Coloring Graphs from Almost Maximum Degree Sized Palettes

by

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ABSTRACT

Every graph can be colored with one more color than its maximum degree. A well-known theorem of Brooks gives the precise conditions under which a graph can be colored with maximum degree colors. It is natural to ask for the required conditions on a graph to color with one less color than the maximum degree; in 1977 Borodin and Kostochka conjectured a solution for graphs with maximum degree at least 9: as long as the graph doesn't contain a maximum-degree-sized clique, it can be colored with one fewer than the maximum degree colors.

This study attacks the conjecture on multiple fronts. The first technique is an extension of a vertex shuffling procedure of Catlin and is used to prove the conjecture for graphs with edgeless high vertex subgraphs. This general approach also bears more theoretical fruit.

The second technique is an extension of a method Kostochka used to reduce the Borodin-Kostochka conjecture to the maximum degree 9 case. Results on the existence of independent transversals are used to find an independent set intersecting every maximum clique in a graph.

The third technique uses list coloring results to exclude induced subgraphs in a counterexample to the conjecture. The classification of such excludable graphs that decompose as the join of two graphs is the backbone of many of the results presented here.

The fourth technique uses the structure theorem for quasi-line graphs of Chudnovsky and Seymour in concert with the third technique to prove the Borodin-Kostochka conjecture for claw-free graphs.

The fifth technique adds edges to proper induced subgraphs of a minimum counterexample to gain control over the colorings produced by minimality.

The sixth technique adapts a recoloring technique originally developed for strong coloring by Haxell and by Aharoni, Berger and Ziv to general coloring. Using this recoloring technique, the Borodin-Kostochka conjectured is proved for graphs where every vertex is in a large clique.

The final technique is naive probabilistic coloring as employed by Reed in the proof of the Borodin-Kostochka conjecture for large maximum degree. The technique is adapted to prove the Borodin-Kostochka conjecture for list coloring for large maximum degree.

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INTRODUCTION

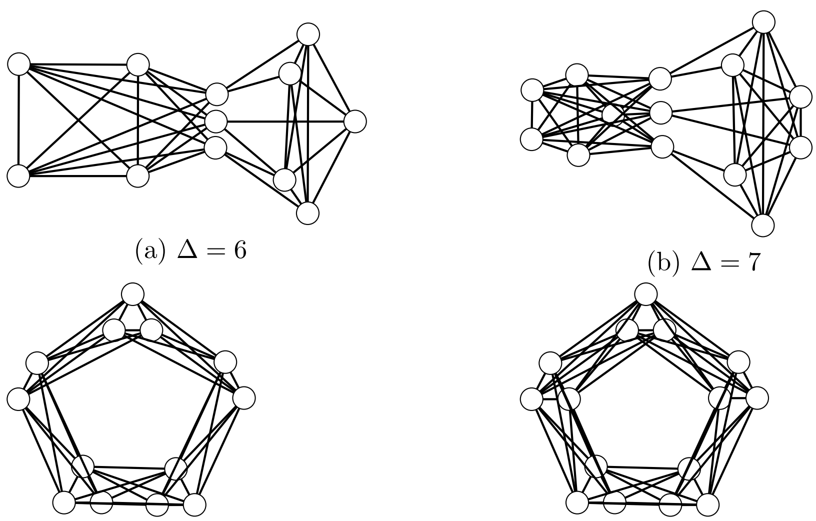
Graph coloring is instantiated by many partitioning problems that arise in practice. Any time we encounter a relation between things of some sort and wish to group those things so that no group contains a related pair, we have to solve a graph coloring problem. On some such encounters, the possible relations between our sorted things are restricted in a way that allows a small number of groups to be used. This is good. If the sort in question were *a sort of tasks to be performed* and the relation were *can't be performed at the same time*, then such a grouping, taken in some order, gives an order in which to perform the tasks. Fewer groups means faster task completion. How many groups are needed for a certain sort of task collection? How can we find a grouping with the minimum number of groups? If we can't find such a grouping, how can we at least determine the number of groups in a minimum grouping? This dissertation is primarily concerned with the first type of question—but an answer to the first type of question can have bearing on the other questions when coupled with a method of sorting. When we move from the instantiation to abstract graph coloring we call the things *vertices* and represent the relation of two vertices by an *edge* between them. The vertices together with the edges constitute a *graph* and one of our groupings corresponds to a labeling of the vertices with $1, 2, \ldots, k$ so that there are no edges between vertices receiving the same label. Such a labeling is called a *coloring* or more precisely a *k-coloring*. For a graph G we write $\chi(G)$ for the minimum k for which G has a k-coloring—this corresponds to the number of groups in a minimum grouping. This study concerns the relation of χ to other *graph parameters*. Our terminology and notation are basically standard (for all notation, see Appendix A).

1.1 A short history

Here we collect statements of the results and conjectures that have bearing on this inquiry woven together with some historical remarks and our improvements. The first non-trivial result about coloring graphs with around Δ colors is Brooks' theorem from 1941.

Theorem 1.1.1 (Brooks [12]). *Every graph with $\Delta \geq 3$ satisfies $\chi \leq \max\{\omega, \Delta\}$.*

In 1977, Borodin and Kostochka conjectured that a similar result holds for $\Delta - 1$ colorings. Counterexamples exist showing that the $\Delta \geq 9$ condition is tight (see Figure 1.1).



(a) $\Delta = 6$ (b) $\Delta = 7$ (c) $\Delta = 7$ (d) $\Delta = 8$

Figure 1.1: Counterexamples to the Borodin-Kostochka Conjecture for small Δ.

Conjecture 1.1.2 (Borodin and Kostochka [10]). *Every graph with $\Delta \geq 9$ satisfies $\chi \leq \max\{\omega, \Delta - 1\}$.*

Note that another way of stating this is that for $\Delta \geq 9$, the only obstruction to $(\Delta - 1)$-coloring is a K_Δ. In the same paper, Borodin and Kostochka prove the following weaker statement.

Theorem 1.1.3 (Borodin and Kostochka [10]). *Every graph satisfying $\chi \geq \Delta \geq 7$ contains a $K_{\left\lfloor \frac{\Delta+1}{2} \right\rfloor}$.*

The proof is quite simple once you have a decomposition lemma of Lovász from the 1960's [51].

Lemma 1.1.4 (Lovász [51]). *Let G be a graph and $r_1, \ldots, r_k \in \mathbb{N}$ such that $\sum_{i=1}^k r_i \geq \Delta(G) + 1 - k$. Then $V(G)$ can be partitioned into sets $V_1, \ldots, V_k$ such that $\Delta(G[V_i]) \leq r_i$ for each $i \in [k]$.*

Proof. For a partition $P := (V_1, \ldots, V_k)$ of $V(G)$ let

$$f(P) := \sum_{i=1}^{k} \left(\|G[V_i]\| - r_i \left|V_i\right| \right).$$

Let $P := (V_1, \ldots, V_k)$ be a partition of $V(G)$ minimizing $f(P)$. Suppose there is $i \in [k]$ and $x \in V_i$ with $d_{V_i}(x) > r_i$. Since $\sum_{i=1}^k r_i \geq \Delta(G) + 1 - k$, there is some $j \neq i$ such that $d_{V_j}(x) \leq r_j$ and thus moving x from V_i to V_j gives a new partition violating minimality of $f(P)$. Hence $\Delta(G[V_i]) \leq r_i$ for each $i \in [k]$. □

Now to prove Borodin and Kostochka's result, let G be a graph with $\chi \geq \Delta \geq 7$ and use $r_1 := \left\lceil \frac{\Delta-1}{2} \right\rceil$ and $r_2 := \left\lfloor \frac{\Delta-1}{2} \right\rfloor$ in Lovász's lemma to get a partition (V_1, V_2) of $V(G)$ with $\Delta(G[V_i]) \leq r_i$ for each $i \in [2]$. Since $r_1 + r_2 = \Delta - 1$ and $\chi \geq \Delta$, it must be that $\chi(G[V_i]) \geq r_i + 1$ for some $i \in [2]$. But $\Delta \geq 7$, so $r_i \geq 3$ and hence by Brooks' theorem $G[V_i]$ contains a $K_{\lfloor \frac{\Delta+1}{2} \rfloor}$.

A decade later, Catlin [14] showed that bumping the $\Delta(G) + 1$ to $\Delta(G) + 2$ allowed for shuffling vertices from one partition set to another and thereby proving stronger decomposition results. A few years later Kostochka [46] modified Catlin's algorithm to show that every triangle-free graph G can be colored with at most $\frac{2}{3}\Delta(G) + 2$ colors. In [64], we generalized Kostochka's modification to prove the following.

Lemma 3.2.1 (Rabern [64])**.** *Let G be a graph and $r_1, \ldots, r_k \in \mathbb{N}$ such that $\sum_{i=1}^{k} r_i \geq \Delta(G) + 2 - k$. Then $V(G)$ can be partitioned into sets $V_1, \ldots, V_k$ such that $\Delta(G[V_i]) \leq r_i$ and $G[V_i]$ contains no incomplete r_i-regular components for each $i \in [k]$.*

Setting $k = \left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ and $r_i = 2$ for each i gives a slightly more general form of Kostochka's triangle-free coloring result.

Corollary 3.2.2 (Rabern [64])**.** *The vertex set of any graph G can be partitioned into $\left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ sets, each of which induces a disjoint union of triangles and paths.*

For coloring, this actually gives the bound $\chi(G) \leq 2\left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ for triangle free graphs. To get $\frac{2}{3}\Delta(G) + 2$, just use $r_k = 0$ when $\Delta \equiv 2 (\text{mod } 3)$. Similarly, for any $r \geq 2$, setting $k = \left\lceil \frac{\Delta(G)+2}{r+1} \right\rceil$ and $r_i = r$ for each i gives the following.

Corollary 3.2.3 (Rabern [64])**.** *Fix $r \geq 2$. The vertex set of any K_{r+1}-free graph G can be partitioned into $\left\lceil \frac{\Delta(G)+2}{r+1} \right\rceil$ sets each inducing an $(r-1)$-degenerate subgraph with maximum degree at most r.*

In fact, we proved a lemma stronger than Lemma 3.2.1 allowing us to forbid a larger class of components coming from any so-called *r-permissible collection*. In section 3.2 we will explore a result that both simplifies and generalizes this latter result.

Also in the 1980's, Kostochka proved the following using a complicated recoloring argument together with a technique for reducing Δ in a counterexample based on hitting every maximum clique with an independent set.

Theorem 1.1.5 (Kostochka [45])**.** *Every graph satisfying $\chi \geq \Delta$ contains a $K_{\Delta-28}$.*

Kostochka [45] proved the following result which shows that graphs having clique number sufficiently close to their maximum degree contain an independent set hitting every maximum clique. In [59] we improved the antecedent to $\omega \geq \frac{3}{4}(\Delta+1)$. Finally, King [42] made the result tight.

Lemma 4.1.1 (Kostochka [45])**.** *If G is a graph satisfying $\omega \geq \Delta + \frac{3}{2} - \sqrt{\Delta}$, then G contains an independent set I such that $\omega(G - I) < \omega(G)$.*

Lemma 4.1.3 (Rabern [59])**.** *If G is a graph satisfying $\omega \geq \frac{3}{4}(\Delta+1)$, then G contains an independent set I such that $\omega(G - I) < \omega(G)$.*

Lemma 4.1.5 (King [42])**.** *If G is a graph satisfying $\omega > \frac{2}{3}(\Delta+1)$, then G contains an independent set I such that $\omega(G - I) < \omega(G)$.*

If G is a vertex critical graph satisfying $\omega > \frac{2}{3}(\Delta+1)$ and we expand the independent set I produced by Lemma 4.1.5 to a maximal independent set M and remove M from G, we see that $\Delta(G - M) \leq \Delta(G) - 1$, $\chi(G - M) = \chi(G) - 1$ and $\omega(G - M) = \omega(G) - 1$. Using this, the proof of many coloring results can be reduced to the case of the smallest Δ for which they work. In Chapter 4, we give three such applications.

A little after Kostochka proved his bound, Mozhan [55] used a function minimization and vertex shuffling procedure different than, but related to Catlin's, to prove the following.

Theorem 1.1.6 (Mozhan [55]). *Every graph satisfying* $\chi \geq \Delta \geq 10$ *contains a* $K_{\left\lfloor \frac{2\Delta+1}{3} \right\rfloor}$.

Finally, in his dissertation Mozhan proved the following. We don't know the method of proof as we were unable to obtain a copy of his dissertation. However, we suspect the method is a more complicated version of the above proof.

Theorem 1.1.7 (Mozhan). *Every graph satisfying* $\chi \geq \Delta \geq 31$ *contains a* $K_{\Delta-3}$.

In [63], we used part of Mozhan's method to prove the following result. For a graph G let $\mathcal{H}(G)$ be the subgraph of G induced on the vertices of degree at least $\chi(G)$.

Theorem 1.1.8 (Rabern [63]). $K_{\chi(G)}$ *is the only vertex critical graph* G *with* $\chi(G) \geq \Delta(G) \geq 6$ *and* $\omega(\mathcal{H}(G)) \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor - 2$.

Setting $\omega(\mathcal{H}(G)) = 1$ proved a conjecture of Kierstead and Kostochka [38].

Corollary 1.1.9 (Rabern [63]). $K_{\chi(G)}$ *is the only vertex critical graph* G *with* $\chi(G) \geq \Delta(G) \geq 6$ *such that* $\mathcal{H}(G)$ *is edgeless.*

In joint work with Kostochka and Stiebitz [48], we generalized and improved this result, again using Mozhan's technique. In section 3.1, we will improve these results further and simplify the proofs by using Catlin's vertex shuffling algorithm in place of Mozhan's.

In 1999, Reed used probabilistic methods to prove that the Borodin-Kostochka conjecture holds for graphs with very large maximum degree.

Theorem 1.1.10 (Reed [67]). *Every graph satisfying $\chi \geq \Delta \geq 10^{14}$ contains a K_Δ.*

A lemma from Reed's proof of the above theorem is generally useful.

Lemma 1.1.11 (Reed [67]). *Let G be a critical graph satisfying $\chi = \Delta \geq 9$ having the minimum number of vertices. If H is a $K_{\Delta-1}$ in G, then any vertex in $G - H$ has at most 4 neighbors in H. In particular, the $K_{\Delta-1}$'s in G are pairwise disjoint.*

In Chapter 7, we improve this lemma by showing that under the same hypotheses, any vertex in $G - H$ has at most 1 neighbor in H. Moreover, we lift the result out of the context of a minimal counterexample to graphs satisfying a certain criticality condition—we refer to such graphs as mules. This allows meaningful results to be proved for values of Δ less than 9. Also in Chapter 7, we prove that the following, prima facie weaker, conjecture is equivalent to the Borodin-Kostochka conjecture.

Conjecture 1.1.12 (Cranston and Rabern [20]). *If G is a graph with $\chi = \Delta = 9$, then $K_3 * \overline{K_6} \subseteq G$.*

At the core of these results are the list coloring lemmas proved in section 5. There we classify graphs of the form $A * B$ that are not f-choosable where $f(v) := d(v) - 1$ for each vertex v. In Chapter 6 we use these list coloring results together with Chudnovsky and Seymour's decomposition theorem for claw-free graphs [18] and our proof in [61] of the Borodin-Kostochka conjecture for line graphs of multigraphs to prove the conjecture for claw-free graphs.

Theorem 1.1.13 (Cranston and Rabern [19]). *Every claw-free graph with $\Delta \geq 9$ satisfies $\chi \leq \max\{\omega, \Delta - 1\}$.*

In Chapter 8, we adapt a recoloring trick previously used for strong coloring and prove the following.

Corollary 8.4.2. *Every graph with* $\chi \geq \Delta \geq 9$ *such that every vertex is in a clique on* $\frac{2}{3}\Delta + 2$ *vertices contains* K_Δ*.*

Using this we show that to prove the Borodin-Kostochka conjecture it is enough to prove it for irregular graphs; more precisely, we prove the following.

Theorem 8.5.1. *Every graph satisfying* $\chi \geq \Delta = k \geq 9$ *either contains* K_k *or contains an irregular critical subgraph satisfying* $\chi = \Delta = k - 1$*.*

In particular, the Borodin-Kostochka conjecture would follow from the following. This is at least somewhat plausible since the only known critical (or connected even) counterexample to Borodin-Kostochka for $\Delta = 8$ is regular (see Figure 7.4).

Conjecture 8.5.2. *Every critical graph satisfying* $\chi \geq \Delta = 8$ *is regular.*

As our final application of the recoloring trick, we prove the following bounds on the chromatic number. The first generalizes the result of Beutelspacher and Hering [5] that the Borodin-Kostochka conjecture holds for graphs with independence number at most two. This result was generalized in another direction in [19] (also Chapter 6) where the conjecture was proved for claw-free graphs.

Theorem 8.7.2. *Every graph satisfies* $\chi \leq \max\{\omega, \Delta - 1, 4\alpha\}$*.*

The second bound shows that the Borodin-Kostochka conjecture holds for graphs with maximum degree on the order of the square root of their order. This improves on prior bounds of $\Delta > \frac{n+1}{2}$ from Beutelspacher and Hering [5] and $\Delta > \frac{n-6}{3}$ of Naserasr [56].

Theorem 8.7.3. *Every graph satisfies* $\chi \leq \max\left\{\omega, \Delta - 1, \left\lceil \frac{15+\sqrt{48n+73}}{4} \right\rceil\right\}$*.*

Borodin and Kostochka also conjectured [47] that their conjecture holds for list coloring. In Chapter 9, we prove that this conjecture holds for large Δ.

Conjecture 1.1.14 (Borodin and Kostochka [47]). *Every graph with* $\Delta \geq 9$ *satisfies* $\chi_l \leq \max\{\omega, \Delta - 1\}$.

Chapter 2

BROOKS' THEOREM

In Chapter 7 we will rely heavily on the technique of adding edges to a proper induced subgraph of a minimum counterexample. We first learned of this technique when reading Reed's proof of the Borodin-Kostochka conjecture for large Δ (see [67]). To introduce the idea we give a short proof of Brooks' theorem. The proof is completely different from Lovász's short proof in [52]. We first reduce to the cubic case and then add edges to a proper induced subgraph to get a coloring we can complete. The reduction to the cubic case is an immediate consequence of more general lemmas on hitting all maximum cliques with an independent set that we prove in Chapter 4 (see also [45], [59] and [42]). Additionally, this reduction was demonstrated by Tverberg in [73]. One interesting feature of the proof is that it doesn't use any connectivity concepts. We'll give two versions of the proof, the first is shorter but uses the extra idea of excluding diamonds (K_4 less an edge).

Theorem 2.0.15 (Brooks [12])**.** *Every graph satisfies* $\chi \leq \max\{3, \omega, \Delta\}$*.*

Proof. Suppose the theorem is false and choose a counterexample G minimizing $|G|$. Put $\Delta := \Delta(G)$. Using minimality of $|G|$, we see that $\chi(G - v) \leq \Delta$ for all $v \in V(G)$. In particular, G is Δ-regular.

First, suppose G is 3-regular. If G contains a diamond D, then we may 3-color $G - D$ and easily extend the coloring to D by first coloring the nonadjacent vertices in D the same. So, G doesn't contain diamonds. Since G is not a forest it contains an induced cycle C. Since $K_4 \not\subseteq G$ we have $|N(C)| \geq 2$. So, we may take different $x, y \in N(C)$ and put $H := G - C$ if x is adjacent to y and $H := (G - C) + xy$ otherwise. Then, H doesn't contain K_4 as G doesn't contain diamonds. By minimality of $|G|$, H is 3-colorable. That is, we have a 3-coloring of $G - C$ where x and y receive different colors. We can easily extend this partial coloring to all of G since each vertex of C has a set of two available colors and some pair of vertices in C get different sets.

Hence we must have $\Delta \geq 4$. Consider a Δ-coloring of $G - v$ for some $v \in V(G)$. Each color must be used on every K_Δ in $G - v$ and hence some color must be used on every K_Δ in G. Let M be such a color class expanded to a maximal independent set. Then $\chi(G-M) = \chi(G)-1 = \Delta > \max\{3, \omega(G-M), \Delta(G-M)\}$, a contradiction. □

Here is the other version, not excluding diamonds and doing the reduction differently.

Theorem 2.0.16 (Brooks [12]). *Every graph G with $\chi(G) = \Delta(G) + 1 \geq 4$ contains $K_{\Delta(G)+1}$.*

Proof. Suppose the theorem is false and choose a counterexample G minimizing $|G|$. Put $\Delta := \Delta(G)$. Using minimality of $|G|$, we see that $\chi(G-v) \leq \Delta$ for all $v \in V(G)$. In particular, G is Δ-regular.

First, suppose $\Delta \geq 4$. Pick $v \in V(G)$ and let $w_1, \ldots, w_\Delta$ be v's neighbors. Since $K_{\Delta+1} \not\subseteq G$, by symmetry we may assume that w_2 and w_3 are not adjacent. Choose a $(\Delta + 1)$-coloring $\{\{v\}, C_1, \ldots, C_\Delta\}$ of G where $w_i \in C_i$ so as to maximize $|C_1|$. Then C_1 is a maximal independent set in G and in particular, with $H := G-C_1$, we have $\chi(H) = \chi(G) - 1 = \Delta = \Delta(H) + 1 \geq 4$. By minimality of $|G|$, we get $K_\Delta \subseteq H$. But $\{\{v\}, C_2, \ldots, C_\Delta\}$ is a Δ-coloring of H, so any K_Δ in H must contain v and hence w_2 and w_3, a contradiction.

Therefore G is 3-regular. Since G is not a forest it contains an induced cycle C. Put $T := N(C)$. Then $|T| \geq 2$ since $K_4 \not\subseteq G$. Take different $x, y \in T$ and put

$H_{xy} := G - C$ if x is adjacent to y and $H_{xy} := (G - C) + xy$ otherwise. Then, by minimality of $|G|$, either H_{xy} is 3-colorable or adding xy created a K_4 in H_{xy}.

Suppose the former happens. Then we have a 3-coloring of $G - C$ where x and y receive different colors. We can easily extend this partial coloring to all of G since each vertex of C has a set of two available colors and some pair of vertices in C get different sets.

Whence adding xy created a K_4, call it A, in H_{xy}. We conclude that T is independent and each vertex in T has exactly one neighbor in C. Hence $|T| \geq |C| \geq 3$. Pick $z \in T - \{x, y\}$. Then x is contained in a K_4, call it B, in H_{xz}. Since $d(x) = 3$, we must have $A - \{x, y\} = B - \{x, z\}$. But then any $w \in A - \{x, y\}$ has degree at least 4, a contradiction. $\square$

Chapter 3

DOING THE VERTEX SHUFFLE

The material in this chapter appeared in [65], [64] and [62].

Let $\mathcal{G}$ be the collection of all finite simple connected graphs. For a graph G, $x \in V(G)$ and $D \subseteq V(G)$ we use the notation $N_D(x) := N(x) \cap D$ and $d_D(x) := |N_D(x)|$. Let $\mathcal{C}_G$ be the components of G and $c(G) := |\mathcal{C}_G|$. If $h\colon \mathcal{G} \to \mathbb{N}$, we define h for any graph as $h(G) := \sum_{D \in \mathcal{C}_G} h(D)$. An *ordered partition* of G is a sequence $(V_1, V_2, \ldots, V_k)$ where the V_i are pairwise disjoint and cover $V(G)$. Note that we allow the V_i to be empty. When there is no possibility of ambiguity, we call such a sequence a *partition*.

3.1 Coloring when the high vertex subgraph has small cliques

In [38] Kierstead and Kostochka investigated the Brooks bound with the Ore-degree θ in place of Δ.

Definition 3.1.1. The *Ore-degree* of an edge xy in a graph G is $\theta(xy) := d(x)+d(y)$. The *Ore-degree* of a graph G is $\theta(G) := \max_{xy \in E(G)} \theta(xy)$.

Theorem 3.1.1 (Kierstead and Kostochka [38] 2010)**.** *If G is a graph with $\chi(G) \geq \left\lfloor \frac{\theta(G)}{2} \right\rfloor + 1 \geq 7$ then G contains $K_{\chi(G)}$.*

This statement about Ore-degree is equivalent to the following statement about vertex critical graphs.

Theorem 3.1.2 (Kierstead and Kostochka [38] 2010)**.** *The only vertex critical graph G with $\chi(G) \geq \Delta(G) \geq 7$ such that $\mathcal{H}(G)$ is edgeless is $K_{\chi(G)}$.*

In [63], we improved the 7 to 6 by proving the following generalization.

Theorem 3.1.3 (Rabern 2012 [63])**.** *The only vertex critical graph G with $\chi(G) \geq \Delta(G) \geq 6$ and $\omega(\mathcal{H}(G)) \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor - 2$ is $K_{\chi(G)}$.*

This result and those in [60] were improved by Kostochka, Rabern and Stiebitz in [48]. In particular, the following was proved.

Theorem 3.1.4 (Kostochka, Rabern and Stiebitz [48] 2012)**.** *The only vertex critical graphs G with $\chi(G) \geq \Delta(G) \geq 5$ such that $\mathcal{H}(G)$ is edgeless are $K_{\chi(G)}$ and O_5.*

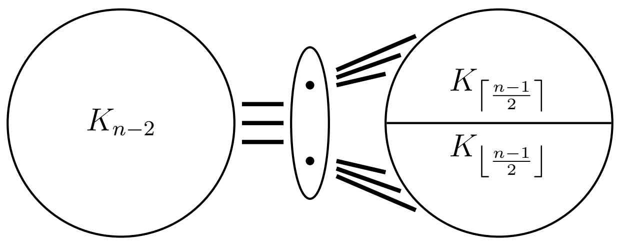


Figure 3.1: The graph O_n.

Here O_n is the graph formed from the disjoint union of $K_n - xy$ and K_{n-1} by joining $\left\lfloor \frac{n-1}{2} \right\rfloor$ vertices of the K_{n-1} to x and the other $\left\lceil \frac{n-1}{2} \right\rceil$ vertices of the K_{n-1} to y (see Figure 3.1). In [65] we proved a result that implies all of the results in [48]. The proof replaces an algorithm of Mozhan [55] with the original, more general, algorithm of Catlin [14] on which it is based. This allows for a considerable simplification. Moreover, we prove two preliminary partitioning results that are of independent interest. All coloring results follow from the first of these, the second is a generalization of a lemma due to Borodin [8] (and independently Bollobás and Manvel [6]) about partitioning a graph into degenerate subgraphs. The following is the main coloring result in [65].

Corollary 3.1.5. *Let G be a vertex critical graph with $\chi(G) \geq \Delta(G) + 1 - p \geq 4$ for some $p \in \mathbb{N}$. If $\omega(\mathcal{H}(G)) \leq \frac{\chi(G)+1}{p+1} - 2$, then $G = K_{\chi(G)}$ or $G = O_5$.*

In the sections that follow we will prove this Corollary. First, we give a non-standard proof of Brooks' theorem to illustrate the technique.

3.1.1 A weird proof of Brooks' theorem

Let G be a graph. A partition $P := (V_0, V_1)$ of $V(G)$ will be called *normal* if it achieves the minimum value of $(\Delta(G) - 1)\,\|V_0\| + \|V_1\|$. Note that if P is a normal partition, then $\Delta(G[V_0]) \leq 1$ and $\Delta(G[V_1]) \leq \Delta(G) - 1$. The *$P$-components* of G are the components of $G[V_i]$ for $i \in [2]$. A P-component is called an *obstruction* if it is a K_2 in $G[V_0]$ or a $K_{\Delta(G)}$ in $G[V_1]$ or an odd cycle in $G[V_1]$ when $\Delta(G) = 3$. A path $x_1 x_2 \cdots x_k$ is called *P-acceptable* if x_1 is contained in an obstruction and for different $i, j \in [k]$, x_i and x_j are in different P-components. For a subgraph H of G and $x \in V(G)$, we put $N_H(x) := N(x) \cap V(H)$.

Lemma 3.1.6. *Let G be a graph with $\Delta(G) \geq 3$. If G doesn't contain $K_{\Delta(G)+1}$, then $V(G)$ has an obstruction-free normal partition.*

Proof. Suppose the lemma is false. Among the normal partitions having the minimum number of obstructions, choose $P := (V_0, V_1)$ and a maximal P-acceptable path $x_1 x_2 \cdots x_k$ so as to minimize k.

Let A and B be the P-components containing x_1 and x_k respectively. Put $X := N_A(x_k)$. First, suppose $|X| = 0$. Then moving x_1 to the other part of P creates another normal partition P' having the minimum number of obstructions. But $x_2 x_3 \cdots x_k$ is a maximal P'-acceptable path, violating the minimality of k. Hence $|X| \geq 1$.

Pick $z \in X$. Moving z to the other part of P destroys the obstruction A, so it must create an obstruction containing x_k and hence B. Since obstructions are complete graphs or odd cycles, the only possibility is that $\{z\} \cup V(B)$ induces an obstruction. Put $Y := N_B(z)$. Then, since obstructions are regular, $N_B(x) = Y$ for all $x \in X$ and $|Y| = \delta(B) + 1$. In particular, X is joined to Y in G.

Suppose $|X| \geq 2$. Then, similarly to above, switching z and x_k in P shows that $\{x_k\} \cup V(A - z)$ induces an obstruction. Since obstructions are regular, we must have $|N_{A-z}(x_k)| = \Delta(A)$ and hence $|X| \geq \Delta(A) + 1$. Thus $|X \cup Y| = \Delta(A) + \delta(B) + 2 = \Delta(G) + 1$. Suppose X is not a clique and pick nonadjacent $v_1, v_2 \in X$. It is easily seen that moving v_1, v_2 and then x_k to their respective other parts violates normality of P. Hence X is a clique. Suppose Y is not a clique and pick nonadjacent $w_1, w_2 \in Y$. Pick $z' \in X - \{z\}$. Now moving z and then w_1, w_2 and then z' to their respective other parts again violates normality of P. Hence Y is a clique. But X is joined to Y, so $X \cup Y$ induces a $K_{\Delta(G)+1}$ in G, a contradiction.

Hence we must have $|X| = 1$. Suppose $X \neq \{x_1\}$. First, suppose A is K_2. Then moving z to the other part of P creates another normal partition Q having the minimum number of obstructions. In Q, $x_k x_{k-1} \cdots x_1$ is a maximal Q-acceptable path since the Q-components containing x_2 and x_k contain all of x_1's neighbors in that part. Running through the above argument using Q gets us to the same point with A not K_2. Hence we may assume A is not K_2.

Move each of $x_1, x_2, \ldots, x_k$ in turn to their respective other parts of P. Then the obstruction A was destroyed by moving x_1 and for $1 \leq i < k$, the obstruction created by moving x_i was destroyed by moving x_{i+1}. Thus, after the moves, x_k is contained in an obstruction. By minimality of k, it must be that $\{x_k\} \cup V(A - x_1)$ induces an obstruction and hence $|X| \geq 2$, a contradiction.

Therefore $X = \{x_1\}$. But then moving x_1 to the other part of P creates an obstruction containing both x_2 and x_k. Hence $k = 2$.

Since $x_1 x_2$ is maximal, x_2 can have no neighbor in the other part besides x_1. But now switching x_1 and x_2 in P creates a partition violating the normality of P. □

Theorem 3.1.7 (Brooks 1941)**.** *If a connected graph G is not complete and not an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Proof. Suppose not and choose a counterexample G minimizing $\Delta(G)$. Plainly, $\Delta(G) \geq 3$. By Lemma 3.1.6, $V(G)$ has an obstruction-free normal partition (V_0, V_1). Since $G[V_0]$ has maximum degree at most one and contains no K_2's, we see that V_0 is independent. Since $G[V_1]$ is obstruction-free, applying minimality of $\Delta(G)$ gives $\chi(G[V_1]) \leq \Delta(G[V_1]) < \Delta(G)$. Hence $\chi(G) \leq \Delta(G)$, a contradiction. □

3.1.2 The partitioning theorems

An *ordered partition* of a graph G is a sequence $(V_1, V_2, \dots, V_k)$ where the V_i are pairwise disjoint and cover $V(G)$. Note that we allow the V_i to be empty. When there is no possibility of ambiguity, we call such a sequence a *partition*. For a vector $\mathbf{r} \in \mathbb{N}^k$ we take the coordinate labeling $\mathbf{r} = (r_1, r_2, \dots, r_k)$ as convention. Define the *weight* of a vector $\mathbf{r} \in \mathbb{N}^k$ as $w(\mathbf{r}) := \sum_{i \in [k]} r_i$. Let G be a graph. An $\mathbf{r}$-*partition* of G is an ordered partition $P := (V_1, \dots, V_k)$ of $V(G)$ minimizing

$$f(P) := \sum_{i \in [k]} \left(\|G[V_i]\| - r_i |V_i| \right).$$

It is a fundamental result of Lovász [51] that if $P := (V_1, \dots, V_k)$ is an $\mathbf{r}$-partition of G with $w(\mathbf{r}) \geq \Delta(G) + 1 - k$, then $\Delta(G[V_i]) \leq r_i$ for each $i \in [k]$. The proof is simple: if there is a vertex in a part violating the condition, then there is some part it can be moved to that decreases $f(P)$. As Catlin [14] showed, with the stronger condition $w(\mathbf{r}) \geq \Delta(G) + 2 - k$, a vertex of degree r_i in $G[V_i]$ can always be moved to some other part while maintaining $f(P)$. Since G is finite, a well-chosen sequence of such moves must always wrap back on itself. Many authors, including

Catlin [14], Bollobás and Manvel [6] and Mozhan [55] have used such techniques to prove coloring results. We generalize these techniques by taking into account the degree in G of the vertex to be moved—a vertex of degree less than the maximum needs a weaker condition on $w(\mathbf{r})$ to be moved.

For $x \in V(G)$ and $D \subseteq V(G)$ we use the notation $N_D(x) := N(x) \cap D$ and $d_D(x) := |N_D(x)|$. Let $\mathcal{C}(G)$ be the components of G and $c(G) := |\mathcal{C}(G)|$. For an induced subgraph H of G, define $\delta_G(H) := \min_{v \in V(H)} d_G(v)$. We also need the following notion of a movable subgraph.

Definition 3.1.2. Let G be a graph and H an induced subgraph of G. For $d \in \mathbb{N}$, the *d-movable subgraph* of H with respect to G is the subgraph H^d of G induced on

$$\{v \in V(H) \mid d_G(v) = d \text{ and } H - v \text{ is connected}\}.$$

We prove two partition theorems of similar form. All of our coloring results will follow from the first theorem, the second theorem is a degeneracy result from which Borodin's result in [8] follows. For unification purposes, define a *t-obstruction* as an odd cycle when $t = 2$ and a K_{t+1} when $t \geq 3$.

Theorem 3.1.8. *Let G be a graph, $k, d \in \mathbb{N}$ with $k \geq 2$ and $\mathbf{r} \in \mathbb{N}^k_{\geq 2}$. If $w(\mathbf{r}) \geq \max\{\Delta(G) + 1 - k, d\}$, then at least one of the following holds:*

1. *$w(\mathbf{r}) = d$ and G contains an induced subgraph Q with $|Q| = d + 1$ that can be partitioned into k cliques $F_1, \dots, F_k$ where*

   a) $|F_1| = r_1 + 1$, $|F_i| = r_i$ for $i \geq 2$,

   b) $\left|F_1^d\right| \geq 2$, $\left|F_i^d\right| \geq 1$ for $i \geq 2$,

   c) for $i \in [k]$, each $v \in V(F_i^d)$ is universal in Q;

2. *there exists an* $\mathbf{r}$*-partition* $P := (V_1, \ldots, V_k)$ *of* G *such that if* C *is an* r_i*-obstruction in* $G[V_i]$*, then* $\delta_G(C) \geq d$ *and* C^d *is edgeless.*

Proof. For $i \in [k]$, call a connected graph C *i-bad* if C is an r_i-obstruction such that C^d has an edge. For a graph H and $i \in [k]$, let $b_i(H)$ be the number of i-bad components of H. For an $\mathbf{r}$-partition $P := (V_1, \ldots, V_k)$ of G let

$$b(P) := \sum_{i \in [k]} b_i(G[V_i]).$$

Let $P := (V_1, \ldots, V_k)$ be an $\mathbf{r}$-partition of $V(G)$ minimizing $b(P)$.

Let $i \in [k]$ and $x \in V_i$ with $d_{V_i}(x) \geq r_i$. Suppose $d_G(x) = d$. Then, since $w(\mathbf{r}) \geq d$, for every $j \neq i$ we have $d_{V_j}(x) \leq r_j$. Moving x from V_i to V_j gives a new partition P^* with $f(P^*) \leq f(P)$. Note that if $d_G(x) < d$ we would have $f(P^*) < f(P)$ contradicting the minimality of P.

Supppose (2) fails to hold. Then $b(P) > 0$. By symmetry, we may assume that there is a 1-bad component A_1 of $G[V_1]$. Put $P_1 := P$ and $V_{1,i} := V_i$ for $i \in [k]$. Since A_1 is 1-bad we have $x_1 \in V(A_1^d)$ which has a neighbor in $V(A_1^d)$. By the above we can move x_1 from $V_{1,1}$ to $V_{1,2}$ to get a new partition $P_2 := (V_{2,1}, V_{2,2}, \ldots, V_{2,k})$ where $f(P_2) = f(P_1)$. Since removing x_1 from A_1 decreased $b_1(G[V_1])$, minimality of $b(P_1)$ implies that x_1 is in a 2-bad component A_2 in $V_{2,2}$. Now, we may choose $x_2 \in V(A_2^d) - \{x_1\}$ having a neighbor in A_2^d and move x_2 from $V_{2,2}$ to $V_{2,1}$ to get a new partition $P_3 := (V_{3,1}, V_{3,2}, \ldots, V_{3,k})$ where $f(P_3) = f(P_1)$. We continue on this way to construct sequences $A_1, A_2, \ldots$, $P_1, P_2, P_3, \ldots$ and $x_1, x_2, \ldots$.

This process can be defined recursively as follows. For $t \in \mathbb{N}$, put $j_t := 1$ for odd t and $j_t := 2$ for even t. Put $P_1 := P$ and $V_{1,i} := V_i$ for $i \in [k]$. Pick $x_1 \in V(A_1^d)$ which has a neighbor in $V(A_1^d)$. Move x_1 from $V_{1,1}$ to $V_{1,2}$ to get a new partition $P_2 := (V_{2,1}, V_{2,2}, \ldots, V_{2,k})$ where $f(P_2) = f(P_1)$ and let A_2 be the

2-bad component in $V_{2,2}$ containing x_1. Then for $t \geq 2$, pick $x_t \in V(A_t^d - x_{t-1})$ which has a neighbor in $V(A_t^d)$. Move x_t from V_{t,j_t} to $V_{t,3-j_t}$ to get a new partition $P_{t+1} := (V_{t+1,1}, V_{t+1,2}, \ldots, V_{t+1,k})$ where $f(P_{t+1}) = f(P_t)$ and let A_{t+1} be the $(3-j_t)$-bad component in $V_{t+1,3-j_t}$ containing x_t.

Since G is finite, at some point we will need to reuse a leftover component; that is, there is a smallest t such that $A_{t+1} - x_t = A_s - x_s$ for some $s < t$. Let $j \in [2]$ be such that in $V(A_s) \subseteq V_{s,j}$. Then $V(A_t) \subseteq V_{t,3-j}$.

Claim 1. *$N(x_t) \cap V(A_s - x_s) = N(x_s) \cap V(A_s - x_s)$.*

This is immediate since A_s is r_j-regular.

Claim 2. *$s = 1$, $t = 2$, both A_s and A_t are complete, A_s^d is joined to $A_t - x_{t-1}$ and A_t^d is joined to $A_s - x_s$.*

Subclaim 2a. *$N(x_s) \cap V(A_s^d) \neq \emptyset$.*

In the construction of the sequence, x_s was chosen such that it had a neighbor in A_s^d.

Subclaim 2b. *For any $z \in N(x_s) \cap V(A_s^d)$ we have $N(z) \cap V(A_t - x_{t-1}) = N(x_{t-1}) \cap V(A_t - x_{t-1})$. Moreover, if x_s is adjacent to x_t, then $N(x_s) \cap V(A_t - x_{t-1}) = N(x_{t-1}) \cap V(A_t - x_{t-1})$ and $x_s = x_{t-1}$.*

In P_s, move z to $V_{s,3-j}$ to get a new partition $P^\gamma := (V_{\gamma,1}, V_{\gamma,2}, \ldots, V_{\gamma,k})$. Then z must create an r_{3-j}-obstruction with $A_t - x_{t-1}$ in $V_{\gamma,3-j}$ since z is adjacent to x_t by Claim 1. In particular, $N(z) \cap V(A_t - x_{t-1}) = N(x_{t-1}) \cap V(A_t - x_{t-1})$. If x_s is adjacent to x_t, the same argument (with x_s in place of z) gives $N(x_s) \cap V(A_t - x_{t-1}) = N(x_{t-1}) \cap V(A_t - x_{t-1})$ and $x_s = x_{t-1}$.

Subclaim 2c. *A_s is complete and x_s is adjacent to x_t.*

By Subclaim 2a, $N(x_s) \cap V(A_s^d) \neq \emptyset$. Pick $z \in N(x_s) \cap V(A_s^d)$ and let P^γ be as in Subclaim 2b. In P^γ, move x_t to $V_{\gamma,j}$ to get a new partition $P^{\gamma*} := (V_{\gamma*,1}, V_{\gamma*,2}, \ldots, V_{\gamma*,k})$. Since x_s has at least two neighbors in A_s, by Claim 1, x_t has a neighbor in $A_s - z$. Hence x_t must create an r_j-obstruction with $A_s - z$ in $V_{\gamma*,j}$. In particular, $N(z) \cap V(A_s - z) = N(x_t) \cap V(A_s - z)$. Thus x_s is adjacent to x_t and we have $N[z] \cap V(A_s) = N[x_s] \cap V(A_s)$. Thus, if A_s is an odd cycle, it must be a triangle. Hence A_s is complete.

Subclaim 2d. *A_s^d is joined to $N(x_{t-1}) \cap V(A_t - x_{t-1})$ and $x_s = x_{t-1}$.*

Since A_s is complete by Subclaim 2c, we have $N(x_s) \cap V(A_s^d) = V(A_s^d - x_s)$. Since x_s is adjacent to x_t by Subclaim 2c, applying Subclaim 2b shows that A_s^d is joined to $N(x_{t-1}) \cap V(A_t - x_{t-1})$ and $x_s = x_{t-1}$.

Subclaim 2e. *$s = 1$ and $t = 2$.*

Suppose $s > 1$. Then, since $x_{s-1} \in V(A_s^d)$, Subclaim 2d shows that x_{s-1} is joined to $N(x_{t-1}) \cap V(A_t - x_{t-1})$ and hence $A_t - x_{t-1} = A_{s-1} - x_{s-1}$ violating minimality of t. Whence, $s = 1$ and $t = 2$.

Subclaim 2f. *A_t is complete and A_s^d is joined to $A_t - x_{t-1}$.*

Pick $z \in N(x_s) \cap V(A_s^d)$. Then z is joined to $A_t - x_t$ by Subclaim 2d. In P_{t+1}, move z to $V_{t+1,3-j}$ to get a new partition $P^\beta := (V_{\beta,1}, V_{\beta,2}, \ldots, V_{\beta,k})$. Then z must create an r_{3-j}-obstruction with $A_t - x_t$ in $V_{\beta,3-j}$. In particular, $V(A_t - x_t) = N(z) \cap V(A_t - x_t) = N(x_t) \cap V(A_t - x_t)$. Thus, if A_t is an odd cycle, it must be a triangle. Hence A_t is complete. Now Subclaim 2d gives that A_s^d is joined to $A_t - x_{t-1}$.

Subclaim 2g. *A_t^d is joined to $A_s - x_s$.*

Since $x_s = x_{t-1}$, the statement is clear for x_{t-1}. Pick $y \in V(A_t^d - x_{t-1})$ and $z \in V(A_s^d)$. In P_t, move y to $V_{t,j}$. Since y is adjacent to z by Subclaim 2f, y must create an r_j-obstruction with $A_s - x_s$ and since A_s is complete, y must be joind to $A_s - x_s$. Hence A_t^d is joined to $A_s - x_s$.

Claim 3. *(1) holds.*

We can play the same game with V_1 and V_i for any $3 \leq i \leq k$ as we did with V_1 and V_2 above. Let $B_1 := A_1$, $B_2 := A_2$ and for $i \geq 3$, let B_i be the r_i-obstruction made by moving x_1 into V_i. Then B_i is complete for each $i \in [k]$. Applying Claim 2 to all pairs B_i, B_j shows that for any distinct $i, j \in [k]$, B_i^d is joined to $B_j - x_1$. Put $F_1 = B_1$ and $F_i = B_i - x_1$ for $i \geq 2$. Let Q be the union of the F_i. Then (a), (b) and (c) of (1) are satisfied. Note that $|Q| = w(\mathbf{r}) + 1$ and since any $v \in B_1^d$ is universal in Q, $|Q| \leq d+1$. By assumption $w(\mathbf{r}) \geq d$, whence $w(\mathbf{r}) = d$. Hence, (1) holds. □

The following result generalizes a lemma due to Borodin [8]. This lemma of Borodin was generalized in another direction in [11]. The proof that follows is basically the same as that of Theorem 3.1.8. For a reader who is only interested in the coloring results, this theorem can be safely skipped.

Theorem 3.1.9. *Let G be a graph, $k, d \in \mathbb{N}$ with $k \geq 2$ and $\mathbf{r} \in \mathbb{N}_{\geq 1}^k$ where at most one of the r_i is one. If $w(\mathbf{r}) \geq \max\{\Delta(G) + 1 - k, d\}$, then at least one of the following holds:*

1. *$w(\mathbf{r}) = d$ and G contains a $K_t * E_{d+1-t}$ where $t \geq d+1-k$, for each $v \in V(K_t)$ we have $d_G(v) = d$ and for each $v \in V(E_{d+1-t})$ we have $d_G(v) > d$; or,*

2. *there exists an $\mathbf{r}$-partition $P := (V_1, \ldots, V_k)$ of G such that if C is an r_i-regular component of $G[V_i]$, then $\delta_G(C) \geq d$ and there is at most one $x \in V(C^d)$ with $d_{C^d}(x) \geq r_i - 1$. Moreover, P can be chosen so that either:*

    a) *for all $i \in [k]$ and r_i-regular component C of $G[V_i]$, we have $\left|C^d\right| \leq 1$; or,* *b) for some $i \in [k]$ and some r_i-regular component C of $G[V_i]$, there is $x \in V(C^d)$ such that $\{y \in N_C(x) \mid d_G(y) = d\}$ is a clique.*

Proof. For $i \in [k]$, call a connected graph C *i-bad* if C is r_i-regular and there are at least two $x \in V(C^d)$ with $d_{C^d}(x) \geq r_i - 1$. For a graph H and $i \in [k]$, let $b_i(H)$ be the number of i-bad components of H. For an $\mathbf{r}$-partition $P := (V_1, \ldots, V_k)$ of G let

$$c(P) := \sum_{i \in [k]} c(G[V_i]),$$

$$b(P) := \sum_{i \in [k]} b_i(G[V_i]).$$

Let $P := (V_1, \ldots, V_k)$ be an $\mathbf{r}$-partition of $V(G)$ minimizing $c(P)$ and subject to that $b(P)$.

Let $i \in [k]$ and $x \in V_i$ with $d_{V_i}(x) \geq r_i$. Suppose $d_G(x) = d$. Then, since $w(\mathbf{r}) \geq d$, for every $j \neq i$ we have $d_{V_j}(x) \leq r_j$. Moving x from V_i to V_j gives a new partition P^* with $f(P^*) \leq f(P)$. Note that if $d_G(x) < d$ we would have $f(P^*) < f(P)$ contradicting the minimality of P.

Suppose $b(P) > 0$. By symmetry, we may assume that there is a 1-bad component A_1 of $G[V_1]$. Put $P_1 := P$ and $V_{1,i} := V_i$ for $i \in [k]$. Since A_1 is 1-bad we have $x_1 \in V(A_1^d)$ with $d_{A_1^d}(x) \geq r_1 - 1$. By the above we can move x_1 from $V_{1,1}$ to $V_{1,2}$ to get a new partition $P_2 := (V_{2,1}, V_{2,2}, \ldots, V_{2,k})$ where $f(P_2) = f(P_1)$. By the minimality of $c(P_1)$, x_1 is adjacent to only one component C_2 in $G[V_{1,2}]$. Let $A_2 := G[V(C_2) \cup \{x_1\}]$. Since removing x_1 from A_1 decreased $b_1(G[V_1])$, minimality of $b(P_1)$ implies that A_2 is 2-bad. Now, we may choose $x_2 \in V(A_2^d) - \{x_1\}$ with $d_{A_2^d}(x) \geq r_2 - 1$ and move x_2 from $V_{2,2}$ to $V_{2,1}$ to get a new partition $P_3 := (V_{3,1}, V_{3,2}, \ldots, V_{3,k})$ where $f(P_3) = f(P_1)$.

Continue on this way to construct sequences $A_1, A_2, \ldots,$ $P_1, P_2, P_3, \ldots$ and $x_1, x_2, \ldots$. Since G is finite, at some point we will need to reuse a leftover component;

that is, there is a smallest t such that $A_{t+1} - x_t = A_s - x_s$ for some $s < t$. Let $j \in [2]$ be such that in $V(A_s) \subseteq V_{s,j}$. Then $V(A_t) \subseteq V_{t,3-j}$. Note that, since A_s is r_j-regular, $N(x_t) \cap V(A_s - x_s) = N(x_s) \cap V(A_s - x_s)$.

We claim that $s = 1$, $t = 2$, both A_s and A_t are complete, A_s^d is joined to $A_t - x_{t-1}$ and A_t^d is joined to $A_s - x_s$.

Put $X := N(x_s) \cap V(A_s^d)$. Since x_s witnesses the j-badness of A_s, $|X| \geq \max\{1, r_j - 1\}$. Pick $z \in X$. In P_s, move z to $V_{s,3-j}$ to get a new partition $P^\gamma := (V_{\gamma,1}, V_{\gamma,2}, \dots, V_{\gamma,k})$. Then z must create an r_{3-j}-regular component with $A_t - x_{t-1}$ in $V_{\gamma,3-j}$ since z is adjacent to x_t. In particular, $N(z) \cap V(A_t - x_{t-1}) = N(x_{t-1}) \cap V(A_t - x_{t-1})$. Since z is adjacent to x_t, so is x_{t-1}.

Suppose $r_j \geq 2$. In P^γ, move x_t to $V_{\gamma,j}$ to get a new partition $P^{\gamma*} := (V_{\gamma*,1}, V_{\gamma*,2}, \dots, V_{\gamma*,k})$. Then x_t must create an r_j-regular component with $A_s - z$ in $V_{\gamma*,j}$. In particular, $N(z) \cap V(A_s - z) = N(x_t) \cap V(A_s - z)$. Thus x_s is adjacent to x_t and we have $N[z] \cap V(A_s) = N[x_s] \cap V(A_s)$. Put $K := X \cup \{x_s\}$. Then $|K| \geq r_j$ and K induces a clique. If $|K| > r_j$, then $A_s = K$ is complete. Otherwise, the vertices of K have a common neighbor $y \in V(A_s) - K$ and again A_s is complete. Also, since x_s is adjacent to x_t, using x_s in place of z in the previous paragraph, we conclude that K is joined to $N(x_{t-1}) \cap V(A_t - x_{t-1})$ and $x_s = x_{t-1}$.

Suppose $s > 1$. Then x_{s-1} is joined to $N(x_{t-1}) \cap V(A_t - x_{t-1})$ and hence $A_t - x_{t-1} = A_{s-1} - x_{s-1}$ violating minimality of t. Whence, if $r_j \geq 2$ then $s = 1$.

Note that $K = V(A_s^d)$ and hence if $r_j \geq 2$ then A_s is complete and A_s^d is joined to $N(x_{t-1}) \cap V(A_t - x_{t-1})$. If $r_{3-j} = 1$, then A_t is a K_2 and $N(x_{t-1}) \cap V(A_t - x_{t-1}) = V(A_t - x_{t-1}) = \{x_t\}$. We already know that x_t is joined to $A_s - x_s$. Thus the cases when $r_j \geq 2$ and $r_{3-j} = 1$ are taken care of. By assumption, at least one of r_j or r_{3-j} is at least two. Hence it remains to handle the cases with $r_{3-j} \geq 2$.

Suppose $r_{3-j} \geq 2$. In P_{t+1}, move z to $V_{t+1,3-j}$ to get a new partition $P^{\beta} := (V_{\beta,1}, V_{\beta,2}, \ldots, V_{\beta,k})$. Then z must create an r_{3-j}-regular component with $A_t - x_t$ in $V_{\beta,3-j}$. In particular, $N(z) \cap V(A_t - x_t) = N(x_t) \cap V(A_t - x_t)$. Since $N(z) \cap V(A_t - x_{t-1}) = N(x_{t-1}) \cap V(A_t - x_{t-1})$, we have $N[x_{t-1}] \cap V(A_t) = N(z) \cap V(A_t) = N[x_t] \cap V(A_t)$. Put $W := N[x_t] \cap V(A_t^d)$. Each $w \in W$ is adjacent to z and running through the argument above with w in place of x_t shows that W is a clique joined to z. Moreover, since x_t witnesses the $(3-j)$-badness of A_t, $|W| \geq r_{3-j}$. As with A_s above, we conclude that A_t is complete. Since $x_s \in V_{t+1,3-j}$ and x_s is adjacent to z, it must be that $x_s \in V(A_t - x_t)$. Thence x_s is joined to W and $x_s = x_{t-1}$.

Suppose that $r_j \geq 2$ as well. We know that $s = 1$, A_s is complete and A_s^d is joined to $N(x_{t-1}) \cap V(A_t - x_{t-1}) = A_t - x_{t-1}$. Also, we just showed that A_t is complete and A_t^d is joined to $A_s - x_s$.

Thus, we must have $r_j = 1$ and $r_{3-j} \geq 2$. Then, since A_s is a K_2, by the above, A_s is joined to W. Since $W = A_t^d$, it only remains to show that $s = 1$. Suppose $s > 1$. Then x_{s-1} is joined to W and hence $A_t - x_{t-1} = A_{s-1} - x_{s-1}$ violating minimality of t.

Therefore $s = 1$, $t = 2$, both A_s and A_t are complete, A_s^d is joined to $A_t - x_{t-1}$ and A_t^d is joined to $A_s - x_s$. But we can play the same game with V_1 and V_i for any $3 \leq i \leq k$ as well. Let $B_1 := A_1$, $B_2 := A_2$ and for $i \geq 3$, let B_i be the r_i-regular component made by moving x_1 into V_i. Then B_i is complete for each $i \in [k]$. Applying what we just proved to all pairs B_i, B_j shows that for any distinct $i, j \in [k]$, B_i^d is joined to $B_j - x_1$. Since $\left|B_i^d\right| \geq r_i$ and $x_1 \in V(B_i^d)$ for each i, this gives a $K_t * E_{w(\mathbf{r})+1-t}$ in G where $t \geq w(\mathbf{r}) + 1 - k$. Take such a subgraph Q maximizing t. Since all the B_i are complete, any vertex of degree d will be in B_i^d; therefore, for each $v \in V(K_t)$ we have $d_G(v) = d$ and for each $v \in V(E_{w(\mathbf{r})+1-t})$ we have $d_G(v) > d$. Note that $|Q| = w(\mathbf{r}) + 1$ and since $d_G(v) = d$ for any $v \in V(K_t)$, $|Q| \leq d + 1$. By assumption $w(\mathbf{r}) \geq d$, whence $w(\mathbf{r}) = d$. Thus if (1) fails, then the first part of (2) holds.

It remains to prove that we can choose P to satisfy one of (a) or (b). Suppose that (1) fails and P cannot be chosen to satisfy either (a) or (b). For $i \in [k]$, call a connected graph C *i-ugly* if C is r_i-regular and $\left|C^d\right| \geq 2$ let $u_i(H)$ be the number of i-ugly components of H. Note that if C is i-bad, then it is i-ugly. For an $\mathbf{r}$-partition $P := (V_1, \ldots, V_k)$ of G let

$$u(P) := \sum_{i \in [k]} u_i(G[V_i]).$$

Choose an $\mathbf{r}$-partition $Q := (V_1, \ldots, V_k)$ of G first minimizing $c(Q)$, then subject to that requiring $b(Q) \leq 1$ and then subject to that minimizing $u(Q)$. Since Q does not satisfy (a), at least one of $b(Q) = 1$ or $u(Q) \geq 1$ holds. By symmetry, we may assume that $G[V_1]$ contains a component D_1 which is either 1-bad or 1-ugly (or both). If D_1 is 1-bad, pick $w_1 \in V(D_1^d)$ witnessing the 1-badness of D_1; otherwise pick $w_1 \in V(D_1^d)$ arbitrarily. Move w_1 to V_2, to form a new $\mathbf{r}$-partition. This new partition still satisfies all of our conditions on Q. As above we construct a sequence of vertex moves that will wrap around on itself. This can be defined recursively as follows. For $t \geq 2$, if D_t is bad pick $w_t \in V(D_t^d - w_{t-1})$ witnessing the badness of D_t; otherwise, if D_t is ugly pick $w_t \in V(D_t^d - w_{t-1})$ arbitrarily. Now move w_t to the part from which w_{t-1} came to form D_{t+1}. Let $Q_1 := Q, Q_2, Q_3, \ldots$ be the partitions created by a run of this process. Note that the process can never create a component that is not ugly lest we violate the minimality of $u(Q)$.

Since G is finite, at some point we will need to reuse a leftover component; that is, there is a smallest t such that $D_{t+1} - x_t = D_s - x_s$ for some $s < t$. First, suppose D_s is not bad, but merely ugly. Then D_{t+1} is not bad and hence $b(Q_{t+1}) = 0$ and $u(Q_{t+1}) < u(Q)$, a contradiction. Hence D_s is bad.

Suppose D_t is not bad. As in the proof of the first part of (2), we can conclude that $x_s = x_{t-1}$. Pick $z \in N(x_s) \cap V(D_s^d)$. Since z is adjacent to x_t, by moving z to the part containing x_t in P_s we conclude $N(z) \cap V(D_t - x_s) = N(x_s) \cap V(D_t - x_s)$. Put $T := \{y \in N_{D_t}(x_s) \mid d_G(y) = d\}$. Suppose T is not a clique and let $w_1, w_2 \in T$ be nonadjacent. Now, in P_t, since z is adjacent to both w_1 and w_2, swapping w_1 and w_2 with z contradicts minimality of $f(Q)$. Hence T is a clique and (b) holds, a contradiction.

Thus we may assume that D_t is bad as well. Now we may apply the same argument as in the proof of the first part of (2) to show that (1) holds. This final contradiction completes the proof.

□

Corollary 3.1.10 (Borodin [8])**.** *Let G be a graph not containing a $K_{\Delta(G)+1}$. If $r_1, r_2 \in \mathbb{N}_{\geq 1}$ with $r_1 + r_2 \geq \Delta(G) \geq 3$, then $V(G)$ can be partitioned into sets V_1, V_2 such that $\Delta(G[V_i]) \leq r_i$ and col$(G[V_i]) \leq r_i$ for $i \in [2]$.*

Proof. Apply Proposition 3.1.9 with $\mathbf{r} := (r_1, r_2)$ and $d = \Delta(G)$. Since G doesn't contain a $K_{\Delta(G)+1}$ and no vertex in G has degree larger than d, (1) cannot hold. Thus (2) must hold. Let $P := (V_1, V_2)$ be the guaranteed partition and suppose that for some $j \in [2]$, $G[V_j]$ contains an r_j-regular component H. Then every vertex of H has degree d in G and hence H^d contains all noncutvertices of H. But H has maximum degree r_j and thus contains at least r_j noncutvertices. If $r_j = 1$, then H is K_2 and hence has 2 noncutvertices. In any case, we have $\left|H^d\right| \geq 2$. Hence (a) cannot hold for P. Thus, by (b), we have $i \in [2]$, an r_i-regular component C of $G[V_i]$ and $x \in V(C)$ such that $N_C(x)$ is a clique. But then C is K_{r_i+1} violating (2), a contradiction.

Therefore, for $i \in [2]$, each component of $G[V_i]$ contains a vertex of degree at most $r_i - 1$. Whence $\mathrm{col}(G[V_i]) \le r_i$ for $i \in [2]$. □

3.1.3 The coloring corollaries

Using Theorem 3.1.8, we can prove coloring results for graphs with only small cliques among the vertices of high degree. To make this precise, for $d \in \mathbb{N}$ define $\omega_d(G)$ to be the size of the largest clique in G containing only vertices of degree larger than d; that is, $\omega_d(G) := \omega\left(G\left[\{v \in V(G) \mid d_G(v) > d\}\right]\right)$.

Corollary 3.1.11. *Let G be a graph, $k, d \in \mathbb{N}$ with $k \ge 2$ and $\mathbf{r} \in \mathbb{N}^k$. If $w\left(\mathbf{r}\right) \ge \max\left\{\Delta(G) + 1 - k, d\right\}$ and $r_i \ge \omega_d(G) + 1$ for all $i \in [k]$, then at least one of the following holds:*

1. *$w\left(\mathbf{r}\right) = d$ and G contains an induced subgraph Q with $|Q| = d + 1$ which can be partitioned into k cliques $F_1, \dots, F_k$ where*

    a) $|F_1| = r_1 + 1$, $|F_i| = r_i$ for $i \ge 2$,

    b) $\left|F_i^d\right| \ge |F_i| - \omega_d(G)$ for $i \in [k]$,

    c) for $i \in [k]$, each $v \in V(F_i^d)$ is universal in Q;

2. *$\chi(G) \le w\left(\mathbf{r}\right)$.*

Proof. Apply Theorem 3.1.8 to conclude that either (1) holds or there exists an $\mathbf{r}$-partition $P := (V_1, \dots, V_k)$ of G such that if C is an r_i-obstruction in $G[V_i]$, then $\delta_G(C) \ge d$ and C^d is edgeless. Since $\Delta(G[V_i]) \le r_i$ for all $i \in [k]$, it will be enough to show that no $G[V_i]$ contains an r_i-obstruction. Suppose otherwise that we have an r_i-obstruction C in some $G[V_i]$. First, if $r_i \ge 3$, then C is K_{r_i+1} and hence C contains a $K_{\omega_d(G)+2}$. But C^d is edgeless, so $\omega_d(G) > \omega_d(G) + 1$, a contradiction. Thus $r_i = 2$ and C is an odd cycle. Since C^d is edgeless, the vertices of C are 2-colored by the properties 'degree is d' and 'degree is greater than d', impossible. □

For a vertex critical graph G, call $v \in V(G)$ *low* if $d(v) = \chi(G) - 1$ and *high* otherwise. Let $\mathcal{H}(G)$ be the subgraph of G induced on the high vertices of G.

Corollary 3.1.12. *Let G be a vertex critical graph with $\chi(G) = \Delta(G)+2-k$ for some $k \geq 2$. If $k \leq \frac{\chi(G)-1}{\omega(\mathcal{H}(G))+1}$, then G contains an induced subgraph Q with $|Q| = \chi(G)$ which can be partitioned into k cliques $F_1, \dots, F_k$ where*

1. *$|F_1| = \chi(G) - (k-1)(\omega(\mathcal{H}(G)) + 1)$, $|F_i| = \omega(\mathcal{H}(G)) + 1$ for $i \geq 2$;*

2. *for each $i \in [k]$, F_i contains at least $|F_i| - \omega(\mathcal{H}(G))$ low vertices that are all universal in Q.*

Proof. Suppose $k \leq \frac{\chi(G)-1}{\omega(\mathcal{H}(G))+1}$. Put $r_i := \omega(\mathcal{H}(G)) + 1$ for $i \in [k] - \{1\}$ and $r_1 := \chi(G) - 1 - (k-1)(\omega(\mathcal{H}(G)) + 1)$. Set $\mathbf{r} := (r_1, r_2, \dots, r_k)$. Then $w(\mathbf{r}) = \chi(G) - 1 = \Delta(G) + 1 - k$. Now applying Corollary 3.1.11 with $d := \chi(G) - 1$ proves the corollary. $\square$

Corollary 3.1.13. *Let G be a vertex critical graph with $\chi(G) \geq \Delta(G) + 1 - p \geq 4$ for some $p \in \mathbb{N}$. If $\omega(\mathcal{H}(G)) \leq \frac{\chi(G)+1}{p+1} - 2$, then $G = K_{\chi(G)}$ or $G = O_5$.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Put $\chi := \chi(G)$, $\Delta := \Delta(G)$ and $h := \omega(\mathcal{H}(G))$. Then $p \geq 1$ and $h \geq 1$ by Brooks' theorem. Hence $\chi \geq 5$. By assumption, we have $h \leq \frac{\chi+1}{p+1} - 2 = \frac{\chi-2p-1}{p+1} \leq \frac{\chi-p-2}{p+1}$ since $p \geq 1$. Thus $p + 1 \leq \frac{\chi-1}{h+1}$ and we may apply Corollary 3.1.13 with $k := p + 1$ to get an induced subgraph Q of G with $|Q| = \chi$ which can be partitioned into $p+1$ cliques $F_1, \dots, F_{p+1}$ where

1. $|F_1| = \chi - p(h + 1)$, $|F_i| = h + 1$ for $i \geq 2$;

2. for each $i \in [p + 1]$, F_i contains at least $|F_i| - h$ low vertices that are all universal in Q.

Let T be the low vertices in Q, put $H := Q - T$ and $t := |T|$. Then $Q = K_t * H$ and $t \geq \chi - p(h+1) + p(h+1) - (p+1)h = \chi - (p+1)h$.

Take any $(\chi - 1)$-coloring of $G - Q$ and let L be the resulting list assignment on Q. Then $|L(v)| = d_Q(v)$ for each $v \in T$ and $|L(v)| \geq d_Q(v) - p$ for each $v \in V(H)$. Since $t \geq \chi - (p+1)h \geq 2p + 1 \geq p + 1$, if there are nonadjacent $x, y \in V(H)$ and $c \in L(x) \cap L(y)$, then we may color x and y both with c and then greedily complete the coloring to the rest of H and then to all of Q, a contradiction. Hence any nonadjacent pair in H have disjoint lists.

Let I be a maximal independent set in H. If there is an induced P_3 in H with ends in I, set $o_I := 1$, otherwise set $o_I := 0$. Since each pair of vertices in I have disjoint lists, we must have

$$\begin{aligned}
\chi - 1 &\geq \sum_{v \in I} |L(v)| \\
&\geq \sum_{v \in I} t + d_H(v) - p \\
&= (t - p)\,|I| + \sum_{v \in I} d_H(v) \\
&\geq (t - p)\,|I| + |H| - |I| + o_I \\
&= (t - (p+1))\,|I| + \chi - t + o_I.
\end{aligned}$$

Hence $|I| \leq \frac{t-1-o_I}{t-(p+1)} = 1 + \frac{p-o_I}{t-(p+1)} \leq 1 + \frac{p-o_I}{2p+1-(p+1)} \leq 2$ as $t \geq 2p+1$. Since G is not K_χ, we must have $|I| = 2$ and thus $t = 2p+1$ and $o_I = 0$. Thence H is the disjoint union of two complete subgraphs. We then have $\frac{\chi-2p-1}{p+1} \geq h \geq \frac{|H|}{2} = \frac{\chi-2p-1}{2}$. Hence $p = 1$, $h = \frac{\chi-3}{2}$ and $Q = K_3 * 2K_h$.

Let $x, y \in V(H)$ be nonadjacent. Then $d_Q(x) + d_Q(y) = \chi + 1$. Let A be the subgraph of G induced on $V(G - Q) \cup \{x, y\}$. Then $d_A(x) + d_A(y) \leq 2\Delta - (\chi + 1) =$

$\chi - 1$. Let A' be the graph obtained by collapsing $\{x, y\}$ to a single vertex v_{xy}. If $\chi(A') \leq \chi - 1$, then we have a $(\chi - 1)$-coloring of A in which x and y receive the same color. This is impossible as then we could complete the $(\chi - 1)$-coloring to all of G greedily as above. Hence $\chi(A') = \chi$ and thus we have a vertex critical subgraph Z of A' with $\chi(Z) = \chi$. We must have $v_{xy} \in V(Z)$ and since $d_A(x) + d_A(y) \leq \chi - 1$, v_{xy} is low. Hence, by minimality of $|G|$, $Z = K_\chi$ or $Z = O_5$.

First, suppose $\chi \geq 6$. Then $h \geq 2$ and thus we have $z \in V(H) - \{x, y\}$ nonadjacent to x. Apply the previous paragraph to both pairs $\{x, y\}$ and $\{x, z\}$. The case $Z = O_5$ cannot happen, for then we would have $\chi = \chi(Z) = 5$, a contradiction. Put $X_1 := N(x) \cap V(G - Q)$, $X_2 := N(y) \cap V(G - Q)$, $X_3 := N(z) \cap V(G - Q)$. Then $|X_i| = \frac{\chi - 1}{2}$ for $i \in [3]$ and X_1 is joined to both X_2 and X_3. Since $|X_i| - h > 0$, each X_i contains a low vertex v_i. But then $N(v_1) = X_1 \cup X_2 \cup \{x\}$ and we must have $X_3 = X_2$. Whence $N(v_2) = X_1 \cup X_2 \cup \{y, z\}$ giving $d(v_2) \geq \chi$, a contradiction.

Therefore $\chi = 5$, $h = 1$ and $V(H) = \{x, y\}$. If $Z = K_5$, then $N[x] \cup N[y]$ induces an O_5 in G and hence $G = O_5$, a contradiction. Thus $Z = O_5$. But $h = 1$, so all of the neighbors of both x and y are low and hence all of the neighbors of v_{xy} in Z are low. But O_5 has no such low vertex v_{xy} with all low neighbors, so this is impossible. □

Question. The condition on k needed in Corollary 3.1.12 is weaker than that in Corollary 3.1.13. What do the intermediate cases look like? What are the extremal examples?

3.2 Destroying incomplete components in vertex partitions

In [46] Kostochka modified an algorithm of Catlin [14] to show that every triangle-free graph G can be colored with at most $\frac{2}{3}\Delta(G) + 2$ colors. In fact, his modification proves that the vertex set of any triangle-free graph G can be partitioned into

$\left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ sets, each of which induces a disjoint union of paths. In [64] we generalized this as follows.

Lemma 3.2.1 (Rabern [64]). *Let G be a graph and $r_1, \ldots, r_k \in \mathbb{N}$ such that $\sum_{i=1}^{k} r_i \geq \Delta(G)+2-k$. Then $V(G)$ can be partitioned into sets $V_1, \ldots, V_k$ such that $\Delta(G[V_i]) \leq r_i$ and $G[V_i]$ contains no incomplete r_i-regular components for each $i \in [k]$.*

Setting $k = \left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ and $r_i = 2$ for each i gives a slightly more general form of Kostochka's theorem.

Corollary 3.2.2 (Rabern [64]). *The vertex set of any graph G can be partitioned into $\left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ sets, each of which induces a disjoint union of triangles and paths.*

For coloring, this actually gives the bound $\chi(G) \leq 2\left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ for triangle free graphs. To get $\frac{2}{3}\Delta(G) + 2$, just use $r_k = 0$ when $\Delta \equiv 2 (\text{mod } 3)$. Similarly, for any $r \geq 2$, setting $k = \left\lceil \frac{\Delta(G)+2}{r+1} \right\rceil$ and $r_i = r$ for each i gives the following.

Corollary 3.2.3 (Rabern [64]). *Fix $r \geq 2$. The vertex set of any K_{r+1}-free graph G can be partitioned into $\left\lceil \frac{\Delta(G)+2}{r+1} \right\rceil$ sets each inducing an $(r-1)$-degenerate subgraph with maximum degree at most r.*

For the purposes of coloring it is more economical to split off $\Delta + 2 - (r+1)\left\lfloor \frac{\Delta+2}{r+1} \right\rfloor$ parts with $r_j = 0$.

Corollary 3.2.4 (Rabern [64]). *Fix $r \geq 2$. The vertex set of any K_{r+1}-free graph G can be partitioned into $\left\lfloor \frac{\Delta(G)+2}{r+1} \right\rfloor$ sets each inducing an $(r-1)$-degenerate subgraph with maximum degree at most r and $\Delta(G) + 2 - (r+1)\left\lfloor \frac{\Delta(G)+2}{r+1} \right\rfloor$ independent sets. In particular, $\chi(G) \leq \Delta(G) + 2 - \left\lfloor \frac{\Delta(G)+2}{r+1} \right\rfloor$.*

For $r \geq 3$, the bound on the chromatic number is only interesting in that its proof does not rely on Brooks' Theorem. Lemma 3.2.1 is of the same form as Lovász's

Lemma 1.1.4, but it gives a more restrictive partition at the cost of replacing $\Delta(G)+1$ with $\Delta(G)+2$. For $r \geq 3$, combining Lovász's Lemma 1.1.4 with Brooks' theorem gives the following better bound for a K_{r+1}-free graph G (first proved in [10], [15] and [50]):

$$\chi(G) \leq \Delta(G) + 1 - \left\lfloor \frac{\Delta(G)+1}{r+1} \right\rfloor .$$

3.2.1 A generalization

Here we prove a generalization of Lemma 3.2.1.

Definition 3.2.1. For $h\colon \mathcal{G} \to \mathbb{N}$ and $G \in \mathcal{G}$, a vertex $x \in V(G)$ is called *h-critical* in G if $G - x \in \mathcal{G}$ and $h(G-x) < h(G)$.

Definition 3.2.2. For $h\colon \mathcal{G} \to \mathbb{N}$ and $G \in \mathcal{G}$, a pair of vertices $\{x, y\} \subseteq V(G)$ is called an *h-critical pair* in G if $G - \{x, y\} \in \mathcal{G}$ and x is h-critical in $G - y$ and y is h-critical in $G - x$.

Definition 3.2.3. For $r \in \mathbb{N}$ a function $h\colon \mathcal{G} \to \mathbb{N}$ is called an *r-height function* if it has each of the following properties:

1. if $h(G) > 0$, then G contains an h-critical vertex x with $d(x) \geq r$;
2. if $G \in \mathcal{G}$ and $x \in V(G)$ is h-critical with $d(x) \geq r$, then $h(G-x) = h(G) - 1$;
3. if $G \in \mathcal{G}$ and $x \in V(G)$ is h-critical with $d(x) \geq r$, then G contains an h-critical vertex $y \notin \{x\} \cup N(x)$ with $d(y) \geq r$;
4. if $G \in \mathcal{G}$ and $\{x, y\} \subseteq V(G)$ is an h-critical pair in G with $d_{G-y}(x) \geq r$ and $d_{G-x}(y) \geq r$, then there exists $z \in N(x) \cap N(y)$ with $d(z) \geq r + 1$.

Lemma 3.2.5. *Let G be a graph and $r_1, \ldots, r_k \in \mathbb{N}$ such that $\sum_{i=1}^{k} r_i \geq \Delta(G)+2-k$. If h_i is an r_i-height function for each $i \in [k]$, then $V(G)$ can be partitioned into sets $V_1, \ldots, V_k$ such that for each $i \in [k]$, $\Delta(G[V_i]) \leq r_i$ and $h_i(D) = 0$ for each component D of $G[V_i]$.*

For each $r \in \mathbb{N}$, it is easy to see that the function $h_r \colon \mathcal{G} \to \mathbb{N}$ defined as follows is an r-height function:

$$h_r(G) := \begin{cases} 1 & G \text{ is incomplete and } r\text{-regular;} \\ 0 & \text{otherwise.} \end{cases}$$

Applying Lemma 3.2.5 with these height functions proves Lemma 3.2.1. Other height functions exist, but we don't yet have a sense of their ubiquity or lack thereof.

Proof of Lemma 3.2.5. For a partition $P := (V_1, \ldots, V_k)$ of $V(G)$ let

$$f(P) := \sum_{i=1}^{k} \left(\|G[V_i]\| - r_i \left|V_i\right| \right),$$

$$c(P) := \sum_{i=1}^{k} c(G[V_i]),$$

$$h(P) := \sum_{i=1}^{k} h_i(G[V_i]).$$

Let $P := (V_1, \ldots, V_k)$ be a partition of $V(G)$ minimizing $f(P)$, and subject to that $c(P)$, and subject to that $h(P)$.

Let $i \in [k]$ and $x \in V_i$ with $d_{V_i}(x) \geq r_i$. Since $\sum_{i=1}^{k} r_i \geq \Delta(G) + 2 - k$ there is some $j \neq i$ such that $d_{V_j}(x) \leq r_j$. Moving x from V_i to V_j gives a new partition P^* with $f(P^*) \leq f(P)$. Note that if $d_{V_i}(x) > r_i$ we would have $f(P^*) < f(P)$ contradicting the minimality of P. This proves that $\Delta(G[V_i]) \leq r_i$ for each $i \in [k]$.

Now suppose that for some i_1 there is a component A_1 of $G[V_{i_1}]$ with $h_{i_1}(A_1) > 0$. Put $P_1 := P$ and $V_{1,i} := V_i$ for $i \in [k]$. By property 1 of height functions, we have an h_{i_1}-critical vertex $x_1 \in V(A_1)$ with $d_{A_1}(x_1) \geq r_{i_1}$. By the above we have $i_2 \neq i_1$ such that moving x_1 from V_{1,i_1} to V_{1,i_2} gives a new partition $P_2 := (V_{2,1}, V_{2,2}, \ldots, V_{2,k})$ where $f(P_2) = f(P_1)$. By the minimality of $c(P_1)$, x_1 is adjacent to only one component C_2 in $G[V_{1,i_2}]$. Let $A_2 := G[V(C_2) \cup \{x_1\}]$. Since x_1 is h_{i_1}-critical, by the minimality of $h(P_1)$, it must be that $h_{i_2}(A_2) > h_{i_2}(C_2)$. By property 2 of height functions we must have $h_{i_2}(A_2) = h_{i_2}(C_2) + 1$. Hence $h(P_2)$ is still minimum. Now, by property 3 of height functions, we have an h_{i_2}-critical vertex $x_2 \in V(A_2) - (\{x_1\} \cup N_{A_2}(x_1))$ with $d_{A_2}(x_2) \geq r_{i_2}$.

Continue on this way to construct sequences $i_1, i_2, \ldots$, $A_1, A_2, \ldots$, $P_1, P_2, P_3, \ldots$ and $x_1, x_2, \ldots$. Since G is finite, at some point we will need to reuse a leftover component; that is, there is a smallest t such that $A_{t+1} - x_t = A_s - x_s$ for some $s < t$. In particular, $\{x_s, x_{t+1}\}$ is an h_{i_s}-critical pair in $Q := G\left[\{x_{t+1}\} \cup V(A_s)\right]$ where $d_{Q-x_{t+1}}(x_s) \geq r_{i_s}$ and $d_{Q-x_s}(x_{t+1}) \geq r_{i_s}$. Thus, by property 4 of height functions, we have $z \in N_Q(x_s) \cap N_Q(x_{t+1})$ with $d_Q(z) \geq r_{i_s} + 1$.

We now modify P_s to contradict the minimality of $f(P)$. At step $t+1$, x_t was adjacent to exactly r_{i_s} vertices in V_{t+1,i_s}. This is what allowed us to move x_t into V_{t+1,i_s}. Our goal is to modify P_s so that we can move x_t into the i_s part without moving x_s out. Since z is adjacent to both x_s and x_t, moving z out of the i_s part will then give us our desired contradiction.

So, consider the set X of vertices that could have been moved out of V_{s,i_s} between step s and step $t+1$; that is, $X := \{x_{s+1}, x_{s+2}, \ldots, x_{t-1}\} \cap V_{s,i_s}$. For $x_j \in X$, since $d_{A_j}(x_j) \geq r_{i_s}$ and x_j is not adjacent to x_{j-1} we see that $d_{V_{s,i_s}}(x_j) \geq r_{i_s}$. Similarly, $d_{V_{s,i_t}}(x_t) \geq r_{i_t}$.

Also, by the minimality of t, X is an independent set in G. Thus we may move all elements of X out of V_{s,i_s} to get a new partition $P^* := (V_{*,1}, \ldots, V_{*,k})$ with $f(P^*) = f(P)$.

Since x_t is adjacent to exactly r_{i_s} vertices in V_{t+1,i_s} and the only possible neighbors of x_t that were moved out of V_{s,i_s} between steps s and $t+1$ are the elements of X, we see that $d_{V_{*,i_s}}(x_t) = r_{i_s}$. Since $d_{V_{*,i_t}}(x_t) \geq r_{i_t}$ we can move x_t from V_{*,i_t} to V_{*,i_s} to get a new partition $P^{**} := (V_{**,1}, \ldots, V_{**,k})$ with $f(P^{**}) = f(P^*)$. Now, recall that $z \in V_{**,i_s}$. Since z is adjacent to x_t we have $d_{V_{**,i_s}}(z) \geq r_{i_s} + 1$. Thus we may move z out of V_{**,i_s} to get a new partition P^{***} with $f(P^{***}) < f(P^{**}) = f(P)$. This contradicts the minimality of $f(P)$. $\square$

Chapter 4

REDUCING MAXIMUM DEGREE

Some of the material in this chapter appeared in [59].

4.1 Hitting all maximum cliques

As part of his proof that every graph with $\chi \geq \Delta$ contains a $K_{\Delta-28}$, Kostochka proved the following lemma.

Lemma 4.1.1 (Kostochka [45])**.** *If G is a graph satisfying $\omega \geq \Delta + \frac{3}{2} - \sqrt{\Delta}$, then G contains an independent set I such that $\omega(G - I) < \omega(G)$.*

To talk about the proof we first need a definition.

Clique Graph. Let G be a graph. For a collection of cliques $\mathcal{Q}$ in G, let $X_{\mathcal{Q}}$ be the intersection graph of $\mathcal{Q}$. That is, the vertex set of $X_{\mathcal{Q}}$ is $\mathcal{Q}$ and there is an edge between $Q_1 \neq Q_2 \in \mathcal{Q}$ iff Q_1 and Q_2 intersect.

Kostochka's proof proceeded in two stages. First show that the vertices in each component of the clique graph have a large intersection. Then find an independent transversal of these intersections. Such a transversal is an independent set hitting every maximum clique. Kostochka used a custom method to find a transversal. In [59], we applied the following lemma of Haxell [31] (proved long after Kostochka's paper) to find the independent transversal.

Lemma 4.1.2. *Let H be a graph and $V_1 \cup \cdots \cup V_r$ a partition of $V(H)$. Suppose that $|V_i| \geq 2\Delta(H)$ for each $i \in [r]$. Then H has an independent set $\{v_1, \ldots, v_n\}$ where $v_i \in V_i$ for each $i \in [r]$.*

Finding the independent transversal using this lemma gives the following.

Lemma 4.1.3 (Rabern [59]). *If G is a graph satisfying $\omega \geq \frac{3}{4}(\Delta + 1)$, then G contains an independent set I such that $\omega(G - I) < \omega(G)$.*

Aharoni, Berger and Ziv [1] showed that Haxell's proof actually gets more than Lemma 4.1.2. From their extension, King [42] proved the following lopsided version of Haxell's lemma.

Lemma 4.1.4 (King [42]). *Let G be a graph partitioned into r cliques $V_1, \dots, V_r$. If there exists $k \geq 1$ such that for each i every $v \in V_i$ has at most $\min\{k, |V_i| - k\}$ neighbors outside V_i, then G contains an independent set with r vertices.*

Using this gives the best possible form of the lemma.

Lemma 4.1.5 (King [42]). *If G is a graph satisfying $\omega > \frac{2}{3}(\Delta + 1)$, then G contains an independent set I such that $\omega(G - I) < \omega(G)$.*

4.1.1 A simple proof of Kostochka's first stage

The proofs for Kostochka's first stage can be made much simpler than the originals and we do so here.

Lemma 4.1.6 (Hajnal [30]). *Let G be a graph and $\mathcal{Q}$ a collection of maximum cliques in G. Then*

$$\left|\bigcup \mathcal{Q}\right| + \left|\bigcap \mathcal{Q}\right| \geq 2\omega(G).$$

Proof. Suppose the lemma is false and let $\mathcal{Q}$ be a counterexample with $|\mathcal{Q}|$ minimal. Put $r := |\mathcal{Q}|$ and say $\mathcal{Q} = \{Q_1, \dots, Q_r\}$.

Consider the set $W := (Q_1 \cap \bigcup_{i=2}^r Q_i) \cup \bigcap_{i=2}^r Q_i$. Plainly, W is a clique. Thus we may derive a contradiction as follows.

$$\begin{aligned}
\omega(G) &\geq |W| \\
&= \left|(Q_1 \cap \bigcup_{i=2}^{r} Q_i) \cup \bigcap_{i=2}^{r} Q_i\right| \\
&= \left|Q_1 \cap \bigcup_{i=2}^{r} Q_i\right| + \left|\bigcap_{i=2}^{r} Q_i\right| - \left|\bigcap_{i=1}^{r} Q_i \cap \bigcup_{i=2}^{r} Q_i\right| \\
&= |Q_1| + \left|\bigcup_{i=2}^{r} Q_i\right| - \left|\bigcup_{i=1}^{r} Q_i\right| + \left|\bigcap_{i=2}^{r} Q_i\right| - \left|\bigcap_{i=1}^{r} Q_i\right| \\
&= \omega(G) + \left|\bigcup_{i=2}^{r} Q_i\right| + \left|\bigcap_{i=2}^{r} Q_i\right| - \left|\bigcup_{i=1}^{r} Q_i\right| - \left|\bigcap_{i=1}^{r} Q_i\right| \\
&\geq \omega(G) + 2\omega(G) - \left(\left|\bigcup_{i=1}^{r} Q_i\right| + \left|\bigcap_{i=1}^{r} Q_i\right|\right) \\
&> \omega(G).
\end{aligned}$$

□

Lemma 4.1.7 (Kostochka [45]). *If $\mathcal{Q}$ is a collection of maximum cliques in a graph G with $\omega(G) > \frac{2}{3}(\Delta(G) + 1)$ such that $X_{\mathcal{Q}}$ is connected, then $\cap\mathcal{Q} \neq \emptyset$.*

Proof. Suppose not and choose a counterexample $\mathcal{Q} := \{Q_1, \dots, Q_r\}$ minimizing r. Plainly, $r \geq 3$. Let A be a noncutvertex in $X_{\mathcal{Q}}$ and B a neighbor of A. Put $\mathcal{Z} := Q - \{A\}$. Then $X_{\mathcal{Z}}$ is connected and hence by minimality of r, $\cap\mathcal{Z} \neq \emptyset$. In particular, $|\cup\mathcal{Z}| \leq \Delta(G) + 1$. Hence $|\cup\mathcal{Q}| \leq |\cup\mathcal{Z}| + |A - B| \leq 2(\Delta(G) + 1) - \omega(G) < 2\omega(G)$. This contradicts Hajnal's lemma. □

With a little more work we can prove the following generalization of Kostochka's lemma which has a Helly feel. We won't use this result here, but it has some independent interest. The following example from King [40] shows that the condition $\omega > \frac{k+1}{2k+1}(\Delta + 1)$ is tight. Take $K_{k+1} * E_{k+1}$ and remove a perfect matching between

K_{k+1} and E_{k+1} to get a graph H. Then $\omega(H) = k+1$ and $\Delta(H) = 2k$ and thus $\omega(H) = \frac{k+1}{2k+1}(\Delta(H)+1)$. Taking $\mathcal{Q}$ to be all $(k+1)$-cliques containing a vertex in the E_{k+1}, we see that $\cap\mathcal{Q} = \emptyset$ but any k elements of $\mathcal{Q}$ have common intersection.

Lemma 4.1.8. *Fix $k \geq 2$. Let G be a graph satisfying $\omega > \frac{k+1}{2k+1}(\Delta+1)$. If $\mathcal{Q}$ is a collection of maximum cliques in G such that any k elements of $\mathcal{Q}$ have common intersection, then $\cap\mathcal{Q} \neq \emptyset$.*

Proof. Suppose not and choose a counterexample $\mathcal{Q} := \{Q_1, \ldots, Q_r\}$ minimizing r. Plainly, $r \geq k+1$. Put $\mathcal{Z}_i := \mathcal{Q} - \{Q_i\}$. Then any k elements of $\mathcal{Z}_i$ have common intersection and hence by minimality $\cap\mathcal{Z}_i \neq \emptyset$. In particular $\cup\mathcal{Z}_i$ contains a universal vertex and thus $|\cup\mathcal{Z}_i| \leq \Delta(G)+1$. Now, by Hajnal's Lemma, $|\cap\mathcal{Z}_i| \geq 2\omega(G) - (\Delta(G)+1) > 2\omega(G) - \frac{2k+1}{k+1}\omega(G) = \frac{1}{k+1}\omega(G)$.

Put $m := \min_i |Q_i - \cup\mathcal{Z}_i|$. Note that the $\cap Z_i$ are pairwise disjoint since $\cap\mathcal{Q} = \emptyset$. Thus $\cup\mathcal{Q}$ contains the disjoint union of the $\cap Z_i$ as well as at least m vertices in each clique outside the rest. In particular,

$$|\cup\mathcal{Q}| \geq \frac{1}{k+1}\omega(G)r + mr \geq \omega(G) + (k+1)m.$$

In addition,

$$|\cup\mathcal{Q}| \leq m + \Delta(G) + 1.$$

Hence,

$$m \leq \frac{\Delta(G) + 1 - \omega(G)}{k} < \frac{1}{k+1}\omega(G).$$

Finally,

$$|\cup\mathcal{Q}| \leq m + \Delta(G) + 1 < \frac{1}{k+1}\omega(G) + \frac{2k+1}{k+1}\omega(G) = 2\omega(G).$$

Applying Hajnal's Lemma gives a contradiction. □

4.1.2 Independent transversals

In [34], Haxell and Szabó develop a technique for dealing with independent transversals. In [33], Haxell used this technique to give simpler proof of her Lemma. The proof gives a bit more as the following lemma shows. This is just slightly more general than the extension given in [1] by Aharoni, Berger and Ziv and either gives enough to prove King's lopsided version of Haxell's lemma. We write $f\colon A \twoheadrightarrow B$ for a surjective function from A to B. Let G be a graph. For a k-coloring $\pi\colon V(G) \twoheadrightarrow [k]$ of G and a subgraph H of G we say that $I := \{x_1, \dots, x_k\} \subseteq V(H)$ is an H-independent transversal of π if I is an independent set in H and $\pi(x_i) = i$ for all $i \in [k]$.

Lemma 4.1.9. *Let G be a graph and $\pi\colon V(G) \twoheadrightarrow [k]$ a proper k-coloring of G. Suppose that π has no G-independent transversal, but for every $e \in E(G)$, π has a $(G-e)$-independent transversal. Then for every $xy \in E(G)$ there is $J \subseteq [k]$ with $\pi(x), \pi(y) \in J$ and an induced matching M of $G[\pi^{-1}(J)]$ with $xy \in M$ such that*

1. *$\bigcup M$ totally dominates $G[\pi^{-1}(J)]$,*
2. *the multigraph with vertex set J and an edge between $a, b \in J$ for each $uv \in M$ with $\pi(u) = a$ and $\pi(v) = b$ is a (simple) tree. In particular $|M| = |J| - 1$.*

Proof. Suppose the lemma is false and choose a counterexample G with $\pi\colon V(G) \twoheadrightarrow [k]$ so as to minimize k. Let $xy \in E(G)$. By assumption π has a $(G-xy)$-independent transversal T. Note that we must have $x, y \in T$ lest T be a G-independent transversal of π.

By symmetry we may assume that $\pi(x) = k-1$ and $\pi(y) = k$. Put $X := \pi^{-1}(k-1)$, $Y := \pi^{-1}(k)$ and $H := G - N(\{x,y\}) - E(X,Y)$. Define $\zeta\colon V(H) \to [k-1]$ by $\zeta(v) := \min\{\pi(v), k-1\}$. Note that since $x, y \in T$, we have $|\zeta^{-1}(i)| \geq 1$ for each $i \in [k-2]$. Put $Z := \zeta^{-1}(k-1)$. Then $Z \neq \emptyset$ for otherwise $M := \{xy\}$ totally dominates $G[X \cup Y]$ giving a contradiction.

Suppose ζ has an H-independent transversal S. Then we have $z \in S \cap Z$ and by symmetry we may assume $z \in X$. But then $S \cup \{y\}$ is a G-independent transversal of π, a contradiction.

Let $H' \subseteq H$ be a minimal spanning subgraph such that ζ has no H'-independent transversal. Now $d(z) \geq 1$ for each $z \in Z$ for otherwise $T - \{x,y\} \cup \{z\}$ would be an H'-independent transversal of ζ. Pick $zw \in E(H')$. By minimality of k, we have $J \subseteq [k-1]$ with $\zeta(z), \zeta(w) \in J$ and an induced matching M of $H'[\zeta^{-1}(J)]$ with $zw \in M$ such that

1. $\bigcup M$ totally dominates $H'[\zeta^{-1}(J)]$,
2. the multigraph with vertex set J and an edge between $a, b \in J$ for each $uv \in M$ with $\zeta(u) = a$ and $\zeta(v) = b$ is a (simple) tree.

Put $M' := M \cup \{xy\}$ and $J' := J \cup \{k\}$. Since H' is a spanning subgraph of H, $\bigcup M$ totally dominates $H[\zeta^{-1}(J)]$ and hence $\bigcup M'$ totally dominates $G[\pi^{-1}(J')]$. Moreover, the multigraph in (2) for M' and J' is formed by splitting the vertex $k-1 \in J$ in two vertices and adding an edge between them and hence it is still a tree. This final contradiction proves the lemma. $\square$

Lemma 4.1.4 (King [42]). *Let H be a graph partitioned into k cliques $V_1, \dots, V_k$. If there exists $r \geq 1$ such that for each i every $v \in V_i$ has at most $\min\{r, |V_i| - r\}$ neighbors outside V_i, then G contains an independent set with k vertices.*

Proof. Suppose not and choose a counterexample H minimizing $\|H\|$. Remove all the edges from each V_i to form a graph G. Pick $xy \in E(G)$ and apply Lemma 4.1.9 on xy with $\pi\colon V(G) \twoheadrightarrow [r]$ given by $\pi(V_i) = i$ to get the guaranteed $J \subseteq [r]$ and induced matching M. Note that by our assumption, the ends of an edge from V_i to V_j together dominate at most $\min\{|V_i|, |V_j|\}$ vertices. Let T be the tree with vertex set J and an edge between $a, b \in J$ for each $uv \in M$ with $\pi(u) = a$ and $\pi(v) = b$. Choose a root c of T. Traversing T in leaf-first order and for each leaf a with parent b picking $|V_a|$ from $\min\{|V_a|, |V_b|\}$ we get that the vertices in M together dominate at most $\sum_{i \in J-c} |V_i|$ vertices, a contradiction. $\square$

4.1.3 Putting it all together

Now we are in a position to prove Lemma 4.1.5.

Lemma 4.1.5 (King [42])**.** *If G is a graph satisfying $\omega > \frac{2}{3}(\Delta + 1)$, then G contains an independent set I such that $\omega(G - I) < \omega(G)$.*

Proof. Put $\Delta := \Delta(G)$ and $\omega := \omega(G)$. Let $\mathcal{Q}$ be all the maximum cliques in G and $\mathcal{Q}_1, \ldots, \mathcal{Q}_k$ the vertex sets of the components of $X_{\mathcal{Q}}$. Since the components of $X_{\mathcal{Q}}$ satisfy the hypotheses of Lemma 4.1.7, we have $F_i := \cap \mathcal{Q}_i \neq \emptyset$ for all $i \in [k]$. Put $D_i := \cup \mathcal{Q}_i$.

Put $r := \frac{1}{3}(\Delta + 1)$. Note that the vertices in F_i are universal in D_i. Since $|D_i| \geq \omega > \frac{2}{3}(\Delta + 1)$, each $v \in F_i$ has at most r neighbors in the rest of the F_j. Applying Lemma 4.1.6 gives $|F_i| + |D_i| \geq 2\omega > \frac{4}{3}(\Delta + 1)$. Thus each $v \in F_i$ has at most $\Delta + 1 - |D_i| > \Delta + 1 - (\frac{4}{3}(\Delta + 1) - |F_i|) = |F_i| - r$ neighbors in the rest of the F_j. Applying Lemma 4.1.4 gives an independent set intersecting each F_i and hence every maximum clique in G. $\square$

4.2 Example reductions

4.2.1 The quintessential reduction example

Reed [66] has conjectured that every graph satisfies

$$\chi \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil.$$

If we could always find an independent set whose removal decreased both ω and Δ, then the conjecture would follow by simple induction since we can give the independent set a single color and use at most $\left\lceil \frac{\omega+\Delta+1}{2} \right\rceil - 1$ colors on what remains. Expanding the independent set given by Lemma 4.1.5 to a maximal one shows that this sort of argument goes through when $\omega > \frac{2}{3}(\Delta + 1)$. Thus a minimum counterexample to Reed's conjecture satisfies $\omega \leq \frac{2}{3}(\Delta + 1)$.

4.2.2 Reducing for Brooks

The proof in Chapter 2 reduces Brooks' theorem to the $\Delta = 3$ case by an ad hoc argument. The reduction follows from the general lemmas on hitting maximum cliques as follows. Let G be a counterexample to Brooks' theorem minimizing $\Delta(G)$. Suppose $\Delta(G) \geq 4$. We may assume G is critical. If $\omega(G) < \Delta(G)$, then removing any maximal independent set from G decreases $\chi(G)$ and $\Delta(G)$ both by one giving a counterexample with smaller Δ. Hence $\omega(G) \geq \Delta(G)$. But then $\omega(G) > \frac{2}{3}(\Delta(G)+1)$ and Lemma 4.1.5 gives us an independent set I such that $\omega(G - I) < \omega(G)$. Let M be a maximal independent set containing I. Then $G - M$ is a counterexample with smaller Δ.

4.2.3 Reducing for Borodin-Kostochka

More generally, we can use the facts on hitting maximum cliques to prove the following reduction lemma.

Definition 4.2.1. For $k, j \in \mathbb{N}$, let $\mathcal{C}_{k,j}$ be the collection of all vertex critical graphs satisfying $\chi = \Delta = k$ and $\omega < k - j$. Put $\mathcal{C}_k := \mathcal{C}_{k,0}$. Note that $\mathcal{C}_{k,j} \subseteq \mathcal{C}_{k,i}$ for $j \geq i$.

Lemma 4.2.1. *Fix $k, j \in \mathbb{N}$ with $k \geq 3j+6$. If $G \in \mathcal{C}_{k,j}$, then there exists $H \in \mathcal{C}_{k-1,j}$ such that $H \lhd G$.*

Proof. Let $G \in \mathcal{C}_{k,j}$. We first show that there exists a maximal independent set M such that $\omega(G - M) < k - (j + 1)$. If $\omega(G) < k - (j + 1)$, then any maximal independent set will do for M. Otherwise, $\omega(G) = k - (j + 1)$. Since $k \geq 3j + 6$, we have $\omega(G) = k - (j+1) > \frac{2}{3}(k+1) = \frac{2}{3}(\Delta(G)+1)$. Thus by Lemma 4.1.5, we have an independent set I such that $\omega(G - I) < \omega(G)$. Expand I to a maximal independent set to get M.

Now $\chi(G - M) = k - 1 = \Delta(G - M)$, where the last equality follows from Brooks' theorem and $\omega(G - M) < k - (j + 1) \leq k - 1$. Since $\omega(G - M) < k - (j + 1)$, for any $(k - 1)$-critical induced subgraph $H \unlhd G - M$ we have $H \in \mathcal{C}_{k-1,j}$. □

As a consequence we get the result of Kostochka that the Borodin-Kostochka conjecture can be reduced to the $k = 9$ case.

Lemma 4.2.2. *Let $\mathcal{H}$ be a hereditary graph property. For $k \geq 5$, if $\mathcal{H} \cap \mathcal{C}_k = \emptyset$, then $\mathcal{H} \cap \mathcal{C}_{k+1} = \emptyset$. In particular, to prove the Borodin-Kostochka conjecture it is enough to show that $\mathcal{C}_9 = \emptyset$.*

Chapter 5

COLORING FROM ALMOST DEGREE SIZED PALETTES

The material in this chapter appeared in [20] and is joint work with Dan Cranston.

In this section we use list-coloring lemmas to forbid a large class of graphs from appearing as induced subgraphs of vertex critical graphs satisfying $\chi = \Delta$. In each case, we assume that such a graph $H \lhd G$ appears as an induced subgraph of such a graph G. By criticality of G, we can color $G \setminus H$ with $\Delta - 1$ colors. If H can be colored regardless of which colors are forbidden by its colored neighbors in $G \setminus H$, then we can clearly extend this coloring to all of G. Such H are precisely the d_1-choosable graphs.

We characterize all graphs $A * B$ with $|A| \geq 2$, $|B| \geq 2$ that are not d_1-choosable. The characterization is somewhat lengthy, so we split it into a number of lemmas. For the case $|A| \geq 4$, $|B| \geq 4$, see Lemma 5.3.44. When $|A| = 3$, we consider the four cases $A = E_3$ (Lemma 5.3.39), $A = \overline{P_3}$ (Lemma 5.3.43), $A = P_3$ (Lemma 5.3.26), and $A = K_3$ (Lemma 5.3.29). When $|A| = 2$, we consider the case $A = E_2$ in Lemma 5.3.22 and the case $A = K_2$ in Lemma 5.3.47.

Let G be a graph. A *list assignment* to the vertices of G is a function from $V(G)$ to the finite subsets of $\mathbb{N}$. A list assignment L to G is *good* if G has a coloring c where $c(v) \in L(v)$ for each $v \in V(G)$. It is *bad* otherwise. We call the collection of all colors that appear in L, the *pot* of L. That is $Pot(L) := \bigcup_{v \in V(G)} L(v)$. For a subgraph H of G we write $Pot_H(L) := \bigcup_{v \in V(H)} L(v)$. For $S \subseteq Pot(L)$, let G_S be the graph $G[\{v \in V(G) \mid L(v) \cap S \neq \emptyset\}]$. We also write G_c for $G_{\{c\}}$. We let $\mathcal{B}(L)$ be the bipartite graph that has parts $V(G)$ and $Pot(L)$ and an edge from $v \in V(G)$ to $c \in Pot(L)$ iff $c \in L(v)$.

For $f\colon V(G) \to \mathbb{N}$, an f-assignment on G is an assignment L of lists to the vertices of G such that $|L(v)| = f(v)$ for each $v \in V(G)$. We say that G is *f-choosable* if every f-assignment on G is good.

5.1 Shrinking the pot

In this section we prove a lemma about bad list assignments with minimum pot size. Some form of this lemma has appeared independently in at least two places we know of—Kierstead [37] and Reed and Sudakov [68]. We will use this lemma repeatedly in the arguments that follow.

Given a graph G and $f\colon V(G) \to \mathbb{N}$, we have a partial order on the f-assignments to G given by $L < L'$ iff $|Pot(L)| < |Pot(L')|$. When we talk of *minimal* f-assignments, we mean minimal with respect to this partial order.

Lemma 5.1.1. *Let G be a graph and $f\colon V(G) \to \mathbb{N}$. Assume G is not f-choosable and let L be a minimal bad f-assignment. Assume $L(v) \neq Pot(L)$ for each $v \in V(G)$. Then, for each nonempty $S \subseteq Pot(L)$, any coloring of G_S from L uses some color not in S.*

Proof. Suppose not and let $\emptyset \neq S \subseteq Pot(L)$ be such that G_S has a coloring ϕ from L using only colors in S. For $v \in V(G)$, let $h(v)$ be the smallest element of $Pot(L) - L(v)$ (this is well defined by assumption). Pick some $c \in S$ and construct a new list assignment L' as follows.

$$L'(v) = \begin{cases} L(v) & \text{if } v \in V(G) - V(G_S) \\ L(v) & \text{if } v \in V(G_S) \text{ and } c \notin L(v) \\ (L(v) - \{c\}) \cup \{h(v)\} & \text{if } v \in V(G_S) \text{ and } c \in L(v) \end{cases}$$

Note that L' is an f-assignment and $Pot(L') = Pot(L) - \{c\}$. Thus, by minimality of L, we can properly color G from L'. In particular, we have a coloring of $V(G) - V(G_S)$ from L using no color from S. We can complete this to a coloring of G from L using ϕ. This contradicts the fact that L is bad. $\square$

When $|S| = 1$, we can say more. We will use the following lemma in the proof that the graph D_8 in Figure 6.2 is d_1-choosable. It should be useful elsewhere as well.

Lemma 5.1.2. *Let G be a graph and $f: V(G) \to \mathbb{N}$. Suppose G is not f-choosable and let L be a minimal bad f-assignment. Then for any $c \in Pot(L)$, there is a component H of G_c such that $Pot_H(L) = Pot(L)$. In particular, $Pot_{G_c}(L) = Pot(L)$.*

Proof. Suppose otherwise that we have $c \in Pot(L)$ such that $Pot_H(L) \subsetneq Pot(L)$ for all components H of G_c. Say the components of G_c are $H_1, \ldots, H_t$. For $i \in [t]$, choose $\alpha_i \in Pot(L) - Pot_{H_i}(L)$. Now define a list assignment L' on G by setting $L'(v) := L(v)$ for all $v \in V(G) - V(G_c)$ and for each $i \in [t]$ setting $L'(v) := (L(v) - c) \cup \{\alpha_i\}$ for each $v \in V(H_i)$. Then $|Pot(L')| < |Pot(L)|$ and hence by minimality L we have an L'-coloring π of G. Plainly $Q := \{v \in V(G_c) \mid \pi(v) = \alpha_i \text{ for some } i \in [t]\}$ is an independent set. Since c doesn't appear outside G_c, we can recolor all vertices in Q with c to get an L-coloring of G. This contradicts the fact that L is bad. $\square$

Definition 5.1.1. A bipartite graph with parts A and B has *positive surplus* (with respect to A) if $|N(X)| > |X|$ for all $\emptyset \neq X \subseteq A$.

Lemma 5.1.3. *Let G be a graph and $f: V(G) \to \mathbb{N}$. Assume G is not f-choosable and let L be a minimal bad f-assignment. Assume $L(v) \neq Pot(L)$ for each $v \in V(G)$. Then $\mathcal{B}(L)$ has positive surplus (with respect to $Pot(L)$).*

Proof. Suppose not and choose $\emptyset \neq X \subseteq Pot(L)$ such that $|N(X)| \leq |X|$ minimizing $|X|$. If $|X| = 1$, then G_X can be colored from X contradicting Lemma 5.1.1. Hence $|X| \geq 2$. By minimality of $|X|$, for any $Y \subset X$, $|N(Y)| \geq |Y| + 1$. Hence, for any $x \in X$, we have $|N(X)| \geq |N(X - \{x\})| \geq |X - \{x\}| + 1 = |X|$. Thus, by Hall's

Theorem, we have a matching of X into $N(X)$, but $|N(X)| \leq |X|$ so this gives a coloring of G_X from X contradicting Lemma 5.1.1. $\square$

Our approach to coloring a graph (particularly a join) will often be to consider nonadjacent vertices u and v and show that their lists contain a common color. By the pigeonhole principle, this follows immediately when $|L(u)| + |L(v)| > |Pot(L)|$. We will use the following lemma frequently throughout the remainder of this paper.

Small Pot Lemma. *Let G be a graph and $f\colon V(G) \to \mathbb{N}$ with $f(v) < |G|$ for all $v \in V(G)$. If G is not f-choosable, then G has a minimal bad f-assignment L such that $|Pot(L)| < |G|$.*

Proof. Suppose not and let L be a minimal bad f-assignment. For each $v \in V(G)$ we have $|L(v)| = f(v) < |G| \leq |Pot(L)|$ and hence $L(v) \neq Pot(L)$. Thus by Lemma 5.1.3 we have the contradiction $|G| \geq |N(Pot(L))| > |Pot(L)|$. $\square$

5.2 Degree choosability

Definition 5.2.1. Let G be a graph and $r \in \mathbb{Z}$. Then G is *d_r-choosable* if G is f-choosable where $f(v) = d(v) - r$.

Note that a vertex critical graph with $\chi = \Delta + 1 - r$ contains no induced d_r-choosable subgraph. Since we are working to prove the Borodin-Kostochka conjecture, we will focus on the case $r = 1$ and primarily study d_1-choosable graphs. For $r = 0$, we have the following well known generalization of Brooks' Theorem due independently to Borodin [9] and Erdős, Rubin and Taylor [24].

5.2.1 Degree-choosable graphs

Definition 5.2.2. A *Gallai tree* is a graph all of whose blocks are complete graphs or odd cycles.

Classification of d_0-choosable graphs. *For any connected graph G, the following are equivalent.*

- *G is d_0-choosable.*
- *G is not a Gallai tree.*
- *G contains an induced even cycle with at most one chord.*

In [49], Kostochka, Stiebitz and Wirth give a short proof of the equivalence of (1) and (2) as well as extending the result to hypergraphs. In [35], Hladkỳ, Král and Schauz gave an algebraic proof (Rubin's block theorem below plus the Alon-Tarsi theorem [3]), it also works for paintability (online list-choosability), see [69].

We give some lemmas about d_0-assignments that will be useful in the later study of general d_k-assignments. Combined with the following structural result, these lemmas give a quick proof of the classification of d_0-choosable graphs. The following lemma from [24] is due to Rubin. For other proofs, see [23] and [35].

Lemma 5.2.1. *Any 2-connected graph is complete, an odd cycle or contains an induced even cycle with at most one chord.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Since G is 2-connected and not complete, it contains an induced cycle C of length at least four. Then C is an induced odd cycle and thus $G-C$ is not empty. Since G is 2-connected, we may choose a shortest C-path in G with distinct ends in C—call it R. Since $G[V(C)\cup V(R)]$ is 2-connected, by minimality of $|G|$, $V(G)=V(C)\cup V(R)$.

First suppose R has length at least 3. Then since R is shortest, $G=C\cup R$ and thus one of the small cycles in $C\cup R$ is an even induced cycle or the large cycle is an even induced cycle with at most one chord, giving a contradiction.

Thus R has length 2. Let z be the vertex on R in $G - C$. If z has only two neighbors in C, then we get a contradiction as in the previous paragraph. Thus z has at least three neighbors $a, b, c \in V(C)$. Now $|C| \geq 4$ since G is not complete. Thus, without loss of generality, the vertices between a and b on C in cyclic order are $w_1, \ldots, w_k$ with $k \geq 1$. But $G - \{w_1, \ldots, w_k\}$ is 2-connected, not complete, and not an odd cycle. Hence, by minimality of $|G|$, $G - \{w_1, \ldots, w_k\}$ contains an induced even cycle with at most one chord. This final contradiction completes the proof. □

The following lemma was used in [49].

Lemma 5.2.2. *A connected graph is d_0-choosable iff it contains a d_0-choosable induced subgraph.*

Proof. The forward direction is plain. For the reverse, let $H \trianglelefteq G$ be d_0-choosable. Since G is connected, we can order $V(G)$ such that each vertex in $V(G - H)$ has a neighbor after it and $V(H)$ comes last. Coloring $V(G - H)$ greedily from the lists in this order leaves a d_0-assignment on H which we can complete by assumption. □

Lemma 5.2.3. *Let L be a bad d_0-assignment on a connected graph G and $x \in V(G)$ a noncutvertex. Then $L(x) \subseteq L(y)$ for each $y \in N(x)$.*

Proof. Suppose otherwise that we have $c \in L(x) - L(y)$ for some $y \in N(x)$. Coloring x with c leaves at worst a d_0-assignment L' on the connected $H := G - x$ where $|L'(y)| > d_H(y)$. But then we can complete the coloring, a contradiction. □

Lemma 5.2.4. *Any even subdivision of a bridgeless d_0-choosable graph is d_0-choosable.*

Proof. Since subdividing an edge cannot create a bridge, it suffices to show that subdividing an edge with two vertices preserves d_0-choosability. Let G be a bridgeless d_0-choosable graph. Suppose there exists $xy \in E(G)$ such that subdividing xy with vertices w and z creates a graph H which is not d_0-choosable. Let L be a bad d_0-assignment on H. Since G is bridgeless, w and z are not cutvertices of H. By Lemma 5.2.3, $L(w) = L(z)$. But L restricted to G is a d_0-assignment, so we have a coloring π of $H - \{w, z\}$ from L such that $\pi(x) \neq \pi(y)$. Now $L(w) - \{\pi(x)\} \neq L(z) - \{\pi(y)\}$ so we can complete the coloring to all of H, a contradiction. □

Using the Small Pot Lemma it is easy to prove that C_4 and K_4^- are d_0-choosable which combined with Lemma 5.2.4 shows that every even cycle with at most one chord is d_0-choosable. It turns out that the conclusion of the Small Pot Lemma holds for general bad d_0-assignments, not just minimal ones. We will use the following lemma often in proofs when we end up with a bad d_0-assignment that may not be minimal.

Lemma 5.2.5. *If L a bad d_0-assignment on a connected graph G, $|Pot(L)| < |G|$.*

Proof. Suppose that the lemma is false and choose a connected graph G together with a bad d_0-assignment L where $|Pot(L)| \geq |G|$ minimizing $|G|$. Plainly, $|G| \geq 2$. Let $x \in G$ be a noncutvertex (any end block has at least one). By Lemma 5.2.3, $L(x) \subseteq L(y)$ for each $y \in N(x)$. Thus coloring x decreases the pot by at most one, giving a smaller counterexample. This contradiction completes the proof. □

Proof of the classification of d_0-choosable graphs. It is easy to construct a bad d_0-assignment on a Gallai tree—hence (1) implies (2). Now if a graph is not a Gallai tree, then some block is neither complete nor an odd cycle. But then, by Lemma 5.2.1, that block contains an induced even cycle with at most one chord. Hence (2) implies (3).

Now we prove that C_4 and K_4^- are d_0-choosable. If not, then we have a bad d_0-assignment L on C_4 or K_4^-. By Lemma 5.2.5, $|Pot(L)| \leq 3$. Hence some nonadjacent pair can be colored the same leaving a d_{-1}-assignment on the components which can be easily completed.

Thus, by Lemma 5.2.4, any even cycle with at most one chord is d_0-choosable. Combining this with Lemma 5.2.2 proves that (3) implies (1). $\square$

5.2.2 Basic properties

We also need a few basic lemmas about how d_r-choosability behaves with respect to induced subgraphs.

Lemma 5.2.6. *Fix $r \geq 0$. Let G be a graph and $H \trianglelefteq G$ a d_r-choosable subgaph. If L is a d_r-assignment on G and $G - H$ is properly colorable from L, then G is properly colorable from L.*

Proof. Color $G - H$ from L. Let L' be the resulting list assignment on H. Since each $v \in V(H)$ must be adjacent to as many vertices as colors in $G - H$ we see that L' is again a d_r-assignment. The lemma follows. $\square$

Lemma 5.2.7. *Fix $r \geq 0$. Let G be a graph and $H \trianglelefteq G$ a d_r-choosable subgaph. If there exists an ordering $v_1, \ldots, v_t$ of the vertices of $G - H$ such that v_i has degree at least $r + 1$ in $G[V(H) \cup \bigcup_{1 \leq j \leq i-1} v_j]$ for each i, then G is d_r-choosable.*

Proof. Let L be a d_r-assignment on G. Go through $G - H$ in order $v_t, \ldots, v_1$ coloring v_i with the smallest available color in $L(v_i)$. Since when we go to color v_i, it has at least $r+1$ uncolored neighbors we succeed in coloring $G - H$. Now the lemma follows from Lemma 5.2.6. $\square$

We will also use the following immediate consequence of the pigeonhole principle.

Lemma 5.2.8. *If $S_1, \ldots, S_m$ are nonempty subsets of a finite set T and $\sum_{i \geq 1} |S_i| > (m-1)|T|$, then $\bigcap_{i \geq 1} S_i \neq \emptyset$.*

5.3 Handling joins

The main result of this section is Lemma 5.3.8, which plays a key role in our classification of bad graphs $A * B$. Specifically, Lemma 5.3.8 is essential to the proof of Lemma 5.3.18, which considers the case when $|A| \geq 4$ and B is arbitrary.

Lemma 5.3.1. *Fix $r \geq 0$. Let A be a graph with $|A| \geq r+1$ and B a nonempty graph. If $A * B$ is d_r-choosable, then $A * C$ is d_r-choosable for any graph C with $B \trianglelefteq C$.*

Proof. Assume $A * B$ is d_r-choosable and let C be a graph with $B \trianglelefteq C$. Put $H = C - B$. For each $v \in V(H)$, $|L(v)| \geq d(v) - r \geq d_H(v) + r + 1 - r = d_H(v) + 1$. Thus we may color H from its lists. By Lemma 5.2.6, we can complete the coloring to all of $A * C$. $\square$

Lemma 5.3.2. *Fix $r \geq 0$. Let A be a graph with $|A| \geq r$ and B a nonempty graph. If $A * B$ is d_r-choosable, then $A * C$ is d_r-choosable for any connected graph C with $B \trianglelefteq C$.*

Proof. Assume $A * B$ is d_r-choosable and let C be a connected graph with $B \trianglelefteq C$. Put $H = C - B$. For each $v \in H$, $|L(v)| \geq d(v) - r \geq d_H(v) + r - r = d_H(v)$. Since C is connected, each component of H has a vertex v that hits a vertex in B and hence has $|L(v)| \geq d_H(v) + 1$. Thus we may color H from its lists. By Lemma 5.2.6, we can complete the coloring to all of $A * C$. $\square$

Lemma 5.3.3. *Fix $r \geq 0$. Let G be a d_{r-1} choosable graph with at least $2r+2$ vertices. Then $E_2 * G$ is d_r-choosable.*

Proof. Let x, y be the vertices in the E_2. Suppose $E_2 * G$ is not d_r-choosable. Then by the Small Pot Lemma, we have a d_r-assignment L with $|Pot(L)| < 2 + |G|$. Now

$|L(x)|+|L(y)| \geq d(x)+d(y)-2r \geq 2\,|G|-2r \geq 2+|G| > |Pot(L)|$, since $|G| \geq 2r+2$. Thus we can use a single common color on x and y, leaving a d_{r-1}-assignment on G. We may now complete the coloring, giving a contradiction. □

Since every graph is d_{-1}-choosable we get the following immediately.

Corollary 5.3.4. *For $r \geq 0$, both E_2^{r+2} and $E_2^{r+1} * K_2$ are d_r-choosable.*

Note that this is equivalent to the following fundamental result of Erdős, Rubin and Taylor [24].

Lemma 5.3.5. *For all $n \geq 1$, E_2^n is n-choosable.*

Lemma 5.3.6. *Fix $r \geq 0$. Let A be a graph with $|A| \geq 3r+2$ and B an arbitrary graph. If $A * B$ is not d_r-choosable, then $\omega(B) \geq |B| - 2r$.*

Proof. Suppose $G := A * B$ is not d_r-choosable and let L be a minimal bad d_r-assignment. Then, by the Small Pot Lemma, $|Pot(L)| \leq |G|-1$. Let $g\colon S \to Pot_S(L)$ be a partial coloring of B from L maximizing $|S| - |im(g)|$ and then minimizing $|S|$. Color S using g and let L' be the resulting list assignment.

Put $H := G - S$ and $C := B - S$. First suppose that $|S| - |im(g)| \geq r+1$. For each $v \in C$ we have $|L'(v)| \geq d_C(v) - r + 3r + 2 > d_C(v)$, so we can complete g to C. This leaves each $v \in V(A)$ with a list of size at least $d_A(v) - r + |S| - |im(g)| > d_A(v)$. Hence, we can complete the coloring to all of G. Thus L is not bad after all, giving a contradiction.

So instead we assume that $|S| - |im(g)| \leq r$. By the minimality condition on $|S|$ we see that g has no singleton color classes. In particular, $|S| \geq 2\,|im(g)|$. By combining this inequality with $|S| - |im(g)| \leq r$, we get $|S| \leq 2r$. Since $|C| = |B| - |S| \geq |B| - 2r$, the conclusion will follow if we can show that C is complete.

By definition, $|Pot(L')| = |Pot(L)| - |im(g)|$. By the maximality condition on g, every pair of nonadjacent vertices in C must have disjoint lists under L' (otherwise we could use a common color on nonadjacent vertices in C and increase $|S| - |im(g)|$). Let I be a maximal independent set in C. To reach a contradiction, we assume that $|I| \geq 2$. Then for all the elements of I to have disjoint lists, we must have

$$
\begin{aligned}
\sum_{v \in I} |L'(v)| &\leq |Pot(L')| \\
\sum_{v \in I} (d_H(v) - r) &\leq |Pot(L')| \\
\sum_{v \in I} (|A| + d_C(v) - r) &\leq |Pot(L')| \\
(|A| - r)\,|I| + \sum_{v \in I} d_C(v) &\leq |Pot(L')| \\
(|A| - r)\,|I| + |C| - |I| &\leq |Pot(L')| \\
(|A| - r - 1)\,|I| + |B| - |S| &\leq |A| + |B| - 1 - |im(g)| \\
(|A| - r - 1)\,|I| &\leq |A| - 1 + |S| - |im(g)| \\
2(|A| - r - 1) &\leq |A| - 1 + |S| - |im(g)| \\
|A| - 2r - 1 &\leq |S| - |im(g)| \\
r + 1 &\leq |S| - |im(g)|\,.
\end{aligned}
$$

This final inequality contradicts our assumption that $|S| - |im(g)| \leq r$. Hence $|I| \leq 1$; that is, C is complete. □

Lemma 5.3.7. *Fix $r \geq 1$. Let A be a connected graph and B an arbitrary graph such that $A * B$ is not d_r-choosable. Let L be a minimal bad d_r-assignment on $A * B$. If B is colorable from L using at most $|B| - r$ colors, then $|Pot(L)| \leq |A| + |B| - 2$.*

Proof. To get a contradiction suppose that $|Pot(L)| \geq |A| + |B| - 1$ and that B is colorable from L using at most $|B| - r$ colors. If $|Pot_A(L)| \geq |Pot(L)| + 1 - r$, then coloring B with at most $|B| - r$ colors leaves at worst a d_0-assignment L' on A with $|Pot(L')| \geq |A|$. Hence the coloring can be completed to A by Lemma 5.2.5, a contradiction.

Thus we may assume that $|Pot_A(L)| \leq |Pot(L)| - r$. Put $S := Pot(L) - Pot_A(L)$. Let π be a coloring of B from L using at most $|B| - r$ colors, say π uses colors C. Then $|C| = |B| - r$ and $S \cap C = \emptyset$ for otherwise coloring B leaves at worst a d_{-1}-assignment on A. Also, $\pi^{-1}(c) \not\subseteq V(G_S)$ for any $c \in C$ since otherwise we could recolor $\pi^{-1}(c)$ with colors from S to get at worst a d_{-1}-assignment on A. In particular, $|G_S| \leq \sum_{c \in C} (|\pi^{-1}(c)| - 1) = |B| - |C| = r \leq |S|$. But this inequality contradicts Lemma 5.1.3. □

We now use Lemma 5.3.7 to strengthen Lemma 5.3.6.

Lemma 5.3.8. *Fix $r \geq 1$. Let A be a connected graph with $|A| \geq 3r + 1$ and B an arbitrary graph. If $A * B$ is not d_r-choosable, then $\omega(B) \geq |B| - 2r$.*

Proof. Suppose $G := A * B$ is not d_r-choosable and let L be a minimal bad d_r-assignment. Then, by the Small Pot Lemma, $|Pot(L)| \leq |G| - 1$. Let $g\colon S \to Pot_S(L)$ be a partial coloring of B from L maximizing $|S| - |im(g)|$ and then minimizing $|S|$. Color S using g and let L' be the resulting list assignment.

Put $C := B - S$. Running through the argument in Lemma 5.3.6 with $3r + 1$ in place of $3r + 2$ shows that we must have $|S| - |im(g)| = r$. But then completing g to C gives a coloring of B from L using at most $|B| - r$ colors. Thus, by Lemma 5.3.7, $|Pot(L)| \leq |G| - 2$. Now running through the argument in Lemma 5.3.6 again completes the proof. □

5.3.1 The $r = 1$ case

Some preliminary tools

The Small Pot Lemma says that if $A * B$ is not d_1-choosable, then $A * B$ has a bad d_1-assignment L such that $|Pot(L)| \leq |A| + |B| - 1$. In this section, we study conditions under which $|Pot(L)| \leq |A| + |B| - 2$. We also prove a key lemma for coloring graphs of the form $K_1 * B$. In the following section, our results here help us to find nonadjacent vertices with a common color.

Lemma 5.3.9. *Let A be a graph with $|A| \geq 2$, B an arbitrary graph and L a d_1-assignment on $A * B$. If B has an independent set I such that $(|A|-1)\,|I|+|E_B(I)| > |Pot(L)|$, then B can be colored from L using at most $|B|-1$ colors.*

Proof. Suppose that B has an independent set I such that $(|A| - 1)\,|I| + |E(I)| > |Pot(L)|$. Now

$$\sum_{v \in I} |L(v)| = \sum_{v \in I} (d(v)-1) = (|A|-1)|I| + \sum_{v \in I} d_B(v) = (|A|-1)|I| + |E_B(I)| > |Pot(L)|.$$

Hence we have distinct $x, y \in I$ with a common color c in their lists. So we color x and y with c. Since $|A| \geq 2$, this leaves at worst a d_{-1}-assignment on the rest of B. Completing the coloring to the rest of B gives the desired coloring of B from L using at most $|B| - 1$ colors. $\square$

Lemma 5.3.10. *Let G be a graph and I a maximal independent set in G. Then $|E(I)| \geq |G| - |I|$. If I is maximum and $|E(I)| = |G| - |I|$, then G is the disjoint union of $|I|$ complete graphs.*

Proof. Each vertex in $G - I$ is adjacent to at least one vertex in I. Hence $|E(I)| \geq |G| - |I|$. Now assume I is maximum and $|E(I)| = |G| - |I|$. Then $N(x) \cap N(y) = \emptyset$

for every distinct pair $x, y \in I$. Also, $N(x)$ must be a clique for each $x \in I$, since otherwise we could swap x out for a pair of nonadjacent neighbors and get a larger independent set. Since we can swap x with any of its neighbors to get another maximum independent set, we see that G has components $\{G[\{v\} \cup N(v)] \mid v \in I\}$. □

Lemma 5.3.11. *Let A be a connected graph with $|A| \geq 4$ and B an incomplete graph. If $A * B$ is not d_1-choosable, then $A * B$ has a minimal bad d_1-assignment L such that $|Pot(L)| \leq |A| + |B| - 2$.*

Proof. Suppose $A * B$ is not d_1-choosable and let L be a minimal bad d_1-assignment on $A * B$. Then, by the Small Pot Lemma, $|Pot(L)| \leq |A| + |B| - 1$. Let I be a maximum independent set in B. Since B is incomplete, $|I| = \alpha(B) \geq 2$. By Lemma 5.3.10, $|E_B(I)| \geq |B| - |I| = |B| - \alpha(B)$. As $|A| \geq 4$ we have $(|A| - 1)|I| + |E_B(I)| \geq (|A| - 1)\alpha(B) + |B| - \alpha(B) \geq (|A| - 2)\alpha(B) + |B| \geq 2|A| - 4 + |B| > |A| + |B| - 1 \geq |Pot(L)|$. Hence by Lemma 5.3.9, B can be colored from L using at most $|B| - 1$ colors. But then we are done by Lemma 5.3.7. □

Lemma 5.3.12. *Let A be a connected graph with $|A| = 3$ and B a graph that is not the disjoint union of at most two complete subgraphs. If $A * B$ is not d_1-choosable, then $A * B$ has a minimal bad d_1-assignment L such that $|Pot(L)| \leq |B| + 1$.*

Proof. Suppose $A * B$ is not d_1-choosable and let L be a minimal bad d_1-assignment on $A * B$. Then, by the Small Pot Lemma, $|Pot(L)| \leq |B| + 2$.

Let I be a maximum independent set in B. Since B is not the disjoint union of at most two complete subgraphs, Lemma 5.3.10 implies that either $|E(I)| > |B| - |I|$ or $|I| \geq 3$. In the first case, $2|I| + |E(I)| > 2|I| + |B| - |I| \geq 2 + |B| \geq |Pot(L)|$. In the second case, $2|I| + |E(I)| \geq 2|I| + |B| - |I| \geq 3 + |B| > |Pot(L)|$.

Thus by Lemma 5.3.9, B can be colored from L using at most $|B| - 1$ colors. But then we are done by Lemma 5.3.7. □

Lemma 5.3.13. *Let B be a graph containing an induced claw, C_4, K_4^-, P_5, bull, or $2P_3$. If $K_2 * B$ is not d_1-choosable, then $K_2 * B$ has a minimal bad d_1-assignment L such that $|Pot(L)| \leq |B|$.*

Proof. Suppose $K_2 * B$ is not d_1-choosable and let L be a minimal bad d_1-assignment on $K_2 * B$. Then, by the Small Pot Lemma, $|Pot(L)| \leq |B| + 1$.

Let H be an induced claw, C_4, K_4^-, P_5, bull or $2P_3$ in B and M a maximum independent set in H. Expand M to a maximal independent set I in B. We can easily verify that in each case $|E_H(M)| \geq |H| - |M| + 2$, which implies that $|E_B(I)| \geq |B| - |I| + 2$. Hence we have $(|K_2| - 1)\,|I| + |E_B(I)| \geq (|K_2| - 2)\,|I| + |B| + 2 = |B| + 2 > |Pot(L)|$. Now by Lemma 5.3.9, B can be colored from L using at most $|B| - 1$ colors. But then we are done by Lemma 5.3.7. □

In the case that $A = K_1$, we might not be able to finish an arbitrary precoloring of B from L to all of B as we did above. However, if there is a precoloring that has our desired properties, then there is a coloring of B from the lists maintaining these properties. The following lemma makes this precise.

Lemma 5.3.14. *Let A and B be graphs such that $G := A * B$ is not d_1-choosable. If either $|A| \geq 2$ or B is d_0-choosable and L is a bad d_1-assignment on G, then*

1. *for any independent set $I \subseteq V(B)$ with $|I| = 3$, we have $\bigcap_{v \in I} L(v) = \emptyset$; and*
2. *for disjoint nonadjacent pairs $\{x_1, y_1\}$ and $\{x_2, y_2\}$ at least one of the following holds*

    a) $L(x_1) \cap L(y_1) = \emptyset$;

b) $L(x_2) \cap L(y_2) = \emptyset$*;*

c) $|L(x_1) \cap L(y_1)| = 1$ *and* $L(x_1) \cap L(y_1) = L(x_2) \cap L(y_2)$*.*

Proof. First, suppose $|A| \geq 2$. If (1) fails for I, then color all vertices in I the same, complete the coloring to the rest of B and then to A. If (2) fails for $\{x_1, y_1\}$ and $\{x_2, y_2\}$, color x_1, y_1 with $c_1 \in L(x_1) \cap L(y_1)$ and x_2, y_2 with $c_2 \in L(x_2) \cap L(y_2) - \{c_1\}$, complete the coloring to the rest of B and then to A. The more difficult case is when $|A| = 1$, we handle it as follows.

For both (1) and (2) we prove the contrapositive.

(1) Suppose that B has an independent set I of size 3 such that there exists a color c that appears in the list of each vertex in I; let $I = \{v_1, v_2, v_3\}$. Since B is d_0-choosable, B has an L-coloring. We will modify this coloring to get an L-coloring that uses c on at least three vertices.

For each v_i in I, if v_i does not have a neighbor with color c, we recolor v_i with c. If c now appears three or more times in our current coloring, then we are done. Assume that c appears on either a single vertex w_1 or on two vertices w_1 and w_2.

If both w_1 and w_2 have two neighbors in I, then we uncolor w_1 and w_2 and use color c on all vertices of I. Otherwise, there exists a single vertex, say w_1, with at least two neighbors in I for which w_1 is their only neighbor with color c_1. Uncolor w_1 and now use color c on all of its neighbors in I_1 that no longer have a neighbor with color c_1. Since each uncolored vertex has at least two neighbors with color c, we can extend the coloring to all of B. Now since color c is used 3 or more times on B, at most $|G| - 2$ colors are used on G, so we can extend the coloring to A.

(2) Suppose that B has two disjoint independent sets I_1 and I_2 each of size 2 and there exist distinct colors c_1 and c_2 such that (for each $i \in \{1, 2\}$) color c_i appears

in the lists of both vertices of I_i. Since B is d_0-choosable, B has an L-coloring. We will show that B has an L-coloring in which colors c_1 and c_2 each appear twice (or one appears at least three times). We will modify our coloring using recoloring arguments similar to that above, although we may need to recolor repeatedly. (If at any point our coloring of B uses a single color three or more times, then we can stop, since we will be able to extend this coloring to A.)

If c_1 does not appear in our coloring, then we recolor some vertex of I_1 with c_1. Suppose that color c_1 appears only once in our coloring, say on vertex u. Either we can recolor some vertex in I_1 with c_1 or else both vertices in I_1 are adjacent to u. In this case, we uncolor u and use c_1 on both vertices of I_1. Now we have some color available for u. Thus, we may assume that our coloring uses c_1 on exactly two vertices. If neither of these vertices with c_1 are in I_2, then we can use the same recoloring trick for color c_2. Neither vertex with c_1 will get recolored, so afterwards both colors c_1 and c_2 will appear on two vertices (and we'll be able to extend the coloring to A).

Suppose instead that both vertices with color c_1 are in I_2. If neither vertex in I_2 is adjacent to a vertex where color c_2 is used, then we can recolor both of them with c_2. Next we can again apply the recoloring trick for color c_1. Since the vertices in I_2 with color c_2 will not get recolored, this will yield the desired coloring that uses each of c_1 and c_2 twice. So suppose that c_1 is used on both vertices in I_2 and c_2 is used on a vertex adjacent to at least one vertex in I_2. Since we may assume that c_2 appears on only one vertex, when we use the recoloring trick for c_2, we will color at least one vertex of I_2 with c_2. Thus, we may assume (up to symmetry of I_1 and I_2) that color c_1 appears on two vertices and that exactly one of them is in I_2; we may also assume that color c_2 appears on exactly one vertex.

We will show that after applying the recoloring trick at most three times we will get a coloring of B that uses c_1 on two vertices and uses c_2 on two vertices. We call a vertex $v \in I_i$ *miscolored* if it is colored with color c_{3-i}. We will see that each time we apply the recoloring trick, either we increase the total number of vertices colored with c_1 and c_2 or else we decrease the number of miscolored vertices. Since we begin with at most two miscolored vertices, after applying the recoloring trick at most three times, our coloring will use colors c_1 and c_2 each twice (and we will be done).

Assume that c_1 appears on two vertices and exactly one of them is miscolored; assume that c_2 appears on exactly one vertex, which may or may not be miscolored. When we apply the recoloring trick for c_2, we increase the number of vertices using c_2. Thus, we are done unless we decrease the number of vertices using c_1. Since we only remove color c_1 from vertices in I_2, we conclude that we've reduced the number of miscolored vertices. We now apply the recoloring trick for c_1. Again, we are done unless we've recolored a miscolored vertex. So assume that we did. Since we have no remaining miscolored vertices, when we now apply the recoloring trick for c_2, we get a coloring that uses each of c_1 and c_2 twice. Thus, we can extend the coloring of B to A. □

A simple variation of the (1) case in the above together with Lemma 5.3.7 gives the following pot-shrinking lemma for $K_1 * H$.

Lemma 5.3.15. *Let H be a d_0-choosable graph such that $G := K_1 * H$ is not d_1-choosable and L a minimal bad d_1-assignment on G. If some nonadjacent pair in H have intersecting lists, then $|Pot(L)| \leq |H| - 1$.*

With the same proof, we have the following.

Lemma 5.3.16. *Let H be a d_0-choosable graph such that $G := K_1 * H$ is not f-choosable where $f(v) \geq d(v)$ for the v in the K_1 and $f(v) \geq d(x) - 1$ for $x \in V(H)$. If L is a minimal bad f-assignment on G, then all nonadjacent pairs in H have disjoint lists.*

Lemma 5.3.17. *Let A be a connected graph, let $G = A * B$, and suppose that either B is d_0-choosable or $|A| \geq 2$. (1) Let L be a d_1-assignment to G. If B contains disjoint independent sets I_1 and I_2 such that $\sum_{v \in I_1}(d(v) - 1) \geq |Pot(L)| + 1$ and $\sum_{v \in I_2}(d(v) - 1) \geq |Pot(L)| + 2$, then $A * B$ has an L-coloring. (2) In particular, if B contains disjoint independent sets I_1 and I_2 such that $\sum_{v \in I_1}(d(v) - 1) \geq |G| - 1$ and $\sum_{v \in I_2}(d(v) - 1) \geq |G|$, then $A * B$ is d_1-choosable.*

Proof. Let L be a bad d_1-list assignment. We prove (1) and (2) simultaneously. By the Small Pot Lemma, $|Pot(L)| < |G|$. Thus, since $\sum_{v \in I_2}(d(v)-1) > |Pot(L)|$, we see that some color α appears on nonadjacent vertices in I_2. Either B is d_0-choosable or $|A| \geq 2$, so using either Lemma 5.3.15 or Lemma 5.3.7, we get that $|Pot(L)| = |G|-2$, so $|G| - 1 \geq |Pot(L)| + 1$.

Since $\sum_{v \in I_1}(d(v) - 1) \geq |Pot(L)| + 1$, we see that two vertices of I_1 have a common color β. If β appears 3 times in I_2, then we are done by Lemma 5.3.14. Otherwise, we use β on the vertices of I_1 where it appears. After deleting β from the lists of I_2, we can find a common color on two vertices of I_2. Again we are done, by Lemma 5.3.14. □

A classification

In this section we classify the d_1-choosable graphs of the form $A * B$ where $|A| \geq 2$ and $|B| \geq 2$. When $|A| \geq 4$ and A is connected (or similarly for B), the characterizations follows from Lemma 5.3.20 and Corollary 5.3.25. The remainder of the section considers the case when each of A and B is small and/or disconnected.

Definition 5.3.1. A graph G is *almost complete* if $\omega(G) \geq |G| - 1$.

Lemma 5.3.18. *Let A be a connected graph with $|A| \geq 4$ and B an arbitrary graph. If $A * B$ is not d_1-choosable, then B is $E_3 * K_{|B|-3}$ or almost complete.*

Proof. Suppose $A * B$ is not d_1-choosable and B is neither $E_3 * K_{|B|-3}$ nor almost complete. Then, by Lemma 5.3.8, we have $\omega(B) = |B| - 2$.

Let L be a minimal bad d_1-assignment on $A * B$. Then, by Lemma 5.3.11, $|Pot(L)| \leq |A| + |B| - 2$. Choose $x_1, x_2 \in V(B)$ so that $B - \{x_1, x_2\}$ is complete. Since B is not $E_3 * K_{|B|-3}$ we have $x_1', x_2' \in V(B)$ such that $\{x_1, x_1'\}$ and $\{x_2, x_2'\}$ are disjoint pairs of nonadjacent vertices. We have $|L(x_i)| + |L(x_i')| \geq d(x_i) + d(x_i') - 2 \geq 2\,|A| + d_B(x_i) + |B| - 5$.

First suppose $d_B(x_i) > 0$ for some $i \in \{1, 2\}$. Without loss of generality, suppose $i = 1$. Then $|L(x_1)| + |L(x_1')| \geq |Pot(L)| + 2$ and $|L(x_2)| + |L(x_2')| \geq |Pot(L)| + 1$. Hence we have different colors c_1, c_2 such that $c_1 \in L(x_1) \cap L(x_1')$ and $c_2 \in L(x_2) \cap L(x_2')$. Coloring the pairs with these colors leaves a list assignment that is easily completable to all of $A * B$.

Hence we must have $d_B(x_1) = d_B(x_2) = 0$. But then $|L(x_i)| + |L(x_i')| \geq |Pot(L)| + 1$ for each $i \in \{1, 2\}$ and thus both $L(x_1) \cap L(x_1')$ and $L(x_2) \cap L(x_2')$ are nonempty. If they have different colors in common, we can finish as above. If they have the same color c in common, then coloring x_1, x_2 and x_1' with c leaves a list assignment that is easily completable to all of $A * B$. $\square$

Lemma 5.3.19. *Let A be a connected graph with $|A| \geq 6$ and B an arbitrary graph. If $A * B$ is not d_1-choosable, then B is almost complete.*

Proof. Suppose $A * B$ is not d_1-choosable. By Lemma 5.3.18, B is $E_3 * K_{|B|-3}$ or almost complete. Suppose that B is $E_3 * K_{|B|-3}$ and let x_1, x_2, x_3 be the vertices in the E_3.

Let L be a minimal bad d_1-assignment on $A * B$. Then, by Lemma 5.3.11, $|Pot(L)| \leq |A| + |B| - 2$. We have $\sum_{i=1}^{3} |L(x_i)| \geq \sum_{i=1}^{3}(d(x_i) - 1) = 3(|A| + |B| - 4)$. Since $|B| \geq 3$ we have $|A| + |B| \geq 9$ and hence $3(|A| + |B| - 4) > 2(|A| + |B| - 2) \geq 2\,|Pot(L)|$. Thus, by Lemma 5.2.8, we have $c \in \bigcap_{i=1}^{3} L(x_i)$. Coloring x_1, x_2 and x_3 with c leaves a list assignment that is easily completable to the rest of $A * B$. This is a contradiction. Hence B is almost complete. $\square$

When A is incomplete we can do much better.

Lemma 5.3.20. *Let A be a connected incomplete graph with $|A| \geq 4$ and B an arbitrary graph. If $A * B$ is not d_1-choosable, then B is complete.*

Proof. By Lemma 5.3.1 it will suffice to show that $A * E_2$ is d_1-choosable. Suppose not and let L be a minimal bad d_1-assignment on $A * E_2$. Then, by Lemma 5.3.11, $|Pot(L)| \leq |A|$. Let x_1 and x_2 be the vertices in the E_2. Then $|L(x_1)| + |L(x_2)| \geq d(x_1) + d(x_2) - 2 = 2\,|A| - 2 \geq |Pot(L)| + 2$. Hence we have different $c_1, c_2 \in L(x_1) \cap L(x_2)$.

First, suppose there exists $y \in V(A)$ such that $\{c_1, c_2\} \not\subseteq L(y)$. Without loss of generality, assume $c_1 \notin L(y)$. Then coloring x_1 and x_2 with c_1 leaves a list assignment L' on A where $|L'(v)| \geq d_A(v)$ for all $v \in V(A)$ and $|L'(y)| > d_A(y)$. Hence the coloring can be completed, a contradiction.

Hence $\{c_1, c_2\} \subseteq L(v)$ for all $v \in V(A)$. If $\alpha(A) \geq 3$, then coloring a maximum independent set all with c_1 leaves an easily completable list assignment. Also, if A

contains two disjoint pairs of nonadjacent vertices, by coloring one with c_1 and one with c_2 we get another easily completable list assignment. Hence A is almost complete.

Let $z \in V(A)$ such that $A - z$ is complete. Since A is incomplete, we have $w \in V(A - z)$ nonadjacent to z. Also, as A is connected we have $w' \in V(A - z)$ adjacent to z. Color x_1 and x_2 with c_1 and w and z with c_2 to get a list assignment L' on $D := A - \{w, z\}$ where $|L'(v)| \geq d_D(v)$ for all $v \in V(D)$ and $|L'(w')| > d_D(w')$. Hence the coloring can be completed, a contradiction. □

Lemma 5.3.21. *$E_2 * 2P_3$ is d_1-choosable.*

Proof. Suppose otherwise. Let the E_2 have vertices x_1 and x_2 and the two P_3's have vertices y_1, y_2, y_3 and y_4, y_5, y_6. By the Small Pot Lemma, we have a minimal bad d_1-assignment on $E_2 * 2P_3$ with $|Pot(L)| \leq 7$. Since $|L(x_1)|+|L(x_2)| = 10 \geq |Pot(L)|+3$, we have three different colors $c_1, c_2, c_3 \in L(x_1) \cap L(x_2)$. Coloring both x_1 and x_2 with any c_i leaves at worst a d_0-assignment on the $2P_3$. If $c_i \notin L(y_1) \cap L(y_2) \cap L(y_3)$ and $c_i \notin L(y_4) \cap L(y_5) \cap L(y_6)$ for some i, then we can complete the coloring. Thus, without loss of generality, we have $\{c_1, c_2\} \subseteq L(y_1) \cap L(y_2) \cap L(y_3)$ and $c_3 \in L(y_4) \cap L(y_5) \cap L(y_6)$. Color y_1 and y_3 with c_1 and y_4 and y_6 with c_3. Then we can easily complete the coloring on the rest of the $2P_3$. We have used at most 4 colors on the $2P_3$ and hence we can complete the coloring. □

At this point we have enough information to completely classify the d_1-choosable graphs of the form $E_2 * B$.

Lemma 5.3.22. *$E_2 * B$ is not d_1-choosable iff B is the disjoint union of complete subgraphs and at most one P_3.*

Proof. Suppose we have B such that $E_2 * B$ is not d_1-choosable. By Lemma 5.3.21, B has at most one incomplete component. Suppose we have an incomplete component

C and let $y_1y_2y_3$ be an induced P_3 in C. If $C \neq P_3$, then $|C| \geq 4$ and Lemma 5.3.20 gives a contradiction. Hence $C = P_3$.

For the other direction, it is easy to see that for any B such that $E_2 * B$ is not d_1-choosable adding a disjoint complete subgraph to B does not make it d_1-choosable. To see that $E_2 * P_3$ is not d_1-choosable, let x_1, x_2 denote the vertices of the E_2 and let y_1, y_2, y_3 denote in order the vertices of the P_3. Let $L(x_1) = \{a, b\}$, $L(x_2) = \{c, d\}$, $L(y_1) = \{a, c\}$, $L(y_2) = \{a, b, c\}$, and $L(y_3) = \{b, d\}$. It is easy to verify that the graph is not colorable from these lists. This proves the lemma. □

For $t \geq 4$, we know that if $K_t * B$ is not d_1-choosable then B is almost complete; or $t = 4$ and B is E_3 or a claw; or $t = 5$ and B is E_3. The following two lemmas show that this completely characterizes the d_1-choosable graphs of this form.

Lemma 5.3.23. *Almost complete graphs are not d_1-choosable.*

Proof. Let G be almost complete and $x \in V(G)$ such that $G-x$ is complete. Consider the d_1-assignment L given by $L(v) = [d(v) - 1]$ for each $v \in V(G)$. Now $G - x$ is a complete graph of size $|G| - 1$, but the union of the lists on $G - x$ is only $[|G| - 2]$, so by Hall's theorem, G has no coloring from these lists. □

Lemma 5.3.24. *$K_t * E_3$ is d_1-choosable iff $t \geq 6$.*

Proof. That if $t \geq 6$, then $K_t * E_3$ is d_1-choosable follows from Lemma 5.3.19. For the other direction it is enough to show that $K_5 * E_3$ is not d_1-choosable. Figure 5.1 shows a bad d_1-assignment on $K_5 * E_3$. □

Corollary 5.3.25. *For $t \geq 4$, $K_t * B$ is not d_1-choosable iff B is almost complete; or $t = 4$ and B is E_3 or a claw; or $t = 5$ and B is E_3.*

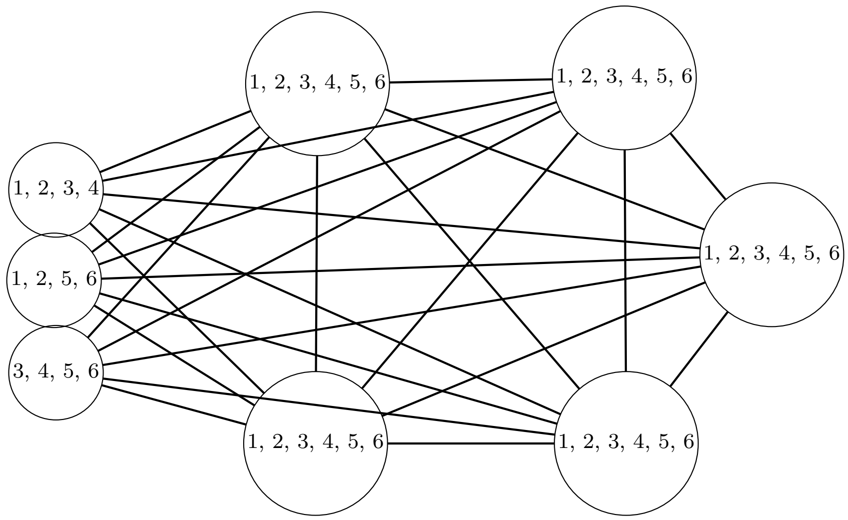


Figure 5.1: A bad d_1-assignment on $K_5 * E_3$.

Lemma 5.3.26. *$P_3 * B$ is not d_1-choosable iff B is E_2 or complete.*

Proof. Moving the center of P_3 to the other side of the join and applying Lemma 5.3.22 proves the lemma. □

Lemma 5.3.27. *$K_3 * P_4$ is d_1-choosable.*

Proof. Suppose otherwise. Denote the vertices of the P_4 as y_1, y_2, y_3, y_4, in order. Note that $|L(y_1)| + |L(y_3)| = 4 + 5 \geq |G| + 1$ and $|L(y_2)| + |L(y_4)| = 5 + 4 \geq |G| + 1$. Now we apply (2) of Lemma 5.3.17 with $I_1 = \{y_1, y_3\}$ and $I_2 = \{y_2, y_4\}$. □

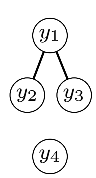


Figure 5.2: The antipaw.

Lemma 5.3.28. *$K_3 *$ antipaw is d_1-choosable.*

Proof. Suppose not. We use the labeling of the antipaw given in Figure 5.2. Since the antipaw is not a disjoint union of at most two complete graphs, Lemma 5.3.12 gives us a minimal bad d_1-assignment L on $K_3 *$antipaw with $|Pot(L)| \leq 5$. Note that $|L(y_1)| + |L(y_4)| \geq 6$ and $|L(y_2)| + |L(y_3)| \geq 6$. Hence, by Lemma 5.3.14, $|L(y_1) \cap L(y_4)| = 1$ and $L(y_1) \cap L(y_4) = L(y_2) \cap L(y_3)$. But then we have $c \in L(y_2) \cap L(y_3) \cap L(y_4)$ and after coloring y_2, y_3, and y_4 with c we can complete the coloring, getting a contradiction. □

Lemma 5.3.29. *$K_3 * B$ is not d_1-choosable iff B is almost complete, $K_t + K_{|B|-t}$, $K_1 + K_t + K_{|B|-t-1}$, $E_3 + K_{|B|-3}$, or $|B| \leq 5$ and $B = E_3 * K_{|B|-3}$.*

Proof. Let $K_3 * B$ be a graph that is not d_1-choosable and let B be none of the specified graphs. Lemma 5.3.12 gives us a minimal bad d_1-assignment L on $K_3 * B$ with $|Pot(L)| \leq |B| + 1$. Furthermore, the proof of Lemma 5.3.12 shows that we can color B with at most $|B|-1$ colors. In particular we have nonadjacent $x, y \in V(B)$ and $c \in L(x) \cap L(y)$. Coloring x and y with c leaves a list assignment L' on $D := B - \{x, y\}$. If $c \in L'(z)$ for some $z \in V(D)$, then $\{x, y, z\}$ is independent and we can color z with c and complete the coloring to get a contradiction. Hence $Pot(L') = Pot(L) - \{c\}$.

Suppose, for a contradiction, that D is not the disjoint union of at most two complete subgraphs. If $\alpha(D) \geq 3$, let J be a maximum independent set in D and set $\gamma := 0$. Otherwise D contains an induced P_3 abc and we let $J \subseteq V(D)$ be a maximal independent set containing $\{a, c\}$ and set $\gamma := 1$. Lemma 5.3.10 implies that $\sum_{v \in J} d_D(v) \geq |D| - |J| + \gamma$. Since L is bad, we must have

$$\sum_{v \in J} |L'(v)| \leq |Pot(L')|$$
$$\sum_{v \in J} |L'(v)| \leq |B|$$
$$2|J| + \sum_{v \in J} d_D(v) \leq |B|$$
$$2|J| + |D| - |J| + \gamma \leq |B|$$
$$|J| + |D| + \gamma \leq |B|$$
$$|J| + |B| - 2 + \gamma \leq |B|.$$

Hence $|J| \leq 2 - \gamma$, a contradiction. Therefore D is indeed the disjoint union of at most two complete subgraphs. (Additionally, if D is not complete then $v \in V(D)$ is not adjacent to both x and y since then we would get the same contradictory degree sum as in the case when $\gamma = 1$.) We now consider the case that D is a complete graph and the case that D is the disjoint union of two complete graphs.

First, suppose D is a complete graph. Plainly, $|D| \geq 2$. Put $X := N(x) \cap V(D)$ and $Y := N(y) \cap V(D)$. Suppose $X - Y \neq \emptyset$ and pick $z \in X - Y$. We have $|L(y)| + |L(z)| \geq d(y) + d(z) - 2 = d_B(y) + d_B(z) + 4 \geq 0 + |B| - 2 + 4 = |B| + 2 > |Pot(L)|$. By repeating the argument given above for $B - \{x, y\}$, we see that $B - \{y, z\}$ is also the disjoint union of at most two complete subgraphs. In particular, x is adjacent to all or none of $D - z$. If all, then B is almost complete, if none then B contains an induced P_4 or antipaw, and both possibilities give contradictions by Lemmas 5.3.27 and 5.3.28. Hence $X - Y = \emptyset$. Similarly, $Y - X = \emptyset$, so $X = Y$. Since B is not $E_2 + K_{|B|-2}$, $|X| > 0$. If $X = V(D)$, then B is almost complete. If $|V(D) - X| \geq 2$, then pick $w_1, w_2 \in V(D) - X$. Now by considering degrees, we see that $L(x) \cap L(w_1)$ and $L(y) \cap L(w_2)$ are both nonempty. Now we can color x, y, w_1, w_2 using only 2 colors, and then complete the coloring. Hence, we must have

$|V(D) - X| = 1$, so let $\{w\} = V(D) - X$. Now x and y are joined to $D - w$ and hence B is $E_3 * K_{|B|-3}$, a contradiction.

Thus D must instead be the disjoint union of two complete subgraphs D_1 and D_2. For each $i \in [2]$, put $X_i := N(x) \cap V(D_i)$ and $Y_i := N(y) \cap V(D_i)$. From our parenthetical remark above, we know that $X_i \cap Y_i = \emptyset$. Suppose we have $z_1 \in V(D_1)$ and $z_2 \in V(D_2)$ such that $L(z_1) \cap L(z_2) \neq \emptyset$. Then, by Lemma 5.3.14, $L(z_1) \cap L(z_2) = L(x) \cap L(y)$. Since no independent set of size three can have a color in common, the edges z_1x and z_2y or z_1y and z_2x must be present. Using the same argument as for $B - \{x, y\}$, we see that $B - \{z_1, z_2\}$ is the disjoint union of at most two complete subgraphs. So each of x and y is adjacent to all or none of each of $V(D_1 - z_1)$ and $V(D_2 - z_2)$. Thus, by symmetry, we may assume that $V(D_1 - z_1) \subseteq X_1$ and $V(D_2 - z_2) \subseteq Y_2$. If $|D_1| = |D_2| = 1$, then B is the disjoint union of two cliques, a contradiction. So, by symmetry, we may assume that $|D_1| \geq 2$. Pick $w \in V(D_1 - z_1)$. If x is not adjacent to z_1, then xwz_1 is an induced P_3 in B. Since $X_1 \cap Y_1 = \emptyset$, this P_3 together with y either induces a P_4 or an antipaw, contradicting Lemmas 5.3.27 and 5.3.28. Hence $X_1 = V(D_1)$. Similarly, if $|D_2| \geq 2$, then $Y_2 = V(D_2)$ and B is the disjoint union of two complete subgraphs, a contradiction. Hence $D_2 = \{z_2\}$. But z_2 must be adjacent to y, so B is again the disjoint union of two cliques, a contradiction.

Thus for every $z_1 \in V(D_1)$ and $z_2 \in V(D_2)$ we have $L(z_1) \cap L(z_2) = \emptyset$. Suppose there exist $z_1 \in V(D_1)$ and $z_2 \in V(D_2)$ such that z_1 and z_2 are each adjacent to at least one of x and y. Then $|L(z_1)| + |L(z_2)| \geq d(z_1) + d(z_2) - 2 \geq d_B(z_1) + d_B(z_2) + 4 \geq |B| - 4 + 2 + 4 = |B| + 2 > |Pot(L)|$. Hence $L(z_1) \cap L(z_2) \neq \emptyset$, a contradiction.

Thus, by symmetry, we may assume that there are no edges between D_1 and $\{x, y\}$. Since no vertex in D_2 is adjacent to both x and y, only one of x or y can have neighbors in D_2 lest B contain an induced P_4 contradicting Lemma 5.3.27.

Without loss of generality, we may assume that y has no neighbors in D_2. Pick $w \in D_1$ and $z \in V(D_2)$.

Suppose that $|D_1| \geq 2$, $|D_2| \geq 2$, and there exists $t \in D_2$ such that x and t are nonadjacent. Now choose $u, v \in V(D_1)$ and $w \in V(D_2) \setminus \{t\}$. Now $\{v, w, y\}$ is independent and $|L(v)| + |L(w)| + |L(y|) \geq d(v) + d(w) + d(y) - 3 \geq d_B(v) + d_B(w) + d_B(y) + 6 \geq |B| + 2 > |Pot(L)|$. Hence either $L(v) \cap L(y) \neq \emptyset$ or $L(w) \cap L(y) \neq \emptyset$. Similarly, either $L(u) \cap L(x) \neq \emptyset$ or $L(t) \cap L(x) \neq \emptyset$. Thus, we can color 4 vertices using only 2 colors, and we can complete the coloring. So now either $|D_1| = 1$, $|D_2| = 1$, or $D_2 \subset N(x)$.

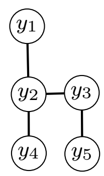
If $|D_2| = 1$, then either $B = K_1 + K_2 + K_{|B|-3}$ or else $B = E_3 + K_{|B|-3}$, both of which are forbidden. Similarly, if $|D_1| = 1$ and x is adjacent to all or none of D_2, then $B = K_1 + K_1 + K_{|B|-2}$ or $E_3 + K_{|B|-3}$. Finally, if x is adjacent to some, but not all of D_2, then B contains an antipaw. By Lemma 5.3.28, this is a contradiction.

It remains to show that $K_3 * B$ is not d_1-choosable for any of the specified B's. For B almost complete, this follows from Lemma 5.3.23 and for $E_3 * K_{|B|-3}$, from Lemma 5.3.25. For all the rest of the options we will give a bad list assignment with lists $[|B| + 1]$ on the K_3. Suppose $K_t + K_{|B|-t}$. On the K_t the lists $[t + 1]$ and on the $K_{|B|-t}$ the lists $[|B| + 1] \smallsetminus [t]$. Then any coloring of $K_3 * B$ from the lists must use three colors on the K_3 and hence at least one of the cliques loses at least two colors leaving it uncolorable. Now suppose $B = K_1 + K_t + K_{|B|-t-1}$. Use the list $\{1, |B| + 1\}$ on the K_1, the lists $[t + 1]$ on the K_t and the lists $[|B + 1|] \smallsetminus [t + 1]$ on the $K_{|B|-t-1}$. This list assignment is clearly bad on $K_3 * B$.

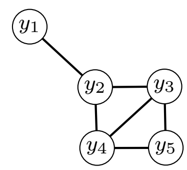
Finally suppose $B = E_3 + K_{|B|-3}$. Give the three K_1's the lists $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and the $K_{|B|-3}$ the list $[|B| + 1] \smallsetminus [3]$. Again, this is clearly a bad list assignment on $K_3 * B$. $\square$

Lemma 5.3.30. *$K_2 * P_5$ is d_1-choosable.*

Proof. Suppose otherwise. By Lemma 5.3.13, we have a minimal bad d_1-assignment L on $P_5 * K_2$ with $|Pot(L)| \leq 5$. Let y_1, y_2, y_3, y_4, y_5 denote the vertices of the P_5 in order. Now $|L(y_2)| + |L(y_4)| \geq 6 \geq |Pot(L)| + 1$ and $|L(y_1)| + |L(y_3)| + |L(y_5)| \geq 7 \geq |Pot(L)| + 2$. So $\{y_2, y_4\}$ and $\{y_1, y_3, y_5\}$ satisfy the hypotheses of Lemma 5.3.17, giving a contradiction. □



(a) The chair.



(b) The antichair.

Figure 5.3: Labelings of the chair and the antichair.

Lemma 5.3.31. *$K_2 *$ chair is d_1-choosable.*

Proof. Suppose otherwise. We use the labeling of the chair given in Figure 5.3a. Since the chair has an induced claw, Lemma 5.3.13 gives us a minimal bad d_1-assignment L on $K_2 *$ chair with $|Pot(L)| \leq 5$. Now $|L(y_2)| + |L(y_5)| \geq 6 \geq |Pot(L)| + 1$ and $|L(y_1)| + |L(y_3)| + |L(y_4)| \geq 7 \geq |Pot(L)| + 2$. Then $\{y_2, y_5\}$ and $\{y_1, y_3, y_4\}$ satisfy the hypotheses of Lemma 5.3.17, giving a contradiction. □

Lemma 5.3.32. *$K_2 *$ antichair is d_1-choosable.*

Proof. Suppose otherwise. We use the labeling of the antichair given in Figure 5.3b. Since the antichair has an induced K_4^-, Lemma 5.3.13 gives us a minimal bad d_1-assignment L on $K_2 *$ antichair with $|Pot(L)| \leq 5$. We have $|L(y_2)| + |L(y_5)| \geq 7$ and hence $|L(y_2) \cap L(y_5)| \geq 2$. But then, by Lemma 5.3.14, we have the contradiction $|L(y_1)| + |L(y_3)| \leq 5$. □

Lemma 5.3.33. *$K_2 * C_5$ is d_1-choosable.*

Proof. Suppose otherwise. By the Small Pot Lemma, we have a minimal bad d_1-assignment L on $C_5 * K_2$ with $|Pot(L)| \leq 6$. Let $y_0, y_1, y_2, y_3, y_4, y_0$ denote in order the vertices of the C_5. Then for $0 \leq i < j \leq 4$ with $i - j \not\equiv 1 (\mathrm{mod}\ 5)$ we have $|L(y_i)| + |L(y_j)| \geq d(y_i) + d(y_j) - 2 = 6$.

First suppose $|Pot(L)| \leq 5$. Then each nonadjacent pair has a color in common and by applying Lemma 5.3.14 multiple times we see that there must exist $c \in \bigcap_{0 \leq i \leq 4} L(y_i)$ and no nonadacent pair can have a color other than c in common. Put $S_i = L(y_i) - \{c\}$ and $T = Pot(L) - \{c\}$. Then we must have $S_0 = T - S_3$, $S_1 = T - S_3 = T - S_4$ and $S_2 = T - S_4$. Hence $S_0 = S_1 = S_2$ contradicting $S_0 \cap S_2 = \emptyset$.

Therefore we must have $|Pot(L)| = 6$. Thus for nonadjacent y_i and y_j, $L(y_i) = Pot(L) - L(y_j)$. We have $L(y_0) = Pot(L) - L(y_3)$, $L(y_1) = Pot(L) - L(y_3) = Pot(L) - L(y_4)$ and $L(y_2) = Pot(L) - L(y_4)$. Hence $L(y_0) = L(y_1) = L(y_2)$. Thus we may color y_0 and y_2 the same and complete this coloring to the rest of B contradicting Lemma 5.3.7. $\square$

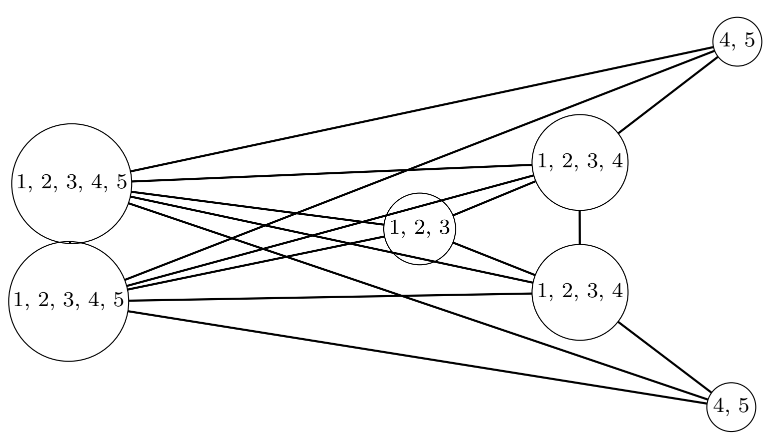


Figure 5.4: A bad d_1-assignment on bull $* K_2$.

Lemma 5.3.34. *$K_2 * 2P_3$ is d_1-choosable.*

Proof. Suppose otherwise. Let y_1, y_2, y_3 and y_4, y_5, y_6 denote in order the vertices of the two P_3's. Lemma 5.3.13 gives us a minimal bad d_1-assignment L on $K_2 * 2P_3$ with $|Pot(L)| \leq 6$.

Since $|L(y_1)| + |L(y_3)| + |L(y_4)| + |L(y_6)| = 8 \geq |Pot(L)| + 2$, either three of these vertices share a common color, or else two pairs of them share distinct common colors. Thus, if $L(y_2) \cap L(y_5) \neq \emptyset$, then we can color G by Lemma 5.3.14. Hence $L(y_2) \cap L(y_5) = \emptyset$.

By summing list sizes, we see that some pair among each of $\{y_1, y_3, y_5\}$ and $\{y_2, y_4, y_6\}$ must have a color in common. Since there are no edges between $\{y_1, y_3\}$ and $\{y_4, y_6\}$, if $L(y_1) \cap L(y_3) \neq \emptyset$ and $L(y_4) \cap L(y_6) \neq \emptyset$, then we get a contradiction. By symmetry, we may assume that the other two options are either $L(y_1) \cap L(y_3) \neq \emptyset$ and $L(y_2) \cap L(y_4) \neq \emptyset$ or else $L(y_1) \cap L(y_5) \neq \emptyset$ and $L(y_2) \cap L(y_4) \neq \emptyset$. In the former case, by Lemma 5.3.14,we must have $L(y_1) \cap L(y_3) \cap L(y_4) \neq \emptyset$, a contradiction. In the latter case, $L(y_1) \cap L(y_5) \neq L(y_2) \cap L(y_4)$ since $L(y_2) \cap L(y_5) = \emptyset$, contradicting Lemma 5.3.14. $\square$

(a) The anticlaw. (b) The antidiamond.

Figure 5.5: Labelings of the anticlaw and the antidiamond.

Note that if L is a bad d_1 assignment on $E_3 * B$ where the E_3 is $\{x_1, x_2, x_3\}$, then $L(x_1) \cap L(x_2) \cap L(x_3) = \emptyset$.

Lemma 5.3.35. *$E_3 *$ anticlaw is d_1-choosable.*

Proof. Suppose otherwise. The Small Pot Lemma gives us a minimal bad d_1-assignment L on $E_3 *$anticlaw with $|Pot(L)| \le 6$. Let the E_3 have vertices x_1, x_2, x_3, and let the anticlaw have vertices y_1, y_2, y_3, y_4, with y_2, y_3, y_4 mutually adjacent. Then $\sum_i |L(x_i)| = 9$ and hence there are three colors c_1, c_2, c_3 such that for each $t \in [3]$, $c_t \in L(x_i) \cap L(x_j)$ for some $1 \le i < j \le 3$.

Suppose there exists $i \in \{2, 3, 4\}$, say $i = 2$, such that y_1 and y_i have a common color c. We use c on y_1 and y_2, and let $L'(v) = L(v) - c$ for each uncolored v; note that c must be absent from some x_i, say x_1. Now since $|L'(x_2)| + |L'(x_3)| \ge 4$, we can color x_2 and x_3 such that at least two colors remain available on y_3. Finally, we greedily color y_4, y_3, x_3.

Otherwise, since $|Pot(L)| \le 6$, we may assume that $L(y_1) = \{a, b\}$ and $L(y_2) = L(y_3) = L(y_4) = \{c, d, e, f\}$. Now we can color x_1, x_2, and x_3 using only two colors, exactly one of which is in $\{a, b\}$. Finally, we greedily color y_1, y_2, y_3, y_4. □

Lemma 5.3.36. *$E_3 * 2K_2$ is d_1-choosable.*

Proof. Suppose otherwise. The Small Pot Lemma gives us a minimal bad d_1-assignment L on $E_3 * 2K_2$ with $|Pot(L)| \le 6$. Let the E_3 have vertices x_1, x_2, x_3, and let the $2K_2$ have vertices y_1 adjacent to y_2 and y_3 adjacent to y_4. Then $\sum_i |L(x_i)| = 9$ and hence there are three colors c_1, c_2, c_3 such that for each $t \in [3]$, $c_t \in L(x_i) \cap L(x_j)$ for some $1 \le i < j \le 3$. If all three c_t appear on all four y_i, then we can 2-color the $2K_2$, and extend the coloring to the E_3. So we may assume instead without loss of generality that c_1 appears on x_1 and x_2, but not y_1. Now use c_1 on x_1 and x_2, then color greedily in the order y_3, y_4, x_3, y_2, y_1. □

Lemma 5.3.37. *$E_3 * E_4$ is d_1-choosable.*

Proof. Suppose otherwise. Let the E_3 have vertices x_1, x_2, x_3 and let the E_4 have vertices y_1, y_2, y_3, y_4. If there exists $c \in \cap_{i=1}^{3} L(x_i)$, then we use c on all x_i and we can finish the coloring, so assume not. By the Small Pot Lemma, $|Pot(L)| \leq 6$, so there exist two y_i, say y_1 and y_2, with a common color c; use c on y_1 and y_2. Now there exists some x_i, say x_3, with $c \notin L(x_i)$. The 4-cycle induced by x_1, x_2, y_3, and y_4 is 2-choosable; then we can extend the coloring to x_3. □

Lemma 5.3.38. *$E_3 *$ antidiamond is d_1-choosable.*

Proof. Suppose otherwise. The Small Pot Lemma gives us a minimal bad d_1-assignment L on $E_3 *$antidiamond with $|Pot(L)| \leq 6$. Let the E_3 have vertices x_1, x_2, x_3, and let the antidiamond have vertices y_1, y_2, y_3, y_4, with y_3 adjacent to y_4. We can assume tht $\cap_{i=1}^{3} L(x_i) = \emptyset$ (since otherwise we use a common color on the x_i and then greedily complete the coloring). If y_3 or y_4 has a common color c with y_1 or y_2, then we can use c on those two vertices and proceed as in the case of $E_3 * E_4$, so assume not. Again $\sum_i |L(x_i)| = 9$ and hence there are three colors c_1, c_2, c_3 such that for each $t \in [3]$, $c_t \in L(x_i) \cap L(x_j)$ for some $1 \leq i < j \leq 3$. So assume that c_1 appears on x_1 and x_2, and use it there. If c_1 appears on neither y_1 or y_2, then we greedily color in the order y_3, y_4, x_3, y_1, y_2. Otherwise c_1 appears on neither y_3 or y_4, so we greedily color in the order y_1, y_2, x_3, y_3, y_4. □

Lemma 5.3.39. *$E_3 * B$ is not d_1-choosable iff $B \in \left\{K_1, K_2, E_2, E_3, \overline{P_3}, K_3, K_4, K_5\right\}$.*

Proof. Suppose we have B such that $E_3 * B$ is not d_1-choosable. By Lemma 5.3.22, B is the disjoint union of complete subgraphs and at most one P_3. If B contained a P_3, then moving its middle vertex to the other side of the join would violate Lemma 5.3.20. By Lemma 5.3.37, B has at most three components. By Lemma 5.3.38, if B

has three components, then $B = E_3$. By Lemma 5.3.36 and Lemma 5.3.35, if B has two components then $B = E_2$ or $B = \overline{P_3}$. Otherwise B is complete and Lemma 5.3.24 shows that $|B| \leq 5$. This proves the forward implication.

For the other direction, it is easy to verify that $E_3 * B$ is not d_1-choosable for the listed graphs. The cases $B \in \{K_1, K_2, E_2\}$ are nearly trivial. For $B = E_3$, we are simply recalling that $K_{3,3}$ is not 2-choosable. For $B \in \{K_3, K_4, K_5\}$, see Figure 5.1. Finally, suppose that $B = \overline{P_3}$. Let x_1, x_2, x_3 denote the vertices of the E_3 and let y_1, y_2, y_3 denote the vertices of the $\overline{P_3}$, where y_2 and y_3 are adjacent. Assign the lists $L(x_1) = \{1, 2\}$, $L(x_2) = \{1, 3\}$, $L(x_3) = \{2, 3\}$, $L(y_1) = \{1, 2\}$, and $L(y_2) = L(y_3) = \{1, 2, 3\}$. To color the $\overline{P_3}$, we clearly use at least two colors, but now some vertex of the E_3 has no remaining colors. □

Lemma 5.3.40. $\overline{P_3} * 2K_2$ *is d_1-choosable.*

Proof. Let x_1, x_2, x_3 be the vertices of $\overline{P_3}$, with x_2 adjacent to x_3, and let y_1, y_2, y_3, y_4 be the vertices of $2K_2$, with y_1 adjacent to y_2 and y_3 adjacent to y_4. By the Small Pot Lemma, $|Pot(L)| \leq 6$, so x_1 and x_2 have a common color c_1. If c_1 is absent from the list of some y_i, say y_1, then we can use c_1 on x_1 and x_2, then greedily color in the order y_4, y_3, x_3, y_2, y_1. Hence c_1 appears on all y_i. If $|Pot(L)| \leq 5$, then x_1 and x_2 have a second common color c_2. Since c_1 and c_2 must appear on all y_i, we can 2-color the $2K_2$, then greedily color x_1, x_2, and x_3. So we can conclude that $L(x_1) \cap L(x_2) = c_1$ and $L(x_1) \cap L(x_3) = c_1$. Similarly, we can 2-color the $2K_2$ if y_1 and y_3 have any common color other than c_1.

Now we use c_1 on y_2 and y_4, and let $L'(v) = L(v) - c_1$ for all uncolored v. Now $|Pot(L')| = |Pot(L)| - 1 = 5$. Let $S = \{x_1, x_2, x_3, y_1, y_3\}$. To show that we can finish the coloring, we use Hall's Theorem. We only need to consider subsets $T \subset S$ of size 3 or 4. If $|T| = 3$, then either $\{y_1, y_3\} \subset T$, so $|\cup_{v \in T} L'(v)| \geq |L'(y_1)| + |L'(y_3)| \geq 4$, or else T contains x_2 or x_3. Since $|L'(x_2)| = |L'(x_3)| = 3$, we are done. If $|T| = 4$, then either $\{y_1, y_3\} \subset T$ or $\{x_1, x_2\} \subset T$ or $\{x_1, x_3\} \subset T$. In each case $|\cup_{v \in T} L'(v)| \geq 4$. □

Lemma 5.3.41. *$\overline{P_3} * $ antidiamond is d_1-choosable.*

Proof. Let x_1, x_2, x_3 be the vertices of $\overline{P_3}$, with x_2 adjacent to x_3, and let y_1, y_2, y_3, y_4 be the vertices of the antidiamond, with y_3 adjacent to y_4. By the Small Pot Lemma, $|Pot(L)| \leq 6$, so x_1 and x_2 have a common color c. If c is absent from y_4, then we use c on x_1 and x_2, then greedily color y_1, y_2, x_3, y_3, y_4. Similarly, if c is absent from y_1 and y_2, then we use c on x_1 and x_2, then greedily color y_3, y_4, x_3, y_2, y_1. So c must appear on y_1 (or y_2) and y_3, and we use it there. Let $L'(v) = L(v) - c$ for all uncolored vertices. Now if there exists $c_2 \in L'(y_2) \setminus L'(x_2)$, then we can use c_2 on y_2 and greedily color x_1, y_4, x_3, x_2. The same argument holds if there exists $c_2 \in L'(y_4) \setminus L'(x_2)$. Thus, we must have $(L'(y_2) \cup L'(y_4)) \subseteq L'(x_2)$, so y_2 and y_4 have a common color c_2. We use it on them and greedily color x_1, x_2, x_3. □

Lemma 5.3.42. *$\overline{P_3} * E_4$ is d_1-choosable.*

Proof. Let x_1, x_2, x_3 be the vertices of $\overline{P_3}$, with x_2 adjacent to x_3, and let y_1, y_2, y_3, y_4 be the vertices of E_4. If three of the y_i's (say y_1, y_2, and y_3) have a common color c, then use c on them, and now greedily color in the order y_4, x_1, x_2, x_3. By the Small Pot Lemma, x_1 and x_2 have a common color c, which we use on them. Now c appears on at most two y_i, say y_1 and y_2, so we can greedily color in the order y_1, y_2, x_3, y_3, y_4. □

Lemma 5.3.43. *$\overline{P_3} * B$ is not d_1-choosable iff B is E_3, $K_{|B|}$, or $K_1 + K_{|B|-1}$.*

Proof. Since $\overline{P_3}$ contains an E_2, Lemma 5.3.22 shows that B is the disjoint union of complete subgraphs and at most one P_3. If B contained a P_3, then moving its middle vertex to the other side of the join would violate Lemma 5.3.20. By Lemma

5.3.40 at most one component of B has more than one vertex. If B has more than two components, then Lemma 5.3.41 shows that B is independent and thus Lemma 5.3.42 shows that $B = E_3$. If B has two components then it is $K_1 + K_{|B|-1}$. Otherwise B is complete. This proves the forward implication.

The reverse implication is easily checked. For $B = E_3$, see Lemma 5.3.39. If $B = K_{|B|}$, then G is almost complete. Suppose that $B = K_{|B|-1|}$. Now $\Delta(G) = \omega(G) = |B| + 1$, so G is not d_1-choosable. □

Lemma 5.3.44. *Let A and B be graphs with $|A| \geq 4$ and $|B| \geq 4$. The graph $A * B$ is not d_1-choosable iff $A * B$ is almost complete, $K_5 * E_3$, or*
$\big(K_1 + K_{|A|-1}\big) * \big(K_1 + K_{|B|-1}\big)$.

Proof. Suppose A and B are graphs with $|A| \geq |B| \geq 4$ such that $A * B$ is not d_1-choosable and not one of the specified graphs.

First suppose A is connected. If A is complete then by Corollary 5.3.25, $|A| = 4$ and B is a claw or B is almost complete. But this implies that $G = K_5 * E_3$ or G is almost complete. Hence A is incomplete. Now Lemma 5.3.20 shows that B is complete. By reversing the roles of A and B in this argument, we get a contradiction; so A is disconnected. The same argument shows that B is also disconnected.

Suppose $\alpha(A) \geq 3$. Then Lemma 5.3.39 shows that B is K_4 or K_5, both impossible as above. Thus $\alpha(A) = 2$ and hence A is the disjoint union of two complete graphs. The same goes for B. Now Lemma 5.3.43 shows that $A = K_1 + K_{|A|-1}$ and $B = K_1 + K_{|B|-1}$. The reverse implication is easily checked. If $A * B$ is almost complete, then clearly it is not d_1-choosable. For $A * B = K_5 * E_3$, see Figure 5.1. So suppose that $A * B = (K_1 + K_{|A|-1}) * (K_1 + K_{|B|-1})$. Now $\Delta(A * B) = \omega(A * B) = |A| + |B| - 2$, so $A * B$ is not d_1-choosable. □

Definition 5.3.2. The *net* is formed by adding one edge incident to each vertex of K_3. The *bowtie* is formed by identifying one vertex in each of two copies of K_3. The M is formed from the bowtie by adding an edge incident to a vertex of degree 2.

Lemma 5.3.45. *The graph $K_2 * A$ is d_1-choosable for all*

$$A \in \{2P_3, C_4, C_5, P_5, chair, antichair, K_1 * antipaw, K_1 * P_4, net, M\}.$$

Proof. For eight of these ten choices of A, we have already proved that $K_2 * A$ is d_1-choosable. Specifically, we have proved this for $2P_3$ (Lemma 5.3.34), C_5 (Lemma 5.3.33), P_5 (Lemma 5.3.30), chair (Lemma 5.3.31), antichair (Lemma 5.3.32), $K_1 *$ antipaw (Lemma 5.3.28), $K_1 * P_4$ (Lemma 5.3.27), and C_4 (since $C_4 = E_2^2$, this is the case $r = 1$ in Corollary 5.3.4). Now we consider the remaining two cases: net and M.

Let $G = K_2 *$net. Let x_1, x_2 denote the vertices of the K_2, let y_1, y_2, y_3 denote the degree-3 vertices in the net, and let z_1, z_2, z_3, denote the leaves of the net, with z_i adjacent to y_i. We consider three cases. (1) If there exists $c_1 \in \cap_{i=1}^{3} L(z_i)$, then we first use c_1 on all three z_i and afterwards color y_1, y_2, y_3, x_1, x_2 greedily. (2) Suppose there exist y_i and z_j, with $i \neq j$, such that there exists $c_1 \in L(y_i) \cap L(z_j)$; by symmetry we assume this is y_1 and z_2. We use c_1 on y_1 and z_2 and let $L'(v) = L(v) - c_1$ for each uncolored vertex v. Now we have $|Pot(L')| < |G \setminus \{y_1, z_2\}| = 6$. Since we have $|L'(z_1)|+|L'(y_2)|+|L'(z_3)| \geq 1+3+2 = 6$, we must have a common color c_2 (different from c_1) on two of z_1, y_2, and z_3. We use this color on these two vertices, then greedily color the remaining vertices of the net before coloring x_1 and x_2. (3) Observe that if $L(z_1)$ and $L(z_2)$ are disjoint, then (since $|Pot(L)| \leq 7$) either $L(z_1) \cap L(y_3) \neq \emptyset$ or $L(z_2) \cap L(y_3) \neq \emptyset$; in each case, we are in (2). Thus, if we are not in (1) or (2) above, then (again, since $|Pot(L)| \leq 7$) by symmetry we have $L(z_1) = \{a, b\}$, $L(z_2) = \{a, c\}$,

$L(z_3) = \{b, c\}$, and $L(y_1) = L(y_2) = L(y_3) = \{d, e, f, g\}$. By symmetry, either $a \notin L(x_1)$ or $d \notin L(x_1)$. Thus, we use a on z_1 and z_2 and we use d on y_3. Now we greedily color z_3, y_1, y_2, x_2, x_1.

Let $G = K_2 * M$ and let x_1, x_2 denote the vertices of the K_2; for the M, let y_1 denote the 1-vertex, y_2 the 3-vertex, y_3 the 2-vertex adjacent to y_2, y_4 the 4-vertex, and y_5 and y_6 the remaining 2-vertices. By the Small Pot Lemma, $|Pot(L)| \leq 7$. Since $|L(y_1)| + |L(y_3)| + |L(y_6)| = 8$, two of them must have a common color c. If all three of y_1, y_3, y_6 have c, then we use c on all three, and afterward we color greedily y_2, y_4, y_5, x_1, x_2. So now we consider three cases. (1) If c appears in $L(y_3) \cap L(y_6)$, then we use c on y_3 and y_6, and let $L'(v) = L(v) - c$ for each uncolored vertex v. By the Small Pot Lemma, $|Pot(L')| \leq 5$. Since $|L'(y_1)| + |L'(y_4)| \geq 2 + 4 > 5$, we have a common color d (different from c) on y_1 and y_4. After we use d on y_1 and y_4, we color greedily y_2, y_5, x_1, x_2. (2) If c appears in $L(y_1) \cap L(y_3)$, then we use c on y_1 and y_3 and let $L'(v) = L(v) - c$ for each uncolored vertex v. Again we have $|Pot(L')| \leq 5$ and $|L'(y_2)| + |L'(y_5)| \geq 3 + 3 > 5$. After using a common color on y_2 and y_5, we greedily color y_4, y_6, x_1, x_2.

(3) Now suppose that c appears in $L(y_1) \cap L(y_6)$. If $c \in L(y_2)$, then we use c on y_2 and y_6, and let $L'(v) = L(v) - c$ for each uncolored vertex v. Again we have $|Pot(L')| \leq 5$ and $|L'(y_1)| + |L'(y_3)| + |L'(y_5)| \geq 1 + 3 + 2$ (since $c \notin L(y_3)$). So again we use a common color on two of y_1, y_3, and y_5, then greedily color the remaining vertices of the M before coloring x_1 and x_2. Suppose instead that $c \notin L(y_2)$. Now we use c on y_1 and y_6, and then use a common color on y_4 and y_5 (since $|Pot(L')| \leq 5 < 6 = 4 + 2 \leq |L'(y_2)| + |L'(y_5)|$). Finally, we greedily color y_3, y_4, x_1, x_2. $\square$

Lemma 5.3.46. *The graph $K_2 * (B + K_t)$ is not d_1-choosable iff $K_2 * B$ is not d_1-choosable.*

Proof. Suppose $K_2 * B$ is not d_1-choosable and let L be a bad list assignment (not using the colors in $[t]$). To form a list assignment for $K_2 * (B + K_t)$, we start with L, then assign $[t]$ to each vertex in the K_t and add $[t]$ to the lists for the vertices in the K_2. Clearly $K_2 * (B + K_t)$ has no coloring from these lists.

Conversely, suppose $K_2 * B$ is d_1-chooable. Given a list assignment for $K_2 * (B + K_t)$, we greedily color the K_t; what remains is a list assignment for $K_2 * B$; thus, we can finish the coloring. □

Since $K_2 * 2P_3$ is d_1-choosable (Lemma 5.3.34) we see that any graph B such that $K_2 * B$ is not d_1-choosable must have at most one incomplete component.

Lemma 5.3.47. *If $K_2 * B$ is not d_1-choosable, then B consists of a disjoint union of complete subgraphs, together with at most one incomplete component H. If H has a dominating vertex v, then $K_2 * H = K_3 * (H - v)$, so by Lemma 5.3.29 we can completely describe H. Otherwise H is formed either by adding an edge between two disjoint cliques or by adding a single pendant edge incident to each of two distinct vertices of a clique. Furthermore, all graphs formed in this way are not d_1-choosable.*

Proof. Let B be a graph such that $K_2 * B$ is not d_1-choosable, and let H be the unique incomplete component of B. Suppose that H does not contain a dominating vertex. We first show that H is a tree of edge-disjoint cliques (clique tree), i.e., every cycle has an edge between every pair of its vertices. Since $K_2 * C_4$, $K_2 * C_5$, and $K_2 * P_5$ are d_1-choosable, we get that H has no induced C_4, C_5, or P_5; thus H is chordal. So if H is not a clique tree, then H contains an induced copy of K_4^-; call it D.

Let w denote a vertex adjacent to D. Each vertex adjacent to D can attach to the vertices of D in 8 possible ways (up to isomorphism); it can attach to 0, 1, or 2 of the vertices of degree 2, and also to 0, 1, or 2 of the vertices of degree 3 (but it must attach to at least one vertex), thus $3*3-1=8$ possibilites. Five of these possibilities yield a graph J such that $K_2 * J$ is d_1-choosable (since J contains an induced copy of either the antichair, $K_1 *$ antipaw, $K_1 * P_4$, or C_4). So we consider the other three possibilities (these are the three possibilities when w is adjacent to both vertices of degree 3 in D).

If D is not dominating, then some vertex x is distance 2 from D, via w. In each case, the subgraph induced by D, w, and x contains an induced d_1-choosable subgraph (in two cases this is a antichair, and in the third case it is $K_1 *$ antipaw). Hence, D is dominating, and all of its neighbors are adjacent to both vertices of degree 3 in D. But now H has two dominating vertices. This contradicts our assumption that H has no dominating vertex. Hence, H is a clique tree.

Since H has no dominating vertex, it must contain an induced P_4, call it P. Since H has neither a P_5 nor a "chair" as an induced subgraph, each vertex adjacent to P must be adjacent to at least two vertices of P. Since C_4 and the antichair and $K_1 * P_4$ are all forbidden, each vertex adjacent to P is adjacent to exactly two consecutive vertices of P. Since both P_5 and the net are forbidden, every vertex in H is adjacent to P. Since $P_1 *$ antipaw is forbidden, every pair of vertices that are adjacent to the same two vertices of P are also adjacent to each other. Finally, since M is forbidden, H must be formed in one of two ways. Either (a) begin with two disjoint cliques and add an edge between them, or else (b) begin with a clique and add exactly one edge incident to exactly two vertices of the clique. Furthermore, all graphs H formed by either (a) or (b) are such that $K_2 * H$ is not d_1-choosable.

In (a), suppose that we begin with a K_r and a K_s. We assign lists as follows: the K_r gets $[r]$, the K_s gets $\{r+1,\ldots,r+s\}$, the dominating vertices (on the other side of the join) get $[r+t]$; finally, the two endpoints of the additional edge also get α added to their lists. $K_2 * H$ is clearly not colorable from these lists, since all but one or $[r+t]$ must be used on H.

In (b), suppose that we begin with a K_r. We assign lists as follows: the K_r gets $[r]$, the two degree 1 vertices get $\{r+1, r+2\}$, the dominating vertices (on the other side of the join) get $[r+2]$; finally, the two vertices in the K_r that are endpoints of the pendant edges also get $r+1$ added to their lists. $K_2 * H$ is clearly not colorable from these lists, since all but one of $[r+2]$ must be used on H. $\square$

Mixed list assignments

Lemma 5.3.48. *Let A be a graph with $|A| \geq 4$. Let L be a list assignment on $G := E_2 * A$ such that $|L(v)| \geq d(v) - 1$ for all $v \in V(G)$ and each component D of A has a vertex v such that $|L(v)| \geq d(v)$. Then L is good on G.*

Proof. By the Small Pot Lemma, $|Pot(L)| \leq |A| + 1$. Say the E_2 has vertices $\{x, y\}$. Then $|L(x)| + |L(y)| \geq 2|A| - 2 > |A| + 1$ since $|A| \geq 4$. Coloring x and y the same leaves at worst a d_0 assignment L' on A where each component D has a vertex v with $|L'(v)| > d_D(v)$. Hence we can complete the coloring. $\square$

Lemma 5.3.49. *Let A be a graph with $|A| \geq 3$. Let L be a list assignment on $G := E_2 * A$ such that $|L(v)| \geq d(v) - 1$ for all $v \in V(G)$, $|L(v)| \geq d(v)$ for some v in the E_2 and each component D of A has a vertex v such that $|L(v)| \geq d(v)$. Then L is good on G.*

Proof. By the Small Pot Lemma, $|Pot(L)| \leq |A| + 1$. Say the E_2 has vertices $\{x, y\}$. Then $|L(x)| + |L(y)| \geq 2|A| - 1 > |A| + 1$ since $|A| \geq 3$. Coloring x and y the same leaves at worst a d_0 assignment L' on A where each component D has a vertex v with $|L'(v)| > d_D(v)$. Hence we can complete the coloring. □

Joins with K_1

Let G be a d_0-choosable graph. If $K_1 * G$ is not d_1-choosable, then we call G *bad*; otherwise we call G *good*. Adding a leaf to a graph does not change whether it is bad, so we focus on bad G such that $\delta(G) \geq 2$. We will also restrict our attention to connected bad graphs.

In this section, we apply Lemma 5.3.14 to characterize all bad triangle-free graphs. An easy special case of this classification for triangle-free graphs is the following lemma. We frequently use the idea of an independent set with a common color, so we call an independent set of size k with a common color an *independent k-set*.

Lemma 5.3.50. *If G is a connected bipartite graph with more edges than vertices, then $K_1 * G$ is d_1-choosable.*

Proof. Let A and B be the parts of G. Let L be a minimal bad d_1-assignment for $K_1 * G$. Since G has more edges than vertices, G has a cycle. Since G is also bipartite, G is d_0-choosable (by the classification of d_0-choosable graphs at the start of Section 5.2). By the Small Pot Lemma, $Pot(L) \leq |G|$. Note that $\sum_{v \in A} d(v) = |E(G)| > |V(G)| \geq |Pot(L)|$. Similarly $\sum_{v \in B} d(v) > |Pot(L)|$. Now we apply Lemma 5.3.17 with $I_1 = A$ and $I_2 = B$. This proves the lemma. □

Lemma 5.3.51. *Let $\mathcal{C}$ be a collection of sets $I_1, \ldots, I_k$, each of size 2. If for all $i \neq j$, we have $I_i \cap I_j \neq \emptyset$, then either there exists $v \in \cap_{i=1}^k I_i$ or there exist v_1, v_2, and v_3 such that each I_i equals either $\{v_1, v_2\}$ or $\{v_1, v_3\}$ or $\{v_2, v_3\}$.*

Proof. Suppose that $\cap_{i=1}^{k} I_i = \emptyset$. Consider distinct sets I_1 and I_2. Let $\{v_1\} = I_1 \cap I_2$, and let $I_1 = \{v_1, v_2\}$ and $I_2 = \{v_1, v_3\}$. Since $\cap_{i=1}^{k} I_i = \emptyset$, there exists I_3 such that $v_1 \notin I_3$. So we must have $I_3 = \{v_2, v_3\}$. Now for all $k \geq 4$, we must have $|I_k \cap \{v_1, v_2, v_3\}| = 2$. □

Using Lemmas 5.3.14 and 5.3.51, we can prove the following classification.

Lemma 5.3.52. *If a d_0-choosable graph G is bad, then $K_1 * G$ has a d_1-list assignment L such that one of the following 5 conditions holds.*

1. *L is a d-clique cover of G of size at most $|G|$.*

2. *There exists $v \in V(G)$ such that L is a d-clique cover of $G - v$ of size at most $|G| - 1$.*

3. *There exists a color c such that the union of all independent 2-sets in c induces P_4 and all other independent 2-sets are the end vertices of the P_4.*

4. *The union of all independent 2-sets is E_3 or E_2.*

5. *All independent 2-sets in L are the same color.*

Proof. Let z denote the K_1. We consider the possible ways for a bad list assignment L to satisfy Lemma 5.3.14. Clearly L has no independent k-sets, for $k \geq 3$. If L has no independent 2-sets, then Condition 1 holds. If all independent 2-sets in L are the same color, then Condition 5 holds. If L has only the same independent 2-set in multiple colors, then the 2-sets induce E_2, so Condition 4 holds. So instead L must have distinct independent 2-sets in distinct colors.

Assume that additionally all independent 2-sets intersect in a common vertex v. If $|Pot_{G-v}(L)| \leq |G| - 1$, then Condition 2 holds. So instead $|Pot_{G-v}(L)| \geq |G|$.

So there exist some $w \in G - v$ and some color $c \in L(w)$ such that $c \notin L(z)$. By Lemma 5.3.14, G has an L-coloring that uses c on w and uses some other common color on two vertices of $G - w$. Now we can extend the coloring to z.

Now suppose that no vertex v lies in all independent 2-sets. If all independent 2-sets are distinct colors, then Lemma 5.3.51 implies that Condition 4 holds. Suppose we have two independent 2-sets $I_1 = \{v_1, v_2\}$ and $I_2 = \{v_1, v_3\}$ in the same color c. Since L has no independent 3-set, v_2 is adjacent to v_3. Recall that L has an independent 2-set I_3 of another color c'. If $v_1 \notin I_3$, then I_3 is disjoint from either I_1 or I_2, so we can finish the coloring, by (2) in Lemma 5.3.14. Hence $v_1 \in I_3$. So the only independent 2-sets not containing v_1 must be of color c, say $\{v_2, v_4\}$. Since L has no independent 3-sets, we must have v_1 adjacent to v_4. Now we see that every independent 2-set in a color other than c must be $\{v_1, v_2\}$. This implies that v_2 and v_3 must be adjacent. Now Condition 3 holds.

Finally, suppose that L has two independent 2-sets $I_1 = \{v_1, v_2\}$ and $I_2 = \{v_3, v_4\}$ in a common color. If we are not in the case above, then $G[v_1, v_2, v_3, v_4] = C_4$. Now every independent 2-set I_3 of another color can intersect at most one of I_1 and I_2, so we can color the graph by (2) in Lemma 5.3.14. $\square$

5.4 Connectivity of complements

As a basic application of our list coloring lemmas, we prove that for $k \geq 5$ any $G \in \mathcal{C}_k$ has maximally connected complement.

Lemma 5.4.1. *Fix $k \geq 5$. If $G \in \mathcal{C}_k$ and $A * B \lhd G$ for graphs A and B with $1 \leq |A| \leq |B|$, then $|A * B| \leq \Delta(G) + 1$.*

Proof. Let $G \in \mathcal{C}_k$ and $A * B \unlhd G$ for graphs A and B with $1 \leq |A| \leq |B|$. Assume $|A * B| > \Delta(G) + 1$. To avoid a vertex with degree larger than $\Delta(G)$, we must have $\Delta(A) \leq |A| - 2$ and $\Delta(B) \leq |B| - 2$. In particular, both A and B are incomplete, so

$2 \leq |A| \leq |B|$ and both A and B contain an induced E_2. Hence, by Lemma 5.3.22, both A and B are the disjoint union of complete subgraphs and at most one P_3.

First, assume $|A| = 2$, say $A = \{x_1, x_2\}$. Since $|B| \geq \Delta(G)$, we conclude that $N(x_1) = N(x_2)$. Thus x_1 and x_2 are nonadjacent twins in a vertex critical graph which is impossible.

Thus we may assume that $|A| \geq 3$. If A contained an induced P_3, then G would have an induced $E_2 * (K_1 * B)$. For $K_1 * B$ to be the disjoint union of complete subgraphs and at most one P_3, B must either be E_2 or complete, both of which are impossible. Hence A is a disjoint union of at least two complete subgraphs. The same goes for B.

Assume that A is edgeless. Then, by Lemma 5.3.39, B must be E_3 or $\overline{P_3}$. Hence $\Delta(G) + 1 < |A| + |B| = 6$, giving the contradiction $\Delta(G) \leq 4$.

Since A is the disjoint union of at least two complete subgraphs and contains an edge, it contains $\overline{P_3}$. By Lemma 5.3.43, B must be either E_3 or the disjoint union of a vertex and a complete subgraph. As above, $B = E_3$ is impossible. In particular B contains $\overline{P_3}$ and using Lemma 5.3.43 again, we conclude that A is the disjoint union of a vertex and a complete subgraph giving the final contradiction $\omega(G) \geq \omega(A * B) \geq \omega(A) + \omega(B) \geq |A| + |B| - 2 \geq \Delta(G)$. □

Lemma 5.4.2. *Fix $k \geq 5$. If $G \in \mathcal{C}_k$, then $\overline{G}$ is maximally connected; that is, $\kappa(\overline{G}) = \delta(\overline{G})$.*

Proof. Let $G \in \mathcal{C}_k$ and let S be a cutset in $\overline{G}$ with $|S| = \kappa(\overline{G})$. To get a contradiction, assume that $|S| < \delta(\overline{G}) = |G| - (\Delta(G)+1)$. Since $\overline{G} - S$ is disconnected, $G - S = A * B$ for some graphs A and B with $1 \leq |A| \leq |B|$. We have $|A| + |B| = \left|\overline{G} - S\right| = |G| - |S| > |G| - (|G| - (\Delta(G) + 1)) = \Delta(G) + 1$, a contradiction by Lemma 5.4.1. □

Chapter 6

CLAW-FREE GRAPHS

Some of the material in this chapter appeared in [19] and is joint work with Dan Cranston.

In [21], Dhurandhar proved the Borodin-Kostochka Conjecture for a superset of line graphs of *simple* graphs defined by excluding the claw, $K_5 - e$ and another graph D as induced subgraphs. Kierstead and Schmerl [39] improved this by removing the need to exclude D. The aim of this chapter is to remove the need to exclude $K_5 - e$; that is, to prove the Borodin-Kostochka Conjecture for claw-free graphs.

Theorem 6.4.5. *Every claw-free graph satisfying $\chi \geq \Delta \geq 9$ contains a K_Δ.*

This also generalizes the result of Beutelspacher and Hering [5] that the Borodin-Kostochka conjecture holds for graphs with independence number at most two. The value of 9 in Theorem 6.4.5 is best possible since the counterexample for $\Delta = 8$ in Figure 1.1 is claw-free. Theorem 6.4.5 is also optimal in the following sense. We can reformulate the statement as: every claw-free graph with $\Delta \geq 9$ satisfies $\chi \leq \max\{\omega, \Delta - 1\}$. Consider a similar statement with $\Delta - 1$ replaced by $f(\Delta)$ for some $f\colon \mathbb{N} \to \mathbb{N}$ and 9 replaced by Δ_0. We show that $f(x) \geq x - 1$ for $x \geq \Delta_0$. Consider $G_t := K_t * C_5$. We have $\chi(G_t) = t + 3$, $\omega(G_t) = t + 2$ and $\Delta(G_t) = t + 4$ and G_t is claw-free. Hence for $t \geq \Delta_0 - 4$ we have $t + 3 \leq \max\{t + 2, f(t + 4)\} \leq f(t + 4)$ giving $f(x) \geq x - 1$ for $x \geq \Delta_0$.

As shown in [61] (also Section 6.1) the situation is very different for line graphs of multigraphs which satisfy $\chi \leq \max\{\omega, \frac{7\Delta+10}{8}\}$. There it was conjectured that $f(x) := \frac{5x+8}{6}$ works for line graphs of multigraphs; this would be best possible. The example of $K_t * C_5$ is claw-free, but it isn't quasi-line.

Question. What is the situation for quasi-line graphs? That is, what is the optimal f such that every quasi-line graph with large enough maximum degree satisfies $\chi \leq \max\{\omega, f(\Delta)\}$.

6.1 Line graphs

The material in this section appeared in [61].

In this section we prove the Bornodin-Kostochka Conjecture for line graphs of *multigraphs.* Moreoever, we prove a strengthening of Brooks' theorem for line graphs of multigraphs and conjecture the best possible such bound.

Lemma 6.1.1. *Fix $k \geq 0$. Let H be a multigraph and put $G = L(H)$. Suppose $\chi(G) = \Delta(G) + 1 - k$. If $xy \in E(H)$ is critical and $\mu(xy) \geq 2k + 2$, then xy is contained in a $\chi(G)$-clique in G.*

Proof. Let $xy \in E(H)$ be a critical edge with $\mu(xy) \geq 2k+2$. Let A be the set of all edges incident with both x and y. Let B be the set of edges incident with either x or y but not both. Then, in G, A is a clique joined to B and B is the complement of a bipartite graph. Put $F = G[A \cup B]$. Since xy is critical, we have a $\chi(G) - 1$ coloring of $G - F$. Viewed as a partial $\chi(G) - 1$ coloring of G this leaves a list assignment L on F with $|L(v)| = \chi(G) - 1 - (d_G(v) - d_F(v)) = d_F(v) - k + \Delta(G) - d_G(v)$ for each $v \in V(F)$. Put $j = k + d_G(xy) - \Delta(G)$.

Let M be a maximum matching in the complement of B. First suppose $|M| \leq j$. Then, since B is perfect, $\omega(B) = \chi(B)$ and we have

$$\begin{aligned}\omega(F) &= \omega(A) + \omega(B) = |A| + \chi(B)\\ &\geq |A| + |B| - j = d_G(xy) + 1 - j\\ &= \Delta(G) + 1 - k = \chi(G).\end{aligned}$$

Thus xy is contained in a $\chi(G)$-clique in G.

Hence we may assume that $|M| \geq j+1$. Let $\{\{x_1, y_1\}, \ldots, \{x_{j+1}, y_{j+1}\}\}$ be a matching in the complement of B. Then, for each $1 \leq i \leq j+1$ we have

$$\begin{aligned}|L(x_i)| + |L(y_i)| &\geq d_F(x_i) + d_F(y_i) - 2k\\ &\geq |B| - 2 + 2|A| - 2k\\ &= d_G(xy) + |A| - 2k - 1\\ &\geq d_G(xy) + 1.\end{aligned}$$

Here the second inequality follows since $\alpha(B) \leq 2$ and the last since $|A| = \mu(xy) \geq 2k+2$. Since the lists together contain at most $\chi(G) - 1 = \Delta(G) - k$ colors we see that for each i,

$$\begin{aligned}|L(x_i) \cap L(y_i)| &\geq |L(x_i)| + |L(y_i)| - (\Delta(G) - k)\\ &\geq d_G(xy) + 1 - \Delta(G) + k\\ &= j + 1.\end{aligned}$$

Thus we may color the vertices in the pairs $\{x_1, y_1\}, \ldots, \{x_{j+1}, y_{j+1}\}$ from L using one color for each pair. Since $|A| \geq k+1$ we can extend this to a coloring of B from L by coloring greedily. But each vertex in A has $j+1$ colors used twice on its neighborhood, thus each vertex in A is left with a list of size at least $d_A(v) - k + \Delta(G) - d_G(v) + j + 1 = d_A(v) + 1$. Hence we can complete the $(\chi(G) - 1)$-coloring to all of F by coloring greedily. This contradiction completes the proof. $\square$

Theorem 6.1.2. *If G is the line graph of a multigraph H and G is vertex critical, then*

$$\chi(G) \leq \max\left\{\omega(G), \Delta(G) + 1 - \frac{\mu(H) - 1}{2}\right\}.$$

Proof. Let G be the line graph of a multigraph H such that G is vertex critical. Say $\chi(G) = \Delta(G) + 1 - k$. Suppose $\chi(G) > \omega(G)$. Since G is vertex critical, every edge in H is critical. Hence, by Lemma 6.1.1, $\mu(H) \leq 2k + 1$. That is, $\mu(H) \leq 2(\Delta(G) + 1 - \chi(G)) + 1$. The theorem follows. $\square$

This upper bound is tight. To see this, let $H_t = t \cdot C_5$ (i.e. C_5 where each edge has multiplicity t) and put $G_t = L(H_t)$. As Catlin [16] showed, for odd t we have $\chi(G_t) = \frac{5t+1}{2}$, $\Delta(G_t) = 3t - 1$, and $\omega(G_t) = 2t$. Since $\mu(H_t) = t$, the upper bound is achieved.

We need the following lemma which is a consequence of the fan equation (see [4, 13, 25, 27]).

Lemma 6.1.3. *Let G be the line graph of a multigraph H. Suppose G is vertex critical with $\chi(G) > \Delta(H)$. Then, for any $x \in V(H)$ there exist $z_1, z_2 \in N_H(x)$ such that $z_1 \neq z_2$ and*

- $\chi(G) \leq d_H(z_1) + \mu(xz_1)$,
- $2\chi(G) \leq d_H(z_1) + \mu(xz_1) + d_H(z_2) + \mu(xz_2)$.

Lemma 6.1.4. *Let G be the line graph of a multigraph H. If G is vertex critical with $\chi(G) > \Delta(H)$, then*

$$\chi(G) \leq \frac{3\mu(H) + \Delta(G) + 1}{2}.$$

Proof. Let $x \in V(H)$ with $d_H(x) = \Delta(H)$. By Lemma 6.1.3 we have $z \in N_H(x)$ such that $\chi(G) \leq d_H(z) + \mu(xz)$. Hence

$$\Delta(G) + 1 \geq d_H(x) + d_H(z) - \mu(xz) \geq d_H(x) + \chi(G) - 2\mu(xz).$$

Which gives

$$\chi(G) \leq \Delta(G) + 1 - \Delta(H) + 2\mu(H).$$

Adding Vizing's inequality $\chi(G) \leq \Delta(H) + \mu(H)$ gives the desired result. □

Combining this with Theorem 6.1.2 we get the following upper bound.

Theorem 6.1.5. *If G is the line graph of a multigraph, then*

$$\chi(G) \leq \max\left\{\omega(G), \frac{7\Delta(G) + 10}{8}\right\}.$$

Proof. Suppose not and choose a counterexample G with the minimum number of vertices. Say $G = L(H)$. Plainly, G is vertex critical. Suppose $\chi(G) > \omega(G)$. By Theorem 6.1.2 we have

$$\chi(G) \leq \Delta(G) + 1 - \frac{\mu(H) - 1}{2}.$$

By Lemma 6.1.4 we have

$$\chi(G) \leq \frac{3\mu(H) + \Delta(G) + 1}{2}.$$

Adding three times the first inequality to the second gives

$$4\chi(G) \leq \frac{7}{2}(\Delta(G)+1) + \frac{3}{2}.$$

The theorem follows. $\square$

Corollary 6.1.6. *If G is the line graph of a multigraph with $\chi(G) \geq \Delta(G) \geq 11$, then G contains a $K_{\Delta(G)}$.*

With a little more care we can get the 11 down to 9. Using Lemma 4.2.2, we can inductively reduce to the $\Delta = 9$ case.

Theorem 6.1.7. *If G is the line graph of a multigraph with $\chi(G) \geq \Delta(G) \geq 9$, then G contains a $K_{\Delta(G)}$.*

Proof. Suppose the theorem is false and choose a counterexample G minimizing $\Delta(G)$. Then G is vertex critical. By Lemma 4.2.2, $\Delta(G) = 9$.

Let H be such that $G = L(H)$. Then by Lemma 6.1.1 and Lemma 6.1.4 we know that $\mu(H) = 3$. Let $x \in V(H)$ with $d_H(x) = \Delta(H)$. Then we have $z_1, z_2 \in N_H(x)$ as in Lemma 6.1.3. This gives

$$\begin{aligned} 9 &\leq d_H(z_1) + \mu(xz_1), && (6.1) \\ 18 &\leq d_H(z_1) + \mu(xz_1) + d_H(z_2) + \mu(xz_2). && (6.2) \end{aligned}$$

In addition, we have for $i = 1, 2$,

$$9 \geq d_H(x) + d_H(z_i) - \mu(xz_i) - 1 = \Delta(H) + d_H(z_i) - \mu(xz_i) - 1.$$

Thus,

$$\Delta(H) \leq 2\mu(xz_1) + 1 \leq 7, \tag{6.3}$$

$$\Delta(H) \leq \mu(xz_1) + \mu(xz_2) + 1. \tag{6.4}$$

Now, let $ab \in E(H)$ with $\mu(ab) = 3$. Then, since G is vertex critical, we have $8 = \Delta(G) - 1 \leq d_H(a) + d_H(b) - \mu(ab) - 1 \leq 2\Delta(H) - 4$. Thus $\Delta(H) \geq 6$. Hence we have $6 \leq \Delta(H) \leq 7$. Thus, by (3), we must have $\mu(xz_1) = 3$.

First, suppose $\Delta(H) = 7$. Then, by (4) we have $\mu(xz_2) = 3$. Let y be the other neighbor of x. Then $\mu(xy) = 1$ and thus $d_H(x) + d_H(y) - 2 \leq 9$. That gives $d_H(y) \leq 4$. Then we have vertices $w_1, w_2 \in N_H(y)$ guaranteed by Lemma 6.1.3. Note that $x \notin \{w_1, w_2\}$. Now $4 \geq d_H(y) \geq 1 + \mu(yw_1) + \mu(yw_2)$. Thus $\mu(yw_1) + \mu(yw_2) \leq 3$. This gives $d_H(w_1) + d_H(w_2) \geq 2\Delta(G) - 3 = 15$ contradicting $\Delta(H) \leq 7$.

Thus we must have $\Delta(H) = 6$. By (1) we have $d_H(z_1) = 6$. Then, applying (2) gives $\mu(xz_2) = 3$ and $d_H(z_2) = 6$. Since x was an arbitrary vertex of maximum degree and H is connected we conclude that $G = L(3 \cdot C_n)$ for some $n \geq 4$. But no such graph is 9-chromatic by Brooks' theorem. □

The graphs $G_t = L(t \cdot C_5)$ discussed above show that the following upper bound would be tight. Creating a counterexample would require some new construction technique that might lead to more counterexamples to Borodin-Kostochka for $\Delta = 8$.

Conjecture 6.1.8. *If G is the line graph of a multigraph, then*

$$\chi(G) \leq \max\left\{\omega(G), \frac{5\Delta(G) + 8}{6}\right\}.$$

6.2 Circular interval graphs

A *representation* of a graph G in a graph H consists of:

- an injection $f\colon V(G) \hookrightarrow V(H)$;
- for each $xy \in E(G)$, a choice of path $p_{xy} \subseteq H$ from $f(x)$ to $f(y)$ such that $f^{-1}(V(p_{xy}))$ is a clique in G.

A graph is a *circular interval graph* if it has a representation in a cycle. We note that this class coincides with the class of proper circular arc graphs. A graph is a *linear interval graph* if it has a representation in a path.

Lemma 6.2.1. *Every circular interval graph satisfying $\chi_l \geq \Delta \geq 9$ contains a K_Δ.*

Proof. Suppose the contrary and choose a counterexample G minimizing $|G|$. Put $\Delta := \Delta(G)$. Then $\chi_l(G) = \Delta$, $\omega(G) \leq \Delta - 1$, $\delta(G) \geq \Delta - 1$ and $\chi_l(G - v) \leq \Delta - 1$ for all $v \in V(G)$. Since G is a circular interval graph, by definition G has a representation in a cycle $v_1 v_2 \dots v_n$. Let K be a maximum clique in G. By symmetry we may assume that $V(K) = \{v_1, v_2, \dots, v_t\}$ for some $t \leq \Delta - 1$; further, if possible we label the vertices so that $v_{t-3} \leftrightarrow v_{t+1}$ and the edge goes through v_{t-2}, v_{t-1}, v_t.

Claim 1. *$v_1 \not\leftrightarrow v_{t+1}$ and $v_2 \not\leftrightarrow v_{t+2}$ and $v_1 \not\leftrightarrow v_{t+2}$.* Suppose the contrary. Clearly we can't have $v_1 \leftrightarrow v_{t+1}$ and have the edge go through $v_2, v_3, \dots, v_t$ (since then we get a clique of size $t+1$). Similarly, we can't have $v_2 \leftrightarrow v_{t+2}$ and have the edge go through $v_3, v_4, \dots, v_{t+1}$. So assume the edge $v_1 v_{t+2}$ exists and goes around the other way. If $v_1 \leftrightarrow v_{t+1}$, then let $G' = G \setminus \{v_1\}$ and if $v_1 \not\leftrightarrow v_{t+1}$, then let $G' = G \setminus \{v_1, v_{t+1}\}$. Now let $V_1 = \{v_2, v_3, \dots, v_t\}$ and $V_2 = V(G') \setminus V_1$. Let $K' = G[V_1]$ and $L' = G[V_2]$; note that K' and L' are each cliques of size at most $\Delta - 2$. Now for each $S \subseteq V_2$, we have $|N_{\overline{G}}(S) \cap V_1| \geq |S|$ (otherwise we get a clique of size t in G' and a clique of size

$t+1$ in G). Now by Hall's Theorem, we have a matching in $\overline{G}$ between V_1 and V_2 that saturates V_2. This implies that $G' \subseteq E_2^{\Delta-2}$, which in turn gives $G \subseteq E_2^{\Delta-1}$. By Lemma 5.3.5, G is $(\Delta-1)$-choosable, which is a contradiction.

Claim 2. *$v_{t-3} \leftrightarrow v_{t+1}$ and the edge passes through v_{t-2}, v_{t-1}, v_t.* Suppose the contrary. If $t \geq 7$, then since $t \leq \Delta - 1$, v_4 has some neighbor outside of K; by (reflectional) symmetry we could have labeled the vertices so that $v_{t-3} \leftrightarrow v_{t+1}$. So we must have $t \leq 6$. Each vertex v that is high has either at least $\lceil \Delta/2 \rceil$ clockwise neighbors or at least $\lceil \Delta/2 \rceil$ counterclockwise neighbors. This gives a clique of size $1 + \lceil \Delta/2 \rceil \geq 6$. If v_3 is high, then either v_3 has at least 4 clockwise neighbors, so $v_3 \leftrightarrow v_7$, or else v_3 has at least 6 counterclockwise neighbors, so $|K| \geq 7$. Thus, we may assume that v_3 is low; by symmetry (and our choice of labeling prior to Claim 1) v_4 is also low. Now since v_4 has only 3 counterclockwise neighbors, we get $v_4 \leftrightarrow v_7$ (in fact, we get $v_4 \leftrightarrow v_9$). Thus, $\{v_3, v_4, v_5, v_6, v_7\}$ induces $K_3 * E_2$ with a low degree vertex in both the K_3 and the E_2, which contradicts Lemma 5.3.49.

Claim 3. *$v_{t-2} \not\leftrightarrow v_{t+2}$.* Suppose the contrary. By Claim 1 the edge goes through v_{t-1}, v_t, v_{t+1}. If $v_{t-3} \leftrightarrow v_{t+2}$, then $\{v_1, v_2, v_{t-3}, v_{t-2}, v_{t-1}, v_t, v_{t+1}, v_{t+2}\}$ induces $K_4 * B$, where B is not almost complete; this contradicts Lemma 5.3.18. If $v_{t-3} \not\leftrightarrow v_{t+2}$, then we get a $K_3 * P_4$ induced by $\{v_1, v_{t-3}, v_{t-2}, v_{t-1}, v_t, v_{t+1}, v_{t+2}\}$, which contradicts Lemma 5.3.27.

Claim 4. *$v_{t-1} \not\leftrightarrow v_{t+2}$.* If not, then $\{v_1, v_{t-3}, v_{t-2}, v_{t-1}, v_t, v_{t+1}, v_{t+2}\}$ induces $K_2 *$ antichair (with v_{t-1}, v_t in the K_2), which contradicts Lemma 5.3.32.

Claim 5. *G is $(\Delta-1)$-choosable.* Let $S = \{v_{t-3}, v_{t-2}, v_{t-1}, v_t\}$. If any vertex of S is low, then $S \cup \{v_1, v_{t+1}\}$ induces $K_4 * E_2$ with a low vertex in the K_4, which contradicts Lemma 5.3.48. So all of S is high. If $v_t \not\leftrightarrow v_{t+2}$, then $\{v_t, v_{t-1}, \ldots, v_{t-\Delta+1}\}$ (subscripts are modulo n) induces K_Δ. So $v_t \leftrightarrow v_{t+2}$. Since $v_{t-1} \not\leftrightarrow v_{t+2}$ and all of

S is high, we get $v_n \in (\cap_{v \in (S \setminus \{v_t\})} N(v)) \setminus N(v_t)$. Now we must have $v_n \not\leftrightarrow v_{t+1}$ (for otherwise G is $(\Delta - 1)$-choosable, as in Claim 1). So we get $K_3 * P_4$ induced by $\{v_{t+1}, v_t, v_{t-1}, v_{t-2}, v_{t-3}, v_1, v_n\}$, which contradicts Lemma 5.3.29. $\square$

6.3 Quasi-line graphs

A graph is *quasi-line* if every vertex is bisimplicial (its neighborhood can be covered by two cliques). We apply a version of Chudnovsky and Seymour's structure theorem for quasi-line graphs from King's thesis [41]. The undefined terms will be defined after the statement.

Lemma 6.3.1. *Every connected skeletal quasi-line graph is a circular interval graph or a composition of linear interval strips.*

A *homogeneous pair of cliques* (A_1, A_2) in a graph G is a pair of disjoint nonempty cliques such that for each $i \in [2]$, every vertex in $G - (A_1 \cup A_2)$ is either joined to A_i or misses all of A_i and $|A_1| + |A_2| \geq 3$. A homogeneous pair of cliques (A_1, A_2) is *skeletal* if for any $e \in E(A, B)$ we have $\omega(G[A \cup B] - e) < \omega(G[A \cup B])$. A graph is *skeletal* if it contains no nonskeletal homogeneous pair of cliques.

Generalizaing a lemma of Chudnovsky and Fradkin [17], King proved a lemma allowing us to handle nonskeletal homogeneous pairs of cliques.

Lemma 6.3.2 (King [41])**.** *If G is a nonskeletal graph, then there is a proper subgraph G' of G such that:*

1. *G' is skeletal;*

2. *$\chi(G') = \chi(G)$;*

3. *If G is claw-free, then so is G';*

4. *If G is quasi-line, then so is G'.*

It remains to define the generalization of line graphs introduced by Chudnovsky and Seymour [18]; this is the notion of *compositions of strips* (for a more detailed introduction, see Chapter 5 of [41]). We use the modified definition from King and Reed [43]. A *strip* (H, A_1, A_2) is a claw-free graph H containing two cliques A_1 and A_2 such that for each $i \in [2]$ and $v \in A_i$, $N_H(v) - A_i$ is a clique. If H is a linear interval graph, then (H, A_1, A_2) is a *linear interval strip*. Now let H be a directed multigraph (possibly with loops) and suppose for each edge e of H we have a strip (H_e, X_e, Y_e). For each $v \in V(H)$ define

$$C_v := \left(\bigcup \{X_e \mid e \text{ is directed out of } v\}\right) \cup \left(\bigcup \{Y_e \mid e \text{ is directed into } v\}\right)$$

The graph formed by taking the disjoint union of $\{H_e \mid e \in E(H)\}$ and making C_v a clique for each $v \in V(H)$ is the composition of the strips (H_e, X_e, Y_e). Any graph formed in such a manner is called a *composition of strips*. It is easy to see that if for each strip (H_e, X_e, Y_e) in the composition we have $V(H_e) = X_e = Y_e$, then the constructed graph is just the line graph of the multigraph formed by replacing each $e \in E(H)$ with $|H_e|$ copies of e.

It will be convenient to have notation and terminology for a strip together with how it attaches to the graph. An *interval* 2*-join* in a graph G is an induced subgraph H such that:

1. H is a (nonempty) linear interval graph,
2. The ends of H are (not necessarily disjoint) cliques A_1, A_2,

3. $G - H$ contains cliques B_1, B_2 (not necessarily disjoint) such that A_1 is joined to B_1 and A_2 is joined to B_2,

4. there are no other edges between H and $G - H$.

Note that A_1, A_2, B_1, B_2 are uniquely determined by H, so we are justified in calling both H and the quintuple (H, A_1, A_2, B_1, B_2) the interval 2-join. An interval 2-join (H, A_1, A_2, B_1, B_2) is *trivial* if $V(H) = A_1 = A_2$ and *canonical* if $A_1 \cap A_2 = \emptyset$. A canonical interval 2-join (H, A_1, A_2, B_1, B_2) with leftmost vertex v_1 and rightmost vertex v_t is *reducible* if H is incomplete and $N_H(A_1) \setminus A_1 = N_H(v_1) \setminus A_1$ or $N_H(A_2) \setminus A_2 = N_H(v_t) \setminus A_2$. We call such a canonical interval 2-join reducible because we can *reduce* it as follows. Suppose H is incomplete and $N_H(A_1) \setminus A_1 = N_H(v_1) \setminus A_1$. Put $C := N_H(v_1) \setminus A_1$ and then $A_1' := C \setminus A_2$ and $A_2' := A_2 \setminus C$. Since H is not complete $v_t \in A_2'$ and hence $H' := G[A_1' \cup A_2']$ is a nonempty linear interval graph that gives the reduced canonical interval 2-join $(H', A_1', A_2', A_1 \cup (C \cap A_2), B_2 \cup (C \cap A_2)$.

Lemma 6.3.3. *If (H, A_1, A_2, B_1, B_2) is an irreducible canonical interval 2-join in a vertex critical graph G with $\chi(G) = \Delta(G) \geq 9$, then $B_1 \cap B_2 = \emptyset$ and $|A_1|, |A_2| \leq 3$. Moreover, if G is skeletal, then H is complete.*

Proof. Let (H, A_1, A_2, B_1, B_2) be an irreducible canonical interval 2-join in a vertex critical graph G with $\chi(G) = \Delta(G) \geq 9$. Put $\Delta := \Delta(G)$.

Note that, since it is vertex critical, G contains no K_Δ and in particular G has no simplicial vertices. Label the vertices of H left-to-right as $v_1, \ldots, v_t$. Say $A_1 = \{v_1, \ldots, v_L\}$ and $A_2 = \{v_R, \ldots, v_t\}$. For $v \in V(H)$, define $r(v) := \max\{i \in [t] \mid v \leftrightarrow v_i\}$ and $l(v) := \min\{i \in [t] \mid v \leftrightarrow v_i\}$. These are well-defined since $|H| \geq 2$ and H is connected by the following claim.

Claim 1. *$A_1, A_2, B_1, B_2 \neq \emptyset$, $B_1 \not\subseteq B_2$, $B_2 \not\subseteq B_1$ and H is connected.* Otherwise G would contain a clique cutset.

Claim 2. *If H is complete, then $R - L = 1$.* Suppose $V(H) \neq A_1 \cup A_2$. Then any $v \in V(H) \setminus A_1 \cup A_2$ would be simplicial in G, which is impossible. Hence $R - L = 1$.

Claim 3. *If H is not complete, then $r(v_L) = r(v_1) + 1$ and $l(v_R) = l(v_t) - 1$. In particular, v_1, v_t are low and $|A_1|, |A_2| \geq 2$.* Suppose otherwise that H is not complete and $r(v_L) \neq r(v_1) + 1$. By definition, $N_H(v_1) \subseteq N_H(v_L)$ and v_1, v_L have the same neighbors in $G \setminus H$. Hence if $r(v_L) > r(v_1) + 1$, then $d(v_L) - d(v_1) \geq 2$, impossible. So we must have $r(v_L) = r(v_1)$ and hence $N_H(A_1) \setminus A_1 = N_H(v_1) \setminus A_1$. Thus the 2-join is reducible, a contradiction. Therefore $r(v_L) = r(v_1) + 1$. Similarly, $l(v_R) = l(v_t) - 1$.

Claim 4. $|A_1|, |A_2| \leq 3$. Suppose otherwise that $|A_1| \geq 4$. First, suppose H is complete. By Claim 2, $V(H) = A_1 \cup A_2$. If v_1 is low, then for any $w_1 \in B_1 \setminus B_2$ the vertex set $\{v_1, \ldots, v_4, v_t, w_1\}$ induces a $K_4 * E_2$ violating Lemma 5.3.48. Hence v_1 is high. If $|A_2| \geq 2$ and $|B_1 \setminus B_2| \geq 2$, then for any $w_1, w_2 \in B_1 \setminus B_2$, the vertex set $\{v_1, \ldots, v_4, v_{t-1}, v_t, w_1, w_2\}$ induces a $K_4 * 2K_2$, which is impossible by Lemma 5.3.18. Hence either $|A_2| = 1$ or $|B_1 \setminus B_2| = 1$. Suppose $|A_2| = 1$. Then, since $A_1 \cup B_1$ induces a clique and $|A_1 \cup B_1| = d(v_1)$, v_1 must be low, impossible. Hence we must have $|B_1 \setminus B_2| = 1$. Thus $|B_1 \cap B_2| = |B_1| - 1$. Hence $V(H) \cup B_1 \cap B_2$ induces a clique with $|A_1| + |A_2| + |B_1| - 1 = d(v_1) = \Delta$ vertices, impossible.

Therefore H must be incomplete. By Claim 3, v_1 is low. But then as above for any $w_1 \in B_1 \setminus B_2$ the vertex set $\{v_1, \ldots, v_4, v_{L+1}, w_1\}$ induces a $K_4 * E_2$ violating Lemma 5.3.48. Hence we must have $|A_1| \leq 3$. Similarly, $|A_2| \leq 3$.

Claim 5. $R - L = 1$. Suppose otherwise that $R - L \geq 2$. Then by Claim 2, H is incomplete. Hence by Claim 3, $r(v_L) = r(v_1) + 1$, $l(v_R) = l(v_t) - 1$, v_1, v_t are low and $|A_1|, |A_2| \geq 2$.

Subclaim 5a. $L + \Delta - 2 \leq r(v_{L+1}) \leq L + \Delta - 1$. Since v_{L+1} has exactly L neighbors to the left, we have $r(v_{L+1}) \leq L + 1 + \Delta - L = \Delta + 1 \leq L + \Delta - 1$. If v_{L+1} is high, the previous computation is exact and $r(v_{L+1}) = \Delta + 1 \geq L + \Delta - 2$. Suppose v_{L+1} is low. If $L = 3$, then for some $w_1 \in B_1$ the vertex set $\{v_1, v_2, v_3, v_4, w_1\}$ induces a $K_3 * E_2$ violating Lemma 5.3.49. Hence $L = 2$ and $r(v_{L+1}) = L + 1 + \Delta - 1 - L = \Delta \geq L + \Delta - 2$.

Subclaim 5b. $L + \Delta - 2 \leq r(v_{L+2}) \leq L + \Delta$. By Subclaim 5a, $r(v_{L+2}) \geq L + \Delta - 2$. Since H contains no Δ-clique, v_{L+2} has at least 2 neighbors to the left if it is high and at least 1 neighbor to the left if it is low. Thus $r(v_{L+2}) \leq L + 2 + \Delta - 2 = L + \Delta$.

Subclaim 5c. *If v_{L+4} is high, then $l(v_{L+4}) \leq L$.* Suppose otherwise. Recall that $v_{L+1} \leftrightarrow v_{L+4}$. Then v_{L+4} has exactly 3 neighbors to the left, so $r(v_{L+4}) = L + \Delta + 1$. Consider the subgraph induced on

$$\{v_{L+1}, v_{L+2}, v_{L+4}, v_{L+5}, v_{L+6}, v_{L+7}, v_{L+9}, v_{L+10}\}.$$

By Subclaim 5a and Subclaim 5b, this induces a subgraph violating Lemma 5.3.18.

Subclaim 5d. $l(v_{L+3}) \leq L$. Suppose otherwise. Since $v_{L+1} \leftrightarrow v_{L+3}$, vertex v_{L+3} has exactly 2 neighbors to the left, so $r(v_{L+3}) \geq L + \Delta$. By Subclaim 5c, v_{L+4} is low. By Subclaim 5a, $L + \Delta - 2 \leq r(v_{L+1}) \leq L + \Delta - 1$. Therefore $\{v_{L+1}, v_{L+3}, v_{L+4}, v_{L+5}, v_{L+6}, v_{L+9}\}$ induces a $K_4 * E_2$ violating Lemma 5.3.48.

Subclaim 5e. $l(v_{L+2}) \leq L - 1$. By Subclaim 5d $r(v_L) \geq L + 3$ and hence by Claim 3, $r(v_1) \geq L + 2$. Hence $l(v_{L+2}) \leq L - 1$.

Subclaim 5f. *Claim 5 is true.* Let π be a $(\Delta-1)$-coloring of $G \setminus H$ and define a list assignment J on H by $J(v) := [\Delta-1] - \pi(N_{G\setminus H}(v))$. Then $|J(v)| \geq d_H(v) - 1$ for all $v \in V(H)$ and since v_1 is low, $|J(v_1)| \geq d_H(v_1)$. Pick $w \in B_1$. Note that $\pi(w) \notin J(v_i)$ for $i \in [L]$. Since $J(v_{L+1}) = [\Delta-1]$, we may color v_{L+1} with $\pi(w)$ to get a new list assignment J' on $H' := H - v_{L+1}$. Then, since $\pi(w) \notin J(v_i)$ for $i \in [L]$, we have $|J'(v_i)| \geq d_{H'}(v_i)$ for $i \in [L]$ and $|J'(v_1)| \geq d_{H'}(v_1)+1$. Now color the vertices of H' greedily from their lists in the order $v_t, v_{t-1}, \ldots, v_1$. Since G has no Δ-clique, we must have $N(v_t) \not\subseteq A_2 \cup B_2$, so $l(v_t) \leq R-1$. Since $l(v_R) = l(v_t) - 1$, each of $v_R, \ldots, v_t$ have at least two neighbors to the left in H'. For $L+4 \leq i < R$, since G doesn't contain K_Δ, every high v_i has at least two neighbors to the left in H' and every low v_i at least one neighbor. By Subclaim 5d, the same holds for v_{L+3} and by Subclaim 5e, it holds for v_{L+2}. Hence each vertex will have a color free to use when we encounter it, so we can complete the $(\Delta-1)$-coloring to all of G, a contradiction.

Claim 6. *$B_1 \cap B_2 = \emptyset$.* Suppose otherwise that we have $w \in B_1 \cap B_2$.

Subclaim 6a. *Each $v \in V(H)$ is low, $|B_1| = |B_2|$, $|B_1 \setminus B_2| = |B_2 \setminus B_1| = 1$, $d(v) = |A_1|+|A_2|+|B_1|-1$ for each $v \in V(H)$ and H is complete.* By Claim 5, we have $d(v) \leq |A_1|+|A_2|+|B_1|-1$ for $v \in A_1$ and $d(v) \leq |A_1|+|A_2|+|B_2|-1$ for $v \in A_2$. Also, as $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$, we have $d(w) \geq \max\{|B_1|, |B_2|\} + |A_1| + |A_2|$. So $d(w) \geq d(v)+1$ for any $v \in V(H)$. This implies that each $v \in V(H)$ is low, $|B_1| = |B_2|$, $|B_1 \setminus B_2| = |B_2 \setminus B_1| = 1$, $d(v) = |A_1|+|A_2|+|B_1|-1$ for each $v \in V(H)$ and hence H is complete.

Subclaim 6b. *$|B_1 \cap B_2| \leq 3$.* Suppose otherwise that $|B_1 \cap B_2| \geq 4$. Pick $w_1 \in B_1 \setminus B_2$, $w_2 \in B_2 \setminus B_1$ and $z_1, z_2, z_3, z_4 \in B_1 \cap B_2$. Then the set $\{z_1, z_2, z_3, z_4, w_1, w_2, v_1, v_t\}$ induces a subgraph violating Lemma 5.3.18. Hence $|B_1 \cap B_2| \leq 3$.

Subclaim 6c. *Claim 6 is true.* By Subclaim 6a and Subclaim 6b we have $3 \geq |B_1 \cap B_2| = |B_1| - 1$ and hence $|B_1| = |B_2| \leq 4$. Suppose $|A_1|, |A_2| \leq 2$. Then $\Delta - 1 = d(v_1) \leq 3 + |B_1| \leq 7$, a contradiction. Hence by symmetry we may assume that $|A_1| \geq 3$. But then for $w_1 \in B_1 \setminus B_2$, the set $\{v_1, v_2, v_3, v_t, w_1\}$ induces a $K_3 * E_2$ violating Lemma 5.3.49.

Claim 7. *If G is skeletal, then H is complete.* Suppose G is skeletal and H is incomplete. By Claim 5, $R - L = 1$. Then, by Claim 3 $r(v_L) = r(v_1) + 1$ and $l(v_R) = l(v_t) - 1$. Since v_1 is not simplicial, $r(v_1) \geq L + 1 = R$. Hence $l(v_R) = 1$ and thus $l(v_t) = 2$. Similarly, $r(v_1) = t - 1$. So, H is K_t less an edge. But (A_1, A_2) is a homogeneous pair of cliques with $|A_1|, |A_2| \geq 2$ and hence there is an edge between A_1 and A_2 that we can remove without decreasing $\omega(G[A_1 \cup A_2])$. This contradicts the fact that G is skeletal. $\square$

Lemma 6.3.4. *An interval* 2*-join in a vertex critical graph satisfying* $\chi = \Delta \geq 9$ *is either trivial or canonical.*

Proof. Let (H, A_1, A_2, B_1, B_2) be an interval 2-join in a vertex critical graph satisfying $\chi = \Delta \geq 9$. Suppose H is nontrivial; that is, $A_1 \neq A_2$. Put $C := A_1 \cap A_2$. Then $(H \setminus C, A_1 \setminus C, A_2 \setminus C, C \cup B_1, C \cup B_2)$ is a canonical interval 2-join. Reduce this 2-join until we get an irreducible canonical interval 2-join $(H', A_1', A_2', B_1', B_2')$ with $H' \trianglelefteq H \setminus C$. Since C is joined to $H - C$, it is also joined to H'. Hence $C \subseteq B_1' \cap B_2' = \emptyset$ by Lemma 6.3.3. Hence $A_1 \cap A_2 = C = \emptyset$ showing that H is canonical. $\square$

Theorem 6.3.5. *Every quasi-line graph satisfying* $\chi \geq \Delta \geq 9$ *contains a* K_Δ.

Proof. We will prove the theorem by reducing to the case of line graphs, i.e., for every strip (H, A_1, A_2) we have $A_1 = A_2$. Suppose not and choose a counterexample G minimizing $|G|$. Plainly, G is vertex critical. By Lemma 6.3.2, we may assume that G is skeletal. By Lemma 6.2.1, G is not a circular interval graph. Therefore, by Lemma 6.3.1, G is a composition of linear interval strips. Choose such a composition representation of G using the maximum number of strips.

Let (H, A_1, A_2) be a strip in the composition. Suppose $A_1 \neq A_2$. Put $B_1 := N_{G \setminus H}(A_1)$ and $B_2 := N_{G \setminus H}(A_2)$. Then (H, A_1, A_2, B_1, B_2) is an interval 2-join. Since $A_1 \neq A_2$, H is canonical by Lemma 6.3.4. Suppose H is reducible. By symmetry, we may assume that $N_H(A_1) \setminus A_1 = N_H(v_1) \setminus A_1$. But then replacing the strip (H, A_1, A_2) with the two strips $(G[A_1], A_1, A_1)$ and $(H \setminus A_1, N_H(A_1) \setminus A_1, A_2)$ gives a composition representation of G using more strips, a contradiction. Hence H is irreducible. By Lemma 6.3.3, H is complete and thus replacing the strip (H, A_1, A_2) with the two strips $(G[A_1], A_1, A_1)$ and $(G[A_2], A_2, A_2)$ gives another contradiction.

Therefore, for every strip (H, A_1, A_2) in the composition we must have $V(H) = A_1 = A_2$. Hence G is a line graph of a multigraph. But this is impossible by Lemma 6.1.7. □

6.4 Claw-free graphs

In this section we reduce the Borodin-Kostochka conjecture for claw-free graphs to the case of quasi-line graphs. We first show that a certain graph cannot appear in the neighborhood of any vertex in our counterexample.

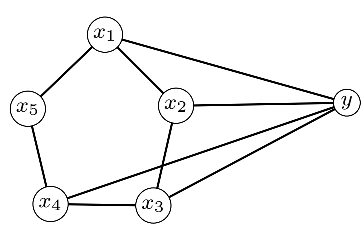


Figure 6.1: The graph N_6.

Lemma 6.4.1. *The graph $K_1 * N_6$ where N_6 is the graph in Figure 6.1 is d_1-choosable.*

Proof. Suppose not and let L be a minimal bad d_1-assignment on $K_1 * N_6$. Then, by the Small Pot Lemma, $|Pot(L)| \leq 6$. Let v be the vertex in the K_1. Note that $|L(v)| = 5$, $|L(y)| = 4$, $|L(x_5)| = 2$, and $|L(x_i)| = 3$ for all $i \in [4]$. Since $\sum_{i=1}^{5} |L(x_i)| = 14 > |Pot(L)|\omega(C_5)$, we see that two nonadjacent x_i's have a common color. Hence, by Lemma 5.3.15, we have $|Pot(L)| \leq 5$. Thus we have $c \in L(y) \cap L(x_5)$. Also, $L(x_1) \cap L(x_4) \neq \emptyset$, $L(x_1) \cap L(x_3) \neq \emptyset$ and $L(x_2) \cap L(x_4) \neq \emptyset$. By Lemma 5.3.14, the common color in all of these sets must be c. Hence c is in all the lists.

Now consider the list assignment L' where $L'(z) = L(z) - c$ for all $z \in N_6$. Then $|Pot(L')| = 4$ and since $\sum_{i=1}^{5} |L'(x_i)| = 9 > |Pot(L')|\omega(C_5)$, we see that that nonadjacent x_i's have a common color different than c. Now appling Lemma 5.3.14 gives a final contradiction. $\square$

By a *thickening* of a graph G, we just mean a graph formed by replacing each $x \in V(G)$ by a complete graph T_x such that $|T_x| \geq 1$ and for $x, y \in V(G)$, T_x is joined to T_y iff $x \leftrightarrow y$.

Lemma 6.4.2. *Any graph H with $\alpha(H) \leq 2$ such that every induced subgraph of $K_1 * H$ is not d_1-choosable can either be covered by two cliques or is a thickening of C_5.*

Proof. Suppose not and let H be a counterexample.

Claim 1. *$K_1 * H$ is d_0-choosable.* Otherwise $K_1 * H$ is a Gallai tree with a universal vertex. Since $\alpha(H) \leq 2$, $K_1 * H$ has at most two blocks and they must be complete. Hence H can be covered by two cliques, a contradiction. Claim 1 will allow us to apply Lemma 5.3.14 below.

Claim 2. *H contains an induced C_4 or an induced C_5.* Suppose not. Then H must be chordal since $\alpha(H) \leq 2$. In particular, H contains a simplicial vertex

x. But then $\{x\} \cup N_H(x)$ and $V(H) - N_H(x) - \{x\}$ are two cliques covering H, a contradiction.

Claim 3. *H does not contain an induced C_5 together with a vertex joined to at least 4 vertices in the C_5.* Suppose the contrary. If the vertex is joined to all of the C_5, then we have a induced $K_2 * C_5$, which is d_1-choosable by Lemma 5.3.47. If the vertex is joined to only four vertices in the C_5, we have an induced $K_1 * N_6$, impossible by Lemma 6.4.1.

Claim 4. *H contains no induced C_4.* Suppose otherwise that H contains an induced C_4, say $x_1x_2x_3x_4x_1$. Put $R := V(H) - \{x_1, x_2, x_3, x_4\}$. Let $y \in R$. As $\alpha(H) \leq 2$, y has a neighbor in $\{x_1, x_3\}$ and a neighbor in $\{x_2, x_4\}$. If y is adjacent to all of $x_1, \dots, x_4$, then $K_1 * H$ contains $K_2 * C_4$ which is d_1-choosable, impossible. If y is adjacent to three of $x_1, \dots, x_4$, then $K_1 * H$ contains $E_2 * \text{paw}$ which is d_1-choosable, impossible.

Thus every $y \in R$ is adjacent to all and only the vertices on one side of the C_4. We show that any two vertices in R must be adjacent to the same or opposite side and this gives the desired covering by two cliques. If this doesn't happen, then by symmetry we may suppose we have $y_1, y_2 \in R$ such that $y_1 \leftrightarrow x_1, x_2$ and $y_2 \leftrightarrow x_2, x_3$. We must have $y_1 \leftrightarrow y_2$ for otherwise $\{y_1, y_2, x_4\}$ is an independent set. But now $x_1y_1y_2x_3x_4x_1$ is an induced C_5 in which x_2 has 4 neighbors, impossible by Claim 3.

Claim 5. *H does not exist.* By Claim 2 and Claim 4, H contains an induced C_5. That H is a thickening of this C_5 is now is immediate from $\alpha(H) \leq 2$ and Claim 3. This final contradiction completes the proof. $\square$

Lemma 6.4.3. *The graph D_8 is d_1-choosable.*

Proof. Suppose not and let L be a minimal bad d_1-assignment on $G := D_8$.

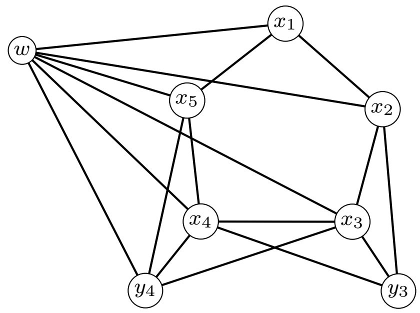


Figure 6.2: The graph D_8.

Claim 1. $|Pot(L)| \leq 6$. By the Small Pot Lemma, we know that $|Pot(L)| \leq 7$. Suppose $|Pot(L)| = 7$. Say $Pot(L) - L(w) = \{a, b\}$.

We must have $L(y_3) = \{a, b\}$. Otherwise we could color y_3 from $L(y_3) - \{a, b\}$ and note that $G - y_3 - w$ is d_0-choosable and hence has a coloring from its lists. Then we can easily modify this coloring to use both a and b at least once. But now we can color w.

If there exist distinct vertices $u, v \in V(G) - y_3$ such that $a \in L(u)$, $b \in L(v)$ and $\{u, v\} \not\subseteq \{x_2, x_3, x_4\}$, then we can color G as follows. Color y_3 arbitrarily to leave a available on u and b available on v. Again, $G - y_3 - w$ has a coloring. We can modify it to use a and b, then color w. Thus, a and b each appear only on some subset of $\{y_3, x_2, x_3, x_4\}$.

If $a \in L(x_2) \cap L(x_4)$, then we use a on x_2 and x_4 and color greedily y_3, x_3, y_4, x_1, x_5, w (actually any order will work if y_3 is first and w is last). If a appears only on y_3 and exactly one neighbor x_i, then we violate Lemma 5.1.2 since $|Pot_{y_3, x_i}(L)| < 7$. So now a appears precisely on either y_3, x_2, x_3 or y_3, x_3, x_4. Similarly b appears precisely on either y_3, x_2, x_3 or y_3, x_3, x_4.

If $\{a, b\} \cap L(x_2) = \emptyset$, then we use a on y_3 and b on x_3, then greedily color y_4, x_4, x_5, x_1, w, x_2. By symmetry, we may assume that $a \in L(x_2)$. But then since $\{a, b\} \subseteq L(x_3)$ we have $|Pot_{y_3,x_2,x_3}(L)| < 7$ violating Lemma 5.1.2. Hence $|Pot(L)| \leq 6$.

Claim 2. $|Pot(L)| \leq 5$. Suppose $|Pot(L)| = 6$. Choose $a \in Pot(L) - L(w)$ and $b \in L(w) \cap L(y_3)$. Put $H := G - y_3 - w$.

First we show that $b \in L(x_2) \cap L(x_3) \cap L(x_4)$. If not, we use b on y_3 and w, then greedily color x_1, x_5, y_4. Now we can finish by coloring last the x_i such that $b \notin L(x_i)$.

We must have $a \in L(y_3)$ or else we color x_2, x_4 with b and something else in H with a (since G_a contains an edge by Lemma 5.1.2) and finish. Now $a \notin L(x_1), L(x_5), L(y_4)$, for otherwise we color x_2, x_4 with b, y_3 with a and then color x_1, x_5, y_4, x_3 in order using a when we can, then color w. Now a is on y_3 and at least two of x_2, x_3, x_4 or else we violate Lemma 5.1.2. Now $a \notin L(x_2) \cap L(x_4)$ since otherwise we color x_2, x_4 with a, then y_3 with b, then greedily color x_1, x_5, y_4, x_3, w. Also $a \notin L(x_2) \cap L(x_3)$ since then $\{a, b\} \subseteq L(y_3) \cap L(x_2) \cap L(x_3)$ and hence $|Pot_{y_3,x_2,x_3}(L)| < 6$ violating Lemma 5.1.2. Therefore $V(G_a) = \{y_3, x_3, x_4\}$.

Now $|Pot_{y_3,x_3,x_4}|(L) \leq 6$ and hence $L(x_3) \cap L(x_4) = \{a, b\}$ for otherwise we violate Lemma 5.1.2. Say $L(x_3) = \{a, b, c, d\}$ and $L(x_4) = \{a, b, e, f\}$. Then by symmetry $L(x_1)$ contains either c or e. If $c \in L(x_1)$, color x_1, x_3 with c, x_4 with a and y_3 with b. Now we can greedily finish. If $e \in L(x_1)$, color x_1, x_4 with e, x_3 with a and y_3 with b, again we can greedily finish. Hence $|Pot(L)| \leq 5$.

Claim 3. *L does not exist.* Since $|Pot(L)| \leq 5$ we see that x_3, x_5 have two colors in common and x_2, x_4 have two colors in common as well. In fact, these sets of common colors must be the same and equal $L(y_3) := \{a, b\}$ or we can finish the coloring. Similarly, we may assume that $a \in L(y_4)$ (if $\{a, b\} \cap L(y_4) = \emptyset$, then we have $L(x_2) \cap L(y_3) \cap (Pot(L) \setminus \{a, b\}) \neq \emptyset$ and color a on x_3, x_5, so we can color y_3

with b and then finish by Lemma 5.3.14). Similarly, $L(x_1)$ contains a or b. But it can't contain a for then we could color y_3, y_4, x_1 with a, and x_2, x_4 with b, and then finish greedily. Say $L(x_4) = \{a, b, c, d\}$. Then as no nonadjacent pair has a color in common that is in $Pot(L) - \{a, b\}$ we have $L(x_2) = \{a, b, e\}$, then by symmetry of c and d we have $L(x_5) = \{a, b, c\}$. Then $L(x_3) = \{a, b, d, e\}$ and hence $L(x_1) = \{a, b\}$, which contradicts that $a \notin L(x_1)$. We conclude that L cannot exist. □

Lemma 6.4.4. *Let H be a thickening of C_5 such that $|H| \geq 6$. Then $K_1 * H$ is f-choosable where $f(v) \geq d(v)$ for the v in the K_1 and $f(x) \geq d(x) - 1$ for $x \in V(H)$.*

Proof. Suppose not and let L be a minimal bad f-assignment on $K_1 * H$. By the Small Pot Lemma, $|Pot(L)| \leq |H|$. Note that H is d_0-choosable since it contains an induced diamond. Let $x_1, \ldots, x_5$ be the vertices of an induced C_5 in H. Then $\sum_i |L(x_i)| = \sum_i d_H(x_i) = 3\,|H| - 5 > 2\,|H| \geq \omega(H[x_1, \ldots, x_5])\,|Pot(L)|$ and hence some nonadjacent pair in $\{x_1, \ldots, x_5\}$ have a color in common. Now applying Lemma 5.3.16 gives a contradiction. □

We are now in a position to finish the proof of Borodin-Kostochka for claw-free graphs.

Theorem 6.4.5. *Every claw-free graph satisfying $\chi \geq \Delta \geq 9$ contains a K_Δ.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Then G is vertex critical and not quasi-line by Lemma 6.3.5. Hence G contains a vertex v that is not bisimplicial. By Lemma 6.4.2, $G_v := G[N(v)]$ is a thickening of a C_5. Also, by Lemma 6.4.4, v is high. Pick a C_5 in G_v and label its vertices $x_1, \ldots, x_5$ in clockwise order. For $i \in [5]$, let T_i be the thickening clique containing x_i. Also, let S be those vertices in $V(G) - N(v) - \{v\}$ that have a neighbor in $\{x_1, \ldots, x_5\}$. First we establish a few properties of vertices in S.

Claim 1. *For each $z \in S$ there is $i \in [5]$ such that $N(z) \cap \{x_1, \ldots, x_5\} \in \{\{x_i, x_{i+1}\}, \{x_i, x_{i+1}, x_{i+2}\}\}$.* Let $z \in S$ and put $N := N(z) \cap \{x_1, \ldots, x_5\}$. If $|N| \geq 4$, then some subset of $\{v, z\} \cup N$ induces the d_1-choosable graph $E_2 * P_4$. Hence $|N| \leq 3$. Since G is claw-free, the vertices in N must be contiguous.

Claim 2. *If $z \in S$ is adjacent to x_i, x_{i+1}, x_{i+2}, then $|T_i| = |T_{i+1}| = |T_{i+2}| = 1$.* Suppose not. First, lets deal with the case when $|T_{i+1}| \geq 2$. Pick $y \in T_{i+1} - x_{i+1}$. If $y \not\leftrightarrow z$, then $\{x_i, y, z, x_{i-1}\}$ induces a claw, impossible. Thus $y \leftrightarrow z$ and $\{v, z, x_i, x_{i+1}, x_{i+2}, y\}$ induces the d_1-choosable graph $E_2 * \text{diamond}$.

Hence, by symmetry, we may assume that $|T_i| \geq 2$. Now, if $y \not\leftrightarrow z$, then $\{v, x_1, \ldots, x_5, y, z\}$ induces a D_8 contradicting Lemma 6.4.3. Hence $y \leftrightarrow z$ and $\{v, z, x_i, x_{i+1}, x_{i+2}, y\}$ induces the d_1-choosable graph $E_2 * \text{paw}$, a contradiction.

Claim 3. *For $i \in [5]$, let B_i be the $z \in S$ with $N(z) \cap \{x_1, \ldots, x_5\} = \{x_i, x_{i+1}\}$. Then $B_i \cup B_{i+1}$ and $B_i \cup T_i \cup T_{i+1}$ both induce cliques for any $i \in [5]$.* Otherwise there would be a claw.

Claim 4. *$|T_i| \leq 2$ for all $i \in [5]$.* Suppose otherwise that we have i such that $|T_i| \geq 3$. Put $A_i := N(x_i) \cap S$. By Claim 2, $A_i \subseteq B_{i-1} \cup B_i$ and A_i is joined to T_i. Thus T_i is joined to $F_i := \{v\} \cup A_i \cup T_{i-1} \cup T_{i+1}$. If $A_i \neq \emptyset$, then F_i induces a graph that is connected and not almost complete, so this is impossible by Lemma 5.3.29. If $A_i = \emptyset$, then x_i must have at least $\Delta - 2$ neighbors in $T_{i-1} \cup T_i \cup T_{i+1}$. But that leaves at most one vertex for $T_{i-2} \cup T_{i+2}$, impossible.

Claim 5. *G does not exist.* Since $d(v) = \Delta \geq 9$, by symmetry we may assume that $|T_i| = 2$ for all $i \in [4]$. As in the proof of Claim 4, we get that T_2 is joined to F_2. Since $|T_i| \leq 2$ for all i, we must have $A_i \neq \emptyset$ (for all i, but in particular for A_2). Since $A_i \subseteq B_{i-1} \cup B_i$, by symmetry, we may assume that $A_2 \cap B_2 \neq \emptyset$.

Pick $z \in A_2 \cap B_2$ and $y_i \in T_i - x_i$ for $i \in [3]$. Then F_2 has the graph in Figure 6.3 as an induced subgraph, but this is impossible by Lemma 5.3.47. □

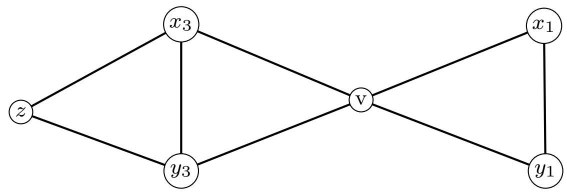


Figure 6.3: K_2 joined to this graph is d_1-choosable

We note that this reduction to the quasi-line case also works for the Borodin-Kostochka conjecture for list coloring; that is, we have the following result.

Theorem 6.4.6. *If every quasi-line graph satisfying $\chi_l \geq \Delta \geq 9$ contains a K_Δ, then the same statement holds for every claw-free graph.*

Chapter 7

MULES

The material in this chapter appeared in [20] and is joint work with Dan Cranston.

In this we chapter carry out an in-depth study of minimum counterexamples to the Borodin-Kostochka conjecture. Our main tool is the classification, in Chapter 5, of graph joins $A * B$ with $|A| \geq 2$, $|B| \geq 2$ which are f-choosable, where $f(v) := d(v) - 1$ for each vertex v. Since such a join cannot be an induced subgraph of a vertex critical graph with $\chi = \Delta$, we have a wealth of structural information about minimum counterexamples to the Borodin-Kostochka conjecture. In Section 7.2, we exploit this information and minimality to improve Reed's Lemma 1.1.11 as follows (see Corollary 7.2.10).

Lemma 7.0.7. *Let G be a vertex critical graph satisfying $\chi = \Delta \geq 9$ having the minimum number of vertices. If H is a $K_{\Delta-1}$ in G, then any vertex in $G - H$ has at most 1 neighbor in H.*

Moreover, we lift the result out of the context of a minimum counterexample to the Borodin-Kostochka conjecture, to the more general context of graphs satisfying a certain criticality condition—we call such graphs mules. This allows us to prove meaningful results for values of Δ less than 9.

Since a graph containing K_Δ as a subgraph also contains $K_{t,\Delta-t}$ as a subgraph for any $t \in [\Delta - 1]$, the Borodin-Kostochka conjecture implies the following conjecture. Our main result in this chapter is that the two conjectures are equivalent.

Conjecture 7.0.8. *Any graph with $\chi = \Delta \geq 9$ contains some $A_1 * A_2$ as an induced subgraph where $|A_1|, |A_2| \geq 3$, $|A_1| + |A_2| = \Delta$ and $A_i \neq K_1 + K_{|A_i|-1}$ for some $i \in [2]$.*

In fact, using Kostochka's reduction (Lemma 4.2.2) to the case $\Delta = 9$, the following conjecture is also equivalent.

Conjecture 7.0.9. *Any graph with $\chi = \Delta = 9$ contains some $A_1 * A_2$ as an induced subgraph where $|A_1|, |A_2| \geq 3$, $|A_1| + |A_2| = 9$ and $A_i \neq K_1 + K_{|A_i|-1}$ for some $i \in [2]$.*

As a special case, we get a couple more palatable equivalent conjectures (see Lemma 7.2.17 and the comment following it).

Conjecture 7.0.10. *Any graph with $\chi = \Delta \geq 9$ contains $K_3 * E_{\Delta-3}$ as a subgraph.*

Conjecture 7.0.11. *Any graph with $\chi = \Delta = 9$ contains $K_3 * E_6$ as a subgraph.*

The condition $A_i \neq K_1 + K_{|A_i|-1}$ is unnatural and by removing it we get a (possibly) weaker conjecture than the Borodin-Kostochka conjecture which has more aesthetic appeal.

Conjecture 7.0.12. *Let G be a graph with $\Delta(G) = k \geq 9$. If $K_{t,k-t} \not\subseteq G$ for all $3 \leq t \leq k-3$, then G can be $(k-1)$-colored.*

Conjecture 7.0.13. *Conjecture 7.0.12 is equivalent to the Borodin-Kostochka conjecture.*

Perhaps it would be easier to attack Conjecture 7.0.12 with $3 \leq t \leq k-3$ replaced by $2 \leq t \leq k-2$? We are unable to prove even this conjecture. Making this change and bringing k down to 5 gives the following conjecture, which, if true, would imply the remaining two cases of Grünbaum's girth problem for graphs with girth at least five.

Conjecture 7.0.14. *Let G be a graph with $\Delta(G) = k \geq 5$. If $K_{t,k-t} \not\subseteq G$ for all $2 \leq t \leq k-2$, then G can be $(k-1)$-colored.*

If G is a graph with with $\Delta(G) = k \geq 5$ and girth at least five, then it contains no $K_{t,k-t}$ for all $2 \leq t \leq k-2$ and hence Conjecture 7.0.14 would give a $(k-1)$-coloring. This conjecture would be tight since the Grünbaum graph and the Brinkmann graph are examples with $\chi = \Delta = 4$ and girth at least five.

Finally, we prove that the following conjecture is equivalent to the Borodin-Kostochka conjecture for graphs with independence number at most 6 (see Theorem 7.2.23).

Conjecture 7.0.15. *Every graph satisfying $\chi = \Delta = 9$ and $\alpha \leq 6$ contains a K_8.*

7.1 What is a mule?

Definition 7.1.1. If G and H are graphs, an *epimorphism* is a graph homomorphism $f\colon G \twoheadrightarrow H$ such that $f(V(G)) = V(H)$. We indicate this with the arrow $\twoheadrightarrow$.

Definition 7.1.2. Let G be a graph. A graph A is called a *child* of G if $A \neq G$ and there exists $H \trianglelefteq G$ and an epimorphism $f\colon H \twoheadrightarrow A$.

Note that the child-of relation is a strict partial order on the set of (finite simple) graphs $\mathcal{G}$. We call this the *child order* on $\mathcal{G}$ and denote it by '$\prec$'. By definition, if $H \triangleleft G$ then $H \prec G$.

Lemma 7.1.1. *The ordering $\prec$ is well-founded on $\mathcal{G}$; that is, every nonempty subset of $\mathcal{G}$ has a minimal element under $\prec$.*

Proof. Let $\mathcal{T}$ be a nonempty subset of $\mathcal{G}$. Pick $G \in \mathcal{T}$ minimizing $|G|$ and then maximizing $\|G\|$. Since any child of G must have fewer vertices or more edges (or both), we see that G is minimal in $\mathcal{T}$ with respect to $\prec$. $\square$

Definition 7.1.3. Let $\mathcal{T}$ be a collection of graphs. A minimal graph in $\mathcal{T}$ under the child order is called a *$\mathcal{T}$-mule*.

With the definition of mule we have captured the important properties (for coloring) of a counterexample first minimizing the number of vertices and then maximizing the number of edges. Viewing $\mathcal{T}$ as a set of counterexamples, we can add edges to or contract independent sets in induced subgraphs of a $\mathcal{T}$-mule and get a non-counterexample. We could do the same with a minimal counterexample, but with mules we have more minimal objects to work with. One striking consequence of this is that many of our proofs naturally construct multiple counterexamples to Borodin-Kostochka for small Δ.

7.2 Excluding induced subgraphs in mules

Our main goal in this section is to prove Lemma 7.2.11, which says that (with only one exception) for $k \geq 7$, no k-mule contains $K_4 * E_{k-4}$ as a subgraph. This result immediately implies that the Borodin-Kostochka conjecture is equivalent to Conjecture 7.2.12. This equivalence is a major step toward our main result. Our approach is based on Lemma 5.3.25, which implies that if G is a counterexample to Lemma 7.2.11, then the vertices of the E_{k-4} induce either E_3, a claw, a clique, or an almost complete graph. Our job in this section consists of showing that each of these four possibilities is, in fact, impossible. Ruling out the clique is easy. The cases of E_3 and the claw are handled in Lemma 7.2.7, and the case of an almost complete graph (which requires the most work) is handled by Corollary 7.2.10.

For $k \in \mathbb{N}$, by a *k-mule* we mean a $\mathcal{C}_k$-mule.

Lemma 7.2.1. *Let G be a k-mule with $k \geq 4$. If A is a child of G with $\Delta(A) \leq k$ then either*

- *A is $(k-1)$-colorable; or,*
- *A contains a K_k.*

Proof. Let A be a child of G with $\Delta(A) \leq k$, $H \trianglelefteq G$ and $f\colon H \twoheadrightarrow A$ an epimorphism. Without loss of generality, A is vertex critical. Suppose A is not $(k-1)$-colorable. Then $\chi(A) \geq k \geq \Delta(A)$. Since $A \prec G$ and G is a mule, $A \notin \mathcal{C}_k$. Thus we have $\chi(A) > \Delta(A) \geq 3$, so Brooks' theorem implies that $A = K_k$. □

Note that adding edges to a graph yields an epimorphism.

Lemma 7.2.2. *Let G be a k-mule with $k \geq 4$ and $H \trianglelefteq G$. Assume $x, y \in V(H)$, $xy \notin E(H)$ and both $d_H(x) \leq k-1$ and $d_H(y) \leq k-1$. If for every $(k-1)$-coloring π of H we have $\pi(x) = \pi(y)$, then H contains $\{x, y\} * K_{k-2}$.*

Proof. Suppose that for every $(k-1)$-coloring π of H we have $\pi(x) = \pi(y)$. Using the inclusion epimorphism $f_{xy}\colon H \twoheadrightarrow H + xy$ in Lemma 7.2.1 shows that either $H + xy$ is $(k-1)$-colorable or $H + xy$ contains a K_k. Since a $(k-1)$-coloring of $H + xy$ would induce a $(k-1)$-coloring of H with x and y colored differently, we conclude that $H + xy$ contains a K_k. But then H contains $\{x, y\} * K_{k-2}$ and the proof is complete. □

We will often begin by coloring some subgraph H of our graph G, and work to extend this partial coloring. More formally, let G be a graph and $H \triangleleft G$. For $t \geq \chi(H)$, let π be a proper t-coloring of H. For each $x \in V(G - H)$, put $L_\pi(x) := \{1, \ldots, t\} - \bigcup_{y \in N(x) \cap V(H)} \pi(y)$. Then π is completable to a t-coloring of G iff L_π admits a coloring of $G - H$. We will use this fact repeatedly in the proofs that follow. The following generalizes a lemma due to Reed [67], the proof is essentially the same.

Lemma 7.2.3. *For $k \geq 6$, if a k-mule G contains an induced $E_2 * K_{k-2}$, then G contains an induced $E_3 * K_{k-2}$.*

Proof. Suppose G is a k-mule containing an induced $E_2 * K_{k-2}$, call it F. Let x, y be the vertices of degree $k-2$ in F and $C := \{w_1, \dots, w_{k-2}\}$ the vertices of degree $k-1$ in F. Put $H := G - F$. Since G is vertex critical, we may $k-1$ color H. Doing so leaves a list assignment L on F with $|L(z)| \geq d_F(z) - 1$ for each $z \in V(F)$. Now $|L(x)| + |L(y)| \geq d_F(x) + d_F(y) - 2 = 2k - 6 > k - 1$ since $k \geq 6$. Hence we have $c \in L(x) \cap L(y)$. Coloring both x and y with c leaves a list assignment L' on C with $|L'(w_i)| \geq k - 3$ for each $1 \leq i \leq k - 2$. Now, if $|L'(w_i)| \geq k - 2$ or $L'(w_i) \neq L'(w_j)$ for some i, j, then we can complete the partial $(k-1)$-coloring to all of G using Hall's Theorem. Hence we must have $d(w_i) = k$ and $L'(w_i) = L'(w_j)$ for all i, j. Let $N := \bigcup_{w \in C} N(w) \cap V(H)$ and note that N is an independent set since it is contained in a single color class in every $(k-1)$-coloring of H. Also, each $w \in C$ has exactly one neighbor in N.

Proving that $|N| = 1$ will give the desired $E_3 * K_{k-2}$ in G. Thus, to reach a contradiction, suppose that $|N| \geq 2$.

We know that H has no $(k-1)$-coloring in which two vertices of N get different colors since then we could complete the partial coloring as above. Let $v_1, v_2 \in N$ be different. Since both v_1 and v_2 have a neighbor in F, we may apply Lemma 7.2.2 to conlcude that $\{v_1, v_2\} * K_{v_1,v_2}$ is in H, where K_{v_1,v_2} is a K_{k-2}.

First, suppose $|N| \geq 3$, say $N = \{v_1, v_2, v_3\}$. We have $z \in K_{v_1,v_2} \cap K_{v_1,v_3}$ for otherwise $d(v_1) \geq 2(k-2) > k$. Since z already has k neighbors among $K_{v_1,v_2} - \{z\}$ and v_1, v_2, v_3, we must have $K_{v_1,v_3} = K_{v_1,v_2}$. But then $\{v_1, v_2, v_3\} + K_{v_1,v_2}$ is our desired $E_3 * K_{k-2}$ in G.

Hence we must have $|N| = 2$, say $N = \{v_1, v_2\}$. For $i \in [2]$, v_i has $k-2$ neighbors in K_{v_1,v_2} and thus at most two neighbors in C. Hence $|C| \leq 4$. Thus we must have $k = 6$.

We may apply the same reasoning to $\{v_1, v_2\} * K_{v_1,v_2}$ that we did to F to get vertices $v_{2,1}, v_{2,2}$ such that $\{v_{2,1}, v_{2,2}\} * K_{v_{2,1},v_{2,2}}$ is in G. But then we may do it again with $\{v_{2,1}, v_{2,2}\} * K_{v_{2,1},v_{2,2}}$ and so on. Since G is finite, at some point this process must terminate. But the only way to terminate is to come back around and use x and y. This graph is 5-colorable since we may color all the E_2's with the same color and then 4-color the remaining K_4 components. This final contradiction completes the proof. □

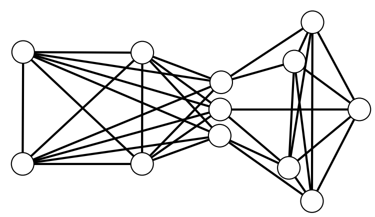


Figure 7.1: The mule $M_{6,1}$.

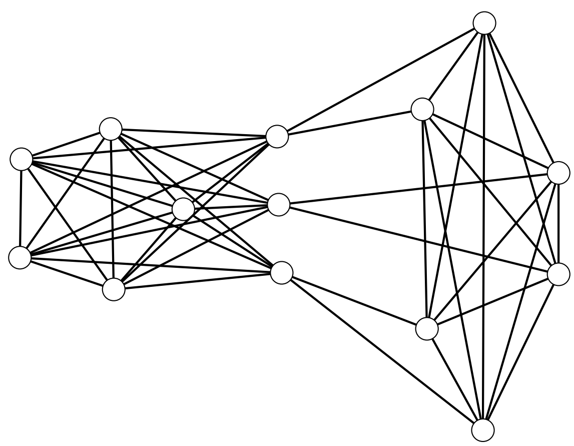


Figure 7.2: The mule $M_{7,1}$.

Lemma 7.2.4. *For $k \geq 6$, the only k-mules containing an induced $E_2 * K_{k-2}$ are $M_{6,1}$ and $M_{7,1}$.*

Proof. Suppose we have a k-mule G that contains an induced $E_2 * K_{k-2}$. Then by Lemma 7.2.3, G contains an induced $E_3 * K_{k-2}$, call it F.

Let x, y, z be the vertices of degree $k-2$ in F and let $C := \{w_1, \ldots, w_{k-2}\}$ be the vertices of degree k in F. Put $H := G - C$. Since each of x, y, z have degree at most 2 in H and G is a mule, the homomorphism from H sending x, y, and z to the same vertex must produce a K_k. Thus we must have $k \leq 7$ and H contains a K_{k-1} (call it D) such that $V(D) \subseteq N(x) \cup N(y) \cup N(z))$. Put $A := G[V(F) \cup V(D)]$. Then A is k-chromatic and as G is a mule, we must have $G = A$. If $k = 7$, then $G = M_{7,1}$. Suppose $k = 6$ and $G \neq M_{6,1}$. Then one of x, y, or z has only one neighbor in D. By symmetry we may assume it is x. But we can add an edge from x to a vertex in D to form $M_{6,1}$ and hence G has a proper child, which is impossible. □

Lemma 7.2.5. *Let G be a k-mule with $k \geq 6$ other than $M_{6,1}$ and $M_{7,1}$ and let $H \lhd G$. If $x, y \in V(H)$ and both $d_H(x) \leq k-1$ and $d_H(y) \leq k-1$, then there exists a $(k-1)$-coloring π of H such that $\pi(x) \neq \pi(y)$.*

Proof. Suppose $x, y \in V(H)$ and both $d_H(x) \leq k-1$ and $d_H(y) \leq k-1$. First, if $xy \in E(H)$ then any $(k-1)$-coloring of H will do. Otherwise, if for every $(k-1)$-coloring π of H we have $\pi(x) = \pi(y)$, then by Lemma 7.2.2, H contains $\{x, y\} * K_{k-2}$. The lemma follows since this is impossible by Lemma 7.2.4. □

Lemma 7.2.6. *Let G be a k-mule with $k \geq 6$ other than $M_{6,1}$ and $M_{7,1}$ and let $F \lhd G$. Put $C := \{v \in V(F) \mid d(v) - d_F(v) \leq 1\}$. At least one of the following holds:*

- *$G - F$ has a $(k-1)$-coloring π such that for some $x, y \in C$ we have $L_\pi(x) \neq L_\pi(y)$; or,*
- *$G - F$ has a $(k-1)$-coloring π such that for some $x \in C$ we have $|L_\pi(x)| = k-1$; or,*
- *there exists $z \in V(G - F)$ such that $C \subseteq N(z)$.*

Proof. Put $H := G - F$. Suppose that for every $(k-1)$-coloring π of H we have $L_\pi(x) = L_\pi(y)$ for every $x, y \in C$. By assumption, the vertices in C have at most one neighbor in H. If some $v \in C$ has no neighbors in H, then for any $(k-1)$-coloring π of H we have $|L_\pi(v)| = k - 1$. Thus we may assume that every $v \in C$ has exactly one neighbor in H.

Let $N := \bigcup_{w \in C} N(w) \cap V(H)$. Suppose $|N| \geq 2$. Pick different $z_1, z_2 \in N$. Then, by Lemma 7.2.5, there is a $(k-1)$-coloring π of H for which $\pi(z_1) \neq \pi(z_2)$. But then $L_\pi(x) \neq L_\pi(y)$ for some $x, y \in C$ giving a contradiction. Hence $N = \{z\}$ and thus $C \subseteq N(z)$. □

By Lemma 5.3.19, no graph in $\mathcal{C}_k$ contains an induced $E_3 * K_{k-3}$ for $k \geq 9$. For mules, we can improve this as follows.

Lemma 7.2.7. *For $k \geq 7$, the only k-mule containing an induced $E_3 * K_{k-3}$ is $M_{7,1}$.*

Proof. Suppose the lemma is false and let G be a k-mule, other than $M_{7,1}$, containing such an induced subgraph F. Let $z_1, z_2, z_3 \in F$ be the vertices with degree $k-3$ in F and C the rest of the vertices in F (all of degree $k-1$ in F). Put $H := G - F$.

First suppose there is not a vertex $x \in V(H)$ which is adjacent to all of C. Let π be a $(k-1)$-coloring of H guaranteed by Lemma 7.2.6 and put $L := L_\pi$. Since $|L(z_1)| + |L(z_2)| + |L(z_3)| \geq 3(k-4) > k-1$ we have $1 \leq i < j \leq 3$ such that $L(z_i) \cap L(z_j) \neq \emptyset$. Without loss of generality, $i = 1$ and $j = 2$. Pick $c \in L(z_1) \cap L(z_2)$ and color both z_1 and z_2 with c. Let L' be the resulting list assignment on $F - \{z_1, z_2\}$. Now $|L'(z_3)| \geq k-4$ and $|L'(v)| \geq k-3$ for each $v \in C$. By our choice of π, either two of the lists in C differ or for some $v \in C$ we have $|L'(v)| \geq k-2$. In either case, we can complete the $(k-1)$-coloring to all of G by Hall's Theorem.

Hence we must have $x \in V(H)$ which is adjacent to all of C. Thus G contains the induced subgraph $K_{k-3} * G[z_1, z_2, z_3, x]$. Therefore $k = 7$ and x is adjacent to each of z_1, z_2, z_3 by Lemma 5.3.25. Hence G contains the induced subgraph $K_5 * E_3$ contradicting Lemma 7.2.4. □

Lemma 7.2.8. *For $k \geq 7$, no k-mule contains an induced $\overline{P_3} * K_{k-3}$.*

Proof. Suppose the lemma is false and let G be a k-mule containing such an induced subgraph F. Note that $M_{7,1}$ has no induced $\overline{P_3} * K_{k-3}$, so $G \neq M_{7,1}$. Let $z \in V(F)$ be the vertex with degree $k-3$ in F, $v_1, v_2 \in F$ the vertices of degree $k-2$ in F and C the rest of the vertices in F (all of degree $k-1$ in F). Put $H := G - F$.

First suppose there is not a vertex $x \in V(H)$ which is adjacent to all of C. Let π be a $(k-1)$-coloring of H guaranteed by Lemma 7.2.6 and put $L := L_\pi$. Then, we have $|L(z)| \geq k-4$ and $|L(v_1)| \geq k-3$. Since $k \geq 7$, $|L(z)|+|L(v_1)| \geq 2k-7 > k-1$. Hence, by Lemma 5.2.8, we may color z and v_1 the same. Let L' be the resulting list assignment on $F - \{z, v_1\}$. Now $|L'(v_2)| \geq k-4$ and $|L'(v)| \geq k-3$ for each $v \in C$. By our choice of π, either two of the lists in C differ or for some $v \in C$ we have $|L'(v)| \geq k-2$. In either case, we can complete the $(k-1)$-coloring to all of G by Hall's Theorem.

Hence we must have $x \in V(H)$ which is adjacent to all of C. Thus G contains the induced subgraph $K_4 * G[z, v_1, v_2, x]$. By Lemma 5.3.25, $G[z, v_1, v_2, x]$ must be almost complete and hence x must be adjacent to both v_1 and v_2. But then $G[v_1, v_2, x] * C$ is a K_k in G, giving a contradiction. □

Reed proved that for $k \geq 9$, a vertex outside a $(k-1)$-clique H in a k-mule can have at most 4 neighbors in H. We improve this to at most one neighbor.

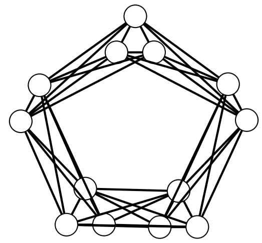


Figure 7.3: The mule $M_{7,2}$.

Lemma 7.2.9. *For $k \geq 7$ and $r \geq 2$, no k-mule except $M_{7,1}$ and $M_{7,2}$ contains an induced $K_r * \big(K_1 + K_{k-(r+1)}\big)$.*

Proof. Suppose the lemma is false and let G be a k-mule, other than $M_{7,1}$ and $M_{7,2}$, containing such an induced subgraph F with r maximal. By Lemma 7.2.4 and Lemma 7.2.8, the lemma holds for $r \geq k-3$. So we have $r \leq k-4$. Now, let $z \in V(F)$ be the vertex with degree r in F, $v_1, v_2, \ldots, v_{k-(r+1)} \in V(F)$ the vertices of degree $k-2$ in F and C the rest of the vertices in F (all of degree $k-1$ in F). Put $H := G - F$.

Let $Z_1 := \{za \mid a \in N(v_1) \cap V(H)\}$. Consider the graph $D := H + z + Z_1$. Since v_1 has at most two neighbors in H, $|Z_1| \leq 2$ and thus to form D from $H + z$, we added $E(A)$ where $A \in \{K_1, K_2, P_3\}$. Since $|C| \geq 2$, $\Delta(D) \leq k$. Hence Lemma 7.2.1 shows that $H + z$ contains a $K_k - E(A)$ or $\chi(D) \leq k-1$. Suppose $\chi(D) \geq k$. If $A = K_1$, $A = K_2$, or $A = P_3$, then we have a contradiction by the fact that $\omega(G) < k$, Lemma 7.2.4, and Lemma 7.2.8, respectively. Thus we must have $\chi(D) \leq k-1$, which gives a $(k-1)$-coloring of $H + z$ in which z receives a color c which is not received by any of the neighbors of v_1 in H. Thus c remains in the list of v_1 and we may color v_1 with c. After doing so, each vertex in C has a list of size at least $k-3$ and v_i for $i > 1$ has a list of size at least $k-4$. If any pair of vertices in C had different lists, then we could complete the partial coloring by Hall's Theorem. Let $N := \bigcup_{w \in C} N(w) \cap V(H)$ and note that N is an independent set since it is contained in a single color class in the $(k-1)$-coloring of H just constructed.

Suppose $|N| \geq 2$. Pick $a_1, a_2 \in N$. Consider the graph $D := H + z + Z_1 + a_1a_2$. Plainly, $\Delta(D) \leq k$. To form D from $H + z$ we added $E(A)$, where $A \in \{K_1, K_2, P_3, K_3, P_4, K_2 + P_3\}$. Hence Lemma 7.2.1 shows that $H + z$ contains a $K_k - E(A)$ or $\chi(D) \leq k - 1$. If $\chi(D) \geq k$, then we have a contradiction since $A = K_1$, $A = K_2$, and $A = P_3$ are impossible as above. To show that $A = K_3$, $A = P_4$, and $A = K_2 + P_3$ are impossible, we apply Lemma 7.2.7 (this is where we use the fact that $G \neq M_{7,1}$), Lemma 5.3.27 (since $K_t - E(P_4) = P_4 * K_{t-4}$), and Lemma 5.3.22, respectively.

Thus we must have $\chi(D) \leq k - 1$, which gives a $(k-1)$-coloring of $H + z$ in which a_1 and a_2 are in different color classes and z receives a color not received by any neighbor of v_1 in H. As above we can complete this partial coloring to all of G by first coloring z and v_1 the same and then using Hall's Theorem.

Hence there is a vertex $x \in V(H)$ which is adjacent to all of C. Note that x is not adjacent to any of $v_1, v_2, \ldots, v_{k-(r+1)}$ by the maximality of r. Let $Z_2 := \{xa \mid a \in N(v_2) \cap V(H)\}$. Consider the graph $D := H + z + Z_1 + Z_2$. As above, both Z_1 and Z_2 have cardinality at most 2. Since $|C| \geq 2$, both x and z have degree at most k in D. Since both xa and za were added only if a was a neighbor of both v_1 and v_2, all the neighbors of v_1 in H have degree at most k in D. Similarly for v_2's neighbors. Hence $\Delta(D) \leq k$. To form D from $H + z$ we added $E(A)$ where $A \in \{K_1, K_2, P_3, K_3, P_4, K_2 + P_3, 2K_2, P_5, 2P_3, C_4\}$. Hence Lemma 7.2.1 shows that $H + z$ contains a $K_k - E(A)$ or $\chi(D) \leq k - 1$.

Suppose $\chi(D) \geq k$. Then $A = K_1$, $A = K_2$, $A = P_3$, $A = K_3$, $A = P_4$, and $A = K_2 + P_3$ are impossible as above. Applying Lemma 5.3.22 shows that $A = 2K_2$, $A = P_5$, and $A = 2P_3$ are impossible. Thus we must have $A = C_4$. If $k \geq 8$, then Lemma 5.3.18 gives a contradiction. Hence we must have $k = 7$. Since $H + z$ contains an induced $K_3 * 2K_2$, we must have $N(v_1) \cap V(H) = N(v_2) \cap V(H)$, say

$N(v_1) \cap V(H) = \{w_1, w_2\}$. Moreoever, $xz \in E(G)$, $w_1w_2 \in E(G)$ and there are no edges between $\{w_1, w_2\}$ and $\{x, z\}$ in G.

Put $Q := \left\{v_1, \ldots, v_{k-(r+1)}\right\}$. Then for $v \in Q$, by the same argument as above, we must have $N(v) \cap V(H) = \{w_1, w_2\}$. Hence Q is joined to $\{w_1, w_2\}$, C is joined to Q, and $\{x, z\}$ and both $\{x, z\}$ and $\{w_1, w_2\}$ are joined to the same K_3 in H. We must have $r = 3$ for otherwise one of x, z, w_1, w_2 has degree larger than 7. Thus we have an $M_{7,2}$ in G and therefore G is $M_{7,2}$, a contradiction.

Thus we must have $\chi(D) \leq k - 1$, which gives a $(k-1)$-coloring of $H + z$ in which z receives a color c_1 which is not received by any of the neighbors of v_1 in H and x receives a color c_2 which is not received by any of the neighbors of v_2 in H. Thus c_1 is in v_1's list and c_2 is in v_2's list. Note that if x and z are adjacent then $c_1 \neq c_2$. Hence, we can 2-color $G[x, z, v_1, v_2]$ from the lists. This leaves $k-3$ vertices. The vertices in C have lists of size at least $k-3$ and the rest have lists of size at least $k-5$. Since the union of any $k-4$ of the lists contains one list of size $k-3$, we can complete the partial coloring by Hall's Theorem. □

Corollary 7.2.10. *For $k \geq 7$, if H is a $(k-1)$-clique in a k-mule G other than $M_{7,1}$ and $M_{7,2}$, then any vertex in $G - H$ has at most one neighbor in H.*

Proof. Let $v \notin H$ be adjacent to r vertices in H. Now $G[H \cup \{v\}] = K_r * (K_1 + K_{k-(r+1)})$. If $r \geq 2$, then $G[H \cup \{v\}]$ is forbidden by Lemma 7.2.9. □

Lemma 7.2.11. *For $k \geq 7$, no k-mule except $M_{7,1}$ contains $K_4 * E_{k-4}$ as a subgraph.*

Proof. Let G be a k-mule other than $M_{7,1}$ and suppose G contains an induced $K_4 * D$ where $|D| = k-4$. Then G is not $M_{7,2}$. By Lemma 5.3.25, D is E_3, a claw, a clique, or almost complete. If D is a clique then G contains K_k, a contradiction. Now Corollary

7.2.10 shows that D being almost complete is impossible. Finally, Lemma 7.2.7 shows that D cannot be E_3 or a claw. This contradiction completes the proof. $\square$

Since $K_4 * E_{\Delta-4} \subseteq K_\Delta$, Lemma 7.2.11 shows that the following conjecture is equivalent to the Borodin-Kostochka conjecture.

Conjecture 7.2.12. *Any graph with $\chi \geq \Delta \geq 9$ contains $K_4 * E_{\Delta-4}$ as a subgraph.*

Lemma 7.2.13. *Let G be a k-mule with $k \geq 8$. Let A and B be graphs with $4 \leq |A| \leq k-4$ and $|B| = k - |A|$ such that $A * B \trianglelefteq G$. Then $A = K_1 + K_{|A|-1}$ and $B = K_1 + K_{|B|-1}$.*

Proof. Note that $|B| \geq 4$. By Lemma 5.3.44, $A * B$ is almost complete, $K_5 * E_3$ or our desired conclusion holds. The first and second cases are impossible by Corollary 7.2.10 and Lemma 7.2.7. $\square$

This shows that the following conjecture is a natural weakening of Borodin-Kostochka.

Conjecture 7.2.14. *Let G be a graph with $\Delta(G) = k \geq 9$. If $K_{t,k-t} \not\subseteq G$ for all $4 \leq t \leq k-4$, then G can be $(k-1)$-colored.*

In the next section we create the tools needed to reduce the 4 in these lemmata to 3.

7.2.1 Tooling up

For an independent set I in a graph G, we write $\frac{G}{[I]}$ for the graph formed by collapsing I to a single vertex and discarding duplicate edges. We write $[I]$ for the resulting vertex in the new graph. If more than one independent set $I_1, I_2, \ldots, I_m$ are collapsed in succession we indicate the resulting graph by $\frac{G}{[I_1][I_2]\cdots[I_m]}$.

Lemma 7.2.15. *Let G be a k-mule other than $M_{7,1}$ and $M_{7,2}$ with $k \geq 7$ and $H \lhd G$. If $x, y \in V(H)$, $xy \notin E(H)$ and $|N_H(x) \cup N_H(y)| \leq k$, then there exists a $(k-1)$-coloring π of H such that $\pi(x) = \pi(y)$.*

Proof. Suppose $x, y \in V(H)$, $xy \notin E(H)$ and $|N_H(x) \cup N_H(y)| \leq k$. Put $H' := \frac{H}{[x,y]}$. Then $H' \prec H$ via the natural epimorphism $f \colon H \twoheadrightarrow H'$. By applying Lemma 7.2.1 we either get the desired $(k-1)$-coloring π of H or a K_{k-1} in H with $V(K_{k-1}) \subseteq N(x) \cup N(y)$. But $k - 1 \geq 6$, so one of x or y has at least three neighbors in K_{k-1} violating Corollary 7.2.10. $\square$

Lemma 7.2.16. *Let G be a k-mule other than $M_{7,1}$ and $M_{7,2}$ with $k \geq 7$ and $H \lhd G$. Suppose there are disjoint nonadjacent pairs $\{x_1, y_1\}, \{x_2, y_2\} \subseteq V(H)$ with $d_H(x_1), d_H(y_1) \leq k-1$ and $|N_H(x_2) \cup N_H(y_2)| \leq k$. Then there exists a $(k-1)$-coloring π of H such that $\pi(x_1) \neq \pi(y_1)$ and $\pi(x_2) = \pi(y_2)$.*

Proof. Put $H' := \frac{H}{[x_2,y_2]} + x_1y_1$. Then $H' \prec H$ via the natural epimorphism $f \colon H \twoheadrightarrow H'$. Suppose the desired $(k-1)$-coloring π of H doesn't exist. Apply Lemma 7.2.1 to get a K_k in H'. Put $z := [x_2, y_2]$. By Lemma 7.2.4 the K_k must contain z and by Lemma 7.2.9, the K_k must contain x_1y_1; hence the K_k contains x_1, y_1, and z. Thus H contains an induced subgraph $A := \{x_1, y_1\} * K_{k-3}$ where $V(A) \subseteq N_H(x_2) \cup N_H(y_2)$. Then x_2 and y_2 each have at most two neighbors in the K_{k-3} by Lemma 7.2.11 and Lemma 5.3.29. Thus $k = 7$ and both x_2 and y_2 have exactly two neighbors in the K_4. One of x_2 or y_2 has at least one neighbor in $\{x_1, y_1\}$, so by symmetry we may assume that x_2 is adjacent to x_1. But then $\{x_2\} \cup V(A)$ induces either a $K_2 *$antichair (if $x_2 \not\leftrightarrow y_1$) or a graph containing $K_2 * C_4$ (if $x_2 \leftrightarrow y_1$), and both are impossible by Lemma 5.3.45. $\square$

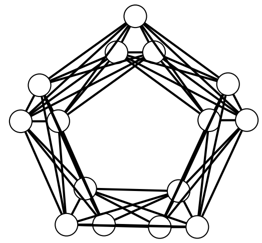


Figure 7.4: The mule M_8.

7.2.2 Using our new tools

Lemma 7.2.17. *For $k \geq 7$, the only k-mules containing $K_3 * E_{k-3}$ as a subgraph are $M_{7,1}$, $M_{7,2}$ and M_8.*

Proof. Suppose not and let G be a k-mule other than $M_{7,1}$, $M_{7,2}$ and M_8 containing $F := C * B$ as an induced subgraph where $C = K_3$ and B is an arbitrary graph with $|B| = k - 3$. By Lemma 5.3.29, B is: $E_3 * K_{|B|-3}$, almost complete, $K_t + K_{|B|-t}$, $K_1 + K_t + K_{|B|-t-1}$, or $E_3 + K_{|B|-3}$. The first two options are impossible by Lemma 7.2.11.

First, suppose there is no $z \in V(G - F)$ with $C \subseteq N(z)$. Let π be the $(k-1)$-coloring of $G - F$ guaranteed by Lemma 7.2.6. Put $L := L_\pi$. Let I be a maximal independent set in B. If there are $x, y \in I$ and $c \in L(x) \cap L(y)$, then we may color x and y with c and then greedily complete the coloring to the rest of F giving a contradiction. Thus we must have

$$
\begin{aligned}
k-1 &\geq \sum_{v \in I} |L(v)| \\
&\geq \sum_{v \in I} (d_F(v) - 1) \\
&= \sum_{v \in I} (d_B(v) + 3 - 1) \\
&= 2|I| + \sum_{v \in I} d_B(v) \\
&= |B| + |I| \\
&= k - 3 + |I|.
\end{aligned}
$$

Therefore $|I| \leq 2$ and hence B is $K_t + K_{|B|-t}$. Put $N := \bigcup_{w \in C} N(w) \cap V(G-F)$. Then $|N| \geq 2$ by assumption. Pick $x_1, y_1 \in N$ and nonadjacent $x_2, y_2 \in V(B)$ and put $H := G[V(G-F) \cup \{x_2, y_2\}]$. Plainly, the conditions of Lemma 7.2.16 are satisfied and hence we have a $(k-1)$-coloring γ of H such that $\gamma(x_1) \neq \gamma(y_1)$ and $\gamma(x_2) = \gamma(y_2)$. But then we can greedily complete this coloring to all of G, a contradiction.

Thus we have $z \in V(G-F)$ with $C \subseteq N(z)$. Put $B' := G[V(B) \cup \{z\}]$ and $F' := G[V(F) \cup \{z\}]$. As above, using Lemma 5.3.29 and Lemma 7.2.11, we see that B' is $K_t + K_{|B'|-t}$, $K_1 + K_t + K_{|B'|-t-1}$ or $E_3 + K_{|B'|-3}$.

Suppose B' is $E_3 + K_{|B'|-3}$, say the E_3 is $\{z_1, z_2, z_3\}$. Since $k \geq 7$, we have $w_1, w_2 \in V(B') - \{z_1, z_2, z_3\}$. Then $d_{F'}(z_3) + d_{F'}(w_1) = k$ and hence we may apply Lemma 7.2.15 to get a $(k-1)$-coloring ζ of $G - F'$ such that there is some $c \in L_\zeta(z_3) \cap L_\zeta(w_1)$. Now $|L_\zeta(z_1)| + |L_\zeta(z_2)| + |L_\zeta(w_2)| \geq 2 + 2 + k - 4 = k$ and hence there is a color c_1 that is in at least two of $L_\zeta(z_1)$, $L_\zeta(z_2)$ and $L_\zeta(w_2)$. If $c_1 = c$, then c appears on an independent set of size 3 in B' and we may color this set with c and greedily complete the coloring. Otherwise, B' contains two disjoint nonadjacent pairs which we can color with different colors and again complete the coloring greedily, a contradiction.

Now suppose B' is $K_1 + K_t + K_{|B'|-t-1}$. By Lemma 7.2.9, we must have $2 \leq t \leq |B'| - 3$. Let x be the vertex in the K_1, $w_1, w_2 \in V(K_t)$ and $z_1, z_2 \in V(K_{|B'|-t-1})$. Then $d_{F'}(w_1) + d_{F'}(z_1) = k + 1$ and hence we may apply Lemma 7.2.15 to get a $(k-1)$-coloring ζ of $G - F'$ such that there is some $c \in L_\zeta(w_1) \cap L_\zeta(z_1)$. Now $|L_\zeta(x)| + |L_\zeta(w_2)| + |L_\zeta(z_2)| \geq 2 + k - 1 = k + 1$ and hence there is are at least two colors c_1, c_2 that are each in at least two of $L_\zeta(x)$, $L_\zeta(w_2)$ and $L_\zeta(z_2)$. If $c_1 \neq c$ or $c_2 \neq c$, then B' contains two disjoint nonadjacent pairs which we can color with different colors and then complete the coloring greedily. Otherwise c appears on an independent set of size 3 in B' and we may color this set with c and greedily complete the coloring, a contradiction.

Therefore B' must be $K_t + K_{|B'|-t}$. By Lemma 7.2.9, we must have $3 \leq t \leq |B'| - 3$. Thus $k \geq 8$. Let X and Y be the two cliques covering B'. Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Put $H := G\left[V(G - F') \cup \{x_1, x_2, y_1, y_2\}\right]$ and $H' := \frac{H}{[x_1,y_1][x_2,y_2]}$. For $i \in [2]$, $d_{F'}(x_i) + d_{F'}(y_i) = k + 2$ and thus $\Delta(H') \leq k$. If $\chi(H') \leq k - 1$, then we have a $(k-1)$-coloring of H which can be greedily completed to all of G, a contradiction. Hence, by Lemma 7.2.1, H' contains K_k. Thence $H - \{x_1, y_1, x_2, y_2\}$ contains a K_{k-2}, call it A, such that $V(A) \subseteq N(x_i) \cup N(y_i)$ for $i \in [2]$. Since $d_{F'}(x_i) + d_{F'}(y_i) = k + 2$, we see that $N_H(x_i) \cap N_H(y_i) = \emptyset$ for $i \in [2]$. But we can play the same game with the pairs $\{x_1, y_2\}$ and $\{x_2, y_1\}$. We conclude that $N(x_1) \cap V(A) = N(x_2) \cap V(A)$ and $N(y_1) \cap V(A) = N(y_2) \cap V(A)$. In fact we can extend this equality to all of X and Y. Put $Q := N(x_1) \cap V(A)$ and $P := N(y_1) \cap V(A)$. Then we conclude that X is joined to Q and Y is joined to P. Moreover, we already know that X and Y are joined to the same K_3. The edges in these joins exhaust the degrees of all the vertices, hence G is a 5-cycle with vertices blown up to cliques. If $k = 8$, then

$|X| = |Y| = 3$ and thus $|Q| = |P| = 3$, but then $G = M_8$, a contradiction. So $k \geq 9$. Since $|X| + |Y| = k - 2 \geq 7$, we have either $|X| \geq 4$ or $|Y| \geq 4$. If $|X| \geq 4$, then for each $q \in Q$, we have $d(q) \geq (k-2) - 1 + |X| \geq k+1$, contradiction. If $|Y| \geq 4$, then for each $p \in P$, we have $d(p) \geq (k-2) - 1 + |Y| \geq k+1$, contradiction. □

Since $K_3 * E_{\Delta-3} \subseteq K_\Delta$, Lemma 7.2.17 shows that Conjecture 7.0.10 is equivalent to the Borodin-Kostochka conjecture.

Lemma 7.2.18. *Let G be a k-mule with $k \geq 7$ other than $M_{7,1}$, $M_{7,2}$ and M_8. Let A and B be graphs with $3 \leq |A| \leq k-3$ and $|B| = k - |A|$ such that $A * B \trianglelefteq G$. Then $A = K_1 + K_{|A|-1}$ and $B = K_1 + K_{|B|-1}$.*

Proof. Suppose the lemma is false and let $A * B \trianglelefteq G$ be a counterexample.

First suppose $|A|, |B| \geq 4$. Then, by Lemma 5.3.44, $A * B$ is almost complete or $K_5 * E_3$. The first and second cases are impossible by Corollary 7.2.10 and Lemma 7.2.7 respectively.

Thus we may assume $|A| = 3$. By Lemma 7.2.17, $A \in \{E_3, P_3, K_1 + K_2\}$. If $A = E_3$, then B is complete by Lemma 5.3.39, but this is impossible by Lemma 7.2.7. If $A = P_3$, then B is complete by Lemma 5.3.20, but this is impossible by Lemma 7.2.4. Hence $A = K_1 + K_2$. By Lemma 5.3.43, B is complete or $K_1 + K_{|B|-1}$. The former is impossible by Lemma 7.2.8 and the latter by supposition. □

Lemma 7.2.18 proves our main result, that Conjecture 7.0.8 is equivalent to the Borodin-Kostochka conjecture.

7.2.3 The low vertex subgraph of a mule

In this section we show that if a mule is not regular, then the subgraph of non-maximum-degree vertices is severely restricted. For a vertex critical graph G we write $\mathcal{L}(G)$ for the subgraph induced on the vertices of degree $\chi(G) - 1$ in G and $\mathcal{H}(G)$ for the subgraph induced on the rest of the vertices. We call $v \in V(G)$ *low* if $v \in V(\mathcal{L}(G))$ and *high* otherwise.

Lemma 7.2.19. *For $k \geq 6$, no k-mule contains an induced $E_2 * K_{k-2}$ with some vertex low.*

Proof. Since $M_{6,1}$ and $M_{7,1}$ contain no such induced subgraph, the lemma follows from Lemma 7.2.4. □

Lemma 7.2.20. *If G is a k-mule with $k \geq 6$, then $\mathcal{L}(G)$ is complete.*

Proof. Let G be a k-mule with $k \geq 6$ and suppose G has nonadjacent low vertices x and y. Then $G + xy \prec G$ and hence, by Lemma 7.2.1, $G + xy$ contains a K_k. But then G contains an $E_2 * K_{k-2}$ with some vertex low, contradicting Lemma 7.2.19. Hence $\mathcal{L}(G)$ is complete. □

Lemma 7.2.21. *If G is a k-mule with $k \geq 6$ other than $M_{6,1}$ and $M_{7,1}$, then $|\mathcal{L}(G)| \leq k - 2$.*

Proof. Let G be a k-mule with $k \geq 6$ other than $M_{6,1}$ and $M_{7,1}$. By Lemma 7.2.20, $\mathcal{L}(G)$ is complete and hence $|\mathcal{L}(G)| \leq k-1$. Suppose $|\mathcal{L}(G)| = k-1$. Since G doesn't contain K_k, no high z is adjacent to all of $\mathcal{L}(G)$. Hence, by Lemma 7.2.6, there is a $(k-1)$-coloring of $\mathcal{H}(G)$ that we can complete to all of G using Hall's Theorem. This contradiction completes the proof. □

Lemma 7.2.22. *Let G be a k-mule with $k \geq 6$. If a high $x \in V(G)$ has at least three low neighbors, then x is adjacent to all low vertices in G.*

Proof. Assume the lemma is false. Let x be a high degree vertex with at least three neighbors in $V(\mathcal{L}(G))$. If $|V(\mathcal{L}(G))| = 3$, then the claim holds. So assume that

$|V(\mathcal{L}(G))| \geq 4$ and choose $y \in V(\mathcal{L}(G)) \setminus N(x)$. Let $A = V(\mathcal{L}(G)) \cap N(x)$. By Lemma 7.2.20, $\mathcal{L}(G)$ is complete. Thus, $G[\{x, y\} \cup A] = E_2 * K_{|A|}$. Since $L(v) = d(v)$ for all $v \in (A \cup \{y\})$, Lemma 5.3.49 implies that $E_2 * K_{|A|}$ cannot appear in G. This contradiction implies the lemma. □

7.2.4 Restrictions on the independence number

The Borodin-Kostochka conjecture has been proven for graphs with independence number at most two [5]. Here we prove that if we wish to prove the Borodin-Kostochka conjecture for graphs with independence number at most a for any $a \leq 6$, it suffices to construct a $K_{\Delta-1}$.

For $a \geq 2$, let $\mathcal{C}_k^a$ be those $G \in \mathcal{C}_k$ with $\alpha(G) \leq a$. By a (k, a)-mule we mean a $\mathcal{C}_k^a$-mule. Note that if $G \in \mathcal{C}_k^a$ and for some $H \in \mathcal{C}_k$ we have $H \prec G$, then $H \in \mathcal{C}_k^a$ as well. Therefore any (k, a)-mule is also a k-mule.

Theorem 7.2.23. *For $k \geq 7$ and $2 \leq a \leq k-3$, no (k, a)-mule except $M_{7,1}$ contains a K_{k-1}.*

Proof. Suppose otherwise and let G be such a (k, a)-mule containing a K_{k-1}, call it H. By Corollary 7.2.10, each vertex in $G - H$ has at most one neighbor in H. Let π be a $(k-1)$-coloring of $G - H$. Then $|L_\pi(v)| \geq k-3$ for all $v \in V(H)$. Since H cannot be colored from L_π, applying Hall's Theorem shows that either $|Pot(L_\pi)| \leq k-2$ or there is some $x \in V(H)$ such that $|Pot_{H-x}(L_\pi)| \leq k-3$. In the former case, π must have some color class to which each vertex of H is adjacent and hence $\alpha(G) \geq k-1$, a contradiction. In the latter case, π must have two color classes to which each vertex of $H - x$ is adjacent and hence G has two disjoint independent sets of size $k-2$. Again we have a contradiction since $\alpha(G) \geq k-2$. □

It follows that Conjecture 7.0.15 is equivalent to the Borodin-Kostochka conjecture for graphs with independence number at most 6.

Chapter 8

STRONG COLORING

Using ideas developed for strong coloring by Haxell [32] and by Aharoni, Berger and Ziv [1], we make explicit a recoloring technique and apply it to the Borodin-Kostochka conjecture.

8.1 Strong coloring

For a positive integer r, a graph G with $|G| = rk$ is called *strongly r-colorable* if for every partition of $V(G)$ into parts of size r there is a proper coloring of G that uses all r colors on each part. If $|G|$ is not a multiple of r, then G is strongly r-colorable iff the graph formed by adding $r\left\lceil \frac{|G|}{r} \right\rceil - |G|$ isolated vertices to G is strongly r-colorable. The *strong chromatic number* $s\chi(G)$ is the smallest r for which G is strongly r-colorable.

Note that a strong r-coloring of G with respect to a partition $V_1, \ldots, V_k$ of $V(G)$ with $|V_i| = r$ must partition $V(G)$ into r independent transversals of $V_1, \ldots, V_k$. In [72], Szabó and Tardos constructed partitioned graphs with part sizes $2\Delta - 1$ that have no independent transversal. So we must have $s\chi(G) \geq 2\Delta(G)$. It is conjectured that this bound is tight.

Haxell [32] proved that $s\chi(G) \leq 3\Delta(G) - 1$. Aharoni, Berger and Ziv [1] gave a simple proof that $s\chi(G) \leq 3\Delta(G)$. It is this latter proof whose recoloring technique we use. First we need a lemma allowing us to pick an independent transversal when one of the sets has only one element.

Lemma 8.1.1. *Let H be a graph and $V_1 \cup \cdots \cup V_r$ a partition of $V(H)$. Suppose that $|V_i| \geq 2\Delta(H)$ for each $i \in [r]$. If a graph G is formed by attaching a new vertex x to fewer than $2\Delta(H)$ vertices of H, then G has an independent set $\{x, v_1, \ldots, v_r\}$ where $v_i \in V_i$ for each $i \in [r]$.*

Proof. Suppose not. Remove $\{x\} \cup N(x)$ from G to form H' with induced partition $V_1', V_2', \dots, V_r'$. Then $V_1', V_2', \dots, V_r'$ has no independent transversal since we could combine one with x to get our desired independent set in G. Note that $|V_i'| \geq 1$. Create a graph Q by removing edges from H' until it is edge minimal without an independent transversal. Pick $yz \in E(Q)$ and apply Lemma 4.1.9 on yz with the induced partition to get the guaranteed $J \subseteq [r]$ and the totally dominating induced matching M with $|M| = |J| - 1$. Now $\left|\bigcup_{i \in J} V_i'\right| > 2\Delta(H)\,|J| - 2\Delta(H) = 2(|J| - 1)\Delta(H)$ and hence M cannot dominate, a contradiction. $\square$

Theorem 8.1.2. *Every graph satisfies $s\chi \leq 3\Delta$.*

Proof. We only need to prove that graphs with $n := 3\Delta k$ vertices have a 3Δ coloring for each $k \geq 1$. Suppose not and choose a counterexample G minimizing $\|G\|$. Put $r := 3\Delta(G)$ and let $V_1, \dots, V_k$ be a partition of G for which there is no acceptable coloring. Then the V_i are independent by minimality of $\|G\|$. By symmetry we may assume there are adjacent vertices $x \in V_1$ and $y \in V_2$. Apply minimality of $\|G\|$ to get an r-coloring π of $G - xy$ with $\pi(V_i) = [r]$ for each $i \in [k]$. We will modify π to get such a coloring of G.

By symmetry, we may assume that $\pi(x) = \pi(y) = 1$. For $2 \leq i \leq k$, let z_i be the unique element of $\pi^{-1}(1) \cap V_i$ and put

$$W_i := V_i - \{v \in V_i \mid \pi(v) = \pi(w) \text{ for some } w \in N(z_i)\}.$$

Then $|W_i| \geq 2\Delta(G)$ and we may apply Lemma 8.1.1 to get a G-independent transversal $w_1, w_2, \dots, w_k$ of $\{x\}, W_2, W_3, \dots, W_k$. Define a new coloring ζ of G by

$$\zeta(v) := \begin{cases} 1 & \text{if } v = w_i \\ \pi(w_i) & \text{if } v = z_i \\ \pi(v) & \text{otherwise.} \end{cases}$$

Then ζ is a proper coloring of G with $\zeta(V_i) = [r]$ for each $i \in [k]$, a contradiction. $\square$

For our application we will need a lopsided version of Lemma 8.1.1 Lemma 4.1.4.

Lemma 8.1.3. *Let H be a graph and $V_1 \cup \cdots \cup V_r$ a partition of $V(H)$. Suppose there exists $t \geq 1$ such that for each $i \in [r]$ and each $v \in V_i$ we have $d(v) \leq \min\{t, |V_i| - t\}$. For any $S \subseteq V(H)$ with $|S| < \min\{|V_1|, \ldots, |V_r|\}$, there is an independent transversal I of $V_1, \ldots, V_r$ with $I \cap S = \emptyset$.*

Proof. Suppose the lemma fails for such an $S \subseteq V(H)$. Put $H' := H - S$ and let $V_1', \ldots, V_r'$ be the induced partition of H'. Then there is no independent trasversal of $V_1', \ldots, V_r'$ and $|V_i'| \geq 1$ for each $i \in [r]$. Create a graph Q by removing edges from H' until it is edge minimal without an independent transversal. Pick $yz \in E(Q)$ and apply Lemma 4.1.9 on yz with the induced partition to get the guaranteed $J \subseteq [r]$ and the tree T with vertex set J and an edge between $a, b \in J$ for each $uv \in M$ with $u \in V_a'$ and $v \in V_b'$. By our condition, for each $uv \in E(V_i, V_j)$, we have $|N_H(u) \cup N_H(v)| \leq \min\{|V_i|, |V_j|\}$.

Choose a root c of T. Traversing T in leaf-first order and for each leaf a with parent b picking $|V_a|$ from $\min\{|V_a|, |V_b|\}$ we get that the vertices in M together dominate at most $\sum_{i \in J-c} |V_i|$ vertices in H. Since $|S| < |V_c|$, M cannot totally dominate $\bigcup_{i \in J} V_i'$, a contradiction. $\square$

We note that the condition on S can be weakened slightly. Suppose we have ordered the V_i so that $|V_1| \leq |V_2| \leq \cdots \leq |V_r|$. Then for any $S \subseteq V(H)$ with $|S| < |V_2|$ such that $V_1 \not\subseteq S$, there is an independent transversal I of $V_1, \ldots, V_r$ with $I \cap S = \emptyset$. The proof is the same except when we choose our root c, choose it so as to maximize $|V_c|$. Since $|J| \geq 2$, we get $|V_c| \geq |V_2| > |S|$ at the end.

8.2 The recoloring technique

We can extract the idea in the proof of Theorem 8.1.2 to get a general recoloring technique. Suppose G is a k-vertex-critical graph and pick $x \in V(G)$ and $(k-1)$-coloring π of $H := G - x$. Let Z be a color class of π, say $Z = \pi^{-1}(1)$. For each $z \in Z$, let O_z be the neighbors of z which get a color that no other neighbor of z gets; that is, put $O_z := \{v \in N_H(z) \mid \pi(v) \notin \pi(N_H(z) - v)\}$. Suppose the O_z are pairwise disjoint. If we could find an independent transversal $\{x\} \cup \{v_z\}_{z \in Z}$ of $\{x\}$ together with the O_z, then recoloring each $z \in Z$ with $\pi(v_z)$ and coloring each vertex in $\{x\} \cup \{v_z\}_{z \in Z}$ with 1 gives a proper $(k-1)$-coloring of G. This is exactly what happens in the above proof of the strong coloring result. To make this work more generally, we need to find situations where each $G[O_z]$ has high minimum degree. Also, intuitively, the O_z intersecting each other should make things easier since recoloring a vertex in the intersection of O_{z_1} and O_{z_2} works for both z_1 and z_2. In our application we will allow some restricted intersections.

8.3 A general decomposition

Let $\mathcal{D}_1$ be the collection of graphs without induced d_1-choosable subgraphs. Plainly, $\mathcal{D}_1$ is hereditary. For a graph G and $t \in \mathbb{N}$, let $\mathcal{C}_t$ be the maximal cliques in G having at least t vertices. We prove the following decomposition result for graphs in $\mathcal{D}_1$ which generalizes Reed's decomposition in [67].

Lemma 8.3.1. *Suppose $G \in \mathcal{D}_1$ has $\Delta(G) \geq 8$ and contains no $K_{\Delta(G)}$. If $\frac{\Delta(G)+5}{2} \leq t \leq \Delta(G) - 1$, then $\bigcup \mathcal{C}_t$ can be partitioned into sets $D_1, \ldots, D_r$ such that for each*

$i \in [r]$ at least one of the following holds:

- *$D_i = C_i \in \mathcal{C}_t$,*
- *$D_i = C_i \cup \{x_i\}$ where $C_i \in \mathcal{C}_t$ and $|N(x_i) \cap C_i| \geq t-1$.*

Moreover, each $v \in V(G) - D_i$ has at most $t-2$ neighbors in C_i for each $i \in [r]$.

Proof. Suppose $|C_i| \leq |C_j|$ and $C_i \cap C_j \neq \emptyset$. Then $|C_i \cap C_j| \geq |C_i| + |C_j| - (\Delta+1) \geq 4$. It follows from Corollary 5.3.25 that $|C_i - C_j| \leq 1$.

Now suppose C_i intersects C_j and C_k. By the above, $|C_i \cap C_j| \geq \frac{\Delta(G)+3}{2}$ and similarly $|C_i \cap C_k| \geq \frac{\Delta(G)+3}{2}$. Hence $|C_i \cap C_j \cap C_k| \geq \Delta(G) + 3 - (\Delta(G) - 1) = 4$. Put $I := C_i \cap C_j \cap C_k$ and $U := C_i \cup C_j \cup C_k$. By maximality of C_i, C_j, C_k, U cannot induce an almost complete graph. Thus, by Corollary 5.3.25, $|U| \in \{4, 5\}$ and the graph induced on $U - I$ is E_3. But then $t \leq 6$ and hence $\Delta(G) \leq 7$, a contradiction.

The existence of the required partition is immediate. □

When $D_i \in \mathcal{C}_t$, we put $K_i := C_i := D_i$ and when $D_i = C_i \cup \{x_i\}$, we put $K_i := N(x_i) \cap C_i$.

8.4 Borodin-Kostochka when every vertex is in a big clique

Let G be a graph. For $v \in V(G)$, we let $\omega(v)$ be the size of a largest clique in G containing v. The proofs of the results in this section go more smoothly when we strengthen the induction in terms of the parameter $\rho(G) := \max_{v \in V(G)} d(v) - \omega(v)$.

Theorem 8.4.1. *For $k \geq 9$, every graph satisfying $\Delta \leq k$, $\omega < k$ and $\rho \leq \frac{k}{3} - 2$ is $(k-1)$-colorable.*

Proof. Suppose the theorem fails for some $k \geq 9$ and choose a counterexample G minimizing $|G| + \|G\|$. Put $\Delta := \Delta(G)$. If $\Delta < k$, then $\Delta = k - 1$ and by Brooks' theorem G contains K_k, a contradiction. Thus $\chi(G) = k = \Delta$. Also, for any $v \in V(G)$ we have $\rho(G - v) \leq \rho(G)$, applying our minimality condition on G implies that G is vertex critical.

Therefore $\delta(G) \geq \Delta - 1$ and $G \in \mathcal{D}_1$. For any $v \in V(G)$, we have $\Delta - 1 - \omega(v) \leq d(v) - \omega(v) \leq \frac{\Delta}{3} - 2$ and hence $\omega(v) \geq \frac{2}{3}\Delta + 1$. Applying Lemma 8.3.1 with $t := \frac{2}{3}\Delta + 1$ we get a partition $D_1, \ldots, D_r$ of $\bigcup \mathcal{C}_t = V(G)$. Note that for $i \in [r]$, if $K_i \neq D_i$ then all vertices in K_i are high by Lemma 5.3.48. Pick $x \in K_1$. Then x has $|C_1| - 1 \leq \Delta - 2$ neighbors in D_1 if $K_i = D_i$ and $|C_1| \leq \Delta - 1$ if $K_i \neq D_i$. Hence, by our note, x has a neighbor $w \in V(G) - D_1$.

We now claim that xw is a critical edge in G. Suppose otherwise that $\chi(G - xw) = \Delta$. Then by minimality of G we must have $\rho(G - xw) > \rho(G)$. Hence there is some vertex $v \in N(x) \cap N(w)$ so that every largest clique containing v contains xw. But v is in some D_j and all largest cliques containing v are contained in D_j and hence do not contain xw, a contradiction.

Let π be a $(\Delta - 1)$-coloring of $G - xw$ chosen so that $\pi(x) = 1$ and so as to minimize $|\pi^{-1}(1)|$. Consider π as a coloring of $G - x$. One key property of π we will use is that since x got 1 in the coloring of $G - xw$ and $x \in K_1$, no vertex of $D_1 - x$ gets colored 1 by π.

Now put $Z := \pi^{-1}(1)$ and for $z \in Z$, let O_z be as defined in Section 8.2. By minimality of $|Z|$, each $z \in Z$ has at least one neighbor in every color class of π. Hence z has two or more neighbors in at most $2 + d(z) - \Delta$ of π's color classes. For each $z \in Z$ we have $i(z)$ such that $z \in D_{i(z)}$. For $z \in Z$ such that $i(z) \notin i(Z - z)$, put $V_z := O_z \cap C_{i(z)}$.

We have $|V_z| \geq \omega(z) - 1 - (2 + d(z) - \Delta)$. Since $\omega(z) \geq d(z) - \frac{1}{3}\Delta + 2$, we have $|V_z| \geq \frac{2}{3}\Delta - 1$. Each $y \in V_z$ is adjacent to all of $C_{i(z)} - \{y\}$ and hence has at most $d(y) + 1 - \left|C_{i(z)}\right|$ neighbors outside $D_{i(z)}$. Since $\omega(y) \geq d(y) + 2 - \frac{1}{3}\Delta$, we conclude that y has at most $d(y) + 1 - (d(y) + 2 - \frac{1}{3}\Delta) = \frac{1}{3}\Delta - 1$ neighbors outside $D_{i(z)}$.

Now let Z' be the $z \in Z$ with $i(z) \in i(Z - z)$. Then Z' can be partitioned into pairs $\{z, z'\}$ such that $i(z) = i(z')$. For such a pair, one of z, z' is $x_{i(z)}$ and the other is in $C_{i(z)} - K_{i(z)}$. Put $V_z := O_z \cap O_{z'} \cap K_{i(z)}$ and don't define $V_{z'}$. We have $|V_z| \geq \min\{\omega(z), \omega(z')\} - 1 - (2 + d(z) - \Delta) - (2 + d(z') - \Delta) \geq -\frac{1}{3}\Delta + 2 - 1 - 2(2 - \Delta) - \max\{d(z), d(z')\} = \frac{5}{3}\Delta - \max\{d(z), d(z')\} - 3 \geq \frac{2}{3}\Delta - 3$. Each $y \in V_z$ is adjacent to all of $D_{i(z)} - \{y\}$ and hence has at most $d(y) + 1 - \left|D_{i(z)}\right|$ neighbors outside $D_{i(z)}$. Since $\left|D_{i(z)}\right| = \omega(y) + 1 \geq d(y) + 3 - \frac{1}{3}\Delta$, we conclude that y has at most $\frac{1}{3}\Delta - 2$ neighbors outside $D_{i(z)}$.

Let H be the subgraph of G induced on the union of the V_z. Put $S := N(x) \cap V(H)$. Since $Z \cap D_1 = \emptyset$, x has at least $|D_1| - 1$ neighbors in D_1 none of which are in S. Hence $|S| \leq d(x) + 1 - |D_1| \leq d(x) + 1 - \omega(x) \leq \frac{\Delta}{3} - 1 < |V_z|$ for all V_z since $\Delta \geq 7$. Hence we may apply Lemma 8.1.3 on H with $t := \frac{1}{3}\Delta - 1$ to get an independent set $\{v_z\}_{z \in Z}$ disjoint from S where $v_z \in V_z$. Recoloring each $z \in Z$ with $\pi(z)$ and coloring $x \cup \{v_z\}_{z \in Z}$ with 1 gives a $(\Delta - 1)$-coloring of G, a contradiction. $\square$

The following special case is a bit easier to digest.

Corollary 8.4.2. *Every graph with $\chi \geq \Delta \geq 9$ such that every vertex is in a clique on $\frac{2}{3}\Delta + 2$ vertices contains K_Δ.*

8.5 Reducing to the irregular case

It is easy to see that if there are irregular counterexamples to the Borodin-Kostochka conjecture, then there are regular examples as well: take an irregular counterexample G clone it add an edge between any vertex with degree less than $\Delta(G)$ and its clone; repeat until you have a regular graph (from [54]).

But what about the converse? If there are regular examples, must there be (connected) irregular examples? We'll see that the answer is yes, but we need to decrease the maximum degree by one.

Theorem 8.5.1. *Every graph satisfying $\chi \geq \Delta = k \geq 9$ either contains K_k or contains an irregular critical subgraph satisfying $\chi = \Delta = k - 1$.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Then G is vertex critical. If every vertex in G were contained in a $(k-1)$-clique, then Corollary 8.4.2 would give a K_k in G, impossible. Hence we may pick $v \in V(G)$ not in a $(k-1)$-clique. If v is high, choose a $(k-1)$-coloring π of $G - v$ so that the color class T of π where v has two neighbors is as large as possible; if v is low, let π be a $(k-1)$-coloring of $G - v$ where some color class T of π is as large as possible. By symmetry, we may assume that $\pi(T) = k - 1$.

Now we have a $(k-1)$-coloring ζ of $H := G - T$ given by $\zeta(x) = \pi(x)$ for $x \neq v$ and $\zeta(v) = k-1$. Since $\chi(H) = k-1$, the maximality condition on T together with Brooks' theorem gives $\Delta(H) = k - 1$. Note that $d_H(v) = k - 2$. Let H' be a $(k-1)$-critical subgraph of H. Then H' must contain v and hence is not K_{k-1}. Since $d_{H'}(v) = k - 2$ and $\Delta(H') = k - 1$ (by Brooks' theorem), H' is an irregular critical subgraph of G satisfying $\chi = \Delta = k - 1$, a contradiction. □

Since the only known critical (or connected even) counterexample to Borodin-Kostochka for $\Delta = 8$ is regular (see Figure 7.4) we might hope that the following strengthened conjecture is true.

Conjecture 8.5.2. *Every critical graph satisfying $\chi \geq \Delta = 8$ is regular.*

8.6 Dense neighborhoods

Here we show that the Borodin-Kostochka conjecture holds for graphs where each neighboorhood has "most" of its possible edges. First, we need to convert high average degree in a neighborhood into a large clique in the neighborhood. We need the following extension of a fundamental result of Mader [53] (see Diestel [22] for some history of this result). We will also need d_1-choosability results from [20] as well as some ideas for dealing with average degree in neighborhoods used in [19].

Lemma 8.6.1. *For $k \geq 1$, every graph G with $d(G) \geq 4k$ has a $(k+1)$-connected induced subgraph H such that $d(H) > d(G) - 2k$.*

Lemma 8.6.2. *If B is a graph with $d(B) \geq \omega(B) + 2$, then B has an induced subgraph H such that $K_1 * H$ is f-choosable where $f(v) \geq d(v)$ for the v in the K_1 and $f(x) \geq d(x) - 1$ for $x \in V(H)$.*

Proof. Let B be such a graph. Applying Lemma 8.6.1 with $k := 1$, we get a 2-connected subgraph H of B with $d(H) > d(B) - 2 \geq \omega(B)$. Since H is 2-connected, if it is not d_0-choosable, then it is either an odd cycle or complete. The former is impossible since $d(H) \geq 3$, hence H would be complete and we'd have the contradiction $\omega(H) > \omega(B)$. Hence H is d_0-choosable.

Suppose $K_1 * H$ isn't f-choosable and let L be a minimal bad f-assignment on $K_1 * B$. By Lemma 5.3.16, no nonadjacent pair in H have intersecting lists and hence we must have $\sum_{v \in V(H)} |L(v)| \leq |Pot(L)|\, \omega(H)$. Since for each $v \in V(H)$ we have $|L(v)| \geq d_H(v)$ and by the Small Pot Lemma we have $|Pot(L)| \leq |H|$, we must have $d(H) \leq \omega(H) \leq \omega(B) < d(H)$, a contradiction. $\square$

Lemma 8.6.3. *If B is a graph with $d(B) \geq \omega(B)+3$, then B has an induced subgraph H such that $K_1 * H$ is d_1-choosable.*

Proof. Let B be such a graph. Applying Lemma 8.6.1 with $k := 1$, we get a 2-connected subgraph H of B with $d(H) > d(B) - 2 \geq \omega(B) + 1$. As in the proof of Lemma 8.6.2, we see that H is d_0-choosable. Suppose $K_1 * H$ is not d_1-choosable and let L be a minimal bad d_1-assignment on $K_1 * H$. Combining Lemma 5.3.15 with the same argument as in the proof of Lemma 8.6.2 shows that $|Pot(L)| \leq |H| - 1$.

Now, for $c \in Pot(L)$, we consider how big the color graphs H_c can be. All of the information comes from Lemma 5.3.14. We have $\alpha(G_c) \leq 2$ for all $c \in Pot(L)$. First, suppose we have $c \in Pot(L)$ such that $|H_c| \geq \omega(H) + 3$. Then, using Lemma 5.3.14, we see that $|H_{c'}| \leq \omega(H)$ for all $c' \in Pot(L) - c$ and hence $\sum_{\gamma \in Pot(L)} |H_\gamma| \leq |H| + (|Pot(L)| - 1)\, \omega(H) \leq |H|\, \omega(H) + |H| - 2\omega(H)$. Now suppose we have $c \in Pot(L)$ such that $|H_c| = \omega(H) + 2$. Then, using Lemma 5.3.14 again, we see that $|H_{c'}| \leq \omega(H) + 1$ for all $c' \in Pot(L) - c$ and hence $\sum_{\gamma \in Pot(L)} |H_\gamma| \leq 1 + |Pot(L)|\, (\omega(H)+1) \leq |H|\, \omega(H) + |H| - \omega(H)$.

Therefore we must have $2\,\|H\| \leq |H|\,(\omega(H) + 1) - \omega(H)$ and hence $d(H) \leq \omega(H) + 1 < d(H)$, a contradiction. $\square$

Theorem 8.6.4. *Every graph G with $\omega(G) < \Delta(G)$ such that $d(G_v) \geq \frac{2}{3}\Delta(G) + 4$ for each $v \in V(G)$ is $(\Delta(G) - 1)$-colorable.*

Proof. Suppose note and let G be a counterexample. Put $\Delta := \Delta(G)$. Let H be a Δ-vertex-critical induced subgraph of G. Then $\delta(H) \geq \Delta - 1$ and H has no d_1-choosable induced subgraphs. By Theorem 8.4.2, we must have $v \in V(H)$ with $\omega(v) < \frac{2}{3}\Delta + 2$. Suppose $d(H_v) < d(G_v)$. Then $d_H(v) = \Delta - 1$ and $\|H_v\| \geq \|G_v\| - (\Delta - 1)$; therefore, $d(H_v) > d(G_v) - 1 \geq \frac{2}{3}\Delta + 3$. Applying Lemma 8.6.2 gives $\omega(v) > d(H_v) - 1 \geq \frac{2}{3}\Delta + 2$, a contradiction.

Hence we must have $d(H_v) = d(G_v) \geq \frac{2}{3}\Delta + 4$. Applying Lemma 8.6.3 gives $\omega(v) > d(H_v) - 2 \geq \frac{2}{3}\Delta + 2$, a contradiction. □

8.7 Bounding the order and independence number

Lemma 8.7.1. *Let G be a vertex critical graph with $\chi(G) = \Delta(G) + 1 - k$. For every $v \in V(G)$ there is $H_v \trianglelefteq G_v$ with:*

1. $|H_v| \geq \Delta(G) - 2k$; *and*
2. $\delta(H_v) \geq |H_v| - (k+1)(\alpha(G) - 1) - 1$; *and*
3. $\|H_v\| \geq |H_v|\left(|H_v| - (k+2)\right) - (k+1)\left(|G| + 2k - (\Delta(G) + 1)\right)$.

Proof. Put $\Delta := \Delta(G)$. Pick $v \in V(G)$ and let π be a $(\Delta - k)$-coloring of $G - v$. Let H_v be the subgraph of G_v induced on $\{x \in N(v) \mid \pi(x) \notin \pi(N(v) - x)\}$. Plainly, $|H_v| \geq \Delta - 2k$.

By the usual Kempe chain argument, any $x, y \in V(H_v)$ must be in the same component of $C_{x,y} := G[\pi^{-1}(\pi(x)) \cup \pi^{-1}(\pi(y))]$. Thus if $xy \notin E(G)$, there must be a path of length at least 3 in $C_{x,y}$ from x to y and hence some vertex of color $\pi(x)$ other than x must have at least two neighbors of color $\pi(y)$ and some vertex of color $\pi(y)$ other than y must have at least two neigbhors of color $\pi(x)$. We say that such an intermediate vertex *proxies* for xy. Each xy with $y \in V(H_v)$ must have some proxy $z_{xy} \in \pi^{-1}(\pi(x)) - x$ such that z_{xy} proxies for at most $k+1$ total xw with

$w \in V(H_v)$, for otherwise we could recolor all of xy's proxies, swap $\pi(x)$ and $\pi(y)$ in x's component of $C_{x,y}$ and then color v with $\pi(x)$ to get a $(\Delta - k)$-coloring of G. We conclude that x has at most $(k+1)(|\pi^{-1}(\pi(x))| - 1)$ non-neighbors in H_v. This gives (2) immediately.

For (3), note that $|\pi(i)| \geq 2$ for each $i \in [\Delta - k] - \pi(V(H_v))$ and hence $\sum_{j \in \pi(V(H_v))} |\pi^{-1}(j)| \leq |G| - 1 - 2(\Delta - k - |H_v|)$. Since

$$\|H_v\| \geq \sum_{j \in \pi(V(H_v))} \left(|H_v| - 1 - (k+1)(\left|\pi^{-1}(j)\right| - 1)\right),$$

we see that (3) follows. □

Theorem 8.7.2. *Every graph satisfies* $\chi \leq \max\{\omega, \Delta - 1, 4\alpha\}$.

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Since none of the terms on the right side increase when we remove a vertex, G is vertex critical. Since the Borodin-Kostochka conjecture holds for graphs with $\alpha = 2$ and $\Delta \geq 9$, we must have $\alpha(G) \geq 3$ and hence $\Delta(G) \geq 13$. By Lemma 8.4.2, there must be $v \in V(G)$ with $\omega(v) < \frac{2}{3}\Delta(G) + 2$. Applying (2) of Lemma 8.7.1, we get $H_v \trianglelefteq G_v$ with $|H_v| \geq \Delta(G) - 2$ and $\delta(H_v) \geq |H_v| - 2\alpha(G) + 1$. Since $\Delta(G) \geq \chi(G) \geq 4\alpha(G) + 1$, we have $\delta(H_v) \geq |H_v| - \frac{\Delta(G)-1}{2} + 1 \geq \frac{|H_v|+1}{2}$. Applying Lemma 9.2.1 shows that either $H_v = K_3 * E_4$ or $\omega(H_v) \geq |H_v| - 1$. The former is impossible since $\Delta(G) > 9$. Therefore $\omega(v) \geq \omega(H_v) + 1 \geq \Delta(G) - 2 \geq \frac{2}{3}\Delta(G) + 2$ since $\Delta(G) \geq 12$, a contradiction. □

Theorem 8.7.3. *Every graph satisfies* $\chi \leq \max\left\{\omega, \Delta - 1, \left\lceil \frac{15+\sqrt{48n+73}}{4} \right\rceil\right\}$.

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Put $\Delta := \Delta(G)$ and $n := |G|$. Since none of the terms on the right side increase when we remove a vertex, G is vertex critical. By Lemma 8.4.2, there must be $v \in V(G)$ with $\omega(v) <$

$\frac{2}{3}\Delta + 2$. Applying (3) of Lemma 8.7.1, we get $H_v \trianglelefteq G_v$ with with $|H_v| \geq \Delta - 2$ and $\|H_v\| \geq |H_v|\left(|H_v| - 3\right) - 2\left(n + 1 - \Delta\right)$. By Lemma 8.6.3, we must have $d(H_v) < \frac{2}{3}\Delta + 4$ and hence we have

$$\begin{aligned}\frac{2}{3}\Delta + 4 &> 2\left(|H_v| - 3\right) - \frac{4\left(n+1-\Delta\right)}{|H_v|} \\ &\geq 2\left(\Delta - 5\right) - \frac{4\left(n+1-\Delta\right)}{\Delta - 2}.\end{aligned}$$

Simplifying a bit, we get $6(n-1) > (2\Delta - 15)(\Delta - 2)$. Since $\Delta \geq \chi(G) \geq \frac{19+\sqrt{48n+73}}{4}$, we have $6(n-1) > (\frac{-11+\sqrt{48n+73}}{2})(\frac{11+\sqrt{48n+73}}{4}) = \frac{48n-48}{8} = 6(n-1)$, a contradiction. $\square$

Chapter 9

LIST BORODIN-KOSTOCHKA FOR LARGE Δ

9.1 The setup

The aim of this chapter is to prove the following result.

Theorem 9.1.1. *There exists Δ_0 such that every graph G with $\chi_l(G) \geq \Delta(G) \geq \Delta_0$ contains a $K_{\Delta(G)}$.*

Suppose the theorem is false and choose a counterexample G minimizing $|G|$. Put $\Delta := \Delta(G)$ and let L be a bad $(\Delta - 1)$-assignment on G. Then, by minimality of $|G|$, any proper induced subgraph of G is L-colorable. In particular, every vertex has degree either Δ or $\Delta - 1$, we call these *high* and *low* vertices respectively.

If G had an induced d_1-choosable subgraph H, then we could L-color $G - H$ by minimality and then complete the L-coloring to all of G. So, G has no d_1-choosable induced subgraphs.

The proof strategy is the same as Reed's [67] for chromatic number, except some more care must be taken when lists have small intersection and $K_{\Delta-1}$'s require special attention.

9.2 The decomposition

We use Lemma 8.3.1 and the notation from Section 8.3.

Definition 9.2.1. The cliques in $\mathcal{C}_{\frac{3}{4}\Delta+1}$ are called *big*.

Let B be all vertices contained in a big clique; that is, $B := \bigcup \mathcal{C}_{\frac{3}{4}\Delta+1}$. For a vertex v, put $G_v := G[N(v)]$.

Definition 9.2.2. A vertex v is called *sparse* if $\|G_v\| < \frac{2}{5}\Delta^2$.

Lemma 9.2.1. *If B is a graph with $\delta(B) \geq \frac{|B|+1}{2}$ such that $K_1 * B$ is not d_1-choosable, then $\omega(B) \geq |B| - 1$ or $B = E_3 * K_4$.*

Proof. Suppose the lemma is false and let L be a minimal bad d_1-assignment on B. First note that if B does not contain disjoint nonadjacent pairs x_1, y_1 and x_2, y_2, then $\omega(B) \geq |B| - 1$ or $B = E_3 * K_4$ by Corollary 5.3.25.

By Dirac's theorem, B is hamiltonian and in particular 2-connected. Since B cannot be an odd cycle or complete, B is d_0-choosable.

By the Small Pot Lemma, $|Pot(L)| \leq |B|$. Since $|L(x_1)| + |L(x_2)| \geq |B| + 1$, the lists intersect and thus Lemma 5.3.15 shows that $|Pot(L)| \leq |B| - 1$. But then $|L(x_i) \cap L(y_i)| \geq 2$ for each i and Lemma 5.3.14 gives a contradiction. □

Note that the neighborhoods we will be looking at are huge, so the $B = E_3 * K_4$ case will never happen here.

Lemma 9.2.2. *Every vertex in $V(G) - B$ is sparse.*

Proof. Suppose $x \in V(G) - B$. By applying Lemma 9.2.1 repeatedly, we get a sequence $y_1, \ldots, y_{\lfloor \frac{\Delta}{4} \rfloor} \in N(x)$ such that

$$|N(y_i) \cap (N(x) - \{y_1, \ldots, y_{i-1}\})| \leq \frac{1}{2}(\Delta + 1 - i).$$

Hence x is sparse since

$$\|G_x\| \leq \binom{\Delta}{2} - \frac{1}{2}\sum_{i=1}^{\lfloor \frac{\Delta}{4} \rfloor}(\Delta - i) < \frac{2}{5}\Delta^2.$$

□

Let $D_1, \ldots, D_r$ be the partition of B guaranteed by Lemma 8.3.1 and put $S := V(G) - B$.

9.3 The random procedure

For each vertex v, pick $c \in L(v)$ at random to get a possibly improper coloring ζ of G from L. Put $U := \{x \in V(G) \mid \zeta(x) = \zeta(y) \text{ for some } y \in N(x)\}$. Put $H := G - U$, $F := G[U]$ and let π be ζ restricted to $V(H)$. We refer to $V(H)$ as the *colored* vertices and $V(F)$ as the *uncolored* vertices. Also, let J be the resulting list assignment on F; that is, $J(x) := L(x) - \bigcup_{y \in N(x) \cap V(H)} \pi(y)$ for $x \in V(F)$.

Definition 9.3.1. A vertex in $v \in V(G)$ is called *safe* if it is colored or $|J(v)| \geq d_F(v) + 1$.

Note that if every vertex is safe, then we can easily complete the L-coloring to all of G. Our goal will be to show that the random procedure will, with positive probability, produce a partial coloring where every sparse vertex is safe and the uncolored nonsparse vertices satisfy conditions that will allow the coloring to be completed. Now we make this precise. Consider the following events:

- S_v, for $v \in S$: the event that v is not safe.
- E_i, for $i \in [r]$ where $|C_i| \leq \Delta - 2$: the event that C_i does not contain two uncolored safe vertices.
- Q_i, for $i \in [r]$ where $|C_i| \leq \Delta - 2$: the event that K_i does not contain two uncolored vertices.
- F_i, for $i \in [r]$ where $|C_i| = \Delta - 1$, every $x \in G - C_i$ has $|N(x) \cap C_i| \leq \sqrt{\Delta}\log(\Delta)$ and there are at most $\log^2(\Delta)$ vertices $x \in G - C_i$ with $|N(x) \cap C_i| > \frac{\sqrt{\Delta}}{\log(\Delta)}$: the event that C_i does not contain two uncolored safe vertices.

- P_i, for $i \in [r]$ where $|C_i| = \Delta - 1$ and either some $x \in G - C_i$ has $|N(x) \cap C_i| > \sqrt{\Delta}\log(\Delta)$ or more than $\log^2(\Delta)$ vertices $x \in G - C_i$ have $|N(x) \cap C_i| > \frac{\sqrt{\Delta}}{\log(\Delta)}$: the event that every $x \in G - C_i$ has at most two "good clumps" in K_i.

It remains to define "good clumps". To do so we need a lemma.

Lemma 9.3.1. *Let K be a $\Delta - 1$ clique in G and $x \in G - K$ with $|N(x) \cap K| \geq 4$. Then every vertex in $|N(x) \cap K|$ is high and there is a partition $\{Z_1, \ldots, Z_m\}$ of $N(x) \cap K$ such that for each $i \in [m]$ we have $|Z_i| \leq 5$ and $L(u) = L(v)$ for all $u, v \in Z_i$. Moreover, $|L(v) - L(w)| \leq 1$ for all $v, w \in N(x) \cap K$.*

Proof. Put $A := N(x) \cap K$ and $Q := G[\{x\} \cup K]$. For any L-coloring γ of $G - Q$, let L_γ be the resulting list assignment on Q.

First, suppose there is an L-coloring γ of $G - Q$ such that $L_\gamma(u) \neq L_\gamma(v)$ for some $u, v \in A$. Pick $y \in K - A$. If $L_\gamma(x) \cap L_\gamma(y) \neq \emptyset$, then coloring x and y the same leaves a list assignment on $K - y$ which is completable by Hall's theorem. Hence we must have $L_\gamma(x) \cap L_\gamma(y) = \emptyset$. Thus $|L_\gamma(x) \cup L_\gamma(y)| \geq \Delta$. Put $Pot(T) := \bigcup_{v \in T} L_\gamma(v)$ for $T \subseteq A$. If there is $c \in (L_\gamma(x) \cup L_\gamma(y)) - Pot(A)$, then coloring x and y so that c is used leaves a list assignment on $K - y$ which is completable by Hall's theorem. In particular, we must have $|Pot(A)| \geq \Delta$. Now, if we color x and y arbitrarily we can complete the coloring unless there exists $T \subseteq A$ with $|T| = |A| - 1$ and $|Pot(T)| \leq \Delta - 2$. Thus we can pick a color in $L_\gamma(x) \cup L_\gamma(y)$ which is not in any of T's lists giving a coloring that is again easily completable.

Therefore $L_\gamma(u) = L_\gamma(v)$ for all $u, v \in A$ for every L-coloring γ of $G - Q$. In particular, no vertex of A is low and $|L(v) - L(w)| \leq 1$ for all $v, w \in A$. Suppose there exists $Z \subseteq A$ with $|Z| \geq 6$ such that $L(u) = L(v)$ for all $u, v \in Z$. Then every $v \in Z$ has exactly one neighbor z_v in $G - Q$. Put $N := \{z_v \mid v \in Z\}$. If $|N| = 1$, then

G contains $K_6 * E_3$ violating Lemma 5.3.25. If some L-coloring γ of $G - Q$ assigned two vertices of N different colors, then L_γ would give different lists for two vertices of A, a contradiction. Hence N is an independent set and adding an edge between two vertices of N in $G - Q$ must create a K_Δ by minimality of $|G|$. By counting degrees this is plainly impossible for $|N| \geq 3$. For $|N| = 2$, both vertices have $\Delta - 2$ neighbors in $G - Q$ and one has at least 3 vertices in Z, impossible.

Now taking maximal subsets of A of vertices all having the same list gives the desired partition. □

The Z_i in the partition in Lemma 9.3.1 are called *clumps* of x in K. Note that there exists Y such that for any $i \neq j$ we have $L(v) \cap L(z) = Y$ for $v \in Z_i$ and $w \in Z_j$. For $i \in [m]$ and $v \in Z_i$ we let α_i be the unique element of $L(v) - Y$. We call α_i the *special* color for Z_i.

Now let $i \in [r]$ where $|C_i| = \Delta - 1$ and some $x \in G - C_i$ has $|N(x) \cap C_i| \geq 4$. A clump $Z_j \subseteq N(x) \cap C_i$ is *good* if there is uncolored $z_i \in Z_i$ such that α_i is not used on any neighbor of z_i and the unique y in $N(z_i) - C_i - \{x\}$ is colored with a color that is either not in $L(z_i)$ or is used on C_i.

Now suppose we have a partial coloring π where none of the bad events S_v, E_i, Q_i, F_i and P_i occur. We color the D_i corresponding to P_i events first. Suppose $x \in G - C_i$ has 3 good clumps Z_1, Z_2, Z_3 in K_i with corresponding vertices z_1, z_2, z_3. Since $\alpha_1 \notin L(z_2), L(z_3)$, coloring z_1 with α_1 leaves a list assignment we can complete greedily by coloring z_2 and z_3 last. However, we need to be careful to not break the other such D_i in the process. So, we first color the respective z_1 in each such D_i. After all of those have been colored, we greedily color the rest of each D_i. It still needs to be checked that when we color z_1 with α_1 we don't lose the ability to do the same with α_1 on some other D_j. To see this, note that x has at least 3 neighbors in

C_i and thus is contain in no other C_j with $|C_j| = \Delta - 1$. Moreover, z_1's only possible other neighbor y outside C_i is already colored by assumption. Now consider the D_i that have two safe uncolored vertices in C_i. If $C_i \neq K_i$, then since Q_i doesn't happen x_i has two uncolored neighbors, color it first. Now color C_i greedily saving the two safe uncolored vertices in C_i for last. Now we can finish the coloring on the sparse vertices greedily. Therefore if we can prevent all the bad events from happening we get our desired contradiction.

It is easy to see that any given event depends on less than $3\Delta^5$ others, so the result will follow by showing that $\Pr(S_v), \Pr(E_i), \Pr(Q_i), \Pr(F_i), \Pr(P_i) \leq \Delta^{-6}$. The following sections prove these bounds.

9.4 $\Pr(S_v) \leq \Delta^{-6}$

We know $\|G_v\| < \frac{2}{5}\Delta^2$. Put $A := \left\{x \in N(v) \mid |L(x) \cap L(v)| \geq \frac{2}{3}\Delta\right\}$ and $B := N(v) - A$. Note that for $x, y \in A$ we have $|L(x) \cap L(y)| \geq \frac{1}{3}\Delta$ and for $x \in B$ we have $|L(x) - L(v)| \geq \frac{1}{3}\Delta$.

Let A_v be the random variable that counts the number of nonadjacent pairs $x, y \in A$ such that, $\zeta(x) = \zeta(y)$ and $\zeta(z) \neq \zeta(x)$ for all $z \in N(v) - \{x, y\} \cup N(x) \cup N(y)$.

Let B_v be the random variable that counts the number of $x \in B$ such that $\zeta(x) \notin L(v)$ and $\zeta(z) \neq \zeta(x)$ for all $z \in N(v) - \{x\} \cup N(x)$.

Put $Z_v := A_v + B_v$. Then $\mathrm{E}(Z_v) = \mathrm{E}(A_v) + \mathrm{E}(B_v)$. We prove the bound $\mathrm{E}(Z_v) \geq \frac{\Delta}{1000}$ and then use Azuma's inequality to prove that $\Pr(|Z_v - \mathrm{E}(Z_v)| > \frac{\Delta}{1000} - 2) \leq \Delta^{-6}$. The conclusion $\Pr(S_v) \leq \Delta^{-6}$ is then immediate.

We know that G_v has at least $\binom{\Delta-1}{2} - \frac{2}{5}\Delta^2 \geq \frac{\Delta^2}{12}$ nonadjacent pairs. Let b be the number of nonadjacent pairs in G_v that intersect B. Plainly, $G[A]$ contains at least $\frac{\Delta^2}{12} - b$ nonadjacent pairs and $b \leq |B|\,\Delta$.

First let's consider $\mathrm{E}(A_v)$. Let $x, y \in A$ be nonadjacent. Since $|L(x) \cap L(y)| \geq \frac{1}{3}\Delta$, the probability that x and y get the same color and this color is not used on any of the rest of $N(v) \cup N(x) \cup N(y)$ is at least $(3\Delta)^{-1}(1-(\Delta-1)^{-1})^{3\Delta-3} \geq (3\Delta)^{-1}3^{-3}$. Thus $\mathrm{E}(A_v) \geq (\frac{\Delta^2}{12} - b)\Delta^{-1}3^{-4} \geq \frac{\Delta}{1000} - \frac{b}{81\Delta}$.

Now consider $\mathrm{E}(B_v)$. Let $x \in B$. Since $|L(x) - L(v)| \geq \frac{1}{3}\Delta$, the probability that x gets a color not in $L(v)$ and this color is not used on any of the rest of $N(v) \cup N(x)$ is at least $\frac{1}{3}(1-(\Delta-1)^{-1})^{2\Delta-2} \geq 3^{-4}$. Hence $\mathrm{E}(B_v) \geq \frac{|B|}{81} \geq \frac{b}{81\Delta}$. Therefore $\mathrm{E}(Z_v) \geq \frac{\Delta}{1000}$.

Now we need Azuma's inequality. The concentration analysis is identical to the coloring case in Reed's proof. We reproduce it here for completeness.

Lemma 9.4.1 (Azuma)**.** *Let X be a random variable determined by n trials $T_1, \ldots, T_n$ such that for each i and any two possible sequences of outcomes $t_1, \ldots, t_i$ and $t_1, \ldots, t_{i-1}, t_i'$:*

$$|\mathrm{E}(X \mid T_1 = t_1, \ldots, T_i = t_i) - \mathrm{E}(X \mid T_1 = t_1, \ldots, T_i = t_i')| \leq c_i,$$

then $\Pr(|X - \mathrm{E}(X)| > t) \leq 2e^{\frac{-t^2}{2\sum c_i^2}}$.

Since we colored the vertices of G independently, we can apply Azuma using any ordering. Order $V(G)$ as $w_1, \ldots, w_n$ so that $N(v)$ comes last and let w_s be the last vertex not in $N(v)$. Changing $\zeta(w_i)$ from β to τ only affects the vertices using β or τ and thus changes the conditional expected value by at most 2. For $w_i \notin N(v)$, the probability that changing w_i's color will affect Z_v is at most the probability that one of w_i's two colors is also assigned to one of its neighbors in $N(v)$. Say w_i has d_i neighbors in $N(v)$. Then the most changing w_i can change $\mathrm{E}(Z_v)$ is $c_i := 2\frac{2d_i}{\Delta-1} = \frac{4d_i}{\Delta-1}$. Now $\sum_{i=1}^{s} d_i \leq \Delta^2$ and thus $\sum_{i=1}^{s} c_i \leq 4\Delta + 4\frac{\Delta}{\Delta-1} \leq 4\Delta + 5$. As each $c_i \leq 5$, we have $\sum_{i=1}^{s} c_i^2 \leq 21\Delta$ and hence $\sum_i c_i^2 \leq 25\Delta$. Now using $t := \frac{\Delta}{1000} - 2$ in Azuma gives $\Pr(Z_v < 2) < 2e^{\frac{-(\frac{\Delta}{1000}-2)^2}{50\Delta}} \leq \Delta^{-6}$ for large enough Δ.

9.5 $\Pr(E_i) \leq \Delta^{-6}$

We first need a couple structural lemmas.

Lemma 9.5.1. *Each $v \in C_i$ has at most one neighbor outside of C_i with more than 4 neighbors in C_i and no such neighbor if v is low.*

Proof. Suppose otherwise that we have $v \in C_i$ with two neighbors $w_1, w_2 \in V(G) - C_i$ each with 5 or more neighbors in C_i. Put $Q := G[\{w_1, w_2\} \cup C_i - v]$, then v is joined to Q and hence $K_1 * Q \trianglelefteq G$. We show that $K_1 * Q$ must be d_1-choosable.

First, suppose there are different $z_1, z_2 \in C_i$ such that $\{w_1, z_1\}$ and $\{w_2, z_2\}$ are independent. Since Q contains an induced diamond, it is d_0-choosable. Let L be a minimal bad d_1-assignment on $K_1 * Q$. Then $|L(w_i)| + |L(z_i)| \geq 4 + |Q| - 3 = |Q| + 1$. By the Small Pot Lemma, $|Pot(L)| \leq |Q|$. Hence $L(w_1) \cap L(z_1) \neq \emptyset$ and Lemma 5.3.15 shows that $|Pot(L)| \leq |Q| - 1$, but then $|L(w_i) \cap L(z_i)| \geq 2$ and Lemma 5.3.14 gives a contradiction.

By maximality of C_i, neither w_1 nor w_2 can be adjacent to all of C_i hence it must be the case that there is $y \in C_i$ such that w_1 and w_2 are joined to $C_i - y$. If w_1 and w_2 aren't adjacent, then G contains $K_6 * E_3$ contradicting Corollary 5.3.25. Hence C_i intersects the larger clique $\{w_1, w_2\} \cup C_i - \{y\}$, this is impossible by the definition of C_i.

When v is low, an argument similar to the above shows that there can be no z_1 in C_i so that $\{w_1, z_1\}$ is independent, and hence $C_i \cup \{w_1\}$ is a clique contradicting maximality of C_i. $\square$

Lemma 9.5.2. *For C_i with $|C_i| \leq \Delta - 2$, there are at least $\frac{3}{28}\Delta$ disjoint P_3's xyz with $y \in C_i$ and $x, z \notin C_i$ such that x and z each have at most 4 neighbors in C_i.*

Proof. Consider a maximal such set of P_3's. Let A be all the central vertices of these P_3's and X all the ends. Then each $v \in X$ has at most 3 neighbors in $C_i - A$ and by Lemma 9.5.1 and maximality, each $v \in C_i - A$ has at most 2 neighbors in $G - C_i - B$ and at most 1 if v is low. Thus $6|A| = 3|X| \geq \|C_i - A, X\| \geq (\Delta - |C_i| - 1)|C_i - A| \geq |C_i| - |A|$. Hence $|A| \geq \frac{3}{28}\Delta$. □

We need to force safe uncolored vertices in C_i. If the lists have small intersections this might not happen with high probability. We handle this case using minimality of $|G|$ instead.

Lemma 9.5.3. *There exists $C_i' \subset C_i$ with $|C_i'| = |C_i| - 1$ such that for $x, y \in C_i'$ we have $|L(x) \cap L(y)| \geq \frac{2}{3}\Delta$.*

Proof. Suppose not and consider an L-coloring of $G - C_i$. Let L' be the resulting list assignment on C_i. Then $|L'(v)| \geq |C_i| - 2$ for all $v \in C_i$. By assumption, for each $v \in C_i$ we have $x, y \in C_i - \{v\}$ with $|L(x) \cap L(y)| < \frac{2}{3}\Delta$. But then $|L'(x) \cup L'(y)| \geq 2(\Delta - 1 - (\Delta + 1 - |C_i|)) - \frac{2}{3}\Delta \geq |C_i|$. Hence we can complete the L-coloring to C_i by Hall's theorem, a contradiction. □

We will find the desired uncolored safe vertices in C_i'. By Lemma 9.5.2, there are at least $\frac{\Delta}{10}$ paths acb where $c \in C_i'$ and $a, b \notin C_i'$ such that a and b each have at most 4 neighbors in C. Let T_i be the union of all the vertices in these paths. For some such fixed path we want to bound the probability that c is uncolored and safe and the colors used on a and b are used on none of the rest of T_i. To do so, we distinguish three cases.

Case 1. $|L(a) \cap L(c)| < \frac{2}{3}\Delta$ *and* $|L(b) \cap L(c)| < \frac{2}{3}\Delta$

For $\alpha \in L(a) - L(c)$, $\beta \in L(b) - L(c)$, $z \in C_i' - T_i$ and $\gamma \in L(c) \cap L(z)$ where α, β, γ are all different, let $A_{\alpha,\beta,\gamma,z}$ be the event that all of the following hold:

1. α is assigned to a and none of the rest of $T_i \cup N(a)$,
2. β is assigned to b and none of the rest of $T_i \cup N(b)$,
3. γ is assigned to c and z and none of the rest of $T_i \cup N(c)$.

Then $\Pr(A_{\alpha,\beta,\gamma,z}) \geq (\Delta-1)^{-1}(1-3(\Delta-1)^{-1})^{|T_i \cup N(a)|}(\Delta-1)^{-1}(1-3(\Delta-1)^{-1})^{|T_i \cup N(b)|}(\Delta-1)^{-2}(1-3(\Delta-1)^{-1})^{|T_i \cup N(c)|} \geq (\Delta-1)^{-4}3^{-18}$. Plainly, the $A_{\alpha,\beta,\gamma,z}$ are disjoint for different sets of indices.

Since $|L(a) - L(c)| \geq \frac{\Delta}{3}$, we have $\frac{\Delta}{3}$ choices for α. Similarly we then have $\frac{\Delta}{3} - 1$ choices for β. For z we have at least $\frac{3}{4}\Delta - \frac{1}{10}\Delta \geq \frac{\Delta}{3}$ choices. Since $|L(z) \cap L(c)| \geq \frac{2}{3}\Delta$, we then have at least $\frac{2}{3}\Delta - 2$ choices for γ for each z. In total we have at least $\Delta^4 3^{-4}$ choices and thus the probability that $A_{\alpha,\beta,\gamma,z}$ holds for some choice of indices is at least 3^{-22}.

Case 2. $|L(a) \cap L(c)| < \frac{2}{3}\Delta$ *and* $|L(b) \cap L(c)| \geq \frac{2}{3}\Delta$

For $y \in C_i' - T_i - N(b)$, $z \in C_i' - T_i$, $\alpha \in L(a) - L(c)$, $\beta \in L(b) \cap L(y)$ and $\gamma \in L(c) \cap L(z)$ where α, β, γ are all different, let $A_{\alpha,\beta,\gamma,y,z}$ be the event that all of the following hold:

1. α is assigned to a and none of the rest of $T_i \cup N(a)$,
2. β is assigned to b and y and none of the rest of $T_i \cup N(b) \cup N(y)$,
3. γ is assigned to c and z and none of the rest of $T_i \cup N(c)$.

Then $\Pr(A_{\alpha,\beta,\gamma,y,z}) \geq (\Delta-1)^{-1}(1-3(\Delta-1)^{-1})^{|T_i\cup N(a)|}(\Delta-1)^{-2}(1-3(\Delta-1)^{-1})^{|T_i\cup N(b)\cup N(y)|}(\Delta-1)^{-2}(1-3(\Delta-1)^{-1})^{|T_i\cup N(c)|} \geq (\Delta-1)^{-5}3^{-21}$.

Again the $A_{\alpha,\beta,\gamma,y,z}$ are disjoint for different sets of indices. For y we have at least $|C_i'|-|T_i\cap C_i'|-|N(b)\cap C_i| \geq \frac{3}{4}\Delta-1-\frac{\Delta}{10}-4 \geq \frac{\Delta}{9}$ choices. For each y we have at least $\frac{2}{3}\Delta$ choices for β. The rest are similar to above and in total we have at least $\Delta^5 3^{-6}$ choices and thus the probability that $A_{\alpha,\beta,\gamma,y,z}$ holds for some choice of indices is at least 3^{-27}.

Case 3. $|L(a)\cap L(c)| \geq \frac{2}{3}\Delta$ *and* $|L(b)\cap L(c)| \geq \frac{2}{3}\Delta$

For $x \in C_i'-T_i-N(a)$, $y \in C_i'-T_i-N(b)$, $z \in C_i'-T_i$, $\alpha \in L(a)\cap L(c)$, $\beta \in L(b)\cap L(y)$ and $\gamma \in L(c)\cap L(z)$ where α,β,γ are all different, let $A_{\alpha,\beta,\gamma,x,y,z}$ be the event that all of the following hold:

1. α is assigned to a and y and none of the rest of $T_i\cup N(a)\cup N(x)$,
2. β is assigned to b and y and none of the rest of $T_i\cup N(b)\cup N(y)$,
3. γ is assigned to c and z and none of the rest of $T_i\cup N(c)$.

Then $\Pr(A_{\alpha,\beta,\gamma,x,y,z}) \geq (\Delta-1)^{-2}(1-3(\Delta-1)^{-1})^{|T_i\cup N(a)\cup N(x)|}(\Delta-1)^{-2}(1-3(\Delta-1)^{-1})^{|T_i\cup N(b)\cup N(y)|}(\Delta-1)^{-2}(1-3(\Delta-1)^{-1})^{|T_i\cup N(c)|} \geq (\Delta-1)^{-6}3^{-24}$.

Again the $A_{\alpha,\beta,\gamma,x,y,z}$ are disjoint for different sets of indices. In total we get at least $\Delta^6 3^{-8}$ choices and thus the probability that $A_{\alpha,\beta,\gamma,x,y,z}$ holds for some choice of indices is at least 3^{-32}.

Now we have at least $\frac{\Delta}{10}$ such triples. So if M_i counts the number of uncolored safe vertices in C_i we have $\mathrm{E}(M_i) \geq 3^{-35}\Delta$. The concentration details are identical to Reed's proof and we conclude $Pr(M_i < 2) < \Delta^{-6}$.

9.6 $\Pr(Q_i) \leq \Delta^{-6}$

If $\zeta(x) = \zeta(y)$ for different $x, y \in K_i$, then x and y will be uncolored and Q_i cannot hold. Thus it is enough to show that all vertices of K_i getting different colors is unlikely. Just like Lemma 9.5.3, we can find $K_i' \subset K_i$ with $|K_i'| = |K_i| - 1$ such that for $x, y \in K_i'$ we have $|L(x) \cap L(y)| \geq \frac{2}{3}\Delta$.

Let $x, y \in K_i'$. The probability that x and y get the same color and this color is used on none of the rest of $N(x) \cup N(y)$ is at least $\frac{2}{3\Delta}(1 - (\Delta - 1)^{-1})^{2\Delta - 2} \geq \frac{2}{3^3\Delta}$. Since there are at least $\frac{1}{2}(\frac{2}{3}\Delta)^2$ such pairs, the expected number of pairs getting the same color is at least $3^{-4}\Delta$. An application of Azuma's inequality very similar to the sparse case now proves $\Pr(Q_i) \leq \Delta^{-6}$.

9.7 $\Pr(F_i) \leq \Delta^{-6}$

In this case we must have $C_i = K_i$ since no vertex outside C_i has $\frac{3}{4}\Delta$ neighbors in C_i. Since low vertices don't make things harder, we will assume there are no low vertices in C_i. In particular, for a low vertex, we don't need a triple as in the follow lemma, but just one good neighbor outside because we only need to save one color on a low vertex's neighborhood to make it safe.

Lemma 9.7.1. *There are at least $\frac{1}{4}\sqrt{\Delta}\log\Delta$ disjoint P_3's xyz with $y \in C_i$ and $x, z \notin C_i$ such that x and z each have at most $\frac{\sqrt{\Delta}}{\log(\Delta)}$ neighbors in C_i.*

Proof. Since there are at most $\log^2(\Delta)$ vertices outside C_i which have more than $\frac{\sqrt{\Delta}}{\log(\Delta)}$ neighbors in C_i and all of these vertices have at most $\sqrt{\Delta}\log(\Delta)$ neighbors in C_i, removing all their neighbors from C_i we are left with a set A of of vertices all of whose neighbors outside C_i have at most $\frac{\sqrt{\Delta}}{\log(\Delta)}$ neighbors in C_i. Now $|A| \geq \Delta - 1 - \log^2(\Delta)\sqrt{\Delta}\log(\Delta) \geq \frac{\Delta}{2}$. Now pick P_3's xyz with $y \in A$ in turn removing the neighbors of x and z each time. We get at least $\frac{|A|}{2\frac{\sqrt{\Delta}}{\log(\Delta)}} \geq \frac{1}{4}\sqrt{\Delta}\log\Delta$ disjoint P_3's.

Now the proof of the expected value is the same as the proof of $\Pr(E_i) \le \Delta^{-6}$, except that we have fewer P_3's to multiply by at the end. So, if M_i counts the number of uncolored safe vertices in C_i, we have $\mathrm{E}(M_i) \ge 3^{-35}(\frac{1}{4})\sqrt{\Delta}\log\Delta \ge 3^{-37}\sqrt{\Delta}\log\Delta$.

Now, the application of Azuma is the same as in the $\Pr(E_i) \le \Delta^{-6}$ case, except we use $t := 3^{-37}\sqrt{\Delta}\log\Delta - 2$, which gives gives $\Pr(M_i < 2) < 2e^{\frac{-(3^{-37}\sqrt{\Delta}\log\Delta-2)^2}{\Delta}} \le \Delta^{-6}$ for large enough Δ. $\square$

9.8 $\Pr(P_i) \le \Delta^{-6}$

Case 1. *Some $x \in G - C_i$ has $|N(x) \cap C_i| > \sqrt{\Delta}\log(\Delta)$.*

If $C_i \neq K_i$, then take x to be x_i. Let $Z_1, \ldots, Z_m$ be the clumps of x in K_i and for $j \in [m]$, let α_j be the color the Z_j clump has that the others do not. By Lemma 9.5.3, we may as well assume that $|L(x) \cap L(y)| \ge \frac{2}{3}\Delta$ for all $x, y \in K_i$ (the cost is one vertex which changes nothing).

Pick $z_j \in Z_j$ arbitrarily. By Lemma 9.5.1, any neighbor of z_j in $G - C_i - x$ (of which there is at most one) has at most 4 neighbors in C_i. Thus, by symmetry, for each $j \in \left[\frac{m}{4}\right]$ we can pick $y_j \in G - C_i - x$ such that $y_j z_j \in E(G)$ and the y_j are all different. Put $A := N(x) \cap K_i$. Then $\frac{m}{4} \ge (\frac{1}{4})(\frac{1}{5})\,|A| \ge \frac{1}{20}\sqrt{\Delta}\log(\Delta)$.

Now, for fixed $j \in \left[\frac{m}{4}\right]$, we bound the probability that z_j is uncolored, α_j is not used on any neighbor of z_j and y_j is colored with a color that is either not in $L(z_j)$ or is used on C_i. Let T_i be the union of all the y_j's and z_j's. We distinguish two cases.

Subcase 1a. $|L(y_j) \cap L(z_j)| < \frac{2}{3}\Delta$

For $\beta \in L(y_j) - L(z_j)$, $w \in C_i - T_i$ and $\gamma \in L(z_j) \cap L(w)$ where $\beta, \gamma \neq \alpha_j$ and $\beta \neq \gamma$, let $F_{\beta,\gamma,w}$ be the event that all of the following hold:

1. β is assigned to y_j and none of the rest of $T_i \cup N(y_j)$,

2. γ is assigned to z_j and w and none of the rest of $T_i \cup N(z_j)$,

3. α_j is assigned to no neighbor of z_j.

The probability of (3) is at least $\left(\frac{\Delta-2}{\Delta-1}\right)^{\Delta} = (1-(1-\Delta)^{-1})^{\Delta} \geq \frac{1}{3}$. Hence $\Pr(F_{\beta,\gamma,w}) \geq \frac{1}{3}(\Delta-1)^{-1}(1-3(\Delta-1)^{-1})^{|T_i \cup N(y_j)|}(\Delta-1)^{-2}(1-3(\Delta-1)^{-1})^{|T_i \cup N(z_j)|} \geq (\Delta-1)^{-3}3^{-13}$.

Now we have at least $\frac{\Delta}{3}$ choices for β, $\frac{\Delta}{2}$ choices for w and $\frac{2}{3}\Delta$ choices for γ for each w. Thus the probability that $F_{\beta,\gamma,w}$ holds for some choice of indices is at least 3^{-15}.

Subcase 1b. $|L(y_j) \cap L(z_j)| \geq \frac{2}{3}\Delta$ For $y \in C_i - T_i - N(y_j)$, $\beta \in L(y_j) \cap L(y)$, $w \in C_i - T_i$ and $\gamma \in L(z_j) \cap L(w)$ where $\beta, \gamma \neq \alpha_j$ and $\beta \neq \gamma$, let $F_{\beta,\gamma,y,w}$ be the event that all of the following hold:

1. β is assigned to y_j and y and none of the rest of $T_i \cup N(y_j) \cup N(y)$,

2. γ is assigned to z_j and w and none of the rest of $T_i \cup N(z_j)$,

3. α_j is assigned to no neighbor of z_j.

We have $\Pr(F_{\beta,\gamma,y,w}) \geq \frac{1}{3}(\Delta-1)^{-2}(1-3(\Delta-1)^{-1})^{|T_i \cup N(y_j) \cup N(y)|}(\Delta-1)^{-2}(1-3(\Delta-1)^{-1})^{|T_i \cup N(z_j)|} \geq (\Delta-1)^{-4}3^{-16}$.

Now for y we have at least $|K_i| - |C_i \cap T_i| - |N(y_j) \cap C_i| \geq \frac{\Delta}{2}$ choices and for each y we have at least $\frac{2}{3}\Delta$ choices for β. Thus the probability that $F_{\beta,\gamma,y,w}$ holds for some choice of indices is at least 3^{-18}.

Therefore the expected number of good clumps is at least $3^{-18}(\frac{1}{20})\sqrt{\Delta}\log(\Delta) \geq 3^{-21}\sqrt{\Delta}\log(\Delta)$. Changing any color will affect the conditional expectations by at most 2 and a similar computation for Azuma shows that $\Pr(F_i) \leq \Delta^{-6}$. The key here is that $(\sqrt{\Delta}\log(\Delta))^2$ grows faster that Δ.

Case 2. *More than* $\log^2(\Delta)$ *vertices* $x \in G - C_i$ *have* $|N(x) \cap C_i| > \frac{\sqrt{\Delta}}{\log(\Delta)}$.

We must have $C_i = K_i$. Let $x_1, \ldots, x_k$ be $k := \lceil \log^2(\Delta) \rceil$ different vertices in $G - C_i$ which have $|N(x_j) \cap C_i| > \frac{\sqrt{\Delta}}{\log(\Delta)}$ for each $j \in [k]$.

The computation for the expected number of good clumps for each x_j is the same as Case 1 and so we expect at least $3^{-21}\frac{\sqrt{\Delta}}{\log(\Delta)}$ good clumps for each x_j. Thus in total we expect $3^{-21}\sqrt{\Delta}\log(\Delta)$ good clumps over the $\log^2(\Delta)$ sets. Let X count this total number of good clumps. We show that $\Pr(X < 3\log^2(\Delta)) \leq \Delta^{-6}$ and hence at least one x_j has at least 3 good clumps with high enough probability.

If we applied Azuma with the information we have now we'd be in trouble because many of the x_j's could use the same special color and hence changing a vertex to that color would change the conditional expectation by a lot. We need one further structural lemma that guarantees at most 4 of the x_j's use any given special color.

Lemma 9.8.1. *Let* K *be a* $\Delta - 1$ *clique in* G *and* $x_1, x_2, x_3, x_4, x_5 \in G - K$ *with* $|N(x_j) \cap K| \geq 5$ *such that the* $N(x_j) \cap K$ *are pairwise disjoint. Then no color is special for all the* x_j.

Proof. Suppose otherwise that some color α is special for all the x_j. Put $A_j := N(x_j) \cap K$. Just like in the proof of Lemma 9.3.1, any L-coloring of $G-(K\cup\{x_1, \ldots, x_5\})$ must not leave α available on any of the vertices in A_j for any $j \in [5]$. Pick $z_j \in A_j$ for each j and let y_j be the neighbor of z_j in $G - (K \cup \{x_1, \ldots, x_5\})$. Put $N := \{y_1, \ldots, y_5\}$. By Lemma 9.5.1, $|N| \geq 2$.

Now just like in Lemma 9.3.1, by using minimality of $|G|$ we see that adding any edge between vertices in N must create a K_Δ and then counting degrees gives a contradiction. $\square$

Now when we change a color we change the conditional expectation by at most 8. A similar computation to before bounds $\sum_j c_j^2 \leq 500\Delta$. Applying Azuma with $t = 3^{-21}\sqrt{\Delta}\log(\Delta) - 3\log^2(\Delta)$ gives $\Pr(X < 3\log^2(\Delta)) < 2e^{\frac{-(3^{-21}\sqrt{\Delta}\log(\Delta)-3\log^2(\Delta))^2}{500\Delta}} \leq \Delta^{-6}$ for large Δ.

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Appendix A

NOTATION

Symbology	Meaning
$\lvert G\rvert$	the number of vertices G has
$\lVert G\rVert$	the number of edges G has
$x \leftrightarrow y$	x and y are adjacent
$G[S]$	the subgraph of G induced on S
$E_G(X,Y)$	the edges in G with one end in X and the other in Y
$E_G(X)$	$E_G(X, V(G)-X)$
$\chi(G)$	the chromatic number of G
$\omega(G)$	the clique number of G
$\alpha(G)$	the independence number of G
$\Delta(G)$	the maximum degree of G
$\delta(G)$	the minimum degree of G
$\kappa(G)$	the vertex connectivity of G
$\overline{G}$	the complement of G
$A+B$	the disjoint union of graphs A and B
$A*B$	the join of graphs A and B (that is, $\overline{\overline{A}+\overline{B}}$)
kG	$\underbrace{G+G+\cdots+G}_{k\text{ times}}$
G^k	$\underbrace{G*G*\cdots*G}_{k\text{ times}}$
$H \subseteq G$	H is a subgraph of G
$H \subset G$	H is a proper subgraph of G
$H \trianglelefteq G$	H is an induced subgraph of G
$H \triangleleft G$	H is a proper induced subgraph of G
$H \prec G$	H is a child of G
$f\colon S \hookrightarrow T$	an injective function from S to T
$f\colon S \twoheadrightarrow T$	a surjective function from S to T
$X := Y$	X is defined as Y
K_k	the complete graph on k vertices
E_k	the edgeless graph on k vertices (that is, $\overline{K_k}$)
P_k	the path on k vertices
C_k	the cycle on k vertices
$K_{a,b}$	the complete bipartite graph with parts of size a and b (that is, $E_a * E_b$)
$[n]$	$\{1, 2, \ldots, n\}$