1:** Symbolize chosen definition for convergence of a sequence of partial sums **Task 2:** Discuss similarities and differences between sequence, sequence of partial sums, and series **Task 3:** Construct a personal written rule for series convergence	12/06	**Task 1:** Symbolize chosen definition for convergence of a sequence of partial sums **Task 2:** Discuss similarities and differences between sequence, sequence of partial sums, and series **Task 3:** Construct a personal written rule for series convergence	12/02
Exit	**Task 1:** Analyze six series for convergence values **Task 2:** Provide a general definition for series convergence	12/09	**Task 1:** Analyze six series for convergence values **Task 2:** Provide a general definition for series convergence	12/09

[8] There were two "review" tasks that students participated in at the beginning of some interviews that are not listed here. First, I had Monica review the personal expressions videos on Days 2-4 and Sylvia review the video on Days 2-3. Second, I had each student review their glossary of inscriptions before beginning the first new task in each interview.

Summary of Data Collection Methods

In this section, I summarize the data collection portion of my dissertation study. I separate my discussion into three major subsections. First, I describe my use of a screening survey to recruit four potential candidates to participate in this study. This section constitutes the only place in this dissertation where I describe my screening survey, so I provide a detailed description of the survey items, dissemination, and analysis of the survey data. I also describe the intake interview tasks and the analysis methods by which I selected the two finalists to serve as study participants for the remainder of the study. Second, I describe the backgrounds of each of the two finalists. In the final subsection, I describe the tasks, learning goals, and research goals for each interview comprising this experiment.

Recruiting Student Participants for the Dissertation Study.

In this section, I address my processes for recruiting two students to participate in the individual teaching experiments comprising the data for study. In brief, I used a screening survey to select four potential candidates from all undergraduate second-semester calculus students interested in my study. I then conducted an intake interview with each of the four possible candidates. Finally, I used the data from the intake interviews to select the two participants for this study.

This section contains four distinct portions. First, I address the sampling theory I utilized with my screening survey, *theoretical sampling* (Patton, 2002). Second, I review the items on my screening survey. Third, I describe the logistics of disseminating the survey. Finally, I discuss my survey analysis. I discuss my survey analysis in this chapter because I want to focus the results chapters (i.e., Chapters 5, 6, 7) solely on the data from the two finalist candidates.

Theoretical Sampling

To select the participants in my study, I conducted *theoretical sampling* (Patton, 2002) through disseminating a screening survey (Lavrakas, 2008). Theoretical sampling occurs according to the theoretical needs of the researcher and the evolving constructs that emerge throughout the various stages of the empirical research process (Coyne, 1997). Theoretical sampling differs from random sampling, where study subjects are spontaneously selected to participate according to criteria determined before the start of the experiment. Theoretical sampling is also different from (1) what Schatzman and Strauss (1973) called *selective sampling*, where a researcher chooses participants in qualitative research according to time and resource constraints (Coyne, 1997).

From my empirical pilot study data analysis, I identified several prerequisite criteria that I envisioned would be necessary for my participants to succeed in my dissertation study. These criteria included: (1) the ability to make additive comparisons between sums of fractions, (2) the ability to recognize the graph or algebraic rule describing a functional relationship, and (3) a working definition for sequence convergence that is not logically equivalent to the formal definition for convergence. From these criteria, I constructed a screening survey to identify undergraduate first-time second-semester calculus students who had the potential to be productive study participants.

Structure and Items on Screening Survey

The purpose of the screening survey was to identify a small group of finalist candidates who portrayed my desired methodological (e.g., first-time second-semester calculus student) and theoretical criteria (e.g., desired meanings for fractions, graphs, and convergence) for the study. I have summarized the structure of the screening survey in Table 4. The survey consisted of (1) a consent form; (2) demographic information regarding students' major and math course experience; and (3) nine survey items addressing topics such as comparing partial sums (Items 1a-1b), understanding of graphs of functions and algebraic rules for functions (Items 2a-2d), interpreting the value of a convergent series (Item 3a), and providing definitions for series convergence (Items 3b-3c). Four survey items (Items 1a, 2a, 2c, 3a) were multiple choice, and the remainder were free response (Items 1b, 2b, 2d, 3b, 3c).

Disseminating the Survey

I prepared the survey in the Qualtrics platform and enlisted the undergraduate mathematics chair to send a link to all students enrolled in second-semester calculus courses at the university (see Appendix B for a copy of the recruitment email). I provided students with approximately one week to complete the survey. I received 14 viable survey responses by the survey close date. The 14 students who completed the screening survey were either (1) currently enrolled in a second-semester calculus course (12 students) or (2) enrolled in another course but had taken some portion of a second-semester calculus course in the current or previous semesters (2 students). Twelve students had previously taken either AP Calculus AB or BC in high school (or both), and six of these students self-reported that they had passed the corresponding AP exam.

Table 4

Screening Survey Structure and Item Description

Item	Description	Item Type
Consent Form	• Information about survey and consent to participate	Single check-box
Preliminary questions	• Demographic information: 1) Name, email, campus student attended 2) Declared major 3) Math courses taken in previous semesters 4) Experience with AP Calculus AB or BC in high school 5) Current mathematics course	Multiple response
Item 1a: Additive comparison of fractions	• Compare the values of two sums of fractions using the inscription >, <, or =. • Students provide justification for their choice of inscription.	Multiple choice Justification
Item 1b: Follow-up questions	• Student provides an explanation of what their chosen inscription for Item 1a means (to them). • Student provides an example of how they might use their chosen inscription for Item 1a in another situation.	Open response
Item 2a: Graph of a function	• Given a function rule and domain, choose the corresponding function graph from a set of four possible graphs. • Students provide justification for their choice of graph.	Multiple choice Justification
Item 2b: Function evaluation	• Describe how to evaluate a function algebraically (given a function rule) at a given value of the independent variable.	Open response
Item 2c: Determining algebraic rule for function graph	• Given a graphical representation of a linear function, choose the appropriate closed-form, explicit rule which produces the graph shown. • Students provide justification for their choice of rule	Multiple choice Justification
Item 2d:	• Given a value of the dependent variable of a function, describe how to find the corresponding value of the independent variable of the function.	Open-ended response
Item 3a	• Determine the truth value of a statement about the convergent value of a geometric series (i.e., $\sum_{n=0}^{\infty}\frac{1}{2^n}$ converges to 2) written in expanded form. • Students provide justification for their choice of truth value	Multiple-choice response Justification
Item 3b	• Students describe what the phrase "a series converges" means (to them).	Open-ended response
Item 3c	• Students complete the statement "A series converges if_____."	Open-ended response
End Screen	• Information about compensation, timeframe for hearing back about interviews, researcher contact information	

The respondents’ major focus of study included engineering (5 students), computer science or a related field (4 students), astrophysics (2 students), applied math for life and social sciences (1 student), mathematics education (1 student), and political science (1 student).

Survey Analysis and Selection of Potential Candidates

I utilized four distinct criteria to sort and rank the students to select four potential candidates for intake interviews. First, I awarded each student one point for providing a normative response to each of the three multiple-choice questions on the survey that I considered to constitute prerequisite knowledge (Items 1a, 2a, 2c). Of the 14 respondents, five students answered at least one of these prerequisite questions incorrectly. I subsequently eliminated these five students from consideration for the study, leaving nine remaining potential candidates.

Second, I examined the students’ responses to multiple-choice Item 3a (decide the truth value of a statement about a specific convergent geometric series) to determine the students’ justification of convergence (all nine remaining candidates provided the normative truth value for the statement). Additionally, I inspected the students’ open responses to Items 3b and 3c (discuss convergence generally) to determine the students’ intuitive meanings for convergence. As I inspected students’ responses, I found that two students had been allowed to complete the survey without providing responses to Items 3a, 3b, and 3c. I eliminated these two students from consideration for the study, leaving seven remaining potential candidates.

The tables below, Table 5 and Table 6, contain samples of the seven remaining students’ justifications for the convergence of a specific series (Table 5) and their intuitive meanings for series convergence (Table 6). Although I only provide sample responses from individual students, the *count* columns in Table 5 and Table 6 indicate how many of the seven potential candidates offered similar responses.

Table 5

Types of Students' Justifications for Specific Series Convergence (Item 3b)

Justification	**Definition**	**Example**	**Count**
Convergence Test	The student justifies the statement by appealing to a convergence test.	*The above sequence is what we consider to be a geometric sequence (in that the equation for the following sequence would be the summation (n=0 to infinity) of 1/(2^n), because the denominator in each term doubles with each subsequent term. Using the geometric series test, [rewrite the equation so that it is in geometric form: an=(1/2)^(n-1)] we are able to determine that the series is convergent because r<1 (r value is 1/2 in the above equation). Therefore, the sum of the sequence can be determined by S=a1/(1-r), in that a1 is the first term. Therefore the sum of the equation is S=2, and the series must therefore converge to 2.*	2
Approaching Asymptote	The student justifies the statement by imagining a dynamic process of approaching 2. Unclear from response what the student believes is approaching 2.	*You are approaching 2 at a decreasing pace but eventually the distance away from 2 would be so small one would effectively arrive at 2. I thought of this as an example of distance in which a person is taking steps towards a doorway in a way such that each step is half the size of the step before it. Eventually they would be taking such small steps, and be so close, that an observer would conclude the individual is in the doorway.*	2
Sum Approaching Asymptote	The student justifies the statement by referencing the sum of something approaching 2.	*If all the fractions were summed, it would slowly reach two.*	2
Unclear	The student's justification was unclear to the researcher.	*It's a series.*	1

The data in Table 5 show that the most common justification for the geometric series $\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{i}$ converging was describing a process of some quantity (e.g., sum) approaching the limit value of 2 (approaching asymptote: 2 students; sum approaching asymptote: 2 students). The other students either appealed to a convergence test (i.e., the geometric series formula; 2 students) or provided an unclear response (1 student). The data in Table 6 also indicate that many students (four responses) envisioned convergence in terms of a potentially infinite process. The remainder of the students described notions of infinite closeness, boundedness, or a conversion process of changing from one number value to another.

Table 6

Students' Intuitive Meanings for Series Convergence (Item 3c)

Intuitive Meaning	Definition	Example	Count
Potential Infinity	The student describes convergence as a process of approaching a value.	*A series converging means that the series is approaching a certain number. In other words, the limit of the series is not infinite and is a real number. It is the opposite of a divergent series, which fails to approach any real number.*	4
Infinite Closeness	The distance between the value of the series and the limit value is infinitesimally small.	*This means that the "series" gets infinitely close to a single real number that the series is basically equal to the number; unlike infinity.*	1
Changing into the limit value	The series dynamically changes from a pre-limit value to the limit value.	*To me, it means how far a number is reached. Almost as if number one was "converting" to number two so we "converge" while we reach number 2.*	1
Bounded	The series is bounded by the limit value.	*As a function goes to infinity, its sum goes to a value that we can determine to be finite*	1

For the third step of the analysis, I assigned binary values of coherence to the students' open responses to Items 3a-3c. I defined a coherent response as one in which the student appeared (to the researcher) to be expressing an idea that was clear (to the student) in a substantive way. In contrast, an incoherent response appeared (to the researcher) to be rambling, betray a lack of confidence, or provide an unclear (to the researcher) justification for any problem. I determined that all but one of the seven responses were coherent. An example of an incoherent response was the response in Table 5 that I coded as “unclear” (i.e., *It’s a series*).

For the final step of the analysis, I assigned an elaboration score (on a scale of 1 to 3) to the students’ justification responses to their choice of truth value for the statement in Item 3a (convergence of geometric series). I assigned an elaboration score of “1” if a student provided a short response (1 sentence) that made a claim but contained no attempt to justify the claim. I assigned an elaboration score of “2” if the student provided a medium-length response (approximately two sentences) that might include an example that the student did not explicitly relate to their response. I provided an elaboration score of “3” if a student provided a detailed response (more than 3 sentences) including an example, explanation, justification, or correlation between the various components of the students’ response. The following table, Table 7, provides an example of each type of elaboration response (the *count* column refers to the number of students who received the same elaboration score, not the number of students providing the same response).

Table 7

Student Screening Survey Response Examples by Elaboration Score

Elaboration Score	Example	Count
1	*The sum of all the values converges to 2.*	4
2	*The ratio between values is 1/2 which is less than one meaning that the series must converge. By setting the series equal to a variable, C, we can algebraically manipulate it to get the formula C-C/2=1 meaning that C must be 2.*	1
3	*You are approaching 2 at a decreasing pace but eventually the distance away from 2 would be so small one would effectively arrive at 2. I thought of this as an example of distance in which a person is taking steps towards a doorway in a way such that each step is half the size of the step before it. Eventually they would be taking such small steps, and be so close, that an observer would conclude the individual is in the doorway.*	2

After concluding the four steps of analysis, I provided a rank to each of the seven remaining student responses on a scale of 0 to 3 (with 3 being the highest rank, i.e., most desirable interviewee). I summarize how I rated and ranked each of the students I analyzed with the criteria in Table 8. I assigned a rank of 1 to students who described a convergence test during their response on Item 3a, 3b, or 3c. My rationale for giving these students a low rank was a concern that these students might procedurally use convergence tests to justify the convergence of series (which I wanted to avoid).

Table 8

Four Criteria Analysis for Screening Survey Data and Ranking of Students

Student	**Criteria 1:** Score (Items 1a, 2a, 3c)	**Criteria 2a:** Justification (Item 3a)	**Criteria 2b:** Meaning for convergence (Items 3b, 3c)	**Criteria 3:** Coherence (Items 3a-3c)	**Criteria 4:** Elaboration Score (Item 3a)	Major	**Rank**
Ann	3	Sum Approaching Asymptote	Changing into Limit Value	Yes	1	Engineering	3
James	3	Approaching Asymptote	Infinite Closeness	Yes	1	Engineering	3
Monica	3	Approaching Asymptote	Potential Infinity	Yes	3	Political Science	3
Justin	3	Unclear	Bounded	No	1	Engineering	2
Sylvia	3	Sum Approaching Asymptote	Potential Infinity	Yes	1	Applied Mathematics for Life and Social Sciences	2
Pablo	3	Convergence Test	Potential Infinity	Yes	3	Engineering	1
Patrick	3	Convergence Test	Potential Infinity	Yes	2	Engineering	1

I ranked the remaining five students (i.e., Ann, James, Monica, Justin, and Sylvia) using two qualifications. First, I wished to achieve the maximal variety between their justification of the geometric series convergence (Item 3a; Criteria 2a) and their meaning for convergence (Items 3b, 3c; Criteria 2b). Second, I wanted to rank students lower who had a deficiency in one of the criteria (e.g., justification, elaboration) compared to the other students. Consequently, I ranked Justin as a two because of his brief response to Item 3a and Sylvia as a two because her justification was similar to Ann's. Her meaning for convergence was also similar to Monica's (and Monica had a much higher elaboration score). I ranked the remaining students—Ann, James, and Monica—as 3's because they represented two different justifications for the geometric series convergence and three meanings for convergence.

Intake Interview and the Selection of Two Participants for the Teaching Experiment

In this section, I describe the process by which I conducted intake interviews and selected the finalists to participate in the teaching experiment. This section comprises three parts. In the first part, I overview the logistical hurdles that I needed to surmount as I attempted to conduct the interviews. In the second part, I briefly describe the content and tasks of the intake interview. Finally, I describe how I chose the official participants for the teaching experiment.

Logistical Issues Conducting Intake Interviews. After completing my analysis of the screening survey data, I contacted each student that I had given a rank of 3 (i.e., Ann, James, Monica) for an intake interview. My original intention was to conduct intake interviews with three students from which I would select my two finalists. Ann and Monica responded to my emails and scheduled intake interviews. However, James did not respond to my requests to schedule an interview (despite repeated reminder emails).

Consequently, I contacted the students I had given a rank of 2 (i.e., Justin, Sylvia). I asked them whether they would like to participate in intake interviews. Both students quickly responded to my message, so I scheduled interviews with both students. As a result, I conducted intake interviews with four students (i.e., Ann, Monica, Justin, Sylvia), two of which I had given a rank of 3 (i.e., Ann, Monica), and two of which I had given a rank of 2 (i.e., Justin, Sylvia).

Content and Tasks of the Intake Interview. The 90-minute intake interview simultaneously functioned as the final vetting stage to select two teaching experiment participants and an opportunity to create an initial model of my participants' thinking about series convergence. The intake interview consisted of two major tasks. In the first task, I presented six series to the students created by a hypothetical student named Abigail (see Table 9).

Table 9

Abigail's Six Series Presented to Students During Intake Interview

Series	Rule	Expanded Form	Series type	Sequence of Partial Sums	Value of Convergence
1	$\sum_{n=0}^{\infty}\frac{3}{\sqrt{n}}$	$\frac{3}{\sqrt{1}}+\frac{3}{\sqrt{2}}+\frac{3}{\sqrt{3}}+\cdots$	p-series $(0<p<1)$	Monotone increasing divergent	
2	$\sum_{n=1}^{\infty}\frac{2(-1)^{n-1}}{n^5}$	$\frac{2}{1^5}-\frac{2}{2^5}+\frac{2}{3^5}-\cdots$	Alternating p-series $(p>1)$	Oscillating convergent	≈ 1.94
3	$\sum_{n=1}^{\infty}\sum_{i=1}^{99}\left[10^{-2n-1}-10^{-2(n+1)-1}i\right]$ $=\sum_{k=0}^{\infty}\frac{495}{10000}\left(\frac{1}{100}\right)^k$	$\frac{99}{10^3}+\frac{98}{10^3}+\cdots+\frac{1}{10^3}+$ $\frac{99}{10^5}+\cdots+\frac{1}{10^5}+$ $\frac{99}{10^7}+\cdots+\frac{1}{10^7}+\cdots$	Geometric	Monotone increasing convergent	$\frac{1}{20}$
4	$\sum_{n=0}^{\infty}\frac{(200-2n)(-1)^n}{n+1}$	$\frac{200}{1}-\frac{198}{2}+\frac{196}{3}-\cdots$	Alternating series	Oscillating divergent	
5	$\sum_{i=0}^{\infty}a_i$ (where a_i corresponds to the i^{th} decimal place of π and $a_0=3$.)	$3+.1+.04+\cdots$	Decimal expansion of irrational number	Monotone increasing convergent	π
6	$\sum_{n=0}^{\infty}(.07)\cdot(-1)^n$	$.07-.07+.07-\cdots$	Alternating series (Grandi's)	Oscillating divergent	

For each series, I asked the student two questions I found in the Larson and Edwards (2015) calculus textbook: (1) *Does the series converge?* and (2) *If the series* *converges, what value does the series converge to?* In the second task, I asked the student to describe their general image of series convergence. I describe Monica and Sylvia's responses to these interview tasks in detail in Chapter 5.

Analysis of Intake Interview Data to Determine two Study Participants. The analysis of the intake interview consisted of three distinct phases. First, I categorized the students' responses to the convergence of each of the six series (see Table 10). Through this stage of the analysis, I realized that the students' responses were very similar for some series (e.g., Series 6) but largely dissimilar for the remainder of the series.

In the second stage of the analysis, I categorized the meanings for series convergence that the students described after reviewing the six series (see Table 11). Through this analysis stage, I recognized that Monica and Sylvia portrayed similar meanings for series convergence as a dynamic partial sum stabilizing toward a particular value (see Eckman & Roh, 2022b for a summary of these meanings). In contrast, Ann always estimated with the same error bound (0.001, which she referred to as "three significant figures). Justin seemed to focus on evaluating large partial sums, which he would then round to a convenient (to him) value (he did not exhibit a consistent pattern with his rounding during the interview). I also found that Ann, Justin, and Sylvia utilized calculators to reason about infinite series, while Monica preferred to reason verbally and graphically about series.

Table 10

Intake Interview Students' Convergence Responses for Abigail's Six Series

<table>
<tr><th colspan="2"></th><th colspan="3">Series 1</th><th colspan="2">Series 2</th><th>Series 3</th><th colspan="2">Series 4</th><th colspan="4">Series 5</th><th colspan="2">Series 6</th></tr>
<tr><td rowspan="2">Monica</td><td>Converge</td><td colspan="2">Y</td><td>N</td><td colspan="2">Y</td><td>N</td><td colspan="2">Y</td><td>Y</td><td>Y</td><td>Y</td><td>Y</td><td colspan="2">N</td></tr>
<tr><td>Value</td><td colspan="2">Unsure</td><td></td><td colspan="2">Unsure</td><td></td><td colspan="2">0</td><td>π</td><td>4</td><td>3.2 and 4</td><td>Any upper bound</td><td colspan="2"></td></tr>
<tr><td rowspan="2">Sylvia</td><td>Converge</td><td>Y</td><td>N</td><td>Y</td><td>N</td><td>Y</td><td>N</td><td colspan="2">Y</td><td colspan="4">Y</td><td>Y</td><td>N</td></tr>
<tr><td>Value</td><td>4</td><td></td><td>Unknown value larger than 4</td><td></td><td>2</td><td></td><td colspan="2">200</td><td colspan="4">π</td><td>0</td><td></td></tr>
<tr><td rowspan="2">Ann</td><td>Converge</td><td colspan="2">Unsure</td><td>N</td><td colspan="2">Y</td><td>Y</td><td colspan="2">N</td><td colspan="4">Y</td><td colspan="2">N</td></tr>
<tr><td>Value</td><td colspan="2"></td><td></td><td colspan="2">1.944</td><td>Unsure</td><td colspan="2"></td><td colspan="4">3.141</td><td colspan="2"></td></tr>
<tr><td rowspan="2">Justin</td><td>Converge</td><td colspan="3">Y</td><td>Y</td><td>Y</td><td>Y</td><td>Y</td><td>Y</td><td colspan="4">Y</td><td colspan="2">N</td></tr>
<tr><td>Value</td><td colspan="3">189000</td><td>1</td><td>0</td><td>0.1</td><td>100</td><td>137.5</td><td colspan="4">3.14</td><td colspan="2"></td></tr>
</table>

Note: Y means the student said "Yes, the series converges," and N means "No, the series does not converge"

Table 11

Intake Interview Students' General Images of Convergence

Student	Meaning for Convergence
Monica	Three implications of an *asymptotic running total* meaning (see Eckman & Roh, 2022b) 1) If the terms in a series perpetually decrease, the series converges. 2) If the value of the running total perpetually increases, the series diverges. 3) If a monotone series can be constructed from an alternating series, the alternating series converges.
Sylvia	Three implications of an *asymptotic running total* meaning (see Eckman & Roh, 2022b) 1) If the terms in a series perpetually decrease, the series converges. 2) If the value of the running total perpetually increases, the series diverges. 3) If a monotone series can be constructed from an alternating series, the alternating series converges.
Ann	Two different meanings: 1) A series converges if it is bounded within a certain tolerance. The most common error bound that Ann used was 0.001, which she referred to as "three significant figures." 2) The value of convergence is the sum of all the series terms.
Justin	Three different meanings: 1) The value of a series (in the general sense) is equivalent to the area under a curve. 2) The running total stays the same (in the long run) for a convergent series and increases for a divergent series. 3) To determine the value of a series, add a large finite number of terms in the series and round off to a convenient value.

In the final stage of the analysis, I evaluated each student's experiences with calculus courses (see Table 12). Although all four students had stated that they had not taken second-semester calculus before during the screening survey, two students revealed during the intake interview that they had previous experiences with university calculus. First, Ann revealed that she had taken an accelerated second-semester calculus over the summer and was re-taking second-semester calculus during the current (Fall) semester. Second, Justin stated that he had taken second-semester calculus twice before (once passing with a low grade, once withdrawing) and was repeating the course a third time in an attempt to improve his grade in the course.

Table 12

Intake Interview Students' Previous Experience with Calculus Courses

Student	Previous Experience with Calculus Courses	Notes
Monica	AP Calculus AB and BC in high school	Passed the AP Calculus AB exam but did not take AP Calculus BC exam
Sylvia	AP Calculus AB in high school	Passed the AP Calculus AB exam
Ann	AP Calculus AB in high school, first-semester calculus, second-semester calculus	Took second-semester university calculus during the Summer 2021 session
Justin	AP Calculus AB in high school, first-semester calculus, second-semester calculus	Took second-semester calculus three times, passing with a low grade, withdrawing, and currently enrolled (respectively)

As I considered how to prioritize the three phases of my analysis toward selecting two finalists, I recalled my commitment after Pilot Study 3 to only interview students with no prior collegiate experience with the sequence and series unit of second-semester calculus. Thus, I placed the highest priority on the third phase of my analysis (i.e., students' previous calculus experience) and a lower priority on the other two phases (i.e., convergence responses, meanings for convergence). As a result, I immediately eliminated Justin as a participant in my study due to his repeated attendance in second-semester calculus courses. I carefully weighed whether to include Ann, who had one previous semester of second-semester calculus experience but whose thinking about convergence was distinctly different than either Monica's or Sylvia's. Ultimately, I decided to eliminate Ann because I was unsure how much of the meanings she exhibited for series convergence during her interview were a result of her previous instruction about sequences and series. Eliminating Justin and Ann left me with two final participants, Monica and Sylvia, who each accepted an invitation to participate in the entire teaching experiment.

Background Information about Study Participants

In this section, I provide background information about the two participants in the teaching experiments, Monica and Sylvia. This information includes their self-reported mathematical background, academic major, and other demographic information for which they gave permission for me to share to contextualize their identities. I devote one subsection to each student.

Monica

Monica was a female, white and Hispanic fourth-year undergraduate student majoring in political science. She was enrolled in the second-semester calculus course because she had recently added a mathematics minor to her degree program (in addition to two other minors in Russian and Spanish). Because she added a mathematics minor, Monica determined she would need an additional (fifth) year to complete her undergraduate degree. Monica aspired to law school. Monica had previously taken both AP Calculus AB (first-semester calculus) and AP Calculus BC (second-semester calculus) in high school, and self-reported passing the AB exam (but not taking the BC exam). At the collegiate level, Monica took College Algebra during her first semester. She claimed that the second-semester calculus course was her first mathematics course since that time (an approximately three-year gap in formal mathematics instruction). Monica reported enjoying collegiate-level calculus more than high-school-level calculus because of her increased access to academic resources and being able to attend her smaller recitation class in conjunction with her large lectures.

Sylvia

Sylvia was a first-year female Hispanic (Mexican-American) student double majoring in (1) applied mathematics for the life and social sciences and (2) disability studies with a minor in psychology. Sylvia had taken an AP Calculus AB course in high school and passed the corresponding AP exam. At the collegiate level, Sylvia was enrolled in a research-based second-semester calculus course and self-reported doing well with her course materials and exams.

Teaching Experiment Days and Interview Tasks for Monica and Sylvia

In this section, I address the nature of each of the seven teaching interviews and the exit interview, including the associated tasks and goals. I am purposefully brief in my descriptions due to the sheer number of tasks comprising the nine interviews of this study.

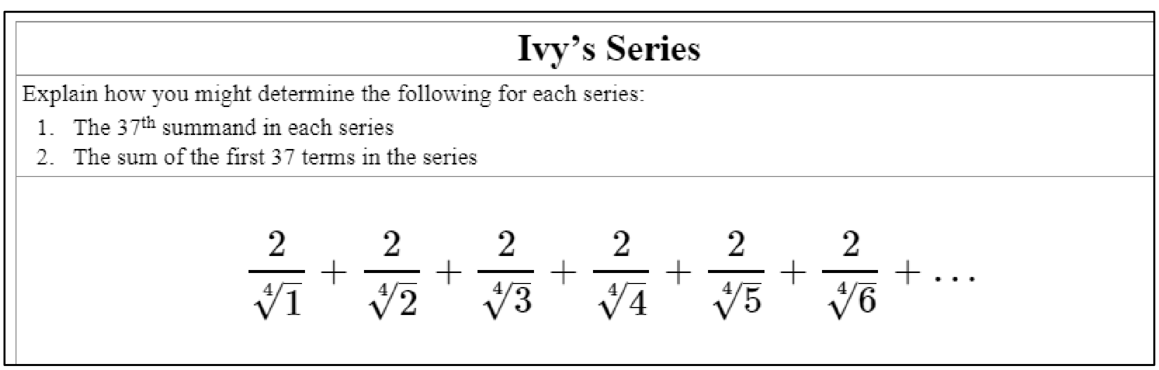
For each interview, my description follows a similar trajectory. First, I provide an overview of the number and types of tasks in each interview. Second, I briefly describe each task and its corresponding learning and research goals. I typically include tables and figures to contextualize further the tasks described in this section. I rarely address specific student data or meanings in this section, except where necessary to contextualize or justify a difference in the tasks presented to a student on a particular day. For a more detailed account of the tasks and student data, please see the full interview protocols in Appendix C.

Day 1

The structure of the Day 1interview was the same for both students and consisted of four major tasks. For the first task, I asked the students to describe how they would determine a specific summand and partial sum for various series in expanded form (with five or six summands visible in each series; see Figure 1 for a screenshot of Series 1 and Table 13 for a list of the six series). The learning goal for this task was for students to repeatedly consider specific summands and partial sums and begin to generalize how they might determine these quantities for an arbitrary series. The research goal for this task was to model students' meanings for partial sums and how these meanings evolved as students reasoned about various series.

Figure 1

Screenshot of Ivy's Series 1



For the second task, I asked each student to construct a written rule (in English) to describe how to determine the value of an arbitrary partial sum (see Figure 2). The learning goal for this task was for students to consciously reflect on their reasoning from Task 1 and create a written description for their actions which they could later reference while creating symbols. The research goal for the second task was to model the process by which students reflected on their thinking from Task 1 to construct their written rule for Task 2.

Table 13

Ivy's Series for the Day 1 Exploratory Teaching Interview

Series	**Expanded Form**	**Series type**	**Partial Sums Behavior**	**Converge**	**Limit Value**
$\sum_{n=0}^{\infty} \frac{2}{\sqrt[4]{n}}$	$\frac{2}{\sqrt[4]{1}} + \frac{2}{\sqrt[4]{2}} + \frac{2}{\sqrt[4]{3}} + \cdots$	p-series	Monotone increasing	No	
$\sum_{n=1}^{\infty} \frac{5}{n}$	$\frac{5}{1} + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} + \cdots$	p-series	Monotone increasing	No	
$\sum_{n=1}^{\infty} \frac{3}{n^5}$	$\frac{3}{1^5} + \frac{3}{2^5} + \frac{3}{3^5} + \frac{4}{3^5} + \cdots$	p-series	Monotone increasing	Yes	≈ 3.11
$\sum_{n=1}^{\infty} (-1)^n \left(\frac{6}{n^2}\right)$	$\frac{6}{1} - \frac{6}{4} + \frac{6}{9} - \frac{6}{16} + \cdots$	Alternating series	Oscillating	Yes	≈ -4.93
$\sum_{n=0}^{\infty} (.04) \cdot (-1)^n$	$.04 - .04 + .04 - \cdots$	Alternating series (Grandi's)	Oscillating	No	
$\sum_{n=0}^{\infty} (-1)^n \left(\frac{n+3}{n^2 - n + 7}\right)$	$\frac{3}{7} - \frac{4}{7} + \frac{5}{9} - \frac{6}{13} + \cdots$	Alternating series	Oscillating	Yes	≈ -0.27

Figure 2

Prompt and Sylvia's Response for Written Rule Creation Task

Construct a written note, say for a fellow study group member, detailing how to determine the nth *partial sum of a series (such as when* n = *12,* when *n = 189, etc.)*

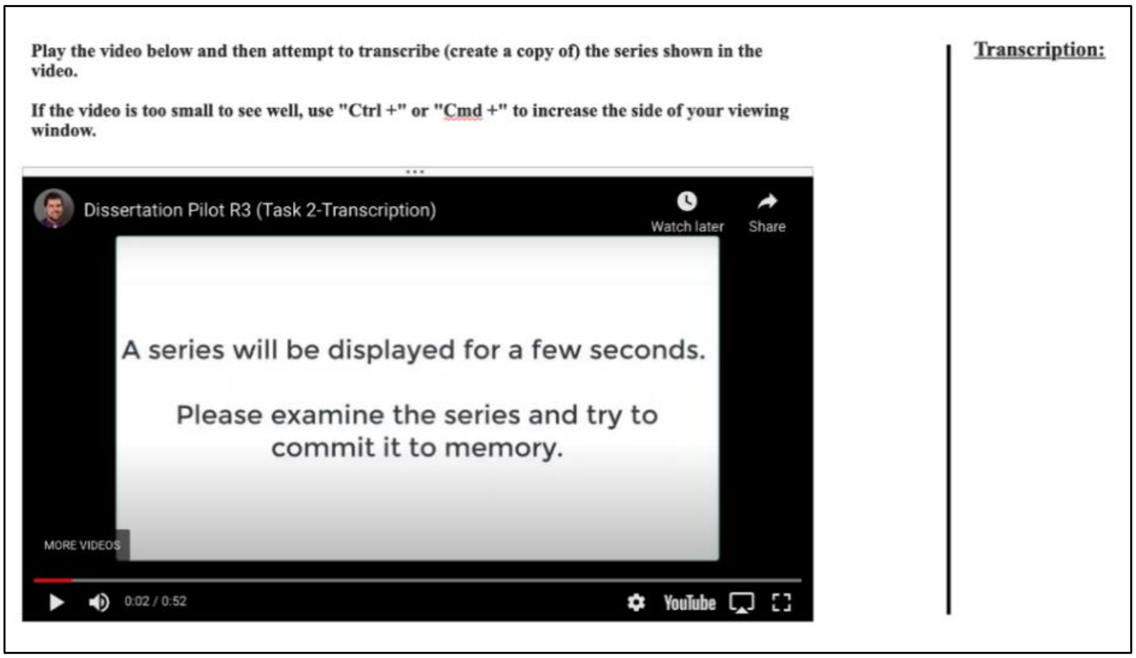
Written note:

First, look for a pattern between the terms of the series (how you get from one term to the next or vice versa). For example, did you add, subtract, multiply, or divide one term to create the next term? Then, apply that pattern to n number of terms, and add with a calculator.

For the third task, I asked the students to create a written transcription of a series that I showed them in a brief video (see Figure 3). There was no explicit learning goal for this task other than for students to (possibly) become aware of the way in which they interpret series written in expanded form. The research goal for this task was to determine the inscriptions (or groups of inscriptions) within the written series on which students focused, and the order in which these students focused on these inscription groups.

Figure 3

Task Prompt for Series Transcription Task



In the final task, I presented a video[9] describing and providing examples of the constructs *inscription* and *personal expression*. I then asked the students to create personal expressions by which to re-present an arbitrary partial sum (see Figures 4, 5; the script for the video is included in the interview protocol in Appendix C and is available to view at https://youtu.be/PdKkhZVPulA). The learning goal for the fourth task was for

[9] There are certain terms that I introduce in the video, such as *operational*, *relational*, and *vicarious* inscriptions, which should not be construed as theoretical constructs. Rather, I employed these terms as a way to categorize a few basic purposes of mathematical notation that seemed relevant to me at the time I created the video.

students to construct a personal expression comprised of one or more inscriptions they defined in a glossary, by which they could re-present the series they encountered during the Day 1 interview (see Figure 6). The research goal for the final task was to determine the portions of each student's reasoning they believed merited symbolizing, whether the students utilized conventional or novel inscriptions in their expression, and how the students combined inscriptions to create expressions.

Figure 4

Task Prompt for Personal Expression Creation Task

Create a personal expression that utilizes inscriptions to describe how to determine the nth *partial sum for any of Ivy's series.*

Please also record any inscriptions that you choose to use in the glossary to the right.

Personal expression(s):

Figure 5

Screenshot from Personal Expressions Video

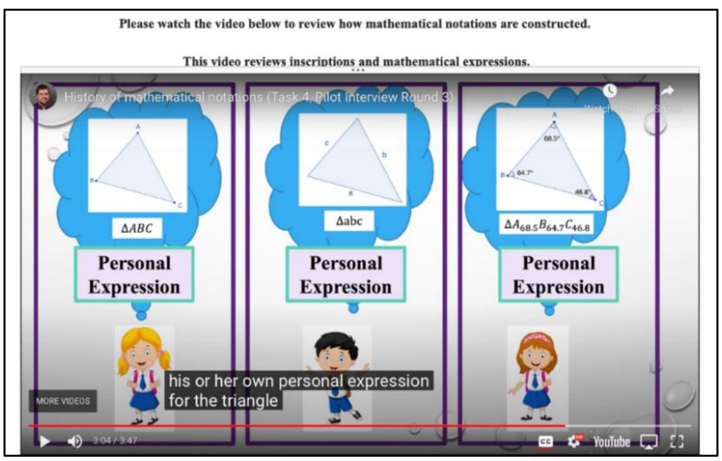


Figure 6

Example of Blank Glossary

Glossary

Inscription	Information inscription conveys

Day 2

The structure of the second interview was similar for both students and consisted of two preliminary tasks and two major tasks. The two preliminary tasks recurred to some degree in many interviews[10]. The first preliminary task was to review the personal expressions video the student saw during the Day 1 interview. The learning goal for this task was for students to repeatedly reflect on the nature of inscriptions and personal expressions to inspire their use and creation of these symbols during the interview. The research goal for the video review was to determine whether students' areas of focus with the video content changed throughout the interviews. The second preliminary task was to review the inscriptions the student had written in their glossary. The learning goal for this task was for students to reinforce the prior meanings they attributed to their inscriptions and modify them (if necessary) to reflect their current thinking about series[11]. The

[10] I had Monica review the personal expressions videos on Days 2-4 and Sylvia review the video on Days 2-3. I had each student review their glossary of inscriptions before beginning the first new task in each interview.

[11] I asked students to write modifications to their glossary in a different color each day.

research goal for this task was to model students' evolution of thinking about their inscriptions and their meta-level reasoning about categories of inscriptions.

For the first major task, I presented each student with an opportunity to reason about a non-normative meaning or symbolization that they had proposed during the Day 1 interview. Monica's task involved comparing two potential definitions of a partial sum as area under a curve, one involving integral notation and one involving summation notation (see Table 14). Sylvia's task involved symbolizing series with various patterns of $+$ and $-$ signs (see Table 15). The learning goals were to help students become more confident in the personal expressions they created during Day 1. The research goals were (1) to determine how Monica would resolve her perturbation about competing expressions for partial sums and (2) find Sylvia's boundary of representation for her non-normative notation. I describe each student's work with their particular tasks in more detail in Chapter 7.

Table 14

Two Definitions and Symbols for Partial Sums Monica Compared on Day 2

Definition Name	Definition
Yolanda	The nth partial sum of Ivy's 1^{st} series can be determined by computing the summation $\sum_1^n \frac{2}{\sqrt[4]{n}}$, which represents the exact area under the curve of the function $f(n) = \frac{2}{\sqrt[4]{n}}$ when it is evaluated at each position from 1 to n.
Zeb	The nth partial sum of Ivy's 1^{st} series can be determined by computing the summation $\sum_1^n \frac{2}{\sqrt[4]{n}}$, which represents the approximate area under the curve of the function $f(n) = \frac{2}{\sqrt[4]{n}}$ using Riemann sums with width 1 from 1 to n.

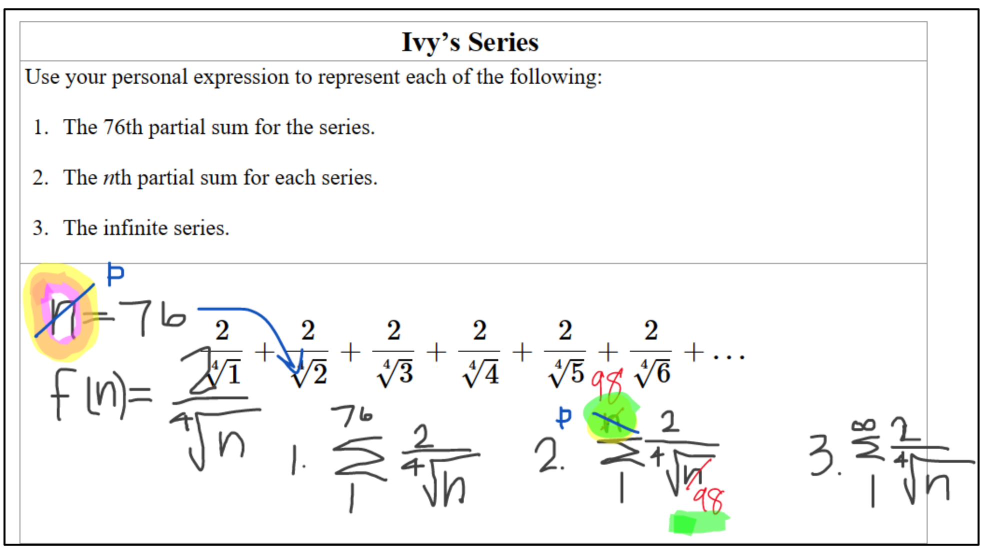
Table 15

The Five Series Sylvia Symbolized with her Novel Notation on Day 2

Label	Series
A	$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\frac{1}{729}+\cdots$
B	$-1-\frac{1}{3}-\frac{1}{9}-\frac{1}{27}-\frac{1}{81}-\frac{1}{243}-\frac{1}{729}-\cdots$
C	$1+\frac{1}{3}+\frac{1}{9}-\frac{1}{27}-\frac{1}{81}+\frac{1}{243}+\frac{1}{729}-\cdots$
D	$1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}-\frac{1}{81}+\frac{1}{243}+\frac{1}{729}-\cdots$
E	$-1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}-\frac{1}{81}+\frac{1}{243}+\frac{1}{729}-\cdots$

Figure 7

Monica's Prompt for Symbolizing Series with Personal Expressions



For the second task, I asked the students to utilize (or modify) their personal expressions to symbolize specific partial sums, arbitrary partial sums, and the infinite series they encountered during the Day 1 interview (see Figure 7). The learning goal for this task was (again) to help students reinforce or modify their personal expressions to utilize their inscriptions to reason about series confidently. The research goal for this task was to determine how students might modify their personal expressions to symbolize various situations (e.g., infinite series, series with no readily discernable general summand).

Day 3

The structure of the third interview was similar for both students. The interview consisted of the same two preliminary tasks from Day 2 (i.e., review personal expressions video, review glossary) and three major tasks. Of the three major tasks I presented during Day 3, the first was different for each student, but the final two were the same. Monica's first task was to investigate three potential graphs of her image of partial sums resembled (1) a smooth curve (see Figure 8, Mario's graph), (2) a step function (see Figure 8, Natalie's graph), and (3) a sequence (see Figure 8, Oscar's graph). The learning goal for this task was for Monica to determine that the graph corresponding to her summation-notation-like personal expressions for partial sums and series would produce a set of dots. The research goal for this task was to determine the role that contrasting prompts (what Halani et al., 2013, called *peer interpretations*) might play in resolving Monica's lingering doubts about whether the "graph" corresponding to the running total of a series would be continuous or disjoint.

Sylvia's first task was to describe how she might symbolize various series that exhibited random behavior in either (1) the number of alternating + and – signs separating summands or (2) the values of the summands in a series (see Figures 9a, 9b). The learning goal for this task was for Sylvia to reinforce or modify her personal expressions she created during the Day 2 interview to symbolize series exhibiting a random behavior. The research goal for this task was to find Sylvia's boundary of representation for her non-normative inscription. I provide additional details regarding Sylvia's actions during this task in Chapter 7.

Figure 8

Three Potential Graphs Related to the "Running Total" of a Series

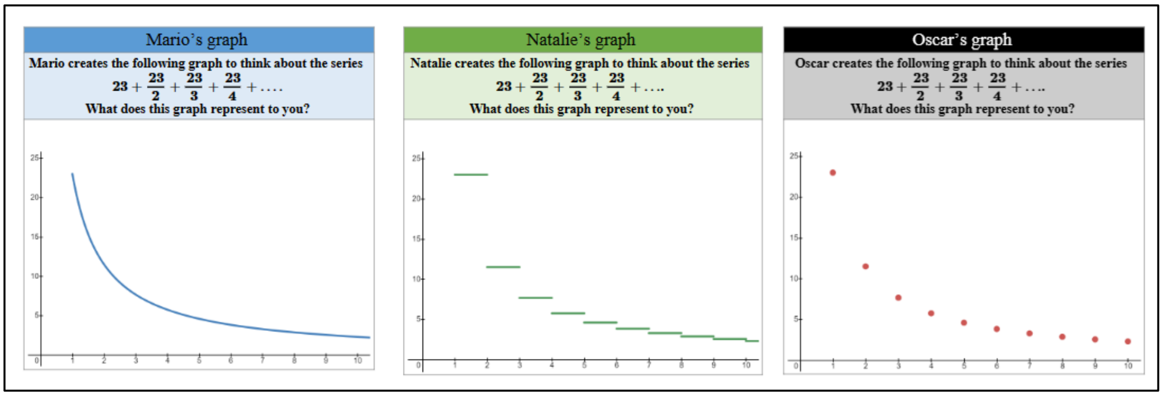
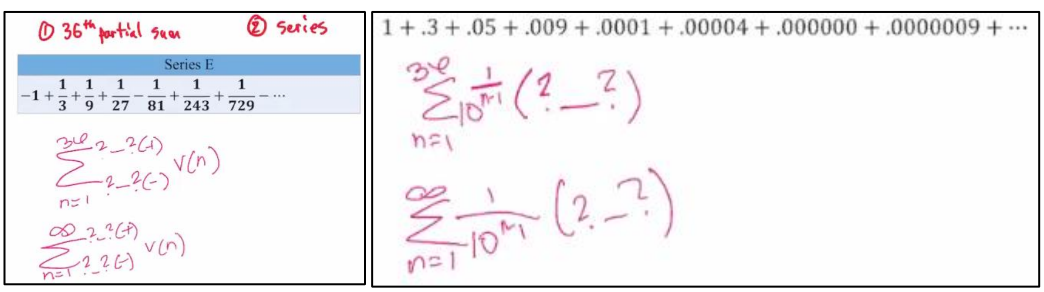


Figure 9a (left), 9b (right)

Sylvia's Symbolization of Series with Random Operator Signs (9a) and Random Summands (9b)

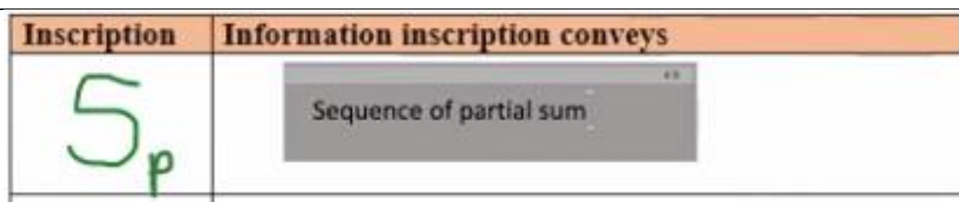


For the second major Day 3 task, I presented a mini-lecture on the normative definitions for a sequence and a sequence of partial sums. The learning goals for this task were to (1) address any unproductive meanings students exhibited for sequences during prior interviews and (2) provide a normative definition for the sequence of partial sums in preparation for students' symbolization of this concept. There was no major research goal for this task other than to monitor and address students' meanings for sequence and sequence of partial sums in preparation for the next interview task.

For the final major Day 3 task, I asked each student to construct personal expressions by which they could re-present the sequence of partial sums (see Table 16). I also asked each student to posit a relationship between a sequence of partial sums and a series. The learning goal for this task was for students to construct or modify a personal expression by which they could re-present the sequence of partial sums. The research goals for this task were to (1) determine whether the students constructed a new expression for the sequence of partial sums or modified an existing expression, (2) determine what components of the sequence of partial sums students believed merited symbolizing, and (3) what relationship (if any) the students perceived between a sequence of partial sums and a series.

Table 16

Students' Personal Expressions for the Sequence of Partial Sums on Day 3

<table>
<tr><td>Student</td><td>Inscriptions for Sequence of Partial Sums</td></tr>
<tr><td>Monica</td><td>
</td></tr>
<tr><td>Sylvia</td><td>
</td></tr>
</table>

Day 4

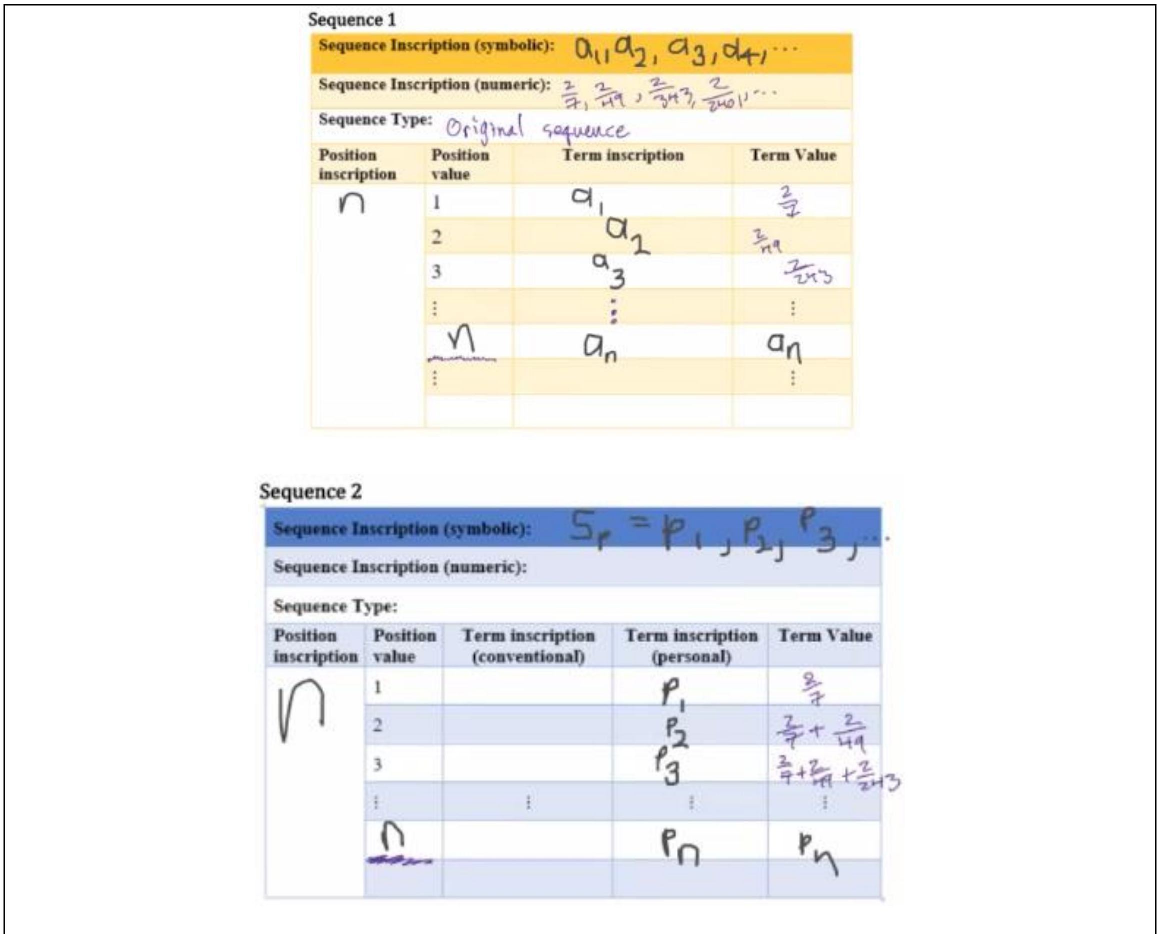
The structure of the fourth interview was similar for both students and consisted of one preliminary task for Monica and the same four major tasks for each student. Monica's preliminary task consisted of a final viewing of the personal expressions video (Sylvia did not view the video on Day 4 because she adeptly summarized the video at the beginning of her Day 4 interview).

The first major task consisted of categorizing the various inscriptions the students had created in their glossary during the Day 1 to Day 3 interviews. The learning goal for this activity was for the students to begin to reason about the types of meanings they attributed to their inscriptions, which I hoped would aid them in their transition from primarily algebraic to graphical reasoning about series convergence at the end of the interview. The research goal for this activity was to examine the symbolization categories each student proposed, how they sorted their inscriptions into these categories, and how students' categorizations differed from the researchers' models of students' symbolization.

The second major task was to reason numerically and symbolically about various components of a given sequence and its corresponding sequence of partial sums (see Figure 10). The learning goal for this task was for students to become comfortable symbolizing various components of the sequence of partial sums, including the index, specific terms, specific term values, and arbitrary terms and values. The research goal for this task was to investigate how students' personal expressions differed for sequences and sequences of partial sums and the relationship between their inscriptions for specific and arbitrary components of the sequences.

Figure 10

Monica's Symbolization of a Sequence and Sequence of Partial Sums

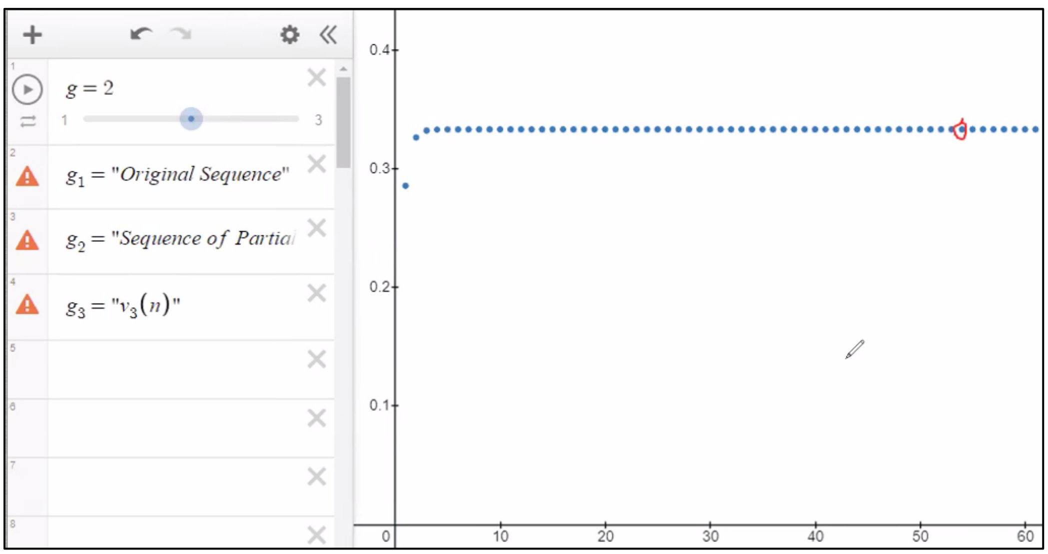


For the third major task, I showed students a Desmos-generated graph of the sequence and sequence of partial sums they had reasoned about during Task 2. I then asked the students to define and symbolize various graph components (see Figure 11). These components included individual points on each graph, the axes, and a rule that would generate the points on the graph. The learning goal for this task was for students to develop productive meanings for various components of the graph of a sequence. The research goal for this task was to determine the degree to which students assimilated the graphical components of the sequences to the personal expressions they created during the previous interviews.

For the final major task, I asked the students to speculate whether each sequence converged based on their interpretation of the graph. This task had no particular learning goal, as its purpose was merely to assess the students' images of convergence after the first four days of the teaching experiment. The research goal of this task was to compare the students' meanings for convergence at the end of Day 4 with the meanings they expressed in the intake interview to determine whether their thinking had evolved.

Figure 11

Screenshot of Graph of Sequence Task from Day 4



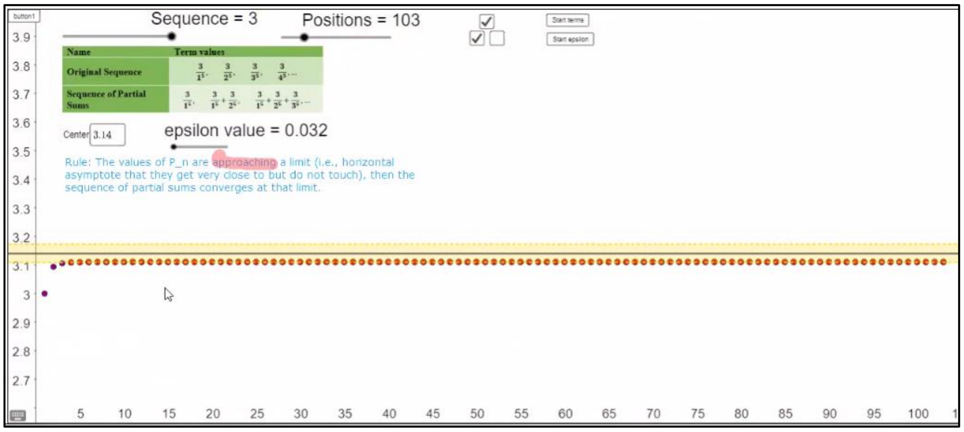
Day 5

The structure of the fifth interview was similar for both students and consisted of the same three major tasks. In the first task, I asked the students to review their symbolization of a sequence and sequence of partial sums from their Day 4 work and state whether they could symbolize the terms of each sequence with any expressions beyond those they wrote during Day 4. The research goals for this task were (1) to determine whether each student believed they could convey the values of sequence terms through their various inscriptions and (2) to discern students' abilities to symbolize the general summand of the sequence of partial sums. Although there was no explicit learning goal for the first task, I conjectured that students might become more confident in their ability to re-present a sequence or sequence of partial sums through their expressions during the activity.

For the second task, I presented a version of Roh's (2010b) ϵ-strip activity for various sequences of partial sums in a dynamic Geogebra applet. I then asked students to (1) state whether the sequence converged, (2) justify their responses, and (3) create an initial rule to describe sequence convergence (see Figure 12; link to applet: https://www.geogebra.org/m/ykrby8du). Due to the sophisticated nature and number of controls within the applet, I controlled the applet and asked the student to provide instructions for how they wished me to modify it. The learning goals for this task were for students to become familiar with the applet and create an informal rule for sequence convergence for reference during the Day 6 interview. The research goals for this task were (1) to model the evolution of students' intuitive thinking about sequence convergence, (2) to monitor what connections (if any) students made between sequence of partial sums convergence and series convergence, and (3) to investigate any spontaneous connections students might make between their personal expressions and the graphical task about sequence convergence.

Figure 12

Example of GeoGebra Applet for ϵ-strip Activity from Day 5



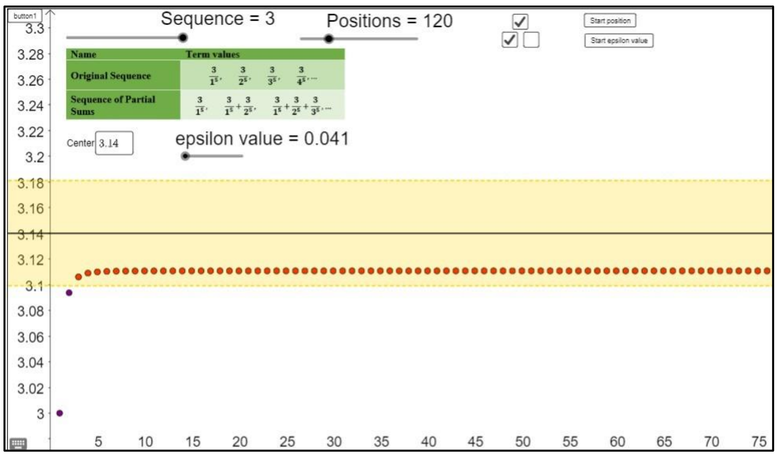
For the final task, I explicitly asked each student whether they wished to create or modify any inscriptions to re-present components of the GeoGebra applet they utilized during Task 2. The learning goal for this task was for students to create or modify inscriptions to re-present to themselves graphical components of a sequence of partial sums and their intuitive rule for sequence convergence. The research goals for this task were to determine (1) which components of the graphs that the students symbolized, (2) the nature of any new inscriptions that students created, and (3) whether students assimilated any of the ideas they reasoned about during the graphical tasks to their existing inscriptions.

Day 6

The structure of the sixth interview was slightly different for each student but consisted of the same two tasks. The first task was for the students to compare their glossary inscriptions that they had created during Days 1-4 (before I introduced graphs of sequences) to a screenshot of the ϵ-strip activity and determine whether they could re-present any components of the screenshot with each inscription (see Figure 13 for the screenshot). The learning goal for this task was for students to connect the inscriptions they created while reasoning about sequences and series symbolically to the graphical representations of sequences they encountered in the ϵ-strip activity. The research goal for this task was to determine which inscriptions (if any) the students appeared able to re-present their non-visual and visual reasoning about sequence and series.

Figure 13

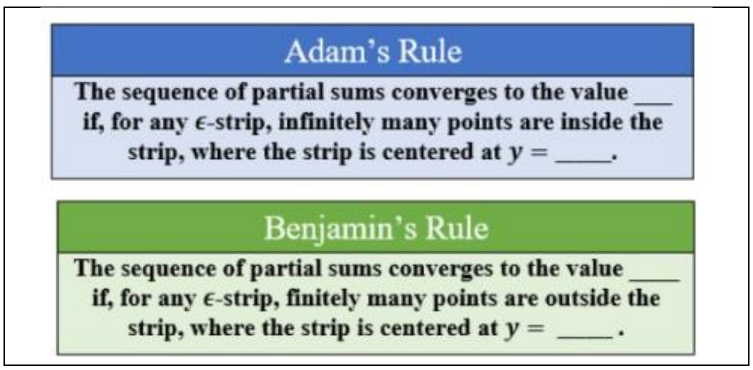
Screenshot of ϵ-strip Activity from Day 6



For the second task, I presented each student with two hypothetical written rules for sequence of partial sums convergence, which I modified from the rules proposed by Roh (2010b). I have included a screenshot of these two rules in Figure 14 and a list of the sequences of partial sums I presented to students in Table 17 below. I then asked the students to evaluate several sequences of partial sums from the perspective of each definition to determine which rules they preferred to describe sequence convergence. The trajectory of this task largely followed the same steps that Roh (2010b) proposed in her report (although the sequences I presented differed from Roh's sequences). The learning goal for this task was for students to determine which rules they preferred to describe sequence of partial sums convergence in preparation for symbolizing their adopted rule during the Day 7 interview. The research goals for this task included (1) attending to students' thinking about the convergence of a sequence, (2) how their thinking changed as they proceeded through the ϵ-strip activity, and (3) whether the students made any connections between their inscriptions and the activity or between sequence convergence and series convergence.

Figure 14

Adam's and Benjamin's Rules for the ϵ-strip Activity on Day 6



Although each student participated in the same tasks during the Day 6 interview, I modified the trajectory of each activity based on each student's actions. For example, in the first task I found that Monica could apply several of her inscriptions to the screenshot during the first task. In contrast, Sylvia could not conceive of her inscriptions conveying any component of the graphical screenshot. Consequently, I was able to spend more time with Sylvia on the ϵ-strip activity and then return to the first task at the end of the interview. In the second task, Monica struggled to apply either Adam or Benjamin's rules to discuss convergence during the ϵ-strip activity. In Monica's case, I used the remainder of the interview time trying (unsuccessfully) to help her make sense of the rules. In contrast, Sylvia could make sense of both rules and eventually selected Benjamin's rule (logically equivalent to the normative definition of sequence convergence) as her preferred rule.

Table 17

Five Sequence for the ϵ-strip Activity on Day 6

Series Number	Series	Partial Sums Expanded form	Converges?
1	$\sum_{n=1}^{\infty} \frac{2}{\sqrt[4]{n}}$	$\frac{2}{\sqrt[4]{1}},\ \frac{2}{\sqrt[4]{1}}+\frac{2}{\sqrt[4]{2}},\ \frac{2}{\sqrt[4]{1}}+\frac{2}{\sqrt[4]{2}}+\frac{2}{\sqrt[4]{3}}, \ldots$	Yes
2	$\sum_{n=1}^{\infty} \frac{5}{n}$	$\frac{5}{1},\ \frac{5}{1}+\frac{5}{2},\ \frac{5}{1}+\frac{5}{2}+\frac{5}{3}, \ldots$	No
3	$\sum_{n=1}^{\infty} \frac{3}{n^5}$	$\frac{3}{1^5},\ \frac{3}{1^5}+\frac{3}{2^5},\ \frac{3}{1^5}+\frac{3}{2^5}+\frac{3}{3^5}, \ldots$	Yes
4	$\sum_{n=1}^{\infty} (-1)^n \left(\frac{6}{n^2}\right)$	$\frac{6}{1},\ \frac{6}{1}-\frac{6}{4},\ \frac{6}{1}-\frac{6}{4}+\frac{6}{9}, \ldots$	Yes
5	$\sum_{n=0}^{\infty} (.04)\cdot(-1)^n$	$.04,\ .04-.04,\ .04-.04+.04, \ldots$	No

Day 7

The structure of the seventh interview consisted of the same three tasks for each student. In the first task, I asked each student to symbolize either Adam's or Benjamin's rules. To give the students more space to write inscriptions, I separated each rule into several lines (see an example of Adam's Rule in Figure 15). The learning goal for the first task on Day 7 was for the students to symbolize their preferred written rule for sequence of partial sums convergence by leveraging inscriptions from their glossary to fulfill these tasks. The research goals for this task were to (1) determine whether each student was able to assimilate the various components of the rule they were trying to symbolize to their existing inscriptions, (2) investigate any modifications that students made to their existing inscriptions to convey their rule, and (3) record any new inscriptions the students created during their symbolization process. Another research goal for this task was to discern how students would combine the various inscriptions they utilized in their symbolizing activity into a cohesive unit to convey their chosen rule holistically.

Figure 15

Separation of Adam' Rule for Symbolization During Day 7

Adam's Rule
The sequence of partial sums
converges to the value ______,
if, for any ϵ-strip,
infinitely many points are inside the strip,
where the strip is centered at $y =$ ______.

In the second task, I asked the students to verbally compare their meanings for sequence, sequence of partial sums, and infinite series. Specifically, I asked each student to discuss their envisioned relationships between each concept. There were two learning goals for this task. First, I wanted to reinforce the relationships students had previously conceived between (1) a sequence and sequence of partial sums and (2) a sequence and series. When I asked students about these relationships, I primarily asked clarifying questions; I did not seriously attempt to perturb student thinking. Second, I hoped the students would conceive a connection between the value to which the sequence of partial sums converges and the value of the corresponding infinite series. If a student could not describe a relationship between a sequence of partial sums and a series, I asked more targeted questions as a minor intervention designed to (potentially) move the student toward conceiving a relationship between these two concepts. The research goal for this task was to determine the overarching relationships between these topics that students had developed throughout the teaching experiment. In particular, I wanted to informally determine whether the students' exposure to the ϵ-strip activity in the context of the sequence of partial sums influenced their thinking about series convergence.

In the final task, I presented the students with a partially filled-out personal written rule for series convergence and asked each student to construct a written rule for series convergence (see an example for Sylvia in Figure 16). The verbiage in the prompt read, "An infinite series converges to the value ___, if _____." I allowed the students to use their inscriptions or written English to complete the rule. The learning goal for this task was for each student to construct their own rule for series convergence for the culmination of the teaching experiment (Sylvia successfully did this, but Monica did not). The research goals for this task were to (1) determine whether a student would construct their rule in written English, symbols, or both; and (2) determine to what degree (if any) the students' previous definitions of sequence of partial sums convergence influenced their definitions of series convergence.

Figure 16

Sylvia's Written Rule Template for Series Convergence for Day 7

Sylvia's Rule
An infinite series converges to the value ____, if ________________________________
____

Exit Interview

I conducted the exit interviews three to seven days after the Day 7 interview. The exit interview was in the clinical interview format, so I did not attempt to perturb or influence student thinking. Consequently, there were no learning goals for the tasks during the exit interview (only research goals). The exit interview contained two preliminary tasks and three major tasks. Of the three major tasks, one was identical to the intake interview (i.e., analyze hypothetical student Abigail's six series), one was highly similar to a task from the intake interview (i.e., describe general convergence), and the final task was not included in the intake interview. I describe each of the preliminary and major tasks in the paragraphs below.

The first preliminary task was finalizing each student's pseudonym, confirming their background information, and ascertaining their willingness to have me report their background information. The second preliminary task was to show each student the final version of their glossary from the end of the Day 7 interview and confirm with the student that the glossary was fully updated and accurate. The research goal for these tasks was to verify that the student data (e.g., glossary) was accurate and to confirm that I had consent to report the student's data and identity at the end of the study.

For the first major task, I asked the students to revisit the hypothetical student Abigail's series from the intake interview and determine whether (1) each series converged and (2) if the series converged, its value of convergence. I also prepared and offered to show a new GeoGebra applet containing (1) the sequence of partial sums corresponding to each of Abigail's series, (2) the ability to reason about the sequences with ϵ-strips, and (3) the rules for convergence (i.e., Adam's rule and Benjamin's rule). I only provided this resource to students if they requested to see the applet while reasoning about one of Abigail's series, and I controlled the applet in these situations (link to applet: https://www.geogebra.org/calculator/nfxc9nvu). The research goals for this task were to (1) determine how the students' responses to the series convergence questions differed from their responses during the intake interviews, (2) evaluate what role the inscriptions the students created emerged in their work, (3) evaluate what role the ϵ-strip activity played in their reasoning, and (4) to assess whether the students could normatively determine series convergence and values after the teaching experiment.

For the second major task, I presented the following two questions (from Larson & Edwards, 2015) to the students and asked them to provide a written response (see Figure 17):

1. How can I tell whether any series converges?

2. If a series converges, how can I determine the value to which it converges?

After the students responded to the tasks, I also asked them to describe their answers symbolically (if possible). The research goal for this task was to assess how students' general meanings for series convergence changed from the intake to the exit interview. In particular, I was interested in whether (1) students' responses were similar to the general rule they created for series convergence at the end of Day 7 and (2) the role of the ϵ-strip activity and their previous inscriptions in their responses.

Figure 17

Screenshot of Exit Interview Task 2

Instructions:

After reviewing each of Abigail's series, I would like you to reflect on the two general questions about series listed below. You may use the open space to provide any writing, drawing, or examples that you feel help to better express your thoughts about infinite series.

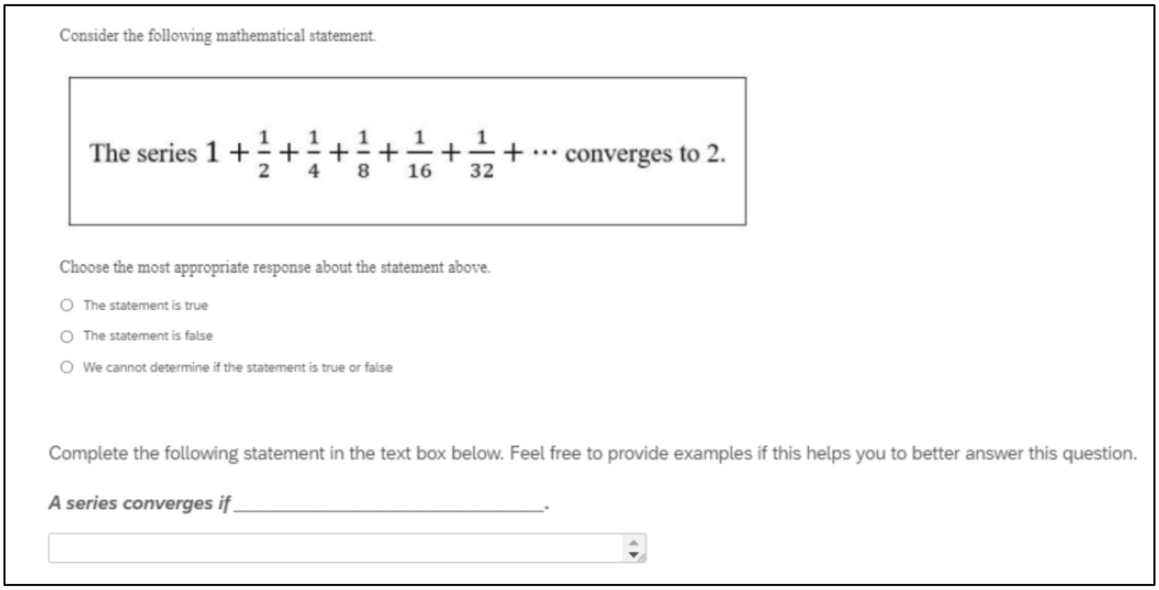
1) How can I tell whether a series converges?

2) If a series converges, how can I determine the value to which it converges?

For the final task, I asked the students to respond to Items 3A and 3C from the screening survey (see Figure 18). After the student completed the two items and explained their responses, I also asked the students to symbolize their general series convergence rule they wrote for Item 3C. The research goals for presenting these survey items again were to (1) investigate whether the students leveraged their general rule from Task 2 to evaluate a claim about the convergence of a specific series and (2) determine whether the students' general rules for series convergence they constructed during Task 3 evolved from their rules they constructed in Task 2.

Figure 18

Screening Survey Items 3A and 3C for Exit Interview Task 3



Consider the following mathematical statement.

The series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$ converges to 2.

Choose the most appropriate response about the statement above.

- ○ The statement is true
- ○ The statement is false
- ○ We cannot determine if the statement is true or false

Complete the following statement in the text box below. Feel free to provide examples if this helps you to better answer this question.

A series converges if ______________________________.

Summary of Data Analysis Methods

This section describes my analysis of the data I collected through the teaching experiment. I analyzed my data in the spirit of grounded theory (Strauss & Corbin, 1998) and conducted two phases of analysis: ongoing and retrospective. This section has two distinct parts. First, I describe the nature of my ongoing analysis throughout the teaching experiment. In the second section, I describe my analysis methods for each of the three results chapters I present in this dissertation (i.e., Chapters 5, 6, and 7).

Ongoing Analysis

My ongoing analysis of data and preparation for subsequent interviews during the teaching experiment consisted of three actions:

- **Action 1:** A 30–60-minute meeting before each interview to overview tasks and goals for each teaching session

- **Action 2:** A 30–60-minute debriefing session immediately following each teaching episode
- **Action 3:** Individual planning session between interviews

In the following paragraphs, I describe how the teacher-researcher (myself) and the witness (Dr. Roh) fulfilled each of these three actions throughout the teaching experiment.

Action 1: Overview Meeting Before Each Teaching Session

The teacher-researcher (myself) and the witness (Dr. Roh) met at least once for 30-60 minutes before each interview session to prepare or finalize our plans for a teaching episode. In each meeting, we (1) reviewed the upcoming interview protocol, (2) tested computerized tasks to ensure they functioned properly, (3) discussed our current models of student thinking and how our tasks would elicit, perturb, or help to reconcile student thinking, and (4) made modifications to our planned tasks to address any insights we had into the students' thinking during or between meetings. We also met directly before each interview session to test recording equipment and review our learning and research goals for a particular session before the student arrived.

Action 2: Debriefing Session Following Each Teaching Episode

The teacher-researcher (myself) and the witness (Dr. Roh) met for 30-60 minutes after each interview. During our post-interview debrief sessions, we discussed (1) the components of the interview that I believed had gone well, (2) any logistical or task-based issues that had emerged during the interview, (3) my model of the student's thinking and challenges I had experienced in creating or confirming my model, and (4) hypothetical tasks to prepare for the next interview session. I recorded each debrief session and referred to them (as needed) in my preparations for upcoming interviews.

Action 3: Individual Planning Between Interviews

The individual planning action of my ongoing analysis comprised several activities. First, I reviewed the video data or field notes after each interview session before preparing the tasks for the next teaching episode. Second, I created updated interview tasks (and a corresponding updated protocol) to reflect our initial interview plan changes. Third, I communicated with the witness (Dr. Roh) to receive feedback on the newly-created or updated tasks. Lastly, I created a final version of the updated tasks based on the feedback I received from the witness and prepared the corresponding OneNote or GeoGebra activities.

Retrospective Analysis

I began my Retrospective analysis after completing the teaching experiment. My analysis was rooted in the principles of grounded theory (Strauss & Corbin, 1998). This section is comprised of five parts. First, I provide an overview of grounded theory emphasizing open and axial coding. Second, I describe the general analysis techniques I used to organize and make an initial pass at the data. In the third section, I describe my fine-grained analysis with regard to the students' intuitive meanings for series convergence, which informed the results I present in Chapter 5. In the fourth section, I describe my fine-grained analysis related to the meanings the students attributed to their symbols, which informed the results I present in Chapter 6. In the final section, I describe my fine-grained analysis with respect to the students' development and modification of personal expressions over multiple teaching interview sessions, which informed the results I present in Chapter 7.

A Brief Overview of Grounded Theory

The purpose of grounded theory is to provide a way to create theory with minimal researcher bias through microanalysis of data. In other words, a researcher who adopts grounded theory to inform her analysis aims to produce a theory grounded in empirical data. Grounded theory takes place in two major stages: open coding and axial coding.

Open coding involves the identification of categories of interest within the data and their ensuing properties. Open coding requires a sizeable period to conduct microanalysis techniques such as (1) line-by-line analysis of students' words in transcript data and (2) the construction of detailed field notes to categorize participants written work. During the microanalysis, a researcher designates categories to represent phenomena in the data by breaking the data "into discrete incidents, ideas, events, and acts" and giving each piece of data a corresponding name or code (Strauss & Corbin, 1998, p. 105). The researcher examines every part of data from multiple viewpoints to elicit as many codes and properties as possible. As the researcher continues to analyze and reanalyze the data, key categories, properties, and dimensions emerge. After the open coding process, the researcher has typically identified a list of key codes and related properties that describe the phenomena of interest within the study. The codes generated during open coding are descriptive but not sufficiently robust to constitute a theory (Strauss & Corbin, 1990).

Axial coding aims to build a theory that can explain phenomena rather than merely describe them. To engage in axial coding, the researcher makes the codes generated during the open coding process the object of his analysis. As the researcher reflects upon his codes and the underlying properties that comprise each code, he will begin to perceive relationships between the codes. For example, some codes constitute a subcategory of another code, or an underlying property informing a code might connect this code with several other codes to create a more general code. In other words, axial coding “begin[s] the process of reassembling data that were fractured during open coding” (Strauss & Corbin, p. 124). As the researcher repeatedly analyzes his codes and revisits the data, he will perceive a small number of overarching categories that subsume or relate to all other categories. The overarching codes generated during axial coding are theoretical and can serve as constructs to explain the phenomena in the data. The researcher continues the open and axial coding processes until his theoretical constructs can explain all phenomena of interest within the data, which Strauss and Corbin (1998) called *theoretical saturation*. After the researcher reaches theoretical saturation, he recontextualizes his theoretical constructs into the data to verify that his theory is based on clear cases of empirical data.

In the following section, I describe my general approach to contextualizing my data during my initial pass at analysis. I consider this next section to constitute the open coding stage of my analysis. In the final three parts of the analysis section, I describe my fine-grained analysis by which I conceived the results that I present in the next three chapters of this dissertation. I consider my descriptions in the final three parts of this section to constitute the axial coding stage of my analysis.

Open Coding: An Initial Deep Pass at the Data

The open coding stage of my data analysis consisted of reviewing all of the video data, creating detailed, moment-by-moment field notes for each interview, and transcribing what I considered to be key moments. During this stage of my data analysis, I focused my attention on three specific narratives. First, I carefully analyzed the intake interview data to categorize the various claims that Monica and Sylvia made about series convergence. Since I had transcribed the entirety of these interviews, I identified each moment in these interviews where the students (1) made a claim about series convergence, (2) posited a convergence value for a series, or (3) introduced a graph to reason about series. I also provided a short description (i.e., open code) for each meaning that the student exhibited in the moments when she was reasoning about the convergence of a particular series. In the next section, Axial Coding 1, I describe my coordination of these open codes into one overarching axial code (i.e., *asymptotic running total* meaning) and three sub-codes (i.e., *decreasing summands convergence*, *monotone running total divergence*, *running total recreation through grouping*) to explain the students' actions throughout the interview. I report the results of this analysis in Chapter 5.

The second narrative on which I focused during my initial moment-by-moment analysis of the video data was the inscriptions that the students created and the meanings they appeared to attribute to these inscriptions. Initially, I created a typewritten version of each students' glossary and the meaning that the students appeared to attribute to their inscription. As I reviewed more and more of the video data, I begin to categorize students' meanings for particular inscriptions (at specific moments) as analogous. As a result, I began to create codes to refer to these inscriptions and categorize moments under inscription categories. Later in this section, I describe how I fit these various moments of inscription meaning into three overarching meanings (i.e., *process*, *concept*, *relational*) and six inscription types (i.e., *command operator*, *create operator*, *indicator*, *placeholder*, *connector, comparator*) Sylvia and Monica constructed to re-present these meanings. I report the results of this analysis in Chapter 6.

The final narrative that I focused on was how students' meanings for their inscriptions changed over time. During the open coding stage of my analysis on this focus, I sorted my notes on students' in-the-moment meanings by inscription and examined the trajectory of students' meanings for various inscriptions across teaching episodes. I also examined how students modified the inscriptions within the personal expressions that they used repeatedly throughout the teaching episodes. Through these processes, I leveraged my axial codes and sub-codes to construct cognitive models for the relationships between Monica's and Sylvia's thinking and their symbolization. In the final section of this chapter, I describe my fine-grained analysis of these two students' symbols over multiple teaching episodes. I report the results of this analysis in Chapter 7.

Axial Coding 1: Students' Intuitive Meanings for Series Convergence

In my preparation to write Chapter 5 (Results Part 1) of this dissertation, I analyzed the open codes that I had created during my preliminary analysis of the intake interview data to organize my descriptive categories into an explanatory framework. During my analysis, an overarching theme emerged: both students repeatedly made reference to the value of the series increasing by adding consecutive summands and seemed to most frequently decide whether a series converged by postulating that the series would (or would not) tend toward a specific value. I considered using the term “partial sum” to describe the students’ image of adding consecutive summands but eventually decided against this idea because neither student indicated (to me) that they were actively coordinating an indexing variable with the result of adding summands. Instead, I decided to use the term *running total* to refer to the dynamic sum the students constructed by adding consecutive summands. I decided to use the term *asymptote* to refer to the value(s) that the students imagined the running total tending toward. My use of the word *asymptote* differs slightly from mathematicians’ conventional use of this term. For example, there were times that the students claimed a series would converge to one or more values (i.e., asymptotic values). Additionally, students did not always claim that the smallest possible bound was the value to which a series converged. Finally, the students would sometimes claim that a series converged to a value that the running total actually achieved. In this study, I use the term *asymptote* to refer to a (possibly unique) value for which the distance between each successive value of the running total and the asymptotic value is less than or equal to the previous value (of the running total). I created an amalgam of the two terms I described in this paragraph, the *asymptotic running total* meaning, to portray students’ intuitive image of successively adding summands in a series and moving closer toward a particular value.

Although Monica and Sylvia exhibited meanings other than the *asymptotic running total* meaning during some moments of the interview, I found that the majority of their reasoning could be explained in terms of (1) saying that a series converged because they perceived that a running total would perpetually move toward a particular value or (2) that a series did not converge because it did not tend toward any values (i.e., was unbounded). In particular, I organized many of Monica and Sylvia’s arguments into three categories: *decreasing summands convergence*, *monotone running total divergence*, and *running total recreation through grouping*. I have previously reported these meanings in a conference proceedings (Eckman & Roh, 2022b) and provide an expanded description of these ideas in Chapter 5 of this dissertation. The purposes of this expanded description are to (1) verify and extend my previously reported findings and (2) provide researchers and instructors with additional insight into how students might initially consider the topic of series convergence.

Axial Coding 2: Students' Meanings they Attribute to their Inscriptions for Series

In my preparation to write Chapter 6 of this dissertation, I examined the various open codes I created to describe the meanings that Monica and Sylvia attributed to their personal expressions throughout the interviews. Some of the students' meanings appeared to match (or at least resemble) conventional meanings for operators, variables, placeholders, and relationships. However, there were instances of unique student inscriptions or meanings that did not appear to follow convention. For example, there were instances where the students would use an inscription as a name or an injunction to create something. As I continued to reflect on the meanings Monica and Sylvia attributed to their inscriptions, three broad categories of meaning began to emerge. First, students would attribute processes that they envisioned to an inscription. Second, students would impute attributes or values of quantities to an inscription. Finally, students would use certain inscriptions to re-present a relationship they perceived between the meanings they attributed to two inscriptions or expressions. Within each of these categories, I also perceived two distinct sub-meanings that Monica and Sylvia re-presented at different times throughout the interview.

Although Monica and Sylvia's attributed meanings to their personal expressions often evolved during their interviews, I found that I could categorize nearly every instance of their symbolization as re-presenting a process, concept, or relationship. In Chapter 6, I provide instances of each type of symbolization and contrast these examples by sub-meaning. The purpose of the results I present in Chapter 6 is to provide a theoretical framework by which instructors and researchers can better comprehend students' symbolizing activity.

Axial Coding 3: The Coevolution of Students' Meanings and Expressions

In preparation to write Chapter 7 of this dissertation, I examined the transcripts, field notes, and open codes I had created for Monica's and Sylvia's personal expression templates that they used repeatedly across several interviews. I found that Monica and Sylvia most frequently (and confidently) used personal expression templates during the first portion of the teaching episodes (i.e., Days 1-3). Through my organization and analysis of my codes for the meaning for each inscription, I perceived two different stories related to Monica's and Sylvia's development of personal expression templates. In Monica's case, she began re-presenting partial sums through a single personal expression template but later constructed two distinct templates through which to re-present different (to her) but similar (to me) mathematical ideas. In Sylvia's case, she developed an initial personal expression template but later found that she could not immediately re-present her thinking about certain types of series through her template. Eventually, Sylvia introduced additional inscriptions to re-present the properties of series that she had not previously considered and incorporated these inscriptions into her existing personal expression template.

As I considered how to align Sylvia's and Monica's stories, I recalled two categories from the theoretical framework I presented for Emily's symbolization in Eckman and Roh (under review). In this framework, I define Category 2 symbolization as an instance where a student constructs two distinct personal expressions for related (in the mind of a researcher) topics. I define Category 3 symbolization as moments where a student attributes two distinct ideas (to them) to a single personal expression. In the context of my framework, I considered Monica's symbolization as Category 2 and Sylvia's symbolization as Category 3. In Chapter 7, I provide further information regarding how these students' personal expressions evolved over several teaching episodes.

CHAPTER 5

RESULTS PART 1: STUDENTS' INTUITIVE MEANINGS FOR INFINITE SERIES CONVERGENCE

In this chapter, I describe Monica and Sylvia's intuitive meanings for infinite series convergence that emerged during the intake interview. The material in Chapter 5 is primarily related to my first research question, *what meanings for series convergence do first-time university calculus students conceive before receiving formal instruction on infinite series?* I have reported about Monica and Sylvia's intuitive meanings for series convergence in a previous conference report (i.e., Eckman & Roh, 2022b). In this chapter, Chapter 5, I provide a more extensive report of both students' thinking and actions throughout their intake interviews to provide additional credence to the results I have previously reported.

In the following table, Table 18, I present definitions for the theoretical constructs I utilized in Eckman and Roh (2022b) to describe students' intuitive meanings for convergence. These definitions include the ideas on which students would focus to determine whether a series converged (i.e., *running total, asymptotic value*) and the meanings of series convergence that informed their arguments. I separated the meanings that informed the students' arguments into two levels through my axial coding of the data from Monica's and Sylvia's intake interviews. At the foundational level, I defined three arguments that students utilized to evaluate series convergence (i.e., *decreasing summands convergence, monotone running total divergence, running total recreation through grouping*). I synthesized these three arguments under a single overarching meaning, which I called an *asymptotic running total* meaning for convergence.

Table 18

Definitions for Constructs Related to Convergence Meanings

Construct	Definition
Running Total	A dynamic sum students construct by adding consecutive summands.
Asymptotic Value	A (possibly unique) number for which the distance between each successive value of the *running total* and the *asymptotic value* is less than or equal to the previous value (of the running total). If such asymptotic value(s) exist, the series converges to these value(s).
Asymptotic Running Total Meaning	Students make a decision about whether a series converges by determining if the *running total* appears to approach an *asymptotic value*.
Decreasing Summands Convergence	Students state that a series converges by arguing that if each consecutive summand in an infinite series is smaller than the previous summand, the *running total* will eventually tend toward an asymptotic value.
Monotone Running Total Divergence	Students claim that a monotone series does not converge by arguing that the *running total* perpetually increases (or decreases) and will eventually surpass every potential bound (i.e., asymptotic value) that might indicate convergence.
Running Total Recreation through Grouping	Students group the terms in an alternating series to create a new series with a monotone *running total*. The students then argue about that the series converges based on their perception of the running total in the new series.

This chapter is comprised of five major sections. In the first section, I describe the *asymptotic running total* meaning as an overarching meaning for series convergence and provide instances of Monica's and Sylvia's exhibition of this meaning. In the second, third, and fourth sections, I describe three arguments that I classified as sub-meanings of ART (i.e., *decreasing summands convergence, monotone running total divergence, running total recreation through grouping*) and examples of Monica's and Sylvia's use of these arguments in their reasoning about series convergence. In Chapter 8, I discuss the relevance of my results to the field of mathematics education and how my findings can influence instructional practices.

Asymptotic Running Total: An Overarching Meaning for Convergence

Eckman and Roh (2022) said the following about Monica and Sylvia's overarching meaning for series convergence:

> Both students exhibited similar meanings for series convergence, which appeared to involve imagining a dynamic *running total* approaching an asymptotic value as additional summands are calculated into the running total. We call this meaning an *asymptotic running total* meaning (p. 1017).

In this excerpt, the term *running total* refers to a dynamic sum that a student creates by iteratively adding consecutive summands in a series and tracking the value of this quantity. The term *asymptotic value* refers to a (possibly unique) number for which the student believes that the distance between each successive value of the *running total* and the value is less than or equal to the previous value (of the running total). The *asymptotic running total* (ART) meaning is analogous to what Martin (2013) called a *dynamic partial sum* image of convergence, or "iteratively adding terms until some condition was reached" (e.g., approaching an asymptote) (p. 273).

I use the term *running total* instead of *dynamic partial sum* for two reasons. First, neither Monica nor Sylvia used the term "partial sum" during the intake interview or gave the impression that they were considering a sequence of partial sums. Rather, they seemed to focus solely on (1) the current value of the dynamic sum and (2) the long-term trajectory they envisioned for this dynamic sum if they continued to add more summands to its current value. In other words, these students did not envision the running total as a covariational relationship (Thompson & Carlson, 2017) between an indexing variable for the summands and the resultant value of the summands. For this reason, using the term “partial sum” (and the corresponding implicit reference to the sequence of partial sums) seemed incongruous (to me) with the ideas that the students were considering during the interview. Second, Martin’s (2013) use of the term *dynamic partial sum* was in the context of Taylor Series (a topic typically taught after infinite series) and I wanted to distinguish between these two related (but distinct) topics. In the following sub-sections, I describe each students’ exhibition of an ART meaning for series convergence during the intake interview.

Monica’s Use of ART Meaning to Make Sense of Series Convergence

Monica leveraged an ART meaning to reason about hypothetical student Abigail’s Series 2, Series 5, and the general series convergence task. In the case of Series 5 and the general series task, she imagined the series converged because (to her) the running total approached one or more asymptotic values. In the case of Series 2, she imagined that the series did not converge because (to her) the running total did not approach any asymptotic values. In the following paragraphs, I address Monica’s actions for these tasks separately.

Abigail’s Fifth Series: $\sum_{i=0}^{\infty} a_i$ ***(where*** a_i ***corresponds to the*** i^{th} ***decimal place of*** π ***and*** $a_0 = 3$***)***

Abigail’s hypothetical fifth series was the infinite decimal expansion of π, which is convergent by definition. I presented the expanded from of this series as $3 + .1 + .04 + .001 + .0005 + .00009 + .000002 + \cdots$. Monica’s initial reaction was to compare the fifth series to her metaphor of approaching a doorway.

Monica: So this one right away made me think of that example I talked about, where you were like walking towards the door frame. Where, these numbers [i.e., summands][12] are so small and even just in one, two, seven, I don't know, like parts of it [i.e., by the 7th summand], we've already seen like a very small number. So if we were to continue on 100 more times or 1000 more times, these numbers [i.e., summands] would be so, so, so insanely small that to, you know, like the naked eye or even like some calculator, it's like, it would be way too small for even them, like they would round up. So my first thought is that it [Abigail's fifth series] does converge.

I then asked Monica to describe whether the series converged and the value to which she believed that the series converged.

Interviewer: Any idea what the series might converge to, if it does converge?

Monica: (…)[13] Like, if I were to take these numbers here and just add these [i.e., first seven summands], it would be 3.141592, right? (…) Is that π? Is that related? Anyway. Oh. I don't know.

Interviewer: I mean, π starts with 3.14, so at least it matches that far.

Monica: OK. (…) so it looks like we're adding on a value almost like to the end is what I'm thinking [i.e., appending another decimal place to the running total]. So I don't think that we would ever equal four. (…) If I had to say it [Abigail's 5th series] converges, then I would say that it does, and it does at four.

[12] Text in brackets [ ] should be read as inclusionary language intended to improve the readability of a quote.
[13] Ellipses (…) denote omitted text. I have generally omitted pauses and repetitious language to improve the readability of the students' comments throughout the results chapters.

Interviewer: (…) So why did you choose four?

Monica: Because the, the more you add on, the larger this number [i.e., running total] gets. And it is getting larger by, you know, not a lot, but it is getting larger. So it's getting further from three and closer to four. But it won't reach four.

Interviewer: OK. (…) What if I were to say the series converges to 3.5 instead of four?

Monica: Oh, well, then maybe you, oh, OK. But then that was my issue, because the number of 3.1 does not round up to four. So. Maybe it converges to three point, I don't know. (*pauses*) The things that I do know, though, about this are that: I would argue we're starting at three and that the number, every time we add on something, it's getting bigger. It's getting bigger at a decreasing rate, and I do not think that we will ever reach four, I don't even think we'll reach (…) 3.2. But I don't know how to decide what it [Abigail's 5th series] converges to.

Interviewer: (…) Could this series converge to both 3.2 and four?

Monica: (pauses) Yes. The way that I'm thinking about it. Yes.

In this excerpt, Monica initially wondered whether the series was equivalent to the irrational number π. After my noncommittal response to her question, she proposed that the fifth series converged to four[14]. Monica justified her choice by saying that she

[14] Although Monica did not explicitly state why she chose four in this transcript, she later said that she chose four because it was a whole number.

imagined that for each summand added to the running total, the running total moves closer to four and further from three. When I proposed 3.5 as a potential convergence value, Monica stated that the running total increased at such a decreasing rate that (she predicted) the running total would never reach the value 3.2. When Monica continued to express uncertainty about the convergence value, I explicitly asked whether Monica believed the fifth series converged to both 3.2 and 4. Monica confirmed that she considered the series to converge to both values. However, she seemed unsure how to justify her answer. Based on this example, I have defined the term *asymptotic value* as one or more values that the student believes the running total approaches, rather than the mathematically conventional definition of asymptote as the limit value of a function as the independent variable approaches infinity.

At this point, I invited Monica to construct a graph to help her explain her reasoning. Monica's graph appeared to be of a running total approaching an asymptotic value (see Figure 19).

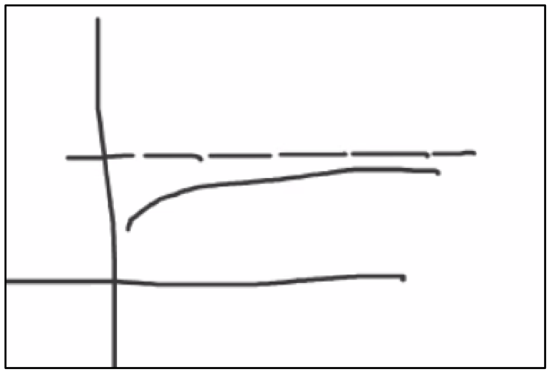
> Monica: So same thing here, we're starting at a value that is not zero (*draws coordinate plane*)[15]. I'm going to say this is three [i.e., the y-coordinate of the starting point of curve that she drew]. And then (*draws smooth, monotone increasing curve that appears to taper off as x-increases*) I haven't added this in the other graphs (…) but I really think that it would help (…) to draw like an asymptote. (…) [I]n the same way where I was like, it [i.e., running total] would never actually hit zero before [a

[15] Italicized text in parentheses generally indicates gestures or student actions performed at the same time (or between) as the words spoken in the quote.

reference to her asymptote from Abigail's 4th series], it is never going to hit 3.2 here [i.e., for Abigail's 5th series].

Figure 19

Monica's Graph of the Running Total for Abigail's 5th Series



After completing her graph of the running total for Abigail's hypothetical 5th series (including drawing an asymptote at the value $y = 3.2$), Monica claimed that the asymptote represented (to her) the value to which the series converged. Additionally, Monica claimed that convergence occurred when the running total approached the asymptote. In this moment, Monica exhibited an overarching meaning for series convergence consistent with an ART meaning.

However, Monica's meaning of convergence as a running total approaching an asymptote did not include a unique convergence value. For instance, when I drew a second (red) horizontal line above Monica's (black) asymptote and asked Monica whether the fifth series also converged to this hypothetical "asymptote" (see Figure 20), she responded:

Monica: I think it's the same.

Interviewer: So what do you mean by that's "the same?"

Monica: Like I would say that the red one [Derek's line] is four and the black one [Monica’s line] is 3.2. But I think it's fair to say that this shape will never, will never touch either of those [values].

Interviewer: OK, so we could think of either of those [lines] as being a value that the series converges to?

Monica: Right. But that doesn't make, then you could just say anything [is an asymptotic value]. So I don't know, because then what's stopping you from putting the number 20 next to that? Nothing. So. I don't know.

Figure 20

Monica’s Graph with Two Asymptotic Values Drawn for Convergence

Monica’s response indicates that she did not consider (at least in this moment) that a convergent series converges to a unique value. Instead, she claimed that any horizontal line with which her graph of the running total did not intersect (i.e., upper bound) was an asymptotic value. Monica further stated that each of these “asymptotic” values could be considered a convergence value for the series, since the running total will “approach” (i.e., consistently move closer to) the asymptotic values but fail to achieve them. Monica recognized the troublesome nature of such a claim but was unable to reconcile her thinking about convergence and asymptotes to posit another relationship between these two concepts.

Abigail's Second Series: $\sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n^5}$

Abigail's hypothetical second series was the alternating convergent p-series $\sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n^5}$, which I presented in the expanded form $\frac{2}{1^5} - \frac{2}{2^5} + \frac{2}{3^5} - \frac{2}{4^5} + \frac{2}{5^5} - \cdots$. Monica's initial reasoning about the second series was to look for patterns across the numerators and denominators of the visible summands. Additionally, Monica noted the alternating signs in the series. She stated that (to her) these signs implied that "numbers [are] canceling each other out." Although she had no idea of the convergence value, Monica eventually guessed that the second series converged.

When I asked Monica to say more about her thoughts on the convergence of the second series, she expressed discomfort with the term "convergence" (implying that she was unsure of the definition of this term).

Interviewer: Tell me what you are thinking.

Monica: So where I'm stuck right now is I'm thinking about like limits. (…) I'm not sure if something approaching a limit is the same thing as it converging to that value. And I remember, like in the first part of this, we had to do that like Qualtrics survey. (…) I wrote about how (…) like if I was walking towards the door and (…) each step I took was half the size of the previous step. (…) I would eventually, effectively get to the door, even though technically on (…) paper, there's always like a smaller fraction of the other stuff I could take. But if somebody just walked in and

> they're like, "Where's [Monica]?" They would say, "She's in the doorway." Like, it would look like I was in the doorway. And so in that, then I would say that it did converge, and I would be like effectively at that point [i.e., the doorway], even though I never technically arrived there.

In this excerpt, Monica shared a metaphor she wrote on her screening survey about approaching a door and her position becoming indistinguishable (to an observer) from being inside the doorway (see Figure 21).

Figure 21

A Recreation of Monica's Screening Survey Justification for Convergence

Explain the truth value of the following statement and your rationale for choosing this truth value.

The series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$ converges to 2.

Monica's Response: [This statement is true.] [I imagine that] you are approaching 2 at a decreasing pace but eventually the distance away from 2 would be so small one would effectively arrive at 2. I thought of this as an example of distance in which a person is taking steps towards a doorway in a way such that each step is half the size of the step before it. eventually they would be taking such small steps, and be so close, that an observer would conclude the individual is in the doorway.

In her metaphor, Monica envisioned herself as the running total and the doorway as the asymptotic value. While Monica acknowledged that she was taking steps (i.e., referencing individual summands in sequence), her focus was on her position (i.e., the value of the running total) and its relationship to the doorway (i.e., the asymptotic value). Monica continued to refer to her "approaching the doorway" metaphor throughout the interview sessions, and this metaphor seemed to comprise her primary image of the mathematical concept of limit (and, eventually, convergence).

General Series Questions

After Monica finished reasoning about Abigail's hypothetical six series (a variation of Grandi's series that I provided in the expanded form $.07 - .07 + .07 - .07 + .07 - \cdots$), I presented two general questions about series: (1) *How can I tell whether a series converges?* and (2) *If a series converges, how can I determine the value to which it converges?* Monica's responses to these questions revealed that her primary meaning for convergence was a running total was approaching an asymptotic value:

> Monica: So for the first question [i.e., *How can I tell whether a series converges?*]. The first thing I thought of was (…) the limits thing, where are we approaching with every (…) next iteration of this series? Are we approaching a value or infinity? (…) Because the thing that stood out to me about the last one [Abigail's 6th series] was that every time you added on one more piece of the series, it did not go the same direction the last one had gone. And even in the one where we were adding and subtracting and then adding and subtracting [i.e., Abigail's 4th series], but the numbers were decreasing, you still saw like a general trend towards, a number, which in that case was zero. So that's what I would say when I'm thinking about, how can I tell whether [a] series converges? (…) Like I'm interested in, what it's approaching and how adding on each next piece of the series is getting it closer to that value.
>
> Interviewer: OK.
>
> Monica: So then the last one [series 6] where it [i.e., subsequent summand] was like essentially undoing what the previous one had done. That was why I

said it [series 6] wasn't converging. Because there was not like one overall direction it was going.

Interviewer: OK. Now, when you say one direction "it" was going. Just to clarify, what do you mean by the "it" that's going somewhere?

Monica: The series as a whole, or like the sum of the series as a whole [i.e., running total]. So like, if I were to take…like one plus two plus three plus four. And on this like, huge number line and I just go up all the way to like a very large number, and that number that was getting big, closer and closer to infinity. And then I went to the 200th part of that series. I can be like, Yeah, that's still getting close to infinity, and I could go to the 1000th part and it would be like, yeah, it still goes to infinity.

In this excerpt, Monica initially compared convergence to her image of limit (similar to her response for Abigail's 2nd series). She then presented three cases of series that she was imagining. In the first case, she referenced Abigail's 6th series and her perception that the running total for this series did not approach any particular values as she added subsequent summands. In the second case, she referenced Abigail's 4th series and stated that the series appeared (to her) to tend toward a particular number (i.e., zero) because of the alternating operator signs and the decreasing summand magnitudes. In the final case, Monica spontaneously introduced the series $1 + 2 + 3 + 4 + 5 + 6 + \cdots$ and stated that, in this instance, the series would not converge because the value of the running total would increase without bound.

Monica's responses to the second question (*If a series converges, how can I determine the value to which it converges?*) were brief and reinforced her earlier comments about a running total approaching an asymptote. She stated “the value to which it [the series] converges is what you're approaching as you're continuing on, every single, like, next part of the series.” She also compared the process of convergence to her “walking toward the door” analogy.

In summary, Monica employed an *asymptotic running total* meaning in three different situations throughout the intake interview. First, she reasoned algebraically and graphically about Abigail’s 5^{th} series to state that the decimal expansion of π converged to 3.2 and 4. In this instance, Monica considered any upper bound of the decimal expansion to constitute a possible convergence value. Second, Monica leveraged her limit metaphor or approaching a doorway to reason about Abigail’s 2^{nd} series and attributed this metaphor to the term “convergence.” Finally, during the general series questions, Monica reiterated her image of convergence as a running total approaching an asymptote. She also presented two instances of non-convergence that failed (in her mind) to meet this criterion: (1) a series whose running total oscillates between two values (i.e., Abigail’s 6^{th} series) and (2) a series whose running total increases without bound (i.e., the sum of the natural numbers). I discuss further instances of Monica’s reasoning about series convergence (and how they relate to the ART meaning) later in this chapter.

Sylvia’s Use of ART Meaning to Make Sense of Series Convergence

Sylvia exhibited clear instances of the ART meaning less frequently than Monica. In the following paragraphs, I describe two places during the interview, Series 2 and Series 6 (both alternating series), where Sylvia explicitly described her reasoning about convergence as a running total approaching an asymptote.

Abigail’s Second Series: $\sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n^5}$

Abigail's hypothetical second series was the alternating convergent p-series $\sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n^5}$, which I presented in the expanded form $\frac{2}{1^5} - \frac{2}{2^5} + \frac{2}{3^5} - \frac{2}{4^5} + \frac{2}{5^5} - \cdots$. Sylvia's initial response to Abigail's second series was that it would not converge because the alternating signs would cause the running total to fluctuate in value and not approach "one final value." From her initial response, I hypothesized that Sylvia was conditioning the convergence of Abigail's 2nd series on whether she could imagine its running total approaching a single value. I then asked Sylvia to clarify why she believed that Series 2 did not converge, and she stated the following:

> Sylvia: Well, I don't know. Because if you have something [i.e., running total] that kind of fluctuates up and down, like I guess in theory, that kind of makes it [i.e., running total] unstable, so it's not going to like lead to one final number [i.e., asymptotic value]. But I guess you could also have like, like a wave that goes up and down, but then it [i.e., running total] kind of stabilizes and the, the max[ima] and the min[ima] get kind of smaller. Um. Because the values [i.e., summands] are getting smaller, because you start with two and then you go to some fraction. I'm sorry, this is confusing.

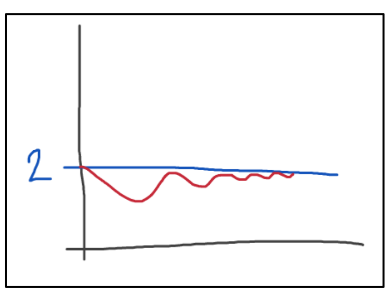
In this excerpt, Sylvia recognized that because the signs of the summands alternate, the running total value will increase and decrease with each additional summand. Her subsequent comment that such an "unstable" running total might not "lead to one final number" indicates that (at this moment) she imagined convergence as a running total approaching an asymptotic value. In the later part of the excerpt, Sylvia began to consider the decreasing magnitudes of the summands and that the distance between each iteration of the running total slowly decreases (which she compared to a wave whose amplitude decreases for each consecutive period).

Sylvia quickly leveraged her image of decreasing amplitudes in the moments following her response above to claim that Series 2 converged. When I asked her whether she would like to draw a graph to help her better describe her thoughts, she created the graph in Figure 22, which she described in the following way:

> Sylvia: So, I'm just going to draw like, this is my graph (*draws two axes representing quadrant 1*), like my (…) x and y axis, and say this is two (*draws horizontal line and writes "2" to the left of the vertical axis as label*). And then this red is what I'm imagining [for the running total]. This looks like you're going to start at two and then go down and then go up a little bit more than you went down (draws small, smooth curve going down toward zero and curving back up to almost 2). So you're almost back at two (…) but not quite. And (…) like each fluctuation is getting smaller and smaller and smaller (*continues to draw waves with smaller and smaller periods that are approaching line y = 2*). That's what I think.

Figure 22

Sylvia's Graph of the Running Total for Abigail's 2nd Series



In this excerpt, Sylvia stated that (to her) the distance between subsequent values of the running total decreases. Although Sylvia did not explicitly state the convergence value in this transcript, she later stated that she believed Series 2 converged to two. This belief is also evidenced in the behavior of the red line (denoting the running total) in Figure 22, which seems to slowly move upward toward the line that Sylvia used to re-present the value of two.

Although Sylvia changed her position on the convergence of Series 2 during the interview, I considered her image of convergence (in these moments) to be grounded in imagining the value of the running total moving progressively closer to two. For instance, before Sylvia recognized that the magnitudes of the summands decreased for consecutive summands, she claimed that Series 2 would not converge because she believed the fluctuations of the running total would not allow its value to move toward a specific value. After Sylvia construed the summand magnitudes as decreasing, she claimed that Series 2 converged (to the first summand's value) because she envisioned that successively decreasing fluctuations in the running total would cause it to approach an asymptotic value. Sylvia employed similar reasoning while considering Abigail's 6th series.

***Abigail's Sixth Series:* $\boldsymbol{\sum_{i=1}^{\infty}(.07)^{i-1}}$.**

Abigail's hypothetical sixth (and final) series was a modification of Grandi's series $\sum_{i=1}^{\infty}(1)^{i-1} = 1 - 1 + 1 - 1 + \cdots$. To reduce the possibility of students recognizing the series, I modified the series to $\sum_{i=1}^{\infty}(.07)^{i-1}$, which I presented in the expanded form $.07 - .07 + .07 - .07 + .07 - \cdots$. Sylvia's initial reaction to Series 6 was that it converged to zero. However, she quickly questioned her claim and determined to draw a graph of the running total to confirm her thinking (see Figure 23). Sylvia described her graph in the following way:

> Sylvia: OK. I, let me draw what I'm thinking right now. (*draws a horizontal and vertical axis to form quadrant 1*) So this is [.07] (*draws a horizontal line and labels the line 0.7*[16] *to the left of the vertical axis*). This is our friend, the series (*draws smooth red curve fluctuating between horizontal line at 0.7 and horizontal axis*). We drop to zero and then we go back and then we drop to zero. Oh. (*Continues to draw more periods of the curve*) OK, I revert my statement. I'm going to say that since this [the drawn curve] fluctuates between zero and 0.7, that it [Abigail's 6th series] doesn't converge. Because it just goes straight up and down from [.07]. I don't know what else to say.

In this excerpt, Sylvia initially created her graph of the running total to confirm her belief that Series 6 converged to zero. However, once she began considering the running total's behavior as more summands were added to its value, she quickly decided that Series 6 did not converge. When I asked Sylvia why her thinking changed, she stated the following:

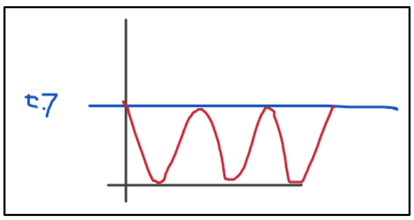
> Sylvia: Um. I think before when I thought it converged, I think I was thinking that like, like [Series 2 and Series 4] that followed the plus, minus [i.e., alternating series]. But those each had fractions [i.e., summands] that were getting smaller and they were like, it was doing this wave thing, and the wave was kind of making its way back up to the initial value. I thought

[16] Sylvia initially stated the value 0.7 instead of 0.07. She later recognized this oversight.

that's what was happening here. But then when I drew the graph, I realized it just strictly bounces up and down from zero to .07 and so it's not going to land on a specific value. Because, in my head, when you're converging to something, like, (…) say you get pretty far on in the series and you're adding and subtracting, the next value that you add is going to get you closer to that number that (…) you think you're converging to. But for this one [i.e., Series 6]? If you say, like, right here (*cursor is placed at the end of drawn curve, aligned with horizontal line at 0.7*), we stop at plus 0.7, (…) the next turn would bring you just straight to zero. And then the next one would bring you straight to 0.7, or .07. So it [i.e., the running total] just bounces up and down.

Figure 23

Sylvia's Graph of the Running Total for Abigail's 6th Series



In this excerpt, Sylvia stated that she initially thought that Series 6 would converge because she anticipated that the magnitudes of the summands would decrease (like her image of Series 2). However, as she began to evaluate the behavior of the running total, she quickly realized that the running total in Series 6 would never approach a specific value.

Sylvia leveraged ART meaning to make claims about the convergence of both Series 2 and Series 6, although her decisions regarding convergence were different in each case. For Series 2, Sylvia claimed that since the fluctuations of the running total would decrease, these fluctuations would eventually become essentially non-existent, allowing the running total to approach a "final value" (i.e., asymptotic value). For Series 6, Sylvia claimed that since the fluctuations of the running total were uniform, the running total would fail to approach a final value. In each instance, Sylvia's image of the behavior of the running total was central to her decision regarding whether she believed a particular series converged.

Monica and Sylvia expressed their *asymptotic running total* meaning in many ways. In the following sections, I explicitly address three of the implications of their ART meaning: *decreasing summands convergence, monotone running total divergence*, and *running total recreation through grouping*. Each implication constitutes a cognitive meaning for convergence and a particular manifestation of the overarching *asymptotic running total* meaning. Consequently, I will alternately use the terms "meaning" and "implication" in the paragraphs below. I further discuss the relevance of these sub-meanings with regard to research and instructional practices in Chapter 8.

ART Implication 1: Decreasing Summands Convergence

Eckman and Roh (2022) defined *decreasing summands convergence* in the following way:

> One implication of an *asymptotic running total* meaning is that a student might believe that if each consecutive summand in an infinite series is smaller than the previous summand, the running total will eventually trend toward one specific value, suggesting that the series will converge. We call this implication *decreasing summands convergence*. In conventional mathematics, the notion of

decreasing summands is a necessary but insufficient property of a convergent series (the most famous example of this principle is the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$; p. 1018).

Additionally, a student reasoning about a series using a *decreasing summands convergence* meaning focuses primarily on the behavior of the summands, and their dialogue may not include explicit references to the running total.

Both students' actions at different moments in their interviews aligned with a *decreasing summands convergence* meaning. In the following paragraphs, I delineate these students' meanings for convergence by series (as opposed to by student, which I did in the previous section). Specifically, I will address both students' reasoning about Abigail's first series and Sylvia's response to the general series convergence question at the end of the first interview.

Abigail's First Series: $\sum_{n=0}^{\infty} \frac{3}{\sqrt{n}}$

Abigail's hypothetical first series was the divergent p-series $\sum_{n=0}^{\infty} \frac{3}{\sqrt{n}}$, which I presented in the expanded form $\frac{3}{\sqrt{1}} + \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{3}} + \frac{3}{\sqrt{4}} + \frac{3}{\sqrt{5}} + \cdots$. Each student exhibited a *decreasing summands convergence* meaning for Series 1 in the moments when they believed that this series converged. In the following paragraphs, I address each student individually.

Monica's verbal comments about the convergence of Series 1 focused on the decreasing nature of the summands.

Monica: In the question, 'Does it converge?' I guess I was thinking [about] where [are] all (…) these positive numbers [i.e., summands] adding up to? Um, and my thought was that as these (…) denominators continue to increase, this number [i.e., summands] will keep getting smaller and smaller, but it will still be positive. (…) But then we're going to add on continually, continually smaller numbers.

Interviewer: OK.

Monica: Wait, maybe not, because if you have the square root of like a really, really big number, it's still going to be a big number, but then that [big number] in the denominator would make it [i.e., the summand] still a small value. (…) So three over a really big number would be a small value. So I would say that it converges because eventually these [summands in the series] would become like, really, really, really small number that you have three over a really, really big number in the denominator. Um. (…) Ooh, I don't know.

Interviewer: (…) So, tell me what you're thinking.

Monica: OK, so I'm stuck now deciding whether or not it [i.e., Series 1] converges. I've decided for sure that we're adding on positive numbers. I've also decided that every number you add on is smaller than the previous number that we added. So in my head of thinking of like a graph that's looking like this (*makes a motion like a decreasing function*) where like the farther you go down the number line [i.e., further out in the sequence of summands], the smaller the value is.

In this excerpt, Monica presented three ideas: (1) every summand in Series 1 would be positive, (2) the value of each subsequent summand would be smaller than the previous, and (3) the process of adding consecutive summands she imagined comprising the infinite series would never end. As she tried to reconcile these three ideas, she initially stated that despite the summands all being positive (idea 1), the values of the summands would become small enough to become essentially zero (idea 2), which implied (to her) that the series would converge. However, when she tried to incorporate the infinite additive process inherent in her image of a series (idea 3), she began to question whether Series 1 converged. Ultimately, Monica determined that Series 1 did not converge. I describe her change in thinking (which relied on her integration of idea 3 into her reasoning about idea 1 and idea 2) in my discussion of *monotone running total divergence*.

Sylvia initially stated that Series 1 converged, stating that “the fact is, it's [i.e., summands] getting smaller, like as you add one, so I'm going to guess that, it does converge.” When I asked to which value Series 1 converged, Sylvia proposed four as the convergence value. However, Sylvia quickly changed her mind, first questioning whether Series 1 converged at all before claiming that the series converged to an unknown value larger than four.

> Sylvia: OK, now I’m thinking that the series doesn’t converge and like it [i.e., running total] keeps adding up and getting bigger and approaches like infinity or something like that.
>
> (*omitted dialogue*)
>
> Interviewer: OK (…) can you say a little bit more about any of the stuff that you’re thinking?

Sylvia: Um (…) If you picture like a perfect square root like three over the square root of nine is three over three, and that's one [$\frac{3}{\sqrt{9}} = \frac{3}{3} = 1$]. But then if you had like three over the square root of 81, that's one-third [$\frac{3}{\sqrt{81}} = \frac{1}{3}$]. So…each item in a series that you're adding on (*brings up hand and mimics placing summands of series sequentially*) is getting smaller. Yes, it is getting smaller. So it's going to… converge to something, but it's not going to be four. Yes, that's my view.

In this excerpt, Sylvia initially claimed that the running total would perpetually increase, which she believed would result in the non-convergence of Series 1. However, when I asked her to clarify her thinking, she reverted to her initial argument that the decreasing summands in the series would eventually cause Series 1 to converge. Although Sylvia could not provide a specific value to which she believed the series converged, she reasoned that the decreasing summands were sufficient for the running total to stabilize towards an asymptotic value.

Sylvia continued to refer to the behavior of the summands in relation to the running total throughout her interactions with Abigail's series. However, she did not make another targeted claim about decreasing summands being a sufficient condition for convergence until I asked her to describe series convergence generally.

General Series Convergence Questions.

Sylvia's response and graph (see Figure 24) that she drew to answer the general series convergence questions revealed that her primary meaning for convergence was of *decreasing summands convergence*, an implication of her *asymptotic total meaning:*

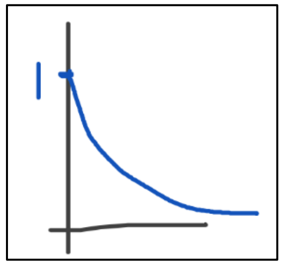
Sylvia: It's hard to determine a rule [for series convergence] because there's so many different examples [of Abigail's series]. But, I think one thing, that could potentially help, determine if a series converges would be, like, if the values of the terms, like in the pattern that they're going, if they, get smaller, I guess, and they keep getting smaller and they follow the same pattern. Because if they're getting smaller, then like, I don't know if this makes sense, but I'm imagining like a graph and it's approaching zero.

Interviewer: OK.

Sylvia: Like the values of the terms, not, not necessarily the sum of the terms, (…) what they converged to, but the values of them [i.e., the summands of the series]. So like if, like, you start here at, so this is like one (*draws a horizontal and vertical axis to make quadrant 1 and marks the value "1" on the vertical axis*) and then each fraction gets smaller and smaller, and smaller and we're approaching zero (*draws monotone decreasing, concave up curve moving from y-intercept (0,1) down toward horizontal axis*). We're not touching it. Like, if that's what the value of the terms are doing, then I'm going to say that that helps you determine if it's converging. But it might not tell you what it [the series] converges to.

Figure 24

Sylvia's Graph for the General Series Convergence Question



In this excerpt, Sylvia claimed that a common theme she perceived across all of Abigail's hypothetical series was that for the convergent series, the magnitude of the summands always decreased. She also drew a graph (whose trace she used to refer to the summands in the series, not the running total) to reinforce her thinking about decreasing summands implying series convergence. However, Sylvia admitted that her *decreasing summands convergence* meaning only allowed her to determine whether a series converged, not the value to which it converged.

ART Implication 2: Monotone Running Total Divergence

Eckman and Roh (2022) defined *monotone running total divergence* in the following way:

> Another implication of an *asymptotic running total* meaning is that a student might believe that since the *running total* perpetually increases (or decreases) in a monotone series, the *running total* will eventually surpass every potential upper bound (i.e., asymptotic value) that might indicate convergence. We call this implication *monotone running total divergence* (p. 1018).

Both students' actions at different moments in their interviews aligned with a *monotone running total divergence* meaning. In the following paragraphs, I address

Monica's transition from *decreasing summands convergence* to *monotone running total divergence* for Abigail's first series and both students' reasoning about Series 3.

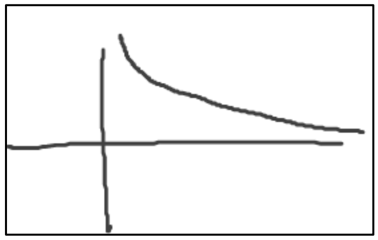
Abigail's First Series: $\sum_{n=0}^{\infty} \frac{3}{\sqrt{n}}$

Abigail's hypothetical first series was the divergent p-series $\sum_{n=0}^{\infty} \frac{3}{\sqrt{n}}$, which I presented in the expanded form $\frac{3}{\sqrt{1}} + \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{3}} + \frac{3}{\sqrt{4}} + \frac{3}{\sqrt{5}} + \cdots$. Monica initially reasoned about this series using a *decreasing summands convergence* meaning (see the previous section for details about this moment). However, she began to question the sufficiency of this meaning when she struggled to reconcile three ideas: (1) every summand in Series 1 is positive, (2) the value of each subsequent summand is smaller than the previous, and (3) the process of adding consecutive summands she imagined comprising the infinite series never ends. In an effort to reconcile her three ideas, she resorted to a graphical argument (see Figure 25).

Monica: OK, so I'm stuck now deciding whether or not [Abigail's first series] converges. I've decided for sure that we're adding on positive numbers. I've also decided that every number you add on is smaller than the previous number that we added. (…) This is what I'm thinking here (*draws graph rapidly decreasing toward zero in Quadrant 1*). (…) So that's the idea that I'm thinking of in the individual values, in, we're looking at, like the y-axis, where they [the summands in the series] remain positive the whole time, they're approaching zero. (…) And so that's going to make the fractions smaller and smaller, but never negative. And also not zero.

Figure 25

Monica's Graph for the Summands of Abigail's 1st Series



In this excerpt, Monica reconciled ideas (1) (i.e., all summands positive) and (2) (i.e., summands decrease toward zero) by drawing a graph with a decreasing curve. However, Monica still seemed unsure how to incorporate idea (3) (i.e., the infinite additive process) into her reasoning about the convergence of Series 1. To redirect her toward idea (3), I asked Monica to state whether she believed that Series 1 converged. In response, she said:

Monica: Um, that it does not [converge]. Because it's approaching infinity because you're constantly adding on more values.

Interviewer: OK, so you're constantly adding on more values and so you're approaching infinity. Can you say a little more about what you mean by that?

Monica: Yes, as we're doing this, even though the fact each fraction is getting smaller and smaller, (…) those [the summands] would still be, although they'd be very small, they'd still be like whole, or they'd still be like positive numbers that you're adding on to the previous part of the expression.

(*omitted dialogue*)

Interviewer: And once again, what do you mean by "approaches infinity?"

Monica: Like, this number [the running total] would just keep getting bigger, and every additional, like three over, like a piece you added on would make it [the running total] an even bigger number. Not by a lot, because you [i.e., the summands] would be so small, but, but it [the running total] would be continually getting more positive.

In this excerpt, Monica applied her notion of positive, decreasing summands (ideas 1 and 2) to the behavior of the running total. In this way, Monica came to envision that if she were to perpetually add the summands of Series 1 (idea 3), then the running total for this series would continually increase. Monica then claimed that the monotone increasing nature of the running total for Series 1 precludes convergence. Monica continued to primarily reason about non-alternating series with her *monotone running total divergence* meaning for the remainder of intake interview.

Inherent in Monica's response is a belief that an increasing running total will eventually surpass every upper bound (i.e., asymptotic value). In other words, a student exhibiting a *monotone running total divergence* meaning will claim that there is no asymptotic value toward which the running total will tend for a particular series. In this case, the student will likely state that the series fails her condition of convergence (i.e., the existence of an asymptotic value for the running total to approach), which implies (to her) that the series does not converge.

Abigail's Third Series: $\sum_{n=1}^{\infty}\sum_{i=1}^{99}\left[\frac{1}{10^{2n+1}}-\frac{i}{10^{2n+3}}\right] = \sum_{k=0}^{\infty}\frac{495}{10000}\left(\frac{1}{100}\right)^{k}$

Abigail's hypothetical third series was the convergent geometric series $\sum_{k=0}^{\infty}\frac{495}{10000}\left(\frac{1}{100}\right)^{k}$. However, before presenting this series to the students, I expanded each term in the geometric series into a finite series of the form $\sum_{i=1}^{99}\left[\frac{1}{10^{2n+1}}-\frac{i}{10^{2n+3}}\right]$, where n corresponded with the index values of the original geometric series. The expanded form of this series that I presented to the students was

$\frac{99}{10^3}+\frac{98}{10^3}+\cdots+\frac{1}{10^3}+\frac{99}{10^5}+\cdots+\frac{1}{10^5}+\frac{99}{10^7}+\cdots+\frac{1}{10^7}+\cdots$.

Monica's initial reasoning about the third series mirrored her strategy for making sense of Series 1: (1) she looked for patterns across the numerators and denominators of the visible summands and (2) claimed that the third series would not converge because the value of the running total would perpetually increase.

I then asked Monica to discuss any similarities and differences she perceived between the methods she used to determine the convergence of the first three series.

Interviewer: What are some similarities and differences that you're seeing between the series that you've looked at so far in terms of determining convergence or not convergence?

Monica: (…) [T]his one [i.e., Abigail's third series], like it seems a lot more straightforward because you're just continually adding on more and more numbers. They're all positive. There's no negatives anywhere. There's no like, like I wrote, like thinking that something might cancel out. There's no thought of that [i.e., terms cancelling out] in these [series] that are all positive [summands]. (…) Like the main thing that's guiding this [my thinking] is that they're all positive numbers that are being added to each other. [This] is really like my main decision-making thing here, which is what made this subtraction one [i.e., Abigail's second series], I guess, like a lot more challenging for me, like deciding an answer.

In this excerpt, Monica again emphasized (similar to Series 1) that her primary focus for determining convergence was thinking about the behavior of the running total as additional summands are computed into this quantity. She also acknowledged her belief that series with monotone running totals (e.g., Abigail's first, third series) will approach infinity. However, she admitted she was unsure of the behavior of alternating series.

Sylvia attempted to reason about the convergence of Series 3 by computing the sum of the first five summands in the series. Her subsequent recognition that the running total would perpetually increase influenced her thinking about whether the series would converge.

Sylvia: I'm just intrigued [by Series 3]. I would like to. Um, 98 over 10 to the fifth plus 97 over 10 to the fifth (*types first three summands in calculator*) is 0.294. (*types in fourth summand*) 0.39 (*indicating sum of first four summands*). Oh, so that's getting greater [i.e., the running total]. Oh, because you're adding them. (*types in fifth summand to yield a partial sum of 0.485*). Yeah, I'm going to say that it [Abigail's third series] doesn't converge because it kind of seems like each value [i.e., partial sum] is getting greater and greater. (*Navigates back to OneNote from the calculator and places cursor on Abigail's third series*) And you're just going to keep adding until you get to some big infinity number. Well, it's not a number [i.e., infinity]. But, yeah.

Interviewer: OK. So when you said each "value" is getting bigger and bigger, what is this "value" that you're, you're referring to?

Sylvia: Oh, like the sum of each of the terms, like just a couple of the times that I did here. (*Navigates back to calculator and places cursor just above the calculated sum of the first five summands*)

In this excerpt, Sylvia relied on a calculator to help her reason about the behavior of the running total. She recognized that each summand in Series 3 would be positive and that the summands were decreasing. Still, based on her calculations, she concluded that (1) the running total would perpetually increase and (2) that this implied that Series 3 did not converge.

Sylvia's decision that Series 3 did not converge conflicted with her earlier statements about Series 1, where she claimed that although the running total perpetually increased, the series would ultimately converge. When I asked Sylvia about this possible contradiction, she said that she recognized that her thinking about Series 3 "kind of broke that pattern" that she used to think about Series 1. However, she was unable to explain why it made sense (to her) to reason about each series in a different way, which is consistent with the findings of Alcock & Simpson (2002) that students can exhibit different meanings for various sequences and series.

ART Implication 3: Running Total Recreation through Grouping

Eckman and Roh (2022b) described Monica and Sylvia's reasoning that corresponded with *running total recreation through grouping* in the following way:

A final implication of an *asymptotic running total* meaning is that a student might believe that if she groups the terms in an alternating series to construct a uni-directional *running total*, the series will converge. We call this implication *running total recreation through grouping* (p. 1019).

In the following paragraphs, I describe how each students' reasoning about Series 4 reflected a *running total recreation through grouping* meaning for series convergence.

Abigail's Fourth Series: $\sum_{n=0}^{\infty} \frac{(200-2n)(-1)^n}{n+1}$.

Abigail's hypothetical fourth series was the alternating divergent series $\sum_{n=0}^{\infty} \frac{(200-2n)(-1)^n}{n+1}$, which I presented in the expanded form $\frac{200}{1} - \frac{198}{2} + \frac{196}{3} - \frac{194}{4} + \frac{192}{5} - \cdots$. While reasoning about this series, Monica claimed that Series 4 converged to zero and Sylvia claimed that the series converged to 200. In each instance, the students justified their conjectures by regrouping the summands in the series. I address each student's responses to this series in the paragraphs below.

Monica (tentatively) claimed that Series 4 converged to zero by combining successive summands into a new series and reasoning about the running total of the new series (see Figure 26):

Monica: I think that if I were to say that it [series 4] does converge, then I would also say that it converges to zero.

Interviewer: OK. Can you say a little bit more about that?

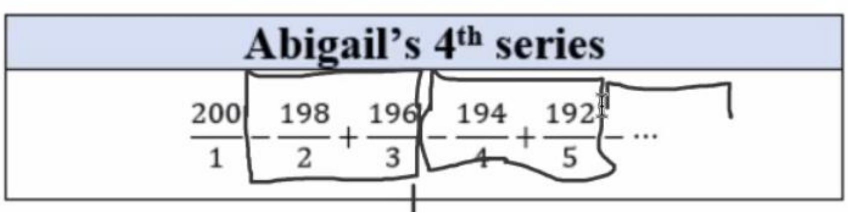
Monica: (…) So in this case, instead of seeing like each individual adding on a fraction is one thing (*places hands close together with small space* *between*), I'm kind of grouping them together, where we are subtracting and adding and that, I'm grouping that together in my head.

Interviewer: OK, could you could you like, mark on the screen what it is that you're grouping together just so that I'm sure that I know?

Monica: Yes. So I would put these together (*draws a bracket above second and third summand*) and then I would put these together (*draws a bracket above the fourth and fifth summand*)… and I take the number 200 and I, do these two things do it (*cursor indicates second and third summands*), I'm going to have a number here (*moves cursor between third summand and minus sign separating third and fourth summand*) that's less than 200, but still very close to it. So basically, what I've decided is if you were to sum these two values (*moves cursor back to indicate second and third summands*), you would have a very small number and you subtract those… So that, that's what makes me think that this [i.e., the running total] is getting smaller and smaller and smaller and smaller.

Figure 26

Monica's Grouping of Summands for Abigail's 4th Series



In this excerpt, Monica acknowledged not imagining the original Series 4 while reasoning about convergence. Instead, she considered the implications of combining each pair of consecutive summands to recreate a new series. In this new series, she began with the first summand, 200, and then appended the sum of each pair of summands to this value. Monica recognized that (according to her grouping of summands) each pairwise sum would be negative, and her newly-constructed version of Abigail's 4th series was of the form $200 + (\text{small negative value}) + (\text{small negative value}) + \cdots$. Monica then reasoned that the running total of her new series would perpetually decrease. Monica concluded that her new series converged to zero, which implied (to her) that Abigail's 4th series also converged to zero. Monica claimed that Series 4 converged to zero (as opposed to some other convergence value) because (1) she believed the running total of the new series would perpetually decrease and (2) she did not think this value would ever become negative.

Similar to Monica, Sylvia's argument hinged on combining the summands of Series 4 to create a new series with a monotone running total (see Figure 27):

Sylvia: So I'm going to say that this one [series 4] converges. And I think I'm going to follow the same logic that I did with [series 2,] that it kind of (…) drops some and increases a little bit less than it dropped (*draws concave-up curve starting from (0, 200) that stops before reaching* $y = 200$), if that makes sense. And then it, like each wave gets smaller (*draws more waves with decreasing amplitudes that progressively move closer and closer to horizontal line at* $y = 200$). And I would say my guess is that it converges to 200.

(*omitted dialogue*)

Interviewer: So, can you explain a little bit more to me about how you're…seeing that come about [convergence to 200]?

Sylvia: So I guess like, if we start at 200, we subtract 99, we'll get 101, and then you add one ninety-six over three [$\frac{196}{3}$], and that number is smaller than 99. Yes. And so you're going to go back up and essentially cancel out some of the, the subtraction that you did. But not like all the way, like you're not going to get back up to 200.

Interviewer: OK.

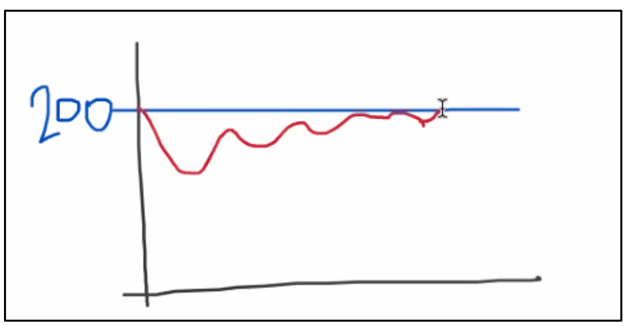
Sylvia: And then you're going to go down a little bit more, but not as great as just went up. And then, like, follow that pattern.

Interviewer: So there. So it's like you're imagining that every time we jump up, we're jumping up farther than we drop down. And so over time, we're slowly moving back up towards 200.

Sylvia: Yes.

Figure 27

Sylvia's Graph of the Running Total for Abigail's 4th Series



In this excerpt, Sylvia imagined computing the sum of the first two summands (i.e., 200 and -99), which resulted in a value of 101. She then imagined that adding the third summand would move the running total value toward 200 (but not reach it). Sylvia then imagined that adding the fourth summand into the running total would cause the value to decrease toward 101 (but not reach it). In Sylvia's graphical re-presentation of her thinking, the oscillating wave she drew (1) decreased in amplitude with each subsequent period and (2) gradually moved upward toward the value of 200. In my clarification of Sylvia's thinking, she confirmed that each pair of consecutive summands would yield (to her) a net positive value. In other words, Sylvia grouped the summands of Series 4 to create a new series of the form $101 + (\text{small positive value}) + (\text{small positive value}) + \cdots$, whose running total she imagined would perpetually increase. Sylvia then claimed that the new series would converge to 200, which implied (to her) that Series 4 converged to 200. Sylvia could not provide a detailed justification for why she claimed that Series 4 converged to 200.[17]

For Monica and Sylvia, their conclusions about the convergence of Series 4 emerged from reasoning about a different series comprised of re-grouped summands from the original series. The students' purpose in performing the regrouping exercise was to create a series with a monotone running total that would perpetually increase (in Sylvia's case) or decrease (in Monica's case) toward a particular asymptotic value. For this reason, I consider *running total recreation through grouping* to be an implication of an overarching ART meaning. Specifically, students' reconception of the series to reflect a simpler (to them) running total and their indication of values toward which they envision the running total moving align with my description of ART.

After Monica and Sylvia performed their regrouping actions, the notion of *monotone running total divergence* disappeared from their reasoning. Instead, the

[17] Sylvia also claimed that Series 2 (another alternating series) converged to the value of the initial summand. Hence, one of Sylvia's rationales for claiming that Series 4 converges to 200 was because she had previously made a similar claim for Series 2.

decreasing summands of the re-grouped series appeared sufficient (in the students' minds) to make claims about convergence and convergence value. Sylvia and Monica only employed the *running total recreation through grouping* meaning while reasoning about alternating series whose summands did not exhibit uniform magnitude. Of the three alternating series I presented during the intake interview (i.e., Abigail's 2nd, 4th, and 6th series), the students showed clear evidence of this reasoning during their work on Series 4. However, other anecdotal evidence of this reasoning exists (i.e., Sylvia's initial claim that Series 6—Grandi's series—converged to zero might have emerged from re-grouping summands).

Summary of Chapter 5 Results

In this chapter, I presented one overarching construct to describe Monica's and Sylvia's intuitive meanings for series convergence (i.e., *asymptotic running total* meaning) and three implications of this meaning (i.e., *decreasing summands convergence*, *monotone running total divergence*, *running total recreation through grouping*) that characterized their actions during the intake interview. The results that I have reported constitute several unique contributions to the literature. For instance, I identified that students with no formal instruction on series convergence might focus on the behavior of a series' *running total* to decide whether they believed a series converged. This construct is similar and different from Martin's (2013) description of a *dynamic partial sum*. My definition of *running total* is similar to a *dynamic partial sum* in that a student's *running total* corresponds to what a mathematician would call a "partial sum." A *running total* differs from a *dynamic partial sum* because students intuitively reasoning about series convergence through a *running total* do not (in the researcher's mind) coordinate the running total with an indexing variable. My extension of the Martin's (2013) idea of a *dynamic partial sum* to students' intuitive reasoning about series convergence provides an additional way for researchers to characterize students' thinking about infinite series.

Another unique contribution to the literature from my results in Chapter 5 is my characterization of *decreasing summands convergence*, *monotone running total divergence*, and *running total recreation through grouping* as students' attempts to reconcile three ideas about series. These three ideas include (1) the signs of the summands (e.g., all positive, alternating), (2) the behavior of the summands (i.e., increasing, decreasing, constant) and its corresponding impact on the running total, and (3) that the process of adding summands into the running total would never terminate (i.e., *potential infinity*). I summarize how Monica's and Sylvia's coordination of these various ideas influenced their exhibited meaning in the paragraphs below.

When the students foregrounded idea (2), focusing primarily on the behavior of the summands, they were most likely to exhibit *decreasing summands convergence*. For instance, Monica's initial conception of Series 1 was that it converged because the magnitude of each summand was smaller than the previous. At the end of her intake interview, Sylvia stated that the common theme she perceived across all convergent series was that the summands decreased. While decreasing summands is necessary for series convergence, Sylvia's comment implies that she also considered decreasing summands a sufficient condition for convergence. Since a student reasoning with *decreasing summands convergence* is focused primarily on the behavior of the summands, the running total is typically present in their thinking but often an afterthought in their verbal explanations of their actions.

When the students considered all three ideas simultaneously about a non-alternating series, they were most likely to focus on the behavior of the running total and exhibit *monotone running total divergence*. For instance, Monica's recognition that the running total in Series 1 would perpetually increase influenced her statement that the running total would increase without bound. Similarly, Sylvia's calculations of the first few values of the running total for Series 3 convinced her that the running total would perpetually increase, which implied (to her) that the series would not converge. In these instances, the students coupled their image of the signs of the summands (idea 1) with their conception of the series as a non-terminating entity (idea 3) to construe the running total as an entity that eventually surpasses all possible bounds. Although Monica and Sylvia often acknowledged that the values of the summands would become incredibly small (idea 2), this notion was subsumed by their image of a monotone increasing running total.

When the students successfully constructed a monotone running total (idea 1) by combining the summands in an alternating series, they would most likely exhibit *running total recreation through grouping*. In these instances, Monica and Sylvia seemed to focus on the monotone nature of the groups of summands (idea 1) and how the relative magnitude of each group behaved (e.g., decreased; idea 2). Although neither student could fully explain their reasoning, Monica and Sylvia believed their reconstructed monotone running totals were bounded. Depending on the signs of the summands (idea

1), the students claimed that the alternating series would either converge to the initial value (increasing summands; Sylvia) or zero (decreasing summands; Monica).

Finally, my categorization of Monica's and Sylvia's intuitive meanings for series convergence serves as a useful instructional tool by which educators might better introduce the topic of infinite series or intervene when students exhibit unconventional meanings for convergence. Specifically, instructors might (1) have students intuitively reason about series convergence as an asymptotic running total and then leverage the corresponding discussion to motivate the need to introduce an indexing variable to track the covariation of summand position and value or (2) ask targeted questions about students' beliefs about the signs, behavior, or process of adding summands in a series during individual discussions to better identify and mitigate students' struggles with comprehending these topics. I summarize the meanings, implications, and focal ideas I presented in Chapter 5 in Table 19 on the next page.

Table 19

Summary of Meanings and Implications Presented in Chapter 5

<table>
<tr><th>Meaning</th><th>Definition</th><th>Implications</th><th>Definition</th><th>Focal Ideas</th></tr>
<tr><td rowspan="3">Asymptotic Running Total Meaning</td><td rowspan="3">Students make a decision about whether a series converges by determining if the running total appears to approach an asymptotic value.</td><td>Decreasing Summands Convergence</td><td>Students state that a series converges by arguing that if each consecutive summand in an infinite series is smaller than the previous summand, the running total will eventually tend toward an asymptotic value.</td><td>• The behavior of the summands for a non-alternating series</td></tr>
<tr><td>Monotone Running Total Divergence</td><td>Students claim that a monotone series does not converge by arguing that the running total perpetually increases (or decreases) and will eventually surpass every potential bound (i.e., asymptotic value) that might indicate convergence.</td><td>• The signs of the summands are uniform
• The behavior of summands for a non-alternating series
• Process of indefinitely adding summands</td></tr>
<tr><td>Running Total Recreation through Grouping</td><td>Students group the terms in an alternating series to create a new series with a monotone running total. The students then argue about that the series converges based on their perception of the running total in the new series.</td><td>• Regrouping summands in an alternating series to create a monotone series of summands
• The behavior of the summands in the regrouped series</td></tr>
</table>

CHAPTER 6

RESULTS PART 2: STUDENTS' CONSTRUCTION OF SYMBOLS TO RE-PRESENT THEIR MEANINGS FOR INFINITE SERIES

In Chapter 6, I propose an explanatory framework for contextualizing the types of meanings that Monica and Sylvia attributed to their various inscriptions and expressions during the teaching experiment. The material in Chapter 6 is related to my second research question, *how do students symbolize their meanings for mathematical topics in the context of infinite series?* The categories for students' symbolization that I present in this chapter, Chapter 6, have emerged through my grounded theory-based data analysis (Strauss & Corbin, 1998). The three central categories of meaning I present constitute the axial codes I created from my analysis of secondary categories I identified through open coding. The primary reasons I share this framework are (1) to categorize the various meanings that students might attribute to their inscriptions during their symbolizing activity and (2) to highlight that students' inscriptions and meanings are distinct entities that do not always align in ways that reflect mathematical convention.

This chapter is comprised of three major sections. In the first section, I present a group of organizational constructs to describe students' general personal expressions and some syntactical properties of these expressions. In the second section, I describe the three meanings Monica and Sylvia attributed to their inscriptions during their symbolizing activity: process, concept, and relational. For each meaning, I also describe two categories of inscriptions that Monica and Sylvia created to re-present these meanings. In sharing these examples, I reference the interview and task related to each interaction to contextualize students' actions with regard to the teaching experiment. If a reader desires further clarification about the purpose, role, and fit of a task within the broad context of the teaching experiment, please refer to Chapter 4 for additional information. In the final section, I summarize the categories and constructs that I present in this chapter. I discuss the implications of these results and future research directions in Chapter 8.

Personal Expression Template, Fixed and Cloze Inscriptions, and Mark Set

In the course of my analysis of Monica's and Sylvia's symbolization, I introduced several terms by which I characterized the syntactic nature of the symbols they created. I describe these constructs before providing empirical examples of student symbolization so that I can use these terms to better contextualize the perceptible artifacts Monica and Sylvia created across examples. These terms include *personal expression template*, *fixed inscription*, *cloze inscription*, and *mark set*. I describe each term individually in the paragraphs below.

Throughout Monica and Sylvia's symbolizing activity, I noticed that they frequently employed similar personal expressions (in the syntactic sense) to symbolize situations they perceived as analogous (e.g., myriad partial sums). In light of this realization, I introduced the term *personal expression template* to describe the general structure of a class of expressions that Monica and Sylvia modified (according to their needs) to symbolize various situations (or quantities) they perceived to have analogous structures or properties. I use the term *template* to refer to Monica's and Sylvia's decisions to fix certain inscriptions across personal expressions (e.g., Σ) and allow others to vary from instantiation to instantiation of their expressions (e.g., indices of summation). I use the term *personal expression* to refer to the perceptible artifacts that

Monica and Sylvia created as instantiations of their personal expression template to re-present their meaning for a particular experience. Although I focus this dissertation on algebraic personal expressions, I consider the construct *personal expression template* to refer to any linguistic (e.g., words), pictorial (e.g., diagrams), symbolic (e.g., notation), visual (e.g., body language), or auditory (e.g., music, spoken language) representation that an individual might use to re-present or convey their meanings in a particular moment.

Since the marks that Monica and Sylvia wrote for the inscriptions in their personal expression templates were sometimes fixed and other times varied, I introduced two terms to differentiate this property of the inscriptions in their templates: *fixed inscriptions* and *cloze inscriptions*. I used the term *fixed inscription* to code instances where Monica and Sylvia used a single written mark uniformly for every instantiation of an inscription in their personal expression template. For example, Sylvia used the inscription Σ in every personal expression she created to re-present a partial sum or series throughout the interviews. In this case, I considered Σ to be a fixed inscription within Sylvia's personal expression template for a partial sum (I describe Sylvia's personal expression template in Chapter 7). I used the term cloze *inscription* to code instances where Monica and Sylvia used different marks across at least two instantiations of a particular inscription for personal expression template. For example, Monica and Sylvia each used various numerical marks to re-present the upper and lower indices of summation for their personal expression templates for partial sums (e.g., the upper bounds in the expressions $\Sigma_1^{37} f(n)$ and $\Sigma_1^{76} f(n)$). My use of the term *cloze inscription* stems from Taylor's (1953) introduction of a *cloze procedure* (also called a *cloze* *activity*). Taylor (1953) introduced the cloze procedure as a psychological measure of reading comprehension in which a student is asked to fill in spaces in a sentence where words have been systematically removed (e.g., I _______ to the store to buy _________.). In the case of student symbolization, I considered a *cloze inscription* to be a component of a personal expression in which a student might write various marks across different instantiations of her expression template.

Once I recognized that Monica and Sylvia sometimes used more than one mark for an inscription across instantiations of their personal expression template, I needed a way to describe the collection of marks that they used (or appeared capable of using) during their symbolizing activity. As a result, I introduced the term *mark set* to refer to the set of marks Monica and Sylvia utilized (or that I imagined they might utilize) across their instantiations of a specific inscription (in the context of their personal expression template). I considered the mark set for the students' fixed inscriptions to be singular since I anticipated that Monica and Sylvia would always use the same mark in each instantiation of a personal expression template. In contrast, I considered the mark set for the students' cloze inscriptions to be nonsingular and comprised of a finite or infinite number of possible marks that Monica and Sylvia used (or I imagined they might use) to re-present the variable (or measurable) attributes of the quantities they imagined.

The following table, Table 20, contains a summary of the major constructs that I introduced in this section. In the following sections, I will use these terms to contextualize how Monica and Sylvia symbolized their meanings for various topics related to infinite series convergence. Specifically, I will address the *personal expression template* at play in each student symbolization example, the nature of the focal inscription (i.e., fixed or cloze), and the character of the corresponding mark set for the inscription. Although I often discuss inscriptions within the context of a personal expression or personal expression template, I do not generally apply these categories to students' meanings they attribute to their expressions (although many such connections seem plausible).

Table 20

Definitions for Personal Expression Template, Inscriptions, and Marks

Term	**Definition**
Personal Expression Template	A representational device that an individual can modify according to her needs to symbolize various situations (or quantities) she perceives to have analogous structures or properties
Fixed Inscription	A single written mark that a student uses uniformly in every instantiation of her personal expression template
Cloze Inscription	A component of a personal expression in which a student might write various marks across different instantiations of her expression template
Mark Set	The set of marks a student utilizes (or the researcher imagines a student might utilize) across her instantiations of a specific inscription (in the context of a personal expression template)

Three Types of Meanings that Students Attribute to their Inscriptions

In the following sections, I describe specific examples of the three types of meanings that emerged through my analysis of Monica's and Sylvia's attribution of meaning to their inscriptions during various moments of their symbolizing activity. These categories included *process*, *concept*, and *relational* meanings. I provide a short description for each category of meaning below.

First, I observed many instances in which Monica or Sylvia would use an inscription to re-present a particular action they carried out (or imagined carrying out) while reasoning about a situation. Some of these actions were relatively simplistic or algorithmic (e.g., add a finite number of summands), while other actions were more complex and required a certain degree of flexibility and creative reasoning (e.g., determine a closed-form rule for the general summand of a series). I coded these meanings, in which Monica and Sylvia attributed the carrying out of an action to an inscription, as a *process* meaning.

Second, I observed instances where Monica and Sylvia used inscriptions to re-present a topic, an attribute of a quantity, or the value of a quantity. The students sometimes used these inscriptions as mnemonic labels to indicate a particular concept (e.g., f is for "function") and other times as placeholders for the value of a quantity (e.g., n in the expression $f(n)$ denotes values of the function's independent variable). I coded these meanings, in which Monica and Sylvia re-presented an attribute or value of a quantity (or a general topic) as a *concept* meaning.

Finally, I observed instances where Monica and Sylvia attempted to re-present a relationship they envisioned between the meanings they attributed to two inscriptions or expressions. In my analysis, I observed two ways these students would symbolize the relationships they envisioned. On the one hand, they introduced new inscriptions (e.g., $=$) that they used to re-present the relationship they envisioned between the (meanings they attributed to the) two expressions. On the other hand, they spatially placed the inscriptions to create an expression (e.g., base with subscript), to which they imputed the relationship they envisioned between the (meanings they attributed to the) inscriptions. I coded these meanings, in which Monica and Sylvia re-presented a relationship between two ideas they had previously symbolized, as a *relational* meaning.

The following table, Table 21, provides a summary for the general categories of meaning that I presented in this section. The table also includes the inscription categories that emerged from my analysis of the inscriptions to which students attributed each meaning. These inscription types included *command* and *create* operators (i.e., *process meaning*), *indicators* and *placeholders* (i.e., *concept meaning*), and *connector* and comparator inscriptions (i.e., *relational meaning*). In the following sections, I provide empirical examples for each meaning and inscription type in Table 21.

Table 21

Meanings Monica and Sylvia Attributed to their Inscriptions

Meaning	Definition	Inscription Types
Process	A student uses an inscription to re-present carrying out an action (or imagining carrying out an action)	• Command Operator • Create Operator
Concept	A student uses an inscription to re-present an attribute or value of a quantity, or a general topic	• Indicator • Placeholder
Relational	A student uses an inscription (or spatial placement of existing inscriptions) to re-present a relationship they envision between two or more ideas they have attributed to other symbols	• Connector • Comparator

Meaning Type 1 for an Inscription: Process

A student who attributes a *process* meaning to an inscription or an expression will re-present a dynamic process through their symbol. In Monica's and Sylvia's symbolization, such inscriptions were often *fixed* (i.e., contained singular mark sets). In the following subsections, I define two inscription types to which Monica and Sylvia attributed a process meaning: (1) command and (2) create operators.

Process Inscription Type 1: Command Operator

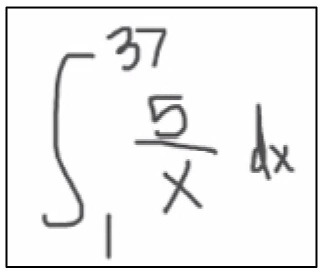
In my analysis of Monica and Sylvia's symbolizing activity, I identified some instances in which they used inscriptions to re-present processes that appeared (to me) to be algorithmic or automatic. In other words, the students used their inscriptions to re-present processes for which the procedural steps were well known (to them), the actions within each step of the process were relatively algorithmic (to them), and the character of the process's result was fairly certain (to them). I used the term *command operator* to categorize the inscriptions to which Monica and Sylvia attributed these predictable processes. In the following paragraphs, I present two examples of Monica's and Sylvia's use of inscriptions as command operators.

Example 1: Monica's Use of $\int$ to Re-present a Command for an Additive Process. Monica spent a significant amount of time during the Day 1 and Day 2 interviews wondering whether she should use an integral sign ($\int$) to symbolize an algorithmic process of adding consecutive values to determine the value of a partial sum. For example, Monica initially proposed the expression $\int_{1}^{37} \frac{5}{x}\, dx$ (see Figure 28) to symbolize the 37th partial sum of Ivy's 2nd series, a divergent p-series which I presented in the expanded form $\frac{5}{1}+\frac{5}{2}+\frac{5}{3}+\frac{5}{4}+\frac{5}{5}+\frac{5}{6}+\cdots$. When I asked her to summarize her reasoning about determining the 37th partial sum for this series, Monica stated the following:

Monica: Ok, so if I were to try to find the sum of the first set of terms and I didn't want to just add them all by hand, because it was a very large number [of summands] that I was trying to find [the sum]. I would make the series into a function like this (*indicates general summand* $\frac{5}{x}$ *in integration notation*) and then I would integrate that function. And the bounds of that integral would be wherever you were starting, what term number you were starting on to what term number you were finishing on. So, if you were like 2nd to 10th, I would do 2 to 10 with whatever function you've made by just looking at your series.

Figure 28

Monica's use of $\int$ *as a Command Operator to Re-present Partial Sums*



In this excerpt, Monica stated that if she did not want to manually compute the sum of a given number of summands in a series, she would construct an expression using an integral sign to re-present performing this computation. Inherent in Monica's response is the notion that the integral symbol constitutes a command to evaluate successive function values and add them together (within the constraints determined by the indices with which she ornamented the integral inscription).

During the Day 2 interview, Monica reasoned about a written rule for determining a partial sum I provided from a hypothetical student named Yolanda (see Figure 29). The purpose of this task was to help Monica select a predominant expression for partial sums by presenting two contrasting definitions for partial sums, one using an integral sign ($\int$; Yolanda's argument) and the other using a summation sign (Σ; Zeb's argument). While reasoning about Yolanda's argument, Monica described her meaning for integration in the following way:

> Monica: Here, with Yolanda (…) every single, infinitely close together point on this line you're also adding together, where there's no room between them. You're not just going, just the y-value at 1, just the y-value at 2. And that's the difference between the two [Yolanda and Zeb].

Figure 29

Yolanda's Definition for Partial Sums from the Day 2 Interview

Yolanda's Argument

The nth partial sum of Ivy's 1st series can be determined by computing the integral $\int_1^n \frac{2}{\sqrt[4]{n}} dn$, which represents the exact area under the curve of the function $f(n) = \frac{2}{\sqrt[4]{n}}$ from 1 to n.

In this excerpt, Monica again described the idea she re-presented through the inscription $\int$ as an additive process. Monica also indicated the algorithmic nature of the process she envisioned as adding together "every single, infinitely close together point on this line."

In these instances, I considered $\int_{\square}^{\square} f(\square)d\square$ to constitute Monica's personal expression template for re-presenting her image of adding together all function outputs between two values of the domain. In each case, it appeared that Monica used the inscription $\int$ as a *fixed inscription* with a singular mark set (comprising only $\int$). At other times during the Day 1 and Day 2 interviews, Monica struggled to decide whether to use the inscription $\int$ or Σ to re-present computing a partial sum and created two distinct personal expressions by which she could re-present this process. I discuss Monica's symbolizing activity with regard to these templates in more in more detail in Chapter 7.

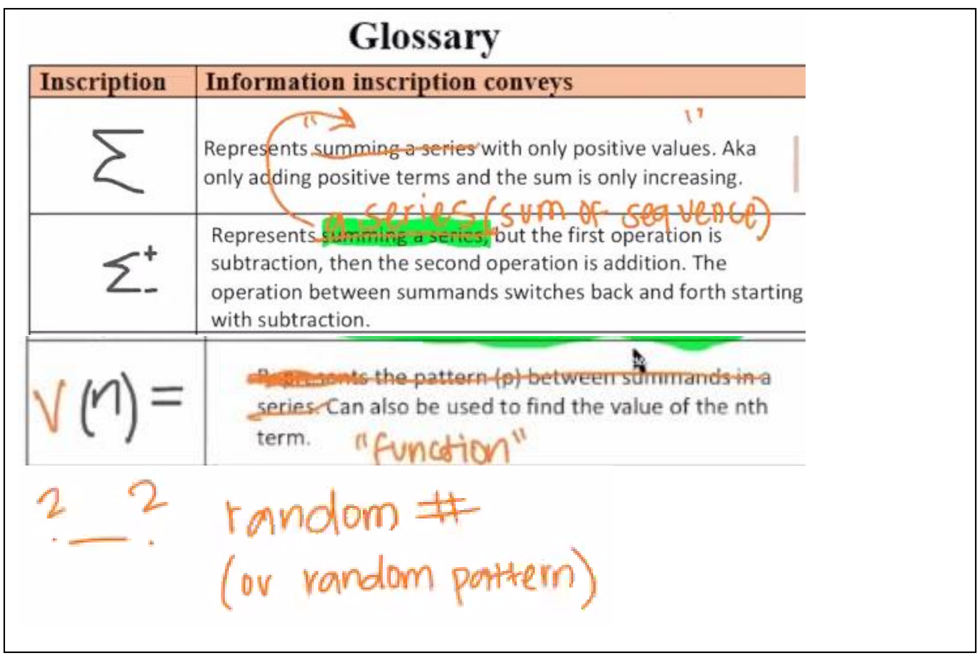
Example 2: Sylvia's Use of ?_? to Re-present the Process of Randomly Generating Values. Sylvia proposed the inscription ?_? to re-present her image of a randomly generated summand value while reasoning about Ivy's 7th series at the end of the Day 2 interview. Ivy's seventh series was the infinite expansion of a non-terminating decimal, which I presented in the expanded form

$$\mathbf{1+.3+.05+.009+.0001+.00004+.000000+.0000009} + \cdots.$$

Sylvia initially symbolized Series 7 with her personal expression template $\Sigma_{\square}^{\square}\square$, which she introduced during the Day 1 interview and consistently utilized during the Day 2 interview to symbolize partial sums and series (inscription definitions in Figure 30[18]).

Figure 30

Sylvia's Glossary Entries[19] *Related to her Expression Template* $\Sigma_{\square}^{\square}\square$



[18] Although Sylvia only wrote the inscription Σ in her glossary, every personal expression she constructed during the Day 1 and Day 2 interview for a partial sum or series included indices and a general summand.
[19] The black type-written text is what Sylvia initially wrote during her creation of the inscriptions during the Day 1 interview. The orange hand-written text reflects changes or additions that Sylvia made to her glossary during the Day 2 interview.

When I asked Sylvia if she could describe a pattern to generate the subsequent summands of the series, Sylvia stated that she could not discern a pattern because the summands appeared (to her) to be randomly generated. I then proposed a hypothetical situation where Ivy constructed Series 7 using a random number generator and asked Sylvia whether she could re-present the random pattern through her expression $v(n)$ (through which she had previously re-presented general summands of several series). Sylvia said that she could not and instead proposed the expression $\Sigma_{n=1}^{\infty}(10^{\square})(random\ \#)$ to re-present the series (see Figure 31):

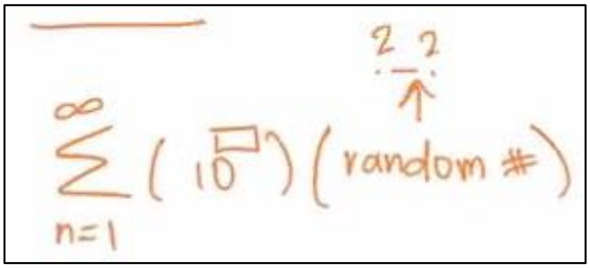
Sylvia: I guess you could still use this (*writes* Σ), you could still write n to 1 and whatever you call it, I'll just keep infinity up here (*writes indices* $n = 1$ *and* ∞). (…) I'm really bad at finding patterns in between series, but what I'm envisioning is you could have some kind of thing here, with like a base 10 and then this would be like, there would be something up here (*draw's a box for an exponent on the base 10*), like a, (…) you could have a negative n. (…) Yeah, I don't really know what the pattern would be. But you would have something that denotes that you would end up with a $\frac{1}{100}$ or $\frac{1}{1000}$ or one over yahdah yahdah yah, and then you would multiply it times something that represents a random number (*writes "random #" in parentheses being multiplied by base 10*). Yeah.

Interviewer: Ok. So, that first ten with the box is being used to represent the appropriate decimal place, if you will.

Sylvia: Yeah, so that is like (…) not constant, but that's the one thing that's not random. Like each decimal point, like you divide by 10.

Figure 31

Sylvia's Personal Expression for Ivy's 7th Series



In this excerpt, Sylvia distinguished between three ideas while constructing her personal expression $\Sigma_{n=1}^{\infty}(10^{\square})(random\ \#)$. First, she used the inscription Σ to denote the summation process inherent in a partial sum or series. Second, she used the indices $n = 1$ and ∞ to describe the summands in the series she imagined comprising the summation. Finally, she used the rule $(10^{\square})(random\ \#)$ to describe the nature of the summands in Series 7. Sylvia further separated her rule for the series into two components: (1) she re-presented the decreasing decimal place values for each subsequent summand using the inscription $10^{\square}$ and (2) she re-presented the use of a random number generator to determine the value of a particular summand using the expression $(random\ \#)$. In other words, Sylvia attributed the algorithmic process of generating a digit value and placing it in a corresponding summand to her expression $(random\ \#)$.

When I asked Sylvia if she could construct an inscription by which to re-present the idea she had attributed to her expression $(random\ \#)$ instead of writing out English words, she proposed the inscription ?_?, which she defined as "random # (or random pattern)." (see Figure 30 above). When I asked Sylvia to describe what she meant by random, she stated: "There's an equal chance for it to be any number, any pattern, any thing. There is no, like, there is no systemic reason as to why this was chosen." Sylvia went on to describe that she envisioned re-presenting values, patterns, or any property of a series that appeared random through her inscription ?_?. Although Sylvia's more general conception of random that she expressed in this moment was likely a conglomeration of more meanings than just an algorithmic command meaning, the impetus for Sylvia's construction of her inscription ?_? was her image of a random number generator systematically generating values that became summands in Ivy's Series 7.

Throughout the remainder of the Day 2 and Day 3 interviews, Sylvia used the inscription ?_? (and only this inscription) to denote components of series (e.g., summand values, patterns in operator signs) that she considered to be random. In these cases, I considered Sylvia's symbol ?_? to constitute a *fixed inscription* with a singular mark set (consisting of only ?_?).

In Chapter 7, I provide insight into how Sylvia's personal expression template $\Sigma_{\square}^{\square}\square$ evolved throughout the interviews for Days 1-3 to include the idea of randomness (and other concepts as well). The example that I have shared of Sylvia's general symbolization of series with random components in this section also aligns with the symbolizing activity of Cedric, whose three distinct symbols for various series (based on his ability to discern a closed-form rule for the general summand) I report in Eckman and Roh (in revision).

Process Inscription Type 2: Create Operator

During certain moments of their symbolizing activity, Monica and Sylvia used inscriptions to re-present processes for which they could describe the procedural steps generally (but not precisely), the action steps of the process appeared (to them) to be investigative and have a high degree of variability, and the character of the result was uncertain (to them). I used the term *create operator* to categorize inscriptions to which the students attributed this sort of inventive process.

A *create operator* differs from a *command operator* in that students believe that the process they re-present through a *create operator* cannot be easily automated and requires the judgment of a reasoning entity to enact the process and decide whether the outcome is appropriate. In contrast, a student using a *command operator* typically believes they can enact the process algorithmically or through a technological medium (i.e., computer, calculator) and accept the result of the process with little question. Since individuals attribute meaning to their inscriptions, no mathematical symbols are inherently *command* or *create* operators. In Chapter 8, I distinguish between the types of conventional expressions that mathematicians use to distinguish between each type of inscription.

Example 1: Emily's Use of ≁ as a Create Operator. I have previously reported one student, Emily, who created the inscription ≁ to re-present her process of constructing a rule to generate the summands of a series (Eckman & Roh, 2022a). In the analysis of my dissertation data, I categorized Emily's inscription ≁ as a *create operator*. I considered Emily's inscription ≁ to constitute a create operator because Emily could re-present a generalized injunction to create a general summand but needed to mentally enact the steps of her process to determine the nature and appropriateness of a general summand for a particular series. Unlike the process she re-presented to find the value of specific partial sums through her *command operator* **Σ**, Emily could not perceive particular attributes of the process that she could symbolize in relation to her operator inscription (e.g., index of summation, an iterative process of evaluating a function fule). Instead, Emily could only re-present the general idea of testing potential patterns until she found one through which she could generate the summands of the series.

Example 2: Sylvia's Use of $\overset{?}{\#}$ as a Create Operator. Sylvia also constructed a *create operator* to re-present the process of discerning the pattern for a particular series. In her case, she proposed a question mark whose lower dot she replaced with a pound sign (see Figure 32). In this paper, I will symbolize Sylvia's inscription using the inscription $\overset{?}{\#}$. Sylvia first proposed this inscription after I asked her about any discrepancies she perceived between her proposed inscriptions and her written rule describing partial sums during the Day 1 interview:

Sylvia: Um, I guess there is not an inscription for finding out how to find the pattern [for the summands in a series]. But I don't know if that's something that you could like represent symbolically, because that's kind of like a brain process, not an operational process. Does that make sense?

Interviewer: Ok, so say a little more about what you mean by that. Why (…) couldn't you make an inscription for this [idea of 'find the pattern']?

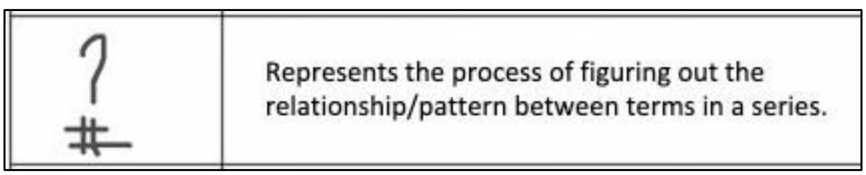
Sylvia: Um, I guess because, it's, well I guess you could because like if you, because I mean adding is also something you do in your brain. But I don't know how you would represent, because (…) there are a lot of different options, there are a lot of different directions that like the pattern could go in, the pattern between the terms. So, I guess it would be hard to (…) I don’t know how to explain it, but it would just be like a question mark, like what’s going on in the series.

Interviewer: Ok, do you want to make question mark an inscription? I mean, you don’t have to but.

Sylvia: Sure. I’ll put a question mark, but instead of the dot it will be pound sign, and that represents (writes “represents the process of figuring out the relationship/pattern between terms in a series”).

Figure 32

Sylvia’s Create Operator $\overset{?}{\#}$ *for Discerning the Pattern of a Series*



In this excerpt, Sylvia initially stated that she was uncertain whether she could symbolize her meanings for the creative process of discerning a pattern for the summands in a series. She also distinguished the investigative process of determining a pattern from an operational process (e.g., adding together summands to calculate the value of a partial sum). Sylvia eventually proposed the inscription $\overset{?}{\#}$ to re-present her overarching uncertainty of the exact process and result she would engage in to determine a pattern for the summands in a particular series.

After Sylvia wrote her inscription in her glossary, I attempted to clarify what she was re-presenting through her inscription. In particular, I wanted to ascertain whether she was able to re-present a closed-form rule for generating summands through her inscription $\overset{?}{\#}$.

Interviewer: So, I’m going to try and restate what I think you, what I understand you to be saying. So, you’re saying that for this fourth inscription, the question mark with the pound sign, that that’s representing to you the process of going out and figuring out the relationship and the pattern.

Sylvia: Mmm-hmm.

Interviewer: So, is that different from, “Oh, now I’ve found it” [i.e., the pattern]?

Sylvia: Um, I guess like the thought “Oh, I know the process” is the result of whatever that inscription entails. But what I’m trying to say is like, there’s a difference between the Sigma [i.e., Σ] and the question/pound sign [i.e., $\overset{?}{\#}$]. Because like the Sigma produces a value, or it produces like something that represents a value, like you get a number. But with the fourth inscription, there’s not really a number, there’s not a tangible outcome. It’s more of like an understanding, or, like an Aha! moment.

Interviewer: Ok, so it’s more of an Aha! moment (…) and then like you found a pattern. So that’s kind of the outcome of the inscription, that fourth one?

Sylvia: Mm-hmm.

Interviewer: Ok, so if you were to actually try and write down the pattern, would you need a different inscription?

Sylvia: Yes.

In this excerpt, Sylvia stated the outcome of the process she re-presented through her inscription $\frac{?}{\#}$ is that she would know a pattern by which she could model the summands of a series. She also distinguished between an earlier inscription, Σ, that she had created to re-present the algorithmic process of computing a sum from consecutive summands, and $\frac{?}{\#}$, which she created to re-present the mental process of investigating and determining a summand pattern for a series. However, when I asked Sylvia whether she could re-present the character of the pattern with her inscription, she stated that this was not possible (for her). Instead, she proposed utilizing a different inscription. Immediately after the episode in the current transcript, Sylvia proposed using the expression $v(n)$ to re-present an open or closed-form rule for the pattern she discovered through the creative process she re-presented through her inscription $\frac{?}{\#}$.

Throughout the remainder of the Day 1 interview, Sylvia continued to use her inscription $\frac{?}{\#}$ (and only this inscription) to denote her creative process of making sense of the summands in a series and determining a formulaic pattern to generate the summands. In this case, I considered the inscription $\frac{?}{\#}$ to be *fixed* with a singular mark set (comprising only the mark $\frac{?}{\#}$). In later interviews (e.g., Day 2), Sylvia incorporated her inscription $\frac{?}{\#}$ into her personal expression template for $\Sigma_{\square}^{\square}\square$ for describing partial sums and series. I address her actions in more detail in Chapter 7.

In summary, Monica used the inscription $\int$ and Sylvia used the inscription ?_? to re-present processes that they considered algorithmic or automated, and whose results they could easily predict and interpret (e.g., the result of an integral computation, the output of a random number generator). In contrast, Emily used the inscription $\nsim$ and Sylvia used the inscription $\frac{?}{\#}$ to re-present creative processes that they (theoretically) knew how to complete but whose exact steps they could not define without actually enacting the process (e.g., determine a pattern by which to describe the general summand of a series). In both instances, the students re-presented processes that guided their actions toward and interpretation of infinite series, an essential cognitive action in learning mathematics (Dubinsky, 1991; Glasersfeld, 1995; Sfard, 1991). Still, their ability to algorithmatize (and symbolize) the steps and components of the process resulted in different inscriptions that served (to them) different purposes for reasoning about series.

Meaning Type 2 for an Inscription: Concept

In my analysis of Monica's and Sylvia's symbolization, I found other instances in which the students appeared to symbolize attributes or values of quantities that they envisioned. I define quantity in the sense of Thompson (1994), who stated that quantities are constructed by individuals and are comprised of three components: (1) an object or entity, (2) an attribute of the object or entity, and (3) a method to measure the attribute (or belief in such a possibility). I decided to use the term *concept* meaning to refer to instances where Monica and Sylvia used an inscription or expression to re-presents a quantity or its values. I also divided the inscriptions to which they attributed concept meanings into two types: indicators and placeholders. In my analysis of Monica's and Sylvia's symbolization, I found that their indicator inscriptions were often *fixed* and their placeholder inscriptions were either *fixed* or *cloze* (i.e., having a nonsingular mark set). In the following subsections, I describe each inscription type individually.

Concept Inscription Type 1: Indicator

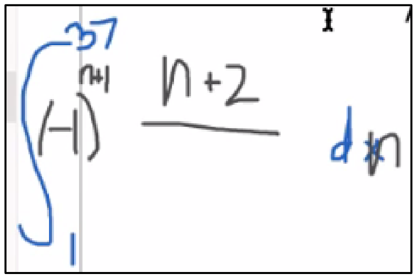
In my analysis, I found that Monica and Sylvia typically re-presented an attribute (but not necessarily the value) of a quantity through their indicator inscriptions. Their most frequent use of indicators was to create a name or a label for an idea they were considering during the interviews. In my review of Monica and Sylvia's data, every inscription I coded as an indicator included a singular mark set (although ornamental inscriptions such as subscripts they attached to their indicators were often *cloze inscriptions*). In the following paragraphs, I share three examples of student's attribution of an indicator meaning to an inscription during the teaching experiment.

Example 1: Monica's Use of dx as a Concept Indicator. Monica used the inscription $\boldsymbol{dx}$ during the Day 1 and Day 2 interviews as a syntactic ornamentation of her personal expression template $\int_{\square}^{\square} \square \boldsymbol{d}\square$ for re-presenting a summation of function values over a finite interval. For example, during her symbolization of Ivy's Series 6 (i.e., $\frac{3}{7} - \frac{4}{7} + \frac{5}{9} - \frac{6}{13} + \frac{7}{19} - \frac{8}{27} + \cdots$) on Day 1, Monica initially symbolized the 37$^{\text{th}}$ partial sum of the series as $\int_{1}^{37} (-1)^{n+1} \frac{(n+2)}{} dx$. When I pointed out Monica's use of both $\boldsymbol{n}$ and $\boldsymbol{x}$ in her personal expression, she quickly changed $\boldsymbol{dx}$ to $\boldsymbol{dn}$, writing $\int_{1}^{37} (-1)^{n+1} \frac{(n+2)}{} dn$ (see Figure 33). When I asked Monica why she had made this change, she said:

> Monica: Just so that they're all the same variable. Cause there's no x's. I don't know what dx even does, I just know it goes at the end of an integral.

Figure 33

Monica's Personal Expression for Ivy's 6th Series



Although Monica's remark was brief and anecdotal, it revealed a profound insight into the meaning she ascribed to her inscription dn. Specifically, she employed $d\square$ as a fixed component of her personal expression template $\int_{\square}^{\square} \square \, d\square$ to indicate the concept of integration. In other words, Monica appended the suffix dx at the end of an integral not to re-present a mathematical meaning but to follow a mathematical convention. While Monica did show that she possessed a nonsingular mark set for $d\square$ (which included the marks dx, dn, and possibly others), her initial writing of the mark dx and her justification for changing dx to dn was purely syntactic. Thus, I consider the expression $d\square$ to be fixed within Monica's personal expression template $\int_{\square}^{\square} \square \, d\square$, but the second inscription of this expression to be a *cloze inscription* (since Monica wrote more than one mark for this inscription across instantiations of her template).

Example 2: Sylvia's Use of CV as a Concept Indicator. After introducing the ϵ-strip activity for sequence of partial sums convergence during the Day 5 interview, I asked Sylvia whether she could symbolize any of the graphical portions of the activity. Sylvia was unable to symbolize anything on her own but eventually proposed the inscription CV to re-present "convergence value" after I explicitly asked her to create an inscription for the value to which a sequence converged.

During the glossary review activity at the beginning of Day 6, I presented Sylvia with a screenshot of the ϵ-strip activity. I then asked her to review what she imagined her inscription CV to refer to on the screenshot. The following transcript and Sylvia's drawing in Figure 34 show her response to my question:

Sylvia: The convergence value? I think that I would label that. I guess that's what you're representing here with this black line (*writes CV next to horizontal line for 'center' of the ϵ-strip*), I think that was like the guesstimate.

Interviewer: Ok, gotcha. Is there anything else in this picture that you would use CV for?

Sylvia: Um, I guess you could maybe label the whole y-axis as CV but that would be like a more general term. And then like.

Interviewer: So can you say a little bit more about that?

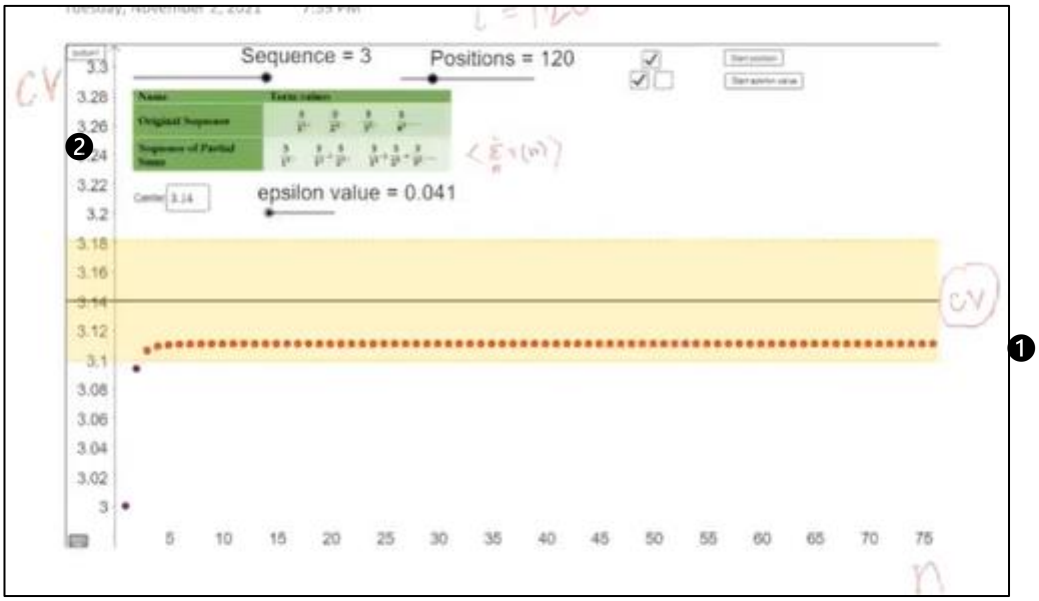
Sylvia: Just that like, if you, for any like general graph if you were to graph it like this and you found the convergence value it would be some value on the y-axis. And, so like, what I just wrote, what I'm circling (*circles CV next to center line*) that's like a specific convergence value, like specific to this sequence. But then, if you wrote over here (*writes CV by vertical axis*) it would be kind of like the general [convergence value].

Interviewer: So, by general convergence value, (…) I'm just not sure what you mean. So, by general convergence value what do you mean?

Sylvia: Um, like those are possibilities, kind of, for a convergence value, (*erases CV from vertical axis*) but like having it as a specific for the graph, it uncomplicates things. So I'd probably just leave it there.

Figure 34

Sylvia's Use of CV to Re-present Components of the ϵ-strip Activity

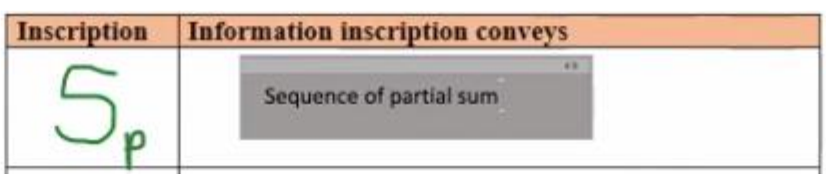


In this excerpt, Sylvia utilized her inscription CV to re-present to herself two distinct ideas. First, she reaffirmed her definition from the conclusion of the Day 5 interview that CV denoted (to her) the value to which a particular series converged. Second, she labeled the vertical axis as CV to indicate a more general sense of the convergence value of an arbitrary series. Unfortunately, I did not understand her general comment during the interview and Sylvia quickly discarded her re-presentational claim when I expressed unsurety regarding her meaning. Still, in the moment of her description, Sylvia recognized that (1) all convergent sequences will converge to values she could re-present through the vertical axis and (2) she could symbolically re-present this property of convergent sequences with her inscription CV. I considered Sylvia's inscription CV to be an *indicator* (and not a *placeholder*) inscription because she indicated that she could re-present the general attribute of convergence value through her inscription but did not reveal whether she believed that she could directly substitute a numerical value for CV in an applied problem. Throughout her symbolizing activity, Sylvia used CV as a *fixed inscription* with a singular mark set (consisting of the lone element CV).

Example 3: Monica's Use of $\boldsymbol{S}$ as a Concept Indicator. During the Day 3 interview, I introduced the concept of the sequence of partial sums to Monica during a mini-lesson. After providing a conventional explanation for this sequence, I asked Monica to construct a personal expression by which she could re-present the sequence of partial sums. Monica subsequently constructed the personal expression $\boldsymbol{S_p}$ to re-present this idea (she claimed that the inscription $\boldsymbol{S}$ stood for "sum" and the inscription $\boldsymbol{p}$ corresponded to the $\boldsymbol{p}$th summand in the series; see Figure 35).

Figure 35

Monica's Expression S_p for the Sequence of Partial Sums

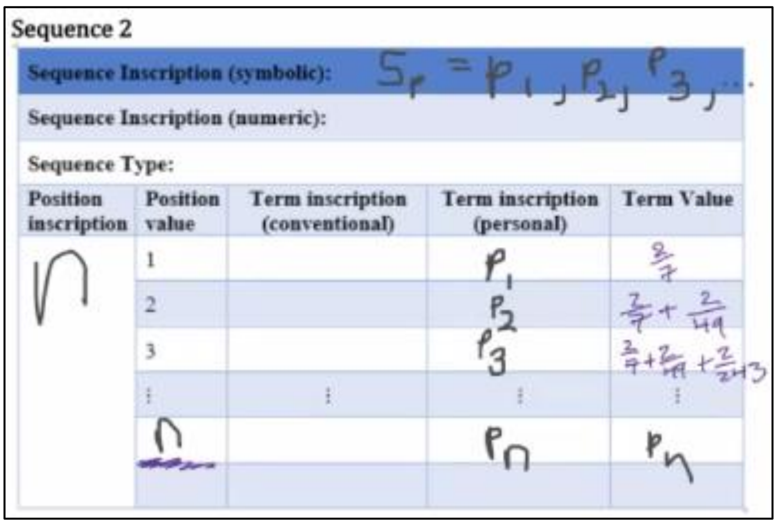


During the Day 4 interview, I presented Monica with an opportunity to reason about and symbolize components of a sequence and its corresponding sequence of partial sums using tables (see Figure 36). During this activity, Monica began attributing the notion of "partial sum" to her inscription p. For example, when I asked Monica why she wrote the personal expression $S_p = p_1, p_2, p_3, \ldots$ to describe the sequence of partial sums, she said,

> Monica: The p on the left means, what p_1 is, or what p_2 is, or what p_3 is. And then the S means they're making a sequence out of all these individual p's. So you wouldn't like put in a number for p.

Figure 36

Monica's Symbolization for Components of a Sequence of Partial Sums



In this excerpt, Monica modified her use of the inscription p, which she used in earlier interviews to denote the upper bound of summation when computing a partial sum, to an indicator for the value of a partial sum (with a corresponding subscript to distinguish which partial sum she was re-presenting at a particular moment). Monica also changed the meaning she re-presented through her inscription S from a "sum" to a "sequence."

Monica's change in attributed meaning for S persisted for the remainder of the Day 4 interview. For instance, when I asked Monica to symbolize a traditional sequence (not a sequence of partial sums), she stated:

Monica: "I guess if I did, it would be, I would use the S because I made, I've decided the S is for sequence. And then I think that I would put, I think that I would put a here (*writes* S_a) to mean a sequence of these terms [i.e., traditional sequence]. So S means that you're going to make a sequence of some kind, and then the subscript is telling you what's going to make up the sequence, like what's going to be between the commas. And in this case I have like a's [i.e., traditional sequence terms], versus p's for the partial sums [i.e., sequence of partial sums terms]."

Monica continued using her personal expression template $S_\square$ throughout the Day 5 interview, saying that for a traditional sequence, she could use any letter for a subscript except for p (i.e., S_p; reserved for sequence of partial sums), n (i.e., S_n; reserved as a variable for the position of the sequence terms), or s (i.e., S_s; reserved for the idea of sequence). In this case, I consider the subscript Monica attached to her inscription S to constitute a *cloze inscription* with a nonsingular mark set (comprising all but a few lowercase English letters) that she employed to distinguish between various sequences she was considering within an example. In contrast, I considered Monica's inscription S to constitute a *fixed inscription* with a singular mark set (comprising only S) that she employed as a mnemonic device to re-present the notion of sequence.

In summary, there were three distinct ways that Monica and Sylvia used indicator inscriptions to re-present an attribute of a concept. In the first example, Monica used her expression dx as a syntactic mechanism by which to indicate (and possibly verify) the concept of integration. In the second example, Sylvia used her expression CV to re-present (1) the horizontal line in the ϵ-strip activity corresponding to the convergence value of a sequence and (2) the vertical axis of the graph, which she used to re-present her envisioned property that all convergent sequences would converge to real numbers. In the final example, Monica used her expression $S_{\square}$ to (eventually) re-present her meaning for a sequence. She also used the subscript of her inscription S to re-present particular sequences she was considering in specific situations. In each of these three examples, Monica and Sylvia used the primary inscriptions that I described (i.e., dx, CV, S) to re-present attributes of quantities (i.e., integration concept, convergence value, sequence) and not necessarily a particular value for these quantities. In the next section, I describe these students' efforts to symbolize values for the attributes of certain quantities they envisioned.

Concept Inscription Type 2: Placeholder

In my analysis, I also identified other instances in which Monica and Sylvia chose to use their inscriptions to re-present one or more values for a particular quantity they envisioned. Consequently, I introduced the term *placeholder* to describe the inscriptions to which these students attributed one (or more) values of a quantity they were considering. In the following subsections, I describe two names by which I further categorize Monica' and Sylvia's inscriptions to which they attributed placeholder meanings: *parameter* and *variable*[20]. I adopt the definitions of these constructs proposed by Thompson et al. (2019), which I share in each section.

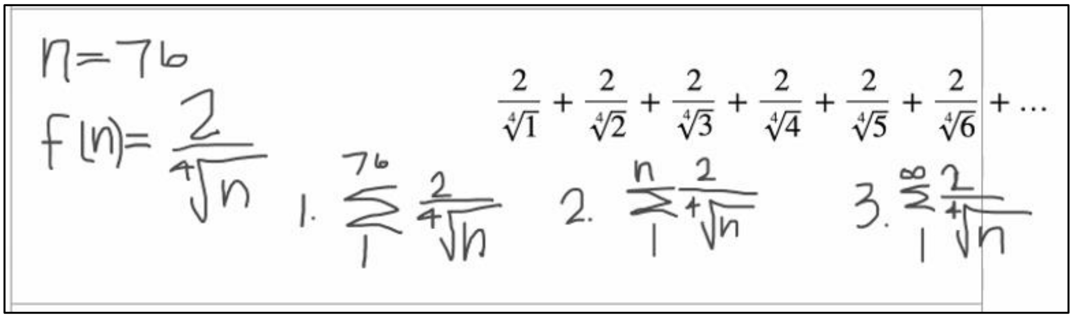
[20] A third category of placeholder inscription might be *constant*. In this instance, students might re-present the value of a quantity that they envision to be uniform in all situations, such as π or e. However, since almost none of my interview tasks focused on these types of values, there is not sufficient data in this study to discuss *constant* placeholders.

Placeholder Inscription Type 1: Parameter. Thompson et al. (2019) described a *parameter* as an inscription to which a student attributes one fixed value of a quantity fixed within a situation but whose value can vary from situation to situation. In Monica's and Sylvia's symbolizing activity, they sometimes symbolized parameters with numeric inscriptions (e.g., 1, 2.5, 1.734) and other times with non-numeric inscriptions (e.g., $\boldsymbol{s}$, $\boldsymbol{p}$). Consequently, I used the terms *fixed* and *cloze* inscriptions to refer to each case of symbolization. In general, Monica and Sylvia used various numeric inscriptions as placeholder parameters across their instantiations of their personal expression templates, which I coded as a *cloze* inscription. However, when the students reasoned about their personal expression templates in the glossary or in open form, they frequently employed the same mark for each instantiation of a placeholder in the arbitrary template. I coded the inscriptions in these arbitrary reasoning situations as *fixed*. In the following paragraphs, I provide two examples from Monica using placeholder parameter inscriptions.

Example 1: Monica's Use of $\boldsymbol{p}$ as as a Placeholder Parameter Inscription. During the Day 2 tasks, Monica used the personal expression template $\boldsymbol{\Sigma_1^n f(n)}$ to symbolize specific and arbitrary partial sums (where she could replace the general inscription $\boldsymbol{n}$ and the general expression $\boldsymbol{f(n)}$ with a particular summand position and general summand rule, respectively). For example, she created the expression $\boldsymbol{\Sigma_1^{76} \frac{2}{\sqrt[4]{n}}}$ for the 76th partial sum of Ivy's 1st series (i.e., $\frac{2}{\sqrt[4]{1}} + \frac{2}{\sqrt[4]{2}} + \frac{2}{\sqrt[4]{3}} + \frac{2}{\sqrt[4]{4}} + \frac{2}{\sqrt[4]{5}} + \frac{2}{\sqrt[4]{6}} + \cdots$) and the expression $\boldsymbol{\Sigma_1^n \frac{2}{\sqrt[4]{n}}}$ for an arbitrary partial sum (see Figure 37).

Figure 37

Monica's Personal Expressions for Ivy's 1st Series



After Monica's construction of personal expressions for Series 1, I decided to address her use of the inscription n to re-present (1) the position of the summands in a series and (2) the upper limit of summation (e.g., the nth partial sum). While Monica had at times indicated that she viewed the upper limit n as a fixed value and the n in her function expression $f(n)$ as a variable taking on multiple values, I was unsure of the degree to which she had authentically reflected on these contrasting uses of the same inscription.

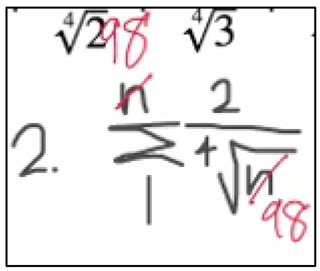
To problematize Monica's potential symbolization issue, I attempted to utilize Monica's personal expression to highlight her use of n to re-present a fixed and varying quantity in the same expression. To this end, I asked her whether she used n to refer to only one summand position while symbolizing the 76th partial sum with her expressions $n = 76$ and $\Sigma_1^{76} \frac{2}{\sqrt[4]{n}}$. Monica confirmed that she intended only to have n denote one value for Question 1 (i.e., symbolize the 76th partial sum). I then stated that I was going to use Monica's personal expression $\Sigma_1^{n} \frac{2}{\sqrt[4]{n}}$ from Question 2 (i.e., symbolize an arbitrary partial sum) to symbolize the 98th partial sum of Series 1 and changed the expression to $\Sigma_1^{98} \frac{2}{\sqrt[4]{98}}$

(see Figure 38). In response to my attempt to adopt Monica's personal expression, she said:

> Monica: I would still leave it as n [the n in the general summand]. (…) Just cause that's just representing the function and you're going to have to (…) do it by hand, not just that specific number [i.e., 98], but, you would (…) have to do it for 1 and for 2 and for 3 and for 4 as well.

Figure 38

Monica's Use of n *for Two Different Purposes*



In response to Monica's comment, I stated that Monica had portrayed n as a single value in her expression for Question 1 but was now saying that n stood for more than one value in her expression for Question 2. In response, Monica stated:

> Monica: Okay, I guess that's not correct. Well, [in Question 1] it [i.e., n] stands for the position that you're evaluating, like, at that time. So, if you wanted to find the 98th term, then n would be 98. But if you want to find the 76th term, n would be 76. That one is not like a fixed definition, I guess.

In this excerpt, Monica portrayed her inscription n for the upper index as a *parameter*, or a value that is fixed in a particular situation (e.g., finding the 98th partial sum) but can vary from situation to situation (e.g., n would be 76 when finding the 76th partial sum).

I considered letting Monica persist with her double use of the inscription n, but ultimately suggested that she create different inscriptions to denote the upper limit of summation and the position of the summands. I justified my intervention because I conjectured that Monica would need a separate inscription to construe as the independent variable for the sequence of partial sums (e.g., $a_i = \Sigma_{n=1}^{i} b_n$) later in the teaching experiment. In response to my suggestion, Monica created a new inscription, p, which she added to her glossary.

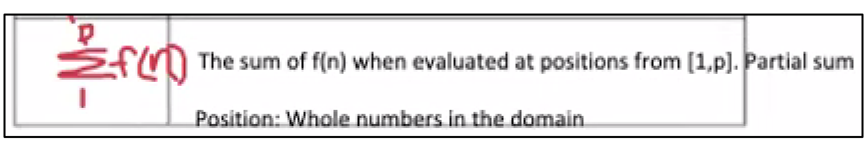
Monica's wrote that the information she wished to re-present through her inscription p was the "upper bound of a partial sum" (see Figure 39). When I asked Monica to clarify what she meant by "upper bound," she proposed an additional symbol, $\Sigma_1^p f(n)$, to re-present "the sum of $f(n)$ when evaluated at positions from $[1, p]$. Partial sum" (see Figure 40).

Figure 39

Monica's Inscription p for the Upper Bound of a Partial Sum

Figure 40

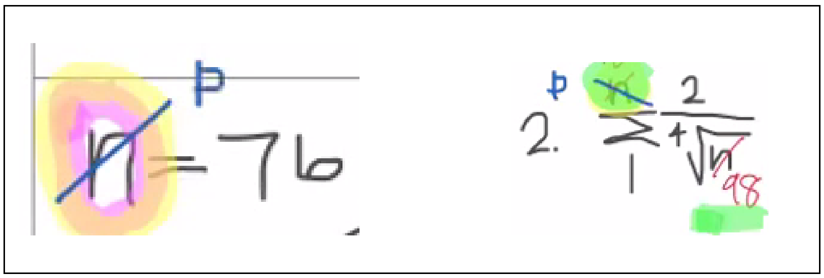
Monica's Expression $\Sigma_1^p f(n)$ for a Partial Sum



I then asked whether Monica wanted to change her personal expressions for Questions 1-3 about Series 1 that she created earlier in the task. In response, Monica proposed changing the "n that was causing issues to a p," modifying her expression $n = 76$ to $p = 76$ (Question 1) and her expression $\Sigma_1^n \frac{2}{\sqrt[4]{n}}$ to $\Sigma_1^p \frac{2}{\sqrt[4]{n}}$ (Question 2; see Figure 41). In this case, Monica's meaning for p was a fixed value corresponded to the position of the partial sum and her meaning for n was the whole number values in the interval $[1, p]$ comprising the input values to $f(n)$.

Figure 41

Monica's Personal Expressions after Introducing the Inscription p



Monica continued to use the inscription p to refer to the upper limit of summation for a partial sum until Day 5, when she attributed the idea of "partial sum" to the inscription p in her expression S_p (see Example 3 from the Concept Indicator section of this chapter). After this change in attributed meaning, Monica proposed using the inscription m to re-present the upper limit of summation for a partial sum. She continued to use the inscription m for the remainder of the teaching experiment.

When describing a placeholder in terms of a *fixed* or *cloze* inscription, I found it necessary to differentiate between Monica's arbitrary reasoning about her personal expression template and her reasoning about specific partial sums in the interview tasks. In the context of the interview tasks, I consider the upper limit of summation in Monica's personal expression template to be a *cloze* inscription because she used various numerical marks for this inscription. In the context of her arbitrary reasoning about her personal expression template, I consider her inscription p (and subsequently, m) to be a *fixed inscription* with a singular mark set (whose lone element changed at various times according to her needs).

Example 2: Monica's Use of Subscripts as Placeholder Parameter Inscriptions. In my description of Monica's use of the personal expression template $\boldsymbol{S}_{\square}$ as an indicator for a particular sequence, I shared a transcript discussion in which Monica used a variety of subscripts by which to differentiate between various sequences. Soon after this discussion, Monica began utilizing the personal expression template $\boldsymbol{S}_{\square_{\boldsymbol{n}}}$ to re-present a particular sequence. Monica's most common instantiation of this template was $\boldsymbol{S}_{\boldsymbol{a}_{\boldsymbol{n}}}$, through which she re-presented her idea of "sequence" with the inscription $\boldsymbol{S}$ and the value of the $\boldsymbol{n}$th term in the sequence with the expression $\boldsymbol{a}_{\boldsymbol{n}}$. When I asked Monica about her predominant use of $\boldsymbol{a}$ as her 1st-level subscript inscription (as opposed to other marks), she said the following:

> Monica: Um, because if you're just working on one sequence and we just defined a as meaning, like, those values, then I would just use a. But I could have picked a different letter for a.
>
> Interviewer: Ok. So, in what situation would you imagine needing to use S_{a_n} and S_{b_n}?
>
> Monica: If you were like, comparing two sequences. And so, S_{a_n} meant something different than what S_{b_n} means, then, that would be, I think, a justified time to use a different letter other than a, that wasn't also being

used for, like, something else. Like, I wouldn't use S_{n_n} because I've already used n.

Interviewer: Ok, so a would work for most problems that you're going to use. But if you have several sequences you're comparing to each other, then you would use further letters as needed.

Monica: Yeah. Yes.

In this excerpt, Monica stated that her first-level subscript referred to the values of a particular sequence. In other words, Monica envisioned that for a specific situation, she would attribute the values of one sequence to the 1st-level subscript inscription in her expression $S_{\square_n}$. For this reason, I classify Monica's 1st-level subscript inscription as a placeholder parameter.

I also classified Monica's 1st-level subscript as a *cloze inscription*, as opposed to my classification of her use of p as a *fixed inscription* for an upper bound of summands for a partial sum. I classified the 1st-level subscript as a *cloze inscription* because although Monica typically re-presented the attribute of sequence values through the mark a, she acknowledged that she could use other marks, such as b, for this inscription. Consequently, Monica's mark set for her first level subscript in her personal expression template $S_{\square_n}$ to be comprised of all lowercase English letters with the exception of s and n.[21]

[21] I considered removing p as a possible mark for this inscription as well. Monica was very clear that S_p denoted (to her) the sequence of partial sums, which she seemed to consider a distinctly different entity than a typical sequence.

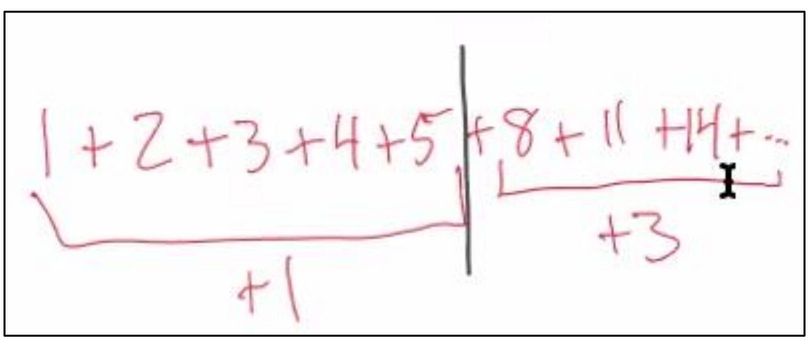
Placeholder Inscription Type 2: Variable. Thompson et al. (2019) refer to a *variable* as an inscription to which a student attributes multiple (varying) values of a quantity within a situation. In the following paragraphs, I provide one example from Monica of her use of inscriptions as placeholder variables.

Example 1: Monica's Use of Placeholder Variable Inscriptions for Independent Variables of Functions. In this excerpt, I describe Monica's use of different marks for her original inscription $\boldsymbol{n}$, which she used to denote the summand positions of a series that would be evaluated in the function $\boldsymbol{f}(\boldsymbol{n})$ in her personal expression template $\boldsymbol{\Sigma_1^p f}(\boldsymbol{n})$ for a partial sum.

At the end of Monica's Day 2 interview, I spontaneously introduced a series whose general summand changed at some point in the series. The specific series I presented was $1 + 2 + 3 + 4 + 5 + 8 + 11 + 14 + \cdots$, which I described to Monica as having a (recursive) pattern of adding 1 for the first five summands and the (recursive) pattern of adding 3 for the next few summands (see Figure 42; the vertical black line was added by Monica). I then asked Monica how she would symbolize this series.

Figure 42

Spontaneously Introduced Series at the End of Day 2 Interview



Monica's initial reaction was that she could not symbolize the spontaneous series with the inscriptions in her glossary. Eventually, she proposed dividing the series into multiple pieces and defining a different function for each piece. The following transcript shows Monica's reasoning process for constructing the expression $(\Sigma_1^5 n) + \left(\Sigma_6^{10} f(x)\right) + \left(\Sigma_{11}^{15} f(k)\right)$ to re-present this spontaneous series (see Figure 43).

Monica: So this would not work [i.e., symbolizing the spontaneous series] with what I have written in my, like, glossary. (…) But if you, if I had to find, like, a sum of some kind, I would break it apart (*creates black dividing line between summands 5 and 6 in the series*) and then I would do all of this, the first half, as one [expression], where my function would be just $f(n)$ is just n, right, so like 1 is just the first position, 2 is the second. And then here [indicates right of black line] this I would have to do another one [i.e., expression] where $f(n)$ is a different function. So, yeah, like if they don't have the same pattern then I couldn't do anything that I've done so far.

(*omitted dialogue*)

Interviewer: Ok, so let's play make believe for a minute, can you just write out what you would do for this [i.e., symbolizing the spontaneous series]? And I mean, let's make this even more consistent, we'll imagine that the [summand] pattern changes every five terms.

Monica: Ok, Ok. (*writes expression* $(\Sigma_1^5 n) + (\Sigma_6^{10} n)$ *and then attempts to determine an explicit rule for generating summands 6 to 10 with which to replace* n) Uh, wait, I don't know what I would even do, this is so bad!

Interviewer: That's ok, let's just pretend, let's just pretend that you know the pattern.

Monica: Ok, (erases n in $\Sigma_6^{10} n$) I will pick a different letter just so that these are clear, these are clearly not the same things going on (writes $(\Sigma_1^5 n) + (\Sigma_6^{10} f(x))$).

Interviewer: Ok, and we haven't even made up what the pattern is for the next five [summands], but yeah.

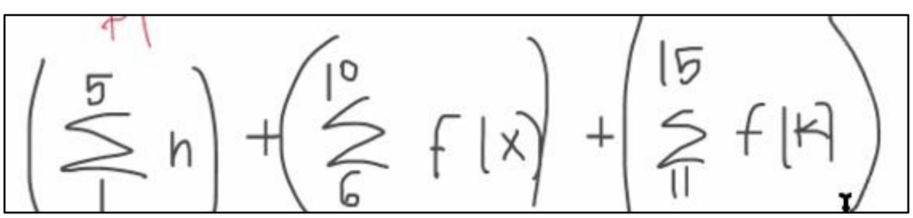
Monica: Right, but and then, I would put, instead of $f(x)$, I would put, yeah, a different function is like the point that I would want to make, and that would be, f of, you know, $f(k)$ or whatever.

Interviewer: Ok, so go ahead and just write that out, what you would imagine the third one [i.e., expression] being.

Monica: (*writes* $(\Sigma_1^5 n) + \left(\Sigma_6^{10} f(x)\right) + \left(\Sigma_{11}^{15} f(k)\right)$). If the pattern changes every five positions, then I would do this, and I would have to know what it was also, what the pattern change was, to make an $f(k)$ or to make an $f(n)$. This should be $f(n)$, I just made $f(n)$ equal to n, so I kind of skipped a step.

Figure 43

Monica's Personal Expression for the Spontaneous Series



In Monica's initial reasoning about symbolizing the spontaneous series, she referred to two functions through which she wished to re-present (1) the first five summands of the series and (2) the remainder of the series. Although Monica recognized that the function rules required to generate these summands would differ, she referred to each function using the expression $f(n)$. However, after Monica failed in her attempt to construct a closed-form, explicit rule for generating the summands of the series in terms of her inscription n, she began to write different expressions to re-present the summands (i.e., $f(n), f(x), f(k)$)[22]. As Monica created the various expressions she used to re-present to the spontaneous series, she stated that she used different inscriptions for each independent function variable because the functions were clearly different (to her).

In this instance, Monica leveraged her personal expression template $\Sigma_1^p f(\square)$ to construct various instantiations for the summands in the spontaneous series. In each instantiation, the numerical mark that she substituted for her inscription p served as a placeholder parameter (in the context of that expression) that was fixed in the situation. In contrast, the non-numerical marks she utilized to denote the summands for each section of the series served as placeholder variables (in the context of each expression) whose values varied according to the constraints of the index values ornamenting the inscription Σ. Additionally, Monica used multiple marks, including $n, x,$ and k in the instantiations of her template. Thus, I consider the placeholder variable through which Monica re-presented the summand positions of the spontaneous series to be a *cloze inscription* with a mark set corresponding to (at least) the lower-case letters $n, x,$ and k.

[22] Monica's symbolization could also be considered an instance of using function notation (or at least the function name) as an idiom (Musgrave & Thompson, 2014).

Monica probably chose to use different inscriptions for the independent variable in each function rule for one of two reasons. First, Monica may have chosen to use different inscriptions in each function rule expression (i.e., $f(n), f(x), f(k)$) because she recognized that the inputs for each function were different values. In this instance, she might have considered it necessary to use different inscriptions to denote the positions 1-5, 6-10, and 11-15 of the spontaneous series. Second, Monica may have chosen to use different inscriptions in each function rule expression because she recognized that each function's explicit, closed-form rules would differ. In this case, it might have made no sense to Monica to use the inscription n twice to write $f(n) = n$ for the first five summands and $f(n) = 3n - 10$ for the next five summands. Instead, Monica may have considered it more authentic to use the function rules $f(n) = n$ and $f(x) = 3x - 10$ to denote what she considered distinct function rules with distinct input values.

Monica’s symbolization also exhibited similarities and differences from the conventional methods that mathematicians use to describe piecewise functions, such as $f_1(n) = n$ for $n = 1, \dots, 5$; $f_2(n) = 3n - 10$ for $n = 6, \dots, 10$. Similar to the conventional example I provided, Monica used the same inscription f in each piece of her rule for the spontaneous series. There was no clear evidence at this moment for the meaning that Monica attributed to the inscription f, although previously in the interview she had expressed that f was a syntactic convention to denote a function (similar to her use of dx with integral notation; see Example 1 of concept indicator inscriptions). Monica also introduced distinct marks to re-present the different rules she perceived for each piece of the function. However, instead of introducing a placeholder parameter ornamentation (e.g., subscript on f_1), Monica attributed this meaning to the marks n, x, and k (her independent variable inscriptions).

Although the data does not permit a rigorous justification of Monica's symbolization, I offer the following conjecture. In the conventional symbolization example for the spontaneous series that I provided in the previous paragraph, there were five distinct ideas in the expression. First, the inscription f is a concept indicator to denote the name of the function (and the function concept). Second, the subscripts in the expressions f_1 and f_2 convey the distinctive nature of each piece of the function. Third, the variable n denotes the values of the independent variable of the function. Fourth, the expression $n = 1, \ldots ,5$ conveys the relevant values of the domain with regard to a particular piece of the function. Finally, the entire expression $f_1(n)$ denoted the output values for one piece of the function.

Monica likely attributed all five ideas I presented in the previous paragraph to her symbolization. For instance, Monica appeared to attribute the concept of function to her inscription f (idea 1), the values of the independent variable to the argument of f (i.e., n, x, k; idea 3), and the entire expression $f(x)$ to the output values for the second piece of the function (idea 5). Monica also likely attributed the relevant values of the domain for each piece of the function (idea 4) to the indices of summation (i.e., $\Sigma_6^{10} f(x)$). However, Monica chose to re-present the distinctive nature of each piece-wise rule (idea 2) by writing different inscriptions for the independent variable of the functions (i.e., n, x, k) instead of through an indexing subscript. While Monica's use of $n, x,$ and k for independent variables allowed her to re-present both idea 2 (different piece-wise rules) and idea 3 (values of the independent variable) through these inscriptions, her simultaneous imputation of both ideas likely made it more difficult for her to convey her thinking. Additionally, the attribution of two distinct concept meanings to the same inscription (i.e., parameter and variable) made it difficult for me (at that moment) to fully comprehend Monica's meanings for her symbolization. Due to time constraints, I could not inquire further into Monica's use of different inscriptions for the independent variable of her function rules.

Meaning Type 3 for an Inscription: Relational

I also identified other moments during the interviews where Monica and Sylvia employed inscriptions (or spatial placements of inscriptions) to re-present relationships between ideas they were envisioning. In Monica's and Sylvia's symbolization, such inscriptions or spatial orientations were often uniform across examples. From the examples I identified, I defined two types of inscriptions to which Monica and Sylvia imputed *relational* meanings: connectors and comparators. In the following paragraphs, I describe two examples of Monica and Sylvia's use of *connector* inscriptions and one example of Monica's use of a *comparator* inscription.

Relational Inscription Type 1: Connector

During certain moments of their symbolizing activity, Monica and Sylvia appeared to use an inscription (or spatial placement of existing inscriptions) to re-present a coordination they envisioned between two components of the same process or quantity. I use the term *connector* to refer to the particular inscription (e.g., =) or spatial placement of the inscriptions in an expression (e.g., base and subscript) through which Monica and Sylvia re-presented this coordinated relationship they envisioned.

In this section, I share two examples of Monica and Sylvia's use of *connectors* to symbolize relationships they constructed between various ideas. I first describe Sylvia's use of a relational inscription to connect an attribute of a quantity and the value of a quantity that she envisioned. I then revisit Monica's use of subscripts to present how a student might spatially orient the inscriptions within an expression to re-present a relationship.

Example 1: Sylvia's Connector Inscription "=" to Relate Process and Result. Sylvia attributed several meanings to her inscription "=" throughout the first five days of the interview. Despite this myriad of meaning, the most consistent idea she attributed to her inscription = was a relationship between a process and its result. In the following paragraphs, I describe Sylvia's various meanings she attributed to her fixed inscription = (and the commonality of the relational meaning) throughout these sessions of the teaching experiment.

Sylvia initially incorporated the inscription = as a component of other expressions in her glossary. For example, at the conclusion of the Day 1 interview, Sylvia proposed the expression $p(n) =$ to (1) denote the pattern to summands in a series (i.e., attribute) and (2) the formula by which to generate the value of this term (i.e., value; see Figure 44). When I asked Sylvia to construct a personal expression for Ivy's 4th Series (i.e., $\frac{6}{1} - \frac{6}{4} + \frac{6}{9} - \frac{6}{16} + \cdots$), Sylvia wrote the expression $\Sigma_{-\,n=1}^{+\,n=37} p(n) = \frac{6}{n^2}$ (see Figure 45). In her description of her expression, Sylvia referred to her entire expression $p(n) = \frac{6}{n^2}$ as the "pattern" for the series. Sylvia also stated that she chose to use the inscription p in her glossary because p is the first letter of the word “pattern,” which is evidence that she considered the inscription p to be an indicator of the concept of pattern.

Figure 44

Sylvia’s Glossary Entry for her Expression $p(n) =$

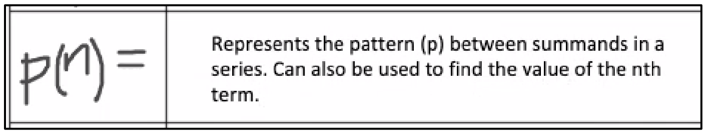
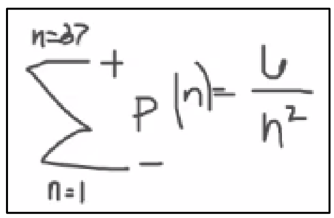


Figure 45

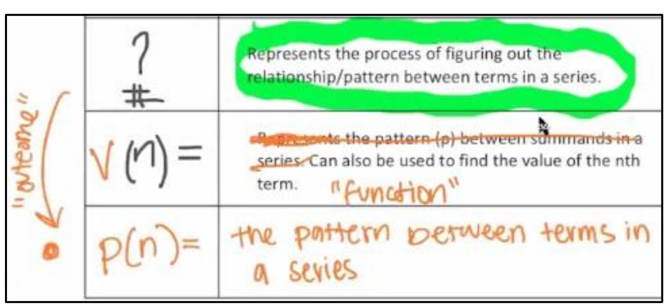
Sylvia’s Personal Expression for Ivy’s 4th Series



During the Day 2 interview, Sylvia introduced the expression $v(n) =$ to differentiate between finding a recursive pattern for generating consecutive summands and an explicit rule for generating a particular summand (see Figure 46)[23].

Figure 46

Sylvia’s Glossary Entry for her Expression $v(n) =$

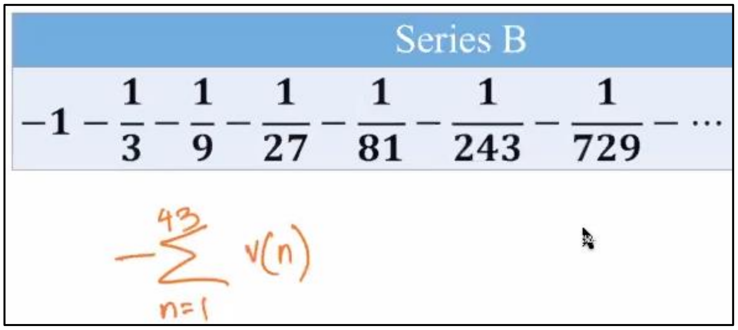


[23] During the Day 3 interview, Sylvia stated that the inscription v in $v(n) =$ stood for “value” (i.e., value of a summand in the series), which implies she still used the inscription v as an indicator of the kind of function she wanted to re-present.

However, when I asked Sylvia to symbolize partial sums and series, she no longer used the compound expression of $v(n)$ and its closed-form rule (separated by the inscription =), as she had during the Day 1 interview (see Figure 47).

Figure 47

Sylvia's Personal Expression for Series B



During the Day 3 interview, I explicitly asked Sylvia to describe why she had included the inscription $=$ in her expression $v(n) =$ in her glossary. Sylvia's response indicated her desire to re-present a connection between the summand values of the series and the formula by which she could generate these values:

Interviewer: My next question for you is, for this $v(n)$ down here and the p, on both of these I see an equal sign. But on some of your other inscriptions, like the first three for example, there are no equal signs. So I was wondering why you put an equal sign on those [symbols].

Sylvia: I don't know if it adds, well, it kind of does add something to the inscription. Because it kind of suggests that there is supposed to be an answer, kind of. Like, for $v(n)$ you're trying to find out the formula. (…) But yeah, the $v(n) =$, I don't know. I guess it just, I kinda just wanted it to be obvious that something is supposed to follow that. Kind of like a 'fill in the blank' moment.

Interviewer: Ok, so it's indicating, or possibly indicating that there is something that is supposed to be on the other side [of the equals sign]?

Sylvia: Mm-hmm.

Interviewer: (…) When you're imagining what is going to be on the other side of that equal sign, what are you imagining? We'll start with $v(n)$.

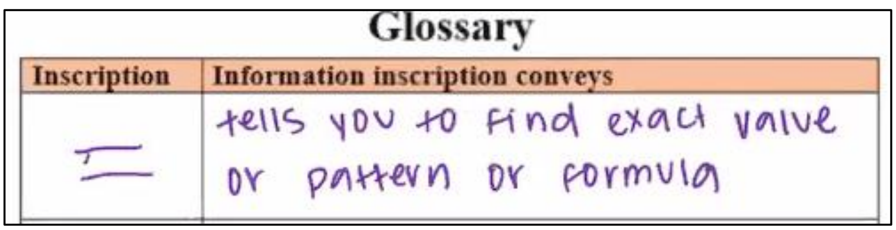
Sylvia: Um, some kind of formula involving n that you can plug in n, so like the 3^{rd} term or the 5^{th} term, and then you'll get the value of that term in the series.

In this excerpt, Sylvia initially questioned her reason for including the inscription $=$ with her expression $v(n) =$ in her glossary. As she reflected, she ultimately stated that she included the inscription $=$ to serve as a reminder that she should complete the expression $v(n) =$ with a closed-form rule for the general summand of a series. In this moment, Sylvia attributed two meanings to the inscription $=$. First, she used the inscription $=$ to re-present the connection she envisioned between her expression $v(n)$ in her personal expression template $\Sigma_{n=\square}^{\square} v(n)$ and the formula by which she would calculate the values of these summands. Second, she used the inscription $=$ as a *command operator* to re-present the process of finding the closed-form explicit rule for the general summand to complete an instantiation of her personal expression template $v(n) = \square$. Still, the primary purpose of Sylvia's inscription $=$ (in the context of her personal expression template $v(n) = \square$) was to re-present the relationship between the summands of a series and a formula by which she could determine these values.

As Sylvia continued to use her inscription $=$ during the Day 4 and Day 5 interviews, her verbal definitions for this inscription aligned more and more with that of a command operator. Finally, on Day 5, Sylvia formally added the inscription $=$ as a line item in her glossary with the definition "tells you to find the exact pattern or formula of something" (see Figure 48). Still, Sylvia self-identified her inscription $=$ as describing a relationship between an inscription such as $v(n)$ that she used to re-present an attribute of a quantity and an expression which she could use to re-present the actual value of the quantity (if she could determine it).

Figure 48

Sylvia's Definition for the Inscription $=$


Glossary

Inscription	Information inscription conveys
=	tells you to find exact value or pattern or formula

Example 2: Monica's Use of Subscripts as Connectors to Spatially Relate an Attribute of a Quantity and its Value. As I discussed previously in this chapter, Monica developed the personal expression template $\boldsymbol{S}_{\square_{\boldsymbol{n}}}$ over the course of the interviews for Days 3-5 to re-present her images of various sequences. In these moments, Monica used the inscription $\boldsymbol{S}$ as an indicator to re-present the general concept (i.e., attribute) of a sequence, her 1^{st}-level subscript as a placeholder parameter to distinguish sequences within a situation, and her inscription $\boldsymbol{n}$ as a placeholder variable to re-present the positions of the terms in the sequence.

Although Monica did not explicitly justify her use of subscripts during the interview, I consider her spatial placement of the various inscriptions in her template $S_{\square_n}$ to constitute a relational inscription. Specifically, Monica used each subscript level to differentiate between specific conditions of a general attribute she was considering. For example, Monica stated that she chose the inscription S to denote the concept of "sequence" but introduced a subscript as a distinguishing agent when she had to reason about more than one sequence in a situation (see Example 2 in the Placeholder: Parameter section of this chapter). In this way, Monica's subscript (1) implied a relationship between her general concept of sequence and a specific sequence she was considering and (2) showed spatial deference to the general concept of sequence.

Monica used the 2nd-level subscript as a placeholder variable to re-present the positions of the various terms within a specific sequence. Thus, when Monica wrote S_{a_n}, she re-presented (1) the broad concept of "sequence" through her inscription S, (2) the narrower notion of a specific sequence through the 1st-level subscript a, and (3) the localized idea of term positions for a specific sequence (with a specific rule for generating term values) through her 2nd-level subscript n. Monica's use of subscripts constitutes her spatial construal of each inscription as a *connector* (in addition to the other meanings she previously ascribed to these inscriptions individually).

Relational Inscription Type 2: Comparator

During my analysis, I found a moment in which Monica introduced an inscription to compare the values of two quantities she envisioned within the same context. In this instance, Monica's symbolizing activity was not oriented toward showing a connection between a process and result (i.e., Sylvia's use of =) or an attribute of a single quantity and its value (i.e., Monica's use of subscripts). Rather, Monica attempted to re-present her method for comparing the values of two quantities for the purpose of making an inference about the quantities' relationship. I used the term *comparator* to code instances in which Monica or Sylvia attributed a comparative process (and resulting implications of the comparison) to a particular inscription. In the paragraphs below, I describe Monica's use of the inscription > to compare the value of the terms in the sequence of partial sums against the lower bound of an ϵ-strip during the Day 5 interview.

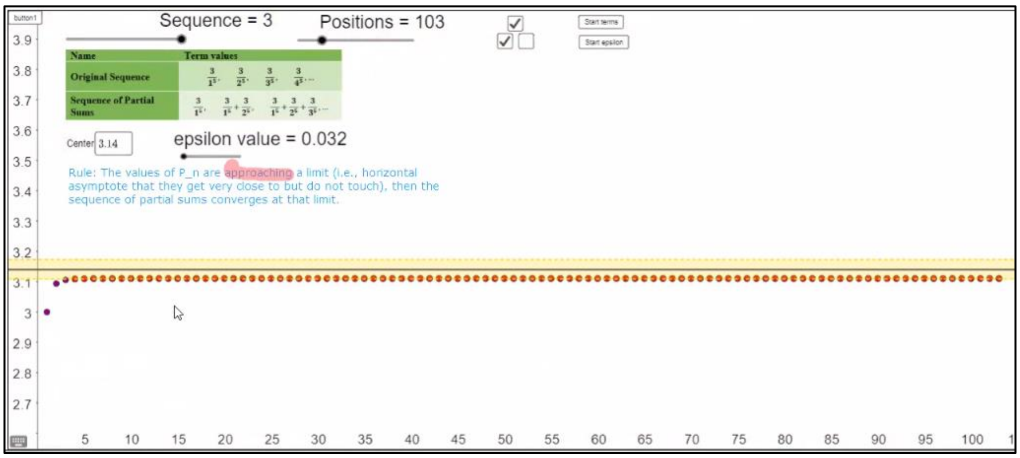
Example 1: Monica's Use of > as a Comparator to Relate Partial Sums and Error Bounds. At the end of the Day 5 interview, I introduced the ϵ-strip activity in the context of the sequence of partial sums. In the activity (see Figure 49 for an example), I portrayed partial sums as dots, a possible convergence value with a black horizontal line, and an error bound ϵ around the potential convergence value as a translucent horizontal yellow region. Additionally, I differentiated between the partial sums inside the ϵ-strip (which I colored red) and the dots outside the strip (which I colored purple). During the task, I chose a center value and presented several values of ϵ to Monica, asking for each value of ϵ how many dots she believed were inside the ϵ-strip and how many dots were outside of the strip. After several iterations of this task for various values of ϵ, I asked Monica to provide a general description for how she might determine the number of dots inside or outside the ϵ-strip. In response, Monica said:

> Monica: Well, I guess you could find what y-value you get, like, if the bottom of our yellow region [i.e., ϵ-strip] was its own line, what the y-value is there, and then make some kind of, like inequality where you have, your sequence with a variable and you solve with this inequality where it's less

than whatever that lower line is. And then you would solve for n, which would be your position, and then that would be the number, I think, of points that were outside of it on the bottom line.

Figure 49

Screenshot from ϵ-strip Activity on Day 5



In Monica's response, she described (1) determining the value corresponding to the lower bound of the ϵ-strip and (2) constructing an inequality by which she could determine the value of n (i.e., position in the sequence) corresponding to the final dot outside of the ϵ-strip. In many ways, Monica's intuitive reasoning about determining the number of dots outside the strip corresponds to the formal definition for limit of a sequence. For instance, the formal definition of sequence uses an inequality to denote the region covered by a particular ϵ-strip. Also, to prove that a sequence converges to a given value, a student must construct a rule for determining a finite number of terms outside the ϵ-strip for any positive value of ϵ.

At this moment, I conjectured that Monica was considering how to symbolize the relationship she conceived between the partial sums, ϵ-strips, and potential convergence value. When I asked Monica how she might write out her image of the points inside and outside the strip, she began to formulate an expression (although she could not initially complete it).

Monica: Um, I guess I would do p sub n is less than (writes $p_n >$). Um, I don't know how to write this, where I want it to be, like. (pauses)

Interviewer: So, why don't you explain to me what it is that you're hoping to write one more time and then maybe we can come up with inscriptions later that will do this work for us.

Monica: Well, I think that what I want is "center minus epsilon" [after $p_n >$].

Interviewer: Ok.

Monica: Because I want, what I was saying earlier, where like if you have, if you could make the (…) lower line created by this like ϵ in our like yellow region, if you could define what that is as like a y-value, or yeah, then your p_n's that are less than that are going to be below it and outside of it.

In this excerpt, Monica began to construct an expression by which to re-present the dots outside of the ϵ-strip. Her expression included the following components: (1) the expression p_n, which she had previously used to re-present the partial sums in a series or terms in the sequence of partial sums and (2) the inscription >, which she used to re-present the relational concept of "less than."[24] Although Monica was unable to complete her personal expression in this moment, she stated that she wanted to re-present the

[24] In conventional mathematics, the inscription > is typically referred to as "greater than." Still, in Monica's verbal explanations, she referred to the inscription > as "less than," which I take to imply that she was re-presenting the idea of "p_n less than (quantity)" rather than the conventional "p_n greater than (quantity)."

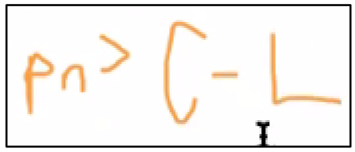
quantity “center minus epsilon” with the remainder of her expression and that she could use this quantity to symbolize the partial sums that were outside of the ϵ-strip.

In response to Monica’s comment, I recommended that she construct inscriptions for the ideas of “center” and “epsilon.” She accepted my suggestion and created the inscription C for center and L for epsilon (she initially wanted to use the letter E for epsilon but had used this inscription in a previous interview for another purpose). Monica then returned to her incomplete personal expression $p_n >$ and included her new inscriptions (see Figure 50):

> Monica: So, this would be all of the terms that are outside of the shaded region. (…) Um, C is your center, L is the epsilon, so this $C - L$ is giving you the lower bound of that shaded region. And then, you’re solving for n which is going to give you the term number [i.e., partial sum] of what is less than that [the lower line on the ϵ-strip with value $C - L$]. So, any, like, terms 0 or 1 through, yeah, term 1 through this n are going to be outside of your shaded region depending on what your L is.
>
> Interviewer: OK. And then, so terms 1 through n will be outside of the shaded region and the implication is that everything else is inside the shaded region?
>
> Monica: Yep, inside, yeah, because then it [i.e., partial sum] will no longer be less than this value [i.e., $C - L$].

Figure 50

Monica's Personal Expression for the Partial Sums Outside of an ϵ-strip



In this excerpt, Monica used the inscription > to show a comparative relationship between the values of two quantities that she envisioned: (1) the last partial sum in the sequence of partial sums whose value was outside the ϵ-strip and (2) the lower bound of the ϵ-strip. Monica used the inscription > to re-present her distinction that if the value of a partial sum were less than $C - L$, then that partial sum would exist outside the ϵ-strip. Monica also used her expression p_n to re-present that there would be n dots outside of the ϵ-strip and that these n dots would correspond with terms $p_1, \ldots, p_n$ of the sequence of partial sums.

Summary of Chapter 6 Results

In this chapter, I presented three broad meanings that Monica and Sylvia attributed to their inscriptions during their symbolizing activity. I categorized these meanings as *process*, *concept*, and *relational*.

I used the term *process* meaning to denote instances where the students re-presented carrying out an action through their inscription. I introduced two inscription types to categorize Monica's and Sylvia's attribution of process meanings: *command* and *create* operators. The students generally attributed an algorithmic and predictable (to them) action to a command operator and an inventive injunction to a create operator.

Monica's and Sylvia's process-oriented inscriptions were also fixed and did not vary from instantiation to instantiation of their corresponding personal expression templates.

I employed the term *concept* meaning to describe moments where Monica and Sylvia used an inscription to re-present an attribute or value of a quantity (or a general topic) they envisioned. I introduced two inscription types to organize their attribution of concept meanings: *indicators* and *placeholders*. The students generally attributed an attribute or topic to an indicator inscription (which often served as a mnemonic device) and the values of a quantity to a placeholder. I further categorized inscriptions to which Monica and Sylvia attributed one value to a placeholder in a given situation as *parameters* and those that they attributed more than one value as *variables*. Monica and Sylvia's indicator inscriptions were generally fixed, although Monica frequently ornamented her indicators with subscripts to re-present multiple instances of a topic within a single example. The students' placeholder inscriptions were typically cloze, and the students employed various marks for their placeholder inscriptions according to their needs (although they typically used the same mark when defining or reasoning about a placeholder arbitrarily).

I utilized the term *relational* meaning to categorize moments in which Monica and Sylvia employed an inscription (or spatial placement of inscriptions) to re-present a relationship between two ideas they envisioned. I introduced two inscription types to categorize Monica's and Sylvia's attribution of relational meanings: *connector* and *comparator*. The students generally re-presented a coordination between (1) a process and result or (2) an attribute of a quantity and its values through a connector. Monica re-presented a contrast between the values of two quantities within a situation using her comparator. Monica’s and Sylvia’s spatial placement of connectors in their personal expression templates were fixed but the marks they used as relational inscriptions were sometimes cloze (e.g., Monica’s use of various marks for subscripts). I summarize the meanings, inscription categories, and properties of these inscriptions in Table 22 on the next page.

Table 22

Summary of Meanings, Inscription Types, and Properties Presented in Chapter 6

Meaning	Definition	Inscription Type	Definition	Examples	Fixed or Cloze?
Process	Re-presenting a particular action a student has previously carried out (or imagines carrying out) while reasoning about a situation	Command Operator	An inscription a student uses to re-present a process for which the procedural steps are well known (to her), the actions within each step of the process are relatively algorithmic (to her), and the character of the process's result is fairly certain (to her)	Σ ?_? $\int$	Fixed
		Create Operator	An inscription a student uses to re-present a process for which she can describe the procedural steps generally (but not precisely), the action steps of the process are investigative and have a high degree of variability (to her), and the character of the result is uncertain (to her)	$\nsim$? #	Fixed
Concept	Re-presenting a topic, an attribute of a quantity, or the value of a quantity	Indicator	An inscription a student uses to re-present an attribute (but not necessarily the value) of a quantity	dx CV S	Fixed
		Placeholder	An inscription a student uses to re-present one or more values for a particular quantity they envision	Parameter: p, m, S_a	Numerical: Cloze Non-numerical: Fixed or Cloze
				Variable: $f(x), f(n), f(k)$	Numerical: Cloze Non-numerical: Fixed or Cloze
Relational	Re-presenting a relationship a student envisions between the meanings they attribute to two inscriptions or expressions	Connector	An inscription (or spatial orientation of inscriptions) a student uses to re-present a coordination they envision between two components of the same process or quantity	$=$ $S_{\square_n}$	Spatial Placement: Fixed Marks: Fixed or Cloze
		Comparator	An inscription a student uses to re-present a comparative process (and resulting implications of the comparison) of the values of two quantities they envision in the same situation	$>$	Spatial Placement: Fixed

CHAPTER 7

RESULTS PART 3: THE COEVOLUTION OF STUDENTS' MEANINGS AND PERSONAL EXPRESSIONS FOR INFINITE SERIES OVER TIME

The material in Chapter 7 is related to my third research question, *how do students' symbols and attribution of meaning to these symbols change as their thinking about infinite series evolves over time?* This chapter is comprised of three major sections. In the first section, I describe Monica's creation of two distinct personal expression templates to re-present related ideas (in her mind) for summing function values over an interval. In the second section, I review Sylvia's iterative construction of a single personal expression template (and corresponding inscriptions) through which she could re-present various types of partial sums and series that she considered to have different properties. I discuss the research and instructional implications of this work, along with an initial theoretical framework for describing the coevolution of students' meanings and personal expression templates (grounded in Piaget's theory of reflected abstraction) in Chapter 8.

The two examples that I share in this chapter largely emanated from my analysis of Monica and Sylvia's symbolization of partial sums and series from Days 1-3 in the teaching experiment. I chose to present data from these interview days because (1) my analysis of the later interview days involved more topics and forms of representation (e.g., convergence definitions, graphs) and (2) Monica and Sylvia did not create standalone personal expression templates for any of the major topics during these days.

Through my analysis of Monica and Sylvia's symbolizing activity about partial sums and series during Days 1-3, I identified (1) the standalone personal expression templates they flexibly used to re-present these topics and (2) the meanings they attributed to each inscription within their templates. In the following sections, I describe each student's chronological development and modification of their personal expression templates to re-present their meanings for partial sums and infinite series.

Example 1: Monica's Construction of Two Personal Expression Templates for Sums

As I have reported in previous chapters in this dissertation, Monica initially attempted during the Day 1 interview to use an integral sign ($\int$) to re-present partial sums and series. The purpose of this section is to chronicle her initial struggle and final resolution of whether to use the integral sign ($\int$), the summation symbol (Σ), or both to re-present her image of a partial sum. In the following sections, I address Monica's symbolizing activity during Day 1 and Day 2 and her eventual decision to create two distinct but related personal expression templates for adding function values.

Day 1: Using Integrals and Summation to Reason about Partial Sums

Monica introduced integral notation while reasoning about how to compute the value of the 37th partial sum for Ivy's 1st Series during the Day 1 interview. Series 1 was the divergent p-series $\sum_{n=0}^{\infty} \frac{2}{\sqrt[4]{n}}$, which I presented in the expanded form $\frac{2}{\sqrt[4]{1}} + \frac{2}{\sqrt[4]{2}} + \frac{2}{\sqrt[4]{3}} + \frac{2}{\sqrt[4]{4}} + \frac{2}{\sqrt[4]{5}} + \frac{2}{\sqrt[4]{6}} + \cdots$. Monica initially proposed manually adding together the first 37 summands to determine the value of the 37th partial sum. As she continued to reflect, she speculated whether she could compute the value of the partial sum more efficiently. She briefly considered using factorial notation (which dynamically decreases by whole number values, e.g., $37! = 37 \cdot 36 \cdot ... \cdot 2 \cdot 1$) but disregarded this idea because she recognized that a partial sum contains an additive (and not multiplicative) operation.

After the interviewer asked her to compute the sum of the first three summands in a calculator, Monica proposed using integral notation (e.g., $\int_1^{37} \frac{2}{\sqrt[4]{x}} dx$) to simplify the calculation of the 37th partial sum (see Figure 51). When the interviewer asked Monica why she adopted integral notation, she said the following:

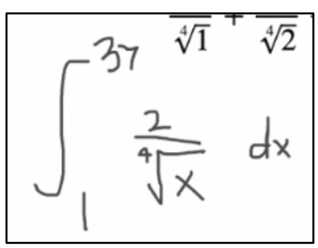
Monica: Ok, (…) I would think of, like an integral being, like in this case would be like the area under a curve. Um, and then in this case I've created like this expression [i.e., $\frac{2}{\sqrt[4]{x}}$], that would be my curve. And then I'm looking for the sum, so it [i.e., $\int_1^{37} \frac{2}{\sqrt[4]{x}} dx$] would be like all the space underneath from 1 to 37.

Interviewer: Ok, when you say "all the space underneath from 1 to 37" is that (*Monica interrupting*: under the curve) the area under the curve?

Monica: Right, right, right, right, right. That's what, I don't know. That would be my other guess. But I'm really not confident in that.

Figure 51

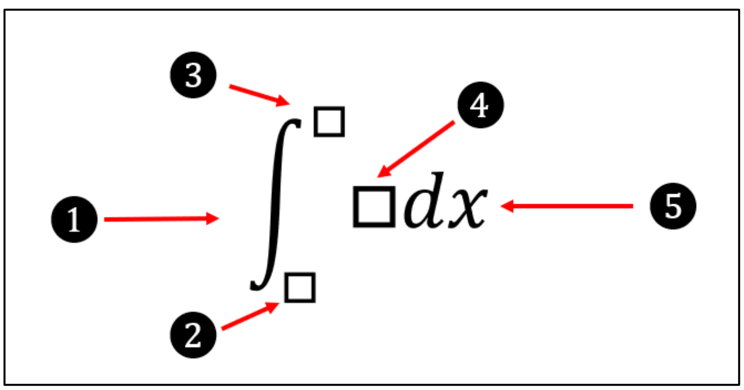
Monica's use of Integral Notation to Re-present the 37th Partial Sum



Monica's comments in this excerpt imply that she was trying to assimilate the notion of computing the value of a partial sum to her personal expression template $\int_{\square}^{\square} \square dx$, which she had previously used in her calculus courses to re-present the idea of function integration. This personal expression template consisted of five inscriptions (see Figure 52): (1) the *fixed command operator* $\int$, which Monica used to alternately re-present determining the area under a curve and adding all function output values between two input values; (2) the *cloze placeholder* subscript of the inscription $\int$, which Monica used to re-present the position of the first summand comprising a partial sum, (3) the *cloze placeholder* superscript of the inscription $\int$, which Monica used to re-present the final summand comprising a partial sum; (4) the *cloze placeholder* argument of $\int$, which Monica used to re-present the function whose area (or output values) she was adding together; and (5) the *fixed indicator* dx, which Monica used as a syntactic convention to re-present the concept of integral.[25]

Figure 52

Monica's Integral-based Personal Expression Template for a Partial Sum



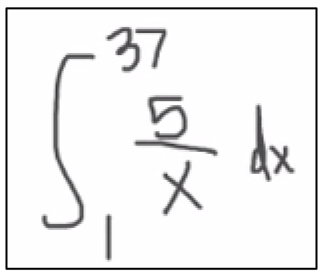
As Monica began to reason about Ivy's 2nd series, she began to question whether it was appropriate for her to symbolize a partial sum with her inscription $\int$. Ivy's second series was the divergent p-series $\sum_{n=1}^{\infty} \frac{5}{n}$, which I presented in the expanded form $\frac{5}{1} + \frac{5}{2} +$

[25] Although there is no explicit evidence of this claim in the previous transcript excerpt, I do provide evidence to substantiate this claim in Chapter 6.

$\frac{5}{3} + \frac{5}{4} + \frac{5}{5} + \frac{5}{6} + \cdots$. When I asked Monica to describe how she would compute the 37th partial sum, she stated that she would either (1) add up the first 37 summands manually or (2) use the integral $\int_1^{37} \frac{5}{x} dx$ (see Figure 53) to compute this value. Although Monica re-presented Monica also stated that she was unsure whether it was "correct" to use an integral (in the normative sense) but that using an integral made sense to her for computing partial sums.

Figure 53

Monica's Use of an Integral to Re-present a Partial Sum



After seeing Monica use an integral to re-present the 37th partial sum for two series in a row, I decided to further clarify her use of this symbol and the relationship she perceived between partial sums and integration. Monica's response to my question revealed that she was still unsure whether her image of integral as "area under the curve" aligned with her conception of a partial sum.

Interviewer: What is it that this integral is going to tell you that helps you to get the sum of the first 37 terms?

Monica[26]: So, if I were to just evaluate, like, just put in 37 [for x in the expression $\frac{5}{x}$], that would not be all of them [i.e., summands], right, that would just be one [summand] (…) but it wouldn't be like, you're not adding the rest of them from 1 to 37. Um, so, the, I got my bounds here from, I want to know [summands] 1 to 37 (…) Which, now that I'm saying this I'm a little bit concerned because I don't know if this [i.e. the integral] would equal the same number. If I were to do it all by hand, and (…) if this was just like a random problem by itself, I don't know if those two numbers [i.e., integral and manually-computed sum] would be the same.

Interviewer: Why, can you say a little bit more about why you might be concerned?

Monica: Um, because (…) in my, like an integral, (…) I'm worried that that might be including more than just, like all the numbers between 1 and 2 as well. I don't know if that is correct though.

Interviewer: Ok, let me see if I can kind of frame what you're thinking. So, you're thinking about the first term to the 37th term. (*Monica:* Mm-hmm) And you're trying to compare just adding them up by hand in Desmos or something with the integral (*Monica:* Right). And you're worried about, if I remember right, you said the numbers between 1 and 2. (*Monica:* Yes). So, what do you mean by that?

[26] This response is a combination of three distinct responses that Monica provided, separated by interviewer questions to clarify the meanings of her terminology. Rather than present the interviewer clarifying questions in the transcript, I added the appropriate terminology in brackets that I gleaned through my questions.

Monica: (*long pause*) Like if I'm thinking about it backwards, if you were to ask me to write out, explain to you what this integral means, I don't think that I would write it out as $\frac{5}{1}+\frac{5}{2}+\frac{5}{3}$. Because I, an integral, like the area under the curve, is like everything, not just on, like, where the whole numbers are.

Interviewer: I see. So I'm going to try to rephrase what I feel like you're saying in terms of a question. What is the connection between (…) area under the curve and $\frac{5}{1}+\frac{5}{2}+\frac{5}{3}$ up to $\frac{5}{37}$?

Monica: Well, I thought they were the same, and now I'm starting to question whether they are or are not the same.

In this excerpt, Monica initially stated that she adopted integral notation to describe the 37th partial sum because she recognized that she needed to evaluate her general summand $\frac{5}{x}$ for each whole number value from 1 to 37. However, she quickly questioned her choice of notation because she was concerned about non-whole number values between 1 and 37. Specifically, she recognized that she would not use non-whole number values of x to compute the value of the 37th partial sum but was uncertain whether these values would be incorporated into a computation utilizing an integral sign. Monica presented two reasons for her uncertainty. First, she stated that if she were to write out a manual computation for an integral[27], she did not believe that she would write the expanded form of Ivy's 2nd series (i.e., $\frac{5}{1}+\frac{5}{2}+\frac{5}{3}+\frac{5}{4}+\frac{5}{5}+\frac{5}{6}+\cdots$). Second, she

[27] An integral, by definition, cannot be manually computed by writing a string of summands and adding them together. However, based on her response, Monica seemed to believe this was possible (or at least imaginable).

presented an image of integral as area under the curve, which she claimed encompassed all values of the independent variable (not merely the whole number values).

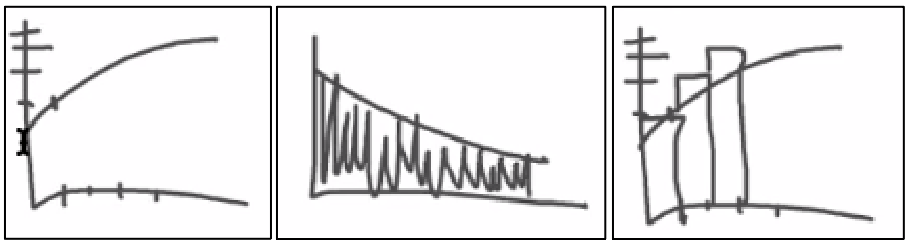
At this point of the interview, I considered Monica to be thoroughly perturbed regarding (1) how to describe a partial sum without resorting to a manual summing of individual summands in a calculator and (2) whether a partial sum was a special instance of an integral. I subsequently asked Monica to say a little more about her confusion, which prompted her to introduce a series of graphical representations to help her further grapple with her cognitive conflict (see Figures 54a, 54b).

> Monica: If I were to draw graphs of both of these things, for this actual, like, series here [i.e., Ivy's 2nd series], the graph would look (*draws two axes to make quadrant 1*) (…) in my head I'm picturing like a Riemann sum more than I'm picturing like, I don't even know if this makes sense. (*draws curve increasing with a decreasing rate*) (…) So, what I see this series as is we're adding up, like, if this is like 1, 2, 3, 4 (*draws a mark on the curve and corresponding marks on the horizontal axes*) we're adding up all these y-values here with whatever $\frac{5}{1}$ is, and $\frac{5}{2}$ is, (*makes marks on vertical axis for summand values*) and I'm realizing I drew this graph very incorrectly. Um, we're like adding up what these numbers are. And I'm not sure that's the same thing, whereas this graph here (*draws a curve that is decreasing with a decreasing rate*) (…) would be like the entire area underneath here (*shades the entire area under the curve in her 2nd graph*). And I'm not sure that those are the same thing. And that's why I thought about Riemann sum because, then, you know, you make like the

rectangles based on the y-value. And I don't know if those are the same because I feel like they're not.

Figures 54a (left), 54b (center), and 54c (right)

Monica's Graphs Related to Partial Sums (10a), Integration (10b), and Riemann Sums (10c)



I then prompted Monica to draw her image of a Riemann sum on her graphs and further compare the difference (in her mind) between a Riemann sum and an integral (see Figure 54c).

Interviewer: Could you draw in what you're imagining with these rectangles and Riemann sums?

Monica: (*draws in rectangles on unshaded graph*) Except that my issue with this is that like with Riemann sums you are adding up the area of these rectangle but in this series I would think that you are adding only the y-value of them, not the entire area. So like, when you're doing a Riemann sum you multiply like, the rectangle, to get like length times width. I wouldn't, that's not how I would describe the series, though, this is just the y-value, like this (*indicates first summand* $\frac{5}{1}$) is the y-value of this function at 1, when x is 1. That's not the same thing as this entire area (*indicates rectangle*).

Interviewer: Ok, and how are they different?

Monica: Actually, if the base is 1, then it is the same, right? Because you're multiplying the number [the y-coordinate] by 1. So maybe they are the same. Ok, I still think that this integral is maybe correct. I think that this is my best like, guess at least.

Interviewer: Ok, can you summarize what your guess is?

Monica: Yeah. So, I. Ok, so if I were to try to find the sum of the first set of terms and I didn't want to just add them all by hand, because it was a very large number that I was trying to find. I would make the series into a function like this (*indicates general summand* $\frac{5}{x}$ *in integration notation*) and then I would integrate that function. And the bounds of that integral would be wherever you were starting, what term number you were starting on to what term number you were finishing on. So, if you were like 2nd to 10th, I would do 2 to 10 with whatever function you've made by just looking at your series.

Monica's response to my question about the relationship between Riemann sums and integrals is significant for several reasons. First, Monica provided a normative explanation of a Riemann sum and (eventually) decided that placing rectangles with a width of 1 on the graph of the general summand[28] would perfectly produce the summands of a series. Second, Monica equated an integral and a Riemann sum using rectangles with width 1, and claimed that for this reason, she would continue to use integral notation to

[28] Monica exclusively drew continuous graphs during the first interview. However, she failed to clarify (and the interviewer did not explicitly ask) regarding the domain of the graphs she drew.

re-present (to herself) specific partial sums. Finally, Monica consciously decided to adopt integral notation as her first personal expression, which necessitated her assimilation of her image of partial sums into her scheme for integration.

Monica also used her personal expression template $\int_{\square}^{\square} \square d\square$ to re-present the 37th partial sum for Ivy's 6th series. After she reasoned about Series 6, I asked Monica to compare the methods she employed to determine specific summands (Question 1) and partial sums (Question 2) across the series. While describing her thinking about partial sums, Monica spontaneously introduced the idea of summation notation. She noted that although she had more experience with integrals than summation, she had started to use summation notation in her calculus coursework[29]. Monica stated she was extremely unsure of the conventional meaning for summation notation but believed this notation might be the appropriate way to symbolize a partial sum. When I asked Monica to contrast integral and summation notation in the context of the graphs she created for Series 2 (see Figures 54a-54c), she stated:

> Monica: The integral to me, I don't think is the same thing as (…) this Riemann sum situation. Because here, we're just using, like, this is the y-value we've got when x is 1 (indicates $\frac{5}{1}$), and this is the y-value when $x = 2$ (indicates $\frac{5}{2}$). And nowhere in this series are we incorporating like the y-value when x is a number between 1 and 2. Because that's not like part of this here. But that would be, I think, part of the integral. And that was

[29] The sequences and series unit for Monica's calculus course took place between her Day 1 and Day 5 interviews.

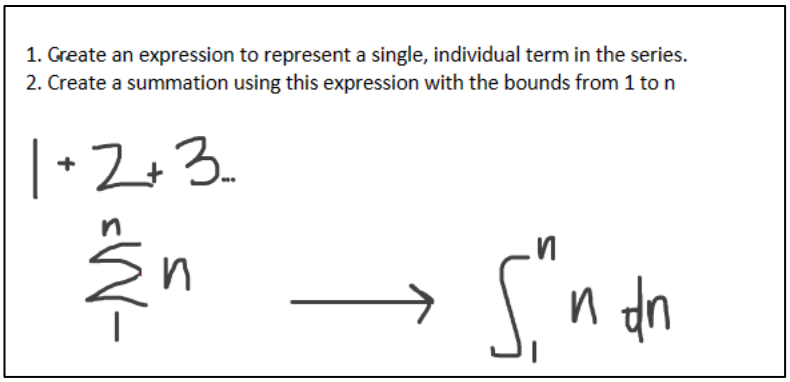
where I was like, these don't feel like they're the same thing. (…) I'm imagining that's why the summation exists, and I'm, I would think that that's the difference between those two things. But I don't really know what the summation, like, is. But that's the thing that felt wrong about the integral, was that I would think that you would have $\frac{5}{1.1}$, $\frac{5}{1.2}$, and (…) you would have like all these numbers in the middle.

In this excerpt, Monica stated that she believed the intention of summation notation was to compute the value of partial sums but was unsure whether this was the conventional meaning for this notation. She continued to contrast her image of a partial sum as a summation of function values corresponding to whole-numbered inputs with an image of an integral as a summation of all function values between the lower and upper bound.

Despite Monica's continued waffling between the command operator she wanted to utilize in her personal expression template for partial sums (i.e., $\int$ or Σ), her meanings for computing partial sums seemed (to me) to have stabilized. For instance, Monica created a concise two-step written rule for determining the value of a partial sum in a series that largely mirrored convention (see Figure 55). When I asked Monica whether she envisioned utilizing an integral symbol ($\int$) or a Σ (Sigma) with her written rule, she again expressed unsurety about the relationship between an integral and a Sigma. In particular, she stated she felt more comfortable using the integral but had recently been introduced to the inscription Σ in her coursework. Additionally, Monica recalled seeing examples where a summation and integral were used in the same problem (a possible reference to the integral test), which lead her to speculate that the expression $\Sigma_1^n n$ might change to $\int_1^n n\,dn$. Monica symbolized this relationship by drawing an arrow between both expressions (see Figure 55). When I asked Monica to clarify how she envisioned symbolizing a partial sum (instead of presenting the methods she saw in her coursework), Monica again stressed that she would prefer to (1) manually compute the value of a partial sum if there were a limited number of summands or (2) use an integral to compute the value of the partial sum instead of a Σ.

Figure 55

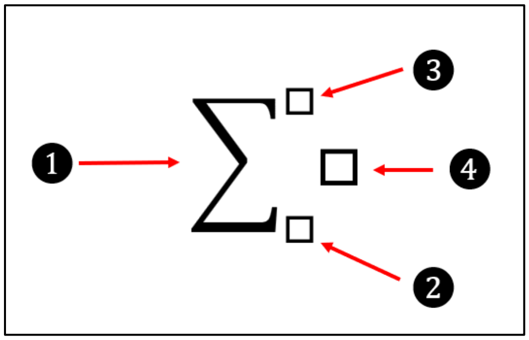
Monica's Written Rule and Symbolization of Partial Sums



During the personal expression creation task, Monica proposed the personal expression template $\Sigma_{\square}^{\square}\square$ to re-present her image of a partial sum (see Figure 56). Her personal expression template was comprised of four distinct inscriptions. First, Monica used the inscription Σ as a *fixed command operator* to re-present the additive process of computing the value of a partial sum from a set of summands. Second, Monica used the subscript of Σ as a *cloze placeholder* to re-present the position of the first summand comprising a partial sum. Third, Monica used the superscript of Σ as a *cloze placeholder* to re-present the position of the final summand comprising a partial sum. Finally, Monica used the argument of Σ as a *cloze placeholder* to re-present the pattern (or function rule) by which she could generate the summands of the series.

Figure 56

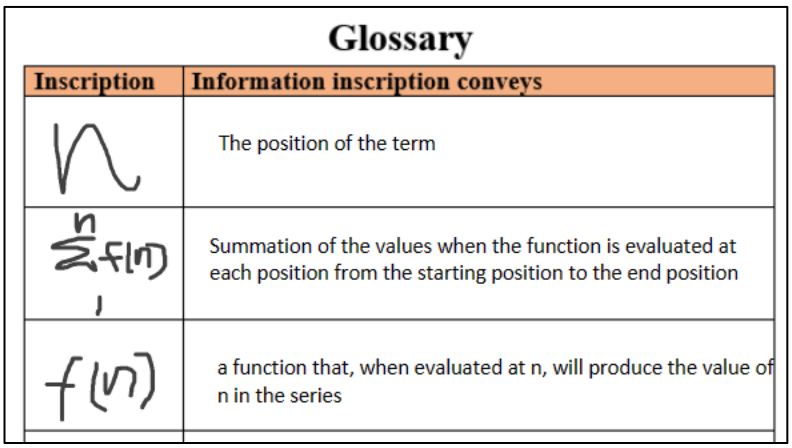
Monica's Summation-based Personal Expression Template for Partial Sums



When Monica included her personal expression template in her glossary, she wrote her template in the form $\Sigma_1^n f(n)$ (see Figure 57). In her glossary, Monica proposed the inscription n (i.e., Inscription 3) to convey "the number of the term that you are looking for" in the context of a partial sum. She initially considered writing "the value of the term" but accepted my proposal of the word "position" in place of "value." When I asked Monica why she preferred "position," she shared that she considered the general summand to be a function, and the "value" of the function referred to the output of the function (in her mind). Second, Monica proposed the expression $f(n)$ (i.e., Inscription 4) to convey "a function that, when evaluated at n, will produce the value of n in the series." Monica stated that the purpose of the function was to "define the pattern that you are seeing in the series." When the interviewer asked what Monica meant by "the pattern," Monica described the general summand of a series. Finally, Monica proposed the expression $\Sigma_1^n f(n)$ (i.e., personal expression template) to re-present a specific partial sum of a series. She stated that the information she wished to convey through $\Sigma_1^n f(n)$ was "evaluating not just n, but we're evaluating the function we've created [i.e., $f(n)$]. And then the bounds there are (…) from 1 to whatever partial sum you are looking for." In other words, Monica wished to use her expression $\Sigma_1^n f(n)$ to re-present her process of iteratively evaluating the function at whole-number values of the index from 1 to n and then adding these function values together to compute the nth partial sum.

Figure 57

Monica's Glossary at the Conclusion of the Day 1 Interview



Glossary

Inscription	Information inscription conveys
n	The position of the term
$\sum_{1}^{n} f(n)$	Summation of the values when the function is evaluated at each position from the starting position to the end position
$f(n)$	a function that, when evaluated at n, will produce the value of n in the series

As a clarifying question, I highlighted the two instances of the inscription n in Monica's expression $\Sigma_1^n f(n)$, and asked her whether these inscriptions had the same meaning (to her). In response to my question, Monica again brought up the notion of an integral. After a few clarifying questions to try and determine why Monica had spontaneously reintroduced the idea of an integral, Monica eventually said she was "representing all of the terms." When I asked Monica what she meant by "representing all of the terms," she said the following:

Monica: I'm treating this like an integral.

Interviewer: And once again, when you're saying "like an integral," what do you mean by that?

Monica: I'm assuming it has the same rules here. Like, I'm assuming it works the same.

Interviewer: (…) When you saying it "has the same rules" as an integral, I'm trying to understand what you mean by that.

Monica: So, what I want to happen is that this $f(n)$ (…) would be $f(1)$ and then also $f(2)$ and then also $f(3)$. That's what I'm trying to say. And I don't know if that is what that symbol means [i.e., $f(n)$], but that's what I want it to mean. That's what, I'm hoping a symbol exists that means that. And that's what I want to be using here.

In this excerpt, Monica revealed that she considered her expression $f(n)$ to denote an iterative process of evaluating the function f at each whole number value between 1 and n. She also continued to compare her images of partial sums and integrals. As I continued to ask clarifying questions about Monica's integral-partial sum connection, she eventually stated that she considered both ideas to constitute similar processes. Specifically, she imagined each concept to imply evaluating a function rule at multiple values of the domain and then performing an operation on the resultant function values.

At the end of the first interview, it was clear that Monica had constructed two distinct personal expression templates: $\int_{\square}^{\square} \square dx$ and $\Sigma_{\square}^{\square}\square$. To Monica, these expressions were highly similar and she attributed analogous (or identical) meanings to corresponding inscriptions in each template. For example, Monica re-presented the idea of adding function values through both $\int$ and Σ. However, Monica was (1) unsure whether she could re-present her image of a partial sum as Reimann rectangles through her inscription $\int$. In the case of the subscript, superscript, and argument in each personal expression template (i.e., Inscriptions 2, 3, and 4), Monica attributed identical (in my mind) meanings to each corresponding inscription. However, Monica's Inscription 5 from her integral expression template (i.e., dx) was unique to this template and did not occur in the other template.

Day 2: Creating Two Distinct Personal Expression Templates for Adding Function Values

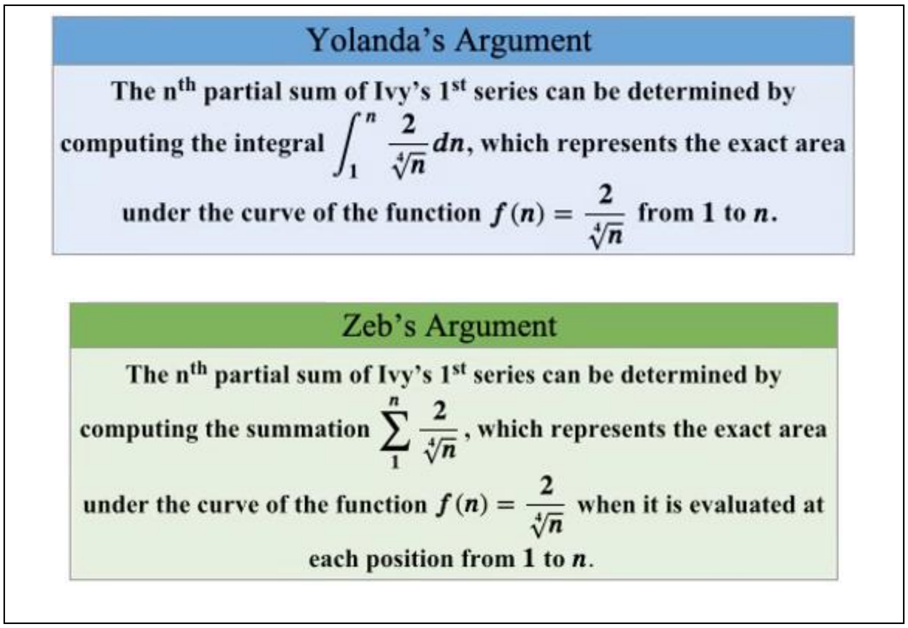
In preparation for the Day 2 interview, I decided to present Monica with contrasting hypothetical definitions (and symbolizations) of partial sums to help her reason through the unsurety she exhibited during Day 1 regarding potential differences between her command operators $\int$ and Σ. The two prompts (presented by hypothetical students) included (1) Yolanda, who utilized integral notation to describe partial sums, and (2) Zeb, who used summation notation to describe partial sums (see Figure 58). I also incorporated Monica's image of "area under the curve" into each definition based on her frequent reference to this idea during the Day 1 interview.

When I presented the two contrasting statements, Monica stated that Yolanda and Zeb's arguments reflected her issue with symbolizing partial sums from the Day 1 interview. She then expressed a slight preference for Zeb's argument. When I asked her why she preferred Zeb's argument, she referenced the inclusion of the word "position" as an indication (to her) that Zeb was only considering whole-number function inputs. In contrast, she claimed that Yolanda's argument included non-whole number inputs

(Monica specifically referred to all real numbers between 1 and 2) and stated that the point of an integral was to add up "infinitely small" Riemann sums.

Figure 58

Yolanda and Zeb's Definitions for Partial Sums



After Monica's initial response, I asked her to draw graphs to help her convey her image of what each student was describing in their argument. For Yolanda's argument, Monica drew a monotone decreasing curve and shaded the region between the curve and the horizontal axis (see Figure 59). Monica claimed that Yolanda's expression $\int_1^n \frac{2}{\sqrt[4]{n}} dn$ represented the "entire space" or area under the curve that she had drawn. For Zeb's argument, Monica drew a similar curve but (1) partitioned the horizontal axis into unit increments and (2) marked the corresponding function values on the trace of the curve (see Figure 60). Monica then stated that for Zeb's argument, only the function values she had marked would be computed and there would be "space" between the various summands. For Yolanda's argument, Monica stated that every function value on the trace of the curve would be added together to constitute the area under the curve:

> Monica: Here, with Yolanda (…) every single, infinitely close together point on this line you're also adding together, where there's no room between them. You're not just going, just the y-value at 1, just the y-value at 2. And that's the difference between the two [Yolanda and Zeb], and that's also, this is what I was wanting to say last week (…) where I was stuck on the idea that somehow they're different [i.e., integral and summation]. This is how they're different, I think.

Figure 59

Monica's Graph Corresponding to Yolanda's Argument

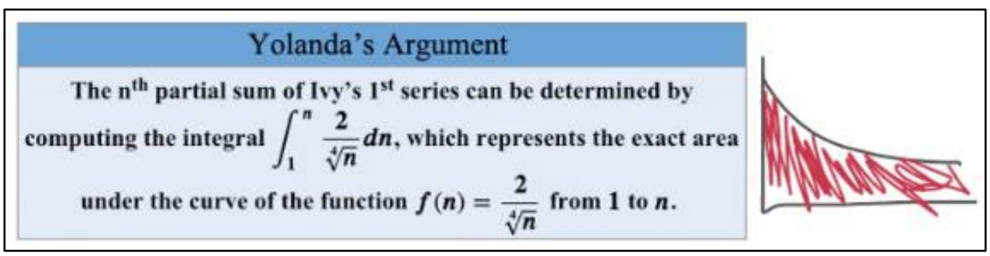
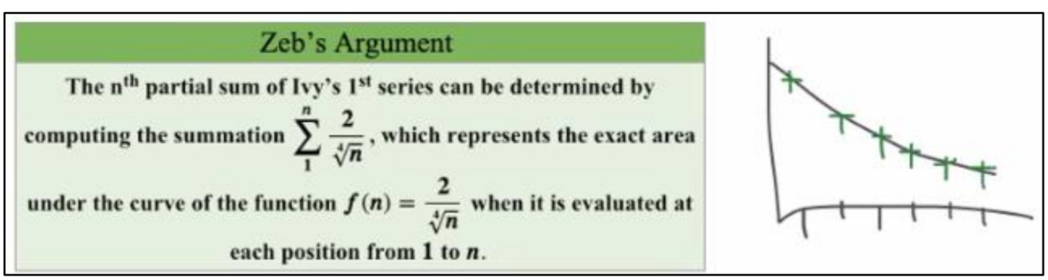


Figure 60

Monica's Graph Corresponding to Zeb's Argument

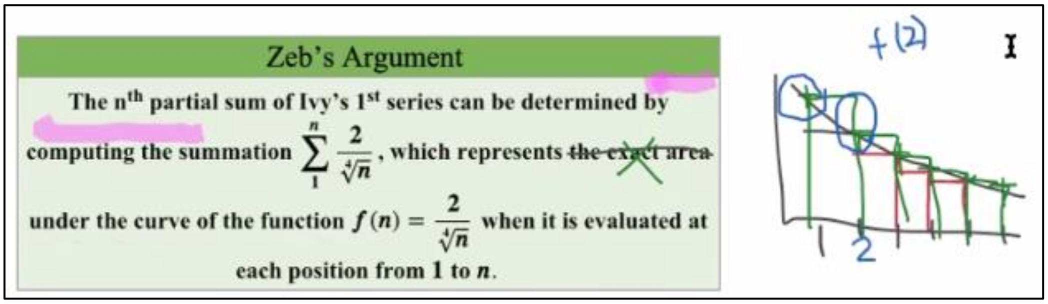


In this excerpt, Monica declared that she could distinguish between the meaning of integration and summation. Specifically, her graphical distinction between Zeb's argument as only incorporating whole-number function inputs (see Figure 59) and Yolanda's argument as incorporating all real-number function inputs (see Figure 60) allowed her to cognitively separate her meanings (and symbolization) for these ideas. In other words, it appeared that Monica had consciously chosen to re-present a Reimann sum through her command operator Σ and an exact area under the curve through her inscription $\int$. Since Monica's meaning for partial sum computation corresponded more closely with her image of a Reimann sum, in this moment she seemed poised to accept her personal expression template $\Sigma_{\square}^{\square}\square$ as her primary method to symbolize partial sums.

To confirm my hypothesis, I highlighted the term "area under the curve" in each argument and asked Monica to describe how she felt about Zeb's use of this phrase. Monica responded that she felt uncomfortable with Zeb's use of "exact area under the curve." The interviewer then asked Monica whether she wished to modify the language in Zeb's argument, which she agreed was a good idea. Monica briefly considered removing the phrase "exact area under the curve" completely, claiming that the final phrase "when it is evaluated at each position from 1 to n" would supersede the idea of area under the curve. Monica then clarified that her biggest problem in Zeb's argument was his use of the word "exact" and that she imagined rectangular Riemann sums to represent the "area" to which Zeb was referring. The interviewer then asked Monica to incorporate the rectangular Riemann sums that she was imagining (see Figure 61).

Figure 61

Monica's Use of Rectangles to Re-present Riemann Sums



As Monica completed her drawing, she discussed three other ideas. First, she briefly questioned whether the top left edge (red rectangles) or top right edge (green rectangles) of the rectangles should intersect the trace of the function. Monica eventually decided the top left edge was most appropriate because she imagined that this side of rectangle corresponded with the input value of the function. Second, Monica stated that she did not believe that Yolanda's and Zeb's expressions would compute the same value. This comment is evidence that she re-presented two distinct (in her mind) processes through the expressions $\int_1^n \frac{2}{\sqrt[4]{n}} dn$ and $\Sigma_1^n \frac{2}{\sqrt[4]{n}}$. Monica's final idea emerged when the interviewer asked whether she still believed the phrase "area under the curve" was appropriate for Zeb's argument. In response, Monica said that she preferred the language "represents the y-values of the function $f(n) = \frac{2}{\sqrt[4]{n}}$, which are added together when it is evaluated at each position from 1 to n." After a final clarification of Monica's meanings for the term "function" and her inscription $f(n)$, I decided to introduce the second task of the interview.

The outcome of Monica's comparison of Yolanda and Zeb's arguments was that she successfully differentiated between the meanings she wished to re-present through her command operators Σ and ∫ . In the graphical sense, Monica determined to re-present the exact area under a curve through her inscription ∫ and a Reimann sum (with rectangles of width 1) through her inscription Σ. In the algorithmic sense, Monica decided to re-present the sum of all function outputs from all real-number inputs between two bounds through her personal expression template $\int_{\square}^{\square} \square d\square$ and the sum of all whole-number values between function inputs through her personal expression template $\Sigma_{\square}^{\square}\square$. In this way, Monica created distinct looking (to her) personal expression templates to re-present two related (to her and to me) ideas about summing function values over an interval of the independent variable.

Example 2: Sylvia's Construction of One Personal Expression Templates for Series

In this section, I overview Sylvia's development of a single robust personal expression template through which she could (eventually) re-present every type of series that we discussed during Days 1-3 of the teaching episodes. Similar to Monica's example in the previous section, I present Sylvia's story linearly. The primary reason that I present Sylvia's symbolizing activity as it unfolded in real time is to highlight her introduction of new inscriptions and updated mark sets over time to re-present more and more examples of infinite series. I present Sylvia's experiences on each interview day in a separate subsection.

Day 1: Developing an Initial Personal Expression Template

Unlike Monica, Sylvia did not introduce a personal expression by which to re-present partial sums until I prompted her to do so. Instead, Sylvia verbally described the values of specific summands for several of Ivy's series (nearly always providing the appropriate value). To determine the value of a partial sum, Sylvia consistently stated that she would (1) generate the summands comprising the partial sum and (2) add the summands in a calculator.

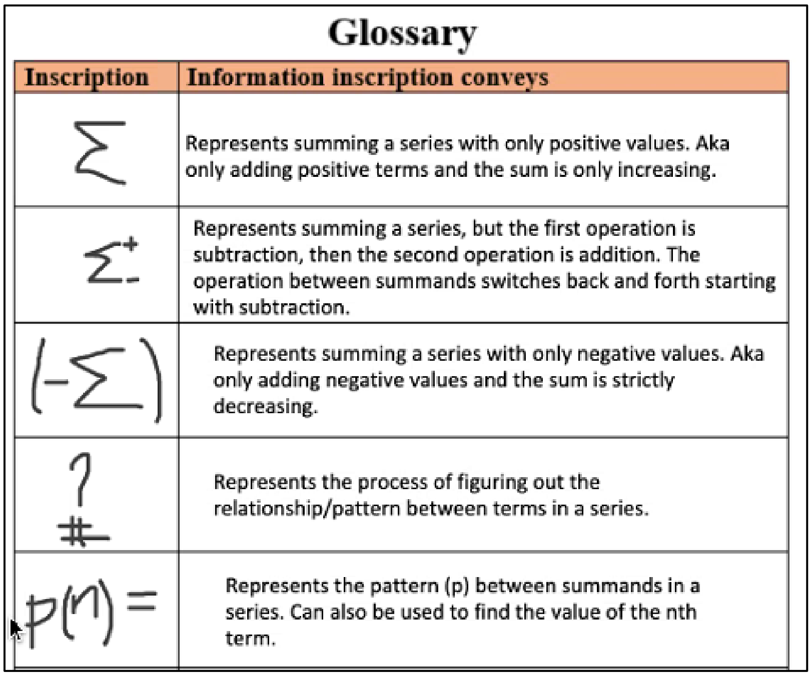
During the personal expression generation activity at the conclusion of the Day 1 interview, Sylvia proposed three different inscriptions by which she could re-present partial sums for various series: Σ, $(-\Sigma)$, and Σ_{-}^{+} (see Figure 62). Sylvia made a conscious decision to reject some notational conventions when constructing her inscriptions, a choice likely emanating from her belief that the story of a mathematician creating a personalized expression for a triangle was "inspirational." As evidence of this claim, I provide two excerpts from Sylvia at different moments in the Day 1 interview. The first excerpt comes from Sylvia's initial reaction to the personal expressions video, in which I presented (1) an example showing mathematicians presenting different inscriptions to symbolize a similar idea and (2) a parting thought that all conventional mathematical symbols were initially introduced by a single mathematician.

> Sylvia: (*While reflecting on the personal expressions video after watching for the first time)* Um, I guess, well what I'm thinking (…) [is] like the expressions and like things can be represented in a lot of different ways. And there are conventional ways to represent them, but those conventional ways used to be just one random person writing something out and then everybody else kind of agreed with them and it became conventional. So it was kind of like an inspirational, motivational, be your own person kind of thing.

In this excerpt, Sylvia stated that a major theme she perceived from the video was that individuals can construct their own inscriptions. She called this idea “inspirational” and implied that the video motivated her to be individualistic with her symbolization.

Figure 62

Sylvia’s Glossary at the Conclusion of the Day 1 Interview



Glossary

Inscription	Information inscription conveys
Σ	Represents summing a series with only positive values. Aka only adding positive terms and the sum is only increasing.
Σ^{+}_{-}	Represents summing a series, but the first operation is subtraction, then the second operation is addition. The operation between summands switches back and forth starting with subtraction.
$(-\Sigma)$	Represents summing a series with only negative values. Aka only adding negative values and the sum is strictly decreasing.
$\frac{?}{\#}$	Represents the process of figuring out the relationship/pattern between terms in a series.
$p(n) =$	Represents the pattern (p) between summands in a series. Can also be used to find the value of the nth term.

The second excerpt comes from Sylvia’s construction of personal expressions at the end of Day 1. Sylvia had constructed an initial inscription Σ to re-present a partial sum but felt that she needed to modify her inscription to differentiate alternating and non-alternating series.

Sylvia: (*After writing her first inscription* Σ *but before writing a definition for this inscription*) When I look at that [i.e., inscription Σ] I think of that, you’re only adding like positive values and so I’m trying to think of something that would represent like you, the terms switch off being positive and negative, so you're kind of adding and subtracting, adding and subtracting. Um.

Interviewer: Ok, (…) so for this Greek letter that you have at the top, that's indicating that you are adding together a bunch of things that are positive but doesn't [include] this idea of switching between addition and subtraction signs, and that's what you're thinking about right now? (*Sylvia: Mm-hmm*) Ok.

Sylvia: Well, I guess you could always just like stick a negative one in there and raise it to a power and have that power switch off. But that's boring. (laughs) Um, what kind of, I guess. Ooh! So what if it (…) (*screen freezes and Sylvia needs to refresh OneNote page; dialogue omitted from this interaction)* Ok, we'll do the same thing (*writes Σ on the second row of her glossary below the Σ on the first row*), but we're going to add (…) a little plus sign here and a little minus sign there (*adds subscript – and superscript + to Σ on second row of glossary*), and that means, uh, kind of like an integral, you start at the bottom with the lower limit so you're going to subtract the second term and then add the third term.

Interviewer: Ok, so the minus sign is referring to what is happening between the first and second term? So like, the first either plus or minus that you see [in the series]?

Sylvia: Mm-hmm.

Interviewer: And then the top [plus sign] is referring to what [operator sign] is coming next?

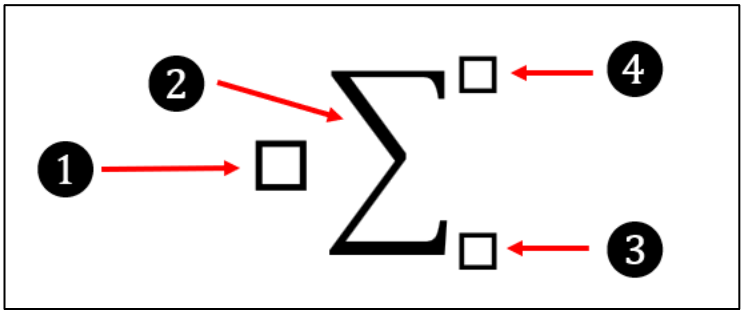
Sylvia: Yeah.

In this excerpt, Sylvia initially proposed using the normative symbolization for the terms of an alternating series (i.e., appending the expression $(-1)^n$ or $(-1)^{n+1}$ to the general summand). However, she quickly noted that such an idea was "boring" to her and instead proposed a novel expression, Σ_{-}^{+}, by which to re-present the alternating terms in a series (where the lower limit denoted the operator signs following each summand in an odd position and the upper limit denoted the sign following each even-positioned summand). When I asked Sylvia to construct definitions for her symbols Σ and Σ_{-}^{+}, she wrote that she considered Σ to denote "summing a series with only positive terms" and Σ_{-}^{+} to convey "summing a series, but (…) the operation between summands switches back and forth starting with subtraction." She then created a third inscription, $-(\Sigma)$, through which she re-presented "summing a series with only negative values." In the case of her symbols Σ and $(-\Sigma)$, she also referred to the behavior of the *running total* for each series perpetually increasing or decreasing (respectively).

Sylvia's symbolization constituted her attempt to " be [her] own person" in her symbolizing activity. As a result of this process, she created a personal expression template $\square\Sigma_{\square}^{\square}$, comprised of four inscriptions: (1) the inscription box to the left of Σ, (2) the inscription Σ, (3) the subscript inscription box for Σ, and (4) the superscript inscription box for Σ (see Figure 63). In the paragraphs below, I briefly summarize the meaning Sylvia attributed to each inscription, the inscription type, and corresponding mark set.

Figure 63

Sylvia's Initial Personal Expression Template for Partial Sums and Series



Sylvia used Inscription 1 (i.e., box to the left of Σ) to re-present whether the operator signs between summands in a series were uniformly positive or negative. Although Sylvia only used the mark $-$ for inscription 1 to denote uniformly negative summands, it is likely that she considered the inscription $+$ could be written (but was conventionally unnecessary) to indicate a series with uniformly positive terms. For this reason, I consider inscription 1 to constitute *cloze placeholder inscription* with the mark set $\{-, +\}$.

Sylvia used Inscription 2 (i.e., Σ) to re-present the process of adding together summands of a series to compute a partial sum. In this case, I consider Sylvia's inscription Σ to constitute a *fixed command operator inscription* with the mark set {Σ}.

Sylvia used Inscription 3 (i.e., lower index of Σ) to re-present the operator sign separating the first and second summands in an alternating series. Although Sylvia only used the mark – for inscription 3 when defining her expression in the glossary, I conjectured that Sylvia would also be able to use the personal expression Σ_{+}^{-} in her symbolizing activity to re-present an alternating series whose first operation sign was +.

For this reason, I coded inscription 3 as a *cloze placeholder inscription* with the mark set $\{-,+\}$.

Sylvia used Inscription 4 (i.e., upper index of Σ) to re-present the operator sign separating the second and third summands in an alternating series. For the same reasons I provided in my description of Inscription 3, I coded Inscription 4 as a a *cloze placeholder inscription* with the mark set $\{-,+\}$.

After having Sylvia compare her symbols in her glossary to her written rule for determining the value of a partial sum (which I describe in Chapter 6), I presented a final interview task, asking her to use her glossary entries to symbolize the 37th partial sum of Ivy's 4th Series. Sylvia had briefly reasoned about Series 4 earlier in the interview, an alternating series which I presented in the expanded form $\frac{6}{1}-\frac{6}{4}+\frac{6}{9}-\frac{6}{16}+\frac{6}{25}-\frac{6}{36}+\cdots$. Sylvia's initially created the expression $\Sigma_{-}^{+}p(n)=\frac{6}{n^2}$ to re-present the 37th partial sum. However, when I asked her to explain how her expression conveyed the idea of the 37th partial sum, she modified her personal expression template (see Figure 64):

> Sylvia: Oh, the 37th [partial sum]. Hmm. Ok, that does not represent the 37th [partial sum].
>
> Interviewer: What, then what does it represent? What were you trying to represent there?
>
> Sylvia: I guess this just represents, the series, well, I don't really know what it represents. But like, I would, if I put (…) n equals 1, and then I'm going to say, at the top we're going to add when n equals 37. (*writes* $n=1$ *below* Σ *and* $n=37$ *above* Σ)

Interviewer: And what are the n equals 1 and n equals 37 showing here?

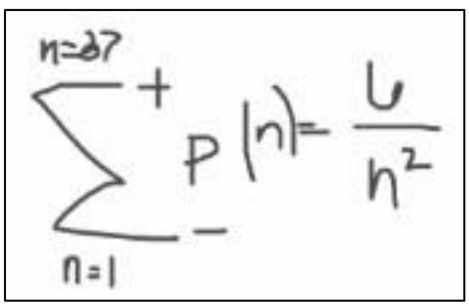
Sylvia: Um, like the, you start at n equals 1 and you stop at n equals 37, the terms.

Interviewer: Ok, all right. I think that makes sense to me. So I'll ask the original question again, uh, why did you write what you did and how does this show the 37th partial sum.

Sylvia: Um, so I wrote the Σ with the plus minus because this one starts with the adding, er, subtracting the second term. And then the pattern I represented with $p(n)$ equals, and the six stays constant, and then, it's the number, like the number of the term that you're on, you square that and you get the denominator. So, yeah. And then it goes from 1 to 37.

Figure 64

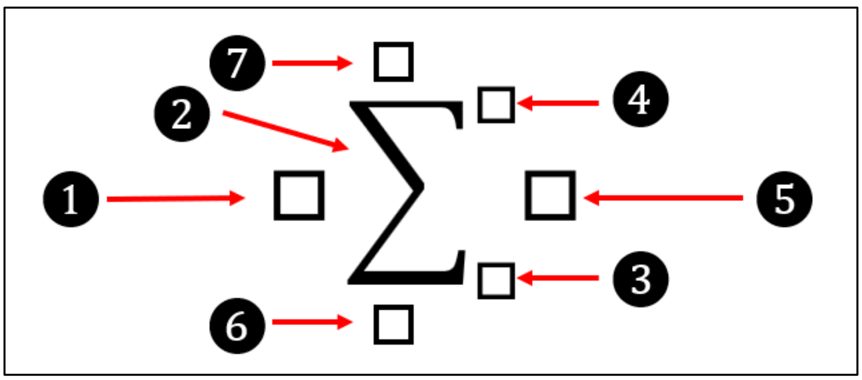
Sylvia's Personal Expression for the 37th Partial Sum of Ivy's 4th Series



In this excerpt, Sylvia believed that her original expression $\Sigma_{-}^{+} p(n) = \frac{6}{n^2}$ did not convey enough information for her to be able to re-present a specific partial sum. To rectify this issue, she introduced two additional inscriptions (i.e., an additional subscript and superscript for Σ) by which to denote the first and last summands necessary to compute the value of a particular partial sum. With Sylvia's additional inclusion of a function rule for the general summand of Series 4, I considered her personal expression template to now be of the form $\square\Sigma_{\square}^{\square}\square$ (see Figure 65 for inscription numbering). I have previously described the meanings I coded for Sylvia's attribution of meaning to Inscriptions 1-4 (which did not appear to change in this instantiation of the template) but describe my codes for her meanings for Inscriptions 5-7 in the paragraphs below.

Figure 65

Sylvia's Updated Personal Expression Template for Partial Sums and Series



Sylvia used Inscription 5 (i.e., box furthest to the right of Σ) to re-present a function rule by which she could generate the summands of the series necessary to compute the value of a partial sum. At this stage of the interview, Sylvia appeared to use the *relational connector inscription* $=$ to re-present the indicator expression $p(n)$ she had created for the summand pattern and an expression for the closed-form rule of this pattern. Although Sylvia used a *relational inscription* for Inscription 5 in this instance, after the Day 1 interview she typically wrote either a general expression for the pattern (e.g., $p(n)$, $v(n)$) or the closed-form rule for the pattern (e.g., $\frac{6}{n^2}$). Based on this occurrence during the other teaching episodes, I classify Inscription 5 as a *cloze concept* *inscription*[30] with a mark set consisting of the expression $p(n)$ and all possible algebraic function rules that Sylvia was capable of creating.

Sylvia used Inscription 6 (i.e., new subscript for Σ) to re-present the position of the first summand in the partial sum. In this moment of the interview, Sylvia wrote an expression of the form $n = \square$ for Inscription 6 in her personal expression template. In this expression, I consider the box to constitute a *fixed placeholder inscription* with a singular mark set $\{1\}$. The reason I coded this inscription with a singular mark set in this moment of the interview is because Sylvia had only created one instantiation of her personal expression template that contained Inscription 6. Although Sylvia utilized other marks for Inscription 6 during Day 2 and Day 3 (which I coded as evidence that Inscription 6 became a *cloze placeholder inscription*), I cannot rigorously claim Sylvia used her inscription in this way using Day 1 data. I also do not have sufficient data to make rigorous claims as to the meanings Sylvia attributed to the inscriptions n and $=$ in Inscription 6, although I think it likely that Sylvia used these inscriptions relationally to connect her function rule to the index values.

Sylvia used Inscription 7 (i.e., new superscript for Σ) to re-present the position of the final summand necessary to compute the value of a partial sum. In this moment, the expression that Sylvia wrote was of the same form as what she wrote for Inscription 6 (i.e., $n = \square$). I also coded Sylvia's Inscription 7 as a *fixed placeholder inscription* with the singular mark set $\{37\}$. Although I consider it highly likely that Sylvia could have immediately replaced her expression 37 with another inscription or expression for a

[30] I use the term *concept inscription* rather than the more specific categories *indicator* and *placeholder* because it was not clear (in this moment of the interview) which of these meanings Sylvia was attributing to Inscription 5.

different partial sum (e.g., 78th partial sum), there is no data to verify this possibility in this moment of the interview. During her symbolization of other partial sums and series during Day 2 and Day 3, Sylvia fluidly modified Inscription 7 in her template to re-present various marks, which I coded as evidence that Inscription 7 became a *cloze placeholder inscription* with a non-singular mark set.

At the conclusion of the Day 1 interview, Sylvia showed evidence that she was able to symbolize the following types of series through her 7-inscription personal expression template $\square\Sigma_{\square_{\square}}^{\square}\square$:

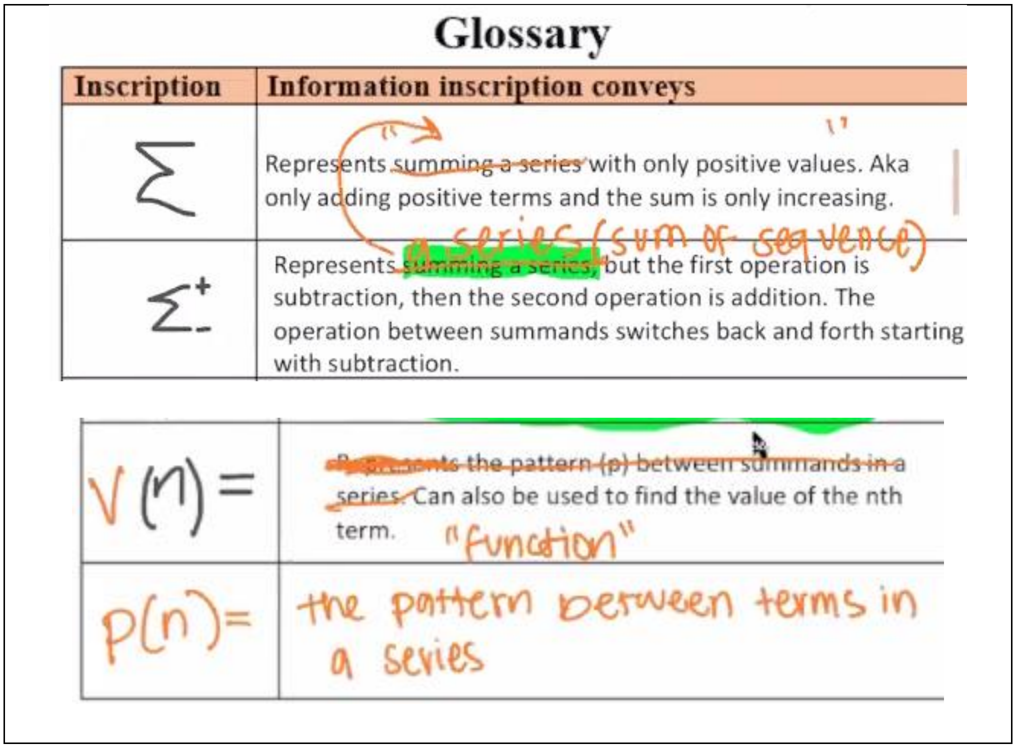
1) Partial sums for a series with only positive summands
2) Partial sums for a series with only negative summands
3) Partial sums for a series whose summands were separated by alternating signs (where each operator occurred 1 time before switching to the other operator).

Day 2: Introducing New Marks and Inscriptions to Re-present Operator Sign Patterns

At the beginning of the Day 2 interview, I had Sylvia review the symbols in her glossary and make any modifications she felt were necessary. During the review, Sylvia made two modifications. First, she modified her definition of her inscription Σ to include the idea of a series as the sum of a sequence (see orange writing in Figure 66). Second, she introduced two expressions to re-present different components of her Day 1 symbol $p(n)$: the expression $v(n) =$ to re-present the function by which she could generate summands in a series and the expression $p(n) =$ (which she later modified to $p =$) to re-present a recursive pattern she could use to generate successive summands (see orange writing in Figure 66). Since I had no teaching-related interactions with Sylvia between the end of the Day 1 interview and this portion of the Day 2 interview, Sylvia's modifications likely resulted from either personal reflection or her experiences in her calculus course during the week between interviews.

Figure 66

Sylvia's Glossary Modifications at the Beginning of her Day 2 Interview



For the first task of the Day 1 interview, I prepared five new series, which I labeled with capital letters (see Table 23). I chose the five series such that each series portrayed the same summand magnitudes (i.e., the summands from the geometric series $\sum_{n=0}^{\infty} \frac{1}{3^n}$) but exhibited increasingly difficult patterns of alternating signs. The purposes of this task were to (1) discern whether Sylvia would continue to utilize her personal expression template from Day 1 (which included distinct inscriptions for summand magnitude and operator signs) and (2) determine whether Sylvia was able to construct instantiations of her template for series with successively more complex operator patterns. When I presented the task to Sylvia, I asked her to create a personal expression for the 43rd partial sum for each of the five series.

Table 23

The Five Series Sylvia Reasoned About During the First Task of Day 2

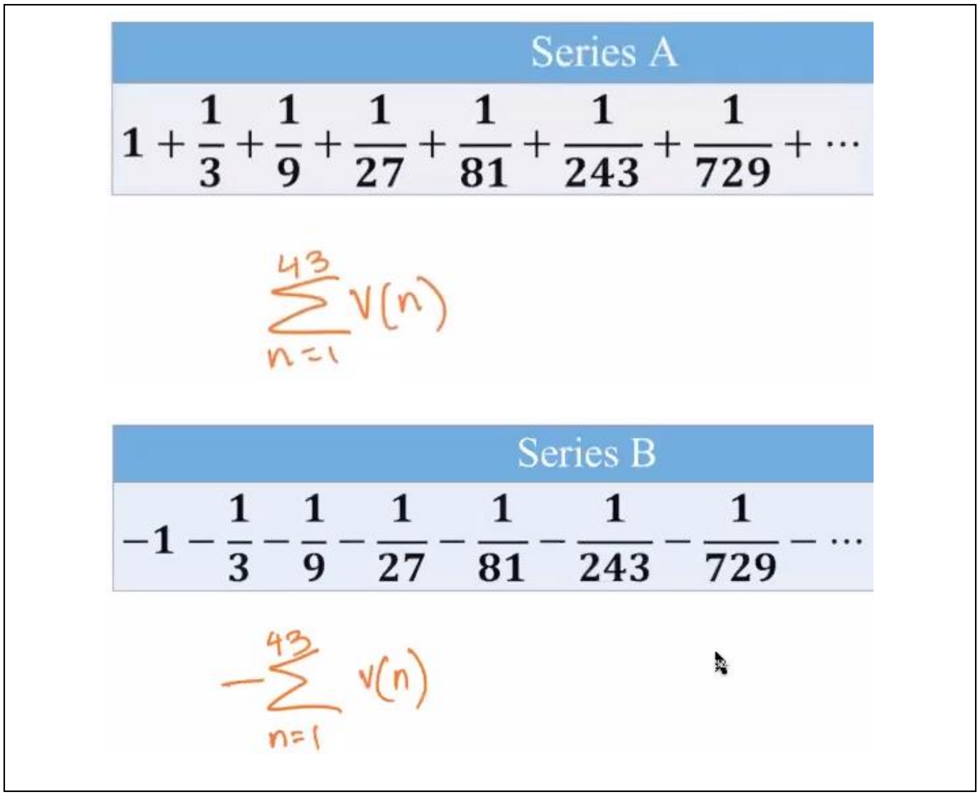
Series Name	Expanded Form	Properties
A	$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729} + \cdots$	Series Rule: $\sum_{n=0}^{\infty} \frac{1}{3^n}$ Convergent p-series Converges to $\frac{3}{2}$
B	$-1 - \frac{1}{3} - \frac{1}{9} - \frac{1}{27} - \frac{1}{81} - \frac{1}{243} - \frac{1}{729} - \cdots$	Same as Series A except all summands are negative
C	$1 + \frac{1}{3} + \frac{1}{9} - \frac{1}{27} - \frac{1}{81} + \frac{1}{243} + \frac{1}{729} - \cdots$	Same as Series A except signs alternate every two summands
D	$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} - \frac{1}{81} + \frac{1}{243} + \frac{1}{729} - \cdots$	Same as Series A except signs alternate for gradually increasing number of summands
E	$-1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \frac{1}{243} + \frac{1}{729} - \cdots$	Same as Series A except signs appear to alternate randomly

Sylvia had little difficulty constructing personal expressions for the 43rd partial sum of either Series A or Series B, and quickly produced the expressions $\Sigma_{n=1}^{43} v(n)$ and $-\Sigma_{n=1}^{43} v(n)$ (respectively; see Figure 67). In these moments, Sylvia's symbolization differed in subtle ways from her personal expression at the end of the Day 1 interview. For example, Sylvia did not write an explicit function rule for the general summand of either Series A or Series B. Also, Sylvia did not write the expression "$n =$" in the upper index of summation. While the Day 1 data was readily available on another OneNote tab, in this moment of the interview I did not think to ask Sylvia about her changes in the style of inscriptions that she used in her template. At best, I can merely conjecture that

Sylvia’s stylistic changes were a result of seeing multiple instantiations of conventional summation notation in her calculus coursework.

Figure 67

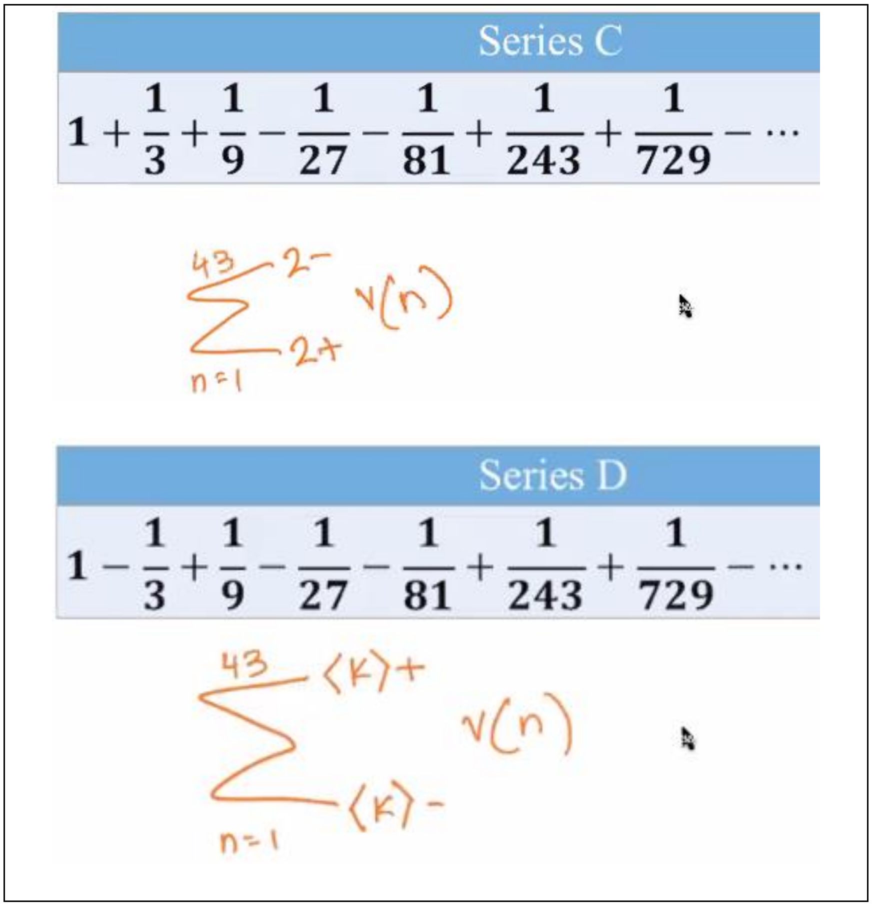
Sylvia’s Personal Expressions for Series A and Series B on Day 2



While reasoning about Series C, Sylvia quickly noted that the series had “two adding, two subtracting” (a reference to the number and type of repeating operators) and wrote the expression $\Sigma_{2+}^{2-}v(n)$. When I asked whether she needed to include “n equals 1 and 43,” she modified her personal expression to $\Sigma_{2+(n=1)}^{2-43}v(n)$ (see Figure 68).

Figure 68

Sylvia's Personal Expressions for Series C and Series D on Day 2



Because Sylvia seemed to assimilate Series C so easily to her personal expression template, I decided to also have her symbolize Series D before rigorously questioning her thinking. After confirming that the number of operator signs would perpetually increase throughout the series, Sylvia proposed using a new mark k for her operator-describing inscriptions (see Figure 68):

> Sylvia: Well that [i.e., changing number of operator signs] in and of itself needs like its own, uh, its own little pattern. (…) (*writes* Σ) so we'll still have this (*writes* $n = 1$ *for lower index*), we're going to the 43rd [summand],

and then I want this [i.e., inscription for re-presenting operator sign pattern] to be its own series [i.e., sequence] of 1, 2, 3, 4, 5, but I don't want it to be n. So, I guess I could put k minus (*writes* $\langle k \rangle -$ *for lower operator sign inscription*). I'll try to explain in a minute. (…) And then k plus (*writes* $\langle k \rangle +$ *for upper operator sign inscription*). And then I also want k to cycle, not cycle, but go from 1, 2, 3, 4, 5, only whole numbers (writes $v(n)$).

Interviewer: Ok.

Sylvia: So, yes, Ok. So the Σ is the same, n equals 1 to 43. The $v(n)$ is the same, that gets you like the actual value of the term you're adding. The k with the chevrons (i.e., $\langle \quad \rangle$) and the minus, the chevrons represent like that this is a sequence.

In this excerpt, Sylvia symbolized two new patterns for the operator signs in an alternating series using her personal expression template $\square\Sigma_{\square_\square}^{\square^\square}\square$. In Series C, Sylvia constructed the expressions 2+ and 2- for Inscriptions 3 and 4 in her template (i.e., the operator sign pattern placeholder inscriptions) to re-present a series whose operator signs alternated in constant sets of two. In this moment, Sylvia attributed a new meaning to Inscriptions 3 and 4 and introduced additional marks by which to re-present these meanings, increasing these inscriptions' mark set from $\{+,-\}$ to (at minimum) $\{+,-,2+,2-\}$. Later, as Sylvia reasoned about Series D, she introduced the expressions $\langle k \rangle -$ and $\langle k \rangle +$ as additional marks by which she could re-present a varying number of operator signs through Inscriptions 3 and 4.

After Sylvia's initial reasoning about Series D, I asked her to summarize what k meant (to her). Her response reiterated her meanings for both Series C and D and confirms the claims I made in the previous paragraph.

Interviewer: Remind me, what does k stand for?

Sylvia: k is going to be similar to n and goes on, only whole numbers. It's kind of like another index, but it's going, it's different from n because when n is, when n is 4, I don't want 4 minuses, if that makes sense.

Interviewer: Ok, I think that makes sense and I think, I think I understand why you feel the need to use a different letter than n. So you said like k is this group of whole numbers, so what are these, what is this group of whole numbers being used to represent, this k?

Sylvia: How many plus or minus signs are going to be used consecutively. So like before we had two plus, two minus, the Series C, and so that represented like two plusses in a row, two minuses in a row, two plusses in a row, two minuses in a row. And so the k here, I don't really know if the use of, like a series is appropriate. But I don't what else to use it, to represent it as, so I'm just going to leave it like that for now. But, so when k is 1, we'll have one minus, one plus. When k is 2, we'll have two minuses, two plusses in a row. When k is 3, and so forth.

Interviewer: Ok, that makes sense. So could you use that idea of k in Series C? I see that you're using slightly different inscriptions on each one of these. And so, I'm wondering if there is a way to potentially combine any of them. And if not, that's okay. I'm just curious.

Sylvia: I think, well Series C is different in that the, the amount of plusses versus minuses, I hate saying the word plusses and minuses, because it sounds so wrong. But, anyway, I don't know how else to say it. But, um, like that's constant. It stays at 2 the entire time. But for Series D it increases each time. So, we'd also need a variable for this second, for Series [D].

Interviewer: So, for Series C because the number of these operators is constant, they always come two at a time, we don't need k.

Sylvia: Yes.

Interviewer: But in Series D, the rate at which they are coming is changing, so we need that variable k.

Sylvia: Yes.

In this excerpt, Sylvia confirmed that she considered Series C and Series D to constitute two distinct situations that required different personal expressions. However, Sylvia was able to use the same personal expression template to symbolize each series. Her symbolizing actions imply that she had successfully assimilated both Series C and D into her general scheme for infinite series, although her use of separate marks for Inscriptions 3 and 4 indicate that she believed each series to possess distinctly different conditions for a particular attribute (i.e., operator sign pattern).

Sylvia was unable to successfully construct a personal expression for Series E during Task 1. In particular, she struggled to symbolize the apparently random (to her) operator sign pattern in this series. When I asked her what she would need to know to complete her personal expression template (which was complete except for Inscriptions 3 and 4) she underlined the ellipses for the series and stated that if she could see the operators for the remainder of the series, then she could likely come up with a pattern in terms of k and write it in the spot of Inscriptions 3 and 4.

At the conclusion of Task 1 in the Day 2 interview, Sylvia showed evidence that she was able to symbolize the following types of series through her 7-inscription personal expression template $\square\Sigma_{\square_{\square}}^{\square}\square$:

1) Partial sums for a series with only positive summands
2) Partial sums for a series with only negative summands
3) Partial sums for a series whose summands were separated by a constant number of alternating signs
4) Partial sums for a series whose summands were separated by a non-constant number of alternating signs which could be modeled using a closed form rule whose input is the set of whole numbers.

Sylvia also indicated that there was one situation that she could not re-present through her personal expression template: a series whose operator signs alternated in a way that appeared random.

Day 3: Introducing New Marks and Inscriptions to Re-present Randomness

At the end of the Day 2 interview, Sylvia created a new inscription, ?_?, by which she re-presented randomly selected (to her) summand values while symbolizing Ivy's 7th Series. She also applied her inscription $\genfrac{}{}{0pt}{}{?}{\#}$ while symbolizing Ivy's 6th Series to re-present a series whose summands appeared to follow a pattern but for which she had been unable to find a closed-form rule for the pattern. I address Sylvia's meanings for these inscriptions in more detail in Chapter 6.

As I prepared for the Day 3 interview, I wondered whether Sylvia would be able to incorporate her new inscription ?_? into the mark set for Inscriptions 3 and 4 to re-present the random operator signs in Series E. As a result, I created two tasks which I considered to have the potential to provide further insight into the situations Sylvia might symbolize through her personal expression template. The first task was to review three scenarios for the general summand of a series: (1) Sylvia could identify a closed form rule for a general summand, (2) Sylvia could not identify a closed form rule for a general summand but believed such a rule could be defined, and (3) Sylvia believed that the summands in a series were randomly generated and could not be modeled with a closed form rule. The second task was to reintroduce Series E (which had random operator signs) and Series 7 (which had random summands) and ask Sylvia whether her new inscriptions (e.g., ?_?) might allow her to re-present these series through her personal expression template.

During the first task, Sylvia reiterated her claims at the end of Day 2 about the types of series that she could re-present through her personal expression template. This data confirmed (to me) that Sylvia could now re-present (at least) three distinct (to her) situations through Inscription 5 in her personal expression template (i.e., general summand box; see Figure 69)

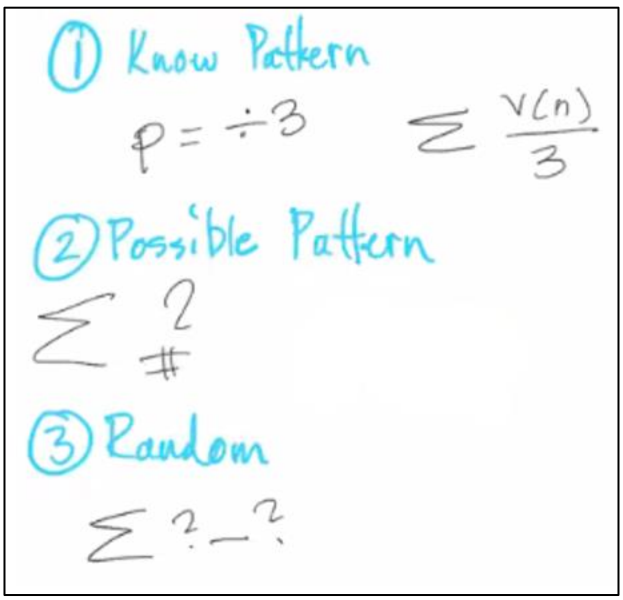
1. A general summand pattern that she could explicitly describe, which she re-presented either through an algebraic expression or the general expression $v(n)$,

2. A general summand pattern that appeared to exhibit a pattern but which she could not (yet) explicitly describe, which she re-presented through her inscription $\frac{?}{\#}$, or
3. A situation in which summands appeared to be randomly generated, which she re-presented through her inscription ?_?.

Since I already described the inscriptions $\frac{?}{\#}$ and ?_? in detail in Chapter 6, I do not reexamine Sylvia's meanings for these inscriptions in any further detail in this section.

Figure 69

Sylvia's Methods for Symbolizing Various Summand Patterns



For the second task, I displayed Series E and Series 7 to Sylvia again and asked whether she could create personal expressions for (1) the 36th partial sum and (2) each series. In my instructions, I told Sylvia that I had picked these particular series because components of these series appeared to be random and that she had struggled at times during Day 2 to symbolize each series. Sylvia quickly produced four personal expressions to answer the two questions for each series, using her indicator inscription for a random attribute of a series in each expression (see Figure 70)

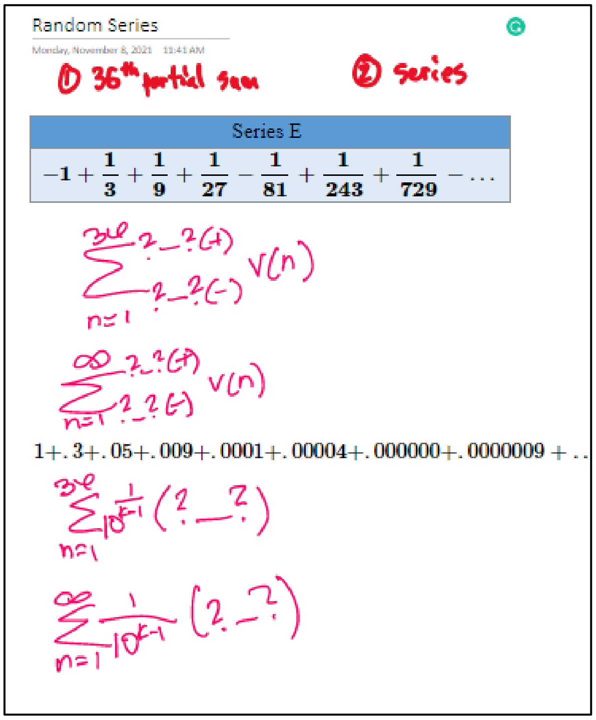
Sylvia: [For Series E,] I would still use this (*writes* Σ), I would still use n equals 1 to 36, and then I might use the question mark, this thing here [indicating inscription ?_?] for the minus and plus signs (*writes ?_?*$(-)$ *for Inscription 3 and ?_?*$(+)$ *for Inscription 4*). And then I would use $v(n)$ here (*writes* $v(n)$ *for Inscription 5*).

(*omitted dialogue*)

Sylvia: And then for this guy [i.e., Series 7], same idea (*writes* $\Sigma_{n=1}^{36}$). Instead of $v(n)$, because we don't, because for Series E it looks like it's 1 over 3 to the n minus 1 [i.e., $\frac{1}{3^{n-1}}$], something like that, yeah. And so you know the formula, you know the value, or like what's happening in the terms. But for this guy, you don't really know what's going on but you know they're all positives [i.e., summands]. So then I would do 1 over ten (*writes* $\frac{1}{10^k}$ *for Inscription 5*), I think that's where I would use k, times a random constant (*writes* $\frac{1}{10^k}$(?_?) *for Inscription 5*). Oh wait, that would be k minus 1, because the first one [i.e., summand] is a whole number (writes $\frac{1}{10^{k-1}}$(?_?) *for Inscription 5*).

Figure 70

Sylvia's Personal Expressions for Series E and Series 7 on Day 3



In these excerpts, Sylvia fluidly used her inscription ?_? to re-present either a random pattern of alternating operator signs or random summand values in the series. Sylvia's response also confirmed my hypothesis that she would assimilate the mark ?_? into her mark set for Inscriptions 3 and 4 to re-present Series E through her personal expression template.

In a final effort to push Sylvia's symbolization, I asked her how she might re-present a new series that exhibited both the random operator sign pattern of Series E and the random summands of Series 7. For this spontaneous task, I did not write an example series posited the hypothetical situation of combining the properties of Series E and Series 7 to create a new series. The transcript below includes our interaction and Sylvia's subsequent expressions (see Figure 71).

Interviewer: Let's pretend for a minute that I copy the random plus and minus signs from Series E and I paste those random signs down into the bottom series. So now we've got a situation where we've got random signs and random terms. How would you answer those two questions, 36th partial sum and the series, if that were to happen where you have random signs and random values?

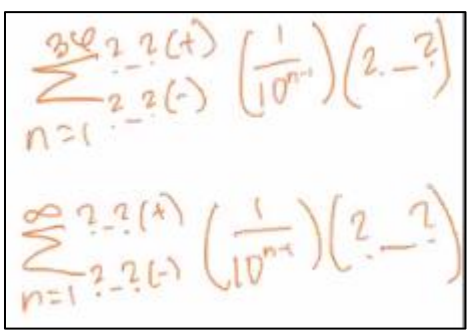
Sylvia: Um, I would just do the same, do you want me to write it?

Interviewer: Yeah, go ahead and write it. You can just write it off to the side somewhere.

Sylvia: Ok. I would do the same thing (*writes* $\Sigma_{n=1}^{36}$) but I would include the random plus and minuses (*writes* $?\_?(-)$ *for subscript Inscription 3 and* $?\_?(+)$ *for superscript Inscription 4*). And this (*writes* $(\frac{1}{10^{n-1}})(?\_?)$ *for Inscription 5*).

Figure 71

Sylvia's Symbolization of Random Operator Signs and Summand Values



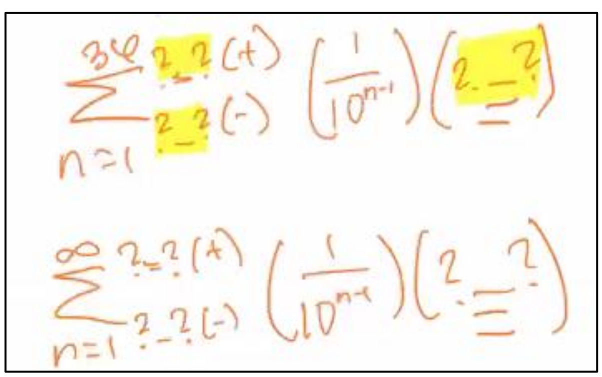
In this excerpt, Sylvia immediately assimilated the situation I presented to her and easily produced a personal expression through which she could re-present a series she imagined having random operator signs and random summand values. This example was especially powerful because Sylvia did not have an exact series written on her screen from which to construct her expression. Rather, her symbolizing activity occurred entirely within her mind, and the only perceivable artifacts from which she could reason were incomplete versions of the situation she was imagining. This final episode confirmed my hypothesis that Sylvia could re-present each of the following series types through her personal expression template $\square\Sigma_{\square_{\square}}^{\square}\square$

1. Partial sums for a series with only positive summands (with a known, knowable, or random summand pattern)
2. Partial sums for a series with only negative summands (with a known, knowable, or random summand pattern)
3. Partial sums for a series whose summands were separated by a constant number of alternating signs (with a known, knowable, or random summand pattern)
4. Partial sums for a series whose summands were separated by a non-constant number of alternating signs which could be modeled using a closed form rule whose input is the set of whole numbers (with a known, knowable, or random summand pattern)
5. Partial sums for a series whose summands were separated by a random number of alternating signs which cannot be modeled with a closed form rule (with a known, knowable, or random summand pattern).

I make one final comment about Sylvia's symbolization before the discussion section. After producing her personal expressions in Figure 71, I asked Sylvia whether it was problematic for her to use the same mark (i.e., ?_?) for Inscription 3, Inscription 4, and Inscription 5 in the same personal expression. Sylvia initially stated that she did not need to distinguish between the inscription marks because each inscription implied randomness. However, she quickly decided that the random numbers generated by each inscription would likely be different. As a result, she decided to use a different number of bars for each instance of the inscription ?_? in her personal expression to distinguish the random processes she was re-presenting (see Figure 72). Sylvia's action in this moment consisted of introducing a metric for differentiating random processes, which required her to (1) reconceive her inscription ?_? as an expression, (2) redefine the bar in her inscription ?_? as a placeholder parameter inscription, and (3) define a new mark set for the bar inscription (i.e., $\{-, =, \equiv, \ldots\}$.This example shows that the evolution of students' symbolizing activity includes not only attributing new meanings to existing inscriptions, but reconceiving inscriptions as expressions to re-present the interrelatedness of inscriptions (and their corresponding meanings) within a personal expression template.

Figure 72

Sylvia's Final Personal Expression for the Hypothetical Random Series



Summary of Chapter 7 Results

In this chapter, I presented two scenarios in which Monica and Sylvia developed and then modified personal expression templates as their meanings for partial sums and series evolved. In the first scenario, I presented Monica's initial attribution of partial sums to her pre-existing template $\int_{\square}^{\square} \square d\square$, which she had used in her coursework to re-present (1) the area under a curve and (2) the sum of a function's output values over all real-number inputs of an interval of its domain. As Monica began to reason about partial sums and integrals graphically, she recognized that she was envisioning a partial sum as a Riemann sum and an integral as an area under a curve. At the end of the Day 1 interview, Monica introduced a new personal expression template, $\Sigma_{\square}^{\square}\square$, to re-present her image of a partial sum but was still unable to rigorously describe specific differences in the meanings she attributed to her two personal expression templates. During the Day 2 interview, I introduced two contrasting prompts by hypothetical students who proposed using Monica's personal expression templates to denote integrals and partial sums. Through this activity, Monica consciously decided to re-present her image of integral through her expression template $\int_{\square}^{\square} \square d\square$ and her image of partial sums through her template $\Sigma_{\square}^{\square}\square$. In other words, Monica created two personal expression templates to distinguish different cases of adding function values she envisioned.

In the second scenario, I presented Sylvia's construction of a personal expression template, $\square\Sigma_{\square}^{\square}\square$, which she used to describe (1) the pattern of the operators in a series and (2) the summands of a series. During the Day 2 interview, I presented series with various predictable and random patterns of operator signs to determine whether Sylvia could re-present partial sums from these series through her template. After Sylvia successfully modified the types of inscriptions she could include in her template to re-present the series from Day 2, I presented more examples of series on Day 3 that focused on randomly generated summands and operator signs. At the conclusion of the Day 3 interview, Sylvia's expression template had evolved to the form $\square\Sigma_{\square}^{\square}\square$, which she could use to re-present series whose patterns and components she (1) knew, (2) did not know but believed could be modeled with a closed-form rule, and (3) appeared (to her) to be randomly generated.

The following tables, Tables 24 and 25, summarize the final versions of each student's personal expression templates at the end of the interview periods that I have described in the chapter. In Table 24, I describe Monica's two personal expression templates, the meanings she re-presented through each template, and my categorization of the various inscriptions within the templates. In Table 25, I provide a similar description of Sylvia's personal expression template and corresponding inscriptions.

Table 24

Monica's Distinct Personal Expression Templates for Re-presenting Addition of Function Values

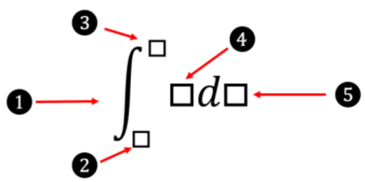
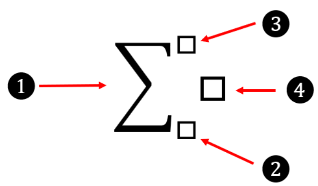
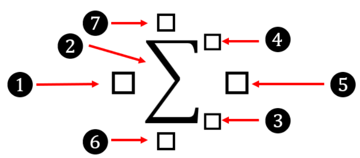
Template	Re-presentable Situations	Meaning of Inscriptions
	1. The addition of the output values of a function with a domain of all real numbers between two bounds 2. The area under a real-valued function curve between two points of its domain	1. *Fixed command operator* denoting the area under a curve and adding all function output values between two input values 2. *Cloze placeholder* subscript denoting the lower bound of the function domain in the context of the addition problem 3. *Cloze placeholder* superscript to denote the upper bound of the domain 4. *Cloze placeholder* argument of $\int$ to denote the function whose output values are being added together 5. *Cloze indicator*, which Monica used as a syntactic convention to re-present the concept of integral and the independent variable of the function
	1. The addition of the output values of a function with a domain of all whole numbers between two bounds	2. *Fixed command operator* to denote the additive process of computing the value of a partial sum from a set of summands. 3. *Cloze placeholder* subscript to denote the position of the first summand comprising a partial sum. 4. *Cloze placeholder* superscript to denote the position of the final summand comprising a partial sum. 5. *Cloze placeholder* to denote the pattern (or function rule) to generate the summands of the series.

Table 25

Sylvia's Personal Expression Template for Partial Sums and Series

Template	Re-presentable Situations	Meaning of Inscriptions
	1. Partial sums for a series with only positive summands 2. Partial sums for a series with only negative summands 3. Partial sums for a series whose summands were separated by a constant number of alternating signs 4. Partial sums for a series whose summands were separated by a non-constant number of alternating signs 5. Partial sums for a series whose summands were separated by random numbers of alternating signs	1. *Cloze placeholder inscription* to denote a series with uniform negative summands (-) or uniform positive summands (no mark written) *2. Fixed command operator* to denote the process of adding together summands in a series to compute a partial sum or reason about a series 3. *Cloze placeholder inscription* to denote (1) the first visible operator in a series or (2) a definable pattern or randomly generated number for the odd-ordered sets of operator signs in a series 4. *Cloze placeholder inscription* to denote (1) the second visible operator in a series or (2) a definable pattern or randomly generated number for the even-ordered sets of operator signs in a series 5. *Cloze placeholder inscription* to denote either an explicit or arbitrary rule for the general summand of a series *6. Cloze placeholder inscription* to denote the position of the first summand in a partial sum or series *7. Cloze placeholder inscription* to denote the position of the final summand in a partial sum or the indefinite ending of a series

CHAPTER 8

DISCUSSION

In this chapter, I contextualize the results of the previous three chapters related to Monica's and Sylvia's intuitive meanings for infinite series (Chapter 5), their creation of inscriptions to re-present ideas related to series and convergence (Chapter 6), and their development of personal expression templates to re-present classes of series (Chapter 7). The purpose of Chapters 5, 6, and 7 was to provide insight into the three research questions motivating my dissertation study:

- RQ1: *What meanings for series convergence do first-time university calculus students conceive before receiving formal instruction on infinite series?*
- RQ2: *How do students symbolize their meanings for mathematical topics in the context of infinite series?*
- RQ3: *How do students' symbols and attribution of meaning to these symbols change as their thinking about infinite series evolves over time?*

In the following sections, I address each research question individually. I have divided each section into three general parts. In the first part of each section, I describe the research implications for my results, including connections to previous literature and the unique contributions of this study to the mathematics education literature. In the second part of each section, I provide examples of how instructors might implement my work to facilitate productive symbolizing activities within their classrooms. In the final part of each section, I briefly address potential future research directions for my work in students' symbolization.

RQ1: Students' Intuitive Meanings for Series Convergence

The purpose of Chapter 5 was to address my first research question, *what meanings for series convergence do first-time university calculus students conceive before receiving formal instruction on infinite series?* In the following subsections, I address the research and teaching implications of my results and the future directions of my studies of students' meaning for infinite series.

Research Implications for Students' Meanings for Series Convergence

The results I presented in Chapter 5 provide unique and relevant contributions to the mathematics education literature on students' meanings for series convergence. For instance, I identified that students with no formal instruction on series convergence might focus on the behavior of a series' *running total* to decide whether they believed a series converged. This construct is similar and different from Martin's (2013) description of a *dynamic partial sum*. My definition of *running total* is similar to a *dynamic partial sum* in that a student's *running total* corresponds to what a mathematician would call a "partial sum." A *running total* differs from a *dynamic partial sum* because students intuitively reasoning about series convergence through a *running total* do not (in the researcher's mind) coordinate the running total with an indexing variable. My extension of the Martin's (2013) idea of a *dynamic partial sum* to students' intuitive reasoning about series convergence provides an additional way for researchers to characterize students' thinking about infinite series.

Many researchers have described students' image of limits or convergence as a function or sequence moving toward a value (e.g., Roh, 2005, 2008; Swinyard & Larsen, 2012). My empirically grounded definition of an *asymptotic value* as one or more values a student believes a running total will perpetually move validates these previous findings. My characterization of three implications of the *asymptotic running total* meaning further extends these results to show empirical examples of how students might reason about particular series to determine whether their *running totals* indicate convergence.

A unique contribution to the literature from my results in Chapter 5 is my characterization of *decreasing summands convergence*, *monotone running total divergence*, and *running total recreation through grouping* as students' attempts to reconcile three ideas about series. These three ideas include (1) the signs of the summands (e.g., all positive, alternating), (2) the behavior of the summands (i.e., increasing, decreasing, constant) and its corresponding impact on the running total, and (3) that the process of adding summands into the running total would never terminate (i.e., *potential infinity*). I summarize how Monica's and Sylvia's coordination of these various ideas influenced their exhibited meaning in the paragraphs below.

When the students foregrounded idea (2), focusing primarily on the behavior of the summands, they were most likely to exhibit *decreasing summands convergence*. For instance, Monica's initial conception of Series 1 was that it converged because the magnitude of each summand was smaller than the previous. At the end of her intake interview, Sylvia stated that the common theme she perceived across all convergent series was that the summands decreased. While decreasing summands is necessary for series convergence, Sylvia's comment implies that she also considered decreasing summands a sufficient condition for convergence. Since a student reasoning with *decreasing summands convergence* is focused primarily on the behavior of the summands, the running total is typically present in their thinking but often an afterthought in their verbal explanations of their actions.

When the students considered all three ideas simultaneously about a non-alternating series, they were most likely to focus on the behavior of the running total and exhibit *monotone running total divergence*. For instance, Monica's recognition that the running total in Series 1 would perpetually increase influenced her statement that the running total would increase without bound. Similarly, Sylvia's calculations of the first few values of the running total for Series 3 convinced her that the running total would perpetually increase, which implied (to her) that the series would not converge. In these instances, the students coupled their image of the signs of the summands (idea 1) with their conception of the series as a non-terminating entity (idea 3) to construe the running total as an entity that eventually surpasses all possible bounds. Although Monica and Sylvia often acknowledged that the values of the summands would become incredibly small (idea 2), this notion was subsumed by their image of a monotone increasing running total.

When the students successfully constructed a monotone running total (idea 1) by combining the summands in an alternating series, they would most likely exhibit *running total recreation through grouping*. In these instances, Monica and Sylvia seemed to focus on the monotone nature of the groups of summands (idea 1) and how the relative magnitude of each group behaved (e.g., decreased; idea 2). Although neither student could fully explain their reasoning, Monica and Sylvia believed their reconstructed monotone running totals were bounded. Depending on the signs of the summands (idea

1), the students claimed that the alternating series would either converge to the initial value (increasing summands; Sylvia) or zero (decreasing summands; Monica).

The results I presented in Chapter 5 provide three unique contributions to the research literature. First, my distinction of a *running total* from a *dynamic partial sum* allows researchers to differentiate between students' use of individual partial sums to reason about series convergence and their coordination of the partial sums with an indexing variable to create a sequence of partial sums. Second, my characterization of Monica and Sylvia's *asymptotic running total* meaning for series convergence and three corresponding implications provides insight into how students' intuitive reasoning strategies to determine series convergence might be grounded within a single overarching meaning. Finally, my description of three foundational ideas Monica and Sylvia attempted to coordinate throughout the interview tasks serves as an organizational tool for the meanings that I proposed in this chapter and might be used by future researchers to describe other student meanings for convergence or prepare interview tasks.

Teaching Implications for Students' Meanings for Series Convergence

In this section, I share two instructional implications of the results I presented in Chapter 5. In both instances, I offer an instructional situation and propose a hypothetical instructional activity by which an educator might implement my results to improve students' construction of meaning for series convergence.

In the first situation, suppose that an instructor introduces the concepts of partial sums and the sequence of partial sums during their initial lecture on infinite series convergence. The results that I presented from Monica and Sylvia imply that many students might focus primarily on the values of partial sums and fail to comprehend the nature of the sequence of partial sums. To address this potential issue, the instructor might consider having her students intuitively reason about the convergence of several infinite series (e.g., Abigail's series I presented during my intake interview tasks). The instructor could then (1) monitor the resultant discussion to determine students' reasoning about a *running total*, (2) present and define the idea of a *running total* to her students, and (3) propose using an indexing variable to track the value of the running total over time, and (4) define the coordination of the indexing variable with the *running total* as the sequence of partial sums. A significant focus of instructor actions (3) and (4) would be to state that the *running total* is a single quantity and the sequence of partial sums denotes a covariational relationship between two quantities (i.e., a whole-number index and the *running total*; Thompson & Carlson, 2017). In this way, the instructor could leverage many students' intuitive focus for reasoning about series (i.e., the *running total*) to define the conventional method to evaluate convergence (i.e., the *sequence of partial sums*).

In the second situation, suppose that an instructor asks his students to discuss whether a series converges in small groups and finds at least one instance of each implication of the ART meaning among the students. In this instance, the instructor may wish to present pre-made hypothetical student arguments (similar to the Yolanda and Zeb task I describe in Chapter 7) for each implication and ask the students to compare these statements. The resulting discussion will make the students more likely to determine whether one or all implications appear viable for the sequence in question. During or after the discussion, the instructor could then introduce the three underlying ideas students coordinate while reasoning about convergence that I presented earlier in this section (i.e., sign of summands, behavior of summands, process of adding summands) and use these ideas to motivate the necessary and sufficient conditions for convergence. The instructor might continue to refer to these three ideas throughout the sequence and series unit to contextualize various convergence tests. In this way, the instructor could make the various meanings students might have for series explicit objects of analysis and provide an easier way for students to map their intuitive thinking about convergence to conventional portrayals of convergence.

Future Research Directions to Investigate Students' Meanings for Series Convergence

In this section, I present two future research directions that I intend to pursue with regard to RQ1. First, in my continued analysis of my dissertation data, I intend to compare the intuitive meanings that Monica and Sylvia exhibited during the intake interview with those they exhibited for series convergence during the exit interview. Specifically, I want to address three major interventions that Monica and Sylvia experienced during the teaching experiment and the impacts these interventions appeared to have on their thinking. First, each student participated in four interview sessions focusing on symbolizing components of infinite series and developing an understanding of the sequence of partial sums. Second, each student participated in three interview sessions focusing on graphical reasoning about infinite series convergence, adopting an appropriate written rule for series convergence, and symbolizing this rule. Finally, each student participated in an instructional unit on sequences and series in their mathematics courses and reviewed these same materials in preparation for their final exam (which they took during the same week as the exit interview). In my future work, I plan to describe (if the data permits) how these interventions emerged as Monica and Sylvia reasoned about Abigail's six series during the exit interview.

The intuitive meanings for series that I have proposed in this dissertation and the additional meanings that I will present through the comparison paper I described in the previous paragraph will prepare me with the theoretical tools to investigate students' meanings for series convergence at other levels of mathematics. In particular, I am interested how secondary students and students with mathematical proof-course experience consider the convergence of series. Through such studies, I hope to confirm the existence of the constructs I present in this dissertation and discover additional meanings individuals have for series convergence.

RQ2: Students' Attribution of Meaning for Series to Inscriptions

The data I presented in Chapter 6 provided a preliminary answer to my second research question, *how do students symbolize their meanings for mathematical topics in the context of infinite series?* In the following subsections, I address the research and teaching implications of my results and the future directions of my studies on students' attribution of meaning to the inscriptions in their personal expressions.

Research Implications for Students' Creation of Inscriptions

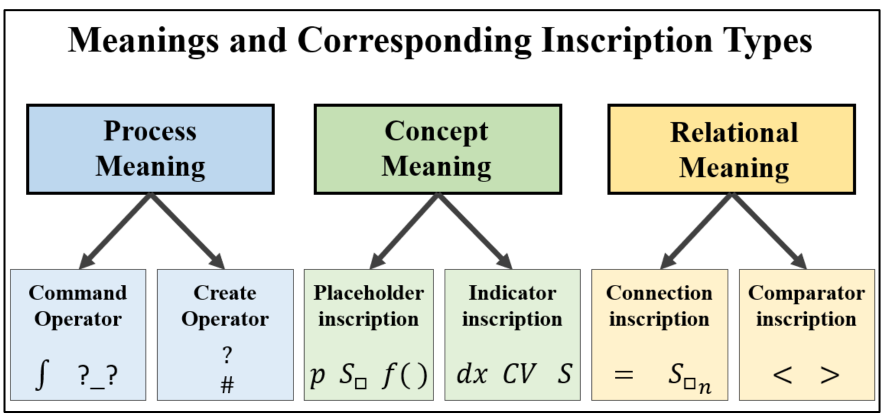
The results I presented in Chapter 6 contributes to the mathematics education literature on students' meanings for series convergence. For example, I have empirically confirmed Gray and Tall's (1994) claim that students can attribute one or more meanings to their symbolic expressions. I have also extended Gray and Tall's work by proposing three distinct meanings Monica and Sylvia attributed to their inscriptions: *process*, *concept*, and *relational*. In the following paragraphs, I describe two unique contributions of my results in Chapter 6 to the research literature: (1) coordinating students' meanings with the types of inscriptions through which they convey their meanings and (2) identifying certain symbolizing norms by comparing my proposed inscription types with mathematical conventions.

Contribution 1: Coordinating Meanings and Inscription Types

A unique contribution of my dissertation study is my coordination of the three meanings I presented in the previous paragraph with six types of inscriptions that Monica and Sylvia utilized during their symbolizing activity (see Figure 73). First, I categorized these students' inscriptions to re-present processes as *command* or *create operators*. For instance, Monica and Sylvia used *command operators* such as $\int$ and ?_? to re-present algorithmic processes such as computing a sum or generating a random value (respectively). Sylvia also used her inscription $\overset{?}{\#}$ as a *create operator* to re-present the process of investigating and determining a rule for the general summand of a series. Second, I labeled the inscriptions through which Monica and Sylvia re-presented topics, attributes, or values of quantities as *indicators* or *placeholders*. For example, Sylvia and Monica used *indicators* to re-present general attributes of a quantity such as integral (dx), a convergence value (CV) and a sequence (S). Monica also used *placeholders* to re-present a single value of a quantity (i.e., *parameter*; p), multiple fixed values across instantiations of a personal expression template in a comparative example (i.e., *parameter*; $S_{\square}$), and multiple quantity values through a single inscription (i.e., *variable*; n, x, k). Finally, I described the inscriptions through which Monica and Sylvia re-presented relationships between one or more meanings they had symbolized as *connector* or *comparator* inscriptions. For instance, Sylvia used the inscription $=$ as a *connector* to re-present the process-result relationship she envisioned between the summands in a series and the closed-form rule of the series that would generate those summands. Additionally, Monica spatially placed subscripts as syntactic *connectors* in her personal expression template $S_{\square_n}$ to re-present different instantiations of a general concept (e.g., sequence) about which she needed to reason within the context of a single example. Finally, Monica used the inscription > as a *comparator* to re-present her metric by which she could designate the number of terms in the sequence of partial sums that fell outside a particular error bound.

Figure 73

The Three Meanings and Inscription Types from Chapter 6



The framework I summarized in the previous paragraph will allow researchers to more clearly attribute (1) the meanings that a student attributes to an inscription and (2) the purpose of the inscription in the students' symbolizing activity. The framework also allows for the flexibility to categorize different meanings that students might attribute to an inscription across or within individual moments. Although I constructed this framework while evaluating student's attributed meanings for individual inscriptions, my constructs have the theoretical potential also to describe students' expressions comprising more than one inscription. I describe several ways in which I hope to refine this framework through my future work later in this section.

Contribution 2: Comparing Students' Symbolization with Mathematical Norms

I constructed the framework I presented in Chapter 6 and the previous section by analyzing Monica's and Sylvia's empirical data from their symbolizing activity in the context of infinite series. Still, my research has the potential to provide insights students' symbolization beyond the context of infinite series. For example, my work with students' symbolization in one context can provide insights into the mathematical norms[31] that students and the mathematical community have for utilizing algebraic representations. In the following sections, I compare my framework for meaning and inscription pairings with the conventional meanings for various mathematical symbols. This discussion aims to highlight how Monica's and Sylvia's symbolization differed from mathematical norms.

Conventional Use of Command and Create Operators to Convey Process Meanings. In the context of conventional expressions, mathematicians have defined a myriad of inscriptions that function as *command operators*, including $+, -, \times, \div, \boldsymbol{\Sigma}, \boldsymbol{\Pi}$, and $\boldsymbol{\frac{d}{dx}}$ (to name a few). In conventional mathematics, command operators are often *fixed* (i.e., have a singular mark set). Still, there are exceptions where mathematicians use then one mark to convey the same concept (e.g., multiplication, function composition, derivative

[31] I use the term *norm* in the sense of Dawkins and Weber (2017).

notation). Monica's and Sylvia's use of command operators generally aligned with mathematical convention.

While mathematicians have developed inscriptions to convey the termination of a creative process (e.g., boxed answers, Q.E.D. or ■ at the end of a proof), there is not (to my knowledge) a conventional inscription to denote the enactment of the creative process. Instead, mathematicians typically use linguistic inscriptions in task prompts as *create operators*, such as *prove*, *construct*, *find*, *determine*, *show*, *simplify*, and *justify*. Thus, the *create operators* that Sylvia and Emily presented during the dissertation and pilot studies (respectively) are truly novel in the context of mathematical symbolization. Additionally, Sylvia and Emily expressed initial discomfort at constructing *create operators* and seemed to require the creative symbolizing license I provided during the interview to justify their use of these inscriptions. As a result, I considered these students' introduction and use of *create operators* to be a violation (in their minds) of a mathematical symbolizing norm.

Conventional Use of Placeholders and Indicator Inscriptions to Convey Concept Meanings. In the realm of conventional expressions, mathematicians have defined myriad inscriptions that serve as *indicators*, such as the capital letters $\boldsymbol{A}, \boldsymbol{B}$, and $\boldsymbol{C}$ for set (or matrix) names, the inscriptions $\boldsymbol{f}, \boldsymbol{g}$, and $\boldsymbol{h}$ for function names, and the inscriptions $\boldsymbol{a_i}, \boldsymbol{b_j}$, and $\boldsymbol{c_k}$ to describe terms in sequences. In conventional mathematics, *indicators* are almost always *cloze inscriptions*. Additionally, mathematicians ascribe different numbers of values of a quantity to an inscription according to their needs. Monica's and Sylvia's creation and use of *indicators* was similar and different from mathematical convention. For example, both students used inscriptions to name or label attributes of quantities they deemed important in a particular task. However, these students (1) sometimes used inscriptions as (1) merely mnemonic recall devices (e.g., Monica's use of $\boldsymbol{f}$ and $\boldsymbol{S}$ for "function" and "sequence") or (2) syntactic conventions of a personal expression template (e.g., Monica's use of $\boldsymbol{dx}$ and $\boldsymbol{dn}$ as a suffix for her integral notation). Additionally, the students typically used fixed inscriptions for their indicators. These students' use of fixed indicators differs from mathematical convention, in which mathematicians often prefer particular marks for their indicators but can utilize unfamiliar or foreign marks for their expressions if the need arises.

Mathematicians have also defined myriad inscriptions that function as *placeholders*, such as x and y for variable real numbers, a and b for fixed components of an exponential function. Similar to *indicators*, mathematicians have developed norms for which marks are typically used for certain *placeholders*. Consequently, most *placeholders* in conventional mathematics are *cloze inscriptions*, meaning that the mark set corresponding to these inscriptions is non-singular (e.g., all lowercase letters, all capital letters. Mathematicians also fluidly assign a particular number of values to a placeholder according to the meaning they wish to convey through their inscription. For example, a mathematician might use a mark to convey exactly one value of a quantity (e.g., the vertical intercept a of a particular exponential function). The mathematician might also use a mark to convey all the values of a quantity within its domain (e.g., the independent variable x of a linear function). Monica's and Sylvia's use of placeholders was similar and different from convention. For example, both students used *placeholders* to re-present one or more values of a quantity they envisioned during a particular task.

However, the students did not always exhibit fluidity in the number of marks they attributed to their inscriptions, often opting to use *fixed placeholders*.

Conventional Use of Connector and Comparator Inscriptions to Convey Relational Meanings. In the context of conventional expressions, mathematicians have defined many relational inscriptions to coordinate or compare mathematical ideas, such as $<, >, =, \neq, \rightarrow, \cap, \cup, \subseteq, \propto, \approx, \cong, \equiv$, and $\in$. Relational inscriptions are generally *fixed*, although mathematicians sometimes employ different relational inscriptions across textbooks or mathematical fields (e.g., congruence notation). Generally speaking, mathematicians use relational inscriptions fluidly to denote the process of determining the relationship between two expressions or the resulting relationship (i.e., as a *connector* and *comparator* simultaneously). For example, a mathematician examining the relationship between two proportional quantities $\boldsymbol{x}$ and $\boldsymbol{y}$ might construct the expression $\boldsymbol{x} \propto \boldsymbol{y}$ to re-present this relationship. In this instance, the mathematician would likely be able to re-present either the process she employed to determine the proportionality of $\boldsymbol{x}$ and $\boldsymbol{y}$ or the proportional relationship itself. Monica's and Sylvia's use of relational inscriptions during some interview moments aligned with convention. For example, Sylvia utilized her inscription = to denote the closed-form rule for a general summand and the resultant summands during the later teaching episodes. Additionally, although I categorized Monica's use of > in her expression $\boldsymbol{p_n > C - L}$ during Day 5 as a *comparator* in Chapter 6, Monica's use of this expression implied that she not only used the inscription to compare partial sums with the bottom border of $\boldsymbol{\epsilon}$-strips but to re-present the partial sums satisfying this condition as well.

In contrast, a student might only re-present a process through the expression $x \propto y$ and demand that the constant of proportionality be set equal to the original expression (e.g., $x \propto y = 2$). In this case, I would say that this student assigned a *connector* meaning to the relational inscription $=$ in which he coordinates the notion of proportionality (an attribute two quantities share) with the constant of proportionality (the measurement of the quantitative relationship). In Sylvia's initial symbolization of a partial sum at the end of the Day 1 interview, she wrote $p(n) = \frac{6}{n^2}$ for the general summand in her personal expression. In this instance, Sylvia appeared (to me) to merely re-present a process-result connection between the general summand rule she constructed and not a comparison (such as substitutional equality) that would have allowed her to only utilize $p(n)$ or $\frac{6}{n^2}$ in her expression.

My comparison between Monica's and Sylvia's inscriptions and the conventional expressions of mathematics reveals three potential norms related to symbolization. First, it seems that mathematicians are reticent to use algebraic symbols to convey creative processes such as proving, simplifying, or constructing rules to describe phenomena. Second, it seems that mathematicians develop the ability to utilize various marks for their *indicator* and *placeholder* inscriptions. In contrast, Monica and Sylvia often used *fixed indicators* and *placeholders* as mnemonic devices to recall particular attributes or values of quantities. Finally, mathematicians tend to use relational inscriptions simultaneously as *connectors* and *comparators*. In contrast, there were some instances where Monica and Sylvia used a relational inscription as only a *connector* or *comparator* (and not both). Although this discussion is limited in data and scope, my description of some potential symbolizing norms in mathematics serves as a productive foundation for future research work to identify the norms and practices of the mathematics community.

Teaching Implications of My Explanatory Framework for Instruction

The conventional examples of symbolization I shared in the previous paragraphs reinforce Gray and Tall's (1994) claim that mathematicians can attribute multiple meanings to a particular inscription within and across situations. However, the data I shared in Chapter 6 indicates that students sometimes attribute only one meaning to their inscriptions at a particular moment. Furthermore, many of the multi-interview examples I shared in Chapters 6 and 7 show that students' attributed meanings to their inscriptions can change over time. In brief, my data and discussion imply that students' symbolizing activity is fragmented and in flux when compared with mathematicians.

The explanatory framework and constructs that I have presented in this chapter provide the linguistic symbols and theoretical ideas by which instructors can begin to (1) describe their students' symbolization and (2) better convey their symbolizing activity during instruction to their students. For example, an instructor who perceives that his student consistently uses the inscription f when using function notation might anticipate (based on my framework) that the student is using their inscription as an *indicator*. The instructor might confirm his conjecture by asking *what does the inscription* f *mean to you?* to discern whether the student is using f merely as a mnemonic device. The instructor might also present a problem situation in which a student must symbolize and compare two distinct function rules to determine the students' mark set for their function name inscription. In either case, the resulting conversation about symbolization norms within mathematics would likely assist the student in constructing future expressions that more closely conform to mathematical convention, thereby ensuring that the student can more clearly communicate their thinking to others in their future symbolizing activity.

Future Research Directions to Investigate Students' Attribution of Meaning to Expressions

In my future work, I intend also to categorize the meanings students ascribe to entire expressions and groups of related expressions. For example, in my pilot study data (see Eckman & Roh, 2022a), I reported Emily's creation of her expression $A \nsim B$ to re-present the process of determining a rule by which she might generate the second summand in a series (B) given the value of the first summand (A). Although Emily re-presented the creative process through her inscription $\nsim$, she specified which iteration of the creative process she envisioned through her inclusion of B and A in her expression.

In future studies, I might also study students' moment-by-moment attribute of meaning to their expressions (as opposed to my analysis in Chapter 6, where I primarily focused on individual inscriptions). For example, I can further review the data in this study to categorize the various personal expressions (and combinations of expressions) that Sylvia and Monica created throughout the interview sessions. My analysis of Sylvia's and Monica's personal expressions related to partial sums in Chapter 7 provides a preliminary example of how I might report such data. Still, my report in Chapter 7 focuses on the evolution of these students' personal expression templates over time (which differs from the moment-by-moment categorization I propose here).

Other avenues of future research I wish to investigate are (1) students' joint symbolizing activity in the context of *communicative* expressions and (2) students' attempts to adopt *conventional* expressions during direct instruction. I have initially attempted to address some of these ideas in the context of students' interpretation of set-builder notation and their construction of Euler diagrams (Eckman et al., 2023). Still, the results of this dissertation study imply that the continued study and categorization of students' symbolization across several grains of analysis is a potentially profitable line of future research work.

RQ3: Coevolution of Student Meanings and Personal Expression Templates

The purpose of Chapter 7 was to describe empirical examples of students' symbolizing activity to provide initial insight into the research question *how do students' symbols and attribution of meaning to these symbols change as their thinking about infinite series evolves over time?* In the following subsections, I address the research and teaching implications of my results and the future directions of my studies on students' coevolution of their meanings and personal expression templates for infinite series topics.

Research Implications for Students' Personal Expression Templates

The results I presented in Chapter 7 contributes to the mathematics education literature on students' symbolization and evolution of meaning. For example, my characterization of Monica's and Sylvia's development of one or more distinct personal expressions to re-present related ideas affirms the relevance of my previous work on Emily's symbolization (Eckman & Roh, 2022a, under review). Additionally, my documentation of the emergence of students' personal expressions from their meanings for series topics lends further credence to other researchers who have studied student cognition and symbolization in tandem (e.g., O'Bryan, 2020; O'Bryan & Carlson, 2016).

In Chapter 7, I described two instances of symbolization. In the first example, Monica created two distinct personal expression templates, $\int_{\square}^{\square} \square d\square$ and $\Sigma_{\square}^{\square}\square$, by which she re-presented the summation of all function values evaluated at (1) all real numbers (i.e., integral) or (2) all whole numbers (i.e., summation) of the independent variable over an interval. In Monica’s case, this single difference in additive process merited a different command operator, which required two distinct personal expression templates. In the second example, Sylvia constructed a single, intricate personal expression of the form $\square\Sigma_{\square_{\square}}^{\square^{\square}}\square$, by which she could eventually symbolize every kind of series that I presented during the interview sequence. In Sylvia’s case, nearly every time she experienced a cognitive conflict in which she was unsure how to apply her personal expression template, she invented a new mark or inscription by which to re-present her new situation.

Although Monica’s symbolization (generally) mirrored convention and Sylvia’s inscriptions were often novel, each student exhibited behaviors common in mathematics. On the one hand, Monica worked hard to create personal expressions which aligned (in her mind) with mathematical convention. As a result, she rarely introduced novel notation. On the other hand, Sylvia perpetually considered how to integrate novel situations into the symbols she had already created. Sylvia’s final personal expression template was robust enough to model many classes of series and intricate enough to show the nuances of a particular series.

In the following section, I apply Monica and Sylvia’s symbolizing examples I described in Chapter 7 to propose a preliminary theoretical framing of the evolution of students’ symbolization. In doing so, I return to the Piagetian notions of scheme, assimilation, and accommodation. This theoretical framing constitutes one of the primary contributions of this dissertation study to the mathematics education field and an initial attempt to (theoretically) generalize the results of my study.

A Theoretical Discussion on Coevolution of Students' Meanings and Personal Expressions

This section aims to theoretically frame a cognitive relationship between the mental actions that students engage in and the perceptible artifacts they create while constructing personal expressions. The major forces at work in students' construction of personal expressions include (1) students' *meanings in the moment* (Thompson et al., 2014), (2) the collection of personal expression templates the student has constructed through their experiences, and (3) the re-presentable meanings the student has imputed to the inscriptions in the template (and the expression itself). In the following paragraphs, I propose various theoretical constructs to contextualize the relationship between these three forces.

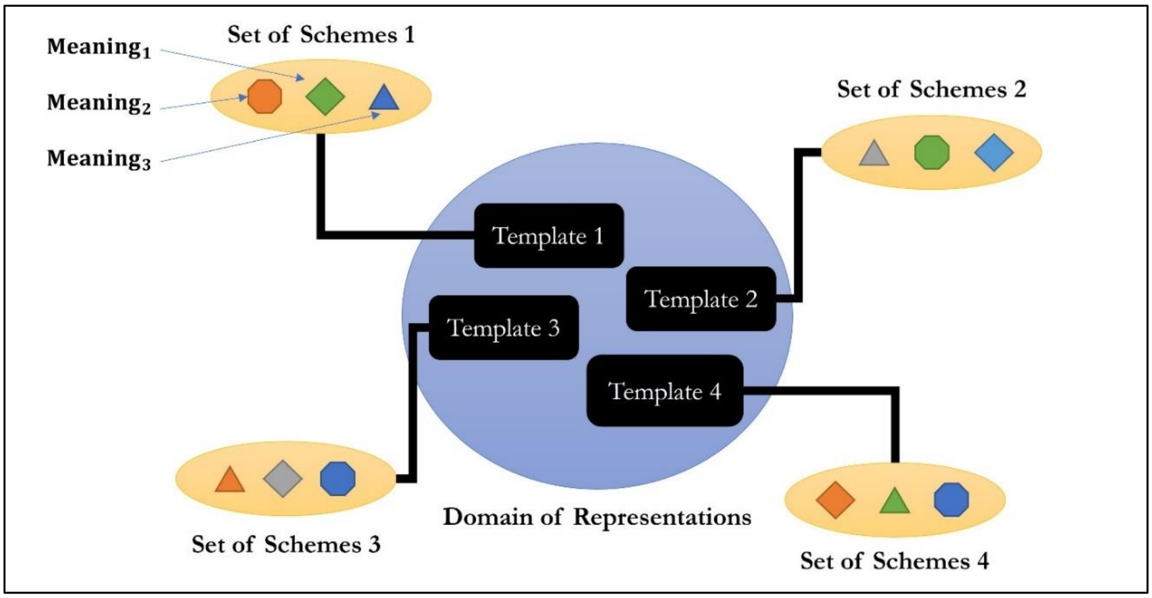
The *domain of representations* consists of all perceptible artifacts (e.g., writing, drawing, gesture, verbalizations) that an individual might use to re-present (to himself) or convey (to others) his meanings. I call each element of the domain of representations a *personal expression template*[32]. A personal expression template is a representational device that an individual can modify according to her needs to symbolize various situations she perceives to have analogous structures or properties. I use the term *personal expression* to refer to an individual's construction of an instantiation of her

[32] Although I focus this dissertation on algebraic personal expressions, the construct *personal expression template* can refer to any linguistic (e.g., words), pictorial (e.g., diagrams), symbolic (e.g., notation), visual (e.g., body language), or auditory (e.g., music, spoken language) representation that an individual might use to re-present or convey their meanings in a particular moment.

personal expression template to reflect on or convey information about her experience. I use the term *template* to refer to students' decisions to use fixed marks for certain inscriptions and allow others to vary from instantiation to instantiation of the expression. Every personal expression template has a corresponding *set of schemes* comprising every meaning an individual might spontaneously re-present through that personal expression template at a given moment. This relationship is summarized in Figure 74 below.

Figure 74

The Domain of Representations, Expression Templates, and Set of Schemes



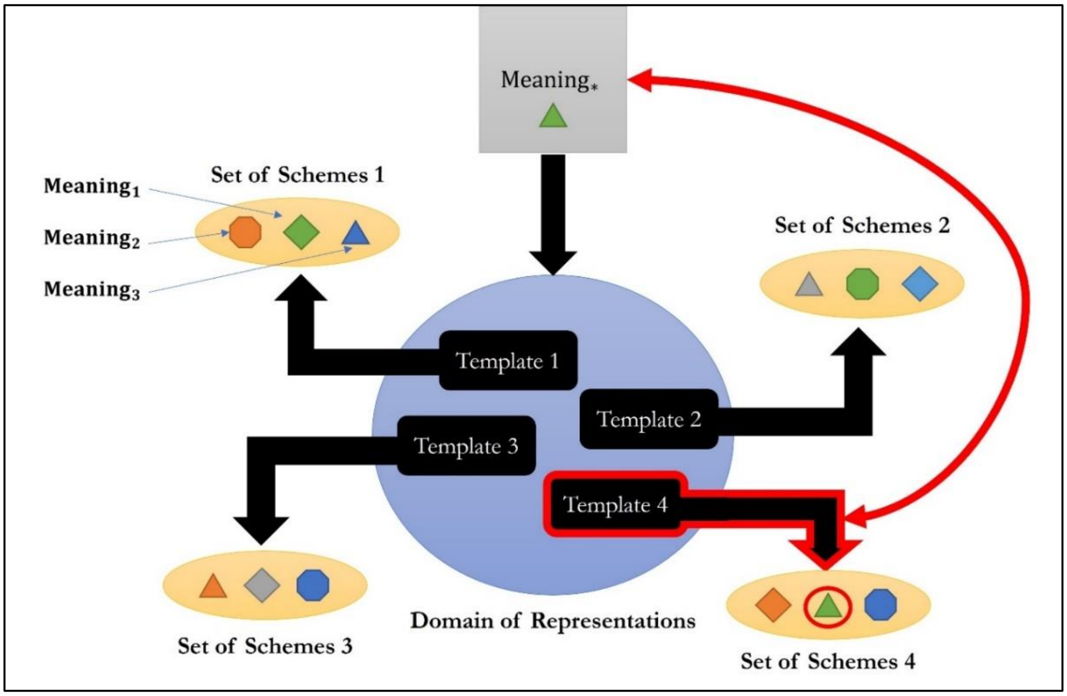
When an individual engages in symbolizing activity, she first compares the meaning she wishes to symbolize with the personal expression templates in her domain of representations (this process is most often completed subconsciously). There are two possible outcomes of this comparative activity. In the first case, the individual might discern a personal expression template whose set of schemes contains a component of her previous experience she believes to be analogous to her current meaning. In this instance, she might construct an instantiation of her identified personal expression template to re-present or convey her meaning. For example, when Sylvia quickly and capably constructed personal expressions for Series A and Series B (which I described in Chapter 7), she assimilated these series to a scheme corresponding to her personal expression template $\square\Sigma_{\square_{\square}}^{\square}\square$. In the second case, the individual might be unable to determine (at least initially) a personal expression template in her domain of representations through which she might re-present or convey her meaning. For example, Sylvia initially struggled to symbolize Series E, and Monica struggled at the end of Day 1 about whether to use summation or integral notation to symbolize a partial sum.

The two outcomes of the comparative symbolizing activity I described in the previous paragraph are analogous to the Piagetian notions of *assimilation* to a scheme and *accommodation*. I use assimilation in the sense of Glasersfeld (1995), who stated that assimilation comprises "treating new material *as an instance of something known*" (Glasersfeld, 1995, p. 62, italics in original). In the first outcome, a student treats her meaning as an instantiation of a previous experience that she has attributed to a personal expression template in her domain of representations. For example, when Sylvia reasoned about Series A, she *assimilated* her experience to a scheme whose space of implications (Thompson et al., 2014) included her personal expression template $\square\Sigma_{\square_{\square}}^{\square}\square$, through which she re-presented her meaning.

In Figure 75, I portray the mental actions Sylvia may have enacted to assimilate her meaning for Series A (i.e., the green triangle, which I call Meaning_*) to her personal expression template $\square\Sigma_{\square_{\square}}^{\square}\square$, (i.e., Template 4) in her domain of representations. The black arrows in Figure 75 denote the mental processes by which Sylvia likely compared her meaning for Series A, Meaning_* to each template in her domain of representations to check whether one of the meanings she previously attributed to each template corresponded with Meaning_* (i.e., her meaning for Series A). The red border around Template 4 indicates Sylvia's successful identification of her personal expression template $\square\Sigma_{\square\square}^{\square\square}\square$ as a medium through which she has re-presented other series she considers analogous to Meaning_* (i.e., Series A). Finally, the bidirectional red arrow connecting Template 4 to the box for Meaning_* denotes the product of the assimilation. The arrow beginning from Template 4 and terminating at Meaning_* represents the Sylvia's ability to re-present Meaning_* (i.e., Series A) through Template 4 (i.e., $\square\Sigma_{\square\square}^{\square\square}\square$). The arrow beginning from Meaning_* and terminating at Template 4 denotes the Sylvia's mapping of Meaning_* to Template 4 to re-present or convey her thinking about other series in her future symbolizing activity. Throughout Sylvia's symbolizing activity for Series A, she exhibited little difficulty constructing or describing her personal expression, which is a common characteristic of *assimilation* to a scheme.

Figure 75

A Cognitive Mapping of Immediate Assimilation in Symbolization

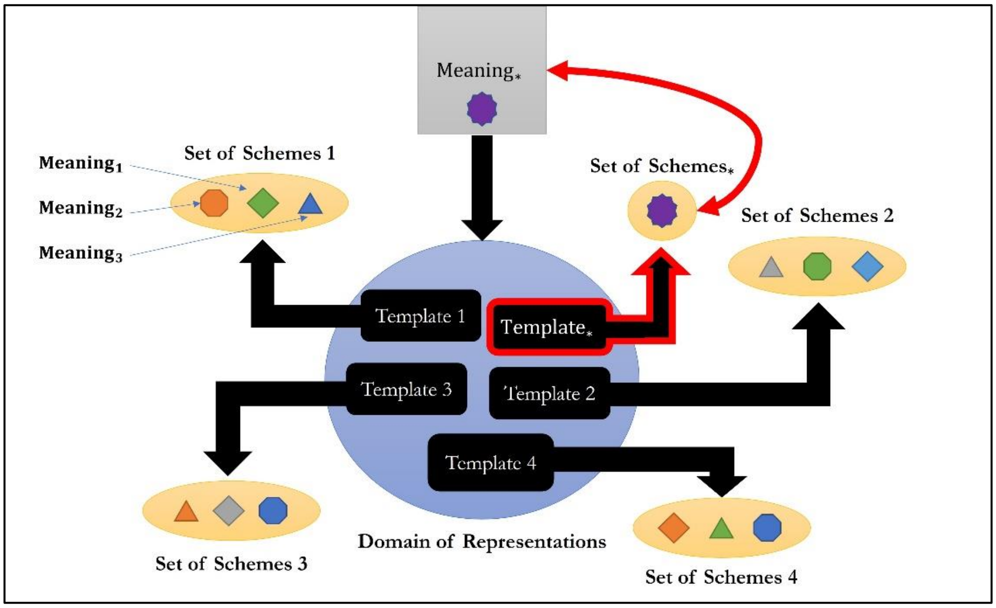


In the second outcome of the meaning-expression comparison activity, the student attempts (but initially fails) to coordinate her meaning with a personal expression template. In this case, the students' recognition that there is no element in her domain of representations through which she can (immediately) re-present her meaning typically induces a state of cognitive conflict, or *perturbation*. To resolve the cognitive conflict, the student might employ one of three tactics: construct a new personal expression template, consciously attribute her meaning to an existing personal expression template, or abandon her attempt to symbolize her idea. Each of these tactics are an instance of the Piagetian notion of *accommodation.* I use the term *accommodation* to refer to an individual's resolution of a perturbation by "modif[ying] or construct[ing] cognitive schemes to accommodate for…unexpected experience[s]" (Tallman, 2015, p. 61). I provide additional details about each accommodation tactic in the paragraphs below.

For the first case of accommodation tactics, a student might construct a new personal expression template or modify an inscription within an existing template to re-present her meaning (see Figure 76). For example, Sylvia constructed symbols such as ?_? and $\frac{?}{\#}$, which she added to the mark sets for various inscriptions of her personal expression template, to re-present additional examples of infinite series she had not previously considered. In Figure 76, I show the mental actions Sylvia may have enacted to introduce her inscription ?_? to re-present Meaning_* (e.g., the random operators of Series E; denoted with a purple star) through her personal expression template $\square\Sigma_{\square\square}^{\square}\square$. In this case, I have used the term Template_* (highlighted in red) to show Sylvia's addition of a new inscription (e.g., ?_?) through which she could re-present Meaning_* (i.e., the randomly generated operators of Series E). The set of schemes corresponding to Template_* is singular, indicating that in the moment of creation, Sylvia could only re-present Meaning_* (i.e., the random operators of Series E) through her expression Template_*.

Figure 76

A Cognitive Mapping of Accommodation by Creating a New Template

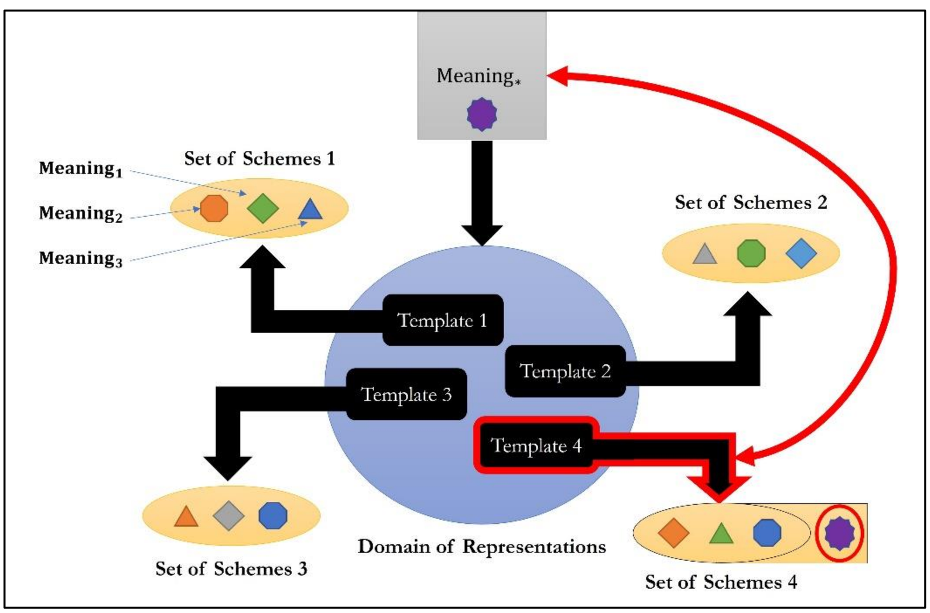


A second symbolizing accommodation tactic a student might employ is consciously attributing her meaning to an existing personal expression template. For example, Monica deliberately chose to attribute her meaning for partial sums to her personal expression template $\int_{\square}^{\square} \square d\square$ during the Day 1 interview, even though she felt a calculator might not return the appropriate answer. Figure 77 shows Monica's likely mental actions, which corresponded with accommodation case (2). In effect, Monica consciously chose to impute Meaning_* (i.e., computing the value of a partial sum) to an already-existing template in her domain of representations (i.e., Template 4, her template for integration). I visually depict Monica's extrapolation of her Template 4 (i.e., $\int_{\square}^{\square} \square d\square$) to include Meaning_* (i.e., the partial sums of Ivy's Series) by appending a rectangular region to Template 4's ovular set of schemes to include the purple star corresponding to Meaning_*.

Monica exhibited less confidence in her symbolization (accommodation case 2) than Sylvia (accommodation case 1), indicating that her imputation of Meaning_* to Template 4 may not have (at least initially) carried the same re-presentational power as the prior meanings she had re-presented through Template 4. After struggling to viably re-present Ivy's Series with her template $\int_{\square}^{\square} \square d\square$ at the beginning of the Day 1 interview, Monica rejected her template $\int_{\square}^{\square} \square d\square$ as a method to re-present computing partial sums. Instead, she introduced a new template, $\Sigma_{\square}^{\square}\square$ (which constituted an instance of accommodation case 1). Monica then used her new personal expression template $\Sigma_{\square}^{\square}\square$ successfully to re-present various partial sums and series for the remainder of the teaching experiment.

Figure 77

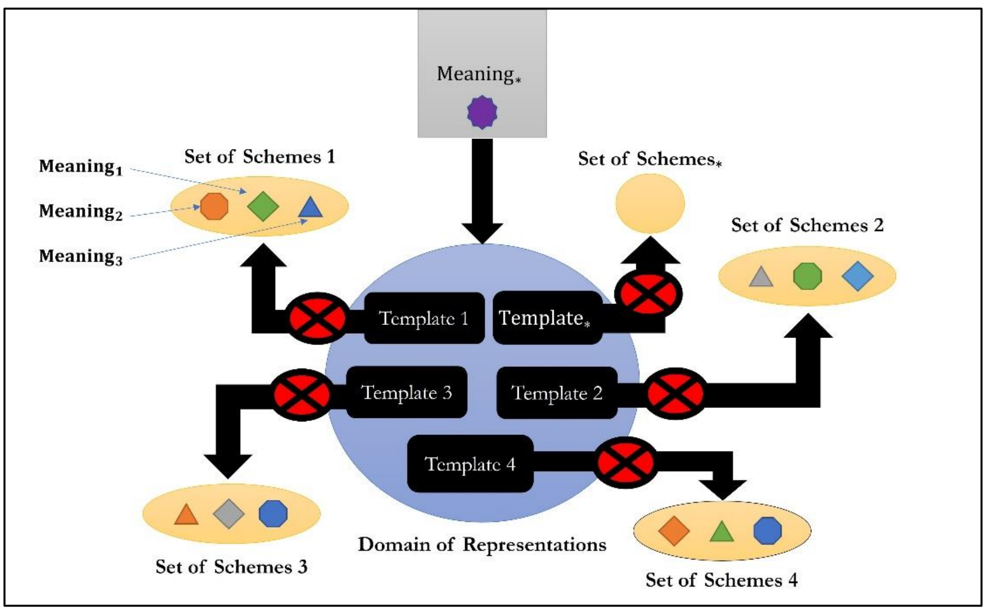
A Cognitive Mapping of Accommodation by Appending Meanings to Templates



A final accommodation tactic a student might employ is to abandon her endeavors to symbolize her meaning through a particular form of expression. For example, I briefly discussed in Chapter 6 how Sylvia was unwilling to construct a personal expression for any component of the ϵ-strip activity until I requested that she create an inscription to re-present the center of the ϵ-strip. If I had not intervened, Sylvia would have left that interview task without a personal expression to re-present her experience in future interview sessions. In Figure 78, I show the mental actions corresponding Sylvia's symbolizing activity, which corresponds to accommodation case (3).

Figure 78

A Cognitive Mapping of Accommodation by Rejecting all Templates



In this figure, I use the black and red symbol ⊗ to denote Sylvia's rejection of each of her personal expression templates as mediums to re-present Meaning$_*$ (e.g., properties of the ϵ-strip). While I do not claim that Sylvia cognitively checked every template in her domain of representations, she likely imagined that if she were to check every template,[33]

[33] I also include Template$_*$ in Figure 78 to highlight that Sylvia might have considered (and subsequently rejected) the idea of creating a new personal expression template by which to re-present Meaning$_*$.

none would be sufficient for her to re-present Meaning_* (i.e., properties of the ϵ-strip). The result of accommodation case (3) was that Sylvia had no readily-available personal expression template by which to re-present her Meaning_* (i.e., properties of an ϵ-strip).

Each case of perturbation resolution I referred to in the previous paragraphs requires a different modification of the elements in the domain of representations and their corresponding sets of schemes. In accommodation case (1), a student constructs a new element in her domain of representations (i.e., a personal expression template) to which she attributes her meaning (which becomes the set of schemes for the new template). I anticipate that a case (1) student could readily utilize her new personal expression to re-present her meaning in future experience. For example, Sylvia employed her new inscription ?_? (and updated template) in later tasks to describe series with randomly generated operators and summand values.

In accommodation case (2), the student identifies an existent element in her domain of representations to which she spontaneously imputes her meaning (thereby creating a new element in the set of schemes for that template). I anticipate that a case (2) student would be capable of re-presenting her new meaning through her expression template in the future. However, her ability to do so may be tenuous or inconsistent due to the existence of other meanings with greater longevity within her set of schemes corresponding to the expression template. For example, Monica used her integral-based personal expression template several times throughout the Day 1 interview to re-present partial sums. However, she eventually rejected her initial template and proposed a Σ-based template instead to re-present the unique components of partial sums that she believed to be incongruous with her meanings for integration.

In accommodation case (3), the student rejects the elements of her domain of representations as potential mediums for re-presenting her meaning and chooses not to create a personal expression template. In such a case, the student's functional accommodation (Steffe & Thompson, 2000) may resolve the perturbation but prove unviable if she encounters a similar (to her) situation during her future symbolizing activity. For example, Sylvia's initial rejection of my invitation to symbolize properties of the ϵ-strips may have allowed her to progress through the end of the Day 5 interview but did not help her to complete the tasks for the Day 6 interview (in which I had her repeatedly reason with and eventually symbolize several components of the ϵ-strips).

I acknowledge that not all accommodations result in valid symbolization (from a mathematically conventional perspective). Still, a student's accommodation results in an "act of learning" that she believes constitutes a viable symbolization (at least in the moment of the accommodation; Glasersfeld, 1995, p. 66).

Teaching Implications of My Explanatory Framework for Students' Symbolizing Activity

Although I do not expect instructors to present my theoretical framing of students' symbolization during their lectures, they can implement the ideas I have discussed into their interactions with their students. I present two such examples in the paragraphs below.

First, instructors who internalize and look for the various types of students' cognitive accommodations I described may gain greater power in orienting students toward appropriate symbolization. For example, instructors might begin to consider a student's inappropriate (in the conventional sense) symbolization as a possible indication that he may be attempting to attribute productive meanings for a topic to a personal expression template he created for a previous idea (i.e., accommodation case 2). In this instance, the instructor could determine the meanings the student intended to convey through his expression and re-orient his (the students') symbolizing activity toward a conventional expression.

Second, instructors might allow individual students the ability (in the context of a classroom discussion) to persist in using a novel inscription or expression to symbolize their thinking (accommodation case 1). For instance, an instructor who allows a student to attempt to symbolize a partial sum or series with integral notation might foster a productive classroom discussion about the similarities between integral and summation notation and why mathematicians. The result of such a discussion would likely be a more explicit understanding (in the student's mind) of the various inscriptions in both conventional expressions and a surer knowledge of how to appropriately symbolize components of infinite series.

Future Research Directions to Investigate Students' Development of Expression Templates

My ultimate goal in conducting this dissertation study was to construct a theoretical framework for categorizing students' attribution of meanings to their personal algebraic representations and the coevolution of these two ideas. In Chapter 6, I presented several constructs to describe the syntactical components of a personal expression template (i.e., *fixed inscription*, *cloze inscription*, *mark set*). In Chapter 8, I introduced another set of constructs to describe the cognitive structures and mechanisms by which individuals maintain and assign meaning to their personal expression templates (i.e., *domain of representations, set of schemes*). Throughout this dissertation, I have also attempted to ground my analysis and results in the Piagetian notions of *assimilation* and *accommodation* and the Glasersfeldian idea of *re-presentation*. As a result, the first research project I would like to pursue is verifying that my constructs can reliably explain individual students' symbolizing activities in other contexts (and other forms of representation). In the algebraic sense, I might study students' construction of personal expressions for ideas conventionally ascribed to fractions or combinatorics counting formulas. In the realm of other forms of symbolization, I might continue my study of students' construction of diagrammatic personal expressions in the context of set-theory and proof (Eckman et al., 2023).

As I stated in my discussion related to my Chapter 6 results, I recognize that I have done very little work related to *communicative* and *conventional* expressions. In my future research, I hope to investigate students' collective creation of expressions to facilitate communication and students' attempts to adopt conventional expressions in the context of direct instruction. I hope that through constructing a grounded theory of individual student symbolization, I can continue to provide explanatory frameworks for students' symbolizing activity across multiple grains of analysis and mathematical contexts.

REFERENCES

Abbott, S. (2015). *Understanding Analysis* (2nd edition). Springer-Verlag New York.

Akgün, L., Işik, C., Tatar, E., Işleyen, T., & Soylu, Y. (2012). Transfer of mathematical knowledge: Series. *Australian Journal of Teacher Education*, *37*(3). https://doi.org/10.14221/ajte.2012v37n3.2

Alcock, L., & Simpson, A. (2002). Definitions: Dealing with Categories Mathematically. *For the Learning of Mathematics*, *22*(2), 28–34.

Alcock, L., & Simpson, A. (2004). Convergence of sequences and series: Interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics*, *57*(1), 1–32. https://doi.org/10.1023/B:EDUC.0000047051.07646.92

Alcock, L., & Simpson, A. (2005). Convergence of sequences and series 2: Interactions between nonvisual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics*, *58*(1), 77–100. https://doi.org/10.1007/s10649-005-2813-5

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, *52*(3), 215–241. https://doi.org/10.1023/A:1024312321077

Austin, J. L., & Howson, A. G. (1979). Language and mathematical education. *Educational Studies in Mathematics*, *10*(2), 161–197.

Azevedo, D. (2021). An alternative integral test for infinite series. *International Journal of Mathematical Education in Science and Technology*, *52*(1), 131–139. https://doi.org/10.1080/0020739X.2020.1730458

Barahmand, A. (2017). The Boundary Between Finite and Infinite States Through the Concept of Limits of Sequences. *International Journal of Science and Mathematics Education*, *15*(3), 569–585. https://doi.org/10.1007/s10763-015-9697-3

Barahmand, A. (2021). Exploring students' consistency in their notions: The case of comparing infinite series. *International Journal of Mathematical Education in Science and Technology*, *52*(7), 1049–1062. https://doi.org/10.1080/0020739X.2020.1736350

Box, G. E. P. (1976). Science and statistics. *Journal of the American Statistical Association*, *71*(356), 791–799.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*(3), 247–285.

Cajori, F. (1993). *A history of mathematical notations* (Vol. 1). Courier Corporation.

Callahan, J., Cox, D., Hoffman, K., O'Shea, D., Pollatsek, H., & Senechal, L. (1995). *Calculus in context: The five college calculus project*. W.H Freeman and Company.

Castillo-Garsow, C. (2010). *Teaching the Verhulst model: A teaching experiment in covariational reasoning and exponential Growth* [Unpublished doctoral dissertation]. Arizona State University.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 547–589). Lawrence Erlbaum.

Cornu, B. (1991). Limits. In D. O. Tall (Ed.), *Advanced Mathematical Thinking* (Vol. 11, pp. 153–166). Springer Science & Business Media.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the Limit Concept: Beginning with a Coordinated Process Schema. *Journal of Mathematical Behavior*, *15*, 167–192.

Coyne, I. T. (1997). Sampling in qualitative research. Purposeful and theoretical sampling; merging or clear boundaries? *Journal of Advanced Nursing*, *26*(3), 623–630. https://doi.org/10.1046/j.1365-2648.1997.t01-25-00999.x

David, E. J., Roh, K. H., & Sellers, M. E. (2019). Value-thinking and location-thinking: Two ways students visualize points and think about graphs. *The Journal of Mathematical Behavior*, *54*, 1–18. https://doi.org/10.1016/j.jmathb.2018.09.004

Dawkins, P. C., & Weber, K. (2017). Values and norms of proof for mathematicians and students. *Educational Studies in Mathematics*, *95*(2), 123–142.

Dawkins, P. C., & Zazkis, D. (2021). Using Moment-by-Moment Reading Protocols to Understand Students' Processes of Reading Mathematical Proof. *Journal for Research in Mathematics Education*, *52*(5), 510–538. https://doi.org/10.5951/jresematheduc-2020-0151

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 95–123). Kluwer. https://doi.org/10.1007/0-306-47203-1_7

Duval, R. (1999). Representation, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 3–27). Universidad Autónoma del Estado de Morelos.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, *61*(1–2), 103–131. https://doi.org/10.1007/s10649-006-0400-z

Eckman, D. C., & Roh, K. H. (2022a). A symbolizing activity for constructing personal expressions and its impact on a student's understanding of the sequence of partial sums. In S. S. Karunakaran & A. Higgins (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 168–176).

Eckman, D. C., & Roh, K. H. (2022b). Students' intuitive meanings for infinite series convergence and corresponding implications. In A. E. Lischka, E. B. Dyer, R. S. Jones, J. N. Lovett, J. Strayer, & S. Drown (Eds.), *Proceedings for the 44th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1014–1023).

Eckman, D. C., & Roh, K. H. (under review). *A symbolizing activity for constructing personal expressions and its impact on a student's understanding of the sequence of partial sums*. Manuscript under review.

Eckman, D. C., & Roh, K. H. (in revision). *Fostering students' development of productive representation systems for infinite series*. Manuscript in revision.

Eckman, D. C., Roh, K. H., Dawkins, P. C., Ruiz, S., & Tucci, A. (2023). Two vignettes on students' symbolizing activity for set relationships. *26th Annual Conference on Research in Undergraduate Mathematics Education*.

Font, V., Godino, J. D., & D'Amore, B. (2007). An onto-semiotic approach to representations in mathematics education. *For the Learning of Mathematics*, *27*(2), 2–14.

Font, V., Godino, J. D., & Gallardo, J. (2013). The emergence of objects from mathematical practices. *Educational Studies in Mathematics*, *82*(1), 97–124. https://doi.org/10.1007/s10649-012-9411-0

Freudenthal, H. (1973). *Mathematics as an educational task*. D. Reidel Publishing Company.

Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Falmer Press.

Godino, J. D., Batanero, C., & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM*, *39*(1–2), 127–135. https://doi.org/10.1007/s11858-006-0004-1

Godino, J. D., & Font, V. (2010). The theory of representations as viewed from the onto-semiotic approach to mathematics education. *Mediterranean Journal for Research in Mathematics Education*, *9*(1), 189–210.

Goldin, G. (2008). Perspectives on representation in mathematical learning and problem solving. In L. D. English (Ed.), *Handbook of International Research in Mathematics Education* (pp. 176–201). Routledge.

González-Martín, A. S., Nardi, E., & Biza, I. (2011). Conceptually driven and visually rich tasks in texts and teaching practice: The case of infinite series. *International Journal of Mathematical Education in Science and Technology*, *42*(5), 565–589. https://doi.org/10.1080/0020739X.2011.562310

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A “proceptual” view of simple arithmetic. *Journal for Research in Mathematics Education*, *25*(2), 116–140. https://doi.org/10.2307/749505

Halani, A., Davis, O., & Roh, K. H. (2013). Critiquing the reasoning of others: Devil’s advocate and peer interpretations as instructional interventions. In S. Brown, G. Karakok, K. H. Roh, & M. Oehrtman (Eds.), *Proceedings of the 16th Annual Conference on Research in Undergraduate Mathematics Education* (Vol. 1, pp. 285–295).

Halliday, M. A. K. (1975). Some aspects of sociolinguistics. *Interactions between Linguistics and Mathematics Education*, 64–73.

Inglis, M., & Alcock, L. (2012). Expert and Novice Approaches to Reading Mathematical Proofs. *Journal for Research in Mathematics Education*, *43*(4), 358–390. https://doi.org/10.5951/jresematheduc.43.4.0358

Iori, M. (2017). Objects, signs, and representations in the semio-cognitive analysis of the processes involved in teaching and learning mathematics: A Duvalian perspective. *Educational Studies in Mathematics*, *94*(3), 275–291. https://doi.org/10.1007/s10649-016-9726-3

Johnckheere, A., Mandelbrot, B. B., & Piaget, J. (1958). *La lecture de l’expérience [Observation and decoding of reality]*. P.U.F.

Kar, T., Işık, A., Öçal, M. F., Çiltaş, A., Güler, G., & Işık, C. (2011). Prospective mathematics teachers abilities’ to construct relations between the different representations of series with complex terms. *Procedia - Social and Behavioral Sciences*, *15*, 356–360. https://doi.org/10.1016/j.sbspro.2011.03.101

Katz, L. (1986). Improvements to the sigma summation notation. *International Journal of Mathematical Education in Science and Technology*, *17*(3), 381–382. https://doi.org/10.1080/0020739860170314

Kidron, I. (2002). Concept definition, concept image, and the notion of infinite sum in old and new environments. In A. D. Cokburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 209–216). PME.

Kidron, I., & Tall, D. (2015). The roles of visualization and symbolism in the potential and actual infinity of the limit process. *Educational Studies in Mathematics*, *88*, 183–199.

Kidron, I., & Vinner, S. (1983). Rational numbers and decimal at the senior high level—Density and comparison. In R. Hershkowitz (Ed.), *Proceedings of hte 7th International Conference for the Psychology of Mathematics Education* (pp. 301–306).

Lakoff, G., & Núñez, R. (2000). *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics Into Being*. Basic Books.

Lannin, J. K., Barker, D. D., & Townsend, B. E. (2006). Recursive and explicit rules: How can we build student algebraic understanding? *The Journal of Mathematical Behavior*, *25*(4), 299–317. https://doi.org/10.1016/j.jmathb.2006.11.004

Larson, R., & Edwards, B. (2015). *Calculus of a Single Variable: Early Transcendental Functions* (6th ed.). Cengage Learning.

Lavrakas, P. J. (2008). *Encyclopedia of survey research methods*. SAGE Publications.

Lindaman, B., & Gay, A. S. (2012). Improving the instruction of infinite series. *Primus : Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *22*(5), 373–385.

Martin, J. (2013). Differences between experts' and students' conceptual images of the mathematical structure of Taylor series convergence. *Educational Studies in Mathematics*, *82*(2), 267–283. https://doi.org/10.1007/s10649-012-9425-7

Martin, J., Oehrtman, M., Roh, K. H., Swinyard, C., & Hart-Weber, C. (2011). Students' reinvention of formal definitions of series and pointwise convergence. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings for the 14th Annual Conference on Research in Undergraduate Mathematics Education* (Vol. 3, pp. 114–119).

Martínez-Planell, R., Gonzalez, A. C., DiCristina, G., & Acevedo, V. (2012). Students' conception of infinite series. *Educational Studies in Mathematics*, *81*(2), 235–249. https://doi.org/10.1007/s10649-012-9401-2

Mazur, J. (2014). *Enlightening symbols: A short history of mathematical notation and its hidden powers*. Princeton University Press.

McDonald, M. A., Mathews, D. M., & Strobel, K. H. (2000). Understanding sequences: A tale of two objects. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education IV* (pp. 77–102). American Mathematical Society.

Moore, K. C. (2010). *The role of quantitative reasoning in precalculus students learning central concepts of trigonometry* [Unpublished doctoral dissertation]. Arizona State University.

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education*, 782–789. http://pat-thompson.net/PDFversions/2015MooreShapeThinking.pdf

Musgrave, S., & Thompson, P. W. (2014). Function notation as an idiom. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allen (Eds.), *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education* (Vol. 4, pp. 281–288). PME.

O'Bryan, A. (2020). Quantitative reasoning and symbolization activity: Do individuals expect calculations and expressions to have quantitative significance? *Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education,* 449–457.

O'Bryan, A., & Carlson, M. (2016). Fostering teacher change through increased noticing: Creating authentic opportunities for teachers to reflect on student thinking. *Proceedings of the Nineteenth Annual Conference on Research in Undergraduate Mathematics Education*, 1192–1200.

Oehrtman, M., Swinyard, C., & Martin, J. (2014). Problems and solutions in students' reinvention of a definition for sequence convergence. *The Journal of Mathematical Behavior, 33*, 131–148. https://doi.org/10.1016/j.jmathb.2013.11.006

Patton, M. (2002). *Qualitative research & evaluation methods*. Sage Publications.

Piaget, J. (1968). *Six psychological studies*. Vintage Books.

Piaget, J. (2001). *Studies in reflecting abstraction* (R. Campbell (trans.)). Psychology Press.

Pimm, D. (2019). *Routledge revivals: Speaking mathematically (1987): Communication in mathematics classrooms* (Vol. 4). Routledge.

Presmeg, N., Radford, L., Roth, W.-M., & Kadunz, G. (2016). *Semiotics in mathematics education*. Springer International Publishing. https://doi.org/10.1007/978-3-319-31370-2

Przenioslo, M. (2006). Conceptions of a sequence formed in secondary schools. *International Journal of Mathematical Education in Science and Technology*, *37*(7), 805–823. https://doi.org/10.1080/00207390600733832

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics*, *42*(3), 237–268.

Radford, L. (2006). Algebraic thinking and generalization of patterns: A semiotic perspective. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, North American Chapter*, *1*, 2–21.

Radford, L. (2008). The ethics of being and knowing: Towards a cultural theory of learning. *Semiotics in Mathematics Education: Epistemology, History, Classroom, and Culture*, 215–234.

Radford, L. (2013). Three key concepts of the theory of objectification: Knowledge, knowing, and learning. *Journal of Research in Mathematics Education*, *1*, 7–44. https://doi.org/10.4471/redimat.2013.19

Roh, K. H. (2005). *College students' intuitive understanding of the limit of a sequence and their levels of reverse thinking* [Unpublished doctoral dissertation]. The Ohio State University.

Roh, K. H. (2008). Students' images and their understanding of definitions of the limit of a sequence. *Educational Studies in Mathematics*, *69*(3), 217–233. https://doi.org/10.1007/s10649-008-9128-2

Roh, K. H. (2010a). An empirical study of students' understanding of a logical structure in the definition of limit via the ε-strip activity. *Educational Studies in Mathematics*, *73*(3), 263–279. https://doi.org/10.1007/s10649-009-9210-4

Roh, K. H. (2010b). How to help students conceptualize the rigorous definition of the limit of a sequence. *PRIMUS*, *20*, 473–287.

Schatzman, L., & Strauss, A. L. (1973). *Field research: Strategies for a natural sociology*. Prentice Hall.

Sellers, M. (2020). *Students' quantifications,interpretations, and negations of complex mathematical statements from calculus* [Unpublished doctoral dissertation]. Arizona State University.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1–36. https://doi.org/10.1007/BF00302715

Shepherd, M. D., & van de Sande, C. C. (2014). Reading mathematics for understanding—From novice to expert. *The Journal of Mathematical Behavior, 35*, 74–86. https://doi.org/10.1016/j.jmathb.2014.06.003

Sierpińska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics, 18*(4), 371–397. https://doi.org/10.1007/BF00240986

Sierpinska, A. (1992). On understanding the notion of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 25–58). Mathematical Association of America.

Sipes, J. G. (2019). *Representations of systems (of equations)* [Unpublished doctoral dissertation]. Arizona State University.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–307). Lawrence Erlbaum.

Stewart, J. (2012). *Calculus: Early transcendentals* (8th ed.). Cengage Learning.

Strand, S., & Larsen, S. (2013). On the plus side: A cognitive model of summation notation. In S. Brown, G. Karakok, K. H. Roh, & Oehrtman, Michael (Eds.), *Proceedings of the 16th Annual Conferene on Research in Undergraduate Mathematics Education* (Vol. 2, pp. 254–260).

Strand, S., Zazkis, D., & Redmond, S. (2012). Summing up students' understandings of sigma notation. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education* (Vol. 2, pp. 551–555).

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Sage Publications.

Swinyard, C., & Larsen, S. (2012). Coming to understand the formal definition of limit: Insights gained From engaging students in reinvention. *Journal for Research in Mathematics Education, 43*(4), 465–493. https://doi.org/10.5951/jresematheduc.43.4.0465

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169. https://doi.org/10.1007/BF00305619

Tallman, M. A. (2015). *An examination of the effect of a secondary teacher's image of instructional constraints on his enacted subject matter knowledge* [Doctoral dissertation]. Arizona State University.

Taylor, W. L. (1953). "Cloze procedure": A new tool for measuring readability. *Journalism Quarterly*, *30*(4), 415–433.

Thompson, P. (1994). The development of the concept of speed and its relationship to concepts of rate. In *The development of multiplicative reasoning in the learning of mathematics* (in G. Harel&J. Confrey (Eds.), pp. 179–234). SUNY Press.

Thompson, P. (2013). In the absence of meaning…. In K. R. Leatham (Ed.), *Vital Directions for Mathematics Education Research* (pp. 57–93). Springer New York. https://doi.org/10.1007/978-1-4614-6977-3_4

Thompson, P., Ashbrook, M., & Milner, F. (2019). *Calculus: Newton, Leibniz, and Robinson meet technology*. Retrieved from http://patthompson.net/ThompsonCalc/.

Thompson, P., & Carlson, M. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In *Compendium for Research in Mathematics Education* (pp. 421–456). National Council for Teachers of Mathematics. http://pat-thompson.net/PDFversions/2016ThompsonCarlsonCovariation.pdf

Thompson, P., Carlson, M., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In L. P. Steffe, K. C. Moore, L. L. Hatfield, & S. Belbase (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing* (pp. 1–24). University of Wyoming. http://pat-thompson.net/PDFversions/2013MagsInAlg.pdf

Thompson, P. W. (2002). Didactic objects and didactic models in radical constructivism. In K. Gravemeijer, R. Lehrer, B. Van Oers, & L. Verschaffel (Eds.), *Symbolizing, Modeling and Tool Use in Mathematics Education* (pp. 197–220). Springer Netherlands. https://doi.org/10.1007/978-94-017-3194-2_12

Tillema, E. (2007). *Students' construction of algebraic symbol systems* [Unpublished doctoral dissertation, University of Georgia]. http://athenaeum.libs.uga.edu/handle/10724/24302

Tillema, E. (2010). Functions of symbolizing activity: A discussion. *For the Learning of Mathematics*, *30*(1), 2–8.

Vergnaud, G. (1998). A comprehensive theory of representation for mathematics education. *The Journal of Mathematical Behavior*, *17*(2), 167–181. https://doi.org/10.1016/S0364-0213(99)80057-3

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development*, *52*(2), 83–94. https://doi.org/10.1159/000202727

Williams, S. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education*, *22*(3), 219–236.

Zandieh, M., Wawro, M., & Rasmussen, C. (2017). An example of inquiry in linear algebra: The roles of symbolizing and brokering. *PRIMUS*, *27*(1), 96–124. https://doi.org/10.1080/10511970.2016.1199618

Zazkis, R., & Hazzan, O. (1998). Interviewing in mathematics education research: Choosing the questions. *The Journal of Mathematical Behavior*, *17*(4), 429–439. https://doi.org/10.1016/S0732-3123(99)00006-1

APPENDIX A

CONSENT/ASSENT FORMS

Invitation to Participate in Full Set of Interviews

[Student],

Thank you for your participation in an initial interview for Derek Eckman's dissertation study. At this time, **we invite you to participate in a series of approximately seven (7) weekly research interviews** as part of Derek Eckman's dissertation study. Please review the following information before committing to the interview sequence:

1) The weekly research interview will take place at a **regularly scheduled time,** and each interview will take **approximately 90 minutes**. Please fill out the WhenIsGood poll below to indicate the times at which you are available.

Link: http://whenisgood.net/dedissertation

2) You will be **compensated for each interview with a $20 Amazon gift card**. If you requested a single payment at the end of the interview cycle, you would receive a single lump-sum gift card ($140 for seven interviews) after the study. **If you withdraw from the study you will only receive gift cards for the interviews that you complete.**

3) The weekly interviews will continue through the **remainder of the semester**. The final interview will likely take place during or after finals week. We want to schedule the last interview at a time convenient to you after you have completed your final exams.

Please reply to this email to confirm whether or not you can **participate in further research interviews.**

Respectfully,

Derek Eckman, M.A.
Ph.D. Candidate
School of Mathematical and Statistical Sciences
Arizona State University

Consent Form-Interview Participants

Infinite Series: Convergence, Properties, and Relationships

Dear Student Volunteer:

I am a graduate student in the Mathematics Education Ph.D. program housed in the School of Mathematical and Statistical Sciences, College of Liberal Arts and Sciences at Arizona State University. I am conducting a research study to explore college students' reasoning about infinite series through task-based interviews.

I am inviting your participation, which will consist of a series of 90-minute interview outside of class meetings over Zoom.

Your participation in this study will allow the mathematics education community to identify necessary changes to develop a more effective curriculum and instruction on infinite series for future students. There are no foreseeable risks or discomforts to your participation.

There will be compensations for you when you want to participate in this study: You will be given a $20 Amazon gift card for each interview that you complete during this study.

The interview will be held remotely via a zoom meeting. For research purposes, the interview will be audio- and video-recorded and your written work during the interviews will be photo-copied. The data collected from you will be retained at the principal investigator's office and then be destroyed when research related to this study ends (no later than 10 years from when the data were collected.)

The responses to your interview will be kept strictly confidential. Your name will not be used in any description or publication of this research. Instead of your real name, a pseudonym might be used in all professional presentations and written papers related to this research.

Your participation in this study is voluntary. You must be 18 or older to participate in the study. The interviews will take place outside of your classes and the results of the interview will not influence your grades. You have the right not to answer any question, and to stop your participation at any time. If you choose not to participate or to withdraw from the study at any time, there will be no penalty, and it will not affect your grade.

If you have any questions concerning the research study, please contact me at: dceckman@asu.edu. If you have any questions about your rights as a subject/participant in this research, or if you feel you have been placed at risk, you can contact the Chair of the Human Subjects Institutional Review Board, through the ASU Office of Research Integrity and Assurance, at (480) 965-6788. Please let me know if you wish to be part of the study.

Respectfully,

Derek Eckman, MA
ECA 302
School of Mathematical and Statistical Sciences
Arizona State University
dceckman@asu.edu

Consent for videotaping in this research

I acknowledge that this study involves videotaping during individual interviews, and I grant permission to videotape during my interviews. I sign it freely and voluntarily.

Date:______________________________ Signed:______________________________
(*Principal Investigator*)

Name:______________________________ Signed:______________________________
(*Participant*) (*Participant*)

Form to Check Out iPad and Apple Pencil for Study Participants

Dear ______________________________:

Thank you for your willingness to serve as a research participant for Derek Eckman's dissertation study. To participate in the interviews, you will need to check out an iPad and Apple Pencil from Dr. Kyeong Hah Roh. The check-out process includes 1) filling out a consent form to participate in the research interviews, 2) filling out this equipment loan form to record your intent to borrow and be responsible for the iPad and Apple Pencil, and 3) an in-person meeting with Dr. Kyeong Hah Roh in her campus office (Wexler 737) to pick up and prepare your iPad.

Timeframe for Loan of the iPad and Apple Pencil:
You will be able to use this iPad as if it were your device for the duration of the **Fall 2021** semester. However, you must return the iPad before **December 15, 2021**, to Derek Eckman or Dr. Kyeong Hah Roh, or you will be charged for the cost of purchasing a new iPad.

The device will be reset to the factory default settings on or after **January 1, 2021**. At that time, all information saved to the iPad that is not backed up to another account or device will be permanently deleted. The research team reserves the right to, at its discretion, determine whether sufficient damages (physical or functional) have occurred to the iPad to warrant financial charges to the student.

Please note that any final financial compensation for participating in research interviews will not be processed until the iPad is returned or financial charges are paid.

Necessary Steps to set up the iPad for interviews:
When you set up an appointment with Derek Eckman to check out the iPad, you will be asked to do the following:
Before your appointment with Dr. Roh--

1) Set up a 60-minute meeting with Dr. Roh in her in Wexler 737 (7th floor) via email (khroh@asu.edu)
2) Set up an Apple ID if you do not have one already (student checking out the iPad)
3) Fill out the consent form to participate in research interviews and this form via DocuSign

At your appointment with Dr. Roh—

1) Dr. Roh will confirm that you have filled out the appropriate forms
2) Sign into the iPad with your ASURITE id and set up the iPad with the student's Apple ID
3) Download the Zoom and OneNote apps for research interviews if necessary

4) Test both apps and the Apple Pencil to ensure that everything connects and works correctly

**Due to coronavirus restrictions and common courtesies, please wear a mask and practice proper sanitation before, during, and after the appointment*

Recommendations for the care of the iPad and Apple Pencil:
Please keep the following in mind as you use the iPad this semester:

- We are not providing you a case to protect your iPad. Please do not leave the iPad in a place where it might be easily dropped or damaged. We need iPads to conduct our research, and you will still be charged if the iPad is broken accidentally.
- We are not providing you a convenient place to keep your Apple Pencil. We encourage you to determine a safe place to keep your Apple Pencil so that you do not lose it. Apple Pencils currently retail for $100, and we will charge you if you lose the pencil.
- Be smart with the content you install, download, or stream on this iPad. This iPad is a school-issued device, so you will not enjoy the same privacy that you would on a personal device. A member of the research team will handle this iPad after you return it to check for functionality issues and perform a hard-reset of the device.
- We strongly recommend that you keep the iPad and Apple Pencil in their boxes when they are not being used and that you secure the devices when you are not using them or transporting them. You will still have to pay for a stolen iPad and Apple Pencil.

Questions:
If you have any questions about this form, please reach out to Dr. Kyeong Hah Roh (khroh@asu.edu) or Derek Eckman (dceckman@asu.edu). We will be able to address specific questions about your use of the iPad during this semester. If you have questions regarding the administration of this equipment loan, student charges for lost or damaged items, or complicated
technological issues, please contact Renate Mittelman (renate@asu.edu) or Luis Gutierrez (lfgutier@mainex1.asu.edu).

My signature below attests that I have read this document and agree to the terms therein.

______________________	______________________	__________
Student Name	Student Signature	Date

______________________	______________________
Principal Investigator Name	Principal Investigator Signature

APPENDIX B

SCREENING SURVEY RECRUITMENT AND ITEMS

Recruitment E-mail Sent to Second-semester Calculus Students

Title: Participants Needed for Student Meanings for Infinite Series Study

Body:

Second-semester calculus students:

My name is Derek Eckman, and I am a Ph.D. candidate in mathematics education under the direction of Dr. Kyeong Hah Roh. For my dissertation study, I plan to conduct a set of individual interviews with a small number of Calculus II students to explore (1) their meanings for infinite sequences and series and (2) their use of various mathematical expressions to describe sequences and series. I anticipate that the students who participate in this study will participate in eight (8) weekly 90-minute interviews, for which they will be compensated with a $20 Amazon gift card per interview.

To determine the participants of this study, I am inviting any student interested in participating in the study to complete a brief survey at the following Survey Link. The survey consists of questions about your mathematical background and interpretations for certain mathematical topics and should take no more than 30 minutes to complete. **Please complete the survey by Wednesday, September 29, to be considered for participation in the study**. All students who complete the survey and are not selected to participate in further interviews will be entered into a raffle to win one of two $20 Amazon gift cards.

If you are selected to participate in a set of interviews, the interviews will take place remotely over Zoom at a regularly scheduled weekly time. Student participants will also sign a consent form to participate in the interviews (see attached) which provides more detailed information on the background of this study.

If you have any questions about the nature or structure of this study, please reach out to me via email (dceckman@asu.edu).

Thank you for your time and I look forward to your participation in my dissertation study.

Derek Eckman
Ph.D. candidate
Mathematics Education
Arizona State University

Screening Survey Items

Consent Form

Welcome to the symbolizing infinite series screening survey!

My name is Derek Eckman, and I am a Ph.D. candidate in mathematics education at Arizona State University. I will conduct a study during the Fall 2021 semester to investigate students' thinking about infinite series and how they utilize symbols to represent their thinking. The purpose of this screening survey is to determine a small number of students to participate in a series of one-on-one interviews.

To participate in this survey, you must be at least 18 years old. The survey should take you up to 30 minutes to complete. If you successfully complete this survey, you will be entered into a raffle to win one of two $20 Amazon gift cards. The raffle winners will be contacted by email to claim their gift cards.

You may be contacted within one to two weeks after completing this survey to participate in a follow-up interview to further refine the candidate pool for this study. For that purpose, we need to collect your name and contact information at the beginning of the survey.

All of your personal information will be deleted from the survey data at the conclusion of this study, which will be no more than three years from now. Please be assured that your responses will be kept completely confidential. Your responses to the survey will be saved on a password-protected and secured server.

Your participation in this research is voluntary. You have the right to withdraw at any point during the study, for any reason, and without any prejudice. If you would like discuss this survey or my research before completing the survey, please email me at dceckman@asu.edu or my advisor, Dr. Kyeong-Hah Roh, at khroh@asu.edu.

By clicking the button below, you acknowledge that:

(1) Your participation in the study is voluntary
(2) You are at least 18 years of age
(3) You are aware that you may choose to terminate your participation in the study at any time and for any reason
(4) You are willing to provide your name and contact information so

that the researchers may follow up with a request for further interviews or provide compensation if you win the gift card raffle

Please note that while you may complete this survey on a mobile device, the questions will be best displayed on a laptop or desktop computer. Some features may be less compatible for use on a mobile device.

This study has been reviewed and approved by the Arizona State University Institutional Review Board. If you have any questions about your rights as a subject/participant in this research, or if you feel you have been placed at risk, you can contact the Institutional Review Board, through the ASU Office of Research Integrity and Assurance, (480) 965-6788.

Thank you for your time in participating in this survey.

Derek Eckman

○ I consent, begin the survey

What is your name?

What is your ASU email address?

Which university campus do you attend?

- ASU Tempe Campus
- ASU West Campus
- ASU Downtown Phoenix Campus
- ASU Polytechnic Campus
- ASU at Lake Havasu
- Other (please explain)

What is your current major?

- Mathematics
- Mathematics Education

- ○ Natural/Life Sciences (biology, chemistry, etc.)
- ○ Physics
- ○ Engineering
- ○ Computer science, programing, networking, or a related field
- ○ Undecided
- ○ ______ Other

Select all of the following mathematics courses that you have taken in previous semesters.

Unless otherwise noted, only indicate courses that you have taken at a university (i.e., Arizona State University).

- ☐ AP Calculus AB in high school
- ☐ AP Calculus BC in high school
- ☐ Multi-variable Calculus (i.e., Calculus 3) in high school
- ☐ Differential Equations in high school
- ☐ College Algebra (e.g., MAT 117)
- ☐ Precalculus (e.g., MAT 170)
- ☐ Calculus 1 (e.g., MAT 265, MAT 270)
- ☐ Calculus 2 (e.g., MAT 266, MAT 271)
- ☐ Calculus 3 (e.g., MAT 267, MAT 272)
- ☐ Differential Equations (e.g., MAT 275)

You indicated that you took AP Calculus AB in high school. Please select one of the following.

- I took the AP Calculus AB exam and passed
- I took the AP Calculus AB exam and did not pass
- I did not take the AP Calculus AB exam

You indicated that you took AP Calculus BC in high school. Please select one of the following.

- I took the AP Calculus BC exam and passed
- I took the AP Calculus BC exam and did not pass
- I did not take the AP Calculus BC exam

Select the mathematics courses that you are currently enrolled in during the Fall 2021 semester.

- College Algebra (e.g., MAT 117)
- Precalculus (e.g., MAT 170)
- Calculus 1 (e.g., MAT 265, MAT 270)
- Calculus 2 (e.g., MAT 266, MAT 271)
- Calculus 3 (e.g., MAT 267, MAT 272)
- Differential Equations (e.g., MAT 275)

You are about to begin the screening survey. This survey will consist of three questions, some of which have multiple parts. As you complete the survey, take your time and describe your thinking with as much detail as possible.

Be aware that because Qualtrics does not have a math equation editor, many of the mathematical content (e.g., equations, graphs) have been included as pictures. If you are on a mobile device, some of the pictures may not display optimally.

Finally, you will not be able to go back and change your answers to any questions. Thank you again for your participation in the survey.

Select "sometimes" to begin the survey.

- ○ Never
- ○ Sometimes
- ○ About half the time
- ○ Most of the time
- ○ Always

Block 3

Question 1:

Determine the appropriate relationship between the expressions below. Use the inscription > to represent *greater than*, the inscription < to represent "less than," or the inscription = to represent "equal."

$$\frac{1}{17}+\frac{1}{33}+\frac{1}{40}+\frac{1}{89} \; \square \; \frac{1}{20}+\frac{1}{37}+\frac{1}{48}+\frac{1}{91}$$

- ○ >
- ○ =
- ○ <

Explain your reasoning for the answer that you chose. Provide as much detail as possible and use complete sentences.

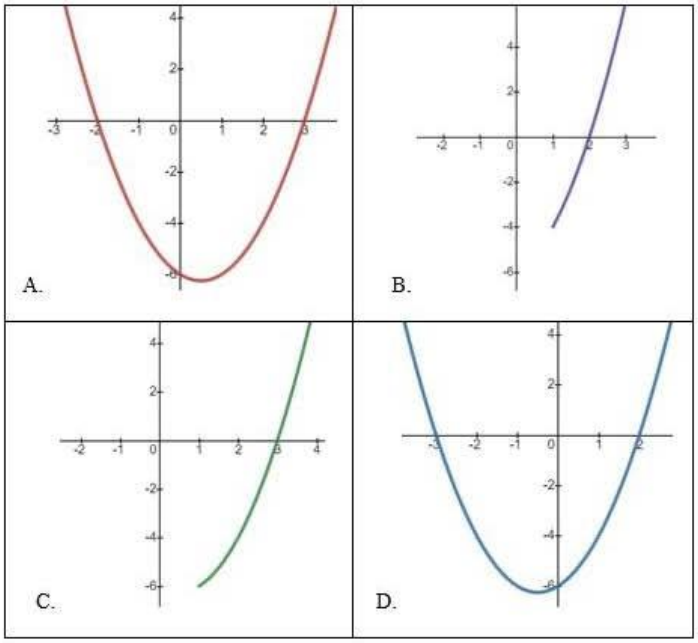
On the previous question, you chose the inscription ${q://QID85/ChoiceGroup/SelectedChoices} to describe the relationship between the fractions. In the box below, please answer the following two questions.

1) Describe what the inscription ${q://QID85/ChoiceGroup/SelectedChoices} means to you.

2) Give an example of how you might use the inscription ${q://QID85/ChoiceGroup/SelectedChoices} to describe a situation.

Question 2, part A

Which of the following graphs represents the function $f(x) = (x - 3)(x + 2)$ with domain $[1, \infty)$?



- Graph A
- Graph B
- Graph C
- Graph D
- We cannot determine which graph corresponds to the function

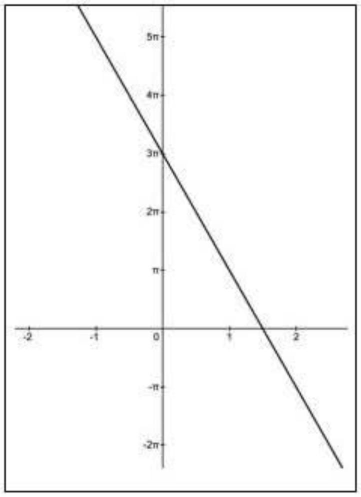
Explain your reasoning for the answer that you chose. Provide as much detail as possible and use complete sentences.

Suppose a student asks you to help them determine the value of the function $f(x) = (x - 3)(x + 2)$ when $x = 5$. Give a hypothetical explanation to the student for how they should determine this value of the function f at $x = 5$. Provide as much detail as possible and use complete sentences.

Question 2, part B

Which of the algebraic expressions could be used to generate the following graph?

(Assume the domain of each function is all real numbers)



A. $g(x) = -2\pi x + 3\pi$

B. $g(x) = -2x + 3\pi$

C. $g(x) = -2\pi x + 1.5$

D. $g(x) = -2x + 1.5$

- Expression A
- Expression B
- Expression C
- Expression D
- We cannot determine which expression corresponds to the graph

Explain your reasoning for the answer that you chose. Provide as much detail as possible and use complete sentences.

Suppose a student tells you they want to find the value of x needed to complete the ordered pair $(x, 3.27\pi)$ such that the resulting point falls on the graph of the function g. Give a hypothetical explanation to the student for how they should determine an appropriate value for x. Provide as much detail as possible and use complete sentences.

Question 3

Consider the following mathematical statement.

The series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$ converges to 2.

Choose the most appropriate response about the statement above.

- ○ The statement is true
- ○ The statement is false
- ○ We cannot determine if the statement is true or false

Explain your reasoning for the answer that you chose. Provide as much detail as possible and use complete sentences.

What does the phrase "a series converges" mean to you? Explain your meaning with as much detail as possible, using complete sentences.

Complete the following statement in the text box below. Feel free to provide examples if this helps you to better answer this question.

A series converges if

________________________________.

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APPENDIX C

INTERVIEW PROTOCOLS FOR ALL INTERVIEWS

Intake Interview Protocol

Intake Interview: Meanings for series convergence

Overview:

The purpose of this interview is to:
(1) Determine students' intuitive meanings for infinite series and their convergence;
(2) Introduce students to Abigail's series

To-do list before interview:

- Set up OneNote file for student
- Send link to OneNote file to student
- Consent form
- Have student's ASU email address handy for sending Amazon gift card

Task 0: Introduction and student information (~5 minutes)

Interviewer: Welcome to the interview, I am happy to see you and appreciate your willingness to participate in my research. The purpose of this interview is to investigate how you think about infinite series. Throughout the interview, I would encourage you to think out loud and describe what is going on inside your head as you work.

Before we begin, could you tell me a little about your background in mathematics and your current program of study at Arizona State University?

Task 1: Reasoning about series convergence (~40 minutes)

Interviewer: In order to complete the interview tasks, you will need to open the OneNote file that I sent you. I will share my screen and demonstrate how to utilize the OneNote file (*interviewer shares* screen). I have put each interview task on a separate page. For each task, you may need to read, write, or watch a video. OneNote will preserve all of your annotations in the place that you make them, which will minimize the need to erase. You are not required to use a calculator for any of the tasks, but one is available on the online version of OneNote if you would like to use it. Simply go to Insert→Math, type in an expression, and then click "Evaluate."

I will ask you to share your desktop while we go through these tasks so that I can see the problem, your work, and the calculator (if you choose to use it). Do you have any questions? Are you able to access the OneNote file? Can you (1) share your screen and (2) go to the "Background information" page for "Student ____" and write answers to the questions on the screen? (*Student shares screen and opens OneNote file*).

Interviewer: Today we are going to look at several series presented by a student named Abigail. For each of Abigail's series, I will ask you the same two questions:

(1) Does the series converge? How can you tell?

(2) If the series converges, what value does it converge to? How can you tell?

Please note that I am more interested in the processes by which you approach these problems than by any numerical results (or "correct answers") that you produce. For example, if I were to give you the problem 1 + 2, I would not want you to merely answer "3." Rather, I would want an explanation such as "I'm thinking of putting one chip together with two other chips and counting the total number of chips, which is 3." I will likely ask you to summarize your methods for examining the series from time to time. Remember, I am not concerned whether or not you are able to produce a textbook "correct" answer; rather, I am only interested in what you are thinking.

Finally, I am not expecting you to be able to answer all of these questions. You are welcome to skip or respond "I don't know" to any question. In this instance, I will ask you "What would you need to know in order to answer the question?" OK, let's get started. Please click on Abigail's first series and answer the two questions.

Series	Expanded Form	Series type	Sequence of Partial Sums	Converge	Limit Value
$\sum_{n=0}^{\infty} \frac{3}{\sqrt{n}}$	$\frac{3}{\sqrt{1}} + \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{3}} + \cdots$	p-series ($0 < p < 1$)	Monotone increasing	No	
$\sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n^5}$	$\frac{2}{1^5} - \frac{2}{2^5} + \frac{2}{3^5} - \cdots$	Alternating p-series ($p > 1$)	Oscillating	Yes	≈ 1.94
$\sum_{n=1}^{\infty} \sum_{i=1}^{99} [10^{-2n-1} - 10^{-2(n+1)-1} i] = \sum_{k=0}^{\infty} \frac{495}{10000} \left(\frac{1}{100}\right)^k$	$\frac{99}{10^3} + \frac{98}{10^3} + \cdots + \frac{1}{10^3} + \frac{99}{10^5} + \cdots + \frac{1}{10^5} + \frac{99}{10^7} + \cdots + \frac{1}{10^7} + \cdots$	Geometric	Monotone increasing	Yes	$\frac{1}{20}$
$\sum_{n=0}^{\infty} \frac{(200-2n)(-1)^n}{n+1}$	$\frac{200}{1} - \frac{198}{2} + \frac{196}{3} - \cdots$	Alternating series	Oscillating	No	
$\sum_{i=0}^{\infty} a_i$ (where a_i corresponds to the i^{th} decimal place of π and $a_0 = 3$.)	$3 + .1 + .04 + \cdots$	Decimal expansion of irrational number	Monotone increasing	Yes	π
$\sum_{n=0}^{\infty} (.07) \cdot (-1)^n$	$.07 - .07 + .07 - \cdots$	Alternating series (Grandi's)	Oscillating	No	

The student attempts to answer the two questions for each series, which are presented one at a time on different OneNote pages. After the student has completed the questions for each series, the interviewer will ask the following questions:

Questions:
1) To confirm, you stated that this series (does/does not) converge, and that the series converges to _____, correct? How did you determine this?
2) What similarities or differences did you perceive between this series and Abigail's other series?
3) *Any other questions that the interviewer feels to ask to clarify students' thinking or meanings.*

Abigail's 1st series
$\frac{3}{\sqrt{1}}+\frac{3}{\sqrt{2}}+\frac{3}{\sqrt{3}}+\frac{3}{\sqrt{4}}+\frac{3}{\sqrt{5}}+\cdots$

Abigail's 2nd series
$\frac{2}{1^5}-\frac{2}{2^5}+\frac{2}{3^5}-\frac{2}{4^5}+\frac{2}{5^5}-\cdots$

Abigail's 3rd series
$\frac{99}{10^3}+\frac{98}{10^3}+\cdots+\frac{1}{10^3}+\frac{99}{10^5}+\cdots+\frac{1}{10^5}+\frac{99}{10^7}+\cdots+\frac{1}{10^7}+\cdots$

Abigail's 4th series
$\frac{200}{1}-\frac{198}{2}+\frac{196}{3}-\frac{194}{4}+\frac{192}{5}-\cdots$

Abigail's 5th series
$3+.1+.04+.001+.0005+.00009+.000002\ldots$

Abigail's 6th series

$$.07 - .07 + .07 - .07 + .07 - \cdots$$

Task 2: Reasoning about general series convergence (~20 minutes)

After the student completes the two questions for all series:

Interviewer: Now that you have completed all the series, I am going to ask you to address the two questions that we have answered for each of Abigail's series in a general way *(Interviewer navigates to "Series Question" page)*. In other words:

(1) How can I tell whether a series converges?
(2) If a series converges, how can I determine the value to which it converges?

I am going to ask you to write an answer to each of these questions on this screen. Take your time, and I will ask you to explain your answers when you are finished.

Student answers questions and interviewer asks the student to explain each response.

Interviewer: Thank you for your participation today. I am interviewing several students on these tasks, and I will ask some of the students to continue to participate in weekly interviews throughout the course of the semester. You would be compensated the same amount for each interview, and we would set up a weekly time to meet. Would you be willing to participate in further interviews? If so, can we set up a time to meet each week? *(Interviewer sets up a time to meet with the student)*. We will meet each week over Zoom and use OneNote during each session to discuss topics related to sequences and series. Do you have any further questions about this interview or my research study? If not, thank you for your time and I look forward to seeing you next week.

Day 1 Interview Protocol

Day 1: Establishing a personal expression

<u>Overview:</u>
The purpose of this interview is to:

(1) Further investigate students' intuitive meanings for partial sums, infinite series and their convergence;
(2) Determine which symbolic components of the longhand series notation that students attend to and their reasons for doing so;
(3) Determine how students might describe the process of determining partial sums and series convergence in words; and
(4) Determine how the students might symbolize their image of partial sums and series convergence.

<u>To-do list before Interview:</u>

- Set up OneNote file for student
- Send link to OneNote file to student
- Consent form
- Have student's ASU email address handy for sending Amazon gift card

<u>Introduction and Student Information (~5 minutes)</u>

Interviewer: Thank you for participating in the interview today. The purpose of this interview is to investigate how you think about infinite series. Throughout the interview, I would encourage you to think out loud and describe what is going on inside your head as you work.

I want to also keep track of the material you are covering in your calculus course. Can you tell me a little bit about the material that your instructor presented over the last week?

I will be reporting the results of this study for my dissertation and in future journal publications. In each publication, I will be referring to you by a pseudonym. If you have a particular pseudonym that you would like to be known by, please let me know today or sometime during the study.

<u>Task 1: Reasoning about Partial Sums (~20 minutes)</u>

Interviewer: In order to complete the interview tasks, you will need to open the OneNote file that I sent you. I will share my screen and demonstrate how to utilize the OneNote file (*interviewer shares* screen). I have put each interview task on a separate page. For each task, you may need to read, write, or watch a video. OneNote will preserve all of your annotations in the place that you make them, which will minimize the need to erase. I will ask you to share your desktop while we go through these tasks so that I can see the problem, your work, and the calculator (if you choose to use it). You are not required to use a calculator for any of the tasks, but you may share your screen and use the Desmos (desmos.com) calculator at any time if you wish. Do you have any questions? Are you able to access the OneNote file and write on the screen? (*Student shares screen and opens OneNote file*).

Interviewer: Please navigate to "Ivy's Series 1" in the OneNote file. For this task, I have included a group of infinite series created by a student named Ivy. For each of Ivy's series, I will ask you the same two questions:

(2) How would you determine the 37th summand (or written term) in the series?
(3) How would you determine the sum of the first 37 terms in the series?

Please note that I am more interested in the processes by which you approach these problems than by any numerical results (or "correct answers") that you produce. For example, if I were to give you the problem 1 + 2, I would not want you to merely answer "3." Rather, I would want an explanation such as "I'm thinking of putting one chip together with two other chips and counting the total number of chips, which is 3." I will likely ask you to summarize your methods for examining the series from time to time. Additionally, I am interested in any similarities or differences that you see between the processes that you use to investigate each of the series. Hence, I will frequently ask you to compare the methods that you use for each new series to the methods that you have used in previous series. Please be open and honest in all your answers. Remember, I am not concerned whether or not you are able to produce a textbook "correct" answer; rather, I am only interested in what you are thinking.

Finally, I am not expecting you to be able to answer all of these questions. You are welcome to skip or respond "I don't know" to any question. In this instance, I will ask you "What would you need to know in order to answer the question?" OK, let's get started.

Series	Expanded Form	Series type	Partial Sums Behavior	Converge	Limit Value
$\sum_{n=0}^{\infty} \frac{2}{\sqrt[4]{n}}$	$\frac{2}{\sqrt[4]{1}}+\frac{2}{\sqrt[4]{2}}+\frac{2}{\sqrt[4]{3}}+\cdots$	p-series	Monotone increasing	No	
$\sum_{n=1}^{\infty} \frac{5}{n}$	$\frac{5}{1}+\frac{5}{2}+\frac{5}{3}+\frac{5}{4}+\cdots$	p-series	Monotone increasing	No	
$\sum_{n=1}^{\infty} \frac{3}{n^5}$	$\frac{3}{1^5}+\frac{3}{2^5}+\frac{3}{3^5}+\frac{4}{3^5}+\cdots$	p-series	Monotone increasing	Yes	≈ 3.11
$\sum_{n=1}^{\infty} (-1)^n \left(\frac{6}{n^2}\right)$	$\frac{6}{1}-\frac{6}{4}+\frac{6}{9}-\frac{6}{16}+\cdots$	Alternating series	Oscillating	Yes	≈ -4.93
$\sum_{n=0}^{\infty} (.04)\cdot(-1)^n$	$.04-.04+.04-\cdots$	Alternating series (Grandi's)	Oscillating	No	

$\sum_{n=0}^{\infty}(-1)^n\left(\frac{n+3}{n^2-n+}\right.$	$\frac{3}{7}-\frac{4}{7}+\frac{5}{9}-\frac{6}{13}+\cdots$	Alternating series	Oscillating	Yes	≈ -0.27

Questions:

1) Can you explain one more time how you determined the 37th summand/partial sum?
2) How was your approach for this series similar or different from the previous series?
3) **Is the student making a distinction between sequence and sequence of partial sums?**

Task 2: Writing a Written Rule for a Partial Sum (~20 minutes)

Interviewer: I appreciate the work that you have done so far and the good job you have done at explaining your thinking. **Could you summarize one more time for me the similarities and differences that you found while working through each of Ivy's series?** Thank you.

When we compute the sum for the first n terms in a series, we determine what is called the nth partial sum of the series. For example, if you add the first 37 summands together, you have found the 37th partial sum. If you add the first 15 terms together, you have found the 15th partial sum. If you add the first 1234 terms together, you have found the 1234th partial sum.

For our next activity, please go to the *Written Rule Task* on the OneNote. I would like you to use the similarities and differences that you have found and create a written note that you could share with a friend describing how to find a partial sum for any finite number of summands. Specifically, I would like you to (*interviewer reads task prompt*)

> *Construct a written note, say for a fellow group member, detailing how to find the determine the sum of any finite number of terms in any series (such as the first 12 terms, the first 189 terms, and so on).*

Your note will be in written English, although you are welcome to write numbers if you would like. Please do not write symbolic notation such as algebra for this task. Do you have any questions? Please think out loud as you construct your written note. After you have finished your written note, I will have you explain it to me.

(*Student proceeds to construct a written rule*)

Questions:

1) Can you explain your written rule to me?
2) Why did you put this word/phrase/sentence in your rule? What does it mean?

3) (*Interviewer selects one of Ivy's* series) Using your rule, could you explain to me how to compute the partial sum when $n = 79$?
4) Will your written rule work for any value of n? Why or why not?
5) It seems that earlier in the interview you were doing _____________________ when thinking about partial sums, but I do not see it included in your plan. Why did you not include it?

Task 3: The Transcription Task (~15 minutes)

Interviewer: For our next activity, I am going to have you transcribe an infinite series. Go ahead and navigate to the *Writing Series Task* in the OneNote file. You should see a YouTube video embedded there. Do you see the video? All right, this is how the activity will work—in a moment, I will have you play the video. The video includes an instruction screen that asks you to memorize an infinite series expression. The video will then show you the expression for about 6 seconds, after which time the expression will disappear. Do not write anything until after the expression disappears from the screen. After the expression disappears, I would like you to recreate what you can remember from the expression on the OneNote page. After you have finished recreating the expression, I will ask you some questions about what you wrote. We might repeat this activity 2-3 times. Do these instructions make sense? Are you ready? Okay, then play the video and get ready to transcribe the expression.

(*Subject plays video and transcribes the expression*)

Interviewer: How do you feel about your expression? (Don't worry, you will have another shot at copying the expression in a minute.) Now, I want to ask you a few questions:

1) Where did you look first when the expression appeared?
2) Why were these components important for you to be able to transcribe the series?
3) I noticed that you paused between writing (*thing 1*) and (*thing 2*). Why did you pause there?
4) Would you like to make another attempt at transcribing the video? Before we replay the video, is there anything that you plan to specifically look for the next time you see the expression?

All right, I am going to have you erase what you have written and attempt a new transcription. Are you ready to try transcribing again? All right, play the video one more time.

(*Subject plays video and transcribes and expression*)

Interviewer: I have a few more questions now that you have completed the activity a second time:

1) How do you feel about your expression now?

2) Did your approach to transcribing the series change when you re-watched the video? How so?
3) I noticed that you paused between writing (*thing 1*) and (*thing 2*). Why did you pause there?
4) Do you want to watch the video one more time? If so, before we replay the video, is there anything that you plan to specifically look for the next time you see the expression?

(Interviewer shows the expression)

1) What similarities and differences are there between your expression and the expression on the screen?
2) What do you think that these components that you did not write might represent in this expression?

Task 4, Activity 1: Personal Expressions Video (~10 min)

Interviewer: (*If there is time left in interview)* I have one final task for you to complete in today's interview. The task involves constructing an algebraic expression from your written expression. **How do you think you might create an algebraic expression that details the same process that you described in words?**

In a few minutes, I will ask you to create an expression. However, first I would like to show you a short video that reviews the ways in which algebraic symbols, which I call *inscriptions*, are used in mathematics.

(*Interviewer plays video*)

Video script
Mathematicians have written down their discoveries for centuries to convey ideas to others and make sense of their own thinking. In the last 700 years, most mathematical topics, including algebra, calculus, and logic have begun to use algebraic inscriptions to simplify communication. But what is an inscription, and what can it be used to represent?

An inscription is a *"a written mark utilized by an author to succinctly represent a property, action, or relationship that the author has envisioned."* There are three types of inscriptions that mathematicians typically use: Operational, Relational, and Proxy.

An operational inscription indicates a task to be performed, such as adding the total number of sheep in a village or dividing 35 pieces of chocolate among you and 5 friends.

A relational inscription indicates a relationship between the values, magnitudes, or properties of two quantities. For example, we can use relational inscriptions to describe the relationship between the number of candies or sheep that two people have.

Vicarious inscriptions serve as representations for stable and complete mathematical concepts, such as function, integral, the set of all real numbers, or even the variable x. Vicarious inscriptions can also be ornamented with superscripts or subscripts to provide greater detail or dimensionality to the original idea represented by the inscription.

Inscriptions are often combined to represent even more complex mathematical ideas. I call these combinations of inscriptions mathematical expressions. Mathematical expressions come in two types: conventional expressions and personal expressions.

Let's illustrate the difference between conventional expressions and personal expressions with an example. Suppose that three mathematicians draw a triangle and want to represent that triangle in their writing without drawing the original triangle. The mathematicians agree to label the vertices of the triangle with the capital letters A, B, C, the sides of the triangle with the lowercase letters a, b, c, and the angles of the triangle with their corresponding measures. The first mathematician decides to represent the triangle using the vertices and writes ΔABC. The second mathematician decides to represent the triangle using the sides and writes Δabc. The third mathematician decides to represent the triangle using the vertices and angle measures. She writes $\Delta A_{68.5}B_{64.7}C_{46.8}$. Each mathematician has created his or her personal expression for the original triangle. However, the mathematicians must collaborate together to determine a conventional expression that they can use to communicate clearly and coherently with each other.

Mathematical expressions are utilized throughout the world in classrooms, textbooks, and homework assignments. Most of these expressions are conventional expressions, but remember: each conventional expression was once a personal expression, used by a single mathematician to represent a mathematical property, action, or relationship in a succinct way.

Questions:

1) What did you understand from the video?
2) Do you want to see another example? (Be prepared with function notation example)

Task 4, Activity 2: Constructing a Personal Expression (~25 min)

Interviewer: Now, I would like you to create a personal expression that describes how to determine the sum for any finite number of terms in any series. Since this is a personal expression, you are welcome to either use inscriptions that you are already familiar with or invent your own. To keep track of the meanings for each of your inscriptions, I have prepared a glossary for you (*interviewer indicates glossary*). I will ask that for each inscription that you decide to use, you include the inscription and your definition for the inscription in the glossary. All right, let's go!

(*Student creates personal expression and fills out glossary*)

Questions:

1) What does (each) inscription represent?
2) How is this inscription correlated to your written response?
3) Why did you place the inscription where you did in your personal expression?
4) Did you exclude any components of your written response from your personal expression? Why?
5) Did you include any components in your personal expression that were not in your written response? Why?

(*If time*) Have the student model a partial sum with their personal expressions.

End of Day 1 Interview

Interviewer: Thank you so much for taking the time to interview today. I appreciate what we have been able to accomplish, and I feel that you did a very good job of expressing your thinking. Please feel free to reach out to me at any time in the future if you have questions regarding my research. Finally, I need to clarify an email address to which I can send your Amazon gift card. Is (student's ASU email address) okay?

Day 2 Interview Protocol

Day 2: Reasoning and Modeling with Personal Expressions

<u>Overview</u>: The purpose of this interview is to have the student reason about and make revisions to their personal expressions while modeling both individual partial sums and the infinite series. The student's revisions to their personal expressions will (potentially) provide greater flexibility from a syntactical standpoint and greater fluency from a semantic standpoint in preparation for creating a personal expression for the sequence of partial sums and beginning to explore convergence during Day 3.

<u>To-do list before interview:</u>

- Set up OneNote file for student
- Send link to OneNote file to student
- Consent form
- Have student's ASU email address handy for sending Amazon gift card

<u>Review Task 1 & 2: Daily video and Glossary review</u>

Interviewer: It has been a week since our first interview. How is your semester going? What topics have you covered in the last week in you calculus class? What do you remember from our interview last week?

(*interviewer addresses questions and provides any necessary background information*)

Interviewer: Before we begin, I wanted to make sure that you are able to access the OneNote file for this week. (*student confirms that they have opened the OneNote file*). I will share my screen to show you the setup for today's interview and then have you share your screen when you do the work just like we did last week. (*interviewer shares screen of OneNote file*).

For our first activity today, I would like you to rewatch the mathematical expressions video and review your inscriptions that you created during the last interview. Remember that you are building your own personal expressions and it is normal to make adjustments to your expression as you proceed. If you choose to change any component of your inscription or introduce a new personal expression, we will simply update the glossary with a new inscription, idea the inscription is designed to convey, or both. In every interview throughout the remainder of the study, we will begin with this activity. (*Interviewer plays video*)

Questions:

1) How has your understanding of the video changed as you watched it this time?

2) (*For each inscription in the student's glossary*) Last week, you constructed this inscription to convey ____________ (*interviewer reads students' definition*). Can you review for me what you mean by this inscription?
3) What is the domain of your inscription $f(n)$? Is this domain the same for all series?

I would also like to review the inscriptions that you created yesterday in your glossary. For each inscription, will you please answer the following questions:

1) What does this inscription mean to you?
2) Would you like to modify your inscription before we begin today's interview tasks?

Task 1a: Instructional Provocation (Monica): Contrasting Prompts

Interviewer: Today's interview will consist of 2 tasks, which are going to focus on ways in which you might use the personal expression that you created last week to give meaning to Ivy's series. For the first task, I would like you to read and interpret the following arguments by students Xavier, Yolanda, and Zeb and tell me which argument most accurately describes how to compute the 127th partial sum of the Ivy's 1st series.

1) **Could you read Xavier's argument?**
2) What does this argument mean to you?
3) Could you draw a graph to represent how you are interpreting this argument?
4) How does your graph relate to Ivy's first series?

5) **Could you read Yolanda's argument?**
6) What does this argument mean to you?
7) Could you draw a graph to represent how you are interpreting this argument?
8) How does your graph relate to Ivy's first series?

9) **Could you read Zeb's argument?**
10) What does this argument mean to you?
11) Could you draw a graph to represent how you are interpreting this argument?
12) How does your graph relate to Ivy's first series?

13) **Which argument do you believe best describes the 129th partial sum of Ivy's first series? Why did you choose this argument?**
14) **Which graph do you believe best describes the 129th partial sum of Ivy's first series? Why did you choose this graph?**

Xavier's Argument

The 129th partial sum of Ivy's 1st series can be determined by computing the integral $\int_1^{129} \frac{2}{\sqrt[4]{n}} dn$, which represents the exact area under the curve of the function $f(n) = \frac{2}{\sqrt[4]{n}}$ from $n = 1$ to $n = 129$.

Yolanda's Argument

The 129th partial sum of Ivy's 1st series can be determined by computing the summation $\sum_1^{129} \frac{2}{\sqrt[4]{n}}$, which represents the exact area under the curve of the function $f(n) = \frac{2}{\sqrt[4]{n}}$ when it is evaluated at each position from $n = 1$ to $n = 129$.

Zeb's Argument

The 129th partial sum of Ivy's 1st series can be determined by computing the summation $\sum_1^{129} \frac{2}{\sqrt[4]{n}}$, which represents the approximate area under the curve of the function $f(n) = \frac{2}{\sqrt[4]{n}}$ using Riemann sums with width 1 from $n = 1$ to $n = 129$.

Task 1b: Instructional Provocation 2 (Sylvia): Devil's Advocate

Interviewer: Today's interview will consist of 2 tasks, which are going to focus on ways in which you might use the personal expression that you created last week to give meaning to Ivy's series. For the first task, I am going to show you several variations of a series that differ only in terms of the signs between the terms. I would like you to create a personal expression for the 43rd partial sum for each of the following series.

1) Can you explain how your personal expression represents the 43rd partial sum? Why did you write it this way?
2) It seems that you have very different personal expressions for each of the series, even though the only differences between the series are the operators. Do you think there is a more efficient way to construct your personal expression? If so, how would you change it?

Series A
$$\mathbf{1}+\frac{\mathbf{1}}{\mathbf{3}}+\frac{\mathbf{1}}{\mathbf{9}}+\frac{\mathbf{1}}{\mathbf{27}}+\frac{\mathbf{1}}{\mathbf{81}}+\frac{\mathbf{1}}{\mathbf{243}}+\frac{\mathbf{1}}{\mathbf{729}}+\cdots$$

Series B
$$-\mathbf{1}-\frac{\mathbf{1}}{\mathbf{3}}-\frac{\mathbf{1}}{\mathbf{9}}-\frac{\mathbf{1}}{\mathbf{27}}-\frac{\mathbf{1}}{\mathbf{81}}-\frac{\mathbf{1}}{\mathbf{243}}-\frac{\mathbf{1}}{\mathbf{729}}-\cdots$$

Series C
$$\mathbf{1}+\frac{\mathbf{1}}{\mathbf{3}}+\frac{\mathbf{1}}{\mathbf{9}}-\frac{\mathbf{1}}{\mathbf{27}}-\frac{\mathbf{1}}{\mathbf{81}}+\frac{\mathbf{1}}{\mathbf{243}}+\frac{\mathbf{1}}{\mathbf{729}}-\cdots$$

Series D
$$\mathbf{1}-\frac{\mathbf{1}}{\mathbf{3}}+\frac{\mathbf{1}}{\mathbf{9}}-\frac{\mathbf{1}}{\mathbf{27}}-\frac{\mathbf{1}}{\mathbf{81}}+\frac{\mathbf{1}}{\mathbf{243}}+\frac{\mathbf{1}}{\mathbf{729}}-\cdots$$

Series E
$$-\mathbf{1}+\frac{\mathbf{1}}{\mathbf{3}}+\frac{\mathbf{1}}{\mathbf{9}}+\frac{\mathbf{1}}{\mathbf{27}}-\frac{\mathbf{1}}{\mathbf{81}}+\frac{\mathbf{1}}{\mathbf{243}}+\frac{\mathbf{1}}{\mathbf{729}}-\cdots$$

Task 2: Using a personal expression to model the SPS with Ivy's series (45-60 min)

Interviewer: Last week you developed the written rule "___________" and the personal expression "___________" to represent the sum of any number of terms in any of Ivy's series. (*interviewer indicates students' rules on the shared screen*). I am going to ask you to use your rules to represent several of Ivy's series. In particular, I will ask you to use your personal expression to represent the following three things:

1) The 76th partial sum in each series
2) The nth partial sum in each series

3) The infinite series itself

After you represent each series, I will ask you what components of your expression were similar or different compared to previous expressions. Finally, after you have looked at all of the series that we have time for, I will ask how you might represent the sum for *any* number of terms in *any* series, and how you might represent the infinite series for *any series*. Do you have any questions? Very good, let's begin.

Possible questions to ask:

1) What does (each) inscription represent?
2) Why did you place the inscription where you did in your personal expression?
3) Does your personal expression coincide with your written rule? What differences (if any) exist between the two?
4) Is your personal expression equivalent to Ivy's series? Why or why not?
5) What are the similarities and differences between the expressions that you are creating to answer question 1 and the expressions that you are using to answer question 2?

<u>Task 3 (optional): Representing components of Patricia's series with personal expression (30 min)</u>

Interviewer: One of the reasons that mathematicians create personal and conventional expressions is to represent a broad class of mathematical topics. In the next part of the interview, I want to show you a different series presented by another student, Patricia. (*Interviewer shows Patricia's series*) I want to highlight several parts of Patricia's series and ask you whether or not you can use your personal expression (or some form of your personal expression) to represent that component of the series. If you believe that you can represent the portion of Patricia's series that I highlight with your personal expression, I will ask you to show me a specific expression that represents the highlighted components.

If you do not believe that you can represent the portion of Patricia's series that I highlighted with your personal expression, I will ask whether you might introduce a new inscription to your expression or otherwise modify your personal expression so that you could represent this highlighted portion of Patricia's series and how you would do this. If you do not believe that you can modify your personal expression to represent the highlighted component of Patricia's series, then I will ask you to create another personal expression to represent the highlighted component of the series.

If you create a new personal expression, we will also write the new inscriptions and the information that they are designed to convey in the glossary. Do you have any questions? This series of activities should take up the remainder of the interview.

Patricia's Series

Create a personal expression to represent the sum of the first 53 terms in Patricia's series.
$13+\frac{13}{4}+\frac{13}{9}+\frac{13}{16}+\frac{13}{25}+\frac{13}{36}+\frac{13}{49}+\frac{13}{64}+\cdots$

Components of Patricia's Series to Highlight

Use a personal expression to model $a_5 = S_5 - S_4$
Use a personal expression to model $a_{10}+a_{11}+a_{12}=S_{12}-S_9$
Use a personal expression to model $\sum_{i=4}^{\infty} a_i = a_4+a_5+a_6+\cdots$
Use a personal expression to model $\sum_{i=1}^{\infty} a_i - a_4 = a_1+a_2+a_3+a_5+a_6+a_7+a_8+\cdots$
(If the student seems to have symbolized the index of the SPS) Use a personal expression with an index shift to represent $\sum_{i=4}^{\infty} a_i = \sum_{j=1}^{\infty} a_{j+3} = a_4+a_5+a_6+\cdots$

Questions:

1) Why did you (or did not) use your personal expression to represent this component of Patricia's series?

2) What does (each) inscription in your expression represent?
3) Why did you place the inscription where you did in your personal expression?

Day 3 Interview Protocol

Day 3: Personal Expression and Graph of SPS

Overview: The purposes of this interview are to (1) encourage students to adopt a normative definition of sequence and the graph of a sequence, (2) construct a personal expression for the sequence of partial sums, and (3) begin to reason about sequence of partial sums convergence using graphs.

To-do list before interview:

- Set up OneNote file for student
- Send link to OneNote file to student
- Check embedded GeoGebra file and make sure that it loads properly

Follow-up from previous week & daily video review

Interviewer: It has been a week since our first interview. How is your semester going? What topics have you covered in the last week in your calculus class? What do you remember from our interview last week?

(*interviewer addresses questions and provides any necessary background information*)

Interviewer: Before we begin, I wanted to make sure that you are able to access the OneNote file for this week. (*student confirms that they have opened the OneNote file*). I will share my screen to show you the setup for today's interview and then have you share your screen when you do the work just like we did last week. (*interviewer shares screen of OneNote file*).
I would like to start by having you rewatch the mathematical expressions video and review your inscriptions that you created during the last interview. Remember, if you choose to change any component of your inscription or introduce a new personal expression, we will update the glossary. (*Interviewer plays video*)

Questions:

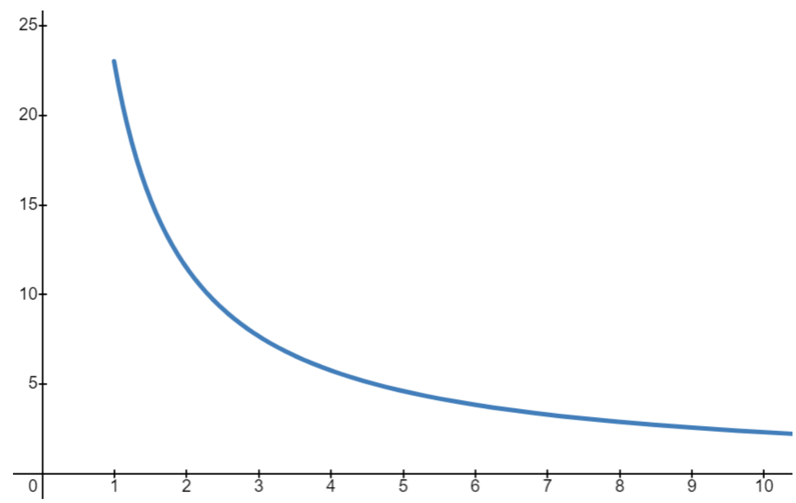
4) How has your understanding of the video changed as you watched it this time?
5) (*For each inscription in the student's glossary*) Last week, you constructed this inscription to convey **____________** (*interviewer reads students' definition*). Can you review for me what you mean by this inscription?
6) (**Monica**) **It seems like last week you were unsure how to distinguish between a sequence and a series. Do you have a better idea now?**
7) **(Sylvia) How would you describe the difference between a sequence and a series?**
8) **(Sylvia) In our last interview, we talked about three different types of series: (a) series with a pattern that we know, (b) series that we believe have a pattern but we are unsure about part of the pattern or the whole pattern,**

and (c) series that we are pretty sure have completely random terms. In our last interview, you used different inscriptions for each of these ideas. Could you clarify for me how you would use each of these inscriptions in terms of a specific series?

Task 1a: Instructional Provocation (Monica): Contrasting Prompts

Interviewer: For our first activity today, I would like to present you with three different graphical interpretations of series from three different students: Mario, Natalie, and Oscar. I wonder whether you agree with some, all, or none of the statements as appropriate ways to think about the series. Please examine the following graphs carefully and share your thoughts with me.

Mario's graph
Mario creates the following graph to think about the series $\mathbf{23 + \frac{23}{2} + \frac{23}{3} + \frac{23}{4} + \cdots}$. **What does this graph represent to you?**

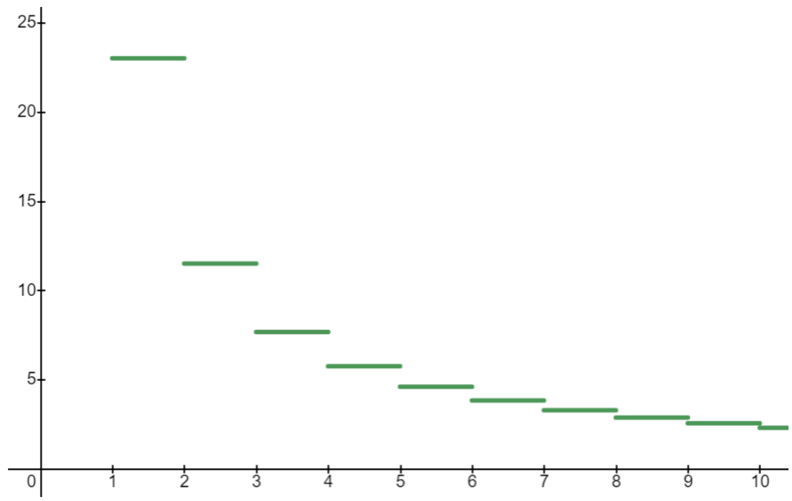


Natalie's graph

Natalie creates the following graph to think about the series

$$23 + \frac{23}{2} + \frac{23}{3} + \frac{23}{4} + \cdots.$$

What does this graph represent to you?

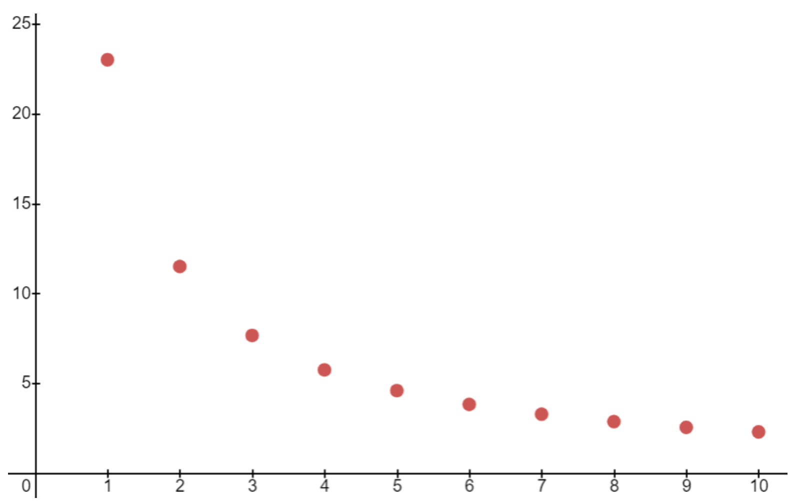


Oscar's graph

Oscar creates the following graph to think about the series

$$23 + \frac{23}{2} + \frac{23}{3} + \frac{23}{4} + \cdots.$$

What does this graph represent to you?



Questions:

1) Which of the following graphs could be used to meaningfully think about the series? Why?
2) What does this point represent (*Interviewer circles a point that falls on the graph of the sequence*)?
3) What does this point represent (*Interviewer circles a point or empty space that is not on the normative graph of the sequence*)?

Task 1b: Instructional Provocation (Sylvia): Symbolizing Random Series

Interviewer: At the end of our last interview, you introduced several inscriptions to describe series with random alternating signs and randomly generated summands. Before we proceed to our next task, I would like to reintroduce two of these series to you, Series E and Ivy's 7th series. Can you construct personal expressions to represent each of these series with the inscriptions from your glossary?

Series E
$$-1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}-\frac{1}{81}+\frac{1}{243}+\frac{1}{729}-\cdots$$

Ivy's 7th Series
$$1+.3+.05+.009+.0001+.00004+.000000+.0000009+\cdots$$

Questions:

1) Why did you use the inscriptions that you did in your personal expressions?
2) What are you trying to convey through each inscription in your personal expression?
3) Suppose a series had random signs like Series E and randomly generated summands like Ivy's 7th Series? What sort of personal expression would you for this kind of a series?
4) If there is more than one instance of randomness in a series, can you use the same inscription for each kind of randomness?

Task 2a: Mini-lecture on sequence (Part 1)

<u>Notes:</u> For Monica, provide all instruction. For Sylvia, determine whether her understanding of sequence seems normative. If it is, then only highlight the crucial

components of a sequence before defining the sequence of partial sums. If Sylvia's meanings for sequence do not appear to be normative, then proceed with the full instruction.

Interviewer: Before we continue, I wanted to take a minute to clarify what we in mathematics mean by a sequence and a series, and how these two terminologies refer to different things in mathematics. Sequences are usually written with the terms separated by commas, as in 1, 1/2, 1/3, 1/4, …., or as in 1/4, 1/3, 1/2, 1, …. Although 1, 1/2, 1/3, 1/4 are in both sequences, we treat 1/4 in the first sequence differently than 1/4 in the second sequence as their positions in the sequence are different. The 1/4 is in the fourth position in the first sequence whereas 1/4 is in the first position in the second sequence. Likely, we in mathematics consider a sequence as an ordered set of numbers. The "ordered" in this description of a sequence means that each number in the set has a specific position. This means that each value, 1, 1/2, 1/3,1/4, etc. in each sequence is assigned a specific position 1, 2, 3, 4, etc. So there are two things that we need to clarify when talking about numbers listed in a sequence: are you talking about the positions 1, 2, 3, 4 or the value at the positions, 1, ½, 1/3, ¼ etc.? For the value at a position, we will use the word "term" in mathematics. For instance, for the "4^{th} term," we are indicating the value that is in the 4^{th} position in the sequence. So in the first sequence, the 4^{th} term will be 1/4 and in the second sequence, the 4^{th} term will be 1.

Another way we can think about a sequence is by using the notion of a function. We can treat the positions 1^{st}, 2^{nd}, 3^{rd}, etc. as values of the independent variable in a function and the terms 1^{st} term, 2^{nd} term, 3^{rd} terms, etc. as value of the dependent variable of the function corresponding to the positions. For instance, the first sequence 1, ½, 1/3, ¼, … can be viewed as a function where all positive integers 1, 2, 3, 4, … are values of the independent variable and unit fractions 1, ½, 1/3, ¼, etc. are values of dependent variable of a function.

In conventional mathematics, the domain of a sequence as a function would be the positive integers (often including 0). In other words, when we define a function on only positive integers, we call the function a sequence.

I noticed that you consider f(n) is a function defined at the position of a non-integer such as here (pick a value between 1 and 2 on the x-axis). But if we follow this mathematical convention about sequence, none of the non-positive integers can be positions (i.e., values of the independent variable) of the sequence. Consequently, the graph of sequence visually appears as a set of "dots," similar to Oscar's graph. For all future sequence graphs that I present during our interviews, the graphs will consist only of a set of dots.

Questions:

1) Do you have any questions about the nature of a sequence and it's graph?
2) Review question 1: What is a sequence?
3) Review question 2: What is the domain of a sequence?
4) Review question 3: Describe the graph of a sequence.
5) Review question 4: What are the differences between a sequence and a series?

Task 2b: Mini-lecture on SPS (Part 2)

Interviewer: I now want to create a new sequence from Oscar's sequence ($v(n) = \frac{23}{n}$ or $f(n) = \frac{23}{n}$).
The positions of the new sequence that we will create are still positive integers.
The term for the 1st position of the new sequence is still the first term of Oscar's sequence.
The second term of the new sequence is then the sum of the first and second terms of Oscar's sequence.
The third term of the new sequence is then the sum of the first, second, and third terms of Oscar's sequence.
In this manner, we can create a new sequence from Oscar's sequence.
Mathematicians call such a new sequence the **sequence of partial sums**.
We may also call the first term of the new sequence as the first partial sum of Oscar's sequence, the second term of the new sequence as the second partial sum of Oscar's sequence, etc.
This relationship holds for all terms in the sequence of partial sums. So, if I wanted to determine the 198^{th} term in the sequence of partial sums, I would determine the 198^{th} partial sum.

Questions:

1) What is the domain of a sequence of partial sums?
2) Will the graph of a sequence of partial sums appear similar to Mario's graph, Natalie's graph, or Oscar's graph?

Task 3a: Constructing a Personal Expression for the Sequence of Partial Sums

Interviewer: Please move to the "Sequence of Partial Sums 1" tab on the OneNote file. On this tab, I have included a copy of Ivy's series and your glossary. For this activity, I am going to ask you to construct a personal expression to represent the sequence of partial sums for each of the series below. We will also add your new personal expression to your glossary. After you create a personal expression, I will ask you to represent the sequence of partial sums for each of Ivy's series with your personal expression.

Question:

1) **Create a personal expression to model the sequence of partial sums for each of Ivy's series.**

Task 3b (optional): Introducing the graph of sequence of partial sums

Interviewer: I appreciate you creating a personal expression for the sequence of partial sums. I would now like to return to the graph of a sequence of partial sums. Please go to the "Sequence of Partial Sums 2" tab. On this tab, I have included a graph of the sequence of partial sums for Ivy's first series along with the first few terms of the sequence of partial sums written out (next week I will try to use your personal expression to represent the sequence of partial sums). Before we look at the graph in more detail, I wanted to ask you about the following:

1) **(*Interviewer circles point on graph*) What does this point mean?**
2) **(*Interviewer indicates each axis on the graph*) How should we label the axes for this graph?**

Interviewer: Thank you. For the rest of this activity, I want you to return to the two questions from our very first interview, this time in terms of the sequence of partial sums.

1) Does the sequence of partial sums converge?
2) If the sequence of partial sums converges, what does it converge to?

Interviewer: We will work through Ivy's first three series today (if we have time), and will look at the rest of Ivy's series in the coming interviews.

The interviewer presents each SPS to the student and asks the following questions:

1) *Does the sequence of partial sums converge?*
2) *If so, what value does the sequence converge to? How can you tell?*
3) *If not, why does the sequence not converge?*
4) *What does it mean for the sequence to converge?*
5) *Does the sequence of partial sums converge to (value relatively far from student's proposed limit, value relatively close to student's proposed limit)? Does the sequence converge to both? How does your rule show your thinking?* (interviewer will tweak the wording of students' rules according to his understanding of students' explanations and ask for students' approval of all changes)
6) (interviewer points out inconsistencies between students' rules and students' reasoning): *I see that you wrote this ("____"), but it seems to me that by your written rule, this sequence should/should not converge to (value). How might you modify your rule so that this is not the case?*

The goal of this section of the teaching session is to thoroughly perturb and confuse the student with their conceptions of limit (*this is highly likely to be the case, but not guaranteed*). In this way, I can promote the necessity of ϵ and the ϵ-strips to give meaning to the concept of limit.

Day 4 Interview Protocol

Day 4: Graph of SPS and Introduction to Convergence

Overview: The learning goals for this interview are to (0) differentiate between series and sequence, (1) differentiate between the SPS and the double summation $a_1 + (a_1 + a_2) + (a_1 + a_2 + a_3) + \cdots$, (2) develop meaning for the graphical representation for the SPS, (3) begin to build a verbal definition for convergence of a sequence of partial sums, and (4) develop inscriptions to represent the SPS and its convergence. The research goals for this interview are (1) to investigate MONICA's separation of inscriptions indicating a process from those indicating a result, (2) model MONICA's preliminary meanings for sequence convergence, and (3) to determine whether MONICA perceives any relationships between the convergence of the sequence of partial sums and infinite series.

To-do list before interview:

- Set up OneNote file for student
- Send link to OneNote file to student
- Open Desmos graphing file

Follow-up from previous week & daily video review

Interviewer: What topics have you covered in the last week in your calculus class? What do you remember from our interview last week?

(*interviewer addresses questions and provides any necessary background information*)

Interviewer: Before we review the personal expressions video today, I want to ask you a question. What do you believe are the important ideas from the video? (*Interviewer plays video*) Was there anything new or different that you noticed in the video this time?

Interviewer: I want to review your inscriptions in a slightly different way this week. Instead of having you describe what you mean by each inscription, I am going to ask you to label each inscription by type. You may label the inscriptions as relational using an "R," operational using an "O," vicarious using a "V," or you may come up with your own label for the type of inscription and write it next to the inscription. After you complete this exercise, I will ask you why you labeled each inscription as that particular type.

Questions:

1) Why did you label this/these inscription(s) with this type?
2) Why did you create this inscription type? What kind of information do inscriptions of this type generally convey?

Task 1: Comparing Inscription Types in Glossary

Interviewer: Over the last three interviews, you have developed several inscriptions to describe partial sums, series, and sequences. Before we begin the first official task of the interview, I wanted to ask you for similarities and differences you imagine between your inscriptions. In particular, I would like to see whether you can group your inscriptions into categories. I will allow you to group and categorize your inscriptions in whatever way you would like. If possible, assign a meaningful name for each inscription category. If you cannot think of a meaningful name, give each category a generic name so that we can use these names later in the interview to talk about your inscriptions.

(*Give the student time to reflect, think, and categorize inscriptions in her glossary*)

Questions:

1) Can you summarize the categories you created for me?
2) Why did you include inscriptions a, b, c, and d in category 1?
3) What is the difference between category 1 and category 2?
4) Were there any inscriptions you felt were difficult to categorize? Why?
5) Were there any inscriptions that you felt belonged in more than one category (or no categories)? Why?

Task 2: Differentiating a Sequence from a Sequence of Partial Sums

Interviewer: For the last three interviews, we have focused primarily on infinite series. For today, I would like to focus on building sequences and representing components of these sequences with your inscriptions. We will build three different sequences using a new series, $2+\frac{2}{7}+\frac{2}{49}+\cdots$. To help make this process easier, I have created a table for each sequence in which you can record your inscriptions and numerical calculations. For an example, I will fill out a portion of the first table. I will begin by filling out the bottom portion of the table. Remember that a sequence coordinates two components: a position value and a term value. In the bottom portion of the table, I have separated the table to indicate these two quantities. In the past you have used the inscription n to indicate position in a sequence and a_n to indicate the value of the nth term in the sequence. I have filled in the "Position value" portion of the table and want you to fill in the "term value" section. For example, based on the new series, I know that the term a_1 is equal to 2, the term a_2 is equal to $\frac{2}{7}\approx .286$, and a_3 is equal to $\frac{2}{49}\approx .041$. I will write these values in the "term value" cells next to the inscriptions $a_1, a_2, and\ a_3$ to show the numerical values for the first three terms of the sequence. **Can you fill out the term value section for the nth term of the sequence?**

(*The student fills in the "Term Value" column*)

Interviewer: Thank you for filling out the table. We will now fill out the top portion of the table. In the first row we will write the sequence using symbolic inscriptions from your glossary. For example, I can fill in the top row by listing the terms of the sequence using your inscription a_n for the term in the sequence at the nth position, separating each term with commas (*Interviewer writes* $a_1, a_2, a_3, \ldots$ *in top row of table)*. In the second row, you will write out the sequence using the numerical values that you calculated in Desmos, such as $2, .286, .041$. In the final row, you will give the sequence a name or type. For the first example, I will write "original sequence" because this is the sequence upon which the other two sequence tables will be based. I will have you name the future sequences (*Interviewer writes "Original sequence" in the second row of the table)*.

New series (presented to the student in expanded form)

Series	Expanded Form	Series type	Partial Sums Behavior	Converge	Limit Value
$\sum_{n=0}^{\infty} 2\left(\frac{1}{7}\right)^n$	$2 + \frac{2}{7} + \frac{2}{49} + \frac{2}{343} + \cdots$	geometric	Monotone increasing	Yes	$\frac{7}{3}$

Student version of Table 1

Sequence Inscription (symbolic):			
Sequence Inscription (numeric):			
Sequence Type:			
Position inscription	**Position value**	**Term inscription**	**Term Value**
	1		
	2		
	3		
	⋮		⋮
	⋮		⋮

Possible completed version of Table 1

Sequence Inscription: $a_1, a_2, a_3, \ldots$			
Sequence Inscription: $3, .094, .012, \ldots$			
Sequence Type: Original Sequence			
Position inscription	**Position value**	**Term inscription**	**Term Value**
	1	a_1	3
	2	a_2	.094
	3	a_3	.012
	$\vdots$	$\vdots$	$\vdots$
$\boldsymbol{n}$	p	a_p	$\frac{3}{p^5}$
	$\vdots$	$\vdots$	$\vdots$
	∞*	a_∞*	$\frac{3}{p^\infty}$*

* Student may or may not write this in, but we wanted to leave the space open just in case.

Interviewer: I have two more tables that I would like you to fill out. In each table, the position values of the sequence stay the same but the way in which the terms values are determined changes. In the next sequence, the nth term of the sequence is determined by adding the first n terms of the original sequence. I have represented the terms of this new sequence using your inscriptions from the original sequence, which I am calling a conventional inscription. However, if you have other inscriptions from your glossary that you would prefer to use, please write them in the "Term inscription (personal)" column. As before, I want you to calculate the term value (where possible) for each position in the sequence. After you finish calculating, you can write the sequence using inscriptions in the first row, using numbers in the second row, and create a written name for the sequence in the third row.

Questions:

1) (*for each inscription* type) Why did you choose this inscription?
2) Can you label each inscription by the types that you created in the during our review activity?
    a. (*for each inscription*) Why did you label this inscription as type "______"?
    b. (*for inscriptions of same type*) What are the relationships between the inscriptions of type "______________"?

Student version of Table 2

Sequence Inscription (symbolic):				
Sequence Inscription (numeric):				
Sequence Type:				
Position inscription	**Position value**	**Term inscription (conventional)**	**Term inscription (personal)**	**Term Value**
	1			
	2			
	3			
	⋮	⋮	⋮	⋮

Possible completed version of Table 2

$m_1, m_2, m_3, \ldots$				
$3, 3.094, 3.106, \ldots$				
Partial Sums Sequence				
Position inscription	**Position value**	**Term inscription (conventional)**	**Term inscription (personal)**	**Term Value**
	1	a_1	m_1	3
	2	$a_1 + a_2$	m_2	3.094
	3	$a_1 + a_2 + a_3$	m_3	3.106
$\boldsymbol{n}$	⋮	⋮	⋮	⋮
	p	$a_1 + a_2 + a_3 + \cdots + a_p$	m_p	$\sum_{n=1}^{p} f(n)$

	∞	$a_1 + a_2 + a_3 + \cdots + a_\infty$	m_∞	$\sum_{n=1}^{\infty} f(n)$

Interviewer: The final table represents the first few terms of a sequence that examines the sum of the first n partial sums. This last sequence doesn't have a name—do you have a recommendation for how we should name this sequence? (If student does not recommend name, state that Dr. Roh and I started calling this sequence "gorilla.") I am going to ask you to fill out the table for the "______________________" sequence as best you can.

Student version of Table 3

Sequence Inscription (symbolic):				
Sequence Inscription (numeric):				
Sequence Type:				
Position inscription	**Position value**	**Term inscription (conventional)**	**Term inscription (personal)**	**Term Value**
	1			
	2			
	3			
	⋮		⋮	⋮

Possible completed version of Table 3

$s_1, s_2, s_3, \ldots$				
$3, 6.094, 9.200, \ldots$				
Sequence of Partial Sums				
Position inscription	**Position value**	**Term inscription (conventional)**	**Term inscription (personal)**	**Term Value**

	1	a_1	s_1	3
	2	$a_1 + (a_1 + a_2)$	s_2	6.094
	3	$a_1 + (a_1 + a_2) + (a_1 + a_2 + a_3)$	s_3	9.200
$\boldsymbol{n}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	p	$a_1 + (a_1 + a_2) + \cdots + (a_1 + \cdots + a_p)$	s_p	??
	∞	$a_1 + \cdots + (a_1 + \cdots + a_\infty)$	s_∞	??

Interviewer: We are not going to focus on "_______________" in the future. Beginning with our next interview, we plan to focus almost exclusively on the second sequence, which you called "_________________". We are now going to start talking about the graphs of these three sequences. **Can you create a graph for what you believe each of these sequences will look like?**

Task 3: Graph of the SPS

(link to Desmos file: Day 4 Graphs (desmos.com))

Interviewer: On the next OneNote tab, there is a link to a Desmos graphing file with the actual graphs of the three sequences. I am going to share my screen to show you how this file works. There is a slider at the top of this file that allows you to change the value of g between 1 and 3. If $g = 1$, you will see the graph of the original sequence. If $g = 2$, you will see the graph of the second sequence, which I will rename "________________" to follow your name. If $g = 3$, you will see the graph of "Gorilla," which I will rename "_____________" to match your name. If you click the little wrench icon in the top right corner you can set the limits of your graphing window, or you can zoom with your mouse or touchpad.

Questions (ask for each graph):

1) We are now going to look at graph ___, which represents the ___________. (*Interviewer circles a point on the graph*). **What does this point represent?**
   a. **What is the term value?**
   b. **How would you use your inscriptions to represent:**
      i. **Term value**
      ii. **Position value**
2) *(Interviewer indicates the axes of the graph)* **How should we label the axes of this graph?**
3) **Which rule would you use to generate this graph, and what inscriptions will you use to represent this rule?** (*Interviewer recommends creating new inscription and adding to the glossary if necessary*)

Task 4 (optional): Introduction to limit of sequence

Interviewer: I am now going to reintroduce the two questions that we have used many times throughout this interview in terms of sequences:

1) Does the (sequence) converge? How can you tell?
2) If the (sequence) converges, what value does it converge to? How can you tell?

For each of the sequence graphs, can you answer these two questions for me?

Interviewer: Now that you have thought about convergence for each of these sequences, I want to end today's activities by asking you to create inscriptions to represent sequence convergence and the value to which the sequence converges.

Day 5 Interview Protocol

Day 5: Convergence of the SPS and the ϵ-strip activity

Overview: The learning goals for this interview are to (1) develop personal criteria for the convergence of the sequence of partial sums, (2) develop inscriptions to represent pertinent (to the student) parts of this criteria, and (3) assimilate to this criterion to determine the value to which a convergent sequence of partial sums does or does not converge. The research goals for this interview are to (1) begin to categorize the ways in which students employ inscriptions (e.g., class of examples, generalized particular, command, process, result, proxy), (2) determine which components of sequence convergence students believe merit creating inscriptions, and (3) model the evolution of students' thinking and their inscriptions as they transition from primarily algebraic to primarily graphical reasoning.

To-do list before interview:

- Set up OneNote file for student
- Send link to OneNote file to student
- Check embedded GeoGebra file and make sure that it loads properly
    - Load file through GeoGebra site as a backup
- Have student's ASU email address handy for sending Amazon gift card

Follow-up from previous week

Interviewer: What topics have you covered in the last week in your calculus class? When is your upcoming calculus exam? What do you believe are the important ideas from the inscriptions video? (*Interviewer will not show inscriptions video this week, but will provide any important information student forgets*).

Interviewer: I want to review your inscriptions this week in the same way we did last week. Instead of having you describe what you mean by each inscription, I am going to ask you to label each inscription by type. After you complete this exercise, I will ask you why you labeled each inscription as that particular type.

Review Task: Debugging inscriptions in the glossary

Student 1: Monica

Interviewer: I remember from our previous interview that you started using the letter "S" with a subscript to talk about various sequences. For example, if I had the sequence $\frac{2}{\sqrt[4]{1}}, \frac{2}{\sqrt[4]{2}}, \frac{2}{\sqrt[4]{3}}, \frac{2}{\sqrt[4]{4}}, \ldots$ (*interviewer writes this sequence next to glossary*), you could write something like $S_a = \frac{2}{\sqrt[4]{1}}, \frac{2}{\sqrt[4]{2}}, \frac{2}{\sqrt[4]{3}}, \frac{2}{\sqrt[4]{4}}, \ldots$ to describe this sequence? Am I remembering correctly? You also stated that if I created a sequence of partial sums based on the original sequence, like $\frac{2}{\sqrt[4]{1}}, \frac{2}{\sqrt[4]{1}} + \frac{2}{\sqrt[4]{2}}, \frac{2}{\sqrt[4]{1}} + \frac{2}{\sqrt[4]{2}} + \frac{2}{\sqrt[4]{3}}, \ldots$ (*interviewer writes this sequence below S_a*) then I could write something like $S_p = \frac{2}{\sqrt[4]{1}}, \frac{2}{\sqrt[4]{1}} + \frac{2}{\sqrt[4]{2}}, \frac{2}{\sqrt[4]{1}} + \frac{2}{\sqrt[4]{2}} + \frac{2}{\sqrt[4]{3}}, \ldots$. Am I remembering this correctly too? It seems like if I had another sequence, then using your inscriptions, I could write other sequences, such as $S_b = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ or $S_c = 1, -1, 2, -2, 3, -3, \ldots$. Am I interpreting your use of this inscription correctly? If I kept writing other sequences (*interviewer makes vertical dots to indicate other unwritten examples*) then I could eventually write other sequences such as $S_o = 3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$, $S_p = 2, 1.7, 1.4, 1.1, 0.8, \ldots$, or $S_q = \frac{2}{7}, \frac{6}{14}, \frac{18}{28}, \frac{54}{56}, \ldots$. Am I still on track with how you are using these inscriptions? (***Student will likely interject at this point of the "debugging." If not, continue to make sequences for S_x, S_y, and S_z. If there are still no student issues at this point, provide one more sequence and ask how you should name it now that you have run out of letter subscripts***).

Interviewer will allow the student to respond to the debugging activity and then attempt to resolve any perturbations that the student has. Such resolutions may include:

- Changing her inscription for the sequence of partial sums so that it does not utilize the S_letter structure
- Designating p as a special subscript that can only be utilized to discuss the sequence of partial sums. MONICA might consequently reject the second sequence S_p.
- In either case, ask about simplifying the original sequence of partial sums to decimal approximations and whether she would still label this sequence as S_p.

Task 1: Construct Inscriptions for General Term of a Sequence

(*Note: in the instructions below, I refer to Monica's inscriptions. Use Sylvia's inscriptions when addressing her symbolization)*

Interviewer: Please navigate to the tab labeled "Sequence Inscriptions." In our last interview, we filled out two tables for an original sequence and another sequence made up of partial sums. I have copied your table from last time for the sequence of partial sums onto this page. If I remember correctly from last time, you stated that the first term in the sequence of partial sums, p_1, is the same as a_1. (*Interviewer points to definition of S_p in* glossary). You also stated that the third term in the sequence of partial sums, p_3, is equivalent to $a_1 + a_2 + a_3$. So it seems that in the "Term Inscription" column of the table you could have written either p_1, p_2, and p_3 or you could have written $a_1, a_1 + a_2$, and $a_1 + a_2 + a_3$.

1) Do you agree? Why or why not?
2) Why did you choose to write p_1, p_2, and p_3?
3) Are there any other inscriptions from your glossary that you could have used instead of p_1, p_2, and p_3? For example, could you have written ____________? (*interviewer picks various inscriptions from the table, but research focus is on presenting summation notation inscription for arbitrary partial sum to student's attention*).

Interviewer will try to get the student to consider whether she can model terms in the sequence of partial sums with summation. Some of the following situations might occur:

1) The student may accept summation notation as a viable replacement for p_1, p_2, and p_3.
2) The student may reject summation notation and state that this inscription can only be used in the context of series (not sequences)
3) The student may reject summation notation and state that this inscription denotes a command or process, whereas p_1, p_2, and p_3 represent values in the sequence (or results of an additive process).
    a. For example, a student might write $p_3 = \sum_{n=1}^{3} a_n = a_1 + a_2 + a_3$ but perceive p_3 as a label for the term, the summation as the rule for determining the term value, and the string of summands as the term value itself.

Task 2: ϵ-strip activity (Iteration 1)

Task 2, Part 1: Reasoning about Convergence without ϵ-strip

Interviewer: Today we are going to explore three sequences of partial sums and try to answer the two questions that I presented last time:

1) Does this sequence of partial sums converge?

2) If the sequence of partial sums converges, what does it converge to?

I am hoping that as you do this activity, you will begin to think of some sort of general rule that you can use to determine whether a sequence of partial sums converges. As you examine the sequences of partial sums, I will attempt to create an initial version of a general rule based on your actions and write it on the screen. We can then change or update the rule as you go through the exercises if you feel that you need to.

I will be using a GeoGebra applet today to dynamically present several sequences of partial sums. For this activity, I will run the applet, but you may annotate in whatever way you would like.

Sequences of partial sums for day 5

Series	**Series Expanded Form**	**Partial Sums Expanded form**	**Iteration**
$\sum_{n=0}^{\infty} \frac{2}{\sqrt[4]{n}}$	$\frac{2}{\sqrt[4]{1}} + \frac{2}{\sqrt[4]{2}} + \frac{2}{\sqrt[4]{3}} + \cdots$	$\frac{2}{\sqrt[4]{1}}, \frac{2}{\sqrt[4]{1}} + \frac{2}{\sqrt[4]{2}}, \frac{2}{\sqrt[4]{1}} + \frac{2}{\sqrt[4]{2}} + \frac{2}{\sqrt[4]{3}}, \ldots$	First Iteration (Week 6)
$\sum_{n=1}^{\infty} \frac{5}{n}$	$\frac{5}{1} + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} + \cdots$	$\frac{5}{1}, \frac{5}{1} + \frac{5}{2}, \frac{5}{1} + \frac{5}{2} + \frac{5}{3}, \ldots$	
$\sum_{n=1}^{\infty} \frac{3}{n^5}$	$\frac{3}{1^5} + \frac{3}{2^5} + \frac{3}{3^5} + \frac{4}{3^5} + \cdots$	$\frac{3}{1^5}, \frac{3}{1^5} + \frac{3}{2^5}, \frac{3}{1^5} + \frac{3}{2^5} + \frac{3}{3^5}, \ldots$	

The interviewer shows the sequences of partial sums to the student individually and asks the following questions:

1) *How would you label the axes for the graph we will produce for the first sequence of partial sums?*
2) **Interviewer starts animation and stops at approximately $n = 30$.**
    a. *Does the sequence of partial sums converge?*
    b. *If so, what value does the sequence converge to? How can you tell?*
        i. **I have created an input box for us to record the value to which you believe that the series converges. The input box is titled "Center," and it will become clearer why I have given the box this name later in the interview. If we put in a value such as Center = 50, then a horizontal line will appear at the value _______________ = 50** (*Interviewer uses student's axis name*). **We can use this value as a way to check whether the values to which we believe that the sequence of partial sums converges.**
        ii. **What inscription can we use to describe the value to which you believe the series converges? Please add this inscription to your glossary.**
    c. (***If student says the sequence does not converge, interviewer picks a value***) *Why does the sequence not converge to the value _________________?*

3) **Interviewer starts and stops the animation at approximately $n = 70$ and $n = 105$ while asking the same questions.**
4) **After finishing the questions for each value of n, the interviewer will ask the following:**
   a. *What does it mean for the sequence to converge?*
      i. **Based on what you have said, I will write a preliminary rule for sequence convergence: "_________________________." As we go through the next sequence of partial sums, we may modify this rule as you see fit. I am hoping that you can create a rule to describe the convergence for any sequence of partial sums.**
   b. ***(If applicable, interviewer points out inconsistencies between students' rules and students' reasoning):*** *I see that you wrote this ("____"), but it seems to me that by your written rule, this sequence should/should not converge to (value). How might you modify your rule so that this is not the case?*
5) **Interviewer shows graphs of SPS 2 and SPS 3 and asks the same questions.**
6) **After SPS, the interviewer adds the following questions:**
   a. *Does the sequence of partial sums converge to* **_______***? (value relatively far from student's proposed limit, value relatively close to student's proposed limit)*
   b. *Does the sequence converge to both values?*
   c. *How does your rule show your thinking?* (interviewer will tweak the wording of students' rules according to his understanding of students' explanations and ask for students' approval of all changes)

The goal of this section of the teaching session is to thoroughly perturb and confuse the student with their conceptions of limit (*this is highly likely to be the case, but not guaranteed*). In this way, I can promote the necessity of ϵ and the ϵ-strips to give meaning to the concept of limit.

<u>Task 2, Part 2: Introducing ϵ-strips (~20 min)</u>

(*Note: The interviewer will present the SPS in the following order: SPS3, SPS2, SPS1)*

Interviewer: It seems like we need something else to help us really give meaning to the idea of limit. So, I am going to introduce an idea called an ϵ-strip (*interviewer shows ϵ-strip, centered at student's chosen limit value on SPS 3 with animated ϵ cycling through interval [.001,2]*). This bar extends horizontally across the graph (not vertically), and has the line "_____=______" (*interviewer uses student's inscriptions and axis name*) running down the center of the strip. The value of ϵ is the distance between the center of the strip (the value of ___) and the top or bottom of the strip. The value of ϵ can be any positive integer (large or small), although I am limited by this program with the span of values that I can display. (*Interviewer runs animation to show ϵ varying.)* Now, I am going to stop the animation at some value of ϵ (*interviewer pauses the animation at a value of ϵ that seems to capture a majority of the dots on the screen*).

1) **How many dots are inside the strip?**
2) **How many dots are outside the strip?**
3) **It would take a long time to actually count all of the dots, so is there a different way that we can refer to the number of dots inside or outside the strip?**

(*Interviewer repeats this process and questions several times while ϵ-strip is centered at the limit value*)

Interviewer: Now, let me move the ϵ-strip another location (*interviewer moves the ϵ-strip so it is centered at another value—not the limit—and starts/pauses animation again*).

1) **How many dots are inside the strip?**
2) **How many dots are outside of the strip?**
3) **How are you determining this, i.e., what is your thought process for determining your answer?**

Task 3: Inscriptions from the ϵ-strip activity

Interviewer: In the last 10 minutes of the interview, I wanted to ask you whether this activity has given you any new ideas for which you would like to create an inscription. You have already created the inscription **_________** for the value to which a series converges. However, are there any other ideas that you have seen in the graphs or during the activities that you want to represent using an inscription?

(*Student creates new inscriptions.)* ***For each inscription, the interviewer asks:***

1) **What does this inscription mean?**
2) **Why did you create this inscription?**
3) **Which category should we give to this inscription?**

Task 4 (optional): The Two Definitions of Convergence (~20 min)

Interviewer: So far during this interview, you have come up with the following rule for determining whether a sequence of partial sums converges and the value that it converges to: (**____**). (***interviewer reads student rule***). I think that it might be difficult for you to discover ϵ-strips and come up with a written rule for sequence of partial sums convergence using ϵ-strips all in the same day. So, rather than have you invent a written rule for sequence convergence using ϵ-strips from scratch, I am going to introduce two rules that have been presented by previous students. I am going to call these written rules "Adam's Rule" and "Benjamin's Rule" (*interviewer shows the two definitions*).

I am going to write your inscription for the value that the sequence converges to each of these rules. (*Interviewer edits rules to reflect student's inscriptions*) Could you read these two rules aloud? What are the similarities and differences between these two rules?

Adam's Rule
The sequence of partial sums converges to the value ___ if, for any ϵ-strip, infinitely many points are inside the strip, where the strip is centered at $y =$ _____.

Benjamin's Rule
The sequence of partial sums converges to the value ____ if, for any ϵ-strip, finitely many points are outside the strip, where the strip is centered at $y =$ _____ .

Interviewer: Thank you. Now I want to look at the graphical representations for each sequence of partial sums and see whether you think either of these definitions (or both) might serve as a good written rule for determining the limit of a sequence of partial sums. Keep in mind the two questions that we typically ask ourselves. First, does the sequence converge? Second, if the sequence converges, what value does it converge to? Once you have chosen a value for ___ (if appropriate), then I will run the ϵ animation to help you think about whether you have found the value of the limit. I will ask similar questions as the ones I asked before as we go through each sequence of partial sums. Finally, after each sequence of partial sums I will ask which of the rules seems most appropriate to you to describe determining the convergence and limit value of a sequence of partial sums.

The student goes through each sequence of partial sums and does the following:

1) *Hypothesizes whether the sequence of partial sums converges*
2) *Hypothesizes a value to which the SPS converges*
3) *Interviewer plays animation to increase students attention on the value of ϵ varying. At regular intervals, the interviewer will pause the animation and ask how many dots are inside the strip and outside the strip.*
4) *Interviewer makes minor changes to the proposed value of S that make no change, little change, and large change to the number of dots that are inside/outside the strip and asks the student whether any of these values could be the limit.*
5) *The interviewer asks the student which rule(s) (in his/her thinking) best convey the idea of limit of SPS.*

Note to interviewer:

- ***This setup will also accompany the next teaching session (2 days total). On day 1, the interviewer will present p-series that have monotone increasing sequences of partial sums. On day 2, the interviewer will introduce oscillating sequences of partial sums, including Grandi's series and the series with a complex general term. Once the student settles on either Adam & Benjamin both being necessary or just Benjamin being necessary (either is sufficient to define limit of sequence), the interviewer might consider doing a comparison task with many types of sequences (e.g., the original series from day 1, day 2) to further reinforce students' thinking before proceeding to the symbolization of components of the rule.***

Day 6 Interview Protocol

Day 6: Convergence of SPS and the ϵ-strip activity (part 2)

<u>Overview</u>: The learning goals for this interview are to (1) develop personal criteria for the convergence of the sequence of partial sums, (2) develop inscriptions to represent pertinent (to the student) parts of this criteria, and (3) assimilate to this criterion to determine the value to which a convergent sequence of partial sums does or does not converge. The research goals for this interview are to (1) categorize the ways in which students correlate inscriptions with graphical representations, (2) determine which graphical and structural components of sequence convergence students believe merit creating inscriptions, and (3) model the evolution of students' thinking and their inscriptions as they transition from primarily algebraic to primarily graphical reasoning.

To-do list before interview:

- Set up OneNote file for student
- Send link to OneNote file to student
- Check embedded GeoGebra file and make sure that it loads properly
  - Load file through GeoGebra site as a backup
- Have student's ASU email address handy for sending Amazon gift card

Follow-up from previous week

Interviewer: What topics have you covered in the last week in your calculus class? What do you believe are the important ideas from the inscriptions video? (*Interviewer will not show inscriptions video this week, but will provide any important information student forgets*).

Task 1: Creating Inscriptions for ϵ-strip Activity

Interviewer: I want to review your inscriptions this week by showing you a screen shot from the GeoGebra activity that we were doing last week. In particular, I want to know whether your inscriptions can be used to represent components of this picture. You may discuss your inscriptions in any order that you would like, but I would like you to think about each inscription.

Questions:
1) Can this inscription be used to describe a portion of the picture? Why or why not?
2) Could I modify this inscription so that it could be used? How so?
3) You have not used your inscriptions to describe this component of the picture. How might you represent this using an inscription?

Student 2: Sylvia

Interviewer: Last week, you told me that your instructor used the graph of the sequence of partial sums in your class sometime during the unit on infinite series. When he showed the graph, what sorts of things did he discuss about the graph? In particular, how did he use the graph to talk about the two big questions that we have discussed: *Does the sequence converge?* and *What value does the sequence converge to?*

Task 2: ϵ-strip activity (Iteration 2)

Task 2, Part 1: Reasoning about Convergence without ϵ-strip

Interviewer: Today we are going to explore three sequences of partial sums and try to answer the two questions that I presented last time:

3) Does this sequence of partial sums converge?
4) If the sequence of partial sums converges, what does it converge to?

In our last session, you developed a general rule that you could use to determine whether a sequence of partial sums converges. Your criteria were:

SYLVIA	If the sequence gets closer and closer to one real number, then it converges to that number. If the sequence gets infinitely large, then it diverges (keeps increasing without limit).
MONICA	If the values of P_n are approaching a horizontal asymptote that they get very close to but do not touch (i.e., limit), the the sequence of partial sums converges to that limit.

Today, I want you to come up with a written rule for sequence of partial sums convergence using ϵ-strips. To help you out, I want to present the criteria that two of my previous students, Adam and Benjamin, developed. (*Interviewer shows rules*). Can you read each of these rules? What do each of these rules mean? What are the differences between the two rules?

Adam's Rule
The sequence of partial sums converges to the value ___ if, for any ϵ-strip, infinitely many points are inside the strip, where the strip is centered at $y =$ _____.

Benjamin's Rule

The sequence of partial sums converges to the value ____ if, for any ϵ-strip, finitely many points are outside the strip, where the strip is centered at $y =$ ____ .

I am hoping that as you do this activity, you will decide which rule or rules work best as a personal criterion to think about sequence of partial sums convergence. I will be using a GeoGebra applet today to dynamically present several sequences of partial sums. For this activity, I will run the applet, but you may annotate in whatever way you would like.

Sequences of partial sums for day 6

Series Number	Series	Partial Sums Expanded form	Converges?
1	$\sum_{n=1}^{\infty} \frac{2}{\sqrt[4]{n}}$	$\frac{2}{\sqrt[4]{1}},\ \frac{2}{\sqrt[4]{1}}+\frac{2}{\sqrt[4]{2}},\ \frac{2}{\sqrt[4]{1}}+\frac{2}{\sqrt[4]{2}}+\frac{2}{\sqrt[4]{3}}, \ldots$	Yes
2	$\sum_{n=1}^{\infty} \frac{5}{n}$	$\frac{5}{1},\ \frac{5}{1}+\frac{5}{2},\ \frac{5}{1}+\frac{5}{2}+\frac{5}{3}, \ldots$	No
3	$\sum_{n=1}^{\infty} \frac{3}{n^5}$	$\frac{3}{1^5},\ \frac{3}{1^5}+\frac{3}{2^5},\ \frac{3}{1^5}+\frac{3}{2^5}+\frac{3}{3^5}, \ldots$	Yes
4	$\sum_{n=1}^{\infty} (-1)^n \left(\frac{6}{n^2}\right)$	$\frac{6}{1},\ \frac{6}{1}-\frac{6}{4},\ \frac{6}{1}-\frac{6}{4}+\frac{6}{9}, \ldots$	Yes
5	$\sum_{n=0}^{\infty} (.04)\cdot(-1)^n$	$.04,\ .04-.04,\ .04-.04+.04, \ldots$	No

The interviewer shows SPS 3 to the student and asks the following questions:

7) *How would you label the axes for the graph we will produce for the first sequence of partial sums?*
8) **Interviewer starts animation and stops at approximately $n = 30$.**
   a. *Does the sequence of partial sums converge?*
   b. *If so, what value does the sequence converge to? How can you tell?*
      i. **Can you explain to me why the sequence does (or does not) converge using your rule? How about Adam's rule? How about Benjamin's rule?**
   c. (***Interviewer picks a value close to the student's chosen value if converges or any value the sequence passes through if the series diverges.***)
      i. *Can you explain to me why the sequence does not converge to the value _________________ using (your rule, Adam's rule, Benjamin's rule)?*

ii. (***If student says sequence converges***) *Does the sequence converge to both values?*

d. **Which rule or rules seem most appropriate to describe sequence convergence?**

9) **Interviewer starts and stops the animation at approximately $n = 70$ and $n = 105$ while asking the same questions.**

10) **Interviewer shows graphs of SPS 4, SPS 5, and SPS 6 (if time) and asks the same questions.**

Task 3: Inscriptions from the ϵ-strip activity

Interviewer: In the last 10 minutes of the interview, I wanted to ask you whether this activity has given you any new ideas for which you would like to create an inscription. You have already created the inscription **_________** for the value to which a series converges. However, are there any other ideas that you have seen in the graphs or during the activities that you want to represent using an inscription?

(*Student creates new inscriptions.)* ***For each inscription, the interviewer asks:***

4) **What does this inscription mean?**
5) **Why did you create this inscription?**
6) **Which category should we give to this inscription?**

Note to interviewer:

- ***This setup will also accompany the next teaching session (2 days total). On day 1, the interviewer will present p-series that have monotone increasing sequences of partial sums. On day 2, the interviewer will introduce oscillating sequences of partial sums, including Grandi's series and the series with a complex general term. Once the student settles on either Adam & Benjamin both being necessary or just Benjamin being necessary (either is sufficient to define limit of sequence), the interviewer might consider doing a comparison task with many types of sequences (e.g., the original series from day 1, day 2) to further reinforce students' thinking before proceeding to the symbolization of components of the rule.***

Day 7 Interview Protocol

Day 7: Symbolizing Convergence of SPS

Overview: The learning goals for this interview are to (1) develop personal criteria for the convergence of the sequence of partial sums, (2) develop inscriptions to represent pertinent (to the student) parts of this criteria, and (3) assimilate to this criterion to determine the value to which a convergent sequence of partial sums does or does not converge. The research goals for this interview are to (1) categorize the ways in which students correlate inscriptions with graphical representations, (2) determine which graphical and structural components of sequence convergence students believe merit creating inscriptions, and (3) model the evolution of students' thinking and their inscriptions as they transition from primarily algebraic to primarily graphical reasoning.

To-do list before interview:

- Set up OneNote file for student
- Send link to OneNote file to student
- Check embedded GeoGebra file and make sure that it loads properly
    - Load file through GeoGebra site as a backup
- Have student's ASU email address handy for sending Amazon gift card

Follow-up from previous week

Interviewer: What topics have you covered in the last week in your calculus class?

Interviewer: Since this is our last interview, I want to be sure that we have your glossary entirely filled out. As you can see, you have not provided information for some of your inscriptions. Can you fill out the rest of the glossary for the inscriptions in your glossary?

Questions:

1) What does this inscription represent?
2) Can you create a graphical representation that also corresponds to this inscription?

Review Task 1a (Monica): Types of Inscriptions (Part I)

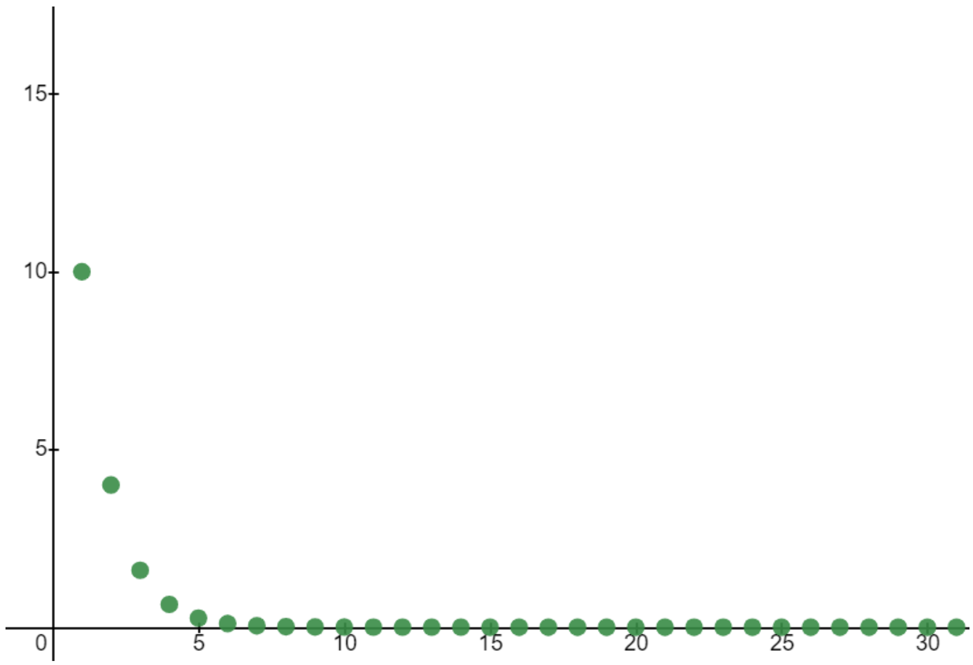
It seems that in our last interview, there were times when you weren't entirely sure how to correlate your inscriptions with some of the graphs that I presented. To take a deeper look at how you might connect your inscriptions with graphs of sequences, I have prepared two graphs that are related in some way to the following sequence (*interviewer indicates sequence*) and a list of seven inscriptions from your glossary that you have used to reference components of various sequences. For each inscription, can you tell me whether you can represent any of the following for me using that inscription?

1) A single point on the graph
2) The 31 dots on the screenshot of the graph
3) A sequence (i.e., the sequence represented by the graph if it were to keep going)

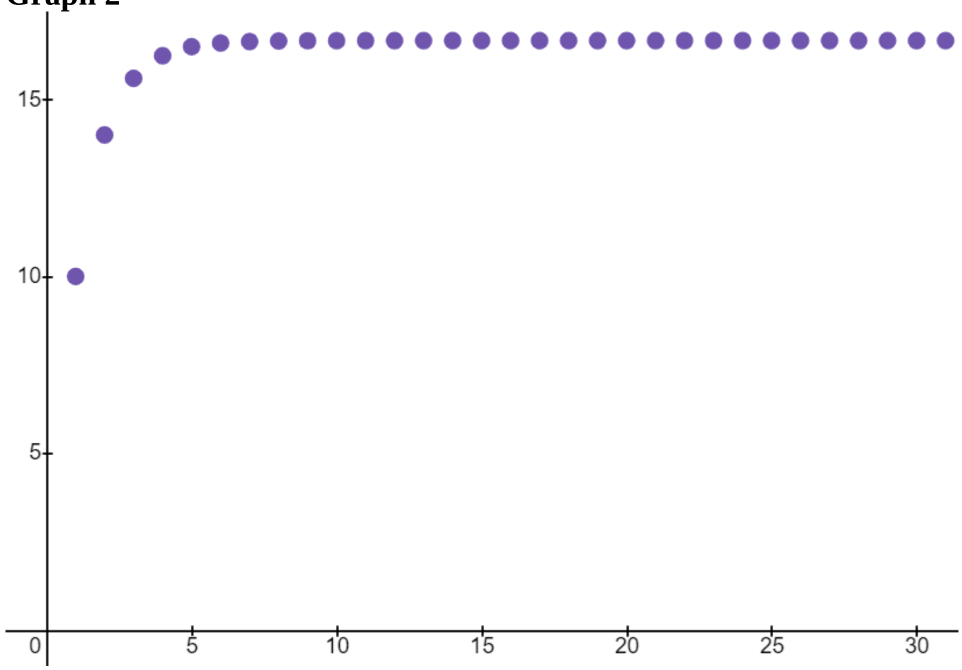
If the student cannot represent one of the ideas with an inscription, recommend that she create a new inscription.

Sequence	$10, \frac{20}{5}, \frac{40}{25}, \frac{80}{125}, \ldots$		
Inscriptions			
$f(n) = 10\left(\frac{2}{5}\right)^{n-1}$	$\sum_{1}^{m} 10\left(\frac{2}{5}\right)^{n-1}$	$\sum_{1}^{m} f(n)$	$\sum_{1}^{m} p_n$
$a_n = a_1, a_2, a_3, \ldots$	$S_p = p_1, p_2, p_3, \ldots$	$\sum_{1}^{\infty} f(n)$	

Graph 1:



Graph 2



<u>Review Task 1b (Monica): Types of Inscriptions (Part II)</u>

Interviewer: As I thought about your inscriptions, I wondered if there were any connections between certain inscriptions. (*Interviewer hand writes inscriptions on OneNote using iPad*).

Example 1: You have an inscription, p_n for a partial sum and another inscription, $\Sigma_1^m f(n)$ also listed as a partial sum. Do you think that we could say something like $\boldsymbol{p_m = \Sigma_1^m f(n)}$**,** or would this not work for your inscriptions? How might you use one (or both) of these inscriptions to represent some portion of the graphs?

Example 2: You have two inscriptions, a_n and $f(n)$, which represent the value of a term given position n. Do you think that we could say something like $\boldsymbol{f(n) = a_n}$**,** or would this not work for your inscriptions? How might you use one (or both) of these inscriptions to represent some portion of the graphs?

Example 3: You have two inscriptions, $S_p = p_1, p_2, p_3, \ldots$ and $\Sigma_1^m p_n$, which incorporate the inscription p_n. Do you think that we could say something like $\boldsymbol{S_p = \Sigma_1^m p_n}$, or would this not work for your inscriptions? How might you use one (or both) of these inscriptions to represent some portion of the graphs?

Task 1: Personal Expression for Convergence (SPS case)

Interviewer: Over the last several interviews, we have looked at the convergence of the sequence of partial sums using ϵ-strips. Today, I am going to return to the idea of personal expressions. Your goal for today will be to create personal expressions to represent components of Benjamin's rule.

I made this table to help you think about creating personal expressions. In the table, I have separated each phrase of Benjamin's rule onto a separate line. You do not have to adopt my method, but I thought this set-up might give you ideas about how to represent Benjamin's rule using your inscriptions. For instance, you may create one personal expression to represent the entire section, or, if you wish, you may construct multiple personal expressions to represent different phrases or subsections. Do you have any questions so far?

Adam's Rule
The sequence of partial sums
converges to the value ______,

if, for any ϵ-strip,
infinitely many points are inside the strip,
where the strip is centered at $y =$ ______.

Benjamin's Rule
The sequence of partial sums
converges to the value ______,
if, for any ϵ-strip,
finitely many points are outside the strip,
where the strip is centered at $y =$ ______.

First, I want to know whether you can use any of the inscriptions in your current glossary to describe portions of Benjamin's rule. After we have discussed all of the elements in your glossary, I will ask if you would like to use any additional inscriptions or expressions. If you wish to create any new inscriptions or expressions, we will add them to the glossary. If you do not wish to create any new inscriptions or expressions, we will move on to the next task. Do you have any questions? Ok, let's begin.

Interviewer proceeds through the student's glossary entries and asks the following questions for each inscription/expression:

1) *Could you use this inscription/expression to describe components of the first section? Why or why not?*
2) (If YES) *Does this inscription/expression represent the entire first section? Why or why not? What might you need to add/delete from the inscription/expression to represent the entire first section?*

3) *Could you use this inscription/expression to describe components of the last section? Why or why not?*
4) (If YES) *Does this inscription/expression represent the entire last section? Why or why not? What might you need to add/delete from the inscription/expression to represent the entire last section?*
5) (AFTER going through entire glossary) *Are there any other inscriptions or expressions that you want to create to represent part of the first section or the last section?*
    a. *What are you trying to represent with this inscription/expression?*
    b. *Why was this idea not represented with the inscriptions/expressions that you already have?*
    c. *Can you update the table to show your new representation for the first/last section with your new inscription/expression?*

Interviewer: Now that you have constructed inscriptions to represent sequence of partial sums convergence, I would you to utilize your personal expression to model the convergence for specific sequences of partial sums. For example, you said that this series (*interviewer indicates series*) converges to the value (______). **Could you represent this sequence of partial sums converging to this particular value with your personal expression?** (*Interviewer repeats this question for all sequences of partial sums that the student claimed converged*).

Task 2: Relationship Between Sequences and Series

Interviewer: We have spent a lot of time over the last two interviews discussing sequence of partial sums convergence. I want to return to the concept of infinite series. In a few minutes, I am going to have you construct a written rule and a personal expression to represent series convergence. Before doing this, I wanted to ask you to reflect on the following two questions:

1) What relationships do you believe exist between a sequence, a sequence of partial sums, and an infinite series?
2) Suppose a sequence of partial sums converges to a value such as 5. Does this tell you anything about the corresponding infinite series?

Sequence	$\mathbf{\frac{3}{1}, \quad -\frac{4}{4}, \quad \frac{5}{9}, \quad -\frac{6}{16}, \dots}$
Sequence of Partial Sums	$\frac{3}{1}, \quad \frac{3}{1}-\frac{4}{4}, \quad \frac{3}{1}-\frac{4}{4}+\frac{5}{9}, \dots$
Infinite Series	$\frac{3}{1}-\frac{4}{4}+\frac{5}{9}-\frac{6}{16}+\cdots$

Interviewer: You claim that the relationship between an infinite series and its corresponding sequence of partial sums is (_____________________). Do you think that this relationship holds between all series and sequences of partial sums? Specifically, do you think that the relationship holds for (*series that the student said converged*) and (*series that the student said did not converge*)? Why or why not?

Interviewer: After discussing similarities between sequences, sequences of partial sums, and series, I would like you to try to construct a written rule to determine infinite series convergence. You may use any language that makes sense to you, but I would like you to write your rule entirely in English, if possible. If you decide to use an inscription in your rule, please provide an explanation for why you chose to include that inscription (as opposed to an English word).

Monica's Rule
An infinite series converges to the value ____, if _______________
____

Sylvia's Rule
An infinite series converges to the value ____, if ________________
____

Interviewer: When we first discussed infinite series convergence, you stated that the series (________) converged to (_____________). Please use Monica's/Sylvia's rule to justify that the series converges to (____________). **Could your rule work for any series? Why or why not?**

Task 3: Personal Expression for Convergence (series case)

Interviewer: We are almost to the end of the interview tasks. In fact, this will be the last task before we begin our final debriefing session. Over the last few days, you have constructed written rules and personal expressions for sequence convergence and series convergence. I am going to put all of your rules and personal expressions on the screen at the same time (*Interviewer organizes screen with all rules and expressions sorted by sequence and series.)*

Interviewer: In our last interview, you stated that the relationship between a sequence of partial sums and a series is (________________________). **Will you review what you meant by this relationship and tell me if you are still thinking the same way?** The purpose of this final task is for you to construct a personal expression for infinite series and use it to model convergent series and the values that the series converge to.

1) Can you use any of your previous personal expressions to model a convergent infinite series?
2) What new inscriptions or personal expressions might you have to create to represent an infinite series that converges to a specific, finite value?
3) Create a personal expression to represent and infinite series that converges to a specific, finite value.
4) I noticed that you had this idea (____________) in Carly's written rule for series convergence but that it does not appear to be represented in your personal expression? Am I interpreting your expression in the way you intend? How might you modify your expression to include the idea (___________)?

(*The student creates a personal expression for infinite series convergence*)

Interviewer: In our previous interviews, you said that the series (________) converged to (____). Can you use your personal expression for series convergence to represent this idea? (*Interviewer presents several series, paying careful attention to whether the values that the series converge to correspond to the values that the sequence of partial sums converged to*).

Final Reflection

At the end of the final task, the student should have constructed several personal expressions and inscriptions. The interviewer will prepare a glossary containing all of the inscriptions that the student has introduced over the course of the interviews and these personal expressions. The student will do the following:

1) Describe the inscriptions in each personal expression

2) Provide a specific instantiation of his personal expression and describe the situation being represented

3) Describe what motivated him to construct each personal expression

4) Indicate whether any of his personal expressions have been subsumed by another, or if each personal expressions represent distinct situations and cannot be substituted.

After the student has responded to these four questions, I will present one final task. In this task, I will present the following two questions to the student, one at a time. I will ask the student to respond to these questions and will only ask clarifying questions. At the conclusion of these questions, I will end the interview.

Question 1: Suppose that an infinite series converges to the value of 4.23. What might this imply about the corresponding sequence of partial sums?

Question 2: Suppose that a sequence of partial sums converges to the value of -1.12. What might this imply about the corresponding infinite series?

Interviewer: I appreciate all the time that you have spent with me during these last couple of months to help me investigate students' thinking about infinite series and sequences of partial sums. I would like to schedule a final exit interview with you, but then we will be finished with the study.

***Interviewer Note: Be sure to ask about compensation and confirm method of payment for after the exit interview!**

Exit Interview Protocol

***Interviewer note: the exit interview will be nearly identical to the intake interview.**

Overview:

The research goals for this interview are to:

(1) Determine how students' meanings for series convergence have changed throughout the interviews.
(2) Determine which ideas from the teaching experiment the students transfer to a general discussion on series
(3) Determine the role that the students' personal expressions play in discussing series convergence.

To-do list before interview:

- Set up OneNote file for student
- Send link to OneNote file to student
- Prepare GeoGebra applet for Abigail's series
- Prepare Qualtrics Screening Survey for student
- Have student's ASU email address handy for sending Amazon gift card

Task 1: Reasoning about series convergence (~40 minutes)

Interviewer: Please navigate to Task 1 in the OneNote file. For each of the tasks, I have included a different infinite series created by a student named Abigail. For each of Abigail's series, I will ask you the same two questions:

(3) Does the series converge? How can you tell?
(4) If the series converges, what value does it converge to? How can you tell?

Please note that I am more interested in the processes by which you approach these problems than by any numerical results (or "correct answers") that you produce. For example, if I were to give you the problem 1 + 2, I would not want you to merely answer "3." Rather, I would want an explanation such as "I'm thinking of putting one chip together with two other chips and counting the total number of chips, which is 3." I will likely ask you to summarize your methods for examining the series from time to time. Remember, I am not concerned whether or not you are able to produce a textbook "correct" answer; rather, I am only interested in what you are thinking.

I have also created a version of the ϵ-strip activity for the sequence of partial sums corresponding to each series. You are welcome to use these graphs if they are helpful to you in making a determination about whether or not a particular series converges. I will not show these graphs to you unless you ask me, so do not feel like you are obligated to use them.

Finally, I am not expecting you to be able to answer all of these questions. You are welcome to skip or respond "I don't know" to any question. In this instance, I will ask you "What would you need to know in order to answer the question?" OK, let's get started. Please answer the two questions for Abigail's first series on Task 1.

Series	Expanded Form	Series type	Sequence of Partial Sums	Converge	Limit Value
$\sum_{n=0}^{\infty} \frac{3}{\sqrt{n}}$	$\frac{3}{\sqrt{1}} + \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{3}} + \cdots$	p-series ($0 < p < 1$)	Monotone increasing	No	
$\sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n^5}$	$\frac{2}{1^5} - \frac{2}{2^5} + \frac{2}{3^5} - \cdots$	Alternating p-series ($p > 1$)	Oscillating	Yes	≈ 1.94
$\sum_{n=1}^{\infty}\sum_{i=1}^{99}[10^{-2n-1} - 10^{-2(n+1)-1}i] = \sum_{k=0}^{\infty} \frac{495}{10000}\left(\frac{1}{100}\right)^k$	$\frac{99}{10^3} + \frac{98}{10^3} + \cdots + \frac{1}{10^3} + \frac{99}{10^5} + \cdots + \frac{1}{10^5} + \frac{99}{10^7} + \cdots + \frac{1}{10^7} + \cdots$	Geometric	Monotone increasing	Yes	$\frac{1}{20}$
$\sum_{n=0}^{\infty} \frac{(200-2n)(-1)^n}{n+1}$	$\frac{200}{1} - \frac{198}{2} + \frac{196}{3} - \cdots$	Alternating series	Oscillating	No	
$\sum_{i=0}^{\infty} a_i$ (where a_i corresponds to the i^{th} decimal place of π and $a_0 = 3$.)	$3 + .1 + .04 + \cdots$	Decimal expansion of irrational number	Monotone increasing	Yes	π
$\sum_{n=0}^{\infty} (.07) \cdot (-1)^n$	$.07 - .07 + .07 - \cdots$	Alternating series (Grandi's)	Oscillating	No	

The student attempts to answer the two questions for each series, which are presented one at a time on different OneNote pages. After the student has completed the questions for each series, the interviewer will ask the following questions:

Questions:

1) To confirm, you stated that this series (does/does not) converge, and that the series converges to ____, correct? How did you determine this?
2) What similarities or differences did you experience between this series and Abigail's other series?
3) How can you use your inscriptions to describe the convergence of the series and the value that the series converges to?
4) *Any other questions that the interviewer feels to ask to clarify students' thinking or meanings.*

If a student asks to examine a sequence of partial sums graph:

1) Why do you want to use this graph? What might it help you to determine?
2) What does this point represent on the graph? How would you determine the values of the point?
3) How does the sequence of partial sums (or its graph) help you to answer the two questions about infinite series convergence?

Task 2: Reasoning about general series convergence (~20 minutes)

After the student completes all series:

Interviewer: Now that you have completed all the series, I will ask you to address the two questions that we have answered for each of Abigail's series in a general way. The following two questions are not about Abigail's series, but some mysterious series that we encounter. In other words,

(4) How can I tell whether the mysterious series converges?
(5) If the mysterious series converges, how can I determine the value to which it converges?
(6) How could you use your inscriptions to represent your answer to questions (1) and (2)?
(7) Could you compare your answer to (3) to Adam's or Benjamin's rule?

(Interviewer navigates to Task 7). I will ask you to write an answer to each of these questions on this screen. Take your time, and I will ask you to explain your answers when you are finished.

Student answers questions and interviewer asks the student to explain each response.

Task 3: Complete the Screening Survey One More Time (~20 minutes)

Interviewer: I would like to ask a favor at the end of this interview. At the beginning of this study, I had you complete a screening survey to determine the students I wanted to interview. Would you be willing to work through the screening survey one more time before you leave to (1) explain your thinking as you take the survey and (2) provide constructive criticism for how I have structured the survey and worded the questions? You will not have to type your responses; rather, I want you to read the questions and respond verbally, telling me how you are reasoning about each question as you go. Are you willing to do this? Thank you.

(*The student takes the screening survey one last time*)

Interviewer: Thank you so much for your efforts during this study. The last thing that I want to do is make sure that we have sorted out how much you will be paid and how I will send you the money. By my records, you have attended (____) interviews, so I owe you $(____). Did I count correctly? Okay. What is an email address or other method for me to send you the money? Thank you. I will send you the money sometime within the next week. Again, I appreciate your sacrifice to participate in these interviews, and I wish you the best in the future. Your input will help me not only finish my dissertation and earn my Ph.D. at ASU, you will help me further research in mathematics education and present new findings that will further humankind's knowledge.

APPENDIX D

HUMAN SUBJECTS APPROVAL LETTER

ASU Knowledge Enterprise Development

EXEMPTION GRANTED

Kyeong Roh
CLAS-NS: Mathematics and Statistical Sciences, School of (SMSS)
480/965-3792
khroh@asu.edu

Dear Kyeong Roh:

On 12/3/2019 the ASU IRB reviewed the following protocol:

Type of Review:	Initial Study
Title:	Connections between Student Meanings and Representations for Sequences and Series
Investigator:	Kyeong Roh
IRB ID:	STUDY00011079
Funding:	None
Grant Title:	None
Grant ID:	None
Documents Reviewed:	• Consent Form (Updated 11/19/2019), Category: Consent Form; • Interview Protocol, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions); • IRB Protocol_DE (Draft 2), Category: IRB Protocol; • Recruitment e-mail, Category: Recruitment Materials;

The IRB determined that the protocol is considered exempt pursuant to Federal Regulations 45CFR46 on 12/3/2019.

In conducting this protocol you are required to follow the requirements listed in the INVESTIGATOR MANUAL (HRP-103).

Sincerely,

IRB Administrator

cc: Derek Eckman
Derek Eckman