Optimal Degree Conditions for Spanning Subgraphs

by

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A Dissertation Presented in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

Approved July, 2011 by the
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August 2011

ABSTRACT

In a large network (graph) it would be desirable to guarantee the existence of some local property based only on global knowledge of the network. Consider the following classical example: how many connections are necessary to guarantee that the network contains three nodes which are pairwise adjacent? It turns out that more than $n^2/4$ connections are needed, and no smaller number will suffice in general. Problems of this type fall into the category of "extremal graph theory."

Generally speaking, extremal graph theory is the study of how global parameters of a graph are related to local properties. This dissertation deals with the relationship between minimum degree conditions of a host graph G and the property that G contains a specified spanning subgraph (or class of subgraphs). The goal is to find the optimal minimum degree which guarantees the existence of a desired spanning subgraph. This goal is achieved in four different settings, with the main tools being Szemerédi's Regularity Lemma; the Blow-up Lemma of Komlós, Sárközy, Szemerédi ; and some basic probabilistic techniques.

DEDICATION

Tom was a good mathematician, and a good man. He was – he was one of us. He was a man who loved the outdoors, and math. And as a teacher he explored the schools of the northeast from New York to Massachusetts, and then down to Arizona. He died, as so many young men of his generation, before his time...

This dissertation is dedicated to the memory of my uncle Tom Wielunski. Regretfully, my period of mathematical enlightenment didn't intersect with your life; however, I remain inspired by you.

ACKNOWLEDGEMENTS

Most importantly, I would like to thank my parents. Their support has helped me achieve goals that, just a short time ago, I never would have imagined.

This dissertation would not be possible without the guidance and assistance of my advisors, Andrzej Czygrinow and Hal Kierstead. I can't thank them enough for the countless hours they spent working with me. I would like to thank the members of my committee: Glenn Hurlbert, Susanna Fishel and Kevin Kadell. In particular, Glenn has had a major influence on my mathematical development. In addition, I am extremely grateful to Yi Zhao, Matthias Kawski, Keyong Hah Roh, and Katie Kolossa for taking the time to write letters of recommendation for me; Anne Gelb for her advice and her work with PFMF; and John Jones for his time during my defense.

Thank you to Kelsey Vance for her love and encouragement; David Smith for being a great friend and a great source of knowledge; and Phong Châu for the many stimulating discussions. Additionally, I've had the pleasure of being friends with the following people, some I met in grad school and others long before: Adam Bland, Katie Carter, Ian Dye, Rosalynn Dye, Molly Etchebarren, Ryan Etchebarren, Tacker Frink, Aviva Halani, Ben Hester, Taylor Hines, Vikram Kamat, Marilyn Murphy, Bruce Rogers, Karin Saoub, Susie Seal, Chris Severs, Wei Shen, Matt Smith, and Heather Sollum.

I must thank the ASU math department as a whole for providing me with many opportunities; I have been very fortunate. A special thanks goes to Debbie Olsen for answering all of my questions and helping me out in a big way with the job application process.

Finally, one BIG thank you to my BIG family who have always been so supportive of me.

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Chapter 1

BACKGROUND MATERIAL

A *hypergraph* is a pair of sets (V, E) with the property that E is a family of subsets of V. A *graph* is a hypergraph in which every element of E has order 2. Given a hypergraph $G = (V, E)$, we refer to the set V as *vertices*, and the set E as *edges*. For any graph G, we will use the notation $V(G)$ to represent the set of vertices of G and the notation $E(G)$ to represent the edges of G. In this dissertation we will only consider graphs with a finite vertex set. Given a set V and a nonnegative integer k, we let $\binom{V}{k} = \{S \subseteq V : |S| = k\}$. For a positive integer k, let $[k] = \{1, 2, \ldots, k\}$. We write edges $\{x, y\}$ as xy.

Let $H = (W, F)$ and $G = (V, E)$ be graphs. We say H is *isomorphic* to G if there exists a function $f : W \to V$ such that $xy \in F$ if and only if $f(x)f(y) \in E$. We say H is a *subgraph* of G, denoted $H \subseteq G$, if there exists some $V' \subseteq V$ and $E' \subseteq \binom{V'}{2}$ such that H is isomorphic to (V', E').

The complete graph is a graph G for which $E(G) = \binom{V(G)}{2}$. We denote the complete graph on r vertices as K_r, and we call K_3 a *triangle*. The starting point of extremal graph theory can be captured in the following question: If G is a graph on n vertices, what is the fewest number of edges that G must have in order to guarantee that G contains a triangle? The answer to this question is Mantel's theorem from 1907.

Theorem 1.0.1 (Mantel [37]). *Let G be a graph on n vertices. If $|E(G)| \geq \left\lfloor \frac{n^2}{4} \right\rfloor + 1$, then $K_3 \subseteq G$. Furthermore, there exists a graph with $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges which is triangle-free.*

Of course the natural follow-up question is: If r is fixed and G is a graph on n vertices, what is the fewest number of edges that G must have in order to guarantee that $K_r \subseteq G$? It appears as if Mantel's result was mostly unknown, since it was not until 1941 when Turán independently asked himself that very question and solved it for all r (for an incredible story of how Turán solved this problem while working in a labor camp during World War II, see [47]). Let $T_r(n)$ be the complete r-partite graph on n vertices such that the sizes of any two parts differ by at most 1; it is clear that $T_r(n)$ does not contain a copy of K_{r+1}. Let $t_r(n)$ be the number of edges in $T_r(n)$. Note that when r divides n, we have $t_r(n) = \binom{r}{2}\left(\frac{n}{r}\right)^2 = \frac{r-1}{r}\frac{n^2}{2}$.

Theorem 1.0.2 (Turán [46])**.** *Let G be a graph on n vertices. If $|E(G)| \geq t_r(n) + 1$, then $K_{r+1} \subseteq G$. Furthermore, there exists a graph with $t_r(n)$ edges which is K_{r+1}-free.*

Starting with Turán's theorem, the subject of extremal graph theory blossomed into a coherent subject with many interesting theorems and powerful techniques.

For the rest of this dissertation we will be focusing on subgraph problems, but of a slightly different type. If H is a subgraph of G, we say H is *spanning* if H and G have the same number of vertices. In this case, it is no longer natural to ask how many edges G must have so that $H \subseteq G$. To see why, let H be any connected graph on n vertices and let G be the complete graph K_{n-1} plus an isolated vertex. On one hand G has almost every possible edge, but on the other hand H is not a subgraph of G. So when studying sufficient conditions for spanning subgraphs, the most natural thing is to restrict the number of edges at each vertex. Let G be a graph and $v \in V(G)$. The neighborhood of v, denoted $N(v)$, is the set $\{u \in V(G) : uv \in E(G)\}$. The degree of v, denoted $\deg(v)$, is the quantity $|N(v)|$. The *minimum degree* of G, denoted $\delta(G)$, is the quantity $\min\{\deg(v) : v \in V(G)\}$. For a set $S \subseteq V(G)$, we write $\deg(v, S)$ for the quantity $|N(v) \cap S|$. We will study the relationship between minimum degree of a graph G and the property $H \subseteq G$. If G has at least as many vertices as H, there is always a relationship between $\delta(G)$ and the property $H \subseteq G$: if $\delta(G) \geq n-1$, then G is complete and $H \subseteq G$. So the goal is to minimize $\delta(G)$ with respect to the condition $H \subseteq G$.

We first define two special types of graphs. Let P_k be a graph with vertex set $\{v_1, v_2, \ldots, v_k\}$ and edge set $\{v_i v_{i+1} : i \in [k-1]\}$. We call P_k a *path* on k vertices and we denote P_k as $v_1 v_2 \ldots v_k$. Let C_k be a graph with vertex set $\{v_1, v_2, \ldots, v_k\}$ and edge set $\{v_i v_{i+1} : i \in [k-1]\} \cup \{v_k v_1\}$. We call C_k a *cycle* on k vertices and we denote C_k as $v_1 v_2 \ldots v_k v_1$.

Let G be a graph on n vertices. To illustrate the title "Optimal Degree Conditions for Spanning Subgraphs", we will fully examine the (well known) relationship between $\delta(G)$ and the property $C_n \subseteq G$. We start with the following basic fact.

Proposition 1.0.3. *If $\delta(G) \geq 2$, then G contains a cycle on at least $\delta(G)+1$ vertices.*

Proof. Let $P = v_1 v_2 \ldots v_k$ be a path of maximum length in G. Since P is maximum, $N(v_1) \subseteq V(P)$. Since $\deg(v_1) \geq \delta(G)$, there exists some $v_i \in N(v_1)$ such that $i \geq \delta(G)+1$. Thus $v_1 v_2 \ldots v_i v_1$ is a cycle on at least $\delta(G)+1$ vertices. $\square$

Unfortunately, we cannot use this result to directly conclude anything about the current problem. All we get is that $\delta(G) \geq n-1$ implies $C_n \subseteq G$, however we already knew this from the discussion above. So we try to do better.

Proposition 1.0.4. *If $\delta(G) \geq \frac{2n}{3}$, then $C_n \subseteq G$.*

Proof. Let $C = v_1v_2\ldots v_kv_1$ be a cycle of maximum length in G. By Proposition 1.0.3, we know $k \geq \frac{2n}{3} + 1$. If $k = n$, we are done, so suppose not. Let $x \in V(G) \setminus V(C)$. If $\deg(x, C) > \frac{k}{2}$, then there exists $i \in [k]$ such that $v_i, v_{i+1} \in N(x)$, but then $v_1v_2\ldots v_ixv_{i+1}\ldots v_kv_1$ is longer cycle than C. So we may suppose that $\deg(x, C) \leq \frac{k}{2}$. However, now we have the following contradiction

$$\frac{2n}{3} \leq \deg(x) \leq \deg(x,C) + \deg(x, G - C) \leq \frac{k}{2} + n - k - 1 = n - 1 - \frac{k}{2} \leq \frac{2n}{3} - \frac{3}{2}.$$

□

So now we ask ourselves if any lower value of $\delta(G)$ will suffice. One thing to do would be to try to construct a graph with minimum degree less than $\frac{2n}{3}$ which does not contain C_n. After trying for a while, two examples might come to mind.

Proposition 1.0.5. *There exists a graph G on n vertices with $\delta(G) = \left\lceil \frac{n}{2} \right\rceil - 1$ such that G does not contain C_n.*

Proof. We give two examples of such a graph. Let G_1 be the union of a complete graph of $\left\lceil \frac{n}{2} \right\rceil$ vertices and a complete graph on $\left\lfloor \frac{n}{2} \right\rfloor + 1$ vertices which intersect in one vertex. First note that G_1 has $\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor + 1 - 1 = n$ vertices. Every vertex in G_1 has degree at least $\min\{n-1, \left\lceil \frac{n}{2} \right\rceil - 1, \left\lfloor \frac{n}{2} \right\rfloor\}$. Since $\left\lfloor \frac{n}{2} \right\rfloor \geq \left\lceil \frac{n}{2} \right\rceil - 1$, we have $\delta(G_1) = \left\lceil \frac{n}{2} \right\rceil - 1$. Since G_1 has a cut vertex, it is not the case that $C_n \subseteq G_1$.

Let G_2 be the complete bipartite graph with parts of size $\left\lfloor \frac{n}{2} \right\rfloor + 1$ and $\left\lceil \frac{n}{2} \right\rceil - 1$. Since $\left\lfloor \frac{n}{2} \right\rfloor + 1 \geq \left\lceil \frac{n}{2} \right\rceil - 1$, we have $\delta(G_2) = \left\lceil \frac{n}{2} \right\rceil - 1$. Since G_2 is bipartite with unequal part sizes, it is not the case that $C_n \subseteq G_2$.

□

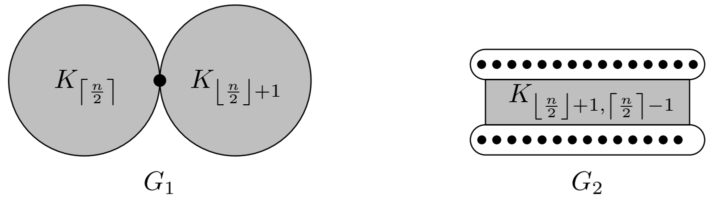


Figure 1.1: Two graphs with $\delta(G) = \lceil \frac{n}{2} \rceil - 1$ which do not contain C_n

In each of the examples above we have $\delta(G) = \lceil \frac{n}{2} \rceil - 1$, so we are not very close to $\frac{2n}{3}$. Perhaps at this point we try to prove that $\delta(G) \geq \frac{n}{2}$ suffices.

Theorem 1.0.6 (Dirac 1952 [15])**.** *If $\delta(G) \geq \frac{n}{2}$, then $C_n \subseteq G$.*

Proof. Let $P = v_1 v_2 \ldots v_k$ be a path of maximum length in G. Note that $k \geq \frac{n}{2} + 1$ by Proposition 1.0.3. We first show that there exists a cycle C with the property that $|C| = k$ and $V(P) \subseteq V(C)$. Since P is a maximum length path, $N(v_1) \subseteq V(P)$ and $N(v_k) \subseteq V(P)$. Let $d_1 := \deg(v_1)$ and $d_k := \deg(v_k)$. We assign d_1 "units of charge" to v_1 and d_k "units of charge" to v_k. Let $N^+(v_k) = \{v_{i+1} : v_i \in N(v_k)\}$. We now distribute the charge according the following rule: v_1 gives one unit of charge to each vertex in $N(v_1)$ and v_k gives one unit of charge to each vertex in $N^+(v_k)$. Note that according to the rule, v_1 necessarily ends up with 0 units of charge. There are now $d_1 + d_k \geq n$ units of charge on at most $k - 1 \leq n - 1$ vertices. So some vertex $v_i \in \{v_2, \ldots, v_k\}$ has two units of charge, which translates to $v_i \in N(v_1)$ and $v_{i-1} \in N(v_k)$. Then $v_1 \ldots v_{i-1} v_k \ldots v_i v_1$ is a cycle with the desired property. If $k = n$, then we have $C_n \subseteq G$, so suppose not. Let $x \in V(G) \setminus V(C)$. Since $k \geq \frac{n}{2} + 1$, we have $n - k \leq \frac{n}{2} - 1$ and thus there exists $v_i \in V(C) \cap N(x)$. But now $x v_i \ldots v_k v_1 \ldots v_{i-1}$ is path which is longer than P, contradicting our assumption.

□

So now we have an optimal result. If $\delta(G) \geq \frac{n}{2}$, then $C_n \subseteq G$, but no smaller value will suffice because of Proposition 1.0.5. This example illustrates the type of results we will prove throughout the dissertation.

Two of the main threads running through the research presented here (in Chapters 2, 5, and 6) can be traced back to Problem 9 of the Proceedings of the Symposium held in Smolenice in June 1963 [16]. Given a graph $G = (V, E)$ let r^{th} *power* of G, denoted G^r, be the graph obtained by adding an edge between every pair of vertices of distance at most r in G. We say G^2 is the *square* of G. Erdős made the following conjecture: If G is a graph on n vertices with $\delta(G) \geq \frac{rn}{r+1}$, then G contains $\left\lfloor \frac{n}{r+1} \right\rfloor$ vertex disjoint copies of K_{r+1}. Erdős goes on to point out that the case $r = 1$ is a consequence of Dirac's Theorem since the graph C_n contains $\left\lfloor \frac{n}{2} \right\rfloor$ copies of K_2. He also mentions that the case $r = 2$ was solved by Corrádi and Hajnal in a paper which appeared in 1963 [9]. Furthermore, he goes on to state that Pósa made the following conjecture which would contain the result of Corrádi and Hajnal: If $\delta(G) \geq \frac{2n}{3}$, then $C_n^2 \subseteq G$. Note that the square of C_n contains $\left\lfloor \frac{n}{3} \right\rfloor$ vertex disjoint copies of K_3.

In 1970, Hajnal and Szemerédi proved Erdős' conjecture from 1963 [21]. Then in 1973, Seymour generalized Pósa's conjecture, making a conjecture which would contain the Hajnal-Szemerédi Theorem [42]: If $\delta(G) \geq \frac{rn}{r+1}$, then $C_n^r \subseteq G$. It would be close to 30 years before there were any results on the Pósa-Seymour conjecture.

One of the most powerful combinatorial tools is Szemerédi's Regularity Lemma [44] (here we will discuss the Regularity Lemma somewhat informally, with precise statements given in Chapter 3). The Regularity Lemma came out of Szemerédi's proof of a conjecture of Erdős and Turán on arithmetic sequences (for which Szemerédi received a \$1000 prize from Erdős).

Theorem 1.0.7 (Szemerédi [45]). *For every $d \in (0, 1)$ and $k \in \mathbb{N}$ there exists N*

such that if $S \subseteq \{1, \ldots, N\}$ and $|S| \geq dN$, then S contains an k-term arithmetic progression.

Here we will only talk about the applications of the Regularity Lemma for graphs. One of the consequences of the Regularity Lemma is that large dense graphs behave like random graphs from the point of view of bounded degree subgraphs. To see what this means more precisely, let $p \in (0, 1)$ and let G_n be a graph on n vertices where each edge exists with probability p – thus the expected number of edges in G is $\Omega(n^2)$ and we say G is *dense*. Let Δ be a positive integer, $\epsilon \in (0, 1)$, and H be a graph on $(1 - \epsilon)n$ vertices with maximum degree $\Delta(H) \leq \Delta$.

Claim 1.0.8. *The probability that $H \subseteq G$ goes to 1 as $n \to \infty$*

Proof. We embed H one vertex at a time. Since there are always at least ϵn vertices left over, the probability that there is no suitable candidate for the next vertex is $(1 - p^{\Delta})^{\epsilon n} \to 0$. □

This shows that it is easy to embed "almost" spanning subgraphs in dense random graphs. The Regularity Lemma and corresponding "Key Lemma" (see [32]) allows one to obtain the same result in any dense enough large graph.

However, we are still at a loss if we want to find spanning subgraphs, which is of course the aim of this dissertation. In the 1990's, Komlós, Sárközy, and Szemerédi proved the Blow-up Lemma [28]. The abstract of their paper read, "Regular pairs behave like complete bipartite graphs from the point of view of bounded degree subgraphs." The Blow-up Lemma works in regular pairs which satisfy an additional minimum degree condition. So using the Blow-up Lemma in conjunction with the Regularity Lemma, it is possible to find spanning subgraphs. In fact, one of the first uses of the Blow-up Lemma was to give a proof of Pósa's conjecture for large graphs.

Theorem 1.0.9 (Komlós, Sárközy, Szemerédi [27]). *Let G be a graph on n vertices. There exists $N_0 \in \mathbb{N}$ such that if $\delta(G) \geq \frac{2n}{3}$ and $n \geq N_0$, then $C_n^2 \subseteq G$.*

They went on to also prove Seymour's conjecture when n is large with respect to r. The Blow-up Lemma has since been used to prove many results and we will give two applications in Chapters 4 and 6. One of the unfortunate aspects of the Regularity-Blow-up method is that the graphs being considered are extremely large. In fact they are so large that they exceed any physical description, i.e. much larger than the number of atoms in the universe. So any result which is proved using the Regularity-Blow-up method leaves open the general statement which has no lower bound on the number of vertices. Lately, there has been increasing success in removing Regularity from certain arguments and we begin with such a result in Chapter 2.

Finally before getting into the main results, we give an example of how the Regularity-Blow-up method is usually applied. Let G be a graph on n vertices with $\delta(G) \geq \frac{n}{2}$. Suppose we are trying to prove that $C_n \subseteq G$. Of course a simple proof was already given above, but imagine for the moment that we are unaware of such a proof. We saw in Proposition 1.0.5, that there exists a graph G with $\delta(G) = \frac{n-1}{2}$ which does not contain C_n. We call G an "extremal example", since any increase in the minimum degree will give us the desired cycle. What is the key property which makes G an extremal example? G has an independent set X of size $\frac{n+1}{2}$, and an independent set Y of size $\frac{n-1}{2}$ with every possible edge between them. G doesn't contain C_n because any two vertices in X must be separated on the cycle by a vertex from Y, which isn't possible since $|X| > |Y|$. Notice that we can in fact add every possible edge to Y and still have an extremal example for the same reason. This tells us that the key property is that G has a slightly too large independent set. Now in the graph with the correct degree condition we can define an appropriate notion of "closeness" to the extremal example. There is no one right way to do this, but we may introduce some parameter $\alpha > 0$ and say that G is in the extremal case if G has a set S of size at least $(1-\alpha)\frac{n}{2}$ which contains fewer than $\alpha\binom{|S|}{2}$ edges. Then we will split the proof into two cases. When G is not in the extremal case, we will apply the Regularity-Blow-up method. When G is in the extremal case, we will use ad hoc techniques which take advantage of the narrow structure imposed by the extremal condition.

Chapter 2

PÓSA'S CONJECTURE FOR GRAPHS ON AT LEAST 2×10^8 VERTICES

This chapter is joint work with Phong Châu and H.A. Kierstead.

2.1 Introduction

The *square* H^2 of a graph H is obtained by joining all pairs $\{x, y\} \subset V(H)$ with distance $dist(x, y) = 2$ in H. If H is a path (cycle) then H^2 is called a *square* path (cycle). Now fix a graph $G = (V, E)$ on n vertices. We say that $v_1 \dots v_t$ is a *square* path (cycle) in G if $v_1 \dots v_t$ is a path (cycle) in G and its square is contained in G. In 1962 Pósa [16] conjectured:

Conjecture 2.1.1. *Every graph G with $\delta(G) \geq \frac{2}{3}|G|$ contains a hamiltonian square cycle.*

During the 90's there were numerous partial results on Pósa's conjecture. Here we review a number that have a direct impact on this paper. Fan and Kierstead [18, 19, 20] proved the following three theorems. The first is a connecting lemma that immediately yields an approximate version of Pósa's conjecture.

Theorem 2.1.2 (Fan and Kierstead [18])**.** *For every $\epsilon > 0$ there exists a constant m such that for every graph G with $\delta(G) \geq (\frac{2}{3} + \epsilon)|G| + m$ and every pair e_1, e_2 of disjoint ordered edges, G has a hamiltonian square path starting with e_1 and ending with e_2. In particular, G has a hamiltonian square cycle.*

We shall need two ideas from this paper—*weak reservoirs* [1], and *optimal* square paths and cycles—which will be presented in the next section. Roughly,

[1]The term reservoir is not mentioned in [18], and the modifiers *weak*, *strong* and *special* are our own invention. However, in light of more recent papers this terminology provides a consistent transition (see Definition 2.2.8).

given a graph G on n vertices, a weak reservoir is a small fraction R of the vertex set $V(G)$ such that $|N \cap R| \approx |N||R|/n$ for any neighborhood $N := N(v)$. Weak reservoirs were used to connect long square paths contained in $V(G) \setminus R$. The second theorem is a path version of Pósa's Conjecture.

Theorem 2.1.3 (Fan and Kierstead [19]). *Every graph G with $\delta(G) \geq \frac{2|G|-1}{3}$ contains a hamiltonian square path.*

The third theorem shows that $V(G)$ can be partitioned into at most two square cycles.

Theorem 2.1.4 (Fan and Kierstead [20]). *Suppose G is a graph with $\delta(G) \geq \frac{2}{3}|G|$. If G has a square cycle of length greater than $\frac{2}{3}|G|$ then G has a hamiltonian square cycle. Moreover, $V(G)$ can be partitioned into at most two square cycles, each of length at least $\frac{1}{3}|G|$.*

The proofs of Theorems 2.1.3 and 2.1.4 are based on optimal paths and cycles, but do not use weak reservoirs. Theorem 2.1.4 is essential to this paper, because it allows our constructions to terminate as soon as we get a square cycle of length greater than $\frac{2}{3}|G|$.

Next came a major breakthrough. Komlós, Sárközy and Szemerédi proved their famous Blow-up Lemma [28], and used it and the Regularity Lemma [44] to prove:

Theorem 2.1.5 (Komlós, Sárközy and Szemerédi [27]). *There exists a constant n_0 such that every graph G with $|G| \geq n_0$ and $\delta(G) \geq \frac{2}{3}|G|$ has a hamiltonian square cycle.*

Their proof has the following structure. First they determine extremal configurations that are very close to being counterexamples, but because of the tightness of the degree condition, cannot achieve this status. (For example, if the independence number $\alpha(G) > \frac{1}{3}|G|$ then G does not have a hamiltonian square cycle, but then also does not satisfy $\delta(G) \geq \frac{2}{3}|G|$. Moreover if G has an almost independent set of size almost $\frac{1}{3}|G|$ and $\delta(G) \geq \frac{2}{3}|G|$, then we will see that G does have a hamiltonian square cycle.) Next they proved that if $|G|$ is sufficiently large, $\delta(G) \geq \frac{2}{3}|G|$, and G has an extremal configuration, then G has a hamiltonian square cycle. When there are no extremal configurations, the Regularity Lemma imposes a pseudo random structure on the graph that can be exploited, using this lack of extremal configurations and the Blow-up Lemma, to construct a hamiltonian square cycle. The use of the Regularity Lemma causes the constant n_0 to be extremely large.

Very recently Rödl, Ruciński and Szemerédi have made another important advance [39, 40]. They proved the following version of Dirac's Theorem for 3-uniform hypergraphs (3-graphs). An *open chain* $P := v_1v_2v_3 \ldots v_{s-2}v_{s-1}v_s$ in a 3-graph H is a sequence of distinct vertices such that $v_iv_{i+1}v_{i+2} \in E(H)$ for all $i \in [s-2]$; P is a *closed chain* if in addition $v_{s-1}v_sv_1, v_sv_1v_2 \in E(H)$.

Theorem 2.1.6 (Rödl, Ruciński and Szemerédi [40]). *There exists an integer n_0 such that for every 3-graph H on at least n_0 vertices, if every pair of vertices of H is contained in at least $\lfloor \frac{1}{2}|H| \rfloor$ edges of H then H contains a hamiltonian closed chain.*

The remarkable proof is very long, but has a similar structure to the proof of Theorem 2.1.5. However, a major difference is that the non-extremal case does not use any version of the Blow-up Lemma, and regularity (weak hypergraph regularity) is only used in a quite generic way to construct various *strong reservoirs*—weak reservoirs with no extreme sets. The Blow-up Lemma is replaced by a construction based on an ingenious *absorbing path* lemma, and a *connecting* lemma, that uses the strong reservoir.

Levitt, Sárközy and Szemerédi [36] applied similar techniques to the non-extremal case of Pósa's Conjecture without using the Regularity Lemma, and thus proved the result for much smaller graphs than those considered in Theorem 2.1.5.

Here we show that Pósa's Conjecture holds for graphs of order at least 2×10^8 without using the Regularity-Blow-up method. In addition, our proof of the extremal case holds for all n. We were influenced by the ideas of [36], but only rely on results from [18, 19, 20], and the idea from [27] of dividing the problem into an extremal case and a non-extremal case. We avoid the Blow-up Lemma and absorbing paths by using Theorem 2.1.4. Our approach is explained fully in the next section.

Notation

Most of our notation is consistent with Diestel's graph theory text [14]. In particular note that P^n is a path on n edges, $|G| = |V(G)|$, $\|G\| = |E(G)|$, and $d(v)$ is the degree of the vertex v. For $A, B \subseteq V(G)$, let $\|A, B\| = |E(A, B)|$, where $E(A, B)$ is the set of edges with one end in A and the other in B, in particular we shall write $\|a, B\|$ if $A = \{a\}$. We also use $\overline{\|A, B\|}$ to denote the number of edges in the complement of G that have one end in A and the other in B. For $a_1, a_2, \dots, a_k \in V(G)$, let $N(a_1, a_2, \dots, a_k) = N(a_1) \cap N(a_2) \cdots \cap N(a_k)$.

2.2 Main theorem and proof strategy

Here is our main result:

Theorem 2.2.1. *Let G be a graph on n vertices with $n \geq n_0 := 2 \times 10^8$. If $\delta(G) \geq \frac{2}{3}n$, then G has a hamiltonian square cycle.*

In this section we organize the structure of the proof. The first step is to define a usable extremal configuration. Our choice is simpler than the choice in [36], which was much simpler than the several extremal configurations used in [27]. A priori, this makes the extremal case easier and the non-extremal case harder.

Definition 2.2.2. *Let G be a graph on n vertices. A set $S \subseteq V(G)$ is α-*extreme *if $|S| \geq (1-\alpha)\frac{n}{3}$ and $\|v, S\| < \alpha\frac{n}{3}$ for all $v \in S$.*

The proof divides into two parts, depending on whether G is $\frac{1}{36}$-extreme, i.e., contains an α-extreme set with $\alpha := \frac{1}{36}$. The extreme case is handled in Section 2.4, where we prove the following theorem without assuming anything about the order of G. Its proof only requires elementary graph theory. Notice that $K_{3t+2} - E(K_{t+1})$ demonstrates that the degree condition is tight.

Theorem 2.2.3 (Extremal Case)**.** *Let G be a graph on n vertices with $\delta(G) \geq \frac{2}{3}n$. If G has a $\frac{1}{36}$-extreme set, then G has a hamiltonian square cycle.*

The non-extremal case is more complicated. In Section 2.3 we will prove:

Theorem 2.2.4 (Non-extremal Case)**.** *Let G be a graph on n vertices with $\delta(G) \geq \frac{2}{3}n$ and $n \geq n_0 := 2 \times 10^8$. If G does not contain a $\frac{1}{36}$-extreme set, then G has a hamiltonian square cycle.*

Note that if G has an α-extreme set $S \subseteq V(G)$ for some $\alpha < \frac{1}{36}$, then S is a $\frac{1}{36}$-extreme set. This explains why we only consider $\frac{1}{36}$-extreme sets in Theorems 2.2.3 and 2.2.4.

The proof of Theorem 2.2.4 has three parts. First we use the Reservoir Lemma (Lemma 2.3.2) to construct a special reservoir R with $|R| < \frac{1}{3}n$. Then we use the Path Cover Lemma (Lemma 2.3.3) to construct two disjoint square paths P_1, P_2 in $G - R$ such that $|P_1| + |P_2| > \frac{2}{3}n$ using techniques and results from [18, 19]. Finally, we use the properties of the special reservoir R, together with our version of the Connecting Lemma (Lemma 2.3.1), to connect the ends of P_1 to the ends of P_2 by disjoint square paths in R so as to form a square cycle of length greater than $\frac{2}{3}n$. Thus by Theorem 2.1.4 we obtain a hamiltonian square cycle.

2.2.1 Reservoirs and the Connecting Lemma

The bottleneck in this line of attack is in determining properties for special reservoirs that are strong enough to prove the Connecting Lemma, yet weak enough to ensure the existence of special reservoirs in moderately sized graphs. In the process of constructing a connecting square path we need to know that certain subsets of the reservoir are nonextreme. Since it is too expensive to ensure that all subsets are nonextreme, we anticipate a limited collection of *special* subsets that might appear in this construction, and construct a reservoir with no extreme special sets.

Definition 2.2.5. *A set $S \subseteq V(G)$ is* special *if there exist (not necessarily distinct) vertices $u, v, w, x, y \in V(G)$ such that*
$S = (N(u, v, w) \cup N(u, v, x)) \cap N(y)$.

A set S of size at least $(1 - \alpha)\frac{n}{3}$ that is not α-extreme has at least one vertex with "large" degree to S, but we will need more than one vertex of "large" degree, so we define a more general notion of extremity.

Definition 2.2.6. *Let G be a graph with n vertices. A set $S \subseteq V(G)$ is* (α, β)-extreme *if $|S| \geq (1 - \alpha + \beta)\frac{n}{3}$ and there are fewer than $\lfloor \beta\frac{n}{3} \rfloor$ vertices $v \in S$ such that $\|v, S\| \geq \alpha\frac{n}{3}$.*

So a set S of size at least $(1-\alpha+\beta)\frac{n}{3}$ that is not (α,β)-extreme has at least $\lfloor \beta\frac{n}{3} \rfloor$ vertices with "large" degree to S. In the non-extremal case we know that G contains no α-extreme sets, but we must ensure for the Connecting Lemma that the reservoir has no (α',β')-extreme special sets. So we use the following simple observation when constructing the reservoir.

Lemma 2.2.7. *Let G be a graph on n vertices and let $\alpha,\beta>0$. If G has no α-extreme sets and $S \subseteq V(G)$ with $|S| \geq (1-\alpha+\beta)\frac{n}{3}$, then S is not (α,β)-extreme.*

Proof. Suppose S is (α,β)-extreme and let $S' = \{v \in S : \|v,S\| \geq \alpha\frac{n}{3}\}$. Since S is (α,β)-extreme, we have $|S'| < \lfloor \beta\frac{n}{3} \rfloor$. Thus $|S \setminus S'| \geq (1-\alpha)\frac{n}{3}$ and $\|v, S\setminus S'\| < \alpha\frac{n}{3}$ for all $v \in S \setminus S'$, contradicting the fact that G has no α-extreme sets. □

Here are the technical definitions of (ϵ,ϱ)-weak, $(\alpha,\epsilon,\varrho)$-strong and $(\alpha,\beta,\epsilon,\varrho)$-special reservoir.

Definition 2.2.8 (Reservoir)**.** *Let G be a graph on n vertices. Let $1 \geq \varrho \geq 0$ and $\epsilon > 0$. An* (ϵ,ϱ)-weak reservoir *is a set $R \subseteq V(G)$ such that $|R| = \lceil \varrho n \rceil$ and for all $u \in V(G)$,*

$$\left(\frac{d(u)}{n} - \epsilon\right)|R| \leq \|u,R\| \leq \left(\frac{d(u)}{n} + \epsilon\right)|R|.$$

An $(\alpha,\epsilon,\varrho)$-strong reservoir *is an (ϵ,ϱ)-weak reservoir R such that $G[R]$ has no α-extreme sets.*

An $(\alpha,\beta,\epsilon,\varrho)$-special reservoir *is an (ϵ,ϱ)-weak reservoir R such that for all special sets $S \subseteq V(G)$, $S \cap R$ is not (α,β)-extreme in $G[R]$.*

A routine application of Chernoff's bound yields (ϵ,ϱ)-weak reservoirs R in moderately large graphs. The reason for this is that we have only polynomially many conditions to preserve. A similar observation allows us to construct $(\alpha, \beta, \epsilon, \varrho)$-special reservoirs. However this standard approach fails for $(\alpha, \epsilon, \varrho)$-strong reservoirs, because there are exponentially many conditions to check.

A connecting lemma should state that any two disjoint ordered edges in $V(G) \setminus R$ can be connected by a short square path whose interior vertices are in R. For example, Fan and Kierstead [18] proved:

Lemma 2.2.9. *If $\delta(G) > \frac{2}{3}|G|$ then there exists a square path connecting any two disjoint edges.*

In the context of Theorem 2.1.2, $(\epsilon/2, \varrho)$-weak reservoirs are sufficient since the degree bounds ensure that $\delta(G[R]) > \frac{2}{3}|R|$. In [36, 40] the authors prove connecting lemmas for strong reservoirs. We use a simpler argument and show that it works for special reservoirs.

2.2.2 Optimal paths

Let $e_1 := v_1v_2$ and $e_2 := v_{s-1}v_s$ be disjoint ordered edges. A square (e_1, e_2)-*path* is a square path of the form $v_1v_2 \ldots v_{s-1}v_s$.

Definition 2.2.10. *An* optimal *square path (or cycle, or (e_1, e_2)-path) is a square path (or cycle, or (e_1, e_2)-path) P such that among all square paths (or cycles, or (e_1, e_2)-paths) (i) P is as long as possible, (ii) subject to (i), P has as many 3-chords as possible, and (iii) subject to (i) and (ii), P has as many 4-chords as possible.*

All the work in [18, 19, 20] starts with lemmas about optimal square paths.

Lemma 2.2.11 (Fan-Kierstead [18], [19] Lemma 1)**.** *Suppose that P is a square path in a graph G and $v \in V(G-P)$. If P is an (e_1, e_2)-optimal square path then $\|v, Q\| \leq \frac{2}{3}|V(Q)| + 1$ for every segment Q of P. Moreover, if P is an optimal square path then $\|v, P\| \leq \frac{2}{3}|P| - \frac{1}{3}$ and if P is an optimal square cycle then $\|v, P\| \leq \frac{2}{3}|P| + \frac{1}{3}$.*

In the extremal case we will take advantage of the following fact.

Corollary 2.2.12. *Pósa's Conjecture is true, if it holds for all G with $|G|$ divisible by* 3*.*

Proof. Suppose $|G| = 3k + r$, where $1 \leq r \leq 2$. Let G' be G with r vertices deleted. Then

$$\delta(G') \geq \lceil \frac{2}{3}(3k + r) \rceil - r = 2k = \frac{2}{3}|G'|.$$

Thus by hypothesis, G' has a hamiltonian square cycle C'. So an optimal square cycle C in G has length at least $3k$. Suppose C is not hamiltonian in G. Then there exists $x \in V(G - C)$. By Lemma 2.2.11, we have the following contradiction:

$$2k + r \leq \delta(G) \leq \|v, C\| + |G| - |C| - 1 \leq |G| - \frac{1}{3}|C| - \frac{2}{3} \leq 2k + r - \frac{2}{3}.$$

$\square$

We will also need:

Lemma 2.2.13 (Fan-Kierstead [19], Lemma 9)**.** *Let P be an optimal square path of G. Let xy be an edge of $G - P$ such that there are square paths, of at least q vertices, starting at xy and yx in $G - P$. If $|P| \geq 2q + 2$, then $\|xy, P\| \leq \frac{4}{3}|P| - \frac{2}{3}q + 2$.*

2.2.3 Probability

If X is a random variable with hypergeometric distribution (and our experiment consists of drawing n items from a collection of N total items, m of which are good and $N-m$ of which are bad) the expected value of X is given by

$$\mathbb{E}X = \sum_{k=0}^{n} k \cdot Pr(X=k) = \sum_{k=0}^{n} k \cdot \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}} = \frac{nm}{N}.$$

Theorem 2.2.14 (Chernoff's bound [8, 24])**.** *Let X be a random variable with binomial or hypergeometric distribution. Then the following hold:*

(i) $Pr(X \geq \mathbb{E}X + t) \leq \exp\left(-\frac{t^2}{2(\mathbb{E}X + t/3)}\right)$, $\quad t \geq 0$;

(ii) $Pr(X \leq \mathbb{E}X - t) \leq \exp\left(-\frac{t^2}{2\mathbb{E}X}\right)$, $\quad t \geq 0$;

(iii) If $0 < \gamma \leq 3/2$, then $Pr(|X - \mathbb{E}X| \geq \gamma \mathbb{E}X) \leq 2\exp\left(-\frac{\gamma^2}{3}\mathbb{E}X\right)$.

2.3 Non-extremal case

In this section we prove Theorem 2.2.4. We have compromised optimality somewhat in our constructions and calculations in favor of clarity of exposition. For instance, we know how to reduce n_0 by a factor of 2. That being said, we can make the reservoir lemma slightly simpler and we can choose "nicer" constants throughout the non-extremal case at the cost of a factor of 3 in n_0.

We first show that if H is a graph with no (α, β)-extreme special sets whose minimum degree is almost $\frac{2}{3}|H|$, then any two disjoint edges in H can be connected by a short square path. Let $xy \in E(H)$; we say that $P\{xy\}Q$ is a square path if one of $PxyQ$ or $PyxQ$ is a square path.

Lemma 2.3.1 (Connecting Lemma)**.** *Let $0 < \beta < \alpha \leq \frac{1}{36}$, $0 < \epsilon \leq \frac{\alpha-\beta}{15.1}$, $l := 10$ and suppose $n \geq \max\{\frac{660}{\epsilon}, \frac{69}{\beta}\}$. Let $H = (V, E)$ be a graph on n vertices with no*

(α,β)-extreme special sets such that $\delta(H)\ge(\frac{2}{3}-\epsilon)n$. Let $L\subseteq V$ such that $|L|\le l$. If ab, cd are any two disjoint ordered edges in $H-L$, then there is a square (ab,cd)-path P of order at most 14 for which $V(P)\subseteq V\setminus L$.

Proof. Let ab, cd be disjoint ordered edges in $H-L$ and set $A:=\{a,b,c,d\}$. Here is our plan. First (a) we find disjoint edges $a'b', c'd'$ in $H-L-A$ such that $\|ab,a'b'\|=4=\|cd,c'd'\|$. Then, setting $A':=\{a',b',c',d'\}$, (b) we construct a square path $\{a'b'\}Q\{c'd'\}$ with $Q\subseteq H':=H\setminus(L\cup A\cup A')$ connecting the unordered edges $a'b', c'd'$. This will yield a square path $ab\{a'b'\}Q\{c'd'\}cd$, where the order of $\{a'b'\}$ and $\{c'd'\}$ is determined by Q.

Let $M\subseteq V$ with $|M|\le l+12$. We will often use the following statement:

$$\text{If } S \text{ is a special set with } |S|\ge(1-\alpha+\beta)\frac{n}{3} \text{ then } \|S\setminus M\|>0. \tag{2.1}$$

To see this, note that since S is not (α,β)-extreme and $n\ge\frac{69}{\beta}$, S has at least $\lfloor\beta\frac{n}{3}\rfloor>l+12$ vertices with degree at least $\alpha\frac{n}{3}>l+12$.

Consider the special set $N(a,b)=(N(a,a,a)\cup N(a,a,a))\cap N(b)$. Since $\delta(H)\ge(\frac{2}{3}-\epsilon)n$, we have

$$|N(a,b)|\ge(1-6\epsilon)\frac{n}{3}\ge(1-\alpha+\beta)\frac{n}{3}.$$

By (2.1), there exists $a'b'\in E(N(a,b)\setminus(L\cup A))$. Likewise there is an edge $c'd'\in E(N(c,d)\setminus(\{a',b'\}\cup L\cup A))$, completing the first goal (a).

Next we show (b). Let $V':=V(H')$. Then $|V'|\ge n-l-8$. We must construct $Q\subseteq H'$. For $i\in[4]$, let $S_i:=S_i(A')=\{v\in V:\|v,A'\|=i\}$. Then

$$\frac{8}{3}n-4\epsilon n=4(\frac{2}{3}-\epsilon)n\le\|A',V\|=\sum_{i\in[4]}i|S_i|\le4|S_4|+3|S_3|+2(n-|S_4|-|S_3|), \tag{2.2}$$

which gives

$$2|S_4|+|S_3|\ge\frac{2}{3}n-4\epsilon n. \tag{2.3}$$

Case 1: $|S_4| > l + 12$. Looking ahead to an application in Case 2.a, we will construct $Q \subseteq H'' := H' - A''$, for any fixed 4-set A''. Set $V'' := V(H'')$. By the case assumption, there exists $x \in S_4 \cap V''$. If there exists $u \in N(x) \cap (S_4 \cup S_3) \cap V''$ then set $Q := \{xu\}$. Otherwise, $|S_4| + |S_3| \leq \frac{1}{3}n + \epsilon n + l + 12$, since $d(x) \geq \frac{2}{3}n - \epsilon n$. Thus by (2.3), and using $\alpha - \beta \geq 15.1\epsilon$ and $n \geq \frac{660}{\epsilon}$, we have

$$|S_4| \geq \frac{1}{3}n - 5\epsilon n - l - 12 \geq (1 - \alpha + \beta)\frac{n}{3}.$$

Moreover, $S_4 = N(a', b', c', d') = (N(a', b', c') \cup N(a', b', c')) \cap N(d')$ is special. Thus by (2.1), there exists an edge $uv \in S_4 \cap V''$, and we set $Q := uv$.

Case 2: $|S_4| \leq l + 12$. Let

$$T_1 := \{v \in S_3 \cup S_4 : \|v, \{a', b'\}\| = 2\} = (N(a', b', c') \cup N(a', b', d')) \cap N(a') \text{ and} \tag{2.4}$$

$$T_2 := \{v \in S_3 \cup S_4 : \|v, \{c', d'\}\| = 2\} = (N(c', d', a') \cup N(c', d', b')) \cap N(c'). \tag{2.5}$$

Then T_1 and T_2 are both special sets. Note that S_3 is partitioned as $(T_1 \setminus S_4) \cup (T_2 \setminus S_4)$ and $T_1 \cap T_2 = S_4$. By (2.3) and the fact that $|T_1| + |T_2| = |S_3| + 2|S_4|$, we have

$$|T_1| + |T_2| \geq \frac{2}{3}n - 4\epsilon n. \tag{2.6}$$

Without loss of generality, $|T_1| \leq |T_2|$, and so $T_2 \neq \emptyset$. Finally, note that by (2.3) and the case assumption we have,

$$|T_1 \cup T_2| = |S_3 \cup S_4| \geq \frac{2}{3}n - 4\epsilon n - l - 12. \tag{2.7}$$

Case 2.a: $|T_1| > l + 8$. If there exists $xy \in E(T_1, T_2) \cap E(H')$, then set $Q := xy$. Otherwise, let $x \in T_1 \cap V'$. Then using, in order, $d(x) \geq (\frac{2}{3} - \epsilon)n$, (2.6), $\alpha - \beta \geq 15.1\epsilon$ and $n \geq \frac{660}{\epsilon}$ we have

$$\frac{n}{3} + \epsilon n + l + 8 \geq |T_2| \geq |T_1| \geq \frac{n}{3} - 5\epsilon n - l - 8 \geq (1 - \alpha + \beta)\frac{n}{3}. \tag{2.8}$$

By (2.1) and (2.8), there exist edges $a''b'' \in E(T_1)$ and $c''d'' \in E(T_2)$ such that $A'' := \{a'', b'', c'', d''\}$ is disjoint from $L \cup A \cup A'$. Note that $A'' \cap S_4 = \emptyset$, since $E(T_1, T_2) \cap E(H') = \emptyset$ as mentioned above.

Set $U := V \setminus (T_1 \cup T_2)$. By (2.7),

$$|U| = n - |T_1 \cup T_2| \leq \frac{n}{3} + 4\epsilon n + l + 12. \tag{2.9}$$

By (2.8), for any $x \in A''$,

$$\|x, U\| \geq \frac{2}{3}n - \epsilon n - |T_2| \geq \frac{n}{3} - 2\epsilon n - l - 8. \tag{2.10}$$

By (2.9), (2.10), and $n \geq \frac{660}{\epsilon}$, we have $\overline{\|x, U\|} \leq 6\epsilon n + 3l + 32 < \frac{1}{5}|U \cap V''|$. Thus there exist more than $l + 12$ vertices in $S_4(A'')$. Thus by Case 1, there exists a square path $Q := \{a''b''\}Q'\{c''d''\}$ with $|Q'| \leq 2$.

Case 2.b: $|T_1| \leq l + 8$. Then $|T_2| \geq \frac{2}{3}n - 4\varepsilon n - l - 8$ by (2.6). Let $x \in N(a', b') \cap V'$, and note that $S := T_2 \cap N(x) = (N(a', c', d') \cup N(b', c', d')) \cap N(x)$ is a special set. Moreover by $\alpha - \beta \geq 15.1\epsilon$ and $n \geq \frac{660}{\epsilon}$ we have

$$|S| \geq |T_2| + |N(x)| - n \geq \frac{n}{3} - 5\epsilon n - l - 8 \geq (1 - \alpha + \beta)\frac{n}{3}.$$

Thus by (2.1), there exists an edge $yz \in E(S \cap V')$. Let $Q := xyz$.

$\square$

Now we prove the reservoir lemma.

Lemma 2.3.2 (Reservoir Lemma)**.** *Let $\alpha \geq \frac{1}{36}$, $c \geq \frac{1}{14}$, $\alpha' := (1 - 3c)\alpha$, $\beta' := c\alpha$, $\epsilon \geq \frac{\alpha' - \beta'}{15.1}$, $\varrho \geq 1 - \frac{2/3 + \epsilon}{5/6 - 2\epsilon}$ and $n \geq n_0 := 2 \times 10^8$. If H is a graph on n vertices such that $\delta(H) \geq \frac{2}{3}n$ and H contains no α-extreme sets, then H contains an $(\alpha', \beta', \epsilon, \varrho)$-special reservoir.*

Proof. Let $\gamma := \frac{2\beta'}{1-\alpha'-\beta'}$. We will show that there exists a set $R \subseteq V(H)$ such that $|R| = \lceil \varrho n \rceil$ which satisfies the following three properties.

(i) For all $u \in V(H)$, $\left(\frac{d(u)}{n} - \epsilon\right)|R| \le \|u, R\| \le \left(\frac{d(u)}{n} + \epsilon\right)|R|$.

(ii) For all special sets $S \subseteq V(H)$, if $|S| \ge (1-\alpha'+\beta')\varrho\frac{n}{3}$, then $|S \cap R| \le 1.05\varrho|S|$ and for all special sets $S \subseteq V(H)$, if $|S \cap R| \ge (1-\alpha'+\beta')\varrho\frac{n}{3}$, then $|S \cap R| \le (1+\gamma)\varrho|S|$.

(iii) For all special sets $S \subseteq V(H)$, if $|S| \ge (1-\alpha'-\beta')\frac{n}{3}$, then there exists a set $T' \subseteq S \cap R$ such that $|T'| \ge \beta'\varrho\frac{n}{3}$ and $\|z, S \cap R\| \ge \alpha'\varrho\frac{n}{3}$ for all $z \in T'$.

Then we will show that these three properties imply that R is an $(\alpha', \beta', \epsilon, \varrho)$-special reservoir.

Let $R \subseteq V(H)$ be a set of size $\lceil \varrho n \rceil =: r$ chosen at random from all $\binom{n}{r}$ possibilities. There are five calculations that follow. In each of these calculations we will need n to be large, specifically $n \ge 2 \times 10^8$ is large enough.

Let $u \in V(H)$. The expected value of $\|u, R\|$ is $\frac{rd(u)}{n} \ge \varrho d(u)$. So by Theorem 2.2.14(iii), we have

$$Pr\left(\left|\|u, R\| - \frac{rd(u)}{n}\right| \ge \frac{\epsilon n}{d(u)}\frac{rd(u)}{n}\right) \le 2\exp\left(-\frac{(\frac{\epsilon n}{d(u)})^2}{3}\frac{rd(u)}{n}\right)$$
$$\le 2\exp\left(\frac{-\epsilon^2\varrho n^2}{3d(u)}\right) < \frac{1}{3n}.$$

There are n vertices in $V(H)$. So by applying Boole's inequality, the probability that there exists a vertex which does not satisfy property (i) is less than 1/3.

Let $S \subseteq V(H)$ be a special set such that $|S| \ge (1-\alpha'+\beta')\varrho\frac{n}{3}$. The expected value of $|S \cap R|$ is $\frac{r|S|}{n} \ge \varrho|S| \ge (1-\alpha'+\beta')\varrho^2\frac{n}{3}$. So by Theorem

2.2.14(i), we have

$$\log Pr(|S\cap R| \geq 1.05\frac{r|S|}{n}) \leq -\frac{(.05\varrho|S|)^2}{2(\varrho|S|+.05\varrho|S|/3)}$$
$$\leq -\frac{.0025\varrho^2(1-\alpha'+\beta')}{2(1+.05/3)}\frac{n}{3} < \log\frac{1}{9n^5}.$$

So with high probability,

$$|S\cap R| \leq 1.05\varrho|S| \text{ for all } S\subseteq V(H) \text{ such that } |S| \geq (1-\alpha'+\beta')\varrho\frac{n}{3}. \qquad (2.11)$$

Now let $S\subseteq V(H)$ be a special set such that $|S\cap R| \geq (1-\alpha'+\beta')\varrho\frac{n}{3}$. Since $|S| \geq |S\cap R|$ we have $|S| \geq (1-\alpha'+\beta')\varrho\frac{n}{3}$ and thus by (2.11), $|S| \geq \frac{|S\cap R|}{1.05\varrho} \geq \frac{(1-\alpha'+\beta')}{1.05}\frac{n}{3}$. The expected value of $|S\cap R|$ is $\frac{r|S|}{n} \geq \varrho|S| \geq \varrho\frac{(1-\alpha'+\beta')}{1.05}\frac{n}{3}$. Using Theorem 2.2.14(i) again, we have

$$\log Pr(|S\cap R| \geq (1+\gamma)\frac{r|S|}{n}) \leq -\frac{(\gamma\varrho|S|)^2}{2(\varrho|S|+\gamma\varrho|S|/3)}$$
$$\leq -\frac{\gamma^2\varrho(1-\alpha'+\beta')}{1.05(2+2\gamma/3)}\frac{n}{3} < \log\frac{1}{3n^5}.$$

There are at most n^5 special sets $S\subseteq V(H)$. So by applying Boole's inequality, the probability that there exists a set S which does not satisfy property (ii) is less than 4/9.

Let $S\subseteq V(H)$ be a special set such that $|S| \geq (1-\alpha'-\beta')\frac{n}{3} = (1-\alpha+2c\alpha)\frac{n}{3}$. Since H has no α-extreme sets, we see by Lemma 2.2.7 that S is not $(\alpha, 2c\alpha)$-extreme. So there exists a set $S'\subseteq S$ having the property that $|S'| = \lfloor 2c\alpha\frac{n}{3}\rfloor$ and for all $v\in S'$, $\|v,S\| \geq \alpha\frac{n}{3}$. Let $T' := S'\cap R$. We first show that with high probability, $|T'| \geq \frac{3\varrho}{4}|S'| \geq \frac{\varrho}{2}(|S'|+1) \geq \beta'\varrho\frac{n}{3}$. The expected value of $|T'|$ is $\varrho|S'| \geq \varrho(2c\alpha\frac{n}{3}-1)$. So by Theorem 2.2.14(ii), we have

$$\log Pr(|T'| \leq \varrho|S'| - \frac{\varrho}{4}|S'|) \leq -\frac{(\frac{\varrho}{4}|S'|)^2}{2(\varrho|S'|)} = -\frac{\varrho|S'|}{32} \leq -\frac{\varrho(2c\alpha\frac{n}{3}-1)}{32} < \log\frac{1}{9n^5}.$$

Next we show that, with high probability, every vertex in S' has at least $(1-3c)\varrho\|v,S\| \geq \alpha'\varrho\frac{n}{3}$ neighbors in $S\cap R$. Let $v\in S'$. The expected value of

$\|v,T\|$ is $\varrho\|v,S\| \geq \varrho\alpha\frac{n}{3}$. So by Theorem 2.2.14(ii), we have

$$\begin{aligned}
&\log Pr(\|v, S\cap R\| \leq (1-3c)\varrho\|v,S\|) \\
&\qquad\qquad \leq -\frac{(3c\varrho\|v,S\|)^2}{2\varrho\|v,S\|} = -\frac{9c^2\varrho\|v,S\|}{2} \leq -\frac{3c^2\varrho\alpha n}{2} < \log\frac{1}{9n^6}.
\end{aligned}$$

There are at most n^5 special sets $S \subseteq V(H)$ and at most n^6 sets defined when we examine the neighborhood of vertices in each special set. So by applying Boole's inequality, the probability that there exists a set S which does not satisfy property (iii) is less than 2/9.

The probability that R doesn't satisfy one of the conditions is less than 1, thus there exists a set $R \subseteq V(H)$ satisfying properties (i)-(iii).

We now show that R is an $(\alpha', \beta', \epsilon, \varrho)$-special reservoir. Since R satisfies property (i), R is a (ϵ, ϱ)-weak reservoir. Let $S \subseteq V(H)$ be a special set such that $|S \cap R| \geq (1-\alpha'+\beta')\varrho\frac{n}{3}$. By property (ii), we have $\varrho|S|(1+\gamma) \geq |S\cap R| \geq (1-\alpha'+\beta')\varrho\frac{n}{3}$, and thus

$$|S| \geq \frac{(1-\alpha'+\beta')}{1+\gamma}\frac{n}{3} = (1-\alpha'-\beta')\frac{n}{3}.$$

Then since $|S| \geq (1-\alpha'-\beta')\frac{n}{3}$ there is, by property (iii), a set of vertices $T' \subseteq S\cap R$ with $|T'| \geq \beta'\varrho\frac{n}{3}$ such that for all $v \in T'$, $\|v, S\cap R\| \geq \alpha'\varrho\frac{n}{3}$. Thus $S \cap R$ is not (α',β')-extreme in $G[R]$. Therefore R is an $(\alpha', \beta', \epsilon, \varrho)$-special reservoir. $\square$

We now prove a lemma which allows us to cover most of the complement of the reservoir with at most two long square paths.

Lemma 2.3.3 (Path Cover Lemma)**.** *Suppose $\epsilon \leq \frac{1}{500}$ and $n \geq 6000$. Let H be a graph on n vertices with $\delta(H) \geq \left(\frac{2}{3}-\epsilon\right)n$. Then*

(a) *H has a square path P with $|P| \geq (\frac{1}{2}-3\epsilon)n$.*

(b) *H has two vertex disjoint square paths P_1 and P_2 so that $|P_1| + |P_2| > (\frac{5}{6} - 2\epsilon)n$.*

Proof. (a) Let $P := u_1u_2...u_p$ be an optimal square path in H and suppose that $p < (\frac{1}{2} - 3\epsilon)n$. We first observe that since $\delta(H) \geq (\frac{2}{3} - \epsilon)n$ we have $N(u_1, u_2) \geq (\frac{1}{3} - 2\epsilon)n$ and thus $p > (\frac{1}{3} - 2\epsilon)n$. Let $H' := H - P$ and set $h := |H'|$. If $\|v, P\| \leq (\frac{2}{3} - 4\epsilon)p$ for all $v \in V(H')$ then we have $\delta(H') \geq (\frac{2}{3} - \epsilon)n - (\frac{2}{3} - 4\epsilon)p \geq \frac{2}{3}h$. Thus by Theorem 2.1.3, H' has a hamiltonian square path of length more than than $\frac{1}{2}n$, contradicting the optimality of P. Thus there is a vertex $x \in V(H')$ such that $\|x, P\| > (\frac{2}{3} - 4\epsilon)p > \frac{1}{2}p + 1$. It follows that x is adjacent to two consecutive vertices of P. Choose $i \in [p]$ as small as possible such that $u_i, u_{i+1} \in N(x)$. Let $Q := u_1u_2...u_{i-1}$ and set $q := i - 1$. Then $\|x, Q\| \leq \frac{1}{2}q$. We claim that $q < (\frac{1}{6} - 2\epsilon)n$. Otherwise,

$$\begin{aligned}
\|x, P - Q\| > (\frac{2}{3} - 4\epsilon)p - \frac{1}{2}q &= \frac{2}{3}(p - q) + \frac{1}{6}q - 4\epsilon p \\
&> \frac{2}{3}|P - Q| + \frac{1}{6}(\frac{1}{6} - 2\epsilon)n - 4\epsilon(\frac{1}{2} - 3\epsilon)n \\
&> \frac{2}{3}|P - Q| + \frac{1}{36}n - \frac{7}{3}\epsilon n \\
&> \frac{2}{3}|P - Q| + 1,
\end{aligned}$$

contradicting Lemma 2.2.11. On the other hand, since $|N(x, u_i)| \geq (\frac{1}{3} - 2\epsilon)n = \frac{2}{3}(\frac{1}{2} - 3\epsilon)n > \frac{2}{3}p$, Lemma 2.2.11 implies x and u_i have a common neighbor y in H'. Also, by Lemma 2.2.11 we have

$$\delta(H') \geq (\frac{2}{3} - \epsilon)n - (\frac{2}{3}p - \frac{1}{3}) > \frac{2}{3}h - \epsilon n,$$

and thus for any edge uv in H', $|N_{H'}(u, v)| \geq \frac{1}{3}h - 2\epsilon n > (\frac{1}{6} - 2\epsilon)n$. Hence, we can find a square path P' of length at least $(\frac{1}{6} - 2\epsilon)n$ starting at xy. Since $|P'| > q$, the square path $P'yxu_iu_{i+1}...u_p$ is longer than P, a contradiction. This completes the proof of part (a).

(b) Let P_1 be an optimal square path in H and let $p := |P_1|$. Note that $p \geq (\frac{1}{2} - 3\epsilon)n$ by Lemma 2.3.3(a). If $p > (\frac{5}{6} - 2\epsilon)n$, then set $P_2 = \emptyset$ and we are done. So we may assume that $p \leq (\frac{5}{6} - 2\epsilon)n$. Set $H' := H - P_1$ and $h := |H'| > n/6$. If $\|v, P_1\| \leq (\frac{2}{3} - 3\epsilon)p$ for all $v \in V(H')$ then $\delta(H') \geq (\frac{2}{3} - \epsilon)n - (\frac{2}{3} - 3\epsilon)p \geq \frac{2}{3}h$. Thus H' has a hamiltonian square path P_2 by Theorem 2.1.3, and we are done. Otherwise, let $x \in V(H')$ such that $\|x, P_1\| > (\frac{2}{3} - 3\epsilon)p$. Note that by Lemma 2.2.11, we have $\delta(H') \geq (\frac{2}{3} - \epsilon)n - (\frac{2}{3}p - \frac{1}{3}) > \frac{2}{3}h - \epsilon n$, and thus there is a square path of length at least $\frac{1}{3}h - 2\epsilon n$ starting at any ordered edge in H'. Set $H'' := G[N_{H'}(x)]$ and $h' := |H''|$. Note that by Lemma 2.2.13, we have that for all $y \in V(H'')$,

$$\|y, P_1\| < \frac{4}{3}p - \frac{2}{3}(\frac{1}{3}h - 2\epsilon n) + 2 - (\frac{2}{3} - 3\epsilon)p = \frac{2}{3}p - \frac{2}{9}h + \frac{4}{3}\epsilon n + 3\epsilon p + 2,$$

so

$$\|y, H'\| > (\frac{2}{3} - \epsilon)n - (\frac{2}{3}p - \frac{2}{9}h + \frac{4}{3}\epsilon n + 3\epsilon p + 2) = \frac{8}{9}h - \frac{7}{3}\epsilon n - 3\epsilon p - 2.$$

So every vertex in H'' has at most $\frac{1}{9}h + \frac{7}{3}\epsilon n + 3\epsilon p + 1$ nonneighbors in H'. Therefore

$$\delta(H'') \geq \frac{\frac{2}{3}h - \epsilon n - (\frac{1}{9}h + \frac{7}{3}\epsilon n + 3\epsilon p + 1)}{\frac{2}{3}h - \epsilon n}h' > \frac{2}{3}h',$$

since $\epsilon \leq \frac{1}{500}$, $n \geq 6000$, and $h > n/6$. Therefore H'' has a hamiltonian square path P_2. Thus

$$|P_1| + |P_2| > p + \frac{2}{3}h - \epsilon n = n - \frac{1}{3}h - \epsilon n \geq n - \frac{1}{3}(\frac{1}{2} + 3\epsilon)n - \epsilon n = (\frac{5}{6} - 2\epsilon)n.$$

□

Now we are ready to finish the nonextreme case.

Proof of Theorem 2.2.4. Let $\alpha := \frac{1}{36}$ and let G be a graph on n vertices. Suppose G has no α-extreme sets, $n \geq n_0 := 2 \times 10^8$, and $\delta(G) \geq \frac{2}{3}n$. Let

$c := \frac{1}{14}$, $\epsilon := \frac{50}{1057}\alpha$, and $\varrho := 1 - \frac{2/3+\epsilon}{5/6-2\epsilon}$. Apply Lemma 2.3.2 to obtain an $(\frac{11}{14}\alpha, \frac{1}{14}\alpha, \epsilon, \varrho)$-special reservoir R. Let $H := G - R$ and let $h := |H|$. Since R is a special reservoir we have $\delta(H) \geq (\frac{2}{3} - \epsilon)h$. Now we apply Lemma 2.3.3 to H, to get disjoint square paths P_1 and P_2 so that

$$|P_1| + |P_2| > (\frac{5}{6} - 2\epsilon)h = (\frac{5}{6} - 2\epsilon)(n - \lceil \varrho n \rceil) \geq (\frac{2}{3} + \epsilon)n - 1 > \frac{2}{3}n.$$

Since R is a special reservoir, every special set $S \subseteq V(G)$ has the property that $S \cap R$ is not $(\frac{11}{14}\alpha, \frac{1}{14}\alpha)$-extreme in $G[R]$. So we apply Lemma 2.3.1 at most twice to connect the paths P_1 and P_2 through R. On the second application, we set $L := V(P_1) \cap R$ to make sure that we avoid the vertices used in the first application. This gives us a square cycle C with $V(P_1) \cup V(P_2) \subseteq V(C)$ and thus $|C| > \frac{2}{3}n$. Therefore G has a hamiltonian square cycle by Theorem 2.1.4. $\square$

2.4 Extremal Case

In this section we prove Theorem 2.2.3. First we need two propositions. Note that the length of an (ordinary) path P is the size $\|P\|$ of its edge set.

Proposition 2.4.1. *Every connected graph H with $|H| \geq 3$ has a path or cycle of length* $\min(2\delta(H), |H|)$.

Proof. Let P be a maximum length path in H. If we are not done, then $\|P\| < 2\delta(H)$. So, as in the proof of Dirac's Theorem [15], G has a cycle C that spans $V(P)$. If C is hamiltonian then we are done; otherwise, using connectivity, we can extend C to a path longer than P, a contradiction. $\square$

Proposition 2.4.2. *If H is a graph with circumference $l > |H| - \delta(H)$, then $l \geq \min(2\delta(H), |H|)$, and moreover, if $|H|$ is also even, then H has an even cycle of length at least* $\min(2\delta(H), |H|)$.

Proof. Let $C \subseteq H$ be a cycle of length l, and fix an orientation of C. If $|C| = |H|$ then we are done, even if $|H|$ is even. Otherwise, let $P := v_1 \dots v_p$ be a maximum path in $H - C$. Then all neighbors of v_p are on $P \cup C$. By hypothesis $\delta(H) > |H| - l \geq p$, and so v_1 has a neighbor $x \in C$ and v_p has a neighbor on $C - x$. Let $y, z \neq x$ be neighbors of v_p on C with y as close as possible to x in the forward direction and z as close as possible in the backward direction (possibly $y = z$). Then $\|zCx\|, \|xCy\| \geq p + 1$, as otherwise we could replace the interior vertices of one of these segments with P to obtain a longer cycle, which would yield a contradiction. Moreover, since C has maximum length, any two neighbors of v_p are separated by at least one vertex on C. Since v_p has at least $d(v_p) - p$ neighbors on $C - x$,

$$|C| = \|xCy\| + \|yCz\| + \|zCx\| \geq (p + 1) + 2(d(v_p) - p - 1) + (p + 1) \geq 2\delta(H).$$

Now suppose $|H|$ is even. If $|C|$ is even we are done, so suppose $|C|$ is odd. Consider the path P and vertices x, y, z defined above. If $\|xCy\|$ and $\|zCx\|$ have different parity, then replace xCy with xPy or replace zCx with zPx to get an even cycle of length at least $2\delta(H)$. So assume $\|xCy\|$ and $\|zCx\|$ have the same parity, and thus $\|yCz\|$ is odd. Now v_p has $k \geq d(v_p) - p$ neighbors on yCz. Let $y = a_1, a_2, \dots, a_k = z$ be the neighbors of v_p on yCz in their natural order. Since $\|yCz\|$ is odd, some segment a_iCa_{i+1} must have odd length. By replacing a_iCa_{i+1} with $a_iv_pa_{i+1}$, we get a cycle C' with even length such that $|C'| \geq (p + 1) + (p + 1) + 2(d(v_p) - p - 1) \geq 2\delta(H)$ as before. $\square$

Proof of Theorem 2.2.3. Let $G = (V, E)$ be a graph on n vertices with $\delta(G) \geq \frac{2}{3}n$. By Corollary 2.2.12 we may assume $n = 3k$, which gives $\delta(G) \geq 2k$. Set $\alpha := \frac{1}{36}$, and suppose G has an α-extreme subset. Let $S \subseteq V$ be an α-extreme set of minimal order, so $|S| = \lceil (1 - \alpha)k \rceil$. Set $T := V \setminus S$. If $k < 1/\alpha$, then $|S| = k$, $|T| = 2k$, $G[S, T]$ is complete and $\delta(G[T]) \geq k$. So by Dirac's theorem T has a hamiltonian cycle $C := y_1 \dots y_{2k} y_1$. Since $G[S,T]$ is complete we can insert the vertices $x_1, x_2, \dots, x_k$ of S into C so that $y_1 y_2 x_1 y_3 y_4 x_2 \dots y_{2k-1} y_{2k} x_k y_1 y_2$ is a hamiltonian square cycle. So for the rest of the proof assume $k \geq 1/\alpha$. Choose $T_0 \subseteq T$ such that $|V \setminus (S \cup T_0)|$ is even, $2\lfloor\sqrt{\alpha} k\rfloor - 1 \leq |T_0| \leq 2\lfloor\sqrt{\alpha} k\rfloor$, and subject to this, $\|T_0, S\|$ is as small as possible. Set $T_1 := T \setminus T_0$, and note that $|T_1|$ is even. We have,

$$\forall x \in S,\ \overline{\|x, T\|} \leq k - (|S| - \|x, S\|) \leq 2\lfloor \alpha k \rfloor. \tag{2.12}$$

Every vertex in T_1 has at most as many nonneighbors in S as every vertex in T_0. Thus, using $\alpha = \frac{1}{36}$, and expressing k as $k = 36q + r$ with $q, r \in \mathbb{Z}$ and $0 \leq r \leq 35$, we have

$$\forall y \in T_1,\ \overline{\|y, S\|} \leq \left\lfloor \frac{2\lfloor \alpha k \rfloor |S|}{|T_0 \cup \{y\}|} \right\rfloor \leq \left\lfloor \frac{2\lfloor \alpha k \rfloor (k - \lfloor \alpha k \rfloor)}{2\lfloor \sqrt{\alpha} k \rfloor} \right\rfloor \leq \left\lfloor \frac{(35q + r)}{6} \right\rfloor \leq \lfloor \sqrt{\alpha} k \rfloor. \tag{2.13}$$

Set $m := k - |T_0| + \lfloor \alpha k \rfloor$ and note that since $k \geq 36$,

$$m \geq \frac{2}{3}k + \lfloor \alpha k \rfloor \geq \frac{2}{3}k + 1. \tag{2.14}$$

Thus we have

$$\delta(G[T_1]) \geq 2k - |S \cup T_0| = k - |T_0| + \lfloor \alpha k \rfloor = m \geq \frac{2}{3}k + 1. \tag{2.15}$$

Case 1: There exists an even cycle $C \subseteq G[T_1]$ of length $2l \geq 2m$; say $C := y_1 \dots y_{2l} y_1$. Looking ahead to an application in Case 2, we prove something slightly more general than what is needed for Case 1. For some $t \leq |T_1|/2$, let $T_1' \subseteq T_1$ such that $|T_1'| = 2t$. Enumerate the vertices of T_1' as $z_1, \dots, z_{2t}$. Let $P := \{p_1, \dots, p_t\}$ be a set of *ports*, where $p_i := \{z_{2i-1}, z_{2i}, z_{2i+1}, z_{2i+2}\}$ and addition of indices is modulo t. We say that a vertex $x \in S$ can be inserted into port p_i if $p_i \subseteq N(x)$.

Claim 2.4.3. *For $S' \subset S$ with $|S'| \geq |S| - 4$, let Γ be the S', P-bigraph with $xp \in E(\Gamma)$ if and only if x can be inserted into p. Then Γ has a matching $M := \{x_i p_i : i \in [t]\}$ that saturates P.*

Proof. Using Hall's Theorem [22], since $|S'| \geq |T_1|/2 \geq |P|$, it suffices to show that

$$\|x, P\|_\Gamma + \|S', p\|_\Gamma \geq |P| \text{ for all } x \in S' \text{ and } p \in P. \tag{2.16}$$

If $x \in S'$, then $\overline{\|x, T\|_G} \leq 2 \lfloor \alpha k \rfloor$ by (2.12). Since each $y \in T_1'$ is in two ports, each nonedge xy contributes to two nonedges in Γ. So $\overline{\|x, P\|_\Gamma} \leq 4 \lfloor \alpha k \rfloor$. Thus

$$\|x, P\|_\Gamma \geq |P| - \overline{\|x, P\|_\Gamma} \geq |P| - 4\alpha k. \tag{2.17}$$

If $p \in P$, then $\overline{\|S', y\|_G} \leq \lfloor \sqrt{\alpha} k \rfloor$ for each $y \in p$ by (2.13). Thus $\overline{\|S', p\|_\Gamma} \leq 4 \lfloor \sqrt{\alpha} k \rfloor$. So

$$\|S', p\|_\Gamma \geq |S'| - \overline{\|S', p\|_\Gamma} \geq (1 - \alpha - \frac{4}{k} - 4\sqrt{\alpha})k. \tag{2.18}$$

Since $4\sqrt{\alpha} + 5\alpha + \frac{4}{k} \leq \frac{33}{36} < 1$, summing (2.17) and (2.18) yields (2.16). □

Let $S' := S$ and $P := \{p_1, \ldots, p_l\}$, where $p_i := \{y_{2i-1}, y_{2i}, y_{2i+1}, y_{2i+2}\}$ and addition of indices is modulo $2l$. By Claim 2.4.3, there exist $x_1, \ldots, x_l$ such that $y_1 y_2 x_1 y_3 y_4 x_2 \ldots y_{2l-1} y_{2l} x_l y_1 y_2$ is a square cycle of length $3l$. By (2.15), $3l \geq 3m > 2k$, and so Theorem 2.1.4 implies that G has a hamiltonian square cycle.

Case 2: Not Case 1. Since $|T_1|$ is even, using Proposition 2.4.2 and (2.15),

$$|D| \leq |T_1| - \delta(G[T_1]) \leq k, \text{ for every cycle } D \subseteq G[T_1]. \tag{2.19}$$

First suppose $G[T_1]$ is connected. By Proposition 2.4.1, there exists a path in $G[T_1]$ of length at least $2m$.

Claim 2.4.4. *Let $P = y_1 \dots y_l$ be a path of maximum length in $G[T_1]$. If $y_i \in N(y_1)$ and $y_j \in N(y_l)$, then $i \leq j$.*

Proof. Suppose there exists $y_i \in N(y_1)$, $y_j \in N(y_l)$ such that $i > j$. With respect to this condition, choose y_i and y_j such that $i - j$ is minimum. If $i - j - 1 \leq \frac{1}{3}k$, set $D := y_1 \dots y_j y_l \dots y_i y_1$. By (2.14), $|D| \geq 2m - \frac{1}{3}k > k$, which contradicts (2.19). If $i - j - 1 > \frac{1}{3}k$, let h be maximum such that $y_h \in N(y_1)$ and set $D := y_1 y_2 \dots y_h y_1$. Since $i - j - 1 > \frac{1}{3}k$ and $i - j$ is minimum, we have $|D| \geq h \geq m + i - j - 1 > k$, which contradicts (2.19). □

Let $P := y_1 \dots y_l$ be a path of maximum length in $G[T_1]$ and with respect to this condition, choose P so that $j - i$ is minimum, where y_j is the smallest indexed neighbor of y_l and y_i the largest indexed neighbor of y_1. Note that by Claim 2.4.4, $j - i \geq 0$. By (2.19) we have,

$$N(y_1) \subseteq \{y_2, \dots y_k\} \text{ and } N(y_l) \subseteq \{y_{l-k+1}, \dots, y_{l-1}\}. \tag{2.20}$$

Set

$$A := \{y_1, \dots, y_{i-1}\},\ B := \{y_i, \dots, y_j\},\ C := \{y_{j+1}, \dots, y_l\}.$$

Without loss of generality we may suppose $|A| \geq |C|$ and thus we have

$$m \leq \delta(G[T_1]) \leq |C| \leq |A| < k \tag{2.21}$$

and $|B| = j - i + 1 \leq l - 2m$.

Next we show that

$$\|A, C\| = 0. \tag{2.22}$$

Suppose $a < i \leq j < b$ and $y_a y_b \in E$. Choose $y_{a'} \in N(y_1)$ and $y_{b'} \in N(y_l)$ such that $a < a' \leq i \leq j \leq b' < b$ and both $a' - a$ and $b - b'$ are minimal. Now $D := y_1 P y_a y_b P y_l y_{b'} P y_{a'} y_1$ is a cycle having the property that

$N(y_1) \cup N(y_l) \subseteq V(D)$ and thus $|D| \geq |N(y_1) \cup N(y_l)| \geq 2m - 1 > k$, contradicting (2.19).

Set $A' := \{y_h \in A : y_{h+1} \in N(v_1)\}$ and $C' := \{y_h \in C : y_{h-1} \in N(y_l)\}$. Note that $|A'| \geq m$ and $|C'| \geq m$. We claim that the vertices in $A' \cup C'$ are good in the sense that

$$\forall a \in A', N(a) \cap (T_1 \setminus (A \cup \{y_i\})) = \emptyset \text{ and } \forall c \in C', N(c) \cap (T_1 \setminus (C \cup \{y_j\})) = \emptyset. \tag{2.23}$$

Without loss of generality, suppose some $y_h \in A'$ has a neighbor $y' \in T_1 \setminus (A \cup \{y_i\})$. If $y' \notin V(P)$, then $y'y_h \dots y_1 y_{h+1} \dots y_l$ is longer than P which is a contradiction. Otherwise, by (2.22), $y' \in B$. However, $y_h \dots y_1 y_{h+1} \dots y_l$ is a path for which $j - i$ is smaller, contradicting the minimality of $j - i$.

Now suppose $G[T_1]$ is not connected. Since $\delta(G[T_1]) \geq m$ and $|T_1| < 3m$, $G[T_1]$ has exactly two components. Call these components A and C, then set $A' := A$ and $C' := C$. Without loss of generality, suppose $|A| \geq |C|$. Since $\delta(G[T_1]) \geq m$, we have $m + 1 \leq |C|$ which implies $|A| < k$, by (2.14) and the fact that $|T_1| = 2k + \lfloor \alpha k \rfloor - |T_0|$. So regardless of whether $G[T_1]$ is connected or not, all of the following hold: (2.21), (2.22), (2.23), and

$$\forall a \in A', \overline{\|a, A\|} \leq |A| - m \;\text{ and }\; \forall c \in C', \overline{\|c, C\|} \leq |C| - m. \tag{2.24}$$

For $Y \in \{A, C\}$, let $Y' = A'$ if $Y = A$ and let $Y' = C'$ if $Y = C$.

Claim 2.4.5. *For all $v \in V \setminus (A \cup C)$, there exists $Y \in \{A, C\}$ such that for all $y \in Y'$, $|(N(v) \cap N(y)) \cap Y| \geq 3$.*

Proof. For all $v \in V \setminus (A \cup C)$, we have

$$\|v, A \cup C\| \geq 2k - (|V| - (|A| + |C|)) = |A| + |C| - k. \tag{2.25}$$

Suppose there exists $v \in V \setminus (A \cup C)$ and $c \in C'$ such that $|(N(v) \cap N(c)) \cap C| \leq 2$. This implies that $\|v, C\| \leq |C| - m + 2$ by (2.24). So we have

$$\|v, A\| \geq |A| + |C| - k - (|C| - m + 2) = |A| + m - k - 2.$$

Let $a \in A'$, then by (2.14),

$$|(N(v) \cap N(a)) \cap A| \geq (|A| + m - k - 2) + m - |A| = 2m - k - 2 \geq \frac{1}{3}k \geq 3.$$

$\square$

Claim 2.4.6. *There exist two disjoint square P^5's connecting edges of A to edges of C.*

Proof. Set $s := \lfloor \frac{|A|}{2} \rfloor$ and $t := \lfloor \frac{|C|}{2} \rfloor$. Choose nonadjacent vertices $x, x' \in S$ and $a_{2s}, c_1 \in N(x)$ with $a_{2s} \in A'$ and $c_1 \in C'$. Since a_{2s} and c_1 are nonadjacent they have at least $k+1$ common neighbors distinct from x, and these common neighbors are not in $A \cup C$. One of them v must also be adjacent to x. By Claim 2.4.5 there exists, without loss of generality, $a_{2s-1} \in A$ such that $a_{2s}, v \in N(a_{2s-1})$. Since $x \in S$, there exists $c_2 \in C$ such that $x, c_1 \in N(c_2)$. Thus $Q := a_{2s-1}a_{2s}vxc_1c_2$ is a square P^5 connecting $a_{2s-1}a_{2s}$ to c_1c_2. Similarly, we can choose $a_1, c_{2t} \in N(x')$ with $a_1 \in A' - a_{2s-1} - a_{2s}$ and $c_{2t} \in C' - c_1 - c_2$. Since a_1 and c_{2t} are nonadjacent, there exist k common neighbors of a_1 and c_{2t} that are distinct from x' and v. One of them v' is adjacent to x', and $v' \neq x$ by the choice of x, x'. Moreover, $v' \notin A \cup C$. So as above, we can choose $a_2 \in A$ and $c_{2t-1} \in C$ so that $Q' := c_{2t-1}c_{2t}\{v'x'\}a_1a_2$, $Q \cap Q' = \emptyset$ and Q' is a square P^5 connecting $c_{2t-1}c_{2t}$ to a_1a_2 (note that we cannot specify the order of v' and x'). $\square$

Finally we claim that there exist paths

$$R := a_1a_2 \ldots a_{2s-1}a_{2s} \subseteq G[A] \text{ and } R' := c_1c_2 \ldots c_{2t-1}c_{2t} \subseteq G[C],$$

such that $|R| = 2s$ and $|R'| = 2t$. If $|A| = m$, then $A = A'$ and thus $G[A]$ is complete by (2.24). Otherwise $|A| \geq m + 1$ and thus by (2.14) we have

$$\frac{1}{3}k + 1 \leq \frac{1}{2}|A|. \tag{2.26}$$

By (2.22) and (2.26), we have

$$\delta(G[A]) \geq 2k - (|V| - (|A| + |C|)) = |A| + |C| - k \geq |A| + 2 - (\frac{k}{3} + 1) \geq \frac{1}{2}|A| + 2.$$

Thus for all $a, a', a'' \in A$,

$$G[A \setminus \{a, a', a''\}] \text{ is hamiltonian connected,}$$

since $\delta(G[A \setminus \{a, a', a''\}]) \geq \frac{1}{2}|A| - 1 > \frac{1}{2}(|A| - 3)$. If $|A| = 2s$, then we use the fact that $G[A \setminus \{a_1, a_{2s}\}]$ is hamiltonian connected to get R. If $|A| = 2s + 1$ we let $a' \in A \setminus \{a_1, a_2, a_{2s-1}, a_{2s}\}$, and we use the fact that $G[A \setminus \{a_1, a_{2s}, a'\}]$ is hamiltonian connected to get R. Since $|A| \geq |C|$, the same argument gives us R' in $G[C]$.

So by Claim 2.4.6, $D := RQR'Q'$ is an even cycle of length $2s + 2t + 4 \geq 2m + 2$ (note that $D \not\subseteq G[T_1]$). Recall that $V(D) \cap S \subseteq \{x, v, x', v'\}$ and set $S' := S \setminus D$. As in Case 1, let $P := \{p_1, \dots, p_s, p'_1, \dots, p'_t\}$ be a set of *ports*, where $p_i := \{a_{2i-1}, a_{2i}, a_{2i+1}, a_{2i+2}\}$ for $1 \leq i \leq s - 1$ and $p'_j := \{c_{2j-1}, c_{2j}, c_{2j+1}, c_{2j+2}\}$ for $1 \leq j \leq t - 1$. By Claim 2.4.3, there exist $x_1, \dots, x_{s-1}, x'_1, \dots, x'_{t-1}$ such that

$$a_1 a_2 x_1 a_3 a_4 x_2 \dots x_{s-1} a_{2s-1} a_{2s} v x c_1 c_2 x'_1 c_3 c_4 x'_2 \dots x'_{t-1} c_{2t-1} c_{2t} \{v' x'\} a_1 a_2$$

is a square cycle of length at least $2s + 2t + 4 + s - 1 + t - 1 \geq 3m - 1 > 2k$. Thus by Theorem 2.1.4, G has a hamiltonian square cycle.

□

2.5 Conclusion

We have established a concrete threshold $n_0 := 2 \times 10^8$ such that Pósa's Conjecture holds for all graphs of order at least n_0, using methods essentially from prior to 1996. It seems in retrospect, that we were blinded by the brilliance of the Regularity-Blow-up method, and missed that the crucial idea of [27] was just to divide the problem into extremal and non-extremal cases. However Pósa's Conjecture remains open. We suspect that our probabilistic methods cannot be used to obtain an improvement of more than a factor of 1000. On the other hand we believe that ordinary graph theoretic methods have not yet been exhausted.

We have also developed the method of special reservoirs, for removing regularity from certain arguments. We believe that this could be used on other problems. The paper [36] was written with the goal of developing methods for a more general set of problems. In particular they used an *absorbing path* lemma which contributes to a much larger value of n_0. However other problems do not (yet) have an analog of Theorem 2.1.4, while the absorbing technique is quite adaptable. Here are some other possible candidates for applying these new techniques, the first of which was discussed in [36].

Conjecture 2.5.1 (Seymour [42]). *For all positive integers k, every graph G with $\delta(G) \geq \frac{k}{k+1}|G|$ contains the k^{th} power of a hamiltonian cycle.*

Komlós, Sárközy and Szemerédi [29, 30] used the Regularity and Blow-up Lemmas to prove that there exists a function $n(k)$ such that Seymour's Conjecture holds for all k and graphs of order at least $n(k)$.

Châu also used the Regularity and Blow-up Lemmas to prove the following Ore-type version of Pósa's Conjecture for graphs of large order.

Theorem 2.5.2 (Châu [7]). *Let G be a graph on n vertices such that $d(x) + d(y) \geq \frac{4}{3}n - \frac{1}{3}$ for all $xy \notin E(G)$.*

(a) *If $\delta(G) = \frac{1}{3}n + 2$ or $\delta(G) = \frac{1}{3}n + \frac{5}{3}$, then G contains a hamiltonian square path.*

(b) *If $\delta(G) > \frac{1}{3}n + 2$, then for sufficiently large n, G contains a hamiltonian square cycle.*

For a directed graph G, the *minimum semi-degree* of G, denoted $\delta^0(G)$, is the minimum of the minimum in-degree $\delta^-(G)$ and the minimum out-degree $\delta^+(G)$. An *oriented graph* is a directed graph with no 2-cycles. Keevash, Kühn, and Osthus proved the following oriented version of Dirac's theorem using the Regularity-Blow-up method (with a directed version of the Regularity Lemma).

Theorem 2.5.3 (Keevash, Kühn, Osthus [25]). *Let G be an oriented graph on n vertices. If $\delta^0(G) \geq \frac{3n-4}{8}$ and n is sufficiently large, then G contains a hamiltonian cycle.*

Finally Treglown conjectured the following oriented version of Pósa's conjecture.

Conjecture 2.5.4 (Treglown [48]). *Let G be an oriented graph on n vertices. If $\delta^0(G) \geq \frac{5n}{12}$, then G contains a the square of a hamiltonian cycle.*

Chapter 3

REGULARITY-BLOW-UP METHOD

In this section we review the Regularity and Blow-up Lemmas and state all the facts needed for our applications in Chapters 4 and 6 (see [32] for a nice reference). Let Γ be a simple graph on n vertices. For two disjoint, nonempty subsets U and V of $V(\Gamma)$, define the density of the pair (U, V) as

$$d(U, V) = \frac{e(U, V)}{|U||V|}.$$

Definition 3.0.5. *A pair (U, V) is called* ϵ-regular *if for every $U' \subseteq U$ with $|U'| \geq \epsilon|U|$ and every $V' \subseteq V$ with $|V'| \geq \epsilon|V|$, $|d(U', V') - d(U, V)| \leq \epsilon$. The pair (U, V) is* (ϵ, δ)-super-regular *if it is ϵ-regular and for all $u \in U$, $\deg(u, V) \geq \delta\,|V|$ and for all $v \in V$, $\deg(v, U) \geq \delta\,|U|$.*

First we note the following facts that we will need about ϵ-regular pairs.

Fact 3.0.6 (Intersection Property)**.** *If (U, V) is an ϵ-regular pair with density d, then for any $Y \subseteq V$ with $(d-\epsilon)^{k-1}|Y| \geq \epsilon|V|$ there are less than $k\epsilon|U|^k$ k-tuples of vertices $(u_1, u_2, \ldots, u_k)$, $u_i \in U$, such that $|Y \cap N(u_1, u_2, \ldots, u_k)| \leq (d-\epsilon)^k|Y|$.*

Fact 3.0.7 (Slicing Lemma)**.** *Let (U, V) be an ϵ-regular pair with density d, and for some $\lambda > \epsilon$ let $U' \subseteq U$, $V' \subseteq V$, with $|U'| \geq \lambda|U|$, $|V'| \geq \lambda|V|$. Then (U', V') is an ϵ'-regular pair of density d' where $\epsilon' = \max\{\frac{\epsilon}{\lambda}, 2\epsilon\}$ and $d' \geq d - \epsilon$.*

Proposition 3.0.8. *If (U, V) is an ϵ-regular pair with density $\delta \geq 2\sqrt{\epsilon} > 0$ and subsets $A, C \subseteq U$, $B, D \subseteq V$ of size at least $\frac{1}{2}\delta|U|$ then there exist $a \in A, b \in B, c \in C, d \in D$ with $abcda = C_4$.*

Lemma 3.0.9 (Augmenting Lemma)**.** *Let (U, V) be an ϵ-regular pair. Suppose that $U' = U \cup S$ and $V' = V \cup T$, where $|S| \leq \mu\,|U|$, $|T| \leq \mu\,|V|$,*

$S \cap V' = \emptyset = T \cap U'$, *and* $0 < \mu < \epsilon$. *Then* (U', V') *is an* ϵ'*-regular pair, where* $\epsilon' = \max\left\{\frac{\mu}{\epsilon}, 6\epsilon\right\}$.

We will use the Regularity Lemma of Szemerédi [44] which we state in its multipartite form.

Lemma 3.0.10 (Regularity Lemma - Bipartite Version)**.** *For every* $\epsilon > 0$ *there exists* $M := M(\epsilon)$ *such that if* $G := G[U, V]$ *is a balanced bipartite graph on* $2n$ *vertices and* $d \in [0, 1]$, *then there is a partition of* U *into clusters* $U_0, U_1, \ldots, U_t$, *a partition of* V *into clusters* $V_0, V_1, \ldots, V_t$, *and a subgraph* $G' := G'[U, V]$ *with the following properties:*

(i) $t \leq M$,

(ii) $|U_0| \leq \epsilon n$, $|V_0| \leq \epsilon n$,

(iii) $|U_i| = |V_i| = \ell \leq \epsilon n$ *for all* $i \in [t]$,

(iv) $\deg_{G'}(x) > \deg_G(x) - (d + \epsilon)n$ *for all* $x \in V(G)$,

(v) *All pairs* (U_i, V_i), $i, j \in [t]$, *are* ϵ*-regular in* G' *each with density either* 0 *or exceeding* d.

We will also use the following stronger version of the Blow-up Lemma of Komlós, Sárközy, and Szemerédi [28].

Lemma 3.0.11 (Blow-up Lemma)**.** *Given* $\delta > 0$, $\Delta > 0$ *and* $\varrho > 0$ *there exist* $\epsilon > 0$ *and* $\eta > 0$ *such that the following holds. Let* $S = (X_1, X_2)$ *be an* (ϵ, δ)*-super-regular pair. with* $|X_1| = n_1$ *and* $|X_2| = n_2$. *If* T *is a* Y_1, Y_2*-bigraph with maximum degree* $\Delta(T) \leq \Delta$ *and* T *is embeddable into the complete bipartite graph* K_{n_1}, n_2 *then it is also embeddable into* S. *Moreover, for all* $\eta|X_i|$*-subsets* $X_i' \subseteq X_i$ *and functions* $f_i : X_i' \to \binom{X_i}{\varrho n_i}$, $i = 1, 2$, T *can be embedded into* S *so that the image of each* $x_i \in X_i'$ *is in the set* $f_i(x_i)$.

Chapter 4

2-FACTORS OF BIPARTITE GRAPHS WITH ASYMMETRIC MINIMUM DEGREES

This chapter is joint work with H.A. Kierstead and Andrzej Czygrinow and was published in SIAM Journal on Discrete Mathematics [12].

4.1 Introduction

This paper is motivated by several lines of research. Let C_n^r (P_n^r) be the r-th power of a cycle (path) on n vertices C_n (P_n). In attempt to inspire a new proof of the Hajnal-Szemerédi theorem, Seymour made the following conjecture:

Conjecture 4.1.1 (Seymour [42])**.** *If G is a graph on n vertices with $\delta(G) \geq \frac{r}{r+1}n$, then $C_n^r \subseteq G$.*

Note that the case $r = 1$ is Dirac's Theorem and the case $r = 2$ is Pósa's Conjecture. Komlós, Sárközy and Szemerédi [29, 30] have used Szemerédi's Regularity Lemma [44] and their own Blow-up Lemma [28] to prove Seymour's conjecture for huge graphs, however even Pósa's Conjecture remains open for small graphs.

Chau generalized the minimum degree condition in Seymour's conjecture to an Ore-type degree condition.

Conjecture 4.1.2 (Chau [7])**.** *Suppose G is a graph on n vertices such that $\deg(x) + \deg(y) \geq \frac{2r}{r+1}n - \frac{r-1}{r+1}$ for all non-adjacent pairs of vertices $x, y \in V(G)$.*

(i) *If $\delta(G) = \frac{r-1}{r+1}n + 2$ or $\delta(G) = \frac{r-1}{r+1}n + \frac{5}{3}$, then $P_n^r \subseteq G$.*

(ii) *If $\delta(G) > \frac{r-1}{r+1}n + 2$, then $C_n^r \subseteq G$.*

When $r = 1$, the condition $\deg(x) + \deg(y) \geq \frac{2r}{r+1}n - \frac{r-1}{r+1}$ is Ore's condition and thus $C_n^r \subseteq G$ with no further restrictions on the minimum degree. Chau proved Conjecture 4.1.2 for huge graphs when $r = 2$.

The following fundamental graph packing conjecture was made independently by Bollobás-Eldridge [5] and Catlin [6]. We state it here in a complementary form.

Conjecture 4.1.3 (Bollobás-Eldridge [5], Catlin [6]). *If G and H are graphs on n vertices with $\Delta(H) \leq r$ and $\delta(G) \geq \frac{rn-1}{r+1}$, then $H \subseteq G$.*

Call a graph on n vertices r-*universal* if it contains every graph H on n vertices with $\Delta(H) \leq r$, then Conjecture 4.1.3 states that G is r-universal if $\delta(G) \geq \frac{rn-1}{r+1}$. The case $r = 1$ follows from the path version of Dirac's Theorem: Since $\delta(G) \geq \frac{n-1}{2}$, G contains the 1-universal graph P_n. Aigner and Brandt [2] proved Conjecture 4.1.3 for the case $r = 2$. Fan and Kierstead [19] proved the path version of Pósa's Conjecture: If $\delta(G) \geq \frac{2n-1}{3}$ then G contains the square P_n^2 of P_n. Since P_n^2 is 2-universal, we have a stronger version of the Aigner-Brandt Theorem: If $\delta(G) \geq \frac{2n-1}{3}$ then G contains a 2-universal graph with maximum degree 4. Csaba, Shokoufandeh and Szemerédi [10] have proved Conjecture 4.1.3 for large graphs when $r = 3$.

Kostochka and Yu generalized the minimum degree condition in the Bollobás-Eldridge conjecture to an Ore-type degree condition.

Conjecture 4.1.4 (Kostochka-Yu [33]). *If G and H are graphs on n vertices with $\Delta(H) \leq r$ and $\deg(x) + \deg(y) \geq \frac{2(rn-1)}{r+1}$ for all non-adjacent pairs of vertices $x, y \in V(G)$, then $H \subseteq G$.*

The case $r = 1$ follows from the path version of Ore's theorem: Since $\deg(x) + \deg(y) \geq n - 1$ for all non-adjacent pairs of vertices $x, y \in V(G)$, G

contains the 1-universal graph P_n. Kostochka and Yu [34] proved Conjecture 4.1.4 for the case $r = 2$.

El-Zahar made the following conjecture.

Conjecture 4.1.5 (El-Zahar [17])**.** *If G is a graph on n vertices with $\delta(G) \geq \sum_{i=1}^{k} \left\lceil \frac{1}{2} n_i \right\rceil$ where $n_i \geq 3$ and $n = \sum_{i=1}^{k} n_i$, then G contains k disjoint cycles of lengths $n_1, \ldots, n_k$.*

El-Zahar proved that if G is a graph on n vertices with $\delta(G) \geq \left\lceil \frac{1}{2} n_1 \right\rceil + \left\lceil \frac{1}{2} n_2 \right\rceil$, where $n_1, n_2 \geq 3$ and $n = n_1 + n_2$, then G contains two disjoint cycles of lengths n_1 and n_2. Abassi [1] used the Blow-up and Regularity Lemmas to prove El-Zahar's Conjecture for huge n.

Now we focus our attention on bipartite graphs. A U, V-bigraph is *balanced* if $|U| = |V|$. We will call a balanced bipartite graph on $2n$ vertices *bi-universal* if it contains every balanced bipartite graph H with $|H| = 2n$ and $\Delta(H) = 2$. Wang made the following conjecture.

Conjecture 4.1.6 (Wang [49])**.** *Every balanced bipartite graph G on $2n$ vertices with $\delta(G) \geq n/2 + 1$ is bi-universal.*

An *n-ladder*, denoted by L_n, is a balanced bipartite graph with vertex sets $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ such that $a_i \sim b_j$ if and only if $|i - j| \leq 1$. We refer to the edges $a_i b_i$ as rungs and the edges $a_1 b_1, a_n b_n$ as the first and last rung respectively. It is easily checked that an n-ladder is a bi-universal graph with maximum degree 3. In this sense, a ladder in a bipartite graph is analogous to a square path in a graph. Czygrinow and Kierstead [13] used the Blow-up and Regularity Lemmas to prove Conjecture 4.1.6 for huge graphs by proving that such graphs contain a spanning ladder.

Finally we consider bipartite graphs with asymmetric minimum degrees. For a U,V-bigraph G, let $\delta_U := \delta_U(G)$ and $\delta_V := \delta_V(G)$ denote the minimum degrees of vertices in U and V respectively. The number of components of G is denoted by $\mathrm{comp}(G)$. Moon and Moser [38] proved that if G is a balanced bipartite graph on $2n$ vertices with $\delta_U + \delta_V \geq n+1$, then G is hamiltonian. Amar [4] proved the following result about more general 2-factors. If G and H are balanced U,V-bigraphs on $2n$ vertices with $\delta_U + \delta_V \geq n+2$, $\Delta(H) \leq 2$ and $\mathrm{comp}(H) \leq 2$ then G contains H. As noted in [4], when $\mathrm{comp}(H) \leq 2$ this result is best possible. Amar then made the following conjecture.

Conjecture 4.1.7 (Amar [4])**.** *Let G and H be balanced U,V-bigraphs on $2n$ vertices with $\Delta(H) \leq 2$. If $\delta_U + \delta_V \geq n + \mathrm{comp}(H)$ then G contains H.*

We will prove the following theorems, strengthening Conjecture 4.1.7 for huge graphs.

Theorem 4.1.8. *Let G and H be balanced U,V-bigraphs on $2n$ vertices with $\Delta(H) \leq 2$. For every integer k there exists $N_0(k)$ such that if $n \geq N_0(k)$, $\delta_U + \delta_V \geq n+2$, and $\mathrm{comp}(H) \leq k$, then G contains H. Furthermore, if $\delta(G) \geq \frac{1}{200k}n + 1$ then G contains a spanning ladder.*

Theorem 4.1.9. *There exists a constant C such that every balanced U,V-bigraph G on $2n$ vertices satisfying $\delta_U + \delta_V \geq n + C$ contains a spanning ladder.*

Theorem 4.1.10. *Let G and H be balanced U,V-bigraphs on $2n$ vertices with $\Delta(H) \leq 2$. There exists an integer N_0 such that if $n \geq N_0$ and $\delta_U + \delta_V \geq n + \mathrm{comp}(H)$ then G contains H.*

We note that there are no known counterexamples to show that the bound in Amar's conjecture is tight when $k \geq 3$. In fact, Wang made the following stronger conjecture:

Conjecture 4.1.11 (Wang [50])**.** *Every balanced U, V-bigraph on $2n$ vertices with $\delta_U + \delta_V \geq n + 2$ is bi-universal.*

In Theorem 4.1.10 we prove Amar's conjecture for huge graphs, but Theorem 4.1.8 gives evidence to suggest that a proof of Conjecture 4.1.11 should ultimately be the goal.

We use the following notation. For $A, B \subseteq V(G)$, $E(A, B)$ is the set of edges with one end in A and the other in B. By $E(A)$ we mean $E(A, V(G) \setminus A)$ and instead of $E(\{a\}, B)$ we will write $E(a, B)$. Let $e(A, B) = |E(A, B)|$, and we will sometimes write $e(a, B)$ as $\deg(a, B)$. For a subgraph $H \subseteq G$, $e(a, H)$ means $e(a, V(H))$. Let $\Delta(A, B) := \max\{e(a, B) : a \in A\}$ and $\delta(A, B) := \min\{e(a, B) : a \in A\}$. We denote the graph induced by A as $G[A]$. Given a tree T, we write xTy for the unique path in T between vertices x and y. We will use the symbol $\oplus$ to denote modular addition, where the modulus will be clear in context.

4.2 Auxiliary facts

We begin with some facts that we will need throughout the paper.

Lemma 4.2.1. *Let G be a connected balanced U, V-bigraph on $2n$ vertices. Then G contains a path of order $t = \min\{2(\delta_U + \delta_V), 2n\}$.*

Proof. Let P be any maximal path with $|P| < t$. It suffices to show that G has a path Q with $|Q| > |P|$. Since P is maximal, the neighborhoods of the ends of P are contained in P. We consider two cases depending on the parity of P.

Case 1: $P = x_1 y_1 \ldots x_l y_l$ is an even path. Then $e(x_1, P) + e(y_l, P) \geq \delta_U + \delta_V > l$. Thus there exists an index $i \in [l]$ such that

$x_1 \sim y_i$ and $y_l \sim x_i$. So $C = x_1 y_i P y_l x_i P x_1$ is a cycle of length $2l$. Since $t \leq 2n$ and G is connected, some vertex $z \in P$ has a neighbor $r \in G - C$. Then $Q = rz(C - z)$ is a longer path.

Case 2: $P = x_1 y_1 \dots x_l y_l x_{l+1}$ is an odd path. Without loss of generality, let $x_1 \in U$. Set $P' = P - x_{l+1}$ and consider the components of $G' = G - P'$. The component containing x_{l+1} has order 1 and thus more vertices from U than V. Since G' is balanced it also has a component D with more vertices from V than U. Since G is connected, there exists a vertex $r \in D$ that is adjacent to a vertex $z \in \{x_j, y_j\} \subseteq V(P')$. If possible, we choose $r \in V$ and with respect to this condition, choose r so that j is maximized. Let w be the predecessor of z on P'. If $|D| = 1$ then $e(r, P') + e(x_1, P') \geq \delta_U + \delta_V > l$, so there exists an index $i \in [l]$ such that $x_1 \sim y_i$ and $r \sim x_i$. Thus $Q = r x_i P x_1 y_i P x_{l+1}$ is a path with $|Q| > |P|$. So we may assume that $|D| \geq 3$. Fix a depth first search tree T of D that is rooted at r. Let b be the number of leaves of T in V. Note that

$$2|T \cap V| - b \leq |E(T)| = |T| - 1 = |D \cap U| + |D \cap V| - 1$$

which implies $b \geq |D \cap V| - |D \cap U| + 1 \geq 2$. Let y be a leaf of T in V that is distinct from r. Since T is a depth first search tree, $N(y) \subseteq V(yTr \cup P')$. Let $m = |V(yTr) \cap U|$ and let i be the largest index with $x_1 \sim y_i$. If $j > l - m$ then $Q = yTrzPx_1$ is a path with $|Q| = 2(j+m) \geq 2(l+1) > |P|$. So suppose $j \leq l - m$. If $i > l - m$ then $Q = yTrzPy_i x_1 Pw$ is a path with $|Q| \geq 2(i+m) \geq 2(l+1) > |P|$. Otherwise $i \leq l - m$. By choice of r we have $e(x_1, Py_{l-m}) + e(y, Px_{l-m}) \geq \delta_U + \delta_V - m > l - m$. So there exists an index $h \in [l-m]$ such that $x_1 \sim y_h$ and $y \sim x_h$. Thus $Q = rTyx_h P x_1 y_h P x_{l+1}$ is a path with $|Q| > |P|$. $\square$

Lemma 4.2.2. *Let G be a balanced U,V-bigraph on $2n$ vertices.*

(i) *If e_s and e_t are independent edges and $\delta(G) \geq \frac{3}{4}n + 1$ then G contains a spanning ladder, starting with e_s and ending with e_t.*

(ii) *If $\Lambda = \{L^1, \ldots, L^s\}$ is a set of disjoint ladders in G such that $\sum_{L \in \Lambda} |L| = 2t$ and $\delta(G) \geq \frac{3n+s+t}{4} + 1$ then G has a spanning ladder starting with the first rung e_1 of L^1, ending with the last rung e_2 of L^s, and containing each $L \in \Lambda$.*

Proof. (i) Let M be a 1-factor of G with $e_s, e_t \in M$. Define an auxiliary graph $H = (M, F)$ on M as follows. If $uv, xy \in M$ with $u, x \in U$ then $uv \sim_H xy$ if and only if $u \sim_G y$ and $v \sim_G x$. There is a natural one-to-one correspondence between ladders $u_1v_1 \ldots u_hv_h$ in G, whose rungs are in M, and paths in H. Also $|H| = n$ and $\delta(H) \geq \frac{1}{2}n + 1$. So H is hamiltonian connected and thus has a Hamilton path, starting with e_s and ending with e_t. This path corresponds to the required ladder in G.

(ii) Note that $\delta(G)$ is large enough to insure that G has a 1-factor M containing all the rungs of the ladders L^i. Form H as in (i). Then each ladder L^i corresponds to a path P_i in H and $\delta(H) \geq \frac{n+s+t}{2} + 1$. Thus any two vertices of H share s non-path neighbors. For $i \in [s-1]$, connect the end c_i of each P_i to the start b_{i+1} of each P_{i+1} with a non-path vertex x_i to form a path $P \subseteq H$ with $|P| = t + s - 1$. Let $H' = H - (P - \{c_{s-1}, x_{s-1}\})$. Then $\delta(H') \geq \frac{1}{2}|H'| + 1$ and so H' is hamiltonian connected. It follows that H' contains a Hamilton path Q starting at c_{s-1} and ending at x_{s-1}. Then the Hamilton path $b_1 P c_{s-1} Q x_{s-1} P c_s$ of H corresponds to the required ladder in G. $\square$

Observe that in the proof of Lemma 4.2.2(ii) we do not need the degrees of "interior" vertices of L^i to be large. More precisely, given a ladder L we define the partition $V(L) = \text{ext}(L) \cup \mathring{L}$, where $\text{ext}(L)$ is the set of *exterior* vertices, and $\mathring{L}$ is the set of *interior* vertices. If L is an *initial* ladder, let $\text{ext}(L)$ be the vertices in the last rung. If L is a *terminal* ladder, let $\text{ext}(L)$ be the vertices in the first rung. If L is not an initial or terminal ladder, let $\text{ext}(L)$ be the vertices in the first and last rung of L. Note that if $L \in \{L_1, L_2\}$, then it is possible for $\mathring{L} = \emptyset$. Set $I := I(\Lambda) = \bigcup_{L\in\Lambda} \mathring{L}$. Then Lemma 4.2.2(ii) still holds if we only require $\deg(v) \geq \frac{3n+s+t}{4} + 1$ for $v \in V(G) \setminus I$.

Lemma 4.2.3. *Let G be a balanced U,V-bigraph on $2n$ vertices and let $\Lambda = \{L^1, \ldots, L^s\}$ be a set of disjoint ladders with initial ladder L^1 and if $s > 1$, terminal ladder L^s such that $\sum_{L\in\Lambda} |L| = 2t$. Suppose $\deg(v) \geq d$ for all $v \notin I(\Lambda)$ and there exists $Q \subseteq U \cup V$ with $|Q| \leq q$ such that $\deg(v) \geq D$ for every $v \notin Q \cup I(\Lambda)$. If*

$$\text{(i)}\quad D \geq \frac{3n+3s+t+4q}{4} + 1 \quad \textit{and} \quad \text{(ii)}\quad d > t + 3q + 2s + n - D.$$

then G has a spanning ladder that starts with the first rung e_1 of L^1, contains each $L \in \Lambda$, and, if $s > 1$, ends with the last rung e_2 of L^s.

Proof. Let M be a matching that saturates $Q' = Q \setminus I$ and avoids the ladders in Λ. This is possible since $q' = |Q'| \leq d - t$ by (ii). We view each edge of M as a 1-ladder. Let $\Lambda^+ = \Lambda \cup M$, $s' = s + q'$ and $t' = t + q'$. Next we extend each ladder $L \in \Lambda^+$ to a new ladder $\phi(L)$ as follows: let $\phi(L^1) = L^1 y_1 z_1$, $\phi(L^s) = a_s b_s L^s$, and $\phi(L^i) = a_i b_i L^i y_i z_i$ for $i \in [s'] \setminus \{1, s\}$ such that $a_h, b_h, y_h, z_h \notin R \cup R'$ for $h \in [s']$, where $R = \bigcup_{L\in\Lambda^+} V(L)$ and R' is the set of all previously chosen extension vertices. For example, suppose we want to find $y_{s'} z_{s'}$ after finding all previous extensions. Let uv be the rung of $L^{s'}$ that we wish to extend, where $u, v \in \text{ext}(L^{s'})$. We have $|(R \cup R') \cap N(v)| < 2s' + t'$, and so it is possible by (ii) to choose $y_{s'} \in N(v) \setminus (R \cup R')$. Note that $Q \cup I(\Lambda) \subseteq R$, and so $\deg(u) \geq D$. Now since $D \leq n$ we have $3s + t + 4q + 4 \leq n$ and thus

$$|(N(u) \cap N(y_{s'})) \setminus (R \cup R')| \geq \frac{1}{2}[n - (s + t + 2q)] + 2 \geq 1. \tag{4.1}$$

So by (i) and (4.1) we may choose $z_{s'} \in (N(u) \cap N(y'_s)) \setminus (R \cup R')$.

Set $\Lambda' = \{\phi(L) : L \in \Lambda^+\}$ and $t'' = t' + 2s' - 2$. Then $s' = |\Lambda'|$ and $2t'' = \sum_{L' \in \Lambda'} |L'|$. By (i)

$$D \geq \frac{3n + 3s + t + 4q}{4} + 1 \geq \frac{3n + (s + q') + (t + q' + 2(s + q'))}{4} + 1 \geq \frac{3n + s' + t''}{4} + 1.$$

Thus by Lemma (4.2.2), $Q \subseteq R \subseteq I(\Lambda')$ and our observation preceding the Lemma, we are done. □

4.3 Set-up and organization of the proof

For the rest of this paper we let G and H be a balanced U, V-bigraphs on $2n$ vertices. Assume $\delta_U + \delta_V \geq n + 2$ and suppose without loss of generality that $\delta_U \leq \delta_V$. Note that this implies $\delta_U \geq 3$. Define γ_1 by $\delta_U = \gamma_1 n + 1$ and γ_2 by $\gamma_1 + \gamma_2 = 1$. Assume $\gamma_1 < \frac{1}{2} < \gamma_2$, since the case where $\gamma_1 = \gamma_2$ was handled in [13]. Also assume $\Delta(H) \leq 2$ and $k = \mathrm{comp}(H)$. Our goal is to show that G contains H.

The rest of the proof is organized as follows. Our main task is to prove Theorem 4.1.8. This proof divides into three main cases. In Section 4 we handle the case that $\gamma_1 < \frac{1}{200k}$. In this case, we will show that G contains H for any value of n, but will not prove the existence of a spanning ladder. Otherwise, we consider two cases, the *extremal* case and the *random* case. The case is determined by whether G is α-*splittable* for a sufficiently small α. In Section 5 we define G to be α-splittable if a certain configuration exists in G. The definition is designed to be most useful in the non-extremal case where G fails to be α-splittable. In the remainder of Section 5 we show that if G is α-splittable and $\beta \geq 2\sqrt{\alpha}$ then G has a much nicer configuration called a β-*partition*. In Section 6, we handle the extremal case by showing that for sufficiently small β, we can obtain a spanning ladder from any β-partition. In Section 7 we introduce the Regularity and Blow-up Lemmas. In Section 8 we use these lemmas to prove that in the non-extremal case, if n is sufficiently large in terms of α, then G contains a spanning ladder. In Section 9 we use our previous results to complete the proofs of Theorem 4.1.9 and Theorem 4.1.10.

4.4 Pre-extremal Case

In this section, we will show that Theorem 4.1.8 is true in the case that one of the minimum degrees is very small.

Lemma 4.4.1. *If $\gamma_1 < \frac{1}{200k}$ then G contains H.*

Proof. Let $S = \{u \in U : \deg(u) < \frac{9}{10}n\}$ and $s = |S|$. Then $\gamma_2 > 1 - \frac{1}{200k}$ and

$$\left(1 - \frac{1}{200k}\right) n^2 \leq \sum_{v \in V} \deg(v) = \sum_{u \in U} \deg(u) < \frac{9}{10}ns + n(n-s)$$

$$s < \frac{1}{20k}n. \tag{4.2}$$

Since $\delta_U + \delta_V \geq n+2$, G contains a Hamilton cycle D. Suppose D orders S as $x_1, \ldots, x_s$, where x_1 is chosen so that $\mathrm{dist}_D(x_1, x_s) > 2$. For each $i \in [s]$, let $w_i x_i y_i \subseteq D$. Since

$$|(N(w_i) \cap N(y_i)) \setminus S| \geq \left(1 - \frac{1}{100k} - \frac{1}{20k}\right) n > s,$$

we can choose distinct $z_i \in U$ such that z_i is adjacent to both y_i and $w_{i\oplus 1}$, if $y_i = w_{i\oplus 1}$ then $z_i = x_{i\oplus 1}$, and otherwise $z_i \notin S$. Note that by the choice of x_1 we have $y_s \neq w_1$ and thus $z_s \neq x_1$. Set $C = w_1 x_1 y_1 z_1 \ldots w_s x_s y_s z_s w_1$. Then C is a cycle with length at most $4s < \frac{2n}{k}$. Let $G' = G - (C - \{w_1, z_s\})$. Then G' is a balanced bipartite graph and $G' \subseteq G - S$. Thus

$$\delta(G') \geq \frac{9}{10}n - 2s \overset{(4.2)}{\geq} \frac{3}{4}n + 1 \geq \frac{3}{4}\frac{|G'|}{2} + 1.$$

So by Lemma 4.2.2(1), G' contains a spanning ladder L with first rung $w_1 z_s$. Since $\mathrm{comp}(H) = k$, some component of H must have size at least $\frac{2n}{k}$ and thus $H \subseteq C \cup L \subseteq G$. $\square$

4.5 Splitting

In this section we define the notions of α-splitting and β-partition. We prove that if G has an α-splitting then it has a β-partition.

Definition 4.5.1. *G is* α-splittable *with* α-splitting (X, Y) *if* $X \subseteq U$ *and* $Y \subseteq V$ *satisfy*

(i) $(\gamma_1 - \alpha)n \leq |X| \leq (\gamma_1 + \alpha)n$ *and* $(\gamma_2 - \alpha)n \leq |Y| \leq (\gamma_2 + \alpha)n$ *and*

(ii) $e(X, Y) \leq \alpha|X||Y|$

Informally, the following lemma asserts that if G is α-splittable then G can *almost* be split into two balanced complete bipartite graphs so that one has order approximately $2\gamma_1 n$ and the other has order approximately $2\gamma_2 n$. Let (X, Y) be an α-splitting of G and set $\overline{X} = U \setminus X$ and $\overline{Y} = V \setminus Y$.

Lemma 4.5.2. *If G is α-splittable for $\alpha \leq \left(\frac{\gamma_1}{4}\right)^2$, then there exist partitions $U = X_0 \cup X_1 \cup X_2$ and $V = Y_0 \cup Y_1 \cup Y_2$ so that*

(i) $X_1 \subseteq X, Y_1 \subseteq \overline{Y}, |X_1| = |Y_1| \geq (\gamma_1 - 2\sqrt{\alpha})n$ *and* $\delta(G[X_1 \cup Y_1]) \geq (\gamma_1 - 4\sqrt{\alpha})n$ *and*

(ii) $X_2 \subseteq \overline{X}, Y_2 \subseteq Y, |X_2| = |Y_2| \geq (\gamma_2 - 2\sqrt{\alpha})n$ *and* $\delta(G[X_2 \cup Y_2]) \geq (\gamma_2 - 4\sqrt{\alpha})n$.

Proof. We will show that there exist $X_1 \subseteq X$ and $Y_1 \subseteq \overline{Y}$ satisfying (i) without using $\gamma_1 < \gamma_2$. Then by the symmetry of γ_1, X and γ_2, Y it will follow that there exists $Y_2 \subseteq Y$ and $X_2 \subseteq \overline{X}$ satisfying (ii).

Let $S = \{x \in X : e(x, \overline{Y}) < (\gamma_1 - \sqrt{\alpha})n\}$. Then

$$\begin{aligned} |S|\sqrt{\alpha}n &< \sum_{x \in X} e(x, Y) = e(X, Y) \le \alpha|X||Y| \\ |S| &\le \sqrt{\alpha}|X|\frac{|Y|}{n} \le \sqrt{\alpha}n. \end{aligned} \tag{4.3}$$

Let $\overline{T} = \{y \in \overline{Y} : e(y, X) < (\gamma_1 - \sqrt{\alpha})n\}$. Then since $\sum_{x \in X} e(x, \overline{Y}) = e(X, \overline{Y}) = \sum_{y \in \overline{Y}} e(y, X)$, we have

$$\gamma_1 n|X| - \alpha|X||Y| \le e(X, \overline{Y}) \le (\gamma_1 - \sqrt{\alpha})n|\overline{T}| + |X|(|\overline{Y}| - |\overline{T}|).$$

Thus

$$\begin{aligned} (|X| - (\gamma_1 - \sqrt{\alpha})n)|\overline{T}| &\le (|\overline{Y}| - \gamma_1 n + \alpha|Y|)|X| \\ (\sqrt{\alpha} - \alpha)n|\overline{T}| &\le ((\gamma_1 + \alpha - \gamma_1)n + \alpha(\gamma_2 + \alpha)n)(\gamma_1 + \alpha)n \\ (1 - \sqrt{\alpha})|\overline{T}| &\le (1 + \gamma_2 + \alpha)(\gamma_1 + \alpha)\sqrt{\alpha}n \\ |\overline{T}| &\le \frac{3}{2}\sqrt{\alpha}n. \end{aligned} \tag{4.4}$$

Choose $X_1 \subseteq X - S$ and $Y_1 \subseteq \overline{Y} - \overline{T}$ such that $|X_1| = |Y_1| \ge (\gamma_1 - 2\sqrt{\alpha})$. This is possible by Definition 4.5.1(i) and the upper bounds (4.3) and (4.4) on $|S|$ and $\left|\overline{T}\right|$. Thus for every $x \in X_1, y \in Y_1$

$$\begin{aligned} e(x, Y_1) &\ge e(x, \overline{Y}) - \left|\overline{T}\right| \ge ((\gamma_1 - \sqrt{\alpha}) - 2\sqrt{\alpha})n \ge (\gamma_1 - 4\sqrt{\alpha})n \quad \text{and} \\ e(y, X_1) &\ge e(y, X) - |S| \ge ((\gamma_1 - \sqrt{\alpha}) - 2\sqrt{\alpha})n \ge (\gamma_1 - 4\sqrt{\alpha})n. \end{aligned}$$

□

Definition 4.5.3. *A* β-partition *of* G *is an ordered partition* $(X_1, S_1, S_2, X_2, Y_1, T_1, T_2, Y_2)$ *with* $U = U_1 \cup U_2, U_1 = X_1 \cup S_1, U_2 = S_2 \cup X_2, V = V_1 \cup V_2, V_1 = Y_1 \cup T_1, V_2 = T_2 \cup Y_2$ *such that for* $g := ||S_i| - |T_i||$ *and* $h \in [2]$ *the following conditions are satisfied*

(i) $(\gamma_h - \beta)n \le |U_h|, |V_h| \le (\gamma_h + \beta)n$;

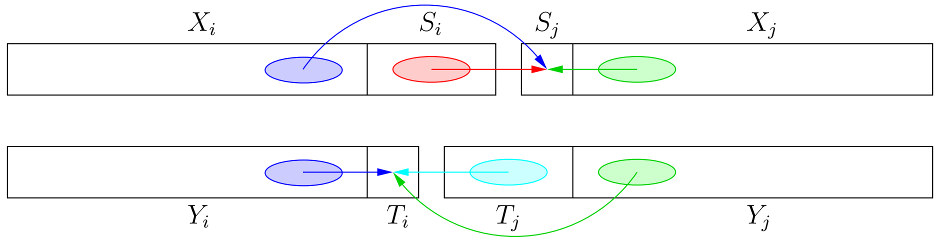


Figure 4.1: Lemma 4.5.4

(ii) $|S_1|, |S_2|, |T_1|, |T_2| \leq 2\beta n$;

(iii) $\delta(X_h, Y_h), \delta(Y_h, X_h) \geq (\gamma_h - 4\beta)n + g$;

(iv) $\delta(S_h, Y_h), \delta(T_h, X_h) \geq 22\beta n + g$;

(v) *if* $|S_i| > |T_i|$ *then* $\Delta(U_i, V_j), \Delta(V_j, U_i) < 24\beta n$ *for* $i \in [2]$ *and* $j = 3 - i$.

Lemma 4.5.4. *If* G *is* α*-splittable and* $2\sqrt{\alpha} \leq \beta \leq \frac{\gamma_1}{268}$ *then* G *has a* β *partition.*

Proof. (See Fig. 4.1.) We start with the partition $U = X_0 \cup X_1 \cup X_2$ and $V = Y_0 \cup Y_1 \cup Y_2$ from Lemma 4.5.2. We describe a process for updating the partition so that conditions (i-v) are satisfied.

Set

$$S_1 = \{x \in X_0 : e(x, Y_1) \geq 24\beta n\},\ S_2 = X_0 \setminus S_1,$$
$$T_1 = \{y \in Y_0 : e(y, X_1) \geq 24\beta n\},\ \text{and } T_2 = Y_0 \setminus T_1.$$

Clearly (i,ii) hold. Also (iii) holds with $2\beta n - g$ to spare. Since $50\beta \leq \gamma_1 \leq \gamma_2$, we have $e(x, Y_2), e(y, X_2) \geq 24\beta n$ for all $x \in S_2$ and $y \in T_2$, and thus (iv) also holds with $2\beta n - g$ to spare. If (v) holds, we are done, so suppose not. Choose i such that $|S_i| > |T_i|$ and set $j = 3 - i$, then $0 < g_0 := |S_i| - |T_i| = |T_j| - |S_j| \leq 2\beta n$. We will now move vertices so that after each move, the difference $|S_i| - |T_i|$ is reduced while (i-iv) continue to hold. Once the difference can no longer be reduced by moving vertices we will claim that (v) holds and then we set $g := |S_i| - |T_i| \geq 0$. On each step we attempt to move vertices $x \in S_i$ with $e(x, Y_j) \geq 24\beta n$ from S_i to S_j and/or vertices $y \in T_j$ with $e(y, X_i) \geq 24\beta n$ from T_j to T_i. If no vertices meet this requirement, then we will attempt to move vertices $x \in X_i$ with $e(x, Y_j) \geq 24\beta n$ from X_i to S_j. Any time a move of this type is made the size of X_i is reduced, so to ensure that $|X_h| = |Y_h|$ we must also move any vertex from Y_i to T_i. Similarly, we may move eligible vertices from Y_j to T_i and compensate by moving any vertex from X_j to S_j. After each move, any of $|X_h|, |Y_h|, \delta(X_i, Y_i), \delta(Y_i, X_i), \delta(S_i, Y_i), \delta(T_i, X_i)$ may decrease, and $|S_j|$ and $|T_i|$ will increase. Note that these parameters may change by only 1 per move. Since we will make at most $g_0 - g$ moves, (iii,iv) will continue to hold. Furthermore, since $|S_i|, |T_j|$ will never be increased, $|U_i|, |V_j|$ may decrease by at most $g_0 - g$ and $|U_j|, |V_i|$ may increase by at most $g_0 - g$, so (i,ii) will continue to hold. When the the process stops, (v) will hold either because $|S_i| = |T_i|$ or because there are no more eligible vertices to move, in which case condition (v) is satisfied.

□

4.6 Extremal case

In this section we prove Theorem 4.1.8 in the case that G is α-splittable for sufficiently small α.

Lemma 4.6.1. *Let $N_1(k) = 408800k + 1$. If $n \geq N_1(k)$, $\gamma_1 \geq \frac{1}{200k}$, and G is α-splittable for $\alpha = \left(\frac{\gamma_1}{584}\right)^2$, then G contains a spanning ladder.*

Proof. Set $\beta = 2\sqrt{\alpha} = \frac{\gamma_1}{292}$, then by Lemma 4.5.4 G has a β-partition $(X_1, S_1, S_2, X_2, Y_1, T_1, T_2, Y_2)$. Since $\gamma_1 \geq \frac{1}{200k}$ we have

$$\beta n = \frac{\gamma_1 n}{292} > 7. \tag{4.5}$$

Set $G_i = G[U_i \cup V_i]$ for $i \in [2]$. For $L \in \{L_2, L_3\}$ we say that L is a *crossing ladder* if its first rung is in G_1 and its last rung is in G_2. Choose i so that $g = |S_i| - |T_i| \geq 0$ and set $j = 3 - i$. Roughly, our plan is to find a crossing ladder L^0 and then find ladders L', L'' spanning G_1, G_2 such that the last rung of L' is the first rung of L^0 and the last rung of L^0 is the first rung of L''. However G_1, G_2 may not be balanced or G_1, G_2 may have been balanced to begin with, but the crossing ladder created an imbalance. In both of these situations we will need a way of moving vertices between G_1 and G_2 so that they may be incorporated into L' and L''.

Formally, our plan is to construct a set of pairwise disjoint ladders $\Lambda = \{L^0, \ldots, L^s\}$ with $s \leq g + 1 \leq 2\beta n + 1$ and $I = I(\Lambda) = \bigcup_{L \in \Lambda} \mathring{L}$ such that

(a) L^0 is a crossing ladder,

(b) for all $p \in [s]$, there exists $h \in [2]$ with $\mathrm{ext}(L^p) \subseteq G_h$ and

(c) $G_1 - I$ is balanced (equivalently, $G_2 - I$ is balanced).

We may also designate one ladder as an initial ladder for each G_h. Then we will apply Lemma 4.2.3 to construct a spanning ladder.

We begin with two useful facts. By our degree conditions we have

$$\forall v, v' \in V \quad |N(v) \cap N(v')| \geq 2\delta_V - n > 2(n/2 + 1) - n = 2 \tag{4.6}$$

Since $\sum_{u \in U} \deg(u) = e(U, V) \geq \delta_V |U|$ and $\delta_U < \delta_V$, there exists $u^* \in U$ with $\deg(u^*) > \delta_V$. Thus

$$\exists u^* \in U \ \forall u \in U \quad |N(u^*) \cap N(u)| \geq \delta_V + 1 + \delta_U - n \geq 3. \tag{4.7}$$

Step 1: (Construct a crossing ladder L^0.) We are done unless

$$\textit{there is no crossing } L_2. \tag{$*$}$$

So suppose not, then by (4.7) there exist vertices $x_1 \in U_1$, $x_2 \in U_2$ such that $|N(x_1) \cap N(x_2)| \geq 3$ and

$$(N(x_1) \cap N(x_2) \subseteq V_1) \vee (N(x_1) \cap N(x_2) \subseteq V_2). \tag{$*1$}$$

Let $y_1, y_2 \in N(x_1) \cap N(x_2)$, by ($*1$) there exists $q \in [2]$ such that $\{y_1, y_2\} \subseteq V_q$. Let $q' = 3 - q$ and $y_3 \in N(x_{q'}) \cap V_{q'}$. By (4.6), y_2 and y_3 have a common neighbor $x_3 \neq x_q, x_{q'}$. By ($*$), $x_3 \in U_{q'}$. Thus $L^0 = x_q y_1 x_{q'} y_2 x_3 y_3$ is a crossing L_3. (See Fig. 4.2)

Step 2: (Construct $L^1, \ldots, L^s$ so that (b) and (c) hold.) For all $u \in U_i$ and $v \in V_j$

$$n+2 \leq \deg(u)+\deg(v) \leq |V_i|+e(u,V_j)+|U_j|+e(v,U_i) \leq n-g+e(u,V_j)+e(v,U_i).$$

Therefore

$$g + 2 \leq \delta(U_i, V_j) + \delta(V_j, U_i). \tag{4.8}$$

Case 1: $g = 0$. If G has a crossing L_2, i.e., ($*$) fails, then there is nothing to do. Otherwise, $L^0 = L_3$ and $y_2 \in \mathring{L}^0 \cap V_q$ thus $|U_q \setminus \mathring{L}^0| = |V_q \setminus \mathring{L}^0| + 1$. Let $x' \in N(y_2) \cap (U_q - x_q)$ and $y' \in N(x_{q'}) \cap (V_{q'} - y_3)$. Since $g = 0$, i and j are interchangeable, so by (4.8), either x' has a neighbor in $V_{q'}$ or y' has a neighbor in U_q and by ($*$), neither of these possible neighbors can be in L^0. Regardless, there exists an edge $xy \in E(U_q, V_{q'})$ whose ends are not in L^0. Let $y^* \in N(x) \cap (V_q \setminus V(L^0))$. By (4.6), y and y^* have a common neighbor x^* with $x^* \neq x, x_h$. By ($*$), $x^* \in U_q$. Set $L^1 = xyx^*y^*$ and specify L^1 as the initial ladder for G_q. Note that $\text{ext}(L^1) \subseteq G_q$ and $|U_q \setminus (\mathring{L}^0 \cup \mathring{L}^1)| = |V_q \setminus (\mathring{L}^0 \cup \mathring{L}^1)|$ so we are done.

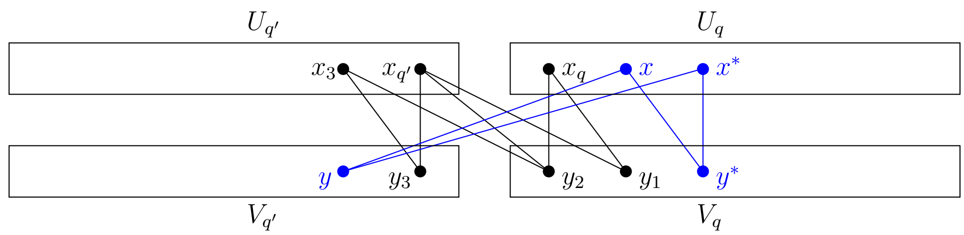


Figure 4.2: Step 1 and Step 2 (Case 1)

Case 2: $g \geq 1$. Using Definition 4.5.3(i,v) and $g \geq 1$ we have

$$\forall v, v^* \in V_j \quad |(N(v) \cap N(v^*)) \cap U_j| \geq 2(\gamma_2 - 24\beta)n - |U_j| \geq |U_j| - 50\beta n > \frac{4}{5}|U_j|. \tag{4.9}$$

If $U_i = U_1$ we have

$$\forall u, u^* \in U_1 \quad |(N(u) \cap N(u^*)) \cap V_1| \geq 2(\gamma_1 - 24\beta)n - |V_1| \geq |V_1| - 50\beta n > \frac{4}{5}|V_1|. \tag{4.10}$$

If $U_i = U_2$ then for all $v \in V_1$, $(\gamma_1 + \beta)n \geq \deg(v, U_1) \geq (\gamma_2 - 24\beta)n$ which implies $\gamma_2 > \gamma_1 \geq \gamma_2 - 25\beta$. In which case we have

$$\begin{aligned} \forall u, u^* \in U_2 \quad |(N(u) \cap N(u^*)) \cap V_2| &\geq 2(\gamma_1 - 24\beta)n - |V_2| \geq 2(\gamma_2 - 49\beta)n - |V_2| \\ &\geq |V_2| - 100\beta n > \frac{13}{20}|V_2|. \end{aligned} \tag{4.11}$$

Let $m = \max\{\delta(U_i, V_j), \delta(V_j, U_i)\}$ and note that by (4.8) and $g \geq 1$, we have $m \geq 2$. Also note that by (4.8), if $g \geq 3$ then $m \geq 3$. It is the case that if $L^0 = L_3$ then $m \geq 3$: if $\delta(V_j, U_i) > 0$, then by (4.6,$*$), we have $\delta(V_j, U_i) \geq 3$ otherwise $\delta(V_j, U_i) = 0$ and thus $\delta(U_i, V_j) \geq 3$ by (4.8).

Case 2a: $m = 2$. Then $L^0 = L_2$, $1 \leq g \leq 2$ and $1 \leq \delta(A, B) \leq \delta(B, A) = 2$ for some choice of $\{A, B\} = \{U_i, V_j\}$. Let $A \cup A', B \cup B' \in \{U, V\}$. By Definition 4.5.3(v) and $g > 0$ there exists $b_1 \in B \setminus V(L^0)$ with no neighbor in $V(L^0) \cap A$ and two neighbors $a_1, a_2 \in A$. By (4.9,4.10,4.11), a_1 and a_2 have a common neighbor $b_2 \in B' \setminus V(L^0)$. Let $L^1 = a_1b_1a_2b_2$ be the initial ladder for G_h, where $b_2 \in G_h$ and $\mathrm{ext}(L^1) \subseteq G_h$. If $g = 1$ then $|U_i \setminus (\mathring{L}^0 \cup \mathring{L}^1)| = |V_i \setminus (\mathring{L}^0 \cup \mathring{L}^1)|$ and we are done. If $g = 2$ then also $\delta(A, B) = 2$ by (4.8), and a similar argument yields an initial ladder $L^2 = a_3b_3a_4b_4$ for G_{h-3} such that $a_3 \in A, b_3, b_4 \in B, a_4 \in A'$ and L^0, L^1, L^2 are disjoint. We have $\mathrm{ext}(L^2) \subseteq G_{h-3}$ and $|U_i \setminus (\mathring{L}^0 \cup \mathring{L}^1 \cup \mathring{L}^2)| = |V_i \setminus (\mathring{L}^0 \cup \mathring{L}^1 \cup \mathring{L}^2)|$ so we are done.

Case 2b: $m \geq 3$. By (4.8) there exists $A \in \{U_i, V_j\} = \{A, B\}$ such that $e(a, B) \geq m \geq 3$ for all $a \in A$. Let $M = \{a_rb_rc_rd_r : r \in [s]\}$ be a maximal set of disjoint claws with root $a_r \in A$ and leaves $b_r, c_r, d_r \in B$. Then every vertex in $\overline{A} = A \setminus \{a_r : r \in [s]\}$ has at least $m - 2$ neighbors in $N = \{b_r, c_r, d_r : r \in [s]\}$. Suppose $s \leq g$. Then using Definition 4.5.3(i,v), $g \leq 2\beta n$ and $g \leq 2m - 2$ (from (4.8)), we note

$$(m-2)((\gamma_1 - \beta)n - s) \leq |E(\overline{A}, N)| \leq 3s \cdot 24\beta n.$$

Thus

$$\gamma_1 \leq 72\beta\frac{g}{m-2} + \beta + \frac{s}{n} \leq 72\beta\frac{2m-2}{m-2} + 3\beta \leq 291\beta < \gamma_1, \tag{4.12}$$

a contradiction. So we conclude that $s \geq g + 1$. Choose B' so that $\{B, B'\} = \{U_l, V_l\}$ for some $l \in [2]$. Let $g' := |B \setminus \mathring{L}^0| - |B' \setminus \mathring{L}^0|$ and note that $g - 1 \leq g' \leq g + 1$. In order to balance $G_l - \mathring{L}^0$ we build a set of disjoint 3-ladders

$$\Lambda(M) = \{x_rb_ra_rc_ry_rd_r : r \in [g'], a_rb_rc_rd_r \in M \text{ and } x_r, y_r \in B'\}.$$

This is possible by $s \geq g + 1$, (4.9,4.10,4.11) and

$$|(N(b_r) \cap N(c_r)) \cap (B' \setminus \mathring{L}^0)|, |(N(c_r) \cap N(d_r)) \cap (B' \setminus \mathring{L}^0)| \geq \frac{13}{20}|B'| - 2 \geq 2g'.$$

Thus $|U_l \setminus (\mathring{L}^0 \cup I(\Lambda(M)))| = |V_l \setminus (\mathring{L}^0 \cup I(\Lambda(M)))|$ and $\mathrm{ext}(L) \subseteq G_l$ for all $L \in \Lambda(M)$ so we are done.

Step 3: (Construct the spanning ladder.) Let Λ be the set of ladders constructed in Steps 1 and 2 and set $I := I(\Lambda)$. Let $\Lambda_h = \{L \in \Lambda : \mathrm{ext}(L) \subseteq G_h\}$ and $G'_h = (G_h - I) \cup \bigcup \Lambda_h$ for $h \in [2]$. Note that G'_1, G'_2 are balanced and $G'_1 \cup G'_2 = G - \mathring{L}^0$. For each ladder $L \in \Lambda_h$ there is a unique vertex $v' \in \mathring{L} \cap V(G_{3-h})$. Since $v' \in \mathring{L}$, we are unconcerned about its degree in G'_h so we add this vertex to the appropriate exceptional set (S_h or T_h) in G'_h.

Let e_1 and e_2 be the first and last rungs of L^0, which we will specify as the terminal ladders in G'_1 and G'_2 respectively. It will suffice to show using Lemma 4.2.3 that each G'_h has a spanning ladder, starting at its initial ladder, if it is specified in Case 1 or Case 2a, and ending at its terminal ladder. Let $s' := |\Lambda_h| \leq g + 1$ and $t' := \frac{1}{2}|\bigcup \Lambda_h| \leq 3(g + 1)$. Recall that $g = |S_i| - |T_i|$. Since we only add vertices to S_j and T_i and $\mathring{L}^0 \cap V(G'_h) = \emptyset$, we have $n' := \frac{1}{2}|G'_h| \leq (\gamma_h + \beta)n$. Let $Q := \{v \in V(G'_h) : \deg(v) < D\}$, where $D := (\gamma_h - 4\beta)n - 1$. By Definition 4.5.3(iii), $Q \subseteq S_h \cup T_h$. Thus, by Definition 4.5.3(ii), $q' := |Q| \leq 4\beta n - g$. By Definition 4.5.3(iii,iv), if $v \in V(G'_h) \smallsetminus I$ then $d := 22\beta n - 1 \leq 22\beta n + g - s' \leq \deg(v, G'_h)$. Thus G'_h has the desired spanning ladder by Lemma 4.2.3, since

$$\frac{3n' + 3s' + t' + 4q'}{4} + 1 \leq \frac{3\gamma_h n + 23\beta n + 10}{4} \leq D$$

and

$$t' + 3q' + 2s' + n' - D \leq 21\beta n + 6 \overset{(4.5)}{<} d.$$

$\square$

4.7 Non-extremal case

In this section, we will show that if the graph is not α-splittable for sufficiently small α then it contains a spanning ladder. The proof uses the Regularity-Blow-up method (see Chapter 3).

Lemma 4.7.1. *Let k be a positive integer and suppose $\gamma_1 \geq \frac{1}{200k}$. There exists $N_2(k) \in \mathbb{N}$ so that if G is not α-splittable for $\alpha = \left(\frac{\gamma_1}{584}\right)^2$, and $n \geq N_2(k)$ then G contains a spanning ladder.*

Proof. Let $0 < d_0 \leq \frac{\alpha\gamma_1\gamma_2}{8}$, $\delta_1 \leq \frac{1}{3072}d_0^2$, $\delta_2 \leq \frac{1}{2}\delta_1$, $\delta_3 \leq \frac{1}{2}\delta_2$, $\delta_4 \leq \frac{1}{2}\delta_3$, $\delta \leq \frac{1}{4}\delta_4$, $\Delta = 4$ and $\varrho = \frac{1}{2}\delta$. For these choices of δ, Δ and ϱ choose $\epsilon < \delta^3$ and η to satisfy the conclusion of Lemma 3.0.11. Now let $\epsilon_5 \leq \left(\frac{\epsilon}{6}\right)^4$, $\epsilon_4 \leq \frac{1}{4}\epsilon_5$, $\epsilon_3 \leq \frac{1}{2}\epsilon_4$, $\epsilon_2 \leq \frac{1}{2}\epsilon_3$, and $\epsilon_1 \leq \frac{1}{2}\epsilon_2$. So

$$0 < \epsilon_1 < \epsilon_2 < \epsilon_3 < \epsilon_4 < \epsilon_5 \ll \epsilon \ll \delta < \delta_4 < \delta_3 < \delta_2 < \delta_1 \ll d_0 \ll \alpha.$$

Let $N_2 = \frac{4M(\epsilon_1)}{\eta}$ where $M(\epsilon_1)$ is the value obtained from Lemma 3.0.10. Apply Lemma 3.0.10 to G with ϵ_1 and δ_1 to obtain a partition $\{U_0, U_1, \ldots, U_t\} \cup \{V_0, V_1, \ldots, V_t\}$ and a subgraph G' satisfying (i-v). For all $i, j \in [t]$, let $\ell := |U_i| = |V_j|$ and note that

$$(1 - \epsilon_1)\frac{n}{t} \leq \ell \leq \frac{n}{t}.$$

Consider the *cluster graph* $\mathcal{G}$ with $V(\mathcal{G}) = \{U_1, \ldots, U_t\} \cup \{V_1, \ldots, V_t\}$ and two clusters W, W' joined by an edge when the pair (W, W') is ϵ_1-regular and $d(W, W') \geq \delta_1$. Then $\mathcal{G}$ is a bipartite graph with bipartition $\{\mathcal{U}, \mathcal{V}\}$, where $\mathcal{U} = \{U_1, \ldots, U_t\}$ and $\mathcal{V} = \{V_1, \ldots, V_t\}$.

Claim 4.7.2. $\delta_{\mathcal{U}} \geq (\gamma_1 - \delta_1 - 2\epsilon_1)t$ *and* $\delta_{\mathcal{V}} \geq (\gamma_2 - \delta_1 - 2\epsilon_1)t$.

Proof. Suppose there exists $Z \in V(\mathcal{G})$ with $\deg_{\mathcal{G}}(Z) < (\gamma_i - \delta_1 - 2\epsilon_1)t$, where $i = 1$ if $Z \in \mathcal{U}$ and $i = 2$ if $Z \in \mathcal{V}$. Then

$$\gamma_i n\ell \leq e_G(Z) < (\gamma_i - \delta_1 - 2\epsilon_1)t\ell^2 + \epsilon_1 n\ell \leq (\gamma_i - \delta_1 - \epsilon_1)n\ell$$

and thus some vertex $z \in Z$ has

$$\deg_{G'}(z) < \gamma_i n - (\delta_1 + \epsilon_1)n \leq \deg_G(z) - (\delta_1 + \epsilon_1)n,$$

contradicting property (iv) of Lemma 3.0.10. $\square$

Claim 4.7.3. *$\mathcal{G}$ contains a path $\mathcal{P}$ on $2q$ vertices with $q \geq (1 - 2\delta_1 - 4\epsilon_1)t$.*

Proof. If $\mathcal{G}$ is connected, then the claim follows immediately from Claim 4.7.2 and Lemma 4.2.1. So suppose that $\mathcal{G}$ is disconnected, we will obtain a contradiction by showing that this implies that $\mathcal{G}$ is α-splittable. Let $\mathcal{A}$ and $\mathcal{B}$ be distinct components of $\mathcal{G}$ and let $X = U \cap \bigcup \mathcal{A}$ and $Y = V \cap \bigcup \mathcal{B}$. Using $e_{\mathcal{G}}(X, Y) = 0$, we have

$$e_G(X, Y) \leq \delta_1|X||Y| + \epsilon_1 t\ell|X| \leq \delta_1|X||Y| + \epsilon_1 3|Y||X| \leq \alpha(\gamma_1 - \alpha)(\gamma_2 - \alpha).$$

Thus Definition 4.5.1(ii) holds. By Claim 4.7.2 we have

$$|X| \geq (\gamma_2 - \delta_1 - 2\epsilon_1)t\ell \geq (\gamma_2 - \delta_1 - 2\epsilon_1)(1 - \epsilon_1)n \geq (\gamma_2 - \delta_1 - 3\epsilon_1)n \geq (\gamma_2 - \alpha)n$$

and

$$|Y| \geq (\gamma_1 - \delta_1 - 2\epsilon_1)t\ell \geq (\gamma_1 - \delta_1 - 2\epsilon_1)(1 - \epsilon_1)n \geq (\gamma_1 - \delta_1 - 3\epsilon_1)n \geq (\gamma_1 - \alpha)n.$$

Thus Definition 4.5.1(i) holds for some $X' \subseteq X$, $Y' \subseteq Y$ and (X', Y') is an α-splitting of $\mathcal{G}$. $\square$

Choose the notation so that $\mathcal{P} = U_1 V_1 \ldots, U_q V_q$. Add all clusters which are not in $\mathcal{P}$ to the exceptional class $U_0 \cup V_0$. As $\delta_1 \gg \epsilon_1$, the exceptional class may now be much larger:

$$|U_0| = |V_0| \leq 3\delta_1 n.$$

Our next task is to reassign the vertices from the exceptional class to $\mathcal{P}$. Since we will need to do this twice, we state the procedure in general terms. Let $\{X_0, X_1, \ldots, X_q\} \cup \{Y_0, Y_1, \ldots, Y_q\}$ be the current partition, where $\bigcup_{i=0}^{q} X_i = U$ and $\bigcup_{i=0}^{q} Y_i = V$. Suppose that (X_i, Y_i) and (X_{i+1}, Y_i) are ϵ'-regular pairs of density at least δ'. Recall that $(1 - \epsilon_1)\frac{n}{t} \leq \ell \leq \frac{n}{t}$ was the common size of the non-exceptional clusters in the initial ϵ_1-regular partition. The procedure takes two parameters σ and τ where $\sigma^2 n$ is an upper bound on the size of the exceptional sets and $2\tau\ell$ is a minimum degree condition which a vertex must meet in order to be reassigned to a cluster. We arbitrarily group the vertices from $X_0 \cup Y_0$ into pairs (u, v) and distribute them one pair at a time. In addition to reassigning vertices from $X_0 \cup Y_0$ we may move a vertex from one cluster to another. This process will be completed after $s := |X_0| = |Y_0| \leq \sigma^2 n$ steps.

We use the following notation. For a cluster Z let Z^r denote Z after the r-th step of the reassignment. So $Z = Z^0$. Let $O(Z^r) := Z^0 \cap Z^r$ denote the original vertices of Z^0 that remain after the r-th step, $T(Z^r) := Z^r \setminus Z^0$ denote the vertices that have been moved to Z during the first r steps, and $F(Z^r) := Z^0 \setminus Z^r$ denote the vertices that have been moved from Z during the first r steps. We say that a cluster Z^r is *full* when $|T(Z^r)| = \sigma\ell$.

Procedure: Reassign

For $r = 1, \ldots, s$ reassign the r-th pair (u, v) as follows:

(i) Choose $i, j \in [q]$ so that each of the following holds:

  (a) None of V_i^{r-1}, U_i^{r-1}, and U_j^{r-1} is full.

  (b) $\deg(v, U_i^0) \geq 2\tau\ell$ and $\deg(u, V_j^0) \geq 2\tau\ell$.

  (c) If $i \neq j$ then $e(U_j^0, V_i^0) \geq 3\tau\ell^2$.

(ii) Reassign u to U_j^{r-1}, v to V_i^{r-1}, and if $i \neq j$ then pick $u' \in O(U_j^{r-1})$ with $\deg(u', V_i^0) \geq 2\tau\ell$ and reassign u' to U_i^{r-1}.

Lemma 4.7.4 (Reassigning Lemma)**.** *Suppose* $\{X_0, X_1, \ldots, X_q\} \cup \{Y_0, Y_1, \ldots, Y_q\}$ *is a partition of* $V(G)$ *in which the pairs* (X_i, Y_i) *and* (X_{j+1}, Y_j) *for* $i \in [q]$ *and* $j \in [q-1]$*, are* ϵ'*-regular with density at*

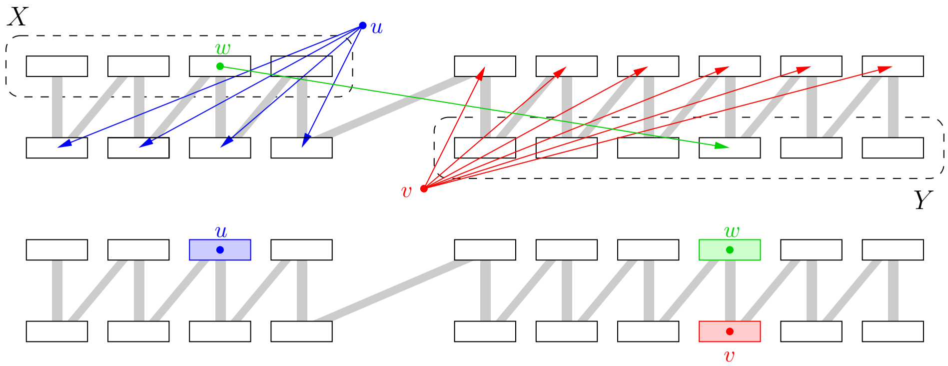


Figure 4.3: Distribution of vertices from $X_0 \cup Y_0$. We write $z \to W_i$ if $\deg(z, W_i^0) \geq 2\tau\ell$.

least δ', where $2\epsilon' \leq \delta'$, $(1-d_0)\ell \leq |X_i|, |Y_i| \leq \ell$ and $s = |X_0| = |Y_0| \leq \sigma^2 n$. If $\epsilon_1 \leq \epsilon' \leq \sigma \leq \frac{1}{4}\tau \leq \frac{1}{4}d_0$, then Reassign *distributes all vertices from $X_0 \cup Y_0$ so that the following conditions are satisfied:*

(i) *If $u \in T(X_i^s)$ then $\deg(u, O(Y_i^s)) \geq \tau\ell$ and if $v \in T(Y_i^s)$ then $\deg(v, O(X_i^s)) \geq \tau\ell$;*

(ii) $|X_i^s| - |Y_i^s| = |X_i^0| - |Y_i^0|$;

(iii) $|T(X_i^s)|, |T(Y_i^s)| \leq \sigma\ell$ *and* $|F(X_i^s)|, |F(Y_i^s)| \leq \sigma\ell$;

(iv) *the pairs $(O(X_i^s), O(Y_i^s))$ and $(O(X_{j+1}^s), O(Y_j^s))$ are $2\epsilon'$-regular with density at least $\frac{1}{2}\delta'$.*

Proof. Suppose that r pairs have been distributed and consider the $(r+1)$-th pair (u, v). Let

$$N'(u) = \{i : \deg(u, Y_i^0) \geq 2\tau\ell\} \quad \text{and} \quad N'(v) = \{i : \deg(v, X_i^0) \geq 2\tau\ell\}.$$

Since

$$\gamma_2 n \leq \deg(v) \leq |N'(v)|\ell + 2\tau\ell t + \sigma^2 n \leq |N'(v)|\frac{n}{t} + 2\tau n + \sigma^2 n,$$

we have

$$|N'(v)| \geq (\gamma_2 - 2\tau - \sigma^2)t \geq (\gamma_2 - 3\tau)t.$$

In the same way we obtain

$$|N'(u)| \geq (\gamma_1 - 3\tau)t.$$

Now let

$$X = \bigcup_{i \in N'(u)} X_i^0 \subseteq U \text{ and } Y = \bigcup_{i \in N'(v)} Y_i^0 \subseteq V.$$

Then we have

$$|Y| \geq |N'(v)|(1-d_0)(1-\epsilon_1)\frac{n}{t} \geq (\gamma_2-3\tau)(1-d_0)(1-\epsilon_1)n \geq (\gamma_2-5d_0)n \geq (\gamma_2-\alpha)n.$$

Similarly

$$|X| \geq (\gamma_1 - \alpha)n.$$

Consequently, as the graph is not α-splittable, we have

$$e(X,Y) > \alpha|X||Y| \geq \alpha(\gamma_1 - \alpha)(\gamma_2 - \alpha)n^2 \geq \alpha\gamma_1\gamma_2 n^2/2. \tag{4.13}$$

Suppose that we are unable to distribute the pair (u, v). We will derive a contradiction by counting edges incident with full clusters and edges in pairs (U_i^r, V_j^r) with $e(U_i^r, V_j^r) < 3\tau\ell^2$. At most $s - 1 \leq \sigma^2 n$ pairs of exceptional vertices have been distributed, and each time a pair is distributed there are at most two indices i such that $|T(X_i^r)|$ or $|T(Y_i^r)|$ increases. Upon distribution, $|T(X_i^r)|$ or $|T(Y_i^r)|$ can increase by at most one. Thus there are at most

$$\frac{2\sigma^2 n}{\sigma\ell} = 2\sigma\frac{n}{\ell}$$

pairs (U_i, V_i) such that either U_i or V_i is full. The total number of edges of G which are incident with vertices in these clusters is at most

$$4\sigma\frac{n}{\ell}\ell n = 4\sigma n^2.$$

There are at most $3\tau n^2$ edges of G in pairs (X_i^0, Y_j^0) with $e(X_i^0, Y_j^0) < 3\tau\ell^2$. Then, since

$$(3\tau + 4\sigma)n^2 \leq 4\tau n^2 \leq \alpha\gamma_1\gamma_2 n^2/2 < e(X, Y)$$

contradicts (4.13), there must exist $i \in N'(v)$ and $j \in N'(u)$ such that none of $X_i^r, Y_i^r, X_j^r, Y_j^r$ is full and $e(X_j^0, Y_i^0) \geq 3\tau\ell^2$. Then since $e(O(X_j^r), Y_i^0) \geq (3\tau - \sigma)\ell^2$ there is $u' \in O(X_j^r)$ with $\deg(u', Y_i^0) \geq 2\tau\ell$. Thus the procedure distributes (u, v).

Conditions (ii) and (iii) hold by design: for (iii) note that a vertex is only reassigned from a cluster if another vertex is reassigned to that cluster. Condition (iv) follows immediately from Lemma 3.0.7. Finally, condition (i) is satisfied since for every $u \in T(U_i^s)$ and $v \in T(V_i^s)$ we have

$$\deg(u, O(V_i^s)) \geq (2\tau - \sigma)\ell \geq \tau\ell \quad \text{and} \quad \deg(v, O(U_i^s)) \geq (2\tau - \sigma)\ell \geq \tau\ell.$$

$\square$

Now we apply Lemma 4.7.4 to the partition $\{U_0, U_1, \ldots, U_q\} \cup \{V_0, V_1, \ldots, V_q\}$ with $\sigma = \sqrt{3\delta_1}$ and $\tau = d_0$, recalling that $\mathcal{P} = U_1V_1 \ldots, U_qV_q$ and $|U_0| = |V_0| \leq 3\delta_1 n$. After the exceptional vertices have been distributed we set $U_i^1 := X_i^s$ and $V_i^1 := Y_i^s$. Then $O(U_i^1) = O(X_i^s)$, etc. By Lemma 4.7.4, each $(O(U_i^1), O(V_i^1))$ is ϵ_2-regular with density at least δ_2 and $\ell \geq |O(U_i^1)| = |O(V_i^1)| \geq (1 - \sqrt{3\delta_1})\ell$. While (U_i^1, V_i^1) may not be ϵ_2-regular, the exceptional parts $T(U_i^1)$ and $T(V_i^1)$ satisfy:

$$\forall u \in T(U_i^1), \forall v \in T(V_i^1),$$
$$\deg(u, O(V_i^1)), \deg(v, O(U_i^1)) \geq d_0\ell > \sqrt{3\delta_1}\ell \geq |T(V_i^1)|, |T(U_i^1)|.$$

Our next goal is to find a small ladder in each pair (U_i, V_i) which will contain all of the exceptional vertices $T(U_i^1)$ and $T(V_i^1)$. Precisely, we will prove the following.

Claim 4.7.5. *For each $i \in [r]$ there exists a ladder $L^i \subseteq U_i^1 \cup V_i^1$ such that:*

(i) $T(U_i^1) \cup T(V_i^1) \subseteq V(L^i)$.

(ii) $|V(L^i)| \leq 16\sqrt{3\delta_1}\ell$.

(iii) *Each $w \in \mathrm{ext}(L^i)$ satisfies* $\deg(w, (O(V_i^1) \cup O(U_i)^1) \setminus L^i) \geq \frac{1}{2}\delta_2\ell$.

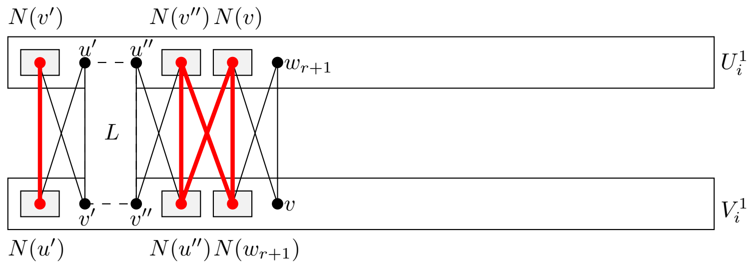


Figure 4.4: Proof of Claim 4.7.5

Proof. Let $w_1, w_2, \ldots, w_s$ be an ordering of $T(U_i^1) \cup T(V_i^1)$. Then $s \leq 2\sqrt{3\delta_1}\ell \leq \frac{1}{16}d_0\ell$. Suppose that we have constructed a ladder $L \subseteq U_i^1 \cup V_i^1$ on $8r$ vertices $(1 \leq r < s)$ that contains exactly the first r vertices of $T(U_i^1) \cup T(V_i^1)$, satisfies (iii), and has first rung $u'v'$ and last rung $u''v''$. Without loss of generality, assume that $w_{r+1} \in T(U_i^1)$.

We will first show how to extend L to L' by attaching a 3-ladder $aba'b'w_{r+1}v$, with $a, a' \in O(U_i^1) \setminus L$ and $b, b', v \in O(V_i^1) \setminus L$, to the end of L so that w_{r+1} and v satisfy (iii). By Lemma 3.0.6, all but at most $\epsilon_2\ell$, vertices $v \in O(V_i^1)$ satisfy $\deg(v, O(V_i^1) \setminus V(L)) \geq \frac{1}{2}\delta_2\ell + 4$. Choose such a vertex $v \in N(w_{r+1}) \setminus V(L)$. Each $x \in \{u'', v'', w_{r+1}, v\}$ has at least $\frac{1}{2}\delta_2\ell$ neighbors in $(O(V_i^1) \cup O(U_i^1)) \setminus L$. So by Proposition 3.0.8 we can find vertices $a, b, a', b' \in (O(V_i^1) \cup O(U_i^1)) \setminus L$ such that $a \sim v'', b \sim u'', a' \sim v, b' \sim w_{r+1}$ and $G[\{a, b, a', b'\}] = C_4$, which completes the extension.

In extending L to L' we may have violated condition (iii) for the first rung $u'v'$ by using up some of its neighbors. So now, in a similar way, we choose

$a'' \in O(U_i^1) \setminus L'$ and $b'' \in O(V_i^1) \setminus L'$ such that $u' \sim b'' \sim a'' \sim v'$ and $\deg(a'', O(V_i^1) \setminus L')$, $\deg(b'', O(U_i^1) \setminus L') \geq \frac{1}{2}\delta_2 \ell + 1$. We then add $a''b''$ to L' as a first rung to obtain L'' satisfying (iii). Continuing in this fashion we obtain the desired ladder L^i satisfying (i-iii). □

For each $i \in [q]$, set $U_i^2 := U_i^1 \setminus L^i$ and $V_i^2 := V_i^1 \setminus L^i$. Then

$$\ell \geq |U_i^2| = |V_i^2| \geq \left(1 - 9\sqrt{3\delta_1}\right) \ell \geq (1 - d_0)\ell.$$

Move one vertex from U_1^2 to U_q^2. By Lemma 3.0.7 each of the pairs (U_i^2, V_i^2) and (U_{i+1}^2, V_i^2) are ϵ_3-regular with density at least δ_3.

Our next goal is to reassign some vertices so that each of the pairs (U_i^2, V_i^2) is (ϵ, δ)-super-regular. Let $Q_i \subseteq U_i^2$ and $R_i \subseteq V_i^2$ be sets of size $\epsilon_3 |V_i^2|$ such that every vertex $w \in U_i^2 \cup V_i^2$ with $\deg(w, U_i^2 \cup V_i^2) \leq (\delta_3 - \epsilon_3)|V_i^2|$ is contained in $Q_i \cup R_i$. This is possible by Lemma 3.0.6.

Move the vertices in $Q_i \cup R_i$ to new exceptional sets to obtain the partition

$$U_0^3 := \bigcup_{i=1}^{q} Q_i, \quad V_0^3 := \bigcup_{i=1}^{q} R_i, \quad U_i^3 := U_i^2 \setminus Q_i, \quad \text{and} \quad V_i^3 := V_i^2 \setminus R_i.$$

Then $|U_0^3| = |V_0^3| \leq \epsilon_3 n$. By Lemma 3.0.7 the pairs (U_i^3, V_i^3) are (ϵ_4, δ_4)-super-regular for $i \in [q]$. The pairs (U_{j+1}^3, V_j^3) may not be super-regular, but they are ϵ_4-regular with density at least δ_4.

Applying Lemma 4.7.4 to the partition $\{U_0^3, U_1^3, \ldots, U_q^3\} \cup \{V_0^3, V_1^3, \ldots, V_q^3\}$ with $\sigma = \sqrt{\epsilon_3}$ and $\tau = \delta_4$, we get a new partition $\{U_1^4, \ldots, U_q^4\} \cup \{V_1^4, \ldots, V_q^4\}$. Note that the pairs $(O(U_i^4), O(V_i^4))$ are $(\frac{1}{2}\epsilon_5, 2\delta)$-super-regular and thus

$$(1 - d_0)\ell \leq (1 - 9\sqrt{3\delta_1} - \epsilon_3 - \sqrt{\epsilon_3})\ell \leq |O(U_i^4)|, |O(V_i^4)| \leq \ell \quad \text{and}$$

$$|T(U_i^4)|, |T(V_i^4)| \leq \sqrt{\epsilon_3}\ell \leq \frac{1}{2}\sqrt{\epsilon_5}\ell \leq \sqrt{\epsilon_5}|O(U_i^4)|, \sqrt{\epsilon_5}|O(V_i^4)|.$$

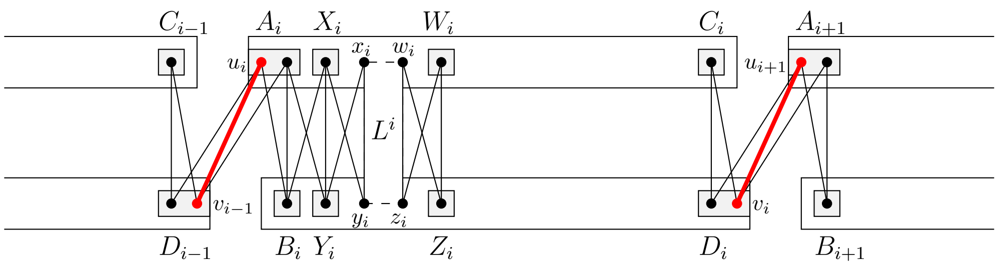


Figure 4.5: Applying Lemma 3.0.11

So by Lemma 3.0.9, since $\deg(u', O(V_i^4)) \geq \delta_4 |O(V_i^4)|$ and $\deg(v', O(U_i^4)) \geq \delta_4 |O(U_i^4)|$, for all $u' \in T(U_i^4)$ and $v' \in T(V_i^4)$, the pairs (U_i^4, V_i^4) are (ϵ, δ)-super-regular (with room to spare). Similarly, each pair (U_{j+1}^4, V_j^4) is ϵ-regular with density at least δ. Also $|U_i^4| = |V_i^4|$, except that $|V_1^4| = |U_1^4| + 1, |U_q^4| = |V_q^4| + 1$.

Using Lemma 3.0.6, for $i \in [q-1]$, choose $v_i \in V_i^4$ such that $|A_{i+1}| \geq \frac{1}{2}\delta\ell$, where $A_{i+1} := U_{i+1}^4 \cap N(v_i)$. Similarly, choose $u_{i+1} \in A_{i+1}$ such that $|D_i| \geq \frac{1}{2}\delta\ell$, where $D_i := V_i^4 \cap N(u_{i+1})$. Set $P := \{v_i, u_{i+1} : i \in [q-1]\}$, $U_i^5 := U_i^4 \setminus P$, and $V_i^5 := V_i^4 \setminus P$. Then (using the spared room) (U_i^5, V_i^5) is still an (ϵ, δ)-super-regular pair. Now set $B_{i+1} := V_i^5 \cap N(u_{i+1})$ and $C_i := U_i^5 \cap N(v_i)$. Let $x_i y_i$ be the first rung of L^i and let $w_i z_i$ be the last rung of L^i, where $x_i, w_i \in U$ and $y_i, z_i \in V$. Finally let $X_i = U_i^5 \cap N(y_i)$, $Y_i = V_i^5 \cap N(x_i)$, $W_i = U_i^5 \cap N(z_i)$, and $Z_i = V_i^5 \cap N(w_i)$. Note that each of X_i, Y_i, W_i, and Z_i has size at least $\frac{1}{2}\delta\ell = \varrho\ell$.

We now apply Lemma 3.0.11 to each pair (U_i^5, V_i^5) to find a spanning ladder M^i whose first rung is contained in $A_i \times B_i$, whose second rung is contained in $X_i \times Y_i$, whose third rung is contained in $W_i \times Z_i$, and whose last rung is contained in $C_i \times D_i$. This is possible since $\eta\ell \geq 4$. Clearly we can insert L^i between the second and third rungs of M^i to obtain a ladder $\mathcal{L}^i$ spanning $U_i^4 \cup V_i^4$. Finally, $\mathcal{L}^1 v_1 u_2 \mathcal{L}^2 \dots v_{r-1} u_r \mathcal{L}^r$ is a spanning ladder of G. $\square$

4.8 Proof of Amar's Conjecture

Theorem 4.1.8 follows immediately from Lemmas 4.4.1, 4.6.1, 4.7.1 with $N_0(k) = \max\{N_1(k), N_2(k)\}$.

Now we prove Theorem 4.1.9.

Proof. Let $N_0(1)$ be the value given when $k = 1$ in Theorem 4.1.8 and set $C := N_0(1)$. Suppose G is a balanced U, V-bigraph on $2n$ vertices with $\delta_U + \delta_V \geq n + C$. We may assume without loss of generality that $\delta_U = \delta(G) =: \delta$. We may assume $\delta < \frac{n}{200} + 1$, otherwise we would have a spanning ladder by Theorem 4.1.8 since the choice of C implies that $n \geq N_0(1)$.

Let $S = \{x \in U : \deg(x) \leq \frac{9n}{10}\}$ and $S' \subseteq S$ be a maximal subset such that $|N(S')| < 3|S'|$. Let $\bar{s} := |S| - |S'|$, then $G[(S \setminus S') \cup (V \setminus N(S'))]$ contains a set of $\bar{s}$ disjoint claws $M = \{a_r b_r c_r d_r : r \in [\bar{s}],\ a_r \in S \setminus S',$ $b_r, c_r, d_r \in V \setminus N(S')\}$. We have the following bound on the cardinality of S,

$$\begin{aligned} (n - \delta + C)n \leq |E(G)| &\leq \frac{9n}{10}|S| + n(n - |S|) \\ |S| &\leq 10\delta - 10C. \end{aligned} \tag{4.14}$$

Note that for all $v_1, v_2 \in V \cap V(M)$ we have

$$|(N(v_1) \cap N(v_2)) \cap (U \setminus S)| \geq 2(n - \delta + C) - n - |S| > \frac{47}{50}n \geq 2\bar{s}. \tag{4.15}$$

Thus by (4.15) there exists a set of 3-ladders

$$\Lambda(M) = \{x_r a_r y_r b_r c_r d_r : r \in [\bar{s}], a_r b_r c_r d_r \in M, x_r, y_r \in U \setminus S\}.$$

Note that $\text{ext}(L) \subseteq V(G) \setminus S$ for all $L \in \Lambda(M)$. Let $R = \bigcup_{L \in \Lambda(M)} V(L)$. For all $v' \in V \setminus N(S')$, we have $\deg(v') \geq n - \delta + C$, thus

$$|S'| \leq \delta - C. \tag{4.16}$$

Now we show that G contains a ladder that spans S'. Let $T = \{x \in U : \deg(x) < n - 29\delta\}$. Then

$$\begin{aligned}(n-\delta+C)n \le |E(G)| &< (n-29\delta)|T| + n(n-|T|)\\ |T| &< \frac{n}{29}.\end{aligned}$$

Let X' be any $(30\delta - |S'|)$-subset of $U \setminus (R \cup S \cup T)$ and $U' = S' \cup X'$. Similarly, let Y' be any $(30\delta - |N(S')|)$-subset of $V \setminus (N(S') \cup V(M))$ and $V' = N(S') \cup Y'$. Let $H := G[U' \cup V']$. Then every vertex in X' is non adjacent to at most 29δ vertices of V and so $\delta_{U'} := \delta_{U'}(H) \ge \delta$. Similarly, $\delta_{V'} := \delta_{V'}(H) \ge 29\delta + C$. Let $m = 30\delta$ and note that $\delta_{U'} + \delta_{V'} \ge m + C$, $\delta(H) \ge \frac{m}{30}$ and by the choice of C, $m \ge N_0(1)$. Thus H contains a spanning ladder $L = u_1v_1 \dots u_{30\delta}v_{30\delta}$ by Lemmas 4.6.1 and 4.7.1. Since $|N(S')| < 3|S'|$ we have $|S' \cup N(S')| < 4\delta$ by (4.16). Thus there exists rungs $u_iv_i, u_{i+1}v_{i+1} \in E(L)$ with $2 \le i \le 30\delta - 2$ such that $u_i, v_i, u_{i+1}, v_{i+1} \in V(H) \setminus (S' \cup N(S'))$. Let $L^1 = u_1v_1 \dots u_iv_i$ and $L^2 = u_{i+1}v_{i+1} \dots u_{30\delta}v_{30\delta}$. We will specify L^1 as the initial ladder and L^2 as the terminal ladder. Let $\Lambda := \Lambda(M) \cup \{L^1, L^2\}$ and let $I = I(\Lambda) = \bigcup_{L \in \Lambda} \mathring{L}$. Set $q' := 0$, $s' := \bar{s} + 2 = |\Lambda|$ and $t' := 30\delta + 3\bar{s}$. Note that for all $z \in V(G) \setminus I$ we have,

$$\deg(z) \ge \frac{9n}{10} \ge \frac{3n + 100\delta}{4} + 1 \ge \frac{3n + 3s' + t' + 4q'}{4} + 1.$$

So we may apply Lemma 4.2.3 to G to obtain a spanning ladder which starts with the first rung of L_1 and ends with the last rung of L_2.

□

Finally, we prove Theorem 4.1.10.

Proof. Let C be the constant from Theorem 4.1.9, let $N_0(1) < N_0(2) < \cdots < N_0(C-1)$ be the values given by Theorem 4.1.8, and let

$N_0 = N_0(C-1)$. Let G be a balanced U,V-bigraph on $2n$ vertices with $n \geq N_0$ which satisfies $\delta_U + \delta_V \geq n + \mathrm{comp}(H)$. By Theorem 4.1.8 and Theorem 4.1.9, we have $H \subseteq G$. $\square$

4.9 Conclusion

A proof of Conjecture 4.1.3 was announced at the end of 2009 by Gábor Kun. In light of this result, it would be interesting to study an analog of Conjecture 4.1.3 for bipartite graphs.

Problem 1. *Let k be a positive integer and let G and H be balanced bipartite graphs on $2n$ vertices with $\Delta(H) \leq k$. Determine the optimal value, $d(k)$, such that $\delta(G) \geq d(k)$ implies $H \subseteq G$.*

For $k = 1$, the answer is $d(1) = \frac{n}{2}$ as implied by Hall's theorem [22]. For $k = 2$, Conjecture 4.1.6 claims that $d(2) = \frac{n}{2} + 1$. As noted in the introduction, Conjecture 4.1.6 was solved by Czygrinow and Kierstead (for large n) in [13].

Chapter 5

TILING IN BIPARTITE GRAPHS: MINIMUM DEGREE

This chapter is joint work with Andrzej Czygrinow.

5.1 Introduction

If G is a graph on $n = sm$ vertices, H is a graph on s vertices and G contains m vertex disjoint copies of H, then we say G can be *tiled* with H. In this language, we state the seminal result of Hajnal and Szemerédi.

Theorem 5.1.1 (Hajnal-Szemerédi [21]). *Let G be a graph on $n = sm$ vertices. If $\delta(G) \geq (s-1)m$, then G can be tiled with K_s.*

For tiling with general H, results of Alon and Yuster [3] and Komlós, Sárközy, and Szemerédi [31] gave sufficient conditions on the minimum degree of a graph G such that G can be tiled with H. Specifically, in [31], it is shown that if G is a graph on n vertices with minimum degree at least $(1 - 1/\chi(H))\, n + K$ for a constant K that only depends on H, then G can be tiled with H. A more delicate minimum degree condition that involves the so-called critical chromatic number of H was conjectured by Komlós and solved by Shokoufandeh and Zhao [43]. Finally, Kühn and Osthus [35] determined exactly when the critical chromatic number or chromatic number is the appropriate parameter and thus settled the problem (for large graphs).

In this paper we study the tiling problem in bipartite graphs. Denote a bipartite graph G with partition sets U and V by $G[U, V]$. We say $G[U, V]$ is *balanced* if $|U| = |V|$. Zhao proved the following Hajnal-Szemerédi type result for bipartite graphs.

Theorem 5.1.2 (Zhao [51]). *For each $s \geq 2$, there exists m_0 such that the* *following holds for all $m \geq m_0$. If G is a balanced bipartite graph on $2n = 2ms$ vertices with*

$$\delta(G) \geq \begin{cases} \frac{n}{2} + s - 1 & \text{if } m \text{ is even} \\ \frac{n+3s}{2} - 2 & \text{if } m \text{ is odd,} \end{cases}$$

then G can be tiled with $K_{s,s}$.

Zhao proved that this minimum degree condition was tight.

Proposition 5.1.3 (Zhao [51])**.** *Let $s \geq 2$, and $n = ms \geq 64s^2$. There exists a balanced bipartite graph, G, on $2n$ vertices with*

$$\delta(G) = \begin{cases} \frac{n}{2} + s - 2 & \text{if } m \text{ is even} \\ \frac{n+3s}{2} - 3 & \text{if } m \text{ is odd} \end{cases}$$

such that G cannot be tiled with $K_{s,s}$.

Hladký and Schacht extended Zhao's result as follows.

Theorem 5.1.4 (Hladký-Schacht [23])**.** *Let $1 \leq s < t$ be fixed integers. There exists m_0 such that the following holds for all $m \geq m_0$. If G is a balanced bipartite graph on $2n = 2m(s+t)$ vertices with*

$$\delta(G) \geq \begin{cases} \frac{n}{2} + s - 1 & \text{if } m \text{ is even} \\ \frac{n+t+s}{2} - 1 & \text{if } m \text{ is odd,} \end{cases}$$

then G can be tiled with $K_{s,t}$.

They proved that this minimum degree condition was tight in all cases except when m is odd and $t > 2s + 1$. Note that since we are dealing with balanced bipartite graphs, in any tiling of $G[U, V]$ with $K_{s,t}$ there must be an equal number of copies of $K_{s,t}$ with s vertices in U as copies of $K_{s,t}$ with t vertices in U. This explains why the authors [23] suppose $2n = 2m(s + t)$ instead of $2n = m(s + t)$.

Proposition 5.1.5 (Hladký-Schacht [23])**.** *Let $1 \leq s < t$ be fixed integers. There exists m_0 such that the following holds for all $m \geq m_0$. There exists a balanced bipartite graph, G, on $2n = 2m(s+t)$ vertices with*

$$\delta(G) = \begin{cases} \frac{n}{2} + s - 2 & \text{if } m \text{ is even} \\ \frac{n+t+s}{2} - 2 & \text{if } m \text{ is odd and } t \leq 2s+1 \end{cases}$$

such that G cannot be tiled with $K_{s,t}$.

Our objective is to give the tight minimum degree condition in the final remaining case, when m is odd and $t > 2s + 1$. We will do this in two parts. First in Section 5.2.3 we prove that when m is odd and $t \geq 2s + 1$, the following minimum degree condition is sufficient.

Theorem 5.1.6. *Let $1 \leq s < t$ be fixed integers with $2s + 1 \leq t$. There exists m_0 such that the following holds for all odd m with $m \geq m_0$. If G is a balanced bipartite graph on $2n = 2m(s+t)$ vertices with*

$$\delta(G) \geq \frac{n + 3s}{2} - 1,$$

then G can be tiled with $K_{s,t}$.

Then in Section 5.3 we prove that the minimum degree condition in Theorem 5.1.6 is tight.

Proposition 5.1.7. *Let $1 \leq s < t$ be fixed integers with $2s + 1 \leq t$. There exists m_0 such that the following holds for all odd m with $m \geq m_0$. There exists a balanced bipartite graph, G, on $2n = 2m(s+t)$ vertices with*

$$\delta(G) = \begin{cases} \frac{n+3s}{2} - \frac{3}{2} & \text{if } t \text{ is odd} \\ \frac{n+3s}{2} - 2 & \text{if } t \text{ is even} \end{cases}$$

such that G cannot be tiled with $K_{s,t}$.

Let $m = 2k + 1$ for some $k \in \mathbb{N}$ and let $n = m(s + t)$. We note that when $t = 2s + 1$, $\frac{n+3s}{2} - 1 = (k + 1)(s + t) - \frac{3}{2}$ and $\frac{n+t+s}{2} - 1 = (k + 1)(s + t) - 1$. So the value for the lower bound in Theorem 5.1.6 is smaller than the value for the lower bound in Theorem 5.1.4 when $t = 2s + 1$, but since $\delta(G)$ only takes integer values the minimum degree condition in Theorem 5.1.6 is not an improvement until $t > 2s + 1$.

5.2 Proof of Theorem 5.1.6

For disjoint sets $A, B \subseteq V(G)$, we define $e(A, B)$ to be the number of edges with one end in A and the other end in B and for $v \in V(G) \setminus A$ we write $\deg(v, A)$ instead of $e(\{v\}, A)$. Also, $d(A, B) = \frac{e(A,B)}{|A||B|}$, $\delta(A, B) = \min\{\deg(v, B) : v \in A\}$ and $\Delta(A, B) = \max\{\deg(v, B) : v \in A\}$. An *$h$-star from A to B*, is a copy of $K_{1,h}$ with the vertex of degree h, *the center*, in A and the vertices of degree 1, *the leaves*, in B.

The following theorem appears in [51].

Theorem 5.2.1 (Zhao [51])**.** *For every $\alpha > 0$ and every positive integer r, there exist $\beta > 0$ and positive integer m_1 such that the following holds for all $n = mr$ with $m \geq m_1$. Given a bipartite graph $G[U, V]$ with $|U| = |V| = n$, if $\delta(G) \geq (\frac{1}{2} - \beta)n$, then either G can be tiled with $K_{r,r}$, or there exist*

$$U_1' \subseteq U,\ V_2' \subseteq V, \quad \textit{such that } |U_1'| = |V_2'| = \lfloor n/2 \rfloor,\ d(U_1', V_2') \leq \alpha. \tag{5.1}$$

If a balanced bipartite graph $G[U, V]$ on $2n$ vertices with n divisible by r satisfies (5.1), we say G is *extremal* with parameter α. In this case we set $U_2' := U \setminus U_1'$ and $V_1' := V \setminus V_2'$.

If we replace r with $s + t$ in Theorem 5.2.1, we see that either G can be tiled with $K_{s+t,s+t}$ or else we are in the extremal case. If it is the case that G

can be tiled with $K_{s+t,s+t}$, we split each copy of $K_{s+t,s+t}$ into two copies of $K_{s,t}$ to give the desired tiling. So we must only deal with the extremal case.

5.2.1 Pre-processing

Claim 5.2.2. *Let $0 < \alpha \ll 1$, $r \in \mathbb{N}$ and let $m_1 \in \mathbb{N}$ be given by Theorem 5.2.1. Let $m \geq m_1$ and suppose that $G[U,V]$ is a balanced bipartite graph on $2n = 2mr$ vertices such that $\delta(G) = \frac{n}{2} + C$, where $0 \leq C \leq 3r/2$. Suppose further that the deletion of any edge of G will cause the resulting graph to have minimum degree less than $\frac{n}{2} + C$. If G is extremal with parameter α, then $d(U_2', V_1') \leq 5\sqrt{\alpha}$.*

Proof. Let $\gamma := 5\sqrt{\alpha}$ and suppose $d(U_2', V_1') > \gamma$. Let
$X' = \{u \in U_2' : \deg(u, V_2') < (1 - \sqrt{\alpha})\frac{n}{2}\}$,
$Y' = \{v \in V_1' : \deg(v, U_1') < (1 - \sqrt{\alpha})\frac{n}{2}\}$. Since $e(U_1', V_2') \leq \alpha\frac{n^2}{4}$ and $e(U_1', V) \geq |U_1'|\frac{n}{2}$, we have $e(U_1', V_1') \geq |U_1'|\frac{n}{2} - \alpha\frac{n^2}{4}$. Thus we can bound the non-edges between U_1' and V_1',

$$\sqrt{\alpha}\frac{n}{2}|Y'| \leq \bar{e}(U_1', V_1') \leq \alpha\frac{n^2}{4},$$

which gives $|Y'| \leq \sqrt{\alpha}\frac{n}{2}$. Similarly we have $|X'| \leq \sqrt{\alpha}\frac{n}{2}$. Let $U_2'' = U_2' \setminus X'$ and $V_1'' = V_1' \setminus Y'$. Since $d(U_2', V_1') > \gamma$, we have

$$e(U_2'', V_1'') \geq \gamma\frac{n^2}{4} - 2\sqrt{\alpha}\frac{n^2}{4} = 3\sqrt{\alpha}\frac{n^2}{4}. \tag{5.2}$$

Let $X'' = \{u \in U_2'' : \deg(u, V_1'') \geq \sqrt{\alpha}\frac{n}{2} + C + 1\}$ and
$Y'' = \{v \in V_1'' : \deg(v, U_2'') \geq \sqrt{\alpha}\frac{n}{2} + C + 1\}$. If there is an edge $uv \in E(X'', Y'')$, then $\deg(u), \deg(y) \geq \frac{n}{2} + C + 1$ which contradicts the edge minimality of G, so suppose $e(X'', Y'') = 0$. Finally, by (5.2) we have

$$3\sqrt{\alpha}\frac{n^2}{4} \leq e(U_2'', V_1'') \leq e(X'', Y'') + e(U_2'' \setminus X'', V_1'') + e(V_1'' \setminus Y'', U_2'') \leq 2(\sqrt{\alpha}\frac{n}{2} + C)\frac{n}{2},$$

which is a contradiction, since n is sufficiently large.

□

Let $1 \le s < t$ be integers so that $2s+1 \le t$, and let $0 < \alpha \ll 1$ (setting $\alpha := \left(\frac{1}{32t(s+t)}\right)^3$ is small enough). Let $G[U,V]$ be a balanced bipartite graph on $2n = 2m(s+t)$ vertices, where $m = 2k+1$ and k is a sufficiently large integer with respect to $(\frac{\alpha}{5})^2$. Suppose that G is extremal with parameter $(\frac{\alpha}{5})^2$ and edge-minimal with respect to the condition $\delta(G) \ge \frac{n+3s}{2} - 1$. By Claim 5.2.2 we have $d(U'_i, V'_{3-i}) \le \alpha$ for $i = 1,2$. Then for $i = 1,2$, we define

$$U_i = \{u \in U : \deg(u, V'_{3-i}) < \alpha^{\frac{1}{3}}\frac{n}{2}\},\ V_i = \{v \in V : \deg(v, U'_{3-i}) < \alpha^{\frac{1}{3}}\frac{n}{2}\},$$

$$U_0 = U - U_1 - U_2, \text{ and } V_0 = V - V_1 - V_2.$$

As a consequence of these definitions, we have the following.

Claim 5.2.3. *For $i = 1,2$*

(i) $(1-\alpha^{2/3})\frac{n}{2} \le |U_i|, |V_i| \le (1+\alpha^{2/3})\frac{n}{2}$, (ii) $|U_0|, |V_0| \le \alpha^{2/3}n$,

(iii) $(1-2\alpha^{1/3})\frac{n}{2} < \delta(U_i, V_i), \delta(V_i, U_i)$, (iv) $(\alpha^{1/3}-\alpha^{2/3})\frac{n}{2} \le \delta(U_0, V_i), \delta(V_0, U_i)$,

(v) $\Delta(U_i, V_{3-i}), \Delta(V_{3-i}, U_i) \le \alpha^{1/3}n$

Proof. A proof of (i)-(iv) can be found in [51] and was also used in [23]. So we prove (v) here.

Let $i \in \{1,2\}$ and note that

$$|U'_i \setminus U_i|\alpha^{1/3}\frac{n}{2} \le e(U'_i \setminus U_i, V'_{3-i}) \le e(U'_i, V'_{3-i}) \le \alpha\frac{n^2}{4} \tag{5.3}$$

and

$$|V'_i \setminus V_i|\alpha^{1/3}\frac{n}{2} \le e(V'_i \setminus V_i, U'_{3-i}) \le e(V'_i, U'_{3-i}) \le \alpha\frac{n^2}{4}. \tag{5.4}$$

Then (5.3) and (5.4) imply

$$|U'_i \setminus U_i|, |V'_i \setminus V_i| \le \alpha^{2/3}\frac{n}{2}, \tag{5.5}$$

which gives

$\Delta(U_i, V_{3-i}) \le \Delta(U_i, V'_{3-i}) + |V_{3-i} \setminus V'_{3-i}| \le \Delta(U_i, V'_{3-i}) + |V'_i \setminus V_i| \le \alpha^{1/3}n$ and

$\Delta(V_i, U_{3-i}) \le \Delta(V_i, U'_{3-i}) + |U_{3-i} \setminus U'_{3-i}| \le \Delta(V_i, U'_{3-i}) + |U'_i \setminus U_i| \le \alpha^{1/3}n.$

□

We need to define some new sets which were not specified in [51].

Definition 5.2.4. *For $i = 1, 2$, let*

$$\tilde{U_i} = \{u \in U_i : \deg(u, V_{3-i}) \geq s\}, \ \tilde{V_i} = \{v \in V_i : \deg(v, U_{3-i}) \geq s\},$$
$$\hat{U_i} = U_i \setminus \tilde{U_i}, \text{ and } \hat{V_i} = V_i \setminus \tilde{V_i}.$$

Note that the following inequalities are satisfied:

$$\delta(\hat{U_1}, V_0) + \delta(\hat{U_2}, V_0) \geq n + 3s - 2 - (|V_1| + s - 1) - (|V_2| + s - 1) = |V_0| + s \quad \text{and} \tag{5.6}$$

$$\delta(\hat{V_1}, U_0) + \delta(\hat{V_2}, U_0) \geq n + 3s - 2 - (|U_1| + s - 1) - (|U_2| + s - 1) = |U_0| + s. \tag{5.7}$$

5.2.2 Preliminary Claims

The following useful lemma appears in [51].

Lemma 5.2.5 (Zhao [51], Fact 5.3)**.** *Let $F[A, B]$ be a bipartite graph with $\delta := \delta(A, B)$ and $\Delta := \Delta(B, A)$ Then F contains f_h vertex disjoint h-stars from A to B, and g_h vertex disjoint h-stars from B to A (the stars from A to B and those from B to A need not be disjoint), where*

$$f_h \geq \frac{(\delta - h + 1)|A|}{h\Delta + \delta - h + 1}, \quad g_h \geq \frac{\delta|A| - (h-1)|B|}{\Delta + h\delta - h + 1}.$$

We now prove three claims that we will need in the main proof.

Claim 5.2.6. *Let $i \in \{1, 2\}$ and $\{A, B\} = \{U_i, V_{3-i}\}$. Let $0 \leq c \leq \alpha^{1/3} n$, $B_0 \subseteq B$ and $A_0 = \{v \in A : \deg(v, B_0) \geq s + c\}$. If $|A_0| \geq \frac{n}{4}$ then there is a set $\mathcal{S}_A$ of at least $\frac{c+1}{8s\alpha^{1/3}}$ vertex disjoint s-stars from A_0 to B_0.*

Proof. Let $\mathcal{S}_A$ be a maximum set of vertex disjoint s-stars from A_0 to B_0 and let $f_s = |\mathcal{S}_A|$. We apply Lemma 5.2.5 to the graph $G[A_0, B_0]$. Recall, by Claim 5.2.3, that $\Delta(B, A) \leq \alpha^{1/3}n$. Then

$$f_s \geq \frac{(c+1)|A_0|}{s\alpha^{1/3}n + c + 1} \geq \frac{(c+1)\frac{n}{4}}{2s\alpha^{1/3}n} = \frac{c+1}{8s\alpha^{1/3}}.$$

□

Note that since $n = (2k+1)(s+t)$, we can write $\delta(G) \geq \frac{n+3s}{2} - 1 = k(s+t) + 2s + \frac{t}{2} - 1$.

Claim 5.2.7. *Let $i \in \{1, 2\}$ and $\{A, B\} = \{U_i, V_{3-i}\}$. Let $|A| = k(s+t) + z$ and $|B| = k(s+t) + y$. Suppose $y \geq z$ and $y \geq \frac{t+1}{2}$. Then there is a set $\mathcal{S}_B$ of y vertex disjoint s-stars with centers $C_B \subseteq B$ and leaves $L_A \subseteq A$. Furthermore if $z \geq 1$, then there is a set $\mathcal{S}_A$ of z vertex disjoint s-stars from $A \setminus L_A$ to $B \setminus C_B$.*

Proof. Let $\beta := 32s\alpha^{1/3}$ and recall that by the choice of α we have $\frac{1}{t} \gg \beta \gg 2\alpha^{1/3}$. We show that the desired set $\mathcal{S}_B$ exists by applying Lemma 5.2.5 to the graph $G[A, B]$. We have $\delta(A, B) \geq k(s+t) + 2s + \frac{t}{2} - 1 - (n - |B|) = y + s - \frac{t}{2} - 1$ and $\Delta(B, A) \leq \alpha^{1/3}n$ by Claim 5.2.3. Let $g_s = |\mathcal{S}_B|$, then

$$\begin{aligned} g_s &\geq \frac{(y - \frac{t}{2} + s - 1)(k(s+t) + z) - (s-1)(k(s+t) + z + y - z)}{\alpha^{1/3}n + s(y - \frac{t}{2} + s - 1) - s + 1} \\ &= \frac{(y - \frac{t}{2})(k(s+t) + z) - (s-1)(y - z)}{\alpha^{1/3}n + s(y - \frac{t}{2}) + s^2 - 2s + 1} \\ &\geq \frac{(y - \frac{t}{2})\frac{n}{3}}{2\alpha^{1/3}n} \qquad \text{(since } y \leq \alpha^{2/3}\frac{n}{2} \text{ and } -\alpha^{2/3}\frac{n}{2} \leq z, \text{ by Claim 5.2.3)} \\ &\geq y \qquad \text{(since } y \geq \frac{t+1}{2} \text{ and } \alpha \ll 1). \end{aligned}$$

Thus the desired set $\mathcal{S}_B$ exists.

Suppose $z \geq 1$. Let $c := \frac{1}{2}y$ if $y \geq 1/\beta$, and let $c := 0$ if $y < 1/\beta$. Let $B_0 = B \setminus C_B$ and $A_0 = \{v \in A \setminus L_A | \deg(v, B_0) \geq s + c\}$ and $\bar{A} = (A \setminus L_A) \setminus A_0$.

Suppose that $|\bar{A}| \geq \frac{n}{16}$. Then there exists $u \in C_B$ such that if $y < 1/\beta$,

$$\deg(u, A) \geq \frac{e(\bar{A}, C_B)}{|C_B|} \geq \frac{\left(y - \frac{t}{2} + s - 1 - (s-1)\right)\frac{n}{16}}{y} = \frac{\left(y - \frac{t}{2}\right)\frac{n}{16}}{y} > \frac{\beta n}{32} \geq \alpha^{1/3} n$$

and if $y \geq 1/\beta$,

$$\begin{aligned} \deg(u, A) &\geq \frac{e(\bar{A}, C_B)}{|C_B|} \\ &> \frac{\left(y - \frac{t}{2} + s - 1 - (s + \frac{1}{2}y)\right)\frac{n}{16}}{y} = \frac{\left(\frac{y}{2} - \frac{t}{2} - 1\right)\frac{n}{16}}{y} > \frac{n}{64} \geq \alpha^{1/3} n, \end{aligned}$$

each contradicting Claim 5.2.3. So $|\bar{A}| < \frac{n}{16}$ and thus $|A_0| \geq |A| - |L_A| - \frac{n}{16} \geq k(s+t) - s\alpha^{2/3}\frac{n}{2} - \frac{n}{16} \geq \frac{n}{4}$. Now let $\mathcal{S}_A$ be a maximum set of disjoint s-stars from A_0 to B_0 and let $f_s = |\mathcal{S}_A|$. By Lemma 5.2.6 we have $f_s \geq \frac{c+1}{8s\alpha^{1/3}}$. Recall that $1 \leq z \leq y$. If $y \geq 1/\beta$, then $f_s \geq \frac{y}{16s\alpha^{1/3}} \geq z$ and if $y < 1/\beta$, then $f_s \geq \frac{1}{8s\alpha^{1/3}} \geq \frac{1}{\beta} \geq z$. So the desired set $\mathcal{S}_A$ exists.

□

Claim 5.2.8. *Suppose $|U_0|, |V_0| \geq s$. If $|\hat{U}_1| \geq \frac{n}{8}$ and $|\hat{U}_2| \geq \frac{n}{8}$ (see Definition 5.2.4), then there is a $K_{s,t} =: K^1$ with s vertices in V_0, $\lceil t/2 \rceil$ vertices in U_1 and $\lfloor t/2 \rfloor$ vertices in U_2. Likewise, if $|\hat{V}_1| \geq \frac{n}{8}$ and $|\hat{V}_2| \geq \frac{n}{8}$ then there is a $K_{s,t} =: K^2$ with s vertices in U_0, $\lceil t/2 \rceil$ vertices in V_1 and $\lfloor t/2 \rfloor$ vertices in V_2.*

Proof. Without loss of generality we will only prove the first statement. Let

$$\ell := s\binom{|U_2|}{\lfloor t/2 \rfloor} \Big/ \binom{\lceil(\alpha^{1/3} - \alpha^{2/3})n/2\rceil}{\lfloor t/2 \rfloor}$$

and recall that $|U_1|, |U_2| \leq (1 + \alpha^{2/3})\frac{n}{2}$ by Claim 5.2.3. Thus we have

$$\begin{aligned} \ell \leq s\left(\frac{|U_2|}{(\alpha^{1/3} - \alpha^{2/3})\frac{n}{2} - \lfloor t/2 \rfloor}\right)^{\lfloor t/2 \rfloor} &\leq s\left(\frac{(1 + \alpha^{2/3})\frac{n}{2}}{(\alpha^{1/3} - \alpha^{2/3})\frac{n}{3}}\right)^{\lfloor t/2 \rfloor} \\ &\leq s\left(\frac{3(1 + \alpha^{2/3})}{2(\alpha^{1/3} - \alpha^{2/3})}\right)^{\lfloor t/2 \rfloor}. \end{aligned}$$

Case 1. $|V_0| \geq \ell\binom{|U_1|}{\lceil t/2 \rceil} / \binom{\lceil(\alpha^{1/3} - \alpha^{2/3})n/2\rceil}{\lceil t/2 \rceil}$. Recall that $\delta(V_0, U_i) \geq (\alpha^{1/3} - \alpha^{2/3})n/2$ for $i = 1, 2$ by Claim 5.2.3 and suppose that there is no $K_{\lceil t/2\rceil,\ell}$ with $\lceil t/2\rceil$ vertices in U_1 and ℓ vertices in V_0. We count the $\lceil t/2\rceil$-stars from V_0 to U_1 in two ways which gives

$$|V_0|\binom{\lceil(\alpha^{1/3}-\alpha^{2/3})n/2\rceil}{\lceil t/2\rceil} < \ell\binom{|U_1|}{\lceil t/2\rceil}$$

contradicting the lower bound for $|V_0|$. Consequently there is a complete bipartite graph $K' = K_{\lceil t/2\rceil,\ell}$ with $\lceil t/2\rceil$ vertices in U_1 and ℓ vertices in V_0. If there is no $K_{\lfloor t/2\rfloor,s}$ with s vertices in $V(K')\cap V_0$ and $\lfloor t/2\rfloor$ vertices in U_2, then a similar counting argument gives

$$\ell\binom{\lceil(\alpha^{1/3}-\alpha^{2/3})n/2\rceil}{\lfloor t/2\rfloor} < s\binom{|U_2|}{\lfloor t/2\rfloor}$$

contradicting the definition of ℓ.

Case 2. $|V_0| < \ell\binom{|U_1|}{\lceil t/2\rceil}/\binom{\lceil(\alpha^{1/3}-\alpha^{2/3})n/2\rceil}{\lceil t/2\rceil}$. By (4.2.2), we have

$$|V_0| < \ell\left(\frac{3(1+\alpha^{2/3})}{2(\alpha^{1/3}-\alpha^{2/3})}\right)^{\lceil t/2\rceil} \leq s\left(\frac{3(1+\alpha^{2/3})}{2(\alpha^{1/3}-\alpha^{2/3})}\right)^{t}.$$

Let $p := \delta(\hat{U}_1, V_0)$, and note that $p \geq s$ by (5.6). We claim that there is a complete bipartite graph $K' := K_{\lceil t/2\rceil,p}$ with $\lceil t/2\rceil$ vertices in $\hat{U}_1$ and p vertices in V_0. Let c be the number of p-stars with centers in $\hat{U}_1$ and leaves in V_0. We have $c \geq |\hat{U}_1| \geq \frac{n}{8}$ and if no p-subset of V_0 is in $\lceil t/2\rceil$ of such stars, i.e. K' does not exist, we have $c \leq (\lceil t/2\rceil - 1)\binom{|V_0|}{p}$ which contradicts the fact that $|V_0|$ is $O(1)$ and n is sufficiently large (with respect to α, t, and consequently $|V_0|$). From (5.6) we have $\delta(\hat{U}_2, V_0) \geq |V_0| - p + s$, so every vertex $u \in \hat{U}_2$ has at least s neighbors in $V(K')\cap V_0$. Repeating the argument above by counting s-stars with centers in $\hat{U}_2$ and leaves in $V(K')\cap V_0$ gives $K'' := K_{s,\lfloor t/2\rfloor}$. Now choose $K^1 \subseteq K' \cup K''$ having the property that $|V_0 \cap V(K^1)| = s$, $|U_1 \cap V(K^1)| = \lceil t/2\rceil$, and $|U_2 \cap V(K^1)| = \lfloor t/2\rfloor$ as desired. $\square$

5.2.3 Extremal Case

Recall that $t \geq 2s+1$, $n = (2k+1)(s+t)$ for some sufficiently large $k \in \mathbb{N}$, and $\delta(G) \geq \frac{n+3s}{2} - 1 = k(s+t) + 2s + \frac{t}{2} - 1$. We start with the partition given in

Section 5.2.1 and we call U_0 and V_0 the *exceptional* sets. Let $i \in \{1, 2\}$. We will attempt to update the partition by moving a constant number (depending only on t) of *special* vertices between U_1 and U_2, denote them by X, and *special* vertices between V_1 and V_2, denote them by Y, as well as partitioning the exceptional sets as $U_0 = U_0^1 \cup U_0^2$ and $V_0 = V_0^1 \cup V_0^2$. Let U_1^*, U_2^*, V_1^* and V_2^* be the resulting sets after moving the special vertices. Our goal is to obtain two graphs, $G_1 := G[U_1^* \cup U_0^1, V_1^* \cup V_0^1]$ and $G_2 := [U_2^* \cup U_0^2, V_2^* \cup V_0^2]$ so that G_1 satisfies

$$|U_1^* \cup U_0^1| = \ell_1(s+t) + as + bt, |V_1^* \cup V_0^1| = \ell_1(s+t) + bs + at$$

and G_2 satisfies

$$|U_2^* \cup U_0^2| = \ell_2(s+t) + bs + at, |V_2^* \cup V_0^2| = \ell_2(s+t) + as + bt,$$

for some nonnegative integers a, b, ℓ_1, ℓ_2. We tile G_1 as follows. We find a copies of $K_{s,t}$, each with t vertices in U_1^*, so that each special vertex in $X \cap U_1^*$ is in a unique copy (some copies may not contain any special vertex). Also, we find b copies of $K_{s,t}$, each with t vertices in V_1^* so that each special vertex in $Y \cap V_1^*$ is in a unique copy (some copies may not contain any special vertex). Note that we only move vertices which will make this step possible. Deleting these $a + b$ copies of $K_{s,t}$ from G_1 gives us a balanced bipartite graph on $2\ell_1(s+t)$ vertices. As noted in [51] and [23], this graph can easily be tiled: By Claim 5.2.3 there are at most $\alpha^{2/3}\frac{n}{2}$ exceptional vertices in U_0^1 (resp. V_0^1), each with degree at least $(\alpha^{1/3} - \alpha^{2/3})\frac{n}{2}$ to V_1 (resp. U_1), so they may greedily be incorporated into unique copies of $K_{s+t,s+t}$. The remaining graph is still balanced, divisible by $s + t$, and almost complete, thus can be tiled.

So if we are able to split G into graphs G_1 and G_2 as detailed above, we will conclude that G can be tiled. However, if it is not possible to carry out this goal, then we will use an alternate method which is explained in Case 2.

Proof of Theorem 1.6. There are two main cases.

Case 1. $\max\{|U_1|, |U_2|, |V_1|, |V_2|\} \geq k(s+t) + \frac{t+1}{2}$. Without loss of generality, suppose $|U_1| = \max\{|U_1|, |U_2|, |V_1|, |V_2|\}$.

Case 1.1. $|V_2 \cup V_0| \geq k(s+t) + s$. We apply Claim 5.2.7 to $G[U_1, V_2]$ with $A = V_2$ and $B = U_1$ to obtain $|U_1| - (k(s+t)+s)$ vertex disjoint s-stars with centers $C_U \subseteq U_1$ and leaves in V_2 and a set of $\max\{0, |V_2| - (k(s+t)+s)\}$ vertex disjoint s-stars with centers $C_V \subseteq V_2$ and leaves in U_1. We move the vertices in C_U to U_2 and the vertices in C_V to V_1. If $|V_2| < k(s+t)+s$, we choose $V_0' \subseteq V_0$ so that $|(V_2 \cup V_0) \setminus V_0')| = k(s+t)+s$ otherwise we set $V_0' = \emptyset$. Then $G_1 := G[U_1 \setminus C_U, V_1 \cup C_V \cup V_0']$ satisfies

$$|U_1| - |C_U| = k(s+t) + s, |V_1| + |V_0'| + |C_V| = k(s+t) + t,$$

and $G_2 := G - G_1$ satisfies

$$|U_2 \cup U_0| + |C_U| = k(s+t) + t, |V_2| + |V_0 \setminus V_0'| - |C_V| = k(s+t) + s.$$

Thus G_1 and G_2 can be tiled, which completes the tiling of G.

Case 1.2. $|V_2 \cup V_0| < k(s+t) + s$.

This implies $|V_1| > k(s+t) + t$. So we apply Claim 5.2.7 to $G[V_1, U_2]$ with $A = U_2$ and $B = V_1$ to obtain a set of $|V_1| - k(s+t)$ vertex disjoint s-stars with centers $C_V \subseteq V_1$ and leaves in U_2. Likewise we apply Claim 5.2.7 to $G[U_1, V_2]$ with $A = V_2$ and $B = U_1$ to obtain a set of $|U_1| - k(s+t)$ vertex s-stars with centers $C_U \subseteq U_1$ and leaves in V_2. We move the vertices in C_U to U_2 and the vertices in C_V to V_2. Then $G_1 := G[U_1 \setminus C_U, V_1 \setminus C_V]$ satisfies

$$|U_1| - |C_U| = k(s+t), |V_1| - |C_V| = k(s+t)$$

and $G_2 := G - G_1$ satisfies

$$|U_2 \cup U_0| + |C_U| = (k+1)(s+t), |V_2 \cup V_0| + |C_V| = (k+1)(s+t).$$

Thus G_1 and G_2 can be tiled, which completes the tiling of G.

Case 2. $\max\{|U_1|, |U_2|, |V_1|, |V_2|\} \leq k(s+t) + \frac{t}{2}$. Note that this implies $|U_0|, |V_0| \geq s$.

Case 2.1. $\max\{|\tilde{U}_1|, |\tilde{U}_2|, |\tilde{V}_1|, |\tilde{V}_2|\} \geq \frac{n}{4}$ (see Definition 5.2.4). Without loss of generality we can assume $|\tilde{U}_1| = \max\{|\tilde{U}_1|, |\tilde{U}_2|, |\tilde{V}_1|, |\tilde{V}_2|\}$. Set $h := \lceil t/(2s) \rceil$. Since $|\tilde{U}_1| > \frac{n}{4}$ and $\frac{1}{8s\alpha^{1/3}} \geq (h-1)(s+t)$, we can apply Claim 5.2.6 to $G[\tilde{U}_1, V_2]$ with $c = 0$ to obtain a set of $(h-1)(s+t)$ vertex disjoint s-stars with centers $C_U \subseteq \tilde{U}_1$ and leaves in V_2. We first move the vertices in C_U from $\tilde{U}_1$ to U_2. Then since

$$\frac{t}{2} = s\frac{t}{2s} \leq sh \leq s\frac{t+2s-1}{2s} = \frac{t}{2} + s - \frac{1}{2},$$

we can choose sets $U_0' \subseteq U_0$ with $|U_0'| = k(s+t) + \lfloor t/2 \rfloor - |U_1| + sh - \lfloor t/2 \rfloor$ and $V_0' \subseteq V_0$ with $|V_0'| = k(s+t) + \lfloor t/2 \rfloor - |V_1| + s + \lceil t/2 \rceil - sh$ so that $G_1 := G[(U_1 \cup U_0') \setminus C_U, V_1 \cup V_0']$ satisfies

$$|U_1| + |U_0'| - |C_U| = (k-h+1)(s+t) + hs, |V_1| + |V_0'| = (k-h+1)(s+t) + ht,$$

and $G_2 := G - G_1$ satisfies

$$|U_2| + |U_0 \setminus U_0'| + |C_U| = k(s+t) + ht, |V_2| + |V_0 \setminus V_0'| = k(s+t) + hs.$$

Thus G_1 and G_2 can be tiled, which completes the tiling of G.

Case 2.2. $\max\{|\tilde{U}_1|, |\tilde{U}_2|, |\tilde{V}_1|, |\tilde{V}_2|\} < \frac{n}{4}$. Thus for $i = 1, 2$, we have

$$|\hat{U}_i|, |\hat{V}_i| \geq (1-\alpha^{2/3})\frac{n}{2} - \frac{n}{4} \geq \frac{n}{8}.$$

So we may apply Claim 5.2.8 to obtain the two special copies of $K_{s,t}$, K^1 and K^2. Note that $|U_i \setminus V(K^1)|$, $|V_i \setminus V(K^2)| \leq k(s+t)$ for $i = 1, 2$. Let $U_0' = U_0 \setminus V(K^2)$ and $V_0' = V_0 \setminus V(K^1)$. We remove the graphs K^1 and K^2, then we partition the vertices $U_0' = U_0^1 \cup U_0^2$ and $V_0' = V_0^1 \cup V_0^2$ so that

$G_1 := G[(U_1 \cup U_0^1) \setminus V(K^1), (V_1 \cup V_0^1) \setminus V(K^2)]$ satisfies

$$|U_1| - \lceil t/2 \rceil + |U_0^1| = k(s+t), |V_1| - \lceil t/2 \rceil + |V_0^1| = k(s+t)$$

and $G_2 = G - G_1 - K^1 - K^2$ satisfies

$$|U_2| - \lfloor t/2 \rfloor + |U_0^2| = k(s+t), |V_2| - \lfloor t/2 \rfloor + |V_0^2| = k(s+t).$$

Thus G_1 and G_2 can be tiled, so along with K^1 and K^2, this completes the tiling of G.

□

5.3 Tightness

In this section we will prove Proposition 5.1.7. We will need to use the graphs $P(m,p)$, where $m, p \in \mathbb{N}$, introduced by Zhao in [51].

Lemma 5.3.1. *For all $p \in \mathbb{N}$ there exists m_0 such that for all $m \in \mathbb{N}$, $m > m_0$, there exists a balanced bipartite graph, $P(m,p)$, on $2m$ vertices, so that the following hold:*

(i) *$P(m,p)$ is p-regular*

(ii) *$P(m,p)$ does not contain a copy of $K_{2,2}$.*

Proof of Proposition 5.1.7. Let $G[U,V]$ be a balanced bipartite graph on $2n$ vertices satisfying the following conditions. Let $n = (2k+1)(s+t)$ for some sufficiently large k (as determined by Lemma 5.3.1 with $p = s-1$). Partition U into $U = U_0 \cup U_1 \cup U_2$ and partition V into $V = V_0 \cup V_1 \cup V_2$ where, $|U_1| = |V_2| = k(s+t) + \left\lfloor \frac{t+1}{2} \right\rfloor$, $|V_1| = |U_2| = k(s+t) + \left\lceil \frac{t+1}{2} \right\rceil$ and $|U_0| = |V_0| = s-1$. Let $G[U_i, V_i]$ be complete for $i \in \{1,2\}$, $G[U_1, V_2] \cong P\left(k(s+t) + \left\lfloor \frac{t+1}{2} \right\rfloor, s-1\right)$ and

$G[U_2, V_1] \cong P\left(k(s+t) + \left\lceil \frac{t+1}{2} \right\rceil, s-1\right)$. Let $G[U_0, V_1 \cup V_2]$ be complete, $G[V_0, U_1 \cup U_2]$ be complete and $G[U_0, V_0]$ be empty. Note that

$$\delta(G) = \begin{cases} \frac{n+3s}{2} - \frac{3}{2} & \text{if } t \text{ is odd} \\ \frac{n+3s}{2} - 2 & \text{if } t \text{ is even.} \end{cases}$$

Finally we reiterate the following properties of $G[U_1, V_2]$ and $G[U_2, V_1]$. For $i = 1, 2$,

$$\Delta(U_i, V_{3-i}) = \Delta(V_i, U_{3-i}) = s - 1 \tag{5.8}$$

and

$$G[U_i, V_{3-i}] \text{ is } K_{2,2}\text{-free.} \tag{5.9}$$

For $i \in \{1, 2\}$ and $A \in \{U_i, V_i\}$, let $A^D := V_{3-i}$ if $A = U_i$ and let $A^D := U_{3-i}$ if $A = V_i$. We call A^D the *diagonal set of* A. Let $A^N := V_i$ if $A = U_i$ and $A^N := U_i$ if $A = V_i$. We call A^N the *non-diagonal set of* A. Finally, we let $A^M := V_0$ if $A = U_i$ and $A^M := U_0$ if $A = V_i$. We call A^M the *opposite middle set of* A.

Suppose $K \cong K_{s,t}$ is a subgraph of G. We say K is a *crossing* $K_{s,t}$ if $V(K) \cap (U_1 \cup V_1) \neq \emptyset$ and $V(K) \cap (U_2 \cup V_2) \neq \emptyset$. Let $\mathcal{W} = \{U_1, U_2, V_1, V_2\}$.

Claim 5.3.2. *If K is a crossing $K_{s,t}$, then*

(i) *$V(K)$ must intersect some member of $\mathcal{W}$ in exactly one vertex, and*

(ii) *there is a unique $A_0 \in \{U_0, V_0\}$ such that $V(K) \cap A_0 \neq \emptyset$.*

Furthermore, if $|V(K) \cap A| = 1$ for some $A \in \mathcal{W}$, then

(iii) *$V(K) \cap A^D \neq \emptyset$, and*

(iv) *either $|V(K) \cap A^N| \geq 2$ and $V(K) \cap (A^N)^D = \emptyset$, or $V(K) \cap A^N = \emptyset$ and $|V(K) \cap (A^N)^D| \geq 2$.*

Proof. *(i)* Suppose not. Then without loss of generality, suppose that $|V(K) \cap V_1| \geq 2$. By (5.9) we have, $|V(K) \cap U_2| \leq 1$ and thus $V(K) \cap U_2 = \emptyset$. Since K is crossing, we have $V(K) \cap V_2 \neq \emptyset$ and thus $|V(K) \cap V_2| \geq 2$. By (5.9) we have, $|V(K) \cap U_1| \leq 1$ and thus $V(K) \cap U_1 = \emptyset$. This is a contradiction, since $K \cong K_{s,t}$ and $|V(K) \cap U| \leq |U_0| = s - 1$.

(ii) Suppose first that $V(K) \cap U_0 = \emptyset = V(K) \cap V_0$. By Claim 5.3.2 (i), we can assume without loss of generality that $|V(K) \cap U_1| = 1$. Then either $|V(K) \cap U_2| = t - 1$ or $|V(K) \cap U_2| = s - 1$. If $|V(K) \cap U_2| = t - 1$, then by (5.8) we must have $V(K) \cap V_1 = \emptyset$ which implies $|V(K) \cap V_2| = s$, contradicting (5.8). If $|V(K) \cap U_2| = s - 1$, then since $t \geq 2s + 1$ we have $|V(K) \cap V_1| \geq s + 1$ or $|V(K) \cap V_2| \geq s + 1$, both of which contradict (5.8). Thus there exists $A_0 \in \{U_0, V_0\}$ such that $V(K) \cap A_0 \neq \emptyset$. Finally since $G[U_0, V_0]$ is empty, A_0 must be unique.

(iii) Suppose that $V(K) \cap A^D = \emptyset$. Since $|V_0| = s - 1$, we have $V(K) \cap A^N \neq \emptyset$ and since K is crossing, we have $V(K) \cap (A^N)^D \neq \emptyset$. Then by (5.8), we have $|V(K) \cap A^N|, |V(K) \cap (A^N)^D| \leq s - 1$. Thus $|V(K) \cap U| \leq 2s - 1$ and $|V(K) \cap V| \leq 2s - 2$, contradicting the fact that $K \cong K_{s,t}$ and $t \geq 2s + 1$.

(iv) We first show that it is not possible for either $|V(K) \cap A^N| = 1$ or $|V(K) \cap (A^N)^D| = 1$. If $|V(K) \cap A^N| = 1$, then by (5.8) and $|U_0| = |V_0| = s - 1$, we have $|V(K) \cap U|, |V(K) \cap V| \leq 2s - 1$, contradicting the fact that $K \cong K_{s,t}$ and $t \geq 2s + 1$. So suppose $|V(K) \cap (A^N)^D| = 1$. If $V(K) \cap U_0 = \emptyset$, then $|V(K) \cap U| = 2$ and since $t \geq 3$ we must have $s = 2$. Then by (5.8) we have $|V(K) \cap V| \leq 3$ contradicting the fact that $K \cong K_{s,t}$ and $t \geq 2s + 1$. If $V(K) \cap U_0 \neq \emptyset$, then $V(K) \cap V_0 = \emptyset$. So $|V(K) \cap U| \leq s + 1$ and by (5.8),

$|V(K) \cap V| \leq 2s - 2$ contradicting the fact that $K \cong K_{s,t}$ and $t \geq 2s + 1$.

Now suppose $V(K) \cap A^N \neq \emptyset$ and $V(K) \cap (A^N)^D \neq \emptyset$. Thus, by the previous paragraph we have $|V(K) \cap A^N|, |V(K) \cap (A^N)^D| \geq 2$, contradicting (5.9).

So suppose that $V(K) \cap A^N = \emptyset = V(K) \cap (A^N)^D$. Then it must be the case that $|V(K) \cap (A^N)^M| = s - 1$ and consequently $|V(K) \cap A^D| = t$, contradicting (5.8).

□

Let $A \in \mathcal{W}$. We say K is *crossing from* A if either $|V(K) \cap A| = 1$ and $|V(K) \cap A^D| \geq 2$, or $|V(K) \cap A| = 1$, $|V(K) \cap A^D| = 1$ and $V(K) \cap A^M \neq \emptyset$. We say that a crossing $K_{s,t}$ from A is *Type 1* if $|V(K) \cap (A^N)^M| = s - 1$, $|V(K) \cap A^N| = t - p$ and $|V(K) \cap A^D| = p$ for some $2 \leq p \leq s - 1$. We say that a crossing $K_{s,t}$ from A is *Type 2* if $|V(K) \cap (A^N)^D| = t - 1$, $|V(K) \cap A^M| = s - p$, and $|V(K) \cap A^D| = p$ for some $1 \leq p \leq s - 1$.

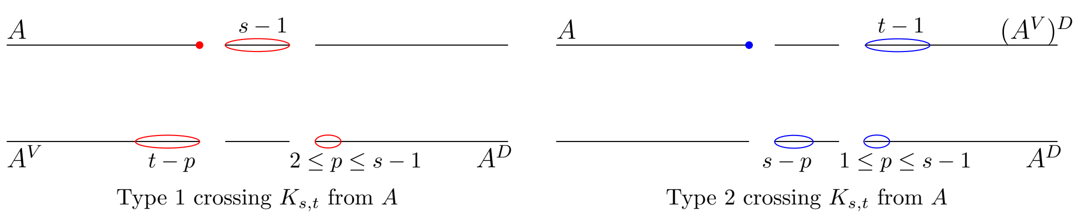


Figure 5.1: Two Types

Claim 5.3.3. *Every crossing $K_{s,t}$ is either Type 1 or Type 2.*

Proof. (See Figure 1) Let K be a crossing $K_{s,t}$ and without loss of generality suppose K is crossing from U_1. Let $p := |V(K) \cap V_2|$. By Claim 5.3.2 (iii) and (5.8) we have $1 \leq p \leq s - 1$. Suppose K is not Type 1. If $V(K) \cap U_2 = \emptyset$, then $|V(K) \cap U_0| = s - 1$ which implies $V(K) \cap V_0 = \emptyset$ by Claim 5.3.2 (ii). Since K is not Type 1, it must be the case that $|V(K) \cap V_2| = 1$ and $|V(K) \cap V_1| = t - 1$ in which case K is not crossing from U_1, contradicting our assumption. So we suppose that $V(K) \cap U_2 \neq \emptyset$. By Claim 5.3.2 (iv) we have $|V(K) \cap U_2| \geq 2$ and $V(K) \cap V_1 = \emptyset$, which implies that $|V(K) \cap V_0| = s - p$. So by Claim 5.3.2 (ii), we have $V(K) \cap U_0 = \emptyset$ and thus $|V(K) \cap U_2| = t - 1$, so K is Type 2.

□

Suppose for a contradiction that G can be tiled with $K_{s,t}$. Let $\mathcal{F}$ be a tiling of G which minimizes the number of crossing $K_{s,t}$'s.

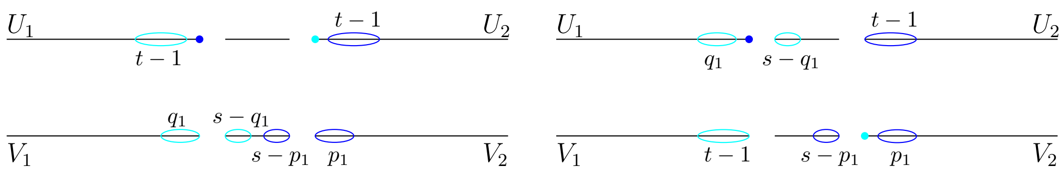


Figure 5.2: Two cases in the proof of Claim 5.3.4

Claim 5.3.4. *For $i = 1, 2$, if there is a crossing $K_{s,t}$ of Type 2 from U_i or V_i, then there is no crossing $K_{s,t}$ of Type 2 from U_{3-i} or V_{3-i}.*

Proof. Without loss of generality suppose K^1 is a crossing $K_{s,t}$ of Type 2 from U_1. Suppose that K^2 is a crossing $K_{s,t}$ of Type 2 from U_2 (See Figure 2). For $i \in \{1, 2\}$, let

$$K_*^i := G[U_i \cap (V(K^1) \cup V(K^2)), V(K^{3-i}) \cap (V_0 \cup V_i)].$$

We have $K_*^1 \cong K_{s,t} \cong K_*^2$, neither of K_*^1, K_*^2 are crossing, and $V(K^1) \cup V(K^2) = V(K_*^1) \cup V(K_*^2)$. Thus we obtain a tiling with fewer crossing $K_{s,t}$'s, contradicting the minimality of $\mathcal{F}$.

Now, suppose K^1 is a crossing $K_{s,t}$ of Type 2 from U_1 and K^2 is a crossing $K_{s,t}$ of Type 2 from V_2 (See Figure 2). Specify an element $L^1 \in \mathcal{F}$, such that $V(L^1) \subseteq U_1 \cup V_1$ and $|V(L^1) \cap V_1| = t$ and specify an element $L^2 \in \mathcal{F}$, such that $V(L^2) \subseteq U_2 \cup V_2$ and $|V(L^2) \cap U_2| = t$. Choose arbitrary vertices $v' \in V(K^1) \cap V_0$ and $u' \in V(K^2) \cap U_0$. We now define four subgraphs of G. Let

$$\begin{aligned}
K^1_* &:= G[V(L^1) \cap V_1, (V(K^1) \cup V(K^2)) \cap ((U_1 \cup U_0) \setminus \{u'\})], \\
L^1_* &:= G[V(L^1) \cap U_1, (V(K^2) \cap V_1) \cup \{v'\}], \\
K^2_* &:= G[V(L^2) \cap U_2, (V(K^1) \cup V(K^2)) \cap ((V_2 \cup V_0) \setminus \{v'\})], \text{ and} \\
L^2_* &:= G[V(L^2) \cap V_1, (V(K^1) \cap U_2) \cup \{u'\}].
\end{aligned}$$

All of $K^1_*, K^2_*, L^1_*, L^2_*$ are isomorphic to $K_{s,t}$, none of $K^1_*, K^2_*, L^1_*, L^2_*$ are crossing, and $V(K^1_*) \cup V(K^2_*) \cup V(L^1_*) \cup V(L^2_*) = V(K^1) \cup V(K^2) \cup V(L^1) \cup V(L^2)$. Thus we obtain a tiling with fewer crossing $K_{s,t}$'s, contradicting the minimality of $\mathcal{F}$. $\square$

For $i \in \{1, 2\}$, let $\mathcal{F}_i$ be the set of all copies of $K_{s,t}$ in $\mathcal{F}$ which touch $U_i \cup V_i$. And let U^*_i (resp. V^*_i) be all the vertices in U (resp. V) which touch elements of $\mathcal{F}_i$. Precisely, let $\mathcal{F}_i = \{K \in \mathcal{F} : V(K) \cap (U_i \cup V_i) \neq \emptyset\}$ for $i = 1, 2$, and let

$$U^*_i = (\cup_{K \in \mathcal{F}_i} V(K)) \cap U \quad \text{and} \quad V^*_i = (\cup_{K \in \mathcal{F}_i} V(K)) \cap V.$$

Note that $U_i \subseteq U^*_i$ and $V_i \subseteq V^*_i$. We will use the following claim to show that all of the remaining possible configurations of crossing $K_{s,t}$'s lead to contradictions.

Claim 5.3.5. *For all $i \in \{1, 2\}$, either*

$$\max\{|U^*_i|, |V^*_i|\} \geq k(s+t) + 2t \quad \textit{or} \quad \min\{|U^*_i|, |V^*_i|\} \geq (k+1)(s+t).$$

Proof. Suppose that $\max\{|U^*_i|, |V^*_i|\} < k(s+t) + 2t$. Then since $U_i \subseteq U^*_i$ and $V_i \subseteq V^*_i$, we have

$$k(s+t) + s < |U^*_i|, |V^*_i| < k(s+t) + 2t, \tag{5.10}$$

and thus

$$||U_i^*| - |V_i^*|| < 2t - s. \tag{5.11}$$

By definition $G[U_i^*, V_i^*]$ can be tiled, thus there exists nonnegative integers ℓ, a, b such that $|U_i^*| = \ell(s+t) + as + bt$ and $|V_i^*| = \ell(s+t) + at + bs$. By choosing ℓ to be maximal, we have $a = 0$ or $b = 0$. If $\ell \leq k - 1$, then in order to satisfy the lower bound in (5.10) we must have $a \geq 3$ or $b \geq 3$. Since $a = 0$ or $b = 0$, we have $||U_i^*| - |V_i^*|| \geq 3t - 3s \geq 2t - s$, which contradicts (5.11). If $\ell = k$, then in order to satisfy the lower bound in (5.10), we must have $a \geq 2$ or $b \geq 2$, but then we violate the upper bound. So $\ell \geq k + 1$ and we have
$\min\{|U_i^*|, |V_i^*|\} \geq (k+1)(s+t)$. □

We will also use the following facts. For $i = 1, 2$, we have

$$|V_i \cup V_0| + s, |U_i \cup U_0| + s \leq k(s+t) + \frac{t+2}{2} + 2s - 1 < (k+1)(s+t). \tag{5.12}$$

which in particular implies

$$|V_i \cup V_0| + t, |U_i \cup U_0| + t < k(s+t) + 2t. \tag{5.13}$$

Let $i \in \{1, 2\}$ and let
$X_i = \{K \in \mathcal{F} : K$ is crossing from U_i and K is Type 2$\}$ and
$Y_i = \{K \in \mathcal{F} : K$ is crossing from V_i and K is Type 2$\}$. Since
$|U_0| = |V_0| = s - 1$, Claim 5.3.2 (ii) implies,

$$0 \leq |X_i|, |Y_i| \leq s - 1. \tag{5.14}$$

Case 0. There are no crossing $K_{s,t}$'s. So $|U_1^*| \leq |U_1 \cup U_0|$ and $|V_1^*| \leq |V_1 \cup V_0|$. Then by (5.12) we have $|U_1^*|, |V_1^*| < (k+1)(s+t)$, contradicting Claim 5.3.5.

Case 1. There is a crossing $K_{s,t}$ of Type 1. Without loss of generality, suppose K^1 is a crossing $K_{s,t}$ of Type 1 from U_1 and let $p := |V(K^1) \cap V_2|$. Since

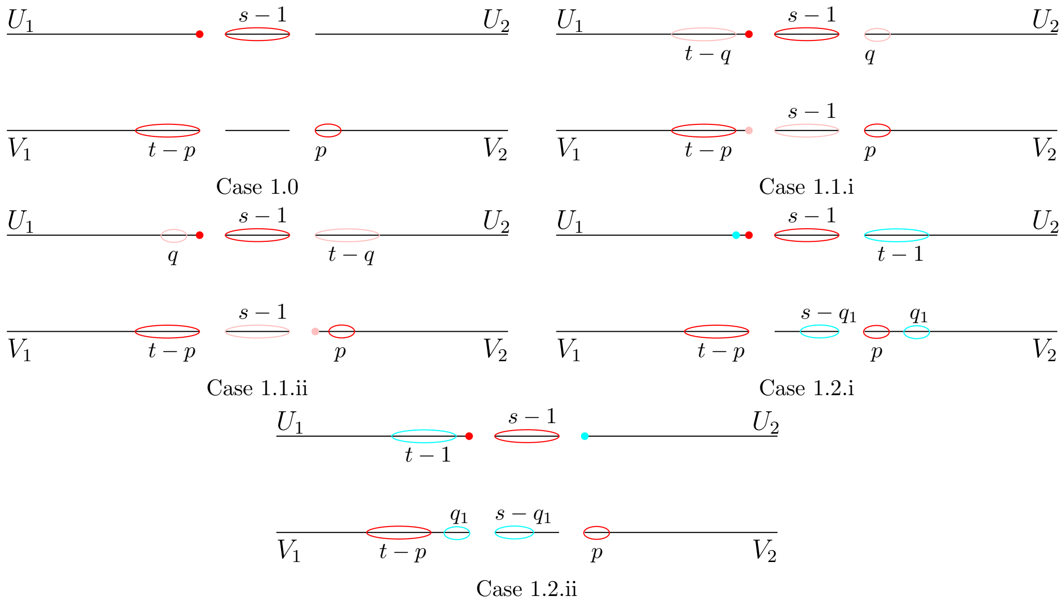


Figure 5.3: Case 1

$U_0 \setminus V(K^1) = \emptyset$, there can be no other crossing $K_{s,t}$'s of Type 1 from U_1 or U_2 and no crossing $K_{s,t}$'s of Type 2 from V_1 or V_2. By Claim 5.3.3, we must only consider five subcases:

Case 1.0. K^1 is the only crossing $K_{s,t}$. So $|U_1^*| \leq |U_1 \cup U_0|$ and $|V_1^*| \leq |V_1 \cup V_0| + p < |V_1 \cup V_0| + s$. Then by (5.12) we have $|U_1^*|, |V_1^*| < (k+1)(s+t)$, contradicting Claim 5.3.5.

Case 1.1.i. There is a crossing $K_{s,t}$ of Type 1 from V_1. Let K^2 be a crossing $K_{s,t}$ from V_1 and let $q := |V(K^2) \cap U_2|$. Since $V_0 \setminus V(K^2) = \emptyset$, K^1 and K^2 are the only crossing $K_{s,t}$'s. So $|U_1^*| \leq |U_1 \cup U_0| + q < |U_1 \cup U_0| + s$ and $|V_1^*| \leq |V_1 \cup V_0| + p < |V_1 \cup V_0| + s$. Then by (5.12) we have, $|U_1^*|, |V_1^*| < (k+1)(s+t)$, contradicting Claim 5.3.5.

Case 1.1.ii. There is a crossing $K_{s,t}$ of Type 1 from V_2. Let K^2 be a crossing $K_{s,t}$ from V_2 and let $q := |V(K^2) \cap U_1|$. Since $V_0 \setminus V(K^2) = \emptyset$, K^1 and K^2 are the only crossing $K_{s,t}$'s. So $|V_1^*| \leq |V_1 \cup V_0| + p + 1 \leq |V_1 \cup V_0| + s$ and $|U_1^*| \leq |U_1 \cup U_0| + t - q < |U_1 \cup U_0| + t$. Then by (5.12) and (5.13) we have

$|V_1^*| < (k+1)(s+t)$ and $|U_1^*| < k(s+t)+2t$, contradicting Claim 5.3.5.

Case 1.2.i. $1 \leq |X_1|$. By Claim 5.3.4, since there exists a crossing $K_{s,t}$ of Type 2 from U_1, there can be no crossing $K_{s,t}$'s of Type 2 from U_2. So
$|U_2^*| \leq |U_2 \cup U_0| + |X_1| + 1 \leq |U_2 \cup U_0| + s$ and
$|V_2^*| \leq |V_2 \cup V_0| + t - p < |V_2 \cup V_0| + t$. Then by (5.12) and (5.13) we have
$|U_2^*| < (k+1)(s+t)$ and $|V_2^*| < k(s+t)+2t$, contradicting Claim 5.3.5.

Case 1.2.ii. $1 \leq |X_2|$. By Claim 5.3.4, since there exists a crossing $K_{s,t}$ of Type 2 from U_2, then there can be no crossing $K_{s,t}$'s of Type 2 from U_1. So
$|U_1^*| \leq |U_1 \cup U_0| + |X_2| < |U_1 \cup U_0| + s$ and $|V_1^*| \leq |V_1 \cup V_0| + p < |V_1 \cup V_0| + s$.
Then by (5.12) we have $|U_1^*|, |V_1^*| < (k+1)(s+t)$, contradicting Claim 5.3.5.

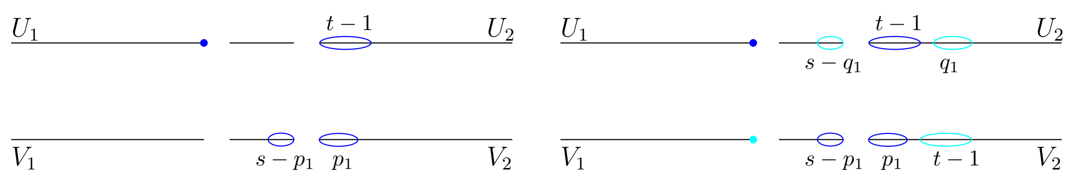


Figure 5.4: Case 2

Case 2. There are no crossing $K_{s,t}$'s of Type 1. By Claim 5.3.3, there can only be crossing $K_{s,t}$'s of Type 2. Without loss of generality suppose that $1 \leq |X_1|$. Then there can be no crossing $K_{s,t}$ of Type 2 from U_2 or V_2. So
$|U_2^*| \leq |U_2 \cup U_0| + |X_1| < |U_2 \cup U_0| + s$ and $|V_2^*| \leq |V_2 \cup V_0| + |Y_1| < |V_2 \cup V_0| + s$.
Then by (5.12) we have $|U_2^*|, |V_2^*| < (k+1)(s+t)$, contradicting Claim 5.3.5.

□

5.4 Conclusion

Seymour conjectured that for any positive integer r, if G is a graph on n vertices with $\delta(G) \geq \frac{r}{r+1}n$, then G contains the r^{th} power of a Hamilton cycle (Conjecture 2.5.1 in Chapter 2). If true, Seymour's conjecture implies Theorem 5.1.1 (with $s = r+1$) since the r^{th} power of a Hamilton cycle contains $\lfloor \frac{n}{r+1} \rfloor$

vertex disjoint copies of K_{r+1}. Define a r-ladder on $2n$ vertices, denoted L_n^r, to be a balanced bipartite graph with vertex sets $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ such that u_iv_j is an edge if $|i - j| \leq r - 1$. Then L_n^r has the property that for all $1 \leq i \leq n - r + 1$, the vertex sets $\{u_i, u_{i+1}, \ldots, u_{i+r-1}\}$ and $\{v_i, v_{i+1}, \ldots, v_{i+r-1}\}$ induce the complete bipartite graph $K_{r,r}$.

Problem 2. *For all $r \in \mathbb{N}$, determine the the optimal value $d(r)$ so that if G is a balanced bipartite graph on $2n$ vertices with $\delta(G) \geq d(r)$, then $L_n^r \subseteq G$.*

A solution to this problem would generalize the tiling results for bipartite graphs as Seymour's conjecture generalizes the Hajnal-Szemerédi theorem. The case $r = 1$ is implied by Hall's theorem [22] which gives $d(1) = \frac{n}{2}$. The case $r = 2$ was solved by Czygrinow and Kierstead (for large n) in [13], giving $d(2) = \frac{n}{2} + 1$. This problem seems like a nice setting to apply the "absorbing" technique (instead of the regularity-blow-up method) developed by Rödl, Ruciński, and Szemerédi [41].

Chapter 6

TILING IN BIPARTITE GRAPHS: ASYMMETRIC MINIMUM DEGREES

This chapter is joint work with Andrzej Czygrinow.

6.1 Introduction

If G is a graph on $n = sm$ vertices, H is a graph on s vertices and G contains m vertex disjoint copies of H, then we say G can be *tiled* with H. We now state two important tiling results which motivate the current research.

Theorem 6.1.1 (Hajnal-Szemerédi [21]). *Let G be a graph on $n = sm$ vertices. If $\delta(G) \geq (s-1)m$, then G can be tiled with K_s.*

Kierstead and Kostochka generalized, and in doing so slightly improved, the result of Hajnal and Szemeredi.

Theorem 6.1.2 (Kierstead-Kostochka [26]). *Let G be a graph on $n = sm$ vertices. If $\deg(x) + \deg(y) \geq 2(s-1)m - 1$, for all non-adjacent $x, y \in V(G)$ then G can be tiled with K_s.*

Both of these results can be shown to be best possible relative to the respective degree condition, i.e. no smaller lower bound on the degree will suffice.

For the rest of the paper we will consider tiling in bipartite graphs. Given a bipartite graph $G[U, V]$ we say G is balanced if $|U| = |V|$. The following theorem is a consequence of Hall's matching theorem [22], and is an early result on bipartite graph tiling.

Theorem 6.1.3. *Let G be a balanced bipartite graph on $2n$ vertices. If $\delta(G) \geq \frac{n}{2}$, then G can be tiled with $K_{1,1}$.*

Zhao determined the best possible minimum degree condition for a bipartite graph to be tiled with $K_{s,s}$ when $s \geq 2$.

Theorem 6.1.4 (Zhao [51])**.** *For each $s \geq 2$, there exists m_0 such that the following holds for all $m \geq m_0$. If G is a balanced bipartite graph on $2n = 2ms$ vertices with*

$$\delta(G) \geq \begin{cases} \frac{n}{2} + s - 1 & \text{if } m \text{ is even} \\ \frac{n+3s}{2} - 2 & \text{if } m \text{ is odd,} \end{cases}$$

then G can be tiled with $K_{s,s}$.

Hladký and Schacht, and Czygrinow and DeBiasio determined the best possible minimum degree condition for a balanced bipartite graph to be tiled with $K_{s,t}$.

Theorem 6.1.5 (Hladký, Schacht [23]; Czygrinow, DeBiasio [11])**.** *For each $t > s \geq 1$, there exists m_0 such that the following holds for all $m \geq m_0$. If G is a balanced bipartite graph on $2n = 2m(s+t)$ vertices with*

$$\delta(G) \geq \begin{cases} \frac{n}{2} + s - 1 & \text{if } m \text{ is even} \\ \frac{n+t+s}{2} - 1 & \text{if } m \text{ is odd and } t \leq 2s \\ \frac{n+3s}{2} - 1 & \text{if } m \text{ is odd and } t \geq 2s+1 \end{cases}$$

then G can be tiled with $K_{s,s}$.

Now we consider a more general degree condition than $\delta(G)$. Given a bipartite graph $G[U, V]$, let $\delta_U(G) := \min\{\deg(u) : u \in U\}$ and $\delta_V(G) := \min\{\deg(v) : v \in V\}$. We will write δ_U and δ_V instead of $\delta_U(G)$ and $\delta_V(G)$ when it is clear which graph we are referring to. The following theorem is again a consequence of Hall's matching theorem and is more general than Theorem 6.1.3.

Theorem 6.1.6. *Let $G[U, V]$ be a balanced bipartite graph on $2n$ vertices. If $\delta_U + \delta_V \geq n$, then G can be tiled with $K_{1,1}$.*

Notice that when $s = 2$, Theorem 6.1.4 says that if $G[U, V]$ is a balanced bipartite graph on $2n$ vertices with $\delta(G) \geq \frac{n}{2} + 1$, then G can be tiled with $K_{2,2}$. Based on this, one might guess that the optimal value of $\delta_U + \delta_V$ which implies that G can be tiled with $K_{2,2}$ is $\delta_U + \delta_V \geq n + 2$. In fact, Wang made the following conjecture about 2-factors in bipartite graphs.

Conjecture 6.1.7 (Wang [50]). *Let $G[U, V]$ and H be balanced bipartite graphs on $2n$ vertices. If $\delta_U + \delta_V \geq n + 2$ and $\Delta(H) \leq 2$, then $H \subseteq G$.*

Czygrinow, DeBiasio, and Kierstead [12] proved Wang's conjecture when $\delta_V \geq \delta_U = \Omega(n)$. However, setting $s = 2$ in Theorems 6.1.8 and 6.1.13, which are stated below, we obtain the result that if $G[U, V]$ is a balanced bipartite graph on $2n$ vertices with $\delta_U + \delta_V \geq n + 1$ and $\delta_V \geq \delta_U = \Omega(n)$, then G can be tiled with $K_{2,2}$.

We prove the following theorems which will generalize the results in [51] for all $s \geq 2$.

Theorem 6.1.8. *For each $s \geq 2$ and $\lambda \in (0, \frac{1}{2})$, there exists m_0 such that the following holds for all $m \geq m_0$. If $G[U, V]$ is a balanced bipartite graph on $2n = 2ms$ vertices with $\delta_V \geq \delta_U \geq \lambda n$ and $\delta_U + \delta_V \geq n + 3s - 5$ then G can be tiled with $K_{s,s}$.*

As mentioned earlier, Zhao gave examples which shows that Theorem 6.1.4 is best possible.

Proposition 6.1.9 (Zhao [51]). *Let $s \geq 2$, and $n = ms \geq 64s^2$. There exists a balanced bipartite graph, G, on $2n$ vertices with*

$$\delta(G) = \begin{cases} \frac{n}{2} + s - 2 & \text{if } m \text{ is even} \\ \frac{n+3s}{2} - 3 & \text{if } m \text{ is odd} \end{cases}$$

such that G cannot be tiled with $K_{s,s}$.

Since there are examples with $\delta(G) = \frac{n+3s}{2} - 3$ such that G cannot be tiled with $K_{s,s}$, this implies that there are examples with $\delta_U + \delta_V = 2\delta(G) = n + 3s - 6$ which cannot be tiled with $K_{s,s}$. This shows that the degree condition in Theorem 6.1.8 is best possible. Notice that Theorem 6.1.4 gives a better bound on $\delta(G)$ when m is even, which might lead you to guess that $\delta_U + \delta_V \geq n + 2s - 3$ suffices when m is even (based on Theorem 6.1.8). However, we show that when m is even (or odd) there are graphs with $\delta_U(G) + \delta_V(G) = n + 3s - 7$ that cannot be tiled with $K_{s,s}$.

Proposition 6.1.10. *Let $s \geq 2$. For every $j \in \mathbb{N}$, there exists an integer m and a balanced bipartite graph $G[U, V]$ on $2n = 2ms$ vertices such that $\delta_U + \delta_V = n + 3s - 7$ and $2sj - s - 1 \leq |\delta_V - \delta_U| \leq 2sj - 1$, but G cannot be tiled with $K_{s,s}$.*

Surprisingly, we show that when δ_U is significantly smaller than δ_V, a smaller sum of degrees will suffice to tile G with $K_{s,s}$, provided $\delta_V \geq \delta_U = \Omega(n)$. First we must give a definition which allows us to precisely state our result.

We make use of the following fact to split the positive integers into two classes.

Fact 6.1.11. *Let s be a positive integer. There exists unique $p, q \in \mathbb{N}$ such that $s = p^2 + q$ and $0 \leq q \leq 2p$.*

Using this fact, we define a function which classifies positive integers depending on their value of q.

Definition 6.1.12. *Let c be a function from $\mathbb{Z}^+$ to $\{0, 1\}$ such that*

$$c(s) = \begin{cases} 0 & \text{if } q = 0 \quad \text{or} \quad p + 1 \leq q \leq 2p \\ 1 & \text{if } 1 \leq q \leq p \end{cases}$$

Theorem 6.1.13. *Let $s \geq 2$ and $\lambda \in (0, \frac{1}{2})$. There exists m_0 such that the following holds for all $m \geq m_0$. Let G be a balanced U, V-bigraph on $2n = 2ms$ vertices with $\delta_V \geq \delta_U \geq \lambda n$, $\delta_U = k_1 s + s + r$ for some $0 \leq r \leq s-1$, $k_2 = m - k_1$, $k_1 \leq (1 - \frac{1}{2s})k_2$, and $0 \leq d \leq s - 2\lceil\sqrt{s}\rceil + c(s) + 1$. If*

(i) $\delta_U + \delta_V \geq n + 3s - 5$ *or*

(ii) $k_2 \geq (s-d)k_1$ *and* $\delta_U + \delta_V \geq n + 2s - 2\lceil\sqrt{s}\rceil + d + c(s)$,

then G can be tiled with $K_{s,s}$.

We also give examples to show that the degree is tight when $d = 0$ in the preceding theorem.

Proposition 6.1.14. *For every $s \geq 2$, there exists a balanced bipartite graph G with $k_2 \geq sk_1$ and*

$$\delta_U + \delta_V = n + 2s - 2\lceil\sqrt{s}\rceil + c(s) - 1$$

such that G cannot be tiled with $K_{s,s}$.

Finally, when δ_U is constant, we first construct two graphs with $\delta_U + \delta_V \geq n + 2s - 2\lceil\sqrt{s}\rceil + c(s)$ which cannot be tiled with $K_{s,s}$. Then we show that there exists graphs (without constructing them) with $\delta_U + \delta_V$ much larger than $n + 3s$ which cannot be tiled with $K_{s,s}$.

Theorem 6.1.15. *There exists $s_0, n_0 \in \mathbb{N}$ such that for all $s \geq s_0$, there exists a graph $G[U, V]$ on $n \geq n_0$ vertices with $\delta_U + \delta_V \geq n + s^{s^{1/3}}$ such that G cannot be tiled with $K_{s,s}$.*

6.2 Extremal Examples

6.2.1 Tightness when $k_2 \approx k_1$

As mentioned in the introduction, Zhao determined the optimal minimum degree condition so that G can be tiled with $K_{s,s}$. If n is an odd multiple of s, then $\delta(G) \geq \frac{n}{2} + \frac{3s}{2} - 2$ is best possible; however, if n is an even multiple of s, then $\delta(G) \geq \frac{n}{2} + s - 1$ is best possible. In Theorem 6.1.8 and Theorem 6.1.13 we show that if $\delta_V \geq \delta_U = \Omega(n)$, then $\delta_U + \delta_V \geq n + 3s - 5$ suffices to give a tiling of G with $K_{s,s}$. We now give an example which shows that even when n is an even multiple of s, we cannot improve the coefficient of the s term in the degree condition.

We will need to use the graphs $P(m,p)$, where $m, p \in \mathbb{N}$, introduced by Zhao in [51].

Lemma 6.2.1. *For all $p \in \mathbb{N}$ there exists m_0 such that for all $m \in \mathbb{N}$, $m > m_0$, there exists a balanced bipartite graph, $P(m,p)$, on $2m$ vertices, so that the following hold:*

(i) *$P(m,p)$ is p-regular*

(ii) *$P(m,p)$ does not contain a copy of $K_{2,2}$.*

First we recall Zhao's example which shows that there exist graphs with $\delta_U + \delta_V = n + 3s - 6$ such that G cannot be tiled with $K_{s,s}$. Let $G[U,V]$ be a balanced bipartite graph on $2n$ vertices with $n = (2k+1)s$. Partition U as $U_1 \cup U_2$ with $|U_1| = ks + 1$, $|U_2| = ks + s - 1$ and partition V as $V_1 \cup V_2$ with $|V_1| = ks + s - 1$, $|V_2| = ks + 1$. Let $G[U_1, V_1]$ and $G[U_2, V_2]$ be complete, let $G[U_1, V_2] \simeq P(ks+1, s-2)$ and let $G[U_2, V_1] \simeq P(ks+s-1, 2s-4)$.

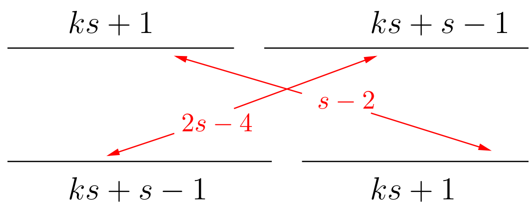


Figure 6.1: m is odd and $\delta_U + \delta_V = n + 3s - 6$

We now recall the argument which shows that G cannot be tiled with $K_{s,s}$. Suppose G can be tiled with $K_{s,s}$ and let $\mathcal{K}$ be such a tiling. For $F \in \mathcal{K}$ and $i = 1, 2$, let $X_i(F) := V(F) \cap U_i$, $Y_i(F) := V(F) \cap V_i$ and $\vec{v}(F) = (|X_1(F)|, |X_2(F)|, |Y_1(F)|, |Y_2(F)|)$. We say $F \in \mathcal{K}$ is *crossing* if $V(F) \cap (U_1 \cup V_1) \neq \emptyset$ and $V(F) \cap (U_2 \cup V_2) \neq \emptyset$. We now claim that if F is crossing then $\vec{v}(F) = (s-1, 1, s, 0)$ or $\vec{v}(F) = (0, s, 1, s-1)$. It is not possible for $X_1(F) \neq \emptyset$ and $Y_2(F) \neq \emptyset$ since $G[U_1, V_2] \simeq P(ks+1, s-2)$ and $G[V_1, U_2]$ is $K_{2,2}$-free. Thus if $X_1(F) \neq \emptyset$, then $|Y_1(F)| = s$, $|X_2(F)| \leq 1$, and $|X_1(F)| \geq s-1$. If $Y_2(F) \neq \emptyset$, then $|X_2(F)| = s$, $|Y_1(F)| \leq 1$, and $|Y_2(F)| \geq s-1$. This shows that if F is crossing then $\vec{v}(F) = (s-1, 1, s, 0)$ or $\vec{v}(F) = (0, s, 1, s-1)$. Finally, since we are supposing that G can be tiled, there exists some $\ell \in \mathbb{N}$ and some subset $\mathcal{K}' \subseteq \mathcal{K}$ such that every $F \in \mathcal{K}'$ is crossing and $\sum_{F \in \mathcal{K}'} |X_1(F)| = \ell s + 1$ and $\sum_{F \in \mathcal{K}'} |Y_1(F)| = \ell s + s - 1$. Let i_1 be the number of $F \in \mathcal{K}'$ with $\vec{v}(F) = (s-1, 1, s, 0)$ and let i_2 be the number of $F \in \mathcal{K}'$ with $\vec{v}(F) = (0, s, 1, s-1)$. Then we have

$$\text{(i)} \;\; (s-1)i_1 = \ell s + 1 \quad \text{ and } \quad \text{(ii)} \;\; si_1 + i_2 = \ell s + s - 1$$

Which implies $i_1 + i_2 = s - 2$. However, (ii) implies that $i_2 \geq s - 1$, a contradiction.

Now we prove Theorem 6.1.10.

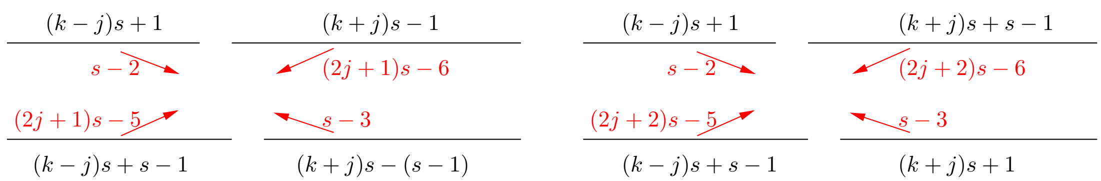
Proof. We give two examples of graphs which cannot be tiled with $K_{s,s}$; one when m is even, one m is odd, and both with $\delta_U + \delta_V = n + 3s - 7$.

Let j be a non-negative integer and let $m = 2k$, where k is sufficiently large. Let U and V be sets of vertices such that $|U| = |V| = 2ks$. Let U be partitioned as $U = U_1 \cup U_2$ and V be partitioned as $V = V_1 \cup V_2$ with $|U_1| = (k-j)s+1$, $|U_2| = (k+j)s-1$, $|V_1| = (k-j+1)s-1$ and $|V_2| = (k+j-1)s+1$. Let $G[U_i, V_i]$ be complete for $i = 1, 2$. Let $G[U_1, V_2]$ be the graph obtained from $G[U_1', V_2] \simeq P((k+j)s-s+1, s-2)$ by deleting $(2j-1)s$ vertices from U_1' while maintaining $\delta(V_2, U_1) \geq s-3$ (note that when $s = 2$, $\delta(V_2, U_1) = 0$). Let $G[U_2, V_1]$ be the graph obtained from $G[U_2, V_1'] \simeq P((k+j)s-1, (2j+1)s-5)$ by deleting $(2j-1)s$ vertices from V_1' while maintaining $\delta(U_2, V_1) \geq (2j+1)s-6$. We have

$$\delta_U = (k-j)s + s - 1 + s - 2 = (k-j+2)s - 3,$$

$$\delta_V = (k+j)s - 1 + s - 3 = (k-j)s + 1 + (2j+1)s - 5 = (k+j+1)s - 4,$$

and thus $\delta_U + \delta_V = 2ks + 3s - 7 = n + 3s - 7$.



Case: m even

Case: m odd

Figure 6.2: $\delta_U + \delta_V = n + 3s - 7$

Let j be a non-negative integer and let $m = 2k+1$, where k is sufficiently large. Let U and V be sets of vertices such that $|U| = |V| = (2k+1)s$. Let U be partitioned as $U = U_1 \cup U_2$ and V be partitioned as $V = V_1 \cup V_2$ with $|U_1| = (k-j)s+1$, $|U_2| = (k+j)s+s-1$, $|V_1| = (k-j)s+s-1$ and $|V_2| = (k+j)s+1$. Let $G[U_i, V_i]$ be complete for $i = 1, 2$. Let $G[U_1, V_2]$ be the graph obtained from $G[U_1', V_2] \simeq P((k+j)s+1, s-2)$ by deleting $2js$ vertices from U_1' while maintaining $\delta(V_2, U_1) \geq s-3$ (note that when $s = 2$, $\delta(V_2, U_1) = 0$). Let $G[U_2, V_1]$ be the graph obtained from

$G[U_2, V_1'] \simeq P((k+j)s+s-1, (2j+2)s-5)$ by deleting $2js$ vertices from V_1' while maintaining $\delta(U_2, V_1) \geq (2j+2)s-6$. We have

$$\delta_U = (k-j)s + s - 1 + s - 2 = (k-j+2)s - 3,$$

$$\delta_V = (k+j)s + s - 1 + s - 3 = (k-j)s + 1 + (2j+2)s - 5 = (k+j+2)s - 4,$$

and thus $\delta_U + \delta_V = (2k+1)s + 3s - 7 = n + 3s - 7$.

The same analysis given before the start of this proof shows that each of these graphs cannot be tiled with $K_{s,s}$.

□

6.2.2 *Tightness when $k_2 \gg k_1$*

Now we prove Theorem 6.1.14.

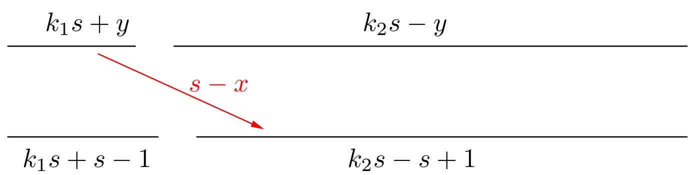


Figure 6.3: $\delta_U + \delta_V = n + 2s - x - y - 1$

Proof. Let $G = (U_1 \cup U_2, V_1 \cup V_2; E)$ be a bipartite graph with $|U_1| = k_1s + y$, $|U_2| = k_2s - y$, $|V_1| = k_1s + s - 1$, $|V_2| = k_2s - s + 1$ such that $G[U_1, V_1]$, $G[U_2, V_2]$, and $G[V_1, U_2]$ are complete. Furthermore suppose $|V_2| \geq (s-x)|U_1|$, every vertex in U_1 has $s - x$ neighbors in V_2, and for all $u, u' \in U_1$, $(N(u) \cap N(u')) \cap V_2 = \emptyset$. Thus we have $0 \leq \delta(V_2, U_1) \leq \Delta(V_2, U_1) \leq 1$ with $\delta(V_2, U_1) = \Delta(V_2, U_1) = 1$ only when $|V_2| = (s-x)|U_1|$ and thus

$$\delta_U + \delta_V \geq k_1s + s - 1 + s - x + k_2s - y = n + 2s - (x+y) - 1 \qquad (6.1)$$

Every copy of $K_{s,s}$ which touches both U_1 and $U_2 \cup V_2$ must have one vertex from U_1, $s - 1$ vertices from U_2, at most $s - x$ vertices from V_2, and at least x vertices from V_1. So if $xy \geq s$, then G cannot be tiled. So in order to maximize $\delta_U + \delta_V$ we minimize $x + y$ subject to the condition that $xy \geq s$. The result is that $x = y = \lceil \sqrt{s} \rceil$, unless $1 \leq q \leq p$ in which case $x = \lceil \sqrt{s} \rceil - 1$, $y = \lceil \sqrt{s} \rceil$ suffices. Thus (6.1) gives $\delta_U + \delta_V = n + 2s - 2\lceil \sqrt{s} \rceil - 1$ in general and $\delta_U + \delta_V = n + 2s - 2\lceil \sqrt{s} \rceil$ when $1 \leq q \leq p$.

□

6.3 Non-extremal Case

In order to prove Theorem 6.1.8 and Theorem 6.1.13 we will first prove the following Theorem.

Theorem 6.3.1. *For every $\alpha > 0$ and every positive integer s, there exist $\beta > 0$ and positive integer m_1 such that the following holds for all $n = ms$ with $m \geq m_1$. Given a bipartite graph $G[U, V]$ with $|U| = |V| = n$, if $\delta_U + \delta_V \geq (1 - 2\beta)n$, $\delta_V \geq \delta_U \gg \alpha n$ and $\delta_U = k_1 s + s + r$ for some $0 \leq r \leq s - 1$ with $k_1 + k_2 = m$, then either G can be tiled with $K_{s,s}$, or*

$$\text{there exist } U_1' \subseteq U,\ V_2' \subseteq V,\ \text{such that } |U_1'| = k_1 s,\ |V_2'| = k_2 s,\ d(U_1', V_2') \leq \alpha. \tag{6.2}$$

If G is a graph for which (6.2) holds, then we say G satisfies the *extremal condition with parameter α*.

6.3.1 Proof of Theorem 6.3.1

Here we prove Theorem 6.3.1. We show that if G is not in the extremal case, we obtain a tiling with $K_{s,s}$; otherwise G is in the extremal case which we deal with in Section 6.4. The proof is adopted from Zhao [51].

Proof. Let ϵ, d, and β be positive real numbers such that

$$\epsilon \ll d \ll \beta \ll \alpha$$

and suppose n is large. Let $G[U, V]$ be a bipartite graph with $|U| = |V| = n$, $\delta_U + \delta_V \geq (1 - \beta)n$, and $\delta_V \geq \delta_U \gg \alpha n$. We also have $\delta_U = k_1 s + s + r$ for some $0 \leq r \leq s - 1$ and we set $k_2 := m - k_1$. Let γ_1, γ_2 be positive real numbers such that $\delta_U \geq (\gamma_1 - \beta)n$, $\delta_V \geq (\gamma_2 - \beta)n$ and $\gamma_1 + \gamma_2 = 1$. Note that $\gamma_2 \geq \gamma_1 \gg \alpha$. We apply Lemma 3.0.10 to G with parameters ϵ and d. We obtain a partition of U into $U_0, U_1, \ldots, U_t$ and V into $V_0, V_1, \ldots, V_t$ such that $|U_i| = |V_i| = \ell \leq \epsilon n$ for all $i \in [t]$ and $|U_0| = |V_0| \leq \epsilon n$. In the graph G' from Lemma 3.0.10, we have (U_i, V_j), is ϵ-regular with density either 0 or exceeding d for all $i, j \in [t]$. We also have $\deg_{G'}(u) > (\gamma_1 - \beta)n - (\epsilon + d)n$ for $u \in U$ and
$\deg_{G'}(v) > (\gamma_2 - \beta)n - (\epsilon + d)n$ for $v \in V$.

We now consider the *reduced graph* of G'. Let G_r be a bipartite graph with parts $\mathcal{U} := \{U_1, \ldots, U_t\}$ and $\mathcal{V} := \{V_1, \ldots, V_t\}$ such that U_i is adjacent to V_j, denoted $U_i \sim V_j$, if and only if (U_i, V_j) is an ϵ-regular pair with density exceeding d. A standard calculation gives the following degree condition in the reduced graph, $\delta_{\mathcal{U}} \geq (\gamma_1 - 2\beta)t$ and $\delta_{\mathcal{V}} \geq (\gamma_2 - 2\beta)t$.

Claim 6.3.2. *If G_r contains two subsets $X \subseteq \mathcal{U}$ and $Y \subseteq \mathcal{V}$ such that $|X| \geq (\gamma_1 - 3\beta)t$, $|Y| \geq (\gamma_2 - 3\beta)t$ and there are no edges between X and Y, then* (6.2) *holds in G.*

Proof. Without loss of generality, assume that $|X| = (\gamma_1 - 3\beta)t$ and $|Y| = (\gamma_2 - 3\beta)t$. Let $U' = \cup_{U_i \in X} U_i$ and $V' = \cup_{V_i \in Y} V_i$. We have

$$(\gamma_1 - 4\beta)n < (\gamma_1 - 3\beta)t\ell = |X|\ell = |U'| \leq (\gamma_1 - 3\beta)n$$

and

$$(\gamma_2 - 4\beta)n < (\gamma_2 - 3\beta)t\ell = |Y|\ell = |V'| \leq (\gamma_2 - 3\beta)n.$$

Since there is no edge between X and Y we have $e_{G'}(U', V') = 0$. Consequently $e_G(U', V') \leq e_{G'}(U', V') + d|U'||V'| + 2\epsilon n|U'| < dk_1 sk_2 s$. By adding at most $4\beta k_1 s$ vertices to U' and $4\beta k_2 s$ vertices to V', we obtain two subsets of size $k_1 s$ and $k_2 s$ respectively, with at most $dk_1 sk_2 s + 4\beta k_1 sk_2 s + 4\beta k_1 sk_2 s < \alpha k_1 sk_2 s$ edges, and thus (6.2) holds in G. □

For the rest of this proof, we suppose that (6.2) does not hold in G.

Claim 6.3.3. *G_r contains a perfect matching.*

Proof. Let M be a maximum matching of G_r. After relabeling indices if necessary, we may assume that $M = \{U_i V_i : i \in [k], k \leq t\}$. If M is not perfect, let $x \in \mathcal{U}$ and $y \in \mathcal{V}$ be vertices which are unsaturated by M. Then the neighborhood $N(x)$ is a subset of $V(M)$, otherwise we can enlarge M by adding an edge xz for any $z \in N(x) \setminus V(M)$. We have $N(y) \subseteq V(M)$ for the same reason. Now let $I = \{i : V_i \in N(x)\}$ and $J = \{j : U_j \in N(y)\}$. If $I \cap J \neq \emptyset$; that is, there exists i such xV_i and yU_i are both edges, then we can obtain a larger matching by replacing $U_i V_i$ in M by xV_i and yU_i. Otherwise, assume that $I \cap J = \emptyset$. Since $|I| \geq (\gamma_1 - 2\beta)t$ and $|J| \geq (\gamma_2 - 2\beta)t$ and (6.2) does not hold in G, then by the contrapositive of Claim 6.3.2 there exists an edge between $\{U_i : i \in I\}$ and $\{V_j : j \in J\}$. This implies that there exist $i \neq j$ such that xV_i, $U_i V_j$, and yU_j are edges. Replacing $U_i V_i$, $U_j V_j$ in M by xV_i, $U_i V_j$ and yU_j, we obtain a larger matching, contradicting the maximality of M.

□

By Claim 6.3.3 we assume that $U_i \sim V_i$ for all $i \in [t]$. If each ϵ-regular pair (U_i, V_i) is also super-regular and s divides ℓ, then the Blow-up Lemma (Lemma 3.0.11) guarantees that $G'[U_i, V_i]$ can be tiled with $K_{s,s}$ (since $K_{\ell,\ell}$ can be tiled with $K_{s,s}$). If we also know that $U_0 = V_0 = \emptyset$, then we obtain a

$K_{s,s}$-tiling of G. Otherwise we do the following steps (details of these steps are given next). *Step 1*: For each $i \geq 1$, we move vertices from U_i to U_0 and from V_i to V_0 so that each remaining vertex in (U_i, V_i) has at least $(d-2\epsilon)\ell$ neighbors. *Step 2:* We eliminate U_0 and V_0 by removing copies of $K_{s,s}$, each of which contains at most one vertex of $U_0 \cup V_0$. *Step 3*: We make sure that for each $i \geq 1$, $|U_i| = |V_i| > (1-d)\ell$ and $|U_i|$ is divisible by s. Finally we apply the Blow-up Lemma to each (U_i, V_i) (which is still super-regular) to finish the proof. Note that we always refer to the clusters as $U_i, V_i, i \geq 0$ even though they may gain or lose vertices during the process.

Step 1. For each $i \geq 1$, we remove all $u \in U_i$ such that $\deg(u, V_i) < (d-\epsilon)\ell$ and all $v \in V_i$ such that $\deg(v, U_i) < (d-\epsilon)\ell$. Fact 3.0.6 (with $k=1$) guarantees that the number of removed vertices is at most $\epsilon\ell$. We then remove more vertices from either U_i or V_i to make sure U_i and V_i still have the same number of vertices. All removed vertices are added to U_0 and V_0. As a result, we have $|U_0| = |V_0| \leq 2\epsilon n$.

Step 2. This step implies that a vertex in U_0, V_0 can be viewed as a vertex in U_i or V_i for some $i \geq 1$. For a vertex $x \in V(G)$ and a cluster C, we say x is adjacent to C, denoted $x \sim C$, if $\deg_G(x, C) \geq d\ell$. We claim that at present, each vertex in U is adjacent to at least $(\gamma_1 - 2\beta)t$ clusters. If this is not true for some $u \in U$, then we obtain a contradiction

$$(\gamma_1 - \beta)n \leq \deg_G(u) \leq (\gamma_1 - 2\beta)t\ell + d\ell t + 2\epsilon n < (\gamma_1 - 3\beta/2)n.$$

Likewise, each vertex in V is adjacent to at least $(\gamma_2 - 2\beta)t$ clusters. Assign an arbitrary order to the vertices in U_0. For each $u \in U_0$, we pick some V_i adjacent to u. The selection of V_i is arbitrary, but no V_i is selected more than $\frac{d\ell}{6s}$ times. Such V_i exists even for the last vertex of U_0 because $|U_0| \leq 2\epsilon n < (\gamma_1 - 2\beta)t\frac{d\ell}{6s}$. For each $u \in U_0$ and its corresponding V_i, we remove a copy of $K_{s,s}$ containing u,

s vertices in V_i, and $s-1$ vertices in U_i. Such a copy of $K_{s,s}$ can always be found even if u is the last vertex in U_0 because (U_i, V_i) is ϵ-regular and $\deg_G(u, V_i) \geq d\ell > \epsilon\ell + \frac{d\ell}{6s}s$ thus Fact 3.0.6 (with $k = s-1$) allows us to choose $s-1$ vertices from U_i and s vertices from $N(u) \cap V_i$ to complete the copy of $K_{s,s}$. As a result, U_i now has one more vertex than V_i, so one may view this process as moving u to U_i. We repeat this process for all $v \in V_0$ as well. By the end of this step, we have $U_0 = V_0 = \emptyset$, and each U_i, V_i, $i \geq 1$ contains at least $\ell - \epsilon\ell - d\ell/3$ vertices (for example, U_i may have lost $\frac{d\ell(s-1)}{6s}$ vertices because of U_0 and $d\ell/6$ vertices because of V_0). As a result, we have $\delta(G[U_i, V_i]) \geq (\frac{2d}{3} - 2\epsilon)\ell$ for all $i \geq 1$. Note that the sizes of U_i and V_i may currently be different.

Step 3. We want to show that for any $i \neq j$, there is a path $U_i V_{i_1} U_{i_1} \dots V_{i_a} U_{i_a} V_j U_j$ (resp. $V_i U_{i_1} V_{i_1} \dots U_{i_a} V_{i_a} U_j V_j$) for some $0 \leq a \leq 2$. If such a path exists, then for each i_b, $1 \leq b \leq a+1$ (assume that $i = i_0$ and $j = i_{a+1}$), we may remove a copy of $K_{s,s}$ containing one vertex from $U_{i_{b-1}}$, s vertices from V_{i_b}, and $s-1$ vertices from U_{i_b}. This removal reduces the size of U_i by one, increases the size of U_j by one but does not change the sizes of other clusters (all modulo s). We may therefore adjust the sizes of U_i and V_i (for $i \geq 1$) such that $|U_i| = |V_i|$ and $|U_i|$ is divisible by s. To do this we will need at most $2t$ paths: (i) Let $r := \lfloor \frac{n}{t} \rfloor \bmod s$. (ii) Pair up the current biggest set U_i and current smallest set U_j and move vertices from U_i to U_j until one of the sets has exactly $\lfloor \frac{n}{t} \rfloor - r$ elements. (iii) Repeat this process until all but one set in $\mathcal{U}$ has exactly $\lfloor \frac{n}{t} \rfloor - r$ elements (there will be one set, say U_t, with as many as $(t-1)^2$ extra vertices) (iv) Do the same for the clusters in $\mathcal{V}$.

Now we show how to find this path from U_1 to U_2. First, if $U_1 \sim V_2$, then $U_1 V_2 U_2$ is a path. Let $I = \{i : U_1 \sim V_i\}$ and $J = \{i : U_i \sim V_2\}$. If there exists $i \in I \cap J$, then we find a path $U_1 V_i U_i V_2 U_2$. Otherwise $I \cap J = \emptyset$. Since both $|I| \geq (\gamma_1 - 2\beta)t$ and $|J| \geq (\gamma_2 - 2\beta)t$, Claim 6.3.2 guarantees that there exists

$i \in I$ and $j \in J$ such that $U_i \sim V_j$. We thus have a path $U_1V_iU_iV_jU_jV_2U_2$. Note that in this step we require that a cluster is contained in at most $\frac{d\ell}{3s}$ paths. This restriction has little impact on the arguments above: we have $|I| > (\gamma_1 - 3\beta)t$ and $|J| > (\gamma_2 - 3\beta)t$ instead, still satisfying the conditions of Claim 6.3.2.

Now $U_0 = V_0 = \emptyset$, and for all $i \geq 1$, $|U_i| = |V_i|$ is divisible by s. Let $\mathcal{K}$ be the union of all vertices in existing copies of $K_{s,s}$ and note that,

$$|U_i \setminus \mathcal{K}| = |V_i \setminus \mathcal{K}| \geq \ell - \epsilon\ell - 2d\ell/3,$$

which implies $\delta(G[U_i, V_i]) \geq (\frac{d}{3} - 2\epsilon)\ell \geq \frac{d}{4}\ell$ for $i \geq 1$. Thus Fact 3.0.7 implies that each pair (U_i, V_i) is $(2\epsilon, \frac{d}{4})$-super-regular. Applying the Blow-up Lemma to each (U_i, V_i), we find the desired $K_{s,s}$-tiling.

□

6.4 Extremal Case

Given $s \geq 2$ and $\lambda \in (0, \frac{1}{2})$, let $\alpha > 0$ be sufficiently small. Let $G[U, V]$ be a balanced bipartite graph on $2n = 2ms$ vertices for sufficiently large n. Without loss of generality suppose $\delta_V \geq \delta_U$ and note that $\delta_U \geq \lambda n$. Suppose G is edge minimal with respect to the condition $\delta_U + \delta_V \geq n + c$, and that G satisfies the extremal condition with parameter α. Let k_1 be defined by $\delta_U = k_1 s + s + r$, where $0 \leq r \leq s - 1$ and let $k_2 s = n - k_1 s$.

The proof will split into cases depending on whether $k_1 \leq (1 - \frac{1}{2s})k_2$ or $k_1 > (1 - \frac{1}{2s})k_2$. When $k_1 > (1 - \frac{1}{2s})k_2$, we have $\delta_U + \delta_V \geq n + 3s - 5$. Since $\delta_U = k_1 s + s + r$, we have $\delta_V \geq k_2 s + 2s - 5 - r$. Since G is edge minimal we have $\delta_V = k_2 s + 2s - 5 - r$, and since $\delta_V \geq \delta_U$, we have $k_2 \geq k_1$. If $\delta_V = \delta_U$, then we have

$$\delta(G) \geq \frac{n + 3s - 5}{2} > \begin{cases} \frac{n}{2} + s - 2 & \text{if } m \text{ is even} \\ \frac{n+3s}{2} - 3 & \text{if } m \text{ is odd,} \end{cases}$$

which is solved in [51]. So we may suppose that $\delta_V > \delta_U$.

Claim 6.4.1. *If $k_2 = k_1$, then $r \leq \frac{s-6}{2}$ and consequently $\delta_V = k_2 s + 2s - 5 - r \geq k_2 s + s$. If $k_2 = k_1 + 1$, then $r \leq s - 3$ and consequently $\delta_V = k_2 s + 2s - 5 - r \geq k_2 s + s - 2$.*

Proof. Both statements are implied the following inequality: $k_2 s + 2s - 5 - r = \delta_V > \delta_U = k_1 s + s + r$. □

When $k_1 \leq (1 - \frac{1}{2s})k_2$, we will show in Theorem 6.1.13 that a smaller degree suffices to tile G with $K_{s,s}$. So Theorem 6.1.13 provides the second half of the proof of Theorem 6.1.8.

6.4.1 Pre-processing

Let $U_2' = U \setminus U_1'$ and $V_1' = V \setminus V_2'$. Let

$$U_1 = \{x \in U : \deg(x, V_2') < \alpha^{1/3} k_1 s\}, \quad V_2 = \{x \in V : \deg(x, U_1') < \alpha^{1/3} k_2 s\},$$

$$U_2 = \{x \in U : \deg(x, V_1') < \alpha^{1/3} k_1 s \vee \deg(x, V_2') > (1 - \alpha^{1/3}) k_2 s\},$$

$$V_1 = \{x \in V : \deg(x, U_2') < \alpha^{1/3} k_2 s \vee \deg(x, U_1') > (1 - \alpha^{1/3}) k_1 s\},$$

$$U_0 = U \setminus (U_1 \cup U_2), \text{ and } V_0 = V \setminus (V_1 \cup V_2).$$

Claim 6.4.2. (i) $k_1 s - \alpha^{2/3} k_2 s \leq |U_1|, |V_1| \leq k_1 s + \alpha^{2/3} k_1 s$

(ii) $k_2 s - \alpha^{2/3} k_1 s \leq |U_2|, |V_2| \leq k_2 s + \alpha^{2/3} k_2 s$

(iii) $|U_0|, |V_0| \leq \alpha^{2/3} n$

(iv) $\delta(U_0, V_1) \geq \alpha^{1/3} k_1 s - \alpha^{2/3} k_2 s$, $\delta(U_0, V_2) \geq \alpha^{1/3} k_1 s - \alpha^{2/3} k_1 s$

(v) $\delta(V_0, U_1) \geq \alpha^{1/3} k_2 s - \alpha^{2/3} k_2 s$, $\delta(V_0, U_2) \geq \alpha^{1/3} k_2 s - \alpha^{2/3} k_1 s$

(vi) $\delta(G[U_i, V_i]) \geq k_i s - \alpha^{1/3} k_i s - \alpha^{2/3} k_{3-i} s \geq (1 - 2\alpha^{1/3}) k_i s$

(vii) $\Delta(U_1, V_2) \leq 2\alpha^{1/3}k_1 s,\ \Delta(V_2, U_1) \leq 2\alpha^{1/3}k_2 s$

Proof. We have

$$\alpha^{1/3}k_1 s|U_1' \setminus U_1| \leq e(U_1' \setminus U_1, V_2') \leq e(U_1', V_2') \leq \alpha k_1 s k_2 s$$

which gives $|U_1' \setminus U_1| \leq \alpha^{2/3}k_2 s$ and thus $|U_1| \geq k_1 s - \alpha^{2/3}k_2 s$.

Also

$$\alpha^{1/3}k_2 s|V_2' \setminus V_2| \leq e(V_2' \setminus V_2, U_1') \leq e(V_2', U_1') \leq \alpha k_1 s k_2 s$$

which gives $|V_2' \setminus V_2| \leq \alpha^{2/3}k_1 s$ and thus $|V_2| \geq k_2 s - \alpha^{2/3}k_1 s$.

Since $e(U_1', V_2') \leq \alpha k_1 s k_2 s$, we have $e(U_2', V_2') \geq k_2 s k_2 s - \alpha k_1 s k_2 s$ and $e(U_1', V_1') \geq k_1 s k_1 s - \alpha k_1 s k_2 s$. Thus

$$\alpha^{1/3}k_2 s|U_2' \setminus U_2| \leq \bar{e}(U_2', V_2') \leq \alpha k_1 s k_2 s$$

which gives $|U_2' \setminus U_2| \leq \alpha^{2/3}k_1 s$ and thus $|U_2| \geq k_2 s - \alpha^{2/3}k_1 s$.

Also

$$\alpha^{1/3}k_1 s|V_1' \setminus V_1| \leq \bar{e}(U_1', V_1') \leq \alpha k_1 s k_2 s$$

which gives $|V_1' \setminus V_1| \leq \alpha^{2/3}k_2 s$ and thus $|V_1| \geq k_1 s - \alpha^{2/3}k_2 s$.

Putting these results together we have $|U_0|, |V_0| \leq \alpha^{2/3}n$, $|U_1|, |V_1| \leq k_1 s + \alpha^{2/3}k_1 s$, and $|U_2|, |V_2| \leq k_2 s + \alpha^{2/3}k_2 s$.

By the definition of U_1, U_2, V_1, V_2 and the lower bounds on their sizes, we have $\delta(U_0, V_1) \geq \alpha^{1/3}k_1 s - \alpha^{2/3}k_2 s$, $\delta(U_0, V_2) \geq \alpha^{1/3}k_1 s - \alpha^{2/3}k_1 s$, $\delta(V_0, U_1) \geq \alpha^{1/3}k_2 s - \alpha^{2/3}k_2 s$, and $\delta(V_0, U_2) \geq \alpha^{1/3}k_2 s - \alpha^{2/3}k_1 s$. By the definition of U_1, V_2 and the upper bounds on their sizes we have $\Delta(U_1, V_2) \leq 2\alpha^{1/3}k_1 s$ and $\Delta(V_2, U_1) \leq 2\alpha^{1/3}k_2 s$.

$\square$

6.4.2 Idea of the Proof

We start with the partition given in Section 6.4.1 and we call U_0 and V_0 the *exceptional* sets. Let $i \in \{1, 2\}$. We will attempt to update the partition by moving a constant number (depending only on s) of *special* vertices between U_1 and U_2, denote them by X, and *special* vertices between V_1 and V_2, denote them by Y, as well as partitioning the exceptional sets as $U_0 = U_0^1 \cup U_0^2$ and $V_0 = V_0^1 \cup V_0^2$. Let U_1^*, U_2^*, V_1^* and V_2^* be the resulting sets after moving the special vertices. Suppose u is a special vertex in the set U_1^*. The degree of u in V_1^* may be small, but u will have a set of at least s neighbors in V_1^* which are disjoint from the neighbors of any other special vertex in U_1^*. Furthermore, these neighbors of u in V_1^* will have huge degree in U_1^*, so it will be easy to incorporate each special vertex into a unique copy of $K_{s,s}$.

Our goal is to obtain two graphs, $G_1 := G[U_1^* \cup U_0^1, V_1^* \cup V_0^1]$ and $G_2 := [U_2^* \cup U_0^2, V_2^* \cup V_0^2]$ so that G_1 satisfies

$$|U_1^* \cup U_0^1| = \ell_1 s, \;\; |V_1^* \cup V_0^1| = \ell_1 s$$

and G_2 satisfies

$$|U_2^* \cup U_0^2| = \ell_2 s, \;\; |V_2^* \cup V_0^2| = \ell_2 s,$$

for some positive integers ℓ_1, ℓ_2. We tile G_1 as follows. We incorporate all of the special vertices into copies of $K_{s,s}$. We now deal with the exceptional vertices: Claim 6.4.2 gives $|U_0|, |V_0| \le \alpha^{2/3} n$ and $\delta(U_0, V_i), \delta(V_0, U_i) \gg s\alpha^{2/3} n$, so they may greedily be incorporated into unique copies of $K_{s,s}$. Then we are left with two balanced "almost complete" graphs, which can be easily tiled.

So throughout the proof, if we can make, say $|U_1^* \cup U_0^1|$ and $|V_1^* \cup V_0^1|$ equal and divisible by s, we simply state that "we are done."

6.4.3 Preliminary Lemma's

In this section we give some lemmas which will be used in the proof of Theorems 6.1.8 and 6.1.13. Recall that in each of those theorems we suppose $k_2 s \geq k_1 s \geq \lambda n$.

Lemma 6.4.3 (Zhao [51], Fact 5.3)**.** *Let F be an A,B-bigraph with $\delta := \delta(A,B)$ and $\Delta := \Delta(B,A)$ Then F contains f_h vertex disjoint h-stars from A to B, and g_h vertex disjoint h-stars from B to A (the stars from A to B and those from B to A need not be disjoint), where*

$$f_h \geq \frac{(\delta - h + 1)|A|}{h\Delta + \delta - h + 1}, \quad g_h \geq \frac{\delta|A| - (h-1)|B|}{\Delta + h\delta - h + 1}.$$

Lemma 6.4.4. *Let $G[A,B]$ be a bipartite graph with $|B| = \ell s + b$ for some positive integers ℓ and b. Let $0 \leq x \leq s-1$ and let γ be a small constant such that $\alpha^{1/3} \ll \gamma \ll \frac{1}{2s}$. If $b < \frac{1}{\gamma}$ and*

(i) *$\delta(B,A) \geq s - x$, $\Delta(A,B) \leq 2\alpha^{1/3}k_2 s$, and $|B| \geq \alpha^{1/6}|A|$*

then there are at least b vertex disjoint $(s-x)$-stars from B to A.

Suppose $k_2 s + \alpha^{2/3}k_2 s \geq |A|, |B| \geq k_1 s - \alpha^{2/3}k_2 s$. If

(ii) *$\delta(A,B) \geq s - 1 + b$ and $k_1 > (1 - \frac{1}{2s})k_2$,*

then there are at least b vertex disjoint s-stars from B to A. If $b < \frac{1}{\gamma}$ and

(iii) *$\delta(A,B) \geq s$, $k_1 > (1 - \frac{1}{2s})k_2$, and $\Delta(B,A) \leq 2\alpha^{1/3}k_2 s$ or*

(iv) *$\delta(A,B) \geq d$, $|A| \geq \frac{s-1/2}{d}|B|$, and $\Delta(B,A) \leq 2\alpha^{1/3}k_2 s$,*

then there are at least b vertex disjoint s-stars from B to A. Furthermore, if $b \geq \frac{1}{\gamma}$ and

(v) $\delta(A,B) \geq b/4$ *and* $\Delta(B,A) < 2\alpha^{1/3}k_2 s$ *or*

(vi) $\delta(B,A) \geq b/4$ *and* $\Delta(A,B) < 2\alpha^{1/3}k_2 s$,

then there are at least b *vertex disjoint* s*-stars from* B *to* A.

Proof. *(i)* Suppose $b < \frac{1}{\gamma}$, $\delta(B,A) \geq s - x$, $\Delta(A,B) \leq 2\alpha^{1/3}k_2 s$, and $|B| \geq \alpha^{1/6}|A|$. Let $\mathcal{S}_B$ be the maximum set of vertex disjoint $(s-x)$-stars from B to A and let $f_{s-x} = |\mathcal{S}_B|$. By Lemma 6.4.3, we have

$$f_{s-x} \geq \frac{|B|}{2(s-x)\alpha^{1/3}k_2 s + 1} \geq \frac{\alpha^{1/6}}{3s\alpha^{1/3}} \geq \frac{1}{\gamma} \geq b$$

(ii) Suppose $\delta(A,B) \geq s - 1 + b$ and $k_1 > (1 - \frac{1}{2s})k_2$. Let $\mathcal{S}_A$ be a maximum set of vertex disjoint s-stars with centers $C \subseteq B$ and leaves $L \subseteq A$. Suppose $|C| \leq b - 1$. Then

$$\begin{aligned} s(|A| - |L|) \leq (s - 1 + b - |C|)(|A| - |L|) &\leq e(B \setminus C, A \setminus L) \\ &\leq (s-1)(|B| - |C|), \end{aligned}$$

which implies

$$s(k_1 s - \alpha^{2/3}k_2 s) \leq (s-1)(k_2 s + \alpha^{2/3}k_2 s) + s|L| - (s-1)|C|.$$

Thus $sk_1 \leq (s - \frac{1}{2})k_2$, contradicting the fact that $k_1 > (1 - \frac{1}{2s})k_2$.

(iii) Suppose $b < \frac{1}{\gamma}$, $\delta(A,B) \geq s$, $k_1 > (1 - \frac{1}{2s})k_2$, and $\Delta(B,A) \leq 2\alpha^{1/3}k_2 s$. Let $\mathcal{S}_A$ be the maximum set of vertex disjoint s-stars from A to B and let $g_s = |\mathcal{S}_A|$. By Lemma 6.4.3, we have

$$\begin{aligned} g_s \geq \frac{s|A| - (s-1)|B|}{2\alpha^{1/3}k_2 s + s^2 - s + 1} &\geq \frac{s(k_1 s - \alpha^{2/3}k_2 s) - (s-1)(k_2 s + \alpha^{2/3}k_2 s)}{3\alpha^{1/3}k_2 s} \\ &\geq \frac{1}{12\alpha^{1/3}} \geq \frac{1}{\gamma} \geq b \end{aligned}$$

Where the third inequality holds since $sk_1 s > (s - \frac{1}{2})k_2 s$.

(iv) Suppose $b < \frac{1}{\gamma}$, $\delta(A,B) \geq d$, $|A| \geq \frac{s-1/2}{d}|B|$, and $\Delta(B,A) \leq 2\alpha^{1/3}k_2 s$. Let $\mathcal{S}_B$ be the maximum set of vertex disjoint s-stars from B to A and let $g_s = |\mathcal{S}_B|$. By Lemma 6.4.3, we have

$$g_s \geq \frac{d|A| - (s-1)|B|}{2\alpha^{1/3}k_2 s + sd - s + 1} \geq \frac{|B|/2}{3\alpha^{1/3}k_2 s} \geq \frac{\lambda}{6\alpha^{1/3}} \geq \frac{1}{\gamma} \geq b$$

(v) Suppose $b \geq \frac{1}{\gamma}$, $\delta(A,B) \geq b/4$ and $\Delta(B,A) < 2\alpha^{1/3}k_2 s$. Let $\mathcal{S}_B$ be the maximum set of vertex disjoint s-stars from B to A and let $g_s = |\mathcal{S}_B|$. By Lemma 6.4.3, we have

$$g_s \geq \frac{\frac{b}{4}|A| - (s-1)|B|}{2\alpha^{1/3}k_2 s + s\frac{b}{4} - s + 1} \geq \frac{b\lambda/4 - (s-1)}{3\alpha^{1/3}} \geq b$$

(vi) Suppose $b \geq \frac{1}{\gamma}$, $\delta(B,A) \geq b/4$ and $\Delta(A,B) < 2\alpha^{1/3}k_2 s$. Let $\mathcal{S}_B$ be the maximum set of vertex disjoint s-stars from B to A and let $f_s = |\mathcal{S}_B|$. By Lemma 6.4.3, we have

$$f_s \geq \frac{(\frac{b}{4} - s + 1)|B|}{2s\alpha^{1/3}k_2 s + \frac{b}{4} - s + 1} \geq \frac{(\frac{b}{4} - s + 1)\lambda}{3\alpha^{1/3}} \geq b$$

□

Lemma 6.4.5. *Let $G[A,B]$ be a bipartite graph with $|A| = \ell_1 s + a$ and $|B| = \ell_2 s + b$ such that $1 \leq b \leq s-1$. Suppose further that $k_2 s + \alpha^{2/3}k_2 s \geq |A|, |B| \geq k_1 s - \alpha^{2/3}k_2 s$ and $\Delta(A,B), \Delta(B,A) \leq 2\alpha^{1/3}k_2 s$. If*

(i) *$a \geq 1$ and $\delta(A,B) + \delta(B,A) \geq 2s - 3 + a + b$ or*

(ii) *$a = 0$ and $\delta(A,B) + \delta(B,A) \geq 2s - 2 + b$,*

then there is a set $\mathcal{S}_A$ of a vertex disjoint s-stars from A to B and a set $\mathcal{S}_B$ of b vertex disjoint s-stars from B to A such that the stars in $\mathcal{S}_A$ are disjoint from the stars in $\mathcal{S}_B$.

Proof. Let γ be a real number such that $\alpha^{1/3} \ll \gamma \ll \frac{1}{2s}$.

Case 1 $a > \frac{1}{\gamma}$. Suppose first $\delta(B, A) \geq \frac{1}{2}(2s - 3 + a + b)$. In this case we apply Lemma 6.4.4(vi) to get a set of b vertex disjoint s-stars with centers $C \subseteq B$ and leaves $L \subseteq A$. Then since $\delta(B, A \setminus L) \geq \frac{1}{2}(2s - 3 + a + b) - bs > \frac{a}{4}$ we apply Lemma 6.4.4(v) to get a set of a vertex disjoint s-stars from $A \setminus L$ to $B \setminus C$. Now suppose $\delta(A, B) > \frac{1}{2}(2s - 3 + a + b)$. As before, we apply Lemma 6.4.4(v) to get a set of b vertex disjoint s-stars with centers $C \subseteq B$ and leaves $L \subseteq A$. Then since $\delta(A, B \setminus C) > \frac{1}{2}(2s - 3 + a + b) - b > \frac{a}{4}$ we apply Lemma 6.4.4(vi) to get a set of a vertex disjoint s-stars from $A \setminus L$ to $B \setminus C$.

Case 2 $1 \leq a \leq \frac{1}{\gamma}$. Suppose first that $\delta(B, A) \geq s - 1 + a$. We apply Lemma 6.4.4(ii) to get a set of a vertex disjoint s-stars with centers $C \subseteq A$ and leaves $L \subseteq B$. We still have $\delta(B \setminus N(C), A \setminus C) \geq s - 1 + a$ and $|B \setminus N(C)| \geq |B| - \frac{2\alpha^{1/3}}{\gamma} k_2 s \geq \alpha^{1/6}|A|$, thus we can apply Lemma 6.4.4(i) to get a set of b vertex disjoint s-stars from $B \setminus N(C)$ to $A \setminus C$. Now suppose $\delta(A, B) \geq s + b$. We apply Lemma 6.4.4(ii) to get a set of b vertex disjoint s-stars with centers $C \subseteq B$ and leaves $L \subseteq A$. We still have $\delta(A \setminus L, B \setminus C) \geq s + b - b = s$ so we apply Lemma 6.4.4(i) to get a vertex disjoint s-stars from $A \setminus L$ to $B \setminus C$.

Case 3 $a = 0$. We have $\delta(A, B) + \delta(B, A) \geq 2s - 2 + b \geq 2s - 1$ and thus $\delta(A, B) \geq s$ or $\delta(B, A) \geq s$. In either case we can apply Lemma 6.4.4(i) or (iii) to get a set of b vertex disjoint s-stars from B to A.

□

Lemma 6.4.6. *Suppose $|U_0| \geq s$. Let $V_1' \subseteq V_1$ and $V_2' \subseteq V_2$ such that $\delta(V_1', U_0) + \delta(V_2', U_0) \geq |U_0| + s$. If $|V_1'| \geq \frac{n}{8}$ and $|V_2'| \geq \frac{n}{8}$, then for any $0 \leq b \leq s$, there is a $K_{s,s} =: K$ with s vertices in U_0, b vertices in V_1 and $s - b$ vertices in V_2.*

For a proof see Chapter 5 Claim 5.2.8.

6.4.4 $k_2 \gg k_1$: Proof of Theorem 6.1.13

In this section we prove Theorem 6.1.13, which at the same time proves Theorem 6.1.8 when $k_1 \leq (1 - \frac{1}{2s})k_2$. Let G be a graph which satisfies the extremal condition and for which $k_1 \leq (1 - \frac{1}{2s})k_2$. Recall the bounds from Claim 6.4.2, specifically $k_1 s - \alpha^{2/3}k_2 s \leq |U_1|, |V_1| \leq k_1 s + \alpha^{2/3}k_1 s$,
$k_2 s - \alpha^{2/3}k_1 s \leq |U_2|, |V_2| \leq k_2 s + \alpha^{2/3}k_2 s$, and $|U_0|, |V_0| \leq \alpha^{2/3}n$. The fact that $\delta_U + \delta_V \geq n$ implies

$$\delta(V_1, U_2) \geq \delta_V - |U_0 \cup U_1| \geq (k_2 - k_1 - 2\alpha^{2/3}k_1)s \geq (\frac{1}{2s}k_2 - 2\alpha^{2/3}k_1)s > \frac{1}{4s}k_2 s. \tag{6.3}$$

First we prove Theorem 6.1.13.

Proof. Note that $s - 2\lceil\sqrt{s}\rceil + c(s) + 1 \geq 0$ with equality if and only if $s = 2$, so d is defined for all $s \geq 2$. Let $\alpha^{1/3} \ll \gamma \ll \frac{1}{2s}$. Let ℓ_1 be maximal so that $|U_1| \geq \ell_1 s$ and $|V_0 \cup V_1| \geq \ell_1 s$. Let $y := |U_1| - \ell_1 s$ and $z := |V_0 \cup V_1| - \ell_1 s$. We note that $n + 3s - 5 \geq n + 2s - 2\lceil\sqrt{s}\rceil + d + c(s)$ with equality if and only if $s = 2$. So for this proof we will assume $\delta_U + \delta_V \geq n + 2s - 2\lceil\sqrt{s}\rceil + d + c(s)$ with one exception that we point out.

Claim 6.4.7. *If there exists ℓ such that $|V_0 \cup V_1| \geq \ell s$ and $|U_1| \leq \ell s$, then G can be tiled with $K_{s,s}$.*

Proof. Suppose there exists such an ℓ. By the choice of ℓ_1, we can assume $|U_1| \leq (\ell_1 + 1)s$ and $|V_0 \cup V_1| \geq (\ell_1 + 1)s$. By (6.3) we have $\delta(V_1, U_2) > \frac{1}{4s}k_2 s \geq 2s\alpha^{2/3}n$ and thus we can greedily choose a set of $z - s$ vertex disjoint s-stars from V_1 to U_2 with centers C_V and leaves L_U. Let $V_1' := V_1 \setminus C_V$ and $U_2' := U_2 \setminus L_U$, since $\delta(V_1', U_2') \geq \frac{1}{8s}k_2 s$ we may apply Lemma

6.4.3 to the graph induced by U_2' and V_1' to get a set of $s-y$ vertex disjoint s-stars from U_2' to V_1'. We move the centers of the stars giving $|U_1|+(s-y)=(\ell_1+1)s=|V_0\cup V_1|-(z-s)$ and we are done. $\square$

If $z\geq s$, then by the maximality of ℓ_1 we have $y<s$ and thus we can apply Claim 6.4.7 to finish. If $y=0$, then we can also apply Claim 6.4.7 to finish. So for the rest of the proof, suppose that $0\leq z\leq s-1$ and $1\leq y$. Our goal is to show that there exists a set $\mathcal{S}_U$ of vertex disjoint $(s-x)$-stars from U_1 to V_2 such that $|V_0\cup V_1|-x|\mathcal{S}_U|\geq|U_1|-|\mathcal{S}_U|=\ell_1 s$ and a set $\mathcal{T}_V$ of vertex disjoint s-stars from V_1 to U_2 so that $|V_0\cup V_1|-x|\mathcal{S}_U|-|\mathcal{T}_V|=\ell_1 s$ for some $0\leq x\leq s-1$. Since $\delta_U+\delta_V\geq n+2s-2\lceil\sqrt{s}\rceil+d+c(s)$, we have

$$\begin{aligned}\delta(U_1,V_2)+\delta(V_2,U_1)&\geq n+2s-2\lceil\sqrt{s}\rceil+d+c(s)-|V_0\cup V_1|-|U_0\cup U_2|\\&\geq 2s-2\lceil\sqrt{s}\rceil+d+c(s)+y-z\end{aligned}\tag{6.4}$$

Case 1 $|U_1|-|V_0\cup V_1|>0$.

Case 1.1 $y\geq\frac{1}{\gamma}$. We have

$$\begin{aligned}\delta(U_1,V_2)+\delta(V_2,U_1)&\geq 2s-2\lceil\sqrt{s}\rceil+d+c(s)+y-z\\&\geq y+s-2\lceil\sqrt{s}\rceil+d+c(s)+1\end{aligned}$$

and thus there are two cases. Either $\delta(U_1,V_2)\geq\frac{1}{2}(y+s-2\lceil\sqrt{s}\rceil+d+c(s)+1)$ and we apply Lemma 6.4.4(vi) to get y vertex disjoint s-stars from U_1 to V_2 or $\delta(V_2,U_1)>\frac{1}{2}(y+s-2\lceil\sqrt{s}\rceil+d+c(s)+1)$ and we apply Lemma 6.4.4(v) to get y vertex disjoint s-stars from U_1 to V_2. We move the centers from U_1 to U_2 to make $|U_1|=\ell_1 s$. Then we move vertices from $V_0\cup V_1$ to V_2 to make $|V_0\cup V_1|=\ell_1 s$.

Case 1.2 $y<\frac{1}{\gamma}$.

Case 1.2.1. $\delta(U_1,V_2)\geq s$. Apply Lemma 6.4.4(i) with $x=0$ to get y vertex disjoint s-stars from U_1 to V_2.

Case 1.2.2. $\delta(U_1, V_2) \leq s-1$. By (6.4) we have $\delta(V_2, U_1) \geq 2s - 2\lceil\sqrt{s}\rceil + d + c(s) + y - z - (s-1) = s - 2\lceil\sqrt{s}\rceil + d + c(s) + 1 + y - z \geq d+1$. Since $k_2 \geq (s-d)k_1$ and thus $|V_2| \geq (s - \frac{1}{2} - d)|U_1| \geq \frac{s-\frac{1}{2}}{d+1}|U_1|$, we can apply Lemma 6.4.4(iv) to get y vertex disjoint s-stars from U_1 to V_2.

Case 2. $|U_1| - |V_0 \cup V_1| \leq 0$. In this case we have $y \leq z$. Rearranging (6.4) gives

$$\delta(U_1, V_2) + \delta(V_2, U_1) \geq 2s - 2\lceil\sqrt{s}\rceil + d + c(s) - (z - y). \tag{6.5}$$

Also since $k_1 \leq \frac{k_2}{s-d}$, we have

$$\begin{aligned}\delta(V_1, U_2) \geq \delta_V - |U_0 \cup U_1| \geq (k_2 - k_1 - 2\alpha^{2/3}k_1)s &\geq (1 - \frac{1+2\alpha^{2/3}}{s-d})k_2 s \\ &\geq \frac{s-d-1-2\alpha^{2/3}}{(s-d)(1+\alpha^{2/3})}|U_2| \\ &\geq \frac{s-d-1-\alpha^{1/3}}{s-d}|U_2| \quad (6.6)\end{aligned}$$

If $\delta_U + \delta_V \geq n + 3s - 5$, then (6.5) gives $\delta(U_1, V_2) + \delta(V_2, U_1) \geq 2s - 3$ since $z - y \leq s - 2$. Thus we have $\delta(V_2, U_1) \geq s-1$ or $\delta(U_1, V_2) \geq s-1$. In either case we can get y vertex disjoint $(s-1)$-stars from U_1 to V_2 by Lemma 6.4.4(iii) or Lemma 6.4.4(i) with $x = 1$. For each $(s-1)$-star we choose a vertex from V_1 and $(s-1)$-vertices in U_2, which is possible by (6.6) and $z \geq y$. So for the rest of the proof we assume $\delta_U + \delta_V \geq n + 2s - 2\lceil\sqrt{s}\rceil + d + c(s)$.

Case 2.1. $z - y \leq s - 2\lceil\sqrt{s}\rceil + c(s) + 1$.

Case 2.1.1. $\delta(U_1, V_2) \geq s-1$. We can get y vertex disjoint $(s-1)$-stars from U_1 to V_2 by Lemma 6.4.4(i) with $x = 1$. For each $(s-1)$-star we choose a vertex from V_1 and $(s-1)$-vertices in U_2, which is possible by (6.6) and $z \geq y$.

Case 2.1.2. $\delta(U_1, V_2) \leq s-2$. So (6.5) and the condition of Case 2.2.1. gives

$$\delta(V_2, U_1) \geq 2s - 2\lceil\sqrt{s}\rceil + d + c(s) - (s - 2\lceil\sqrt{s}\rceil + c(s) + 1) - (s-2) = d+1.$$

We can get y vertex disjoint s-stars from U_1 to V_2 by Lemma 6.4.4(iv) as in Case 1.2.2.

Case 2.2. $z - y \geq s - 2\lceil\sqrt{s}\rceil + c(s) + 2$. If $\delta(U_1, V_2) \geq s - 1$ or $\delta(V_2, U_1) \geq d + 1$, then we would be done as in the previous two cases. So suppose $\delta(U_1, V_2) \leq s - 2$ and $\delta(V_2, U_1) \leq d$. By (6.5), we have

$$\begin{aligned} s - 2 \geq s - x = \delta(U_1, V_2) &\geq 2s - 2\lceil\sqrt{s}\rceil + d + c(s) - (z - y) - \delta(V_2, U_1) \quad (6.7) \\ &\geq s - 2\lceil\sqrt{s}\rceil + c(s) + 2 \geq d + 1 \end{aligned}$$

for some $2 \leq x \leq s - d - 1$.

Let $\mathcal{S}_U$ be a set of y vertex disjoint $(s-x)$-stars from U_1 to V_2, which exists by Lemma 6.4.4(i). For each $(s-x)$-star in $\mathcal{S}_U$ we will choose $s - 1$ vertices from U_2 and x vertices from V_1 to complete a copy of $K_{s,s}$. Let u_1 be the center of a star in $\mathcal{S}_U$ and let $v_1^1, v_1^2, \ldots, v_1^x$ be a set of x vertices in $N(u_1) \cap V_1$. By (6.6), we have $|N(v_1^1, v_1^2, \ldots, v_1^x) \cap U_2| \geq \left(1 - \frac{x(1+\alpha^{1/3})}{s-d}\right)|U_2|$. Let $v_2^1, v_2^2, \ldots, v_2^{s-x}$ be a set of $s - x$ vertices in V_2. By Claim 6.4.2, we have $|N(v_2^1, v_2^2, \ldots, v_2^{s-x}) \cap U_2| \geq (1 - (s-x)\alpha^{1/3})|U_2|$. Thus

$$\begin{aligned} |N(v_1^1, v_1^2, \ldots, v_1^x, v_2^1, v_2^2, \ldots, v_2^{s-x}) \cap U_2| &\geq \left(1 - \frac{x(1+\alpha^{1/3})}{s-d} - (s-x)\alpha^{1/3}\right)|U_2| \\ &\geq \alpha|U_2| \end{aligned}$$

and we can choose x vertices from V_1 and $s - 1$ vertices from U_2 to turn each $s - x$ star into a copy of $K_{s,s}$.

Finally we must be sure that $|V_0 \cup V_1| - xy \geq \ell s$, i.e. $z \geq xy$. There are two cases.

Case 2.2.1. $1 \leq q \leq p$ and consequently $c(s) = 1$. By (6.7) and $\delta(V_2, U_1) \leq d$, we get

$$x + y \leq z - (s - 2\lceil\sqrt{s}\rceil + 1) \quad (6.8)$$

and thus

$$xy \le \left(\frac{z-(s-2\lceil\sqrt{s}\rceil+1)}{2}\right)^2 \le z.$$

The first inequality is by (6.8) and the arithmetic mean-geometric mean inequality. To verify the second inequality, let $F(z) = z - \left(\frac{z-(s-2\lceil\sqrt{s}\rceil+1)}{2}\right)^2$ and note $s-2\lceil\sqrt{s}\rceil+3 \le z \le s-1$. Using calculus, we see that F achieves a maximum at $s-2\lceil\sqrt{s}\rceil+3$, F is decreasing on the interval $[s-2\lceil\sqrt{s}\rceil+3, s-1]$ and $F(s-1) = s-1-(\lceil\sqrt{s}\rceil-1)^2 = p^2+q-1-p^2 \ge 0$.

Case 2.2.2. $q = 0$ or $p+1 \le q \le 2p$ and consequently $c(s) = 0$. By (6.7) and $\delta(V_2, U_1) \le d$, we get

$$x+y \le z-(s-2\lceil\sqrt{s}\rceil). \tag{6.9}$$

If $z = s-1$, then (6.9) gives $x+y \le 2\lceil\sqrt{s}\rceil-1$. Since $2\lceil\sqrt{s}\rceil-1$ is odd, we have

$$xy \le \left(\frac{2\lceil\sqrt{s}\rceil}{2}\right)\left(\frac{2\lceil\sqrt{s}\rceil-2}{2}\right) = \lceil\sqrt{s}\rceil(\lceil\sqrt{s}\rceil-1) \le s-1 = z$$

where the last inequality holds by the assumption of this case. So we may assume $z \le s-2$. So we have

$$xy \le \left(\frac{z-(s-2\lceil\sqrt{s}\rceil)}{2}\right)^2 \le z.$$

The first inequality holds by (6.9) and the arithmetic mean-geometric mean inequality. To verify the second inequality, let $F(z) = z - \left(\frac{z-(s-2\lceil\sqrt{s}\rceil)}{2}\right)^2$ and note $s-2\lceil\sqrt{s}\rceil+2 \le z \le s-2$. Using calculus, we see that F achieves a maximum at $s-2\lceil\sqrt{s}\rceil+2$, F is decreasing on the interval $[s-2\lceil\sqrt{s}\rceil+2, s-2]$ and $F(s-2) = s-2-(\lceil\sqrt{s}\rceil-1)^2$. When $q = 0$ we have $p \ge 2$, and thus $F(s-2) = s-2-(\lceil\sqrt{s}\rceil-1)^2 = p^2-2-(p^2-2p+1) = 2p-3 \ge 1$. When $q \ge p+1$, we have $F(s-2) = s-2-(\lceil\sqrt{s}\rceil-1)^2 = p^2+q-2-p^2 = q-2 \ge 0$.

□

6.4.5 $k_2 \approx k_1$: *Proof of Theorem 6.1.8*

In this section we prove Theorem 6.1.8 when $k_1 > (1 - \frac{1}{2s})k_2$. Recall that $k_1 \leq k_2$. We first give a proof when $s = 2$ since this is often a special case in the general argument. Also, the case $s = 2$ may be of independent interest considering Conjecture 6.1.7.

We start with a graph which satisfies the extremal condition after pre-processing. For $i = 1, 2$, let $U_i^M = \{u \in U_i : \deg(u, V_{3-i}) > \alpha^{1/3}n\}$ and $V_i^M = \{v \in V_i : \deg(v, U_{3-i}) > \alpha^{1/3}n\}$. We call these vertices *movable*. Note that $U_1^M = \emptyset = V_2^M$ by Claim 6.4.2.

s=2

Let γ be a real number such that $\alpha^{1/3} \ll \gamma \ll \frac{1}{2s}$. We assume that $n = 2m$ and $\delta_V > \delta_U$, thus $\delta_V \geq \frac{n}{2} + 1$. As a result

$$\forall v, v' \in V, |N(v) \cap N(v')| \geq 2 \tag{6.10}$$

Furthermore, since $\delta_V \geq \frac{n}{2} + 1$, and since there is some vertex $u \in U$ with $\deg(u, V) \leq \frac{n}{2}$,

$$\exists u^* \in U \text{ such that } \deg(u^*, V) \geq \frac{n}{2} + 2. \tag{6.11}$$

Case 1. $U_0 \cup U_2^M \neq \emptyset$ or $|U_2|$ is even. There are two cases: (i) $|V_0 \cup V_1| > |U_1|$ or (ii) $|V_2| \geq |U_0 \cup U_2|$. If (i) is the case there exists some $\ell_1 \in \mathbb{N}$, $X \subseteq U_0 \cup U_2^M$, and $Y \subseteq V_0 \cup V_1^M$ such that $|U_1 \cup X| = \ell_1 s$, $|(V_0 \cup V_1) \setminus Y| \geq \ell_1 s$ and $|(V_0 \cup V_1) \setminus Y| - |U_1 \cup X|$ is as small as possible. If $|(V_0 \cup V_1) \setminus Y| - |U_1 \cup X| = 0$, then we are done. Otherwise there are no movable vertices left in $(V_0 \cup V_1) \setminus Y$. If (ii) is the case, then there exists some $\ell_2 \in \mathbb{N}$ and $X \subseteq U_0 \cup U_2^M$ with $|X| \leq 1$ such that $|(U_0 \cup U_2) \setminus X| = \ell_2 s$, $|V_2| \geq \ell_2 s$ and $|V_2| - |(U_0 \cup U_2) \setminus X|$ is as small as possible.

Notice that in either case, we are either done or there are no movable vertices left in $(V_0 \cup V_1) \setminus Y$ or V_2. Because of this symmetry we can suppose without loss of generality that that (i) is the case. We reset $U_1 := U_1 \cup X$, $U_0 := (U_0 \cup U_2^M) \setminus X$, $U_2 := U_2 \setminus U_2^M$, $V_1 := V_1 \setminus Y$, and $V_0 := V_0 \cup Y$. Let $\ell_2 = m - \ell_1$. Let $a := |V_1| - \ell_1 s$. If $a = 0$, then we are done, so suppose $a \geq 1$. Note that there are no movable vertices in V_1 or U_2. We have

$$\delta(V_1, U_0 \cup U_2) + \delta(U_0 \cup U_2, V_1) \geq a + 1. \tag{6.12}$$

Case 1.1. $a > \frac{1}{\gamma}$. We know that $|U_0| \leq 1$, otherwise we could make a smaller by moving 2 vertices from U_0 to U_1 while maintaining the fact that $|U_1|$ is even. Either $\delta(V_1, U_2) \geq \delta(V_1, U_0 \cup U_2) - 1 \geq \frac{a+1}{2} - 1$ and we apply Lemma 6.4.4(vi) to get a vertex disjoint 2-stars from V_1 to U_2 or else $\delta(U_0 \cup U_2, V_1) > \frac{a+1}{2}$ and we apply Lemma 6.4.4(v) to get a vertex disjoint 2-stars from V_1 to U_2. We move the centers from V_1 to V_2 to make $|V_1| = \ell_1 s$.

Case 1.2. $a \leq \frac{1}{\gamma}$. If $\delta(U_0 \cup U_2, V_1) \geq 2$, then we apply Lemma 6.4.4(iii) to get a set of a vertex disjoint 2-stars from V_1 to U_2. So suppose $\delta(U_0 \cup U_2, V_1) \leq 1$ and thus

$$\delta(V_1, U_0 \cup U_2) \geq a. \tag{6.13}$$

Case 1.2.1. $a \geq 3$. We know that $|U_0| \leq 1$, otherwise we could make a smaller by moving 2 vertices from U_0 to U_1 while maintaining the fact that $|U_1|$ is even. Since $a \geq 3$, we have $\delta(V_1, U_2) \geq \delta(V_1, U_0 \cup U_1) - 1 \geq 2$ by (6.13), and thus we can apply Lemma 6.4.4(i) to get a set of a vertex disjoint 2-stars from V_1 to U_2. So we only need to deal with the case $a \leq 2$.

Case 1.2.2. $a = 2$. If $U_0 = \emptyset$, then we can use (6.13) and apply Lemma 6.4.4(i) to get a set of a vertex disjoint 2-stars from V_1 to U_2. So suppose $U_0 = \{u_0\}$. If there is a vertex $u \in U_2$ with $\deg(u, V_1) = 0$, then by (6.12) we have $\delta(V_1, U_0 \cup U_2) \geq 3$ and we are done since $\delta(V_1, U_2) \geq \delta(V_1, U_0 \cup U_1) - 1 \geq 2$. So suppose $\delta(U_0 \cup U_2) \geq 1$. If there is a vertex $u \in U_2$ with $\deg(u, V_1) \geq 2$, then we can move u_0 and u to U_1, thus for all $u \in U_2$, $\deg(u, V_1) = 1$. Now suppose there is a vertex $v_1 \in V_1$ with $\deg(v_1, U_2) \geq 2$ and let $u_2, u_2' \in N(v) \cap U_2$. Let $v_1' \in N(u_0) \cap (V_1 \setminus \{v_1\})$. Since $\Delta(U_2, V_1) \leq 1$, there exists some $u' \in (U_2 \setminus \{u_2, u_2'\}) \cap N(v_1')$. Thus we can move v_1 and v_1'. So for all $v \in V_1$, $\deg(v, U_2) = 1$. This implies that $\ell_2 s - 1 = |U_2| = |V_1| = \ell_1 s + 2$, a contradiction.

Case 1.2.3. $a = 1$. If $U_0 \neq \emptyset$, then let $u_0 \in U_0$. Let $u_2 v_1 \in E(V_1, (U_0 \cup U_2) \setminus \{u_0\})$, which exists be (6.12). Let $v_2 \in N(u_2) \cap V_2$. By (6.10), v_1 and v_2 have a common neighbor u' different than u_2. If $u' \in U_0 \cup U_2$, then we are done by simply moving v_1, so we have $u' \in U_1$ which completes a $K_{2,2}$. Now we move u_0 to U_1 to finish.

Finally, suppose $U_0 = \emptyset$. If there exists a vertex $v \in V_1$ such that $\deg(v, U_2) \geq 2$, then we can move v and be done. So suppose $\Delta(V_1, U_2) \leq 1$. Furthermore if there was a vertex $v \in V_1$ such that $\deg(v, U_2) = 0$, then (6.12) would imply $\delta(U_2, V_1) \geq 2$ contradicting the fact that $\Delta(V_1, U_2) \leq 1$. So every vertex in V_1 has exactly one neighbor in U_2 and (6.12) implies $\delta(U_2, V_1) \geq 1$. Since $|U_2|$ is even and $|V_1|$ is odd, we must have $|V_1| \neq |U_2|$. If $|U_2| > |V_1|$, then $\delta(U_2, V_1) \geq 1$ would imply that there was a vertex in V_1 with two neighbors in U_2, so suppose $|V_1| > |U_2|$. This implies that there exists some $u_0 \in U_2$ such that $\deg(u_0, V_1) \geq 2$. Let $u_2 v_1 \in E(V_1, U_2 \setminus \{u_0\})$, which exists be (6.12). Let $v_2 \in N(u_2) \cap V_2$. By (6.10), v_1 and v_2 have a common neighbor u' different than u_2. If $u' \in U_2$, then we are done by simply moving v_1, so we have $u' \in U_1$ which completes a $K_{2,2}$. Now we move u_0 to U_1 to finish.

Case 2. $U_0 \cup U_2^M = \emptyset$ and $|U_2|$ is odd. Now there are no movable vertices in U_1 or U_2. So choose ℓ_1, ℓ_2 such that $|U_1| = \ell_1 s + 1$, $|U_2| = \ell_2 s - 1$. If it is not the case that $|V_0 \cup V_1| \geq \ell_1 s + 2$ or $|V_0 \cup V_2| \geq \ell_2 s$, then $V_0 = \emptyset$, $|V_1| = \ell_1 s + 1$,

$|V_2| = \ell_2 s - 1$, and $V_1^M = \emptyset$. Without loss of generality, suppose $|V_0 \cup V_1| \geq \ell_1 s + 1$. Let $b := |V_1 \cup V_0| - |U_1|$.

Case 2.1. $b = 0$. Note that since $b = 0$, $U_0 = V_0 = U_2^M = V_1^M = \emptyset$ for $i = 1, 2$. We first show that if there is a vertex $u_i \in U_i$ such that $\deg(u_i, V_{3-i}) \geq 2$, then we would be done. Without loss of generality, suppose there exists $u_1 \in U_1$ such that $\deg(u_1, V_2) \geq 2$. Let $v, v' \in N(u_1) \cap V_2$. Since $\delta(V_1, U_2) + \delta(U_2, V_1) \geq 1$, there is an edge $v_1 u_2 \in E(V_1, U_2)$. Let $v_2 \in V_2 \cap N(u_2) \setminus \{v, v'\}$. By (6.10) we know that v_1 and v_2 have a common neighbor u_0 which is different than u_2. If $u_0 \in U_1$, then we have a copy of $K_{2,2}$ with one vertex in each of U_1, U_2, V_1, V_2 and we are done, so suppose $u_0 \in U_2$. Then we choose $u' \in (N(v) \cap N(v')) \cap (U_2 \setminus \{u_0\})$. Thus we can move u and v_1 to finish. So we may suppose that

$$\Delta(U_1, V_2), \Delta(U_2, V_1) \leq 1. \tag{6.14}$$

By (6.11), there is a vertex $u^* \in U$ such that $\deg(u^*, V) \geq \frac{n}{2} + 2$. Without loss of generality, suppose $u^* \in U_1$. Then by (6.14) we have $|U_1| = |V_1| \geq \frac{n}{2} + 1$, which in turn implies that $|U_2| = |V_2| \leq \frac{n}{2} - 1$. However, now we have $\delta(V_2, U_1) \geq 2$, and thus there exists $u \in U_1$ such that $\deg(u, V_2) \geq 2$, contradicting (6.14).

Case 2.2. $b \geq 1$. Suppose first that $|V_1 \setminus V_1^M| \geq \ell_1 s + 3$. Let $b_1' := |V_1 \setminus V_1^M| - (\ell_1 s + 2)$. We have

$$\delta(V_1 \setminus V_1^M, U_2) + \delta(U_2, V_1 \setminus V_1^M) \geq n + 1 - (\ell_1 s + 1 + \ell_2 s - 2 - b_1') = b_1' + 2.$$

So we apply Lemma 6.4.5(i) with $A = V_1 \setminus V_1^M$ and $B = U_2$ to get a set of b_1' vertex disjoint s stars from $V_1 \setminus V_1^M$ to U_2 and one s-star from U_2 to $V_1 \setminus V_1^M$.

So we may suppose $|V_1 \setminus V_1^M| \leq \ell_1 s + 2$. Reset $V_1 := V_1 \setminus V_1^M$ and $V_0 := V_0 \cup V_1^M$, then partition $V_0 = V_0^1 \cup V_0^2$ so that $|V_1 \cup V_0^1| = l_1 s + 2$ and

$|V_2 \cup V_0^2| = l_2 s - 2$. We have

$$\delta(V_1 \cup V_0^1, U_2) + \delta(U_2, V_1 \cup V_0^1) \geq n + 1 - (\ell_1 s + 1 + \ell_2 s - 2) = 2. \qquad (6.15)$$

We first observe that if $\delta(V_1 \cup V_0^1, U_2) \geq 2$, then there will be a vertex $u_2 \in U_2$ such that $\deg(u_2, V_1) \geq 2$ in which case we would be done, so suppose not. This implies that $|U_1| \geq \frac{n}{2}$.

First assume that $|V_0^1| \leq 1$. By (6.15), one of $\delta(U_2, V_1 \cup V_0^1) \geq 2$ or $\delta(V_1 \cup V_0^1, U_2) \geq 1$ must hold. Since $|V_1 \cup V_0^1| > |U_2|$, in either case there is a vertex $u \in U_2$ such that $\deg(u, V_1 \cup V_0^1) \geq 2$, in which case we are done since $|V_0^1| \leq 1$.

So suppose $|V_0^1| \geq 2$. Now if $\delta(V_2 \cup V_0^2, U_1) \geq 2$, then there will be a vertex $u_1 \in U_1$ such that $\deg(u_1, V_2) \geq 2$ in which case we would be done, since we can also move two vertices from V_0^2, so suppose not. This implies that $|U_2| \geq \frac{n}{2}$ and since $|U_1| \geq \frac{n}{2}$, we have $|U_1| = |U_2| = \frac{n}{2}$. So let $v_2 \in V_2$ with $\deg(v_2, U_1) = 1$ and let $v_1 \in N(u_1) \cap V_1$. By (6.10), v_1 and v_2 have a common neighbor in U_2 (since $\deg(v_2, U_1) = 1$) which completes a $K_{2,2}$. We finish by moving one additional vertex from V_0^1 to V_2.

$$s \geq 3$$

The following proof has many cases, so we provide an outline for reference.

1. $|V_1| \leq k_1 s$ and $|V_0 \cup V_1| \leq k_1 s + r$

2. $\exists \ell_1 \geq k_1$, $\exists Y \subseteq V_1^M$ and $\exists V_0' \subseteq V_0$ such that $|(V_1 \setminus Y) \cup V_0'| = \ell_1 s$.

2.1. $|V_1| \leq k_1 s$

2.1.1. $|V_0 \cup V_1| \geq k_1 s + s$

2.1.2. $|V_0 \cup V_1| < k_1 s + s$

2.2. $|V_1| > k_1 s$

2.2.1. $|V_1 \setminus V_1^M| \le k_1 s$

2.2.1.1. $|U_0 \cup U_2| \ge k_2 s$

2.2.1.2. $|U_0 \cup U_2| < k_2 s$

2.2.1.2.1. $|V_0 \cup V_1| \ge k_1 s + s$

2.2.1.2.1.1. $|U_0 \cup U_1| \ge k_1 s + s$

2.2.1.2.1.2. $|U_0 \cup U_1| < k_1 s + s$

2.2.1.2.2. $|V_0 \cup V_1| < k_1 s + s$

2.2.2. $|V_1 \setminus V_1^M| > k_1 s$

2.2.2.1. $\exists \ell_1$, $\exists Y \subseteq V_1^M$ such that $|V_1 \setminus Y| = \ell_1 s$

2.2.2.1.1. $|U_0 \cup U_2| < \ell_2 s$ (i.e. $|U_1| > \ell_1 s$)

2.2.2.1.2. $|U_0 \cup U_2| \ge \ell_2 s$

2.2.2.2. $\exists \ell_1$, $\exists V_0' \subseteq V_0$ such that $|V_1 \cup V_0'| = \ell_1 s$

2.2.2.2.1. $|U_0 \cup U_2| < \ell_2 s$

2.2.2.2.2. $|U_0 \cup U_2| \ge \ell_2 s$

3. For some $\ell_1 \ge k_1$ we have $\ell_1 s < |V_1 \setminus V_1^M| \le |V_1 \cup V_0| < \ell_1 s + s$

3.1. $|U_2 \setminus U_2^M| \ge \ell_2 s$

3.2. $|U_2 \setminus U_2^M| < \ell_2 s$

3.2.1. $|U_0 \cup U_1| \ge \ell_1 s + s$

3.2.1.1. $|U_1| \le \ell_1 s$

3.2.1.2. $|U_1| > \ell_1 s$

3.2.1.2.1. $\ell_1 > k_1$

3.2.1.2.2. $\ell_1 = k_1$

3.2.2. $\ell_1 s < |U_0 \cup U_1| < \ell_1 s + s$

3.2.2.1. $|U_1| \leq \ell_1 s$

3.2.2.2. $|U_1| > \ell_1 s$

3.2.2.2.1. For some $i \in \{1, 2\}$ we have $\delta(V_i, U_{3-i}) \geq s$ or $\delta(U_{3-i}, V_i) \geq s$

3.2.2.2.2. For all $i \in \{1, 2\}$ we have $\delta(V_i, U_{3-i}) < s$ and $\delta(U_{3-i}, V_i) < s$

Recall the following definitions. For $i = 1, 2$,
$U_i^M = \{u \in U_i : \deg(u, V_{3-i}) > \alpha^{1/3} n\}$ and $V_i^M = \{v \in V_i : \deg(v, U_{3-i}) > \alpha^{1/3} n\}$.
Also recall $U_1^M = \emptyset = V_2^M$ by Claim 6.4.2.

Case 1 $|V_1| \leq k_1 s$ and $|V_0 \cup V_1| \leq k_1 s + r$. Let $b_2 := |V_2| - k_2 s$ and note that $b_2 \geq -r$. We have

$$\delta(U_1, V_2) \geq k_1 s + s + r - (k_1 s - b_2) \geq s + r + b_2 \geq s. \tag{6.16}$$

Claim 6.4.8. *If $|V_0 \cup V_1| \geq k_1 s$, then there exists $V_0' \subseteq V_0$ such that $|V_1 \cup (V_0 \setminus V_0')| = k_1 s$. If $|V_0 \cup V_1| < k_1 s$, then there exists a set of vertex disjoint s-stars with centers $C \subseteq V_2$ and leaves in U_1 such that $|V_0 \cup V_1| + |C| = k_1 s$.*

Proof. If $|V_0 \cup V_1| \geq k_1 s$, we just choose $V_0' \subseteq V_0$ such that $|V_1 \cup (V_0 \setminus V_0')| = k_1 s$. Otherwise $b_2 \geq 0$ and thus by (6.16) and $\Delta(V_2, U_1) < 2\alpha^{1/3} k_2 s$, we can apply Lemma 6.4.4(ii) to get a set of b_2 vertex disjoint s-stars from V_2 to U_1 with centers C. So we have $|V_0 \cup V_1 \cup C| = k_1 s$. □

Let $a_2 := |U_2| - k_2 s$. We have two cases.

Suppose $a_2 \geq 0$. Claim 6.4.1 gives
$\delta(V_1, U_2) \geq k_2 s + 2s - 5 - r - (k_1 s - a_2) \geq s + a_2$. So by Lemma 6.4.4(ii) there

are a_2 vertex disjoint s-stars from U_2 to V_1 with centers C_U. So we can make $|U_0 \cup U_1 \cup C_U| = k_1 s$ and apply Claim 6.4.8 to finish.

Suppose $a_2 < 0$. Then $|U_0 \cup U_1| > k_1 s$. If $|U_1| \leq k_1 s$, then there exists $U_0' \subseteq U_0$ such that $|U_1 \cup (U_0 \setminus U_0')| = k_1 s$ and we apply Claim 6.4.8 to finish. Otherwise $|U_1| > k_1 s$ and let $a_1 := |U_1| - k_1 s > 0$. If $b_2 > 0$, then we have

$$\delta(U_1, V_2) + \delta(V_2, U_1) \geq 3s - 5 + a_1 + b_2,$$

and we use Lemma 6.4.5(i) to get a set of a_1 vertex disjoint s-stars from U_1 to V_2 with centers C_U and a set of b_2 vertex disjoint s-stars from V_2 to U_1 with centers C_V. Thus $|U_1 \setminus C_U| = k_1 s$ and $|V_0 \cup V_1 \cup C_V| = k_1 s$. Finally suppose $b_2 \leq 0$, i.e. $|V_0 \cup V_1| \geq k_1 s$. If there exists a set of a_1 vertex disjoint s-stars from U_1 to V_2, then we can apply Claim 6.4.8 to finish. We show that such a set exists. We have

$$\delta(V_2, U_1) \geq k_2 s + 2s - 5 - r - (k_2 s - a_1) = 2s - 5 - r + a_1 \geq s - 4 + a_1. \quad (6.17)$$

If $a_1 \leq 3$, we use (6.16) and Lemma 6.4.4(i) with $x = 0$ to get a set of a_1 vertex disjoint s-stars from U_1 to V_2 with centers C_U. Otherwise $a_1 \geq 4$ and we use (6.17) and Lemma 6.4.4(iii) or (v) to get a set of a_1 vertex disjoint s-stars from U_1 to V_2 with centers C_U.

Case 2. There exists $\ell_1 \geq k_1$, $Y \subseteq V_1^M$ and $V_0' \subseteq V_0$ such that $|(V_1 \setminus Y) \cup V_0'| = \ell_1 s$. Let $\ell_1 \geq k_1$ be minimal.

Case 2.1. $|V_1| \leq k_1 s$. By Case 1 we have $|V_0 \cup V_1| > k_1 s + r$. This implies that there exists $V_0' \subseteq V_0$ such that $|V_1 \cup V_0'| = k_1 s$ and $|(V_0 \cup V_2) \setminus V_0'| = k_2 s$. We now try to make $|U_1| = k_1 s$ or $|U_2| = k_2 s$. Reset $U_2 := U_2 \setminus U_2^M$ and $U_0 := U_0 \cup U_2^M$. Let $a_1 := |U_1| - k_1 s$ and $a_2 := |U_2| - (k_2 s - s)$. We have

$$\delta(V_2, U_1) \geq k_2 s + 2s - 5 - r - (k_2 s - a_1) = 2s - 5 - r + a_1 \quad (6.18)$$

and

$$\delta(V_1, U_2) \geq k_2 s + 2s - 5 - r - (k_1 s + s - a_2) = (k_2 - k_1)s + s - 5 - r + a_2. \quad (6.19)$$

If $|U_2| \geq k_2 s$ i.e. $a_2 \geq s$, then by (6.19) and Claim 6.4.1 we have $\delta(V_1, U_2) \geq s - 1 + (a_2 - s)$ and thus Lemma 6.4.4(ii) gives $a_2 - s$ vertex disjoint s-stars from U_2 to V_1 with centers C_U such that $|U_2 \setminus C_U| = k_2 s$. Otherwise we have $|U_0 \cup U_1| > k_1 s$. If $|U_1| \leq k_1 s$, then we choose $U_0' \subseteq U_0$ such that $|U_1 \cup (U_0 \setminus U_0')| = k_1 s$. So suppose $|U_1| > k_1 s$, i.e. $a_1 > 0$.

Case 2.1.1. $|V_0 \cup V_1| \geq k_1 s + s$. If $|U_0 \cup U_1| \geq k_1 s + s$, then we are done: either $a_1 \leq s$ and we just choose $U_0' \subseteq U_0$ and $V_0' \subseteq V_0$ such that $|V_1 \cup (V_0 \setminus V_0')| = k_1 s + s$ and $|U_1 \cup (U_0 \setminus U_0')| = k_1 s + s$ or else $a_1 > s$ and thus (6.18) gives $\delta(V_2, U_1) \geq 2s - 4 + (a_1 - s) \geq s - 1 + (a_1 - s)$ and thus Lemma 6.4.4(ii) allows us to find $a_1 - s$ vertex disjoint s-stars from U_1 to V_2. So suppose $|U_0 \cup U_1| < k_1 s + s$ and thus $a_2 > 0$.

$k_2 = k_1$. By Claim 6.4.1, $r \leq \frac{s-6}{2}$ which implies $\delta(V_2, U_1) \geq s - 1 + a_1$ by (6.18). So there are a_1 vertex disjoint s-stars from U_1 to V_2 by Lemma 6.4.4(ii).

$k_2 = k_1 + 1$. By Claim 6.4.1, $r \leq s - 3$ which implies $\delta(V_2, U_1) \geq s - 2 + a_1$ by (6.18). If $a_1 \geq 2$ or $r \leq s - 4$, then there are a_1 vertex disjoint s-stars from U_1 to V_2 by Lemma 6.4.4(iii), so suppose $a_1 = 1$ and $r = s - 3$. Furthermore we have $\delta(V_1, U_2) \geq s - 2 + a_2$ by (6.19). If $a_2 \geq 2$, then there are a_2 vertex disjoint s-stars from U_2 to V_1 by Lemma 6.4.4(iii), so suppose $a_2 = 1$. Note that we would be done unless $\Delta(U_1, V_2) \leq s - 1$ and $\Delta(U_2, V_1) \leq s - 1$. Let $d_1 := k_1 s - |V_1|$ and let $d_2 := k_2 s - |V_2|$. Note that $|V_0| = d_1 + d_2 \geq s$. Let $\hat{U}_1 = \{u \in U_1 : \deg(u, V_1) \leq k_1 s - d_1 - 4\}$ and suppose that $\hat{U}_1 \neq \emptyset$. So we have

$$\delta(\hat{U}_1, V_0) + \delta(U_2, V_0) \geq 2(k_1 s + s + r) - (k_1 s - d_1 - 4 + s - 1) - (k_2 s - d_2 + s - 1) \geq |V_0| + s.$$

This implies that we can find a $K_{s,s}$ with one vertex in U_1, $s - 1$ vertices in U_2 and s vertices in V_0. So we may suppose that $\hat{U}_1 = \emptyset$. Note that $\delta(U_1, V_1) \geq k_1 s - d_1 - 3 = |V_1| - 3$. Since $\delta(V_1, U_2) \geq s - 1$, there exists a set of $3s - 2$ vertex disjoint $(s-1)$-stars from U_2 to V_1 with centers C_U. Let

$v_2 \in N(C_U) \cap V_2$. Since $\delta(V_2, U_1) \geq s-1$, we can let $L_U \subseteq N(v_2) \cap U_1$ such that $|L_U| = s-1$. Since $\delta(U_1, V_1) \geq |V_1| - 3$, the leaves of at least one of the $(s-1)$-stars from U_2 to V_1 forms a $K_{s-1,s-1}$ with L_U. This allows us to move a vertex $u_2 \in U_2$ to U_1 and v_2 to V_1. This makes $|U_2 \setminus \{u_2\}| = k_2 s - s$, and we choose $V_0' \subseteq V_0$ such that $|V_0' \cup V_2 \setminus \{v_2\}| = k_2 s - s$.

$k_2 \geq k_1 + 2$. In this case, we see from (6.19) that $\delta(V_1, U_2) \geq 2s - 4 + a_2 \geq s - 1 + a_2$. So there are a_2 vertex disjoint s-stars from U_2 to V_1 by Lemma 6.4.4(ii). Then we choose $V_0' \subseteq V_0$ such that $|V_1 \cup (V_0 \setminus V_0')| = k_1 s + s$.

Case 2.1.2. $|V_0 \cup V_1| < k_1 s + s$. Let $b_2 := |V_2| - (k_2 s - s)$ and note that $b_2 > 0$.

$k_2 = k_1$. Then $r \leq \frac{s-6}{2}$ which implies $\delta(V_2, U_1) \geq s - 1 + a_1$ by (6.18). So by Lemma 6.4.4(ii) there are a_1 vertex disjoint s-stars from U_1 to V_2.

$k_2 = k_1 + 1$. Then $r \leq s - 3$ which implies $\delta(V_2, U_1) \geq s - 2 + a_1$ by (6.18). If $a_1 \geq 2$, then there are a_1 vertex disjoint s-stars from U_1 to V_2, so suppose $a_1 = 1$. We have $|V_2| = k_2 s - s + b_2 = k_1 s + b_2$. If $b_2 \geq 2$, then $|V_2| > |U_1|$ which together with $\delta(V_2, U_1) \geq s - 1$ implies that there is a vertex in U_1 with at least s neighbors in V_2, in which case we are done. So suppose $b_1 = 1$ and thus $|V_2| = |U_1|$. So if there is a vertex in V_2 with s neighbors in U_1, then there is a vertex in U_1 with s neighbors in V_2, so suppose not. Together with $\delta(V_2, U_1) \geq s - 1$, this implies that $G[U_1, V_2]$ is $(s-1)$-regular. So we have $\delta(V_2, U_0 \cup U_2) \geq k_2 s + 2s - 5 - r - (s-1) \geq k_2 s - 1 = |U_0 \cup U_2|$ which implies that $G[V_2, U_0 \cup U_2]$ is complete, and thus we can choose a vertex $u_1 \in U_1$ and a vertex $v_1 \in N(u_1) \cap V_1$. Since $\deg(u_1, V_2) = s - 1$ and $\deg(v_1, U_0 \cup U_2) \geq k_2 s + 2s - 5 - r - (k_1 s + 1) \geq 2s - 3 \geq s$ we can move u_1 and v_1. Then we replace v_1 with a vertex from V_0 as $V_0 \neq \emptyset$.

$k_2 \geq k_1 + 2$.

Claim 6.4.9. *If $|U_0 \cup U_1| \geq k_1 s + s$ and $|U_1| \leq k_1 s + s$, then there exists $U_0' \subseteq U_0$ such that $|(U_0 \cup U_1) \setminus U_0'| = k_1 s + s$. If $|U_0 \cup U_1| < k_1 s + s$, then there exists a set of vertex disjoint s-stars with centers $C \subseteq U_2$ and leaves in V_1 such that $|U_0 \cup U_1| + |C| = k_1 s + s$.*

Proof. Suppose first that $|U_0 \cup U_1| \geq k_1 s + s$ and $|U_1| \leq k_1 s + s$. Let $U_0' \subseteq U_0$ so that $|(U_0 \cup U_1) \setminus U_0'| = k_1 s + s$. Now suppose $|U_0 \cup U_1| < k_1 s + s$ and let $a_2 := |U_2| - (k_2 s - s)$. Since $k_2 \geq k_1 + 2$, (6.19) gives $\delta(V_1, U_2) \geq 2s - 4 + a_2 \geq s - 1 + a_2$ and thus by Lemma 6.4.4(ii) there is a set of a_2 vertex disjoint s-stars with centers $C \subseteq U_2$ and leaves in V_1 such that $|U_0 \cup U_2| + |C| = k_1 s + s$. □

We have

$$\delta(U_1, V_2) \geq k_1 s + s + r - (k_1 s + s - b_2) = r + b_2. \tag{6.20}$$

If $r \geq s - b_2$, then $\delta(U_1, V_2) \geq s$ and we apply Lemma 6.4.4(iii) to get a set of b_2 vertex disjoint s-stars from V_2 to U_1. So suppose $r \leq s - 1 - b_2$. By (6.18) we have

$$\delta(V_2, U_1) \geq s - 4 + a_1 + b_2. \tag{6.21}$$

We would be done unless $2 \leq a_1 + b_2 \leq 3$. Note also that we have

$$\delta(V_1, U_0 \cup U_2) \geq k_2 s + 2s - 5 - r - (k_1 s + a_1) \geq (k_2 - k_1)s + s - 4 + b_2 - a_1 \geq 3s - 4 + b_2 - a_1. \tag{6.22}$$

First suppose $b_2 = 2$ and $a_1 = 1$. By (6.21) we have $\delta(V_2, U_1) \geq s - 1$, and since $|V_2| > |U_1|$ there exists some $u \in U_1$ such that $\deg(u, V_2) \geq s$. Thus we can move one vertex from U_1.

Now suppose $b_2 = 1$. If there is a vertex $v_2 \in V_2$ such that $\deg(v_2, U_1) \geq s$, then $|(V_0 \cup V_1) \cup \{v_2\}| = k_1 s + s$ and we apply Claim 6.4.9 to finish. So suppose $\Delta(V_2, U_1) \leq s - 1$.

If $a_1 = 2$, we have $\delta(V_2, U_0 \cup U_2) \geq k_2 s + 2s - 5 - r - (s-1) \geq k_2 s - 2 = |U_0 \cup U_2|$ which implies that $G[V_2, U_0 \cup U_2]$ is complete. Since $\delta(V_2, U_1) \geq s - 1$ and $|V_2| > |U_1|$, there is a vertex $u_1 \in U_1$ such that $\deg(u_1, V_2) \geq s$ and since $\delta(V_2, U_1) \geq s - 1$ and $\Delta(U_1, V_2) < 2\alpha^{1/3} k_1 s$, there is another vertex $u_1' \in U_1$ such that $\deg(u_1', V_2) \geq s - 1$ and the neighborhoods of u_1 and u_1' in V_2 are disjoint. Let $v_1' \in N(u_1') \cap V_1$; by (6.22) $\deg(v_1', U_0 \cup U_2) \geq s - 1$ and thus since $G[V_2, U_0 \cup U_2]$ is complete we can move u_1, u_1' to make $|U_1| = k_1 s$.

If $a_1 = 1$, we have $\delta(V_2, U_0 \cup U_2) \geq k_2 s + 2s - 5 - r - (s-1) \geq k_2 s - 2 = |U_0 \cup U_2| - 1$. Since $\delta(V_2, U_1) \geq s - 2$ and $|V_2| > |U_1|$, there is a vertex $u_1 \in U_1$ such that $\deg(u_1, V_2) \geq s - 1$. Let $v_1 \in V_1 \cap N(u_1)$; by (6.22) we have $\deg(v_1, U_0 \cup U_2) \geq 3s - 4 \geq 2s - 1$. Since $\delta(V_2, U_0 \cup U_2) \geq |U_0 \cup U_2| - 1$, $K_{s-1,s-1} \subseteq G[N(u_1) \cap V_2, N(v_1) \cap (U_0 \cup U_2)]$. Thus we can move u_1.

Case 2.2 $|V_1| > k_1 s$.

Case 2.2.1. $|V_1 \setminus V_1^M| \leq k_1 s$. Let $Y \subseteq V_1^M$ such that $|V_1 \setminus Y| = k_1 s$.

Case 2.2.1.1. $|U_0 \cup U_2| \geq k_2 s$. If $|U_2| \leq k_2 s$, then there exists $U_0' \subseteq U_0$ such that $|U_1 \cup U_0'| = k_1 s = |V_1 \setminus Y|$ and we are done. If not, then we have $|U_2| > k_2 s$. So let $a_2 := |U_2| - k_2 s$. We have $\delta(V_1, U_2) \geq k_2 s + 2s - 5 - r - (k_1 s - a_2) = (k_2 - k_1)s + 2s - 5 - r + a_2 \geq s - 1 + a_2$ by Claim 6.4.1, and thus we can apply Lemma 6.4.4(ii) to get a set of a_2 vertex disjoint s-stars from U_2 to V_1. Since $|(V_0 \cup V_2) \cup Y| = k_2 s$, we are done.

Case 2.2.1.2. $|U_0 \cup U_2| < k_2 s$. Set $a_1 := |U_1| - k_1 s$ and note that $a_1 \geq 1$.

We have

$$\delta(V_2, U_1) \geq k_2 s + 2s - 5 - r - (k_2 s - a_1) = 2s - 5 - r + a_1. \tag{6.23}$$

Case 2.2.1.2.1. $|V_0 \cup V_1| \geq k_1 s + s$.

Case 2.2.1.2.1.1. $|U_0 \cup U_1| \geq k_1 s + s$. If $a_1 \leq s$, we can choose $U_0' \subseteq U_0$ and $Y' \subseteq V_1^M \cup V_0$ so that $|(U_0 \cup U_1) \setminus U_0'| = |(V_0 \cup V_1) \setminus Y'| = k_1 s + s$. If $a_1 > s$, then (6.23) implies $\delta(V_2, U_1) \geq 2s - 4 + (a_1 - s) \geq s - 1 + (a_1 - s)$ and thus we can apply Lemma 6.4.4(ii) to get $a_1 - s$ vertex disjoint s-stars from U_1 to V_2. Now let $Y' \subseteq V_1^M \cup V_0$ so that $|U_1| - (a_1 - s) = |(V_0 \cup V_1) \setminus Y'| = k_1 s + s$.

Case 2.2.1.2.1.2. $|U_0 \cup U_1| < k_1 s + s$. Let $a_2 = |U_2| - (k_2 s - s)$. We have

$$\delta(V_1, U_2) \geq k_2 s + 2s - 5 - r - (k_1 s + s - a_2) = (k_2 - k_1)s + s - 5 - r + a_2. \tag{6.24}$$

If $k_2 = k_1$, then $r \leq \frac{s-6}{2}$. By (6.23) we have $\delta(V_2, U_1) \geq \frac{3s-4}{2} + a_1 \geq s - 1 + a_1$. So by Lemma 6.4.4(ii), we can move a_1 vertices from U_1 so that $|U_1| - a_1 = k_1 s = |V_1 \setminus Y|$.

If $k_2 = k_1 + 1$, then $r \leq s - 3$. By (6.24) and (6.23) we have $\delta(V_1, U_2) \geq s - 2 + a_2$ and $\delta(V_2, U_1) \geq s - 2 + a_1$. We would be done if either $\delta(V_1, U_2) \geq s$ or $\delta(V_2, U_1) \geq s$, because $|V_0 \cup V_1| \geq k_1 s + s$ and $|V_1 \setminus V_1^M| \leq k_1 s$. So we may suppose $a_1 = a_2 = 1$ and $r = s - 3$. We have $|V_1| \geq |U_2|$, $\delta(V_1, U_2) \geq s - 1$, and at least one vertex $v_1 \in V_1^M$ such that $\deg(v_1, U_2) \geq \alpha^{1/3} n$. Thus there is a vertex $u_2 \in U_2$ such that $\deg(u_2, V_1) \geq s$. So we have $|(U_0 \cup U_2) \cup \{u_2\}| = k_1 s + s$ and $|V_0 \cup V_1| \geq k_1 s + s$ with $|V_1 \setminus V_1^M| \leq k_1 s$ so we are done.

Finally, suppose that $k_2 \geq k_1 + 2$. We have $\delta(V_1, U_2) \geq (k_2 - k_1)s + s - 5 - r + a_2 \geq 2s - 4 + a_2 \geq s - 1 + a_2$ since $s \geq 3$. Thus we can find a_2 vertex disjoint s-stars from U_2 to V_1 by Lemma 6.4.4(ii) and we have $|(U_0 \cup U_1)| + a_2 = k_1 s + s$. Since $|V_0 \cup V_1| \geq k_1 s + s$ and $|V_1 \setminus V_1^M| \leq k_1 s$ we are done.

Case 2.2.1.2.2. $|V_0 \cup V_1| < k_1 s + s$. Set $b_2 := |V_2| - (k_2 s - s)$ and $b_1 := |V_1| - k_1 s$. Note that $1 \leq b_1, b_2 \leq s - 1$.

If $k_2 = k_1$, then $r \leq \frac{s-6}{2}$ by Claim 6.4.1. So by (6.23) we have $\delta(V_2, U_1) \geq \frac{3s-4}{2} + a_1 \geq s - 1 + a_1$. By Lemma 6.4.4(ii), we can move a_1 vertices from U_1 so that $|U_1| - a_1 = k_1 s = |V_1 \setminus Y|$.

If $k_2 = k_1 + 1$, then $r \leq s - 3$ and by (6.23) we have

$$\delta(V_2, U_1) \geq s - 2 + a_1. \tag{6.25}$$

If $a_1 \geq 2$ or $r \leq s - 4$, then (6.25) gives $\delta(V_2, U_1) \geq s - 2 + a_1 \geq s$ in which case we can apply Lemma 6.4.4(iii) to get a set of a_1 vertex disjoint s-stars from U_1 to V_2. So suppose $a_1 = 1$ and $r = s - 3$. We have $\delta(U_1, V_2) \geq k_1 s + s + r - (k_1 s + s - b_2) = r + b_2 \geq s - 3 + b_2$. If $b_2 \geq 3$, then we have $\delta(U_1, V_2) \geq s$ and thus we can move a single vertex from U_1 to make $|U_1| - a_1 = k_1 s = |V_1 \setminus Y|$. So suppose $1 \leq b_2 \leq 2$. By (6.25), we have $\delta(V_2, U_1) \geq s - 1$. If $b_2 = 2$, then $|V_2| = k_1 s + 2 > k_1 s + 1 = |U_1|$ and since $\delta(V_2, U_1) \geq s - 1$ there exists $u \in U_1$ such that $\deg(u, V_2) \geq s$. So we move u to U_2 and $|U_1 \setminus \{u\}| = k_1 s = |V_1 \setminus Y|$. So we may suppose that $b_2 = 1$. Since $\delta(V_2, U_1) \geq s - 1$, if there was a vertex $v \in V_2$ such that $\deg(v, U_1) \geq s$, then there exists $u \in U_1$ such that $\deg(u, V_2) \geq s$ in which case we would be done. So we can suppose $\Delta(U_1, V_2), \Delta(V_2, U_1) \leq s - 1$. Then since $\delta(V_2, U_1) \geq s - 1$ by (6.25), we have that $G[U_1, V_2]$ is $(s-1)$-regular. So we have $\delta(V_2, U_0 \cup U_2) \geq k_2 s + 2s - 5 - r - (s - 1) \geq k_2 s - 1 = |U_0 \cup U_2|$ and thus $G[V_2, U_0 \cup U_2]$ is complete. Since $|V_1| = k_1 s + 1$ and $|V_1 \setminus V_1^M| \leq k_1 s$, there exists some $v_1 \in V_1$ with $\deg(v_1, U_2) > \alpha^{1/3} n$. Let $u_1 \in U_1 \cap N(v_1)$. Since $\deg(u_1, V_2) = s - 1$ and $G[V_2, U_0 \cup U_2]$ is complete there is a copy of $K_{s,s}$ which contains u_1 and v_1. Thus $|U_1 \setminus \{u_1\}| = k_1 s = |V_1 \setminus Y|$.

Finally, suppose $k_2 \geq k_1 + 2$. We first prove the following claim.

Claim 6.4.10. *If $|U_0 \cup U_1| \geq k_1 s + s$ and $|U_1| \leq k_1 s + s$, then there exists $U_0' \subseteq U_0$ such that $|(U_0 \cup U_1) \setminus U_0'| = k_1 s + s$. If $|U_0 \cup U_1| < k_1 s + s$, then there exists a set of vertex disjoint s-stars with centers $C \subseteq U_2$ and leaves in V_1 such that $|U_0 \cup U_1| + |C| = k_1 s + s$.*

Proof. Suppose first that $|U_0 \cup U_1| \geq k_1 s + s$ and $|U_1| \leq k_1 s + s$. Let $U_0' \subseteq U_0$ so that $|(U_0 \cup U_1) \setminus U_0'| = k_1 s + s$. Now suppose $|U_0 \cup U_1| < k_1 s + s$ and let $a_2 := |U_2| - (k_2 s - s)$. Equation (6.24) holds in this case. Since $k_2 \geq k_1 + 2$, (6.24) gives $\delta(V_1, U_2) \geq 2s - 4 + a_2 \geq s - 1 + a_2$ and thus by Lemma 6.4.4(ii) there is a set of a_2 vertex disjoint s-stars with centers $C \subseteq U_2$ and leaves in V_1 such that $|U_0 \cup U_2| + a_2 = k_1 s + s$. □

We have

$$\delta(U_1, V_2) \geq k_1 s + s + r - (k_1 s + s - b_2) = r + b_2. \tag{6.26}$$

If $r \geq s - b_2$, then $\delta(U_1, V_2) \geq s$ and we can apply Lemma 6.4.4(iii) to get a set of a_1 vertex disjoint s-stars from U_1 to V_2 giving $|U_1| - a_1 = k_1 s = |V_1 \setminus Y|$. So suppose $r \leq s - 1 - b_2$. By (6.23) we have

$$\delta(V_2, U_1) \geq s - 4 + a_1 + b_2. \tag{6.27}$$

If $\delta(V_2, U_1) \geq s$, we would be done by moving a_1 vertices from U_1, so suppose $2 \leq a_1 + b_2 \leq 3$.

If $b_2 = 2$ and $a_1 = 1$, then $\delta(V_2, U_1) \geq s - 1$ and since $|V_2| > |U_1|$, there is a vertex $u \in U_1$ with $\deg(u, V_2) \geq s$, which we can move
$|U_1| - a_1 = k_1 s = |V_1 \setminus Y|$.

If $a_1 = 2$ and $b_2 = 1$, then $\delta(V_2, U_1) \geq s - 1$ by (6.27). If $r \leq s - 3$, then (6.23) would give $\delta(V_2, U_1) \geq s$ in which case we would be done by moving two vertices from U_1, so suppose $r = s - 2$. If there is a vertex $v_2 \in V_2$ with $\deg(v_2, U_1) \geq s$, we can move v_2 so that $|(V_0 \cup V_2) \cup \{v_2\}| = k_1 s + s$ and apply Claim 6.4.10 to finish. So suppose $\Delta(V_2, U_1) \leq s - 1$. So for all $v \in V_2$, $\deg(v, U_0 \cup U_2) \geq k_2 s + 2s - 5 - r - (s - 1) = k_2 s - 2 = |U_0 \cup U_2|$, which implies $G[V_2, U_0 \cup U_2]$ is complete. Since $|V_2| > |U_1|$ and $\delta(V_2, U_1) \geq s - 1$, there is a vertex $u_1 \in U_1$ with $\deg(u_1, V_2) \geq s$. Let L be a subset of $N(u_1) \cap V_2$ of size s. Let $v_1 \in V_1^M$ and note that $\delta(U_1, V_2) \geq s - 1$ by (6.26) and the fact that $r = s - 2$. Since $\Delta(V_2, U_1) \leq s - 1$ there must be a vertex $u_1' \in U_1 \cap N(v_1)$ such that $\deg(u_1', V_2 \setminus L) \geq s - 1$. Then since $G[V_2, U_0 \cup U_2]$ is complete, u_1 and v_1 are contained in a copy of $K_{s,s}$. Thus $|U_1 \setminus \{u_1, u_1'\}| = k_1 s = |V_1 \setminus Y|$.

Now in the final case we have $a_1 = 1 = b_2$. If there were a vertex $v_2 \in V_2$ such that $\deg(v_2, U_1) \geq s$, then $|(V_0 \cup V_1) \cup \{v_2\}| = k_1 s + s$ and we apply Claim 6.4.10 to finish. So suppose $\Delta(V_2, U_1) \leq s - 1$. Since $r \leq s - 2$, we have $\delta(V_2, U_0 \cup U_2) \geq k_2 s + 2s - 5 - r - (s - 1) \geq k_2 s - 2 = |U_0 \cup U_2| - 1$. Also $\delta(V_1, U_0 \cup U_2) \geq k_2 s + 2s - 5 - r - (k_1 s + 1) \geq (k_2 - k_1)s + s - 4 \geq 3s - 4 \geq 2s - 2$. Since $\delta(V_2, U_1) \geq s - 2$ and $|V_2| > |U_1|$, there exists $u_1 \in U_1$ with $\deg(u_1, V_2) \geq s - 1$. Let $v_1 \in N(u_1) \cap V_1$. Since v_1 has $2s - 2$ neighbors in $U_0 \cup U_2$ and $\delta(V_2, U_0 \cup U_2) \geq |U_0 \cup U_2| - 1$ there is a copy of $K_{s,s}$ which contains u_1 and v_1 with $s - 1$ vertices in $U_0 \cup U_2$ and $s - 1$ vertices in V_2. If $v_1 \in V_1^M$, then $|U_1 \setminus \{u_1\}| = k_1 s = |V_1 \setminus Y|$. If $v_1 \notin V_1^M$, then let $Y' \subseteq Y$ with $|Y'| = |Y| - 1$ and thus $|U_1 \setminus \{u_1\}| = k_1 s = |(V_1 \setminus \{v_1\}) \setminus Y'|$.

Case 2.2.2. $|V_1 \setminus V_1^M| > k_1 s$.

Case 2.2.2.1. $\exists \ell_1$, $\exists Y \subseteq V_1^M$ such that $|V_1 \setminus Y| = \ell_1 s$. Choose ℓ_1 minimal and note that $\ell_1 > k_1$ by Case 2.2.2. Let $\ell_2 := m - \ell_1$.

Case 2.2.2.1.1. $|U_0 \cup U_2| < \ell_2 s$. Let $a_1 := |U_1| - \ell_1 s$. We have $\delta(V_2, U_1) \geq k_2 s + 2s - 5 - r - (\ell_2 s - a_1) = (k_2 - \ell_2)s + 2s - 5 - r + a_1 \geq 2s - 4 + a_1 \geq s - 1 + a_1$, and thus we can find a set of a_1 vertex disjoint s-stars from U_1 to V_2. This gives $|U_1| - a_1 = \ell_1 s = |V_1 \setminus Y|$.

Case 2.2.2.1.2. $|U_0 \cup U_2| \geq \ell_2 s$. If $|U_2| \leq \ell_2 s$, then there exists $U_0' \subseteq U_0$ such that $|U_1 \cup U_0'| = \ell_1 s = |V_1 \setminus Y|$. Otherwise $|U_2| > \ell_2 s$. Set $a_2 := |U_2| - \ell_2 s$.

We have $|V_1 \setminus Y| = \ell_1 s$ and since $\ell_1 > k_1$ and ℓ_1 is minimal, we have $|V_1^M \setminus Y| < s$. Set $b_1 := |V_1 \setminus V_1^M| - (\ell_1 s - s)$. We have

$$\delta(V_1 \setminus Y, U_2) + \delta(U_2, V_1 \setminus Y) \geq n + 3s - 5 - (\ell_1 s - a_2 + \ell_2 s) = 3s - 5 + a_2 \quad (6.28)$$

and

$$\delta(V_1 \setminus V_1^M, U_2) + \delta(U_2, V_1 \setminus V_1^M) \geq n + 3s - 5 - (\ell_1 s - a_2 + \ell_2 s + s - b_1) = 2s - 5 + b_1 + a_2. \quad (6.29)$$

If $\delta(V_1 \setminus Y, U_2) \geq s$, then there are a_2 vertex disjoint s-stars from U_2 to V_1 by Lemma 6.4.4(iii) and we are done. Otherwise by (6.28) we have $\delta(U_2, V_1 \setminus Y) \geq 2s - 4 + a_2 \geq s$. If $\delta(U_2, V_1 \setminus V_1^M) \geq s$, then since $\Delta(V_1 \setminus V_1^M, U_2) < \alpha^{1/3} n$ we can apply Lemma 6.4.4(iii) to get a set of a_2 vertex disjoint s-stars from U_2 to V_1. Likewise if $\delta(V_1 \setminus V_1^M, U_2) \geq s$. These two facts, together with (6.29) imply $2 \leq a_2 + b_1 \leq 3$. If $a_2 = 1$, then since $\delta(U_2, V_1 \setminus Y) \geq 2s - 3 \geq s$ and we only need to move one vertex, we are done. So we only need to deal with the case when $a_2 = 2$, $b_1 = 1$, and $\delta(U_2, V_1 \setminus V_1^M) = s - 1 = \delta(V_1 \setminus V_1^M, U_2)$. Since $b_1 = 1$ we have $|V_1^M \setminus Y| = s - 1$. If there exists a vertex $u_2 \in U_2$ such that $\deg(u_2, V_1 \setminus V_1^M) \geq s$, then since $\delta(U_2, V_1 \setminus Y) \geq s$, we either have another vertex disjoint s-star and we are done, or every vertex in U_2 must have a neighbor in $N(u_2) \cap (V_1 \setminus V_1^M)$. However this implies that some vertex in $v' \in N(u_2) \cap (V_1 \setminus V_1^M)$ has $\deg(v', U_2) > \alpha^{1/3} n$ contradicting the fact that vertices in $V_1 \setminus V_1^M$ are not movable. So we have

$\Delta(U_2, V_1 \setminus V_1^M) \leq s-1$. Since $\delta(U_2, V_1 \setminus Y) \geq 2s-4+a_2 = 2s-2$, $\Delta(U_2, V_1 \setminus V_1^M) \leq s-1$ and $|V_1^M \setminus Y| = s-1$, every vertex in U_2 is adjacent to every vertex in $V_1^M \setminus Y$. Since $\delta(V_1 \setminus V_1^M, U_2) = s-1$, we can choose $v_1 \in V_1 \setminus V_1^M$ and $u_2, u_2' \in N(v_1) \cap U_2$. Thus $\{v_1\} \cup (V_1^M \setminus Y)$ and $\{u_2, u_2'\}$ form a $K_{s,2}$ and thus we can move u_2, u_2' from U_2, giving $|U_0 \cup U_1| + 2 = \ell_1 s = |V_1 \setminus Y|$.

Case 2.2.2.2. $\exists \ell_1$, $\exists V_0' \subseteq V_0$ such that $|V_0' \cup V_1| = \ell_1 s$. Choose ℓ_1 to be minimal and note that since we are in Case 2.2.2. but not Case 2.2.2.1. we have $|V_1 \setminus V_1^M| > \ell_1 s - s$ and thus

$$\ell_1 \geq k_1 + 1. \tag{6.30}$$

Set $\ell_2 := m - \ell_1$. Since $|V_1 \setminus V_1^M| > \ell_1 s - s$, we reset $V_1 := V_1 \setminus V_1^M$, $V_0 := V_0 \cup V_1^M$ and set $b_1 := |V_1| - (\ell_1 s - s)$.

Case 2.2.2.2.1. $|U_0 \cup U_2| < \ell_2 s$. Set $a_1 := |U_1| - \ell_1 s$. Then we have $\delta(V_2, U_1) \geq k_2 s + 2s - 5 - r - (\ell_2 s - a_1) = (k_2 - \ell_2)s + 2s - 5 - r + a_1 \geq 2s - 4 + a_1 \geq s - 1 + a_1$, and thus we are done by Lemma 6.4.4(ii).

Case 2.2.2.2.2. $|U_0 \cup U_2| \geq \ell_2 s$. If $|U_2| \leq \ell_2 s$, then there exists $U_0' \in U_0$ such that $|U_1 \cup U_0'| = \ell_1 s = |V_1 \cup Y|$. Otherwise $|U_2| > \ell_2 s$. Set $a_2 := |U_2| - \ell_2 s$. Note that if $\ell_2 \geq \ell_1$, then $\ell_2 s \geq \frac{n}{2}$ and consequently $\delta(V_1, U_2) \geq \frac{n+3s-4}{2} - (\ell_1 s - a_2) \geq \frac{3s-4}{2} + a_2 \geq s - 1 + a_2$. Then by Lemma 6.4.4(ii) we can move a_2 vertices from U_2 and we are done. So for the rest of this case we may suppose that

$$\ell_2 \leq \ell_1 - 1. \tag{6.31}$$

Since $|U_2| = \ell_2 s + a_2$, we have

$$\delta(V_1, U_2) + \delta(U_2, V_1) \geq n + 3s - 5 - (\ell_1 s - a_2 + \ell_2 s + s - b_1) = 2s - 5 + a_2 + b_1. \tag{6.32}$$

If $\delta(V_1, U_2) \geq s$ or $\delta(U_2, V_1) \geq s$, then we can apply Lemma 6.4.4(i) or (iii) to get a set of a_2 vertex disjoint s-stars from U_2 to V_1, giving

$|U_1| + a_2 = \ell_1 s = |V_0' \cup V_1|$. So suppose for the rest of the case that

$$\delta(V_1, U_2) \leq s - 1 \text{ and } \delta(U_2, V_1) \leq s - 1. \tag{6.33}$$

Thus (6.32) and (6.33) imply $2 \leq a_2 + b_1 \leq 3$. Furthermore, if $\delta(V_1, U_2) + \delta(U_2, V_1) = 2s - 2$, then we have $\delta(V_1, U_2) = s - 1$ and $\delta(U_2, V_1) = s - 1$.

Claim 6.4.11. *If $|U_1| \leq \ell_1 s - s$, then there exists $U_0' \subseteq U_0$ such that $|U_1 \cup U_0'| = \ell_1 s - s$. If $|U_1| \geq \ell_1 s - s + 1$, then there exists a set of vertex disjoint s-stars with centers $C \subseteq U_1$ and leaves in V_2 such that $|U_1 \setminus C| = \ell_1 s - s$ or else $\delta(V_1, U_2) \geq s - 2 + a_2$.*

Proof. First suppose $|U_1| \leq \ell_1 s - s$. Since $|U_2| = \ell_2 s + a_2 \leq \ell_2 s + 2 < \ell_2 s + s$, there exists $U_0' \subseteq U_0$ such that $|U_0' \cup U_1| = \ell_1 s - s$. Now suppose $|U_1| \geq \ell_1 s - s + 1$ and set $a_1 := |U_1| - (\ell_1 s - s)$. If $\ell_1 \geq k_1 + 2$, then

$$\begin{aligned}\delta(V_2, U_1) \geq k_2 s + 2s - 5 - r - (\ell_2 s + s - a_1) &= (k_2 - \ell_2) + s - 5 - r + a_1 \\ &\geq 2s - 4 + a_1 \geq s - 1 + a_1. \end{aligned} \tag{6.34}$$

Thus we may apply Lemma 6.4.4(ii) to get a set of a_1 vertex disjoint s-stars from U_1 to V_2 giving $|U_1| - a_1 = \ell_1 s - s$. So suppose $\ell_1 \leq k_1 + 1$, which implies $\ell_1 = k_1 + 1$ by (6.30). Consequently $\ell_2 = k_2 - 1$. By (6.31), we have $k_2 - 1 = \ell_2 \leq \ell_1 - 1 = k_1$. By (6.34), we have $\delta(V_2, U_1) \geq 2s - 5 - r + a_1$. If $k_2 = k_1$, then $r \leq \frac{s-6}{2}$ and thus $\delta(V_2, U_1) \geq s - 1 + a_1$. So suppose $k_2 = k_1 + 1$, which implies $r \leq s - 3$ by Claim 6.4.1. If $r \leq s - 4$, then (6.34) gives $\delta(V_2, U_1) \geq s - 1 + a_1$. So suppose $r = s - 3$. If $a_1 \geq 2$, we have $\delta(V_2, U_1) \geq s$. Otherwise $a_1 = 1$ and $\delta(V_1, U_2) \geq k_2 s + 2s - 5 - r - (\ell_1 s - a_2) \geq s - 2 + a_2$. $\square$

$a_2 = 1$, $b_1 = 2$. In this case, $|V_1| > |U_2|$ by (6.31) and since $\delta(V_1, U_2) \geq s - 1$, there is a vertex $u_2 \in U_2$ such that $\deg(u, V_1) \geq s$ and we are done.

$a_2 = 2$, $b_1 = 1$. If there is a vertex $v \in V_1$ with $\deg(v, U_2) \geq s$, then we apply Claim 6.4.11 to either finish or get $\delta(V_1, U_2) \geq s - 2 + a_2$. However, if $\delta(V_1, U_2) \geq s - 2 + a_2$, then the fact that $a_2 = 2$, contradicts (6.33). So suppose $\Delta(V_1, U_2) \leq s - 1$. Since $\delta(U_2, V_1) \leq s - 1$, there exists $u \in U_2$ such that for all $v \in V_1$ we have

$$n + 3s - 5 \leq \deg(v) + \deg(u) \leq \ell_1 s + s - 1 + s - 1 + \ell_2 s - 2 + s - 1 = n + 3s - 5,$$

thus $G[V_1, U_0 \cup U_1]$ is complete. Let $v_0, v_0' \in V_0$. Let $u_2 \in N(v_0) \cap U_2$ and choose a set of $s - 1$ vertices $L \subseteq N(u_2) \cap V_1$. Since $\Delta(V_1, U_2) \leq s - 1$, there exists $u_2' \in N(v_0') \cap U_2$ such that $\deg(u_2', V_1 \setminus L) \geq s - 1$. Let L' be a set of $s - 1$ vertices in $N(u_2') \cap (V_1 \setminus L)$. Since $G[V_1, U_0 \cup U_1]$ is complete we can move u_2 and u_2'.

$a_2 = 1$, $b_1 = 1$. If there is a vertex $v_1 \in V_1$ with $\deg(v_1, U_2) \geq s$, then we apply Claim 6.4.11 to either finish or get $\delta(V_1, U_2) \geq s - 2 + a_2$. Since $a_2 = 1$, we have $\delta(V_1, U_2) \geq s - 1$. Since $|V_1| \geq |U_2|$, $\delta(V_1, U_2) \geq s - 1$, and $\deg(v_1, U_2) \geq s$, there exists a vertex $u_2 \in U_2$ such that $\deg(u_2, V_1) \geq s$ and we are done. So we may suppose $\Delta(V_1, U_2), \Delta(U_2, V_1) \leq s - 1$. This implies that $\delta(U_2, V_0 \cup V_2) \geq |V_0 \cup V_2| - 1$ and $\delta(V_1, U_0 \cup U_1) \geq |U_0 \cup U_1| - 1$. Since $\delta(V_1, U_2) + \delta(U_2, V_1) \geq 2s - 3$, we can choose $u_2 \in U_2$ such that $\deg(u_2, V_1) \geq s - 1$. Let $v_0 \in V_0 \cap N(u_2)$, which exists since $|V_0| \geq s - 1$ and $\delta(U_2, V_0 \cup V_2) \geq |V_0 \cup V_2| - 1$. We have $\deg(v_0, U_1) > 2s - 2$ and thus $G[N(u_2) \cap V_1, N(v_0) \cap U_1]$ contains a copy of $K_{s-1,s-1}$. This allows us to move one vertex from U_2 as needed.

Case 3 For some $\ell_1 \geq k_1$, we have $\ell_1 s < |V_1 \setminus V_1^M| \leq |V_0 \cup V_1| < \ell_1 s + s$. Set $b_1 := |V_1 \setminus V_1^M| - \ell_1 s > 0$ and $b_2 := |V_2| - (\ell_2 s - s)$. Reset $V_1 := V_1 \setminus V_1^M$ and $V_0 := V_0 \cup V_1^M$. Set $\ell_2 = m - \ell_1$.

Case 3.1 $|U_2 \setminus U_2^M| \geq \ell_2 s$. Let $a_2 := |U_2 \setminus U_2^M| - \ell_2 s$. Reset $U_2 := U_2 \setminus U_2^M$ and

$U_0 := U_0 \cup U_2^M$. We have

$$\delta(V_1, U_2) + \delta(U_2, V_1) \geq 3s - 5 + a_2 + b_1 \geq 2s - 2 + a_2 + b_1. \tag{6.35}$$

Note that $a_2 \geq 0$, $b_1 > 0$, so we are done by Lemma 6.4.5.

Case 3.2 $|U_2 \setminus U_2^M| < \ell_2 s$. Reset $U_2 := U_2 \setminus U_2^M$ and $U_0 := U_0 \cup U_2^M$. We have $|U_1 \cup U_0| > \ell_1 s$.

Case 3.2.1. $|U_0 \cup U_1| \geq \ell_1 s + s$.

Case 3.2.1.1. First suppose that $|U_1| \leq \ell_1 s$. Let $\bar{V}_i = \{v \in V_i : \deg(v, U_{3-i}) \geq s\}$. If $|\bar{V}_1| \geq \frac{n}{8}$ or $|\bar{V}_2| \geq \frac{n}{8}$, then we either get a set of b_1 vertex disjoint s-stars from $\bar{V}_1$ to U_2 or a set of b_2 vertex disjoint s-stars from $\bar{V}_2$ to U_1 by Lemma 6.4.4(i). Since $|U_1| \leq \ell_1 s$ and $\ell_1 s + s \leq |U_0 \cup U_1|$ we can choose a set $U_0' \subseteq U_0$ such that $|(U_0 \cup U_1) \setminus U_0'| = \ell_1 s$ or we can choose a set $U_0' \subseteq U_0$ such that $|(U_0 \cup U_1) \setminus U_0'| = \ell_1 s + s$. For $i = 1, 2$, let $\tilde{V}_i = \{v \in V_i \setminus \bar{V}_i : \deg(v, U_1 \cup U_2) \leq |U_i| + s - 2\}$. We have

$$\delta(\tilde{V}_1, U_0) + \delta(\tilde{V}_2, U_0) \geq n + 3s - 4 - (|U_1| + s - 2 + |U_2| + s - 2) = |U_0| + s \tag{6.36}$$

If $|\tilde{V}_1| \geq \frac{n}{8}$ and $|\tilde{V}_2| \geq \frac{n}{8}$, then by (6.36) and Lemma 6.4.6 we can find a $K_{s,s}$ with b_1 vertices in V_1 and $s - b_1$ vertices in V_2. Then we choose $U_0' \subseteq U_0$ such that $|V_1| - b_1 = \ell_1 s = |(U_0 \cup U_1) \setminus U_0'|$. Otherwise we have $|\tilde{V}_1| < \frac{n}{8}$ or $|\tilde{V}_2| < \frac{n}{8}$. Suppose that $|\tilde{V}_1| < \frac{n}{8}$. First note that for all $v \in V_1 \setminus (\bar{V}_1 \cup \tilde{V}_1)$, $\deg(v, U_2) = s - 1$. Since $|V_1 \setminus (\bar{V}_1 \cup \tilde{V}_1)| > \frac{n}{8}$, we can apply Lemma 6.4.4(i) to get a set of b_1 vertex disjoint $(s-1)$-stars from $V_1 \setminus (\bar{V}_1 \cup \tilde{V}_1)$ to U_2. Let $v_1, v_2, \ldots, v_{b_1}$ be the centers and $L(v_i)$ be the leaf sets for each star.

If $|\tilde{V}_2| \geq \frac{n}{8}$, then for every star we have $|N(L(v_i)) \cap \tilde{V}_2| > \frac{n}{16}$ and for all $\tilde{v} \in N(L(v_i)) \cap \tilde{V}_2$ we have

$$n + 3s - 4 \leq \deg(v_i) + \deg(\tilde{v}) \leq |U_1| + s - 1 + \deg(v_i, U_0) + |U_2| + s - 2 + \deg(\tilde{v}, U_0),$$

which implies $\deg(v_i, U_0) + \deg(\tilde{v}, U_0) \geq |U_0| + s - 1$. So for each v_i, we can find a $K_{s-1,s-1}$ with $s-1$ vertices in $N(v_i) \cap U_0$ and $s-1$ vertices in $N(L(v_i)) \cap \tilde{V}_2$. Since we only need to move at most $s-1$ vertices from V_1, we can always choose a unique vertex from U_0 for each center in V_1 to complete the copy of $K_{s,s}$.

If $|\tilde{V}_2| < \frac{n}{8}$, then $|V_i \setminus (\bar{V}_i \cup \tilde{V}_i)| > \frac{n}{8}$ for $i = 1, 2$. Set $V_i' := V_i \setminus (\bar{V}_i \cup \tilde{V}_i)$ for $i = 1, 2$. We know that $\min\{b_1, s - b_1\} \leq \frac{s}{2}$ and since $s \geq 3$, $\min\{b_1, s - b_1\} \leq s - 2$. Without loss of generality, suppose $b_1 \leq s - b_1$. Since $|V_1'| > \frac{n}{8}$, we start by taking a set of b_1 vertex disjoint $(s-1)$-stars from V_1' to U_2. Let $v_1, v_2, \ldots, v_{b_1}$ be the centers and $L(v_i)$ be the leaf sets for each star. For every star we have $|N(L(v_i)) \cap V_2'| > \frac{n}{16}$ and for all $v' \in N(L(v_i)) \cap V_2'$ we have

$$n+3s-4 \leq \deg(v_i)+\deg(v') \leq |U_1|+s-1+\deg(v_i, U_0)+|U_2|+s-1+\deg(v', U_0),$$

which implies $\deg(v_i, U_0) + \deg(v', U_0) \geq |U_0| + s - 2$. So for each v_i, we can find a $K_{s-2,s-1}$ with $s-2$ vertices in $U_0 \cap N(v_i)$ and $s-1$ vertices in $N(L(v_i)) \cap V_2'$. Since we only need to move at most $s-2$ vertices from V_1, we can always choose a unique vertex from U_0 for each center in V_1 to complete the copy of $K_{s,s}$.

Case 3.2.1.2. $|U_1| > \ell_1 s$. Let $a_1 := |U_1| - \ell_1 s$. In this case we have

$$\delta(V_2, U_1) \geq k_2 s + 2s - 5 - r - (\ell_2 s - a_1) = (k_2 - \ell_2)s + 2s - 5 - r + a_1. \quad (6.37)$$

Case 3.2.1.2.1. $\ell_1 > k_1$. Then $\ell_2 < k_2$ and (6.37) gives $\delta(V_2, U_1) \geq s - 1 + a_1$ and we are done by moving vertices to V_1.

Case 3.2.1.2.2. $\ell_1 = k_1$ and so $\ell_2 = k_2$.

Suppose $k_2 = k_1$. Then $r \leq \frac{s-6}{2}$ and we have $\delta(V_2, U_1) \geq s - 1 + a_1$ so we are done by moving vertices to V_1.

Suppose $k_2 = k_1 + 1$. This implies $r \leq s - 3$. Now we have $\delta(V_2, U_1) \geq s - 2 + a_1$. If $\delta(V_2, U_1) \geq s$, then we would be done by moving vertices to V_1. So suppose $a_1 = 1$ and $r = s - 3$. Recall $b_2 = |V_2| - (k_2 s - s)$. We have $\delta(U_1, V_2) \geq k_1 s + s + r - (k_1 s + s - b_2) = s - 3 + b_2$, so we would be done by moving vertices to V_1 unless $1 \leq b_2 \leq 2$. Furthermore, we have

$$\delta(U_2, V_1) \geq k_1 s + s + r - (k_1 s + s - b_1) = s - 3 + b_1 \tag{6.38}$$

Suppose $b_2 = 2$. Since $a_1 = 1$ and $k_2 = k_1 + 1$ we have $|V_2| > |U_1|$. Since $\delta(V_2, U_1) \geq s - 1$, there exists a vertex $u_1 \in U_1$ such that $\deg(u_1, V_2) \geq s$. If $b_1 \geq 3$, then (6.38) implies $\delta(U_2, V_1) \geq s$ and thus we can move b_1 vertices from V_1 by Lemma 6.4.4(iii). Otherwise let $V_2' = \{v \in V_2 : \deg(v, U_1) \leq s - 1\}$. If $|V_2 \setminus V_2'| > 2s\alpha^{1/3}k_2 s$, then since $\Delta(U_1, V_2) \leq 2\alpha^{1/3}k_2 s$ there would be two vertex disjoint s-stars from $V_2 \setminus V_2'$ to U_1. So suppose $|V_2'| > \frac{n}{4}$. Note that for all $v \in V_2'$, $\deg(v, U_0 \cup U_2) \geq k_2 s + 2s - 5 - r - (s - 1) = k_2 s - 1 = |U_0 \cup U_2|$, so $G[V_2', U_0 \cup U_2]$ is complete. If $b_1 = 1$, then since $\delta(V_1, U_0 \cup U_2) \geq 2s - 3 \geq s$ we can move a vertex from V_1, giving $|U_1 \setminus \{u_1\}| = k_1 s = |V_1| - 1$. So suppose $b_1 = 2$. If there is a vertex $v_1 \in V_1$ such that $\deg(v_1, U_0 \cup U_2) \geq 2s$, then we would be done since $\delta(V_1, U_0 \cup U_2) \geq 2s - 3 \geq s$ and $G[V_2', U_0 \cup U_2]$ is complete so we can move two vertices from V_1. So suppose $\Delta(V_1, U_0 \cup U_2) \leq 2s - 1$. Then $\delta(V_1, U_1) \geq k_2 s + 2s - 5 - r - (2s - 1) = k_2 s - s - 1 = k_1 s - 1 = |U_1| - 2$. Since $b_1 = 2$, we have $\delta(U_2, V_1) \geq s - 1$ by (6.38). Thus there are two vertex disjoint s-stars from U_2 to V_1 with leaf sets L_1 and L_2. Let $\tilde{U}_1 := U_1 \cap (N(L_1) \cap N(L_2))$ and note that since $\delta(V_1, U_1) \geq |U_1| - 2$, we have $|\tilde{U}_1| \geq |U_1| - 4s$. Now since $\delta(V_2', U_1) \geq s - 1$ and $\Delta(U_1, V_2) \leq 2\alpha^{1/3}k_2 s$, there exist two vertex disjoint $(s-1)$-stars from V_2' to $\tilde{U}_1$. Since $G[\tilde{U}_1, L_1 \cup L_2]$ and $G[V_2', U_0 \cup U_2]$ are complete, we can move two vertices from V_2 to V_1 and U_2 to U_1. We finish by moving $s - 3$ vertices from U_0 to U_1 and $s - 4$ vertices from V_0 to V_1, giving $|U_1| + 2 + s - 3 = k_1 s + s = |V_1| + 2 + s - 4$.

Suppose $b_2 = 1$. If there exists a vertex $v_2 \in V_2$ such that $\deg(v_2, U_1) \geq s$, then we would be done by moving v_2 to V_1. So suppose $\Delta(V_2, U_2) \le s-1$ and thus $\delta(V_2, U_0 \cup U_2) \ge k_2 s + 2s - 5 - r - (s-1) = k_2 s - 1 = |U_0 \cup U_2|$. Let $v_2 \in V_2$ and let L be the set of leaves in U_1 of an $(s-1)$-star with center v_2. Let $V_1' = N(L) \cap V_1$ and note that $|V_1'| \ge |V_1| - 2s\alpha^{1/3} k_1 s$. Since $\delta(V_1', U_0 \cup U_2) \ge k_2 s + 2s - 5 - r - (k_1 s + 1) = 2s - 3 \ge s$, there exists a vertex $u_2 \in U_0 \cup U_2$ such that $\deg(u, V_1') \ge s - 1$. Since $G[V_2, U_0 \cup U_2]$ is complete, we can move v_2 and u_2. We finish by moving $s-2$ vertices from U_0 to U_1 and $s - 1 - b_1$ vertices from V_0 to V_1 giving $|U_1| + 1 + s - 2 = k_1 s + s = |V_1| + 1 + s - 1 - b_1$.

Finally, suppose $k_2 \ge k_1 + 2$. Here we have $\delta(U_1, V_2) \ge k_1 s + s + r - (k_1 s + s - b_2) = r + b_2$. If $r \ge s - b_2$, then $\delta(U_1, V_2) \ge s$ and we would be done by moving vertices from V_2 to V_1, so suppose $r \le s - 1 - b_2$. Then we have

$$\delta(V_2, U_1) \ge k_2 s + 2s - 5 - r - (k_2 s - a_1) \ge s - 4 + a_1 + b_2. \tag{6.39}$$

We would have $\delta(V_2, U_1) \ge s$ and be done unless $2 \le a_1 + b_2 \le 3$.

Suppose $a_1 = 2$, $b_2 = 1$. If $r \le s - 3$, then $\delta(V_2, U_1) \ge s$ by (6.39), so suppose $r = s - 2$. We have $\delta(U_1, V_2), \delta(V_2, U_1) \ge s - 1$ and $\delta(V_1, U_0 \cup U_2) \ge k_2 s + 2s - 5 - r - (k_1 s + 2) \ge 3s - 5$. If there was a vertex $v_2 \in V_2$ such that $\deg(v_2, U_1) \ge s$, then we would be done by moving v_2 to V_1. So suppose $\Delta(V_2, U_1) \le s - 1$ and thus $\delta(V_2, U_0 \cup U_2) \ge k_2 s + 2s - 5 - r - (s-1) = k_2 s - 2 = |U_0 \cup U_2|$. Let $v_2 \in V_2$ and let $L := N(v_2) \cap U_1$. Every vertex in $N(L) \cap V_1 =: V_1'$ has at least $3s - 5 \ge s$ neighbors in $U_0 \cup U_2$, so there exists a vertex $u_2 \in U_0 \cup U_2$ such that $\deg(u_2, V_1') \ge 3s - 5 \ge s - 1$. Then since $G[V_2, U_0 \cup U_2]$ is complete, we have a copy of $K_{s,s}$ which allows us to move v_2. We finish by moving $s - 3$ vertices from U_0 to U_1 and $s - 1 - b_1$ vertices from V_0 to V_1 giving

$|U_1| + 1 + s - 3 = k_1 s + s = |V_1| + 1 + s - 1 - b_1$.

Suppose $a_1 = 1$, $b_2 = 2$. If $r \leq s - 4$, then $\delta(V_2, U_1) \geq s$ by (6.39), so suppose $r = s - 3$. We have $\delta(U_1, V_2), \delta(V_2, U_1) \geq s - 1$ and $\delta(V_1, U_0 \cup U_2) \geq k_2 s + 2s - 5 - r - (k_1 s + 1) \geq 3s - 3$. Let $V_2' = \{v \in V_2 : \deg(v, U_1) \leq s - 1\}$. If $|V_2 \setminus V_2'| > 2s\alpha^{1/3} k_2 s$, then since $\Delta(U_1, V_2) \leq 2\alpha^{1/3} k_2 s$ there would be two vertex disjoint s-stars from $V_2 \setminus V_2'$ to U_1, so suppose not. Then $|V_2'| > \frac{n}{4}$. Note that $G[V_2', U_0 \cup U_2]$ is complete. Since $|V_2| > |U_1|$ and $\delta(V_2, U_1) \geq s - 1$, there exists a vertex $u_1 \in U_1$ such that $\deg(u_1, V_2) \geq s$. Now we must move b_1 vertices from V_1. If say $\frac{n}{8}$ vertices in V_1 have at least s neighbors in U_0, then we can find a $K_{s,s}$ with s vertices in U_0, b_1 vertices in V_1 and $s - b_1$ vertices in V_2 by Lemma 6.36 and the fact that $G[V_2', U_0 \cup U_2]$ is complete. Otherwise we have $\frac{n}{4}$ vertices with at most $s - 1$ neighbors in U_0 and consequently at least $3s - 3 - (s - 1) \geq s$ neighbors in U_2. Either way there exists b_1 vertex disjoint s-stars from V_1 to U_2.

Suppose $a_1 = 1 = b_2$. If there is a vertex in V_2 with s neighbors in U_1, then we would be done, so suppose not. Since $b_2 = 1$, we have $r \leq s - 2$. If $r = s - 2$, then $\delta(U_1, V_2) \geq s - 1$. If $r \leq s - 3$, then $\delta(V_2, U_1) \geq s - 1$. So either way there is a vertex $v_2 \in V_2$ such that $\deg(v_2, U_1) = s - 1$. Let $L := N(v_2) \cap U_1$. We have $\delta(V_2, U_0 \cup U_2) \geq k_2 s + 2s - 5 - r - (s - 1) \geq k_2 s - 2 = |U_0 \cup U_1| - 1$. Since $\delta(V_1, U_0 \cup U_2) \geq 3s - 4$, every vertex in $N(L) \cap V_1 =: V_1'$ has at least $3s - 5$ neighbors in $N(v_2) \cap (U_0 \cup U_2)$. So there exists a vertex $u_2 \in N(v_2) \cap (U_0 \cup U_2)$ with at least $3s - 5 \geq s - 1$ neighbors in V_1'. This gives us a copy of $K_{s,s}$ which allows us to move v_2.

Case 3.2.2. $\ell_1 s < |U_0 \cup U_1| < \ell_1 s + s$.

Case 3.2.2.1. $|U_1| \leq \ell_1 s$. Thus there exists $U_0' \subseteq U_0$ such that $|(U_0 \cup U_1) \setminus U_0'| = \ell_1 s$. So we try to make $|V_1| = \ell_1 s$ or $|V_2| = \ell_2 s$. Recall

$\ell_2 = m - \ell_1$ and $b_1 = |V_1| - \ell_1 s$. Let $a_2 := |U_2| - (\ell_2 s - s)$. We have

$$\delta(V_1, U_2) + \delta(U_2, V_1) \geq n + 3s - 5 - (\ell_1 s + s - a_2 + \ell_2 s - b_1) = 2s - 5 + a_2 + b_1. \quad (6.40)$$

If $\delta(V_1, U_2) \geq s$ or $\delta(U_2, V_1) \geq s$, then we would be able to find b_1 vertex disjoint s-stars from V_1 to U_2 by Lemma 6.4.4(i) or (iii) and we are done. So suppose $\delta(V_1, U_2) \leq s - 1$ and $\delta(U_2, V_1) \leq s - 1$, thus $2 \leq a_2 + b_1 \leq 3$. If $\delta(V_1, U_2) + \delta(U_2, V_1) = 2s - 2$, then we have $\delta(V_1, U_2) = s - 1$ and $\delta(U_2, V_1) = s - 1$. Furthermore, we have

$$\begin{aligned}\delta(U_0 \cup U_1, V_0 \cup V_2) + \delta(V_0 \cup V_2, U_0 \cup U_1) &\geq n + 3s - 5 - (\ell_1 s + b_1 + \ell_2 s - s + a_2) \\ &= 4s - 5 - a_2 - b_1. \quad (6.41)\end{aligned}$$

Let $U_2' := \{u \in U_2 : \deg(u, V_1) \leq s - 1\}$.

Suppose $a_2 = 2$, $b_1 = 1$. If there is a vertex $v_1 \in V_1$ with $\deg(v_1, U_2) \geq s$, then we are done by moving v_1 to V_2. If $e(U_2, V_1) > (s-1)|V_1|$, then there exists a vertex $v_1 \in V_1$ such that $\deg(v_1, U_2) \geq s$, so suppose not. If $|U_2 \setminus U_2'| > 3\alpha^{2/3} k_2 s$, then since $|V_1| - |U_2| \leq 2\alpha^{2/3} k_2 s$ we have $e(U_2, V_1) > (s-1)|V_1|$, so suppose not. Then $|U_2'| \geq |U_2| - 3\alpha^{2/3} k_2 s$. For all $v \in V_1$ and $u \in U_2'$ we have

$$n + 3s - 5 \leq \deg(v) + \deg(u) \leq \ell_1 s + s - 1 + s - 1 + \ell_2 s - 2 + s - 1 = n + 3s - 5, \quad (6.42)$$

thus $G[V_1, U_0 \cup U_1]$ is complete and $G[U_2', V_0 \cup V_2]$ is complete. Since $\delta(U_2', V_1) \geq s - 1$, there exists a vertex $v_1 \in V_1$, such that $\deg(v_1, U_2') = s - 1$. Let $u_0 \in U_0$ and note that $\deg(u_0, V_2) > s$. Since $G[V_1, U_0 \cup U_1]$ is complete we can move v_1 from V_1 along with u_0.

Suppose $a_2 = 1, b_1 = 2$. First suppose that there exists $v_1 \in V_1$ with at least s neighbors in U_2. Let $L \subseteq N(v_1) \cap U_2$ with $|L| = s$. In this case we can apply the argument of the previous paragraph to the sets $V_1 \setminus v_1$ and $U_2 \setminus L$. So suppose that $\Delta(V_1, U_2) \le s-1$ and $|U_2'| \ge |U_2| - 2\alpha^{2/3}k_2 s$. Equation (6.42) holds which implies that $G[V_1, U_0 \cup U_1]$ is complete and $G[U_2', V_0 \cup V_2]$ is complete. Every vertex in U_2' has $s-1$ neighbors in V_1, so there are two vertex disjoint $(s-1)$-stars from V_1 to U_2' with centers v_1 and v_1'. Since $G[V_1, U_0 \cup U_1]$ is complete and $|U_0| \ge s-1 \ge 2$, there exist $u_0, u_0' \in U_0$. Since $\deg(u_0, V_2), \deg(u_0', V_2) > 2s$, we can move v_1 and v_1' by taking u_0 and u_0'. Then let $U_0' \subseteq U_0$ so that $|U_1| + |U_0'| = \ell_1 s = |V_1| - 2$.

Suppose $a_2 = 1, b_1 = 1$. If there is a vertex $v_1 \in V_1$ such that $\deg(v_1, U_2) \ge s$, then we can move v_1 to V_2 and be done, so suppose $\Delta(V_1, U_2) \le s-1$. First suppose that $\Delta(U_2, V_1) \le s-1$. For all $v \in V_1$ and $u \in U_2$ we have
$n + 3s - 5 \le \deg(u) + \deg(v) \le \ell_1 s + s - 1 + s - 1 + \ell_2 s - 1 + s - 1 = n + 3s - 4$. Thus $\delta(V_1, U_0 \cup U_1) \ge |U_0 \cup U_1| - 1$ and $\delta(U_2, V_0 \cup V_2) \ge |V_0 \cup V_2| - 1$. Let $v_1 \in V_1$ such that $\deg(v_1, U_2) = s-1$, which exists since $\delta(V_1, U_2) \ge s-1$ or $\delta(U_2, V_1) \ge s-1$. Let $L := N(v_1) \cap U_2$ and $V_2' := N(L) \cap V_2$; note that $|V_2'| \ge |V_2| - s$ since $\delta(U_2, V_0 \cup V_2) \ge |V_0 \cup V_2| - 1$. Finally let $u_0 \in U_0 \cap N(v_1)$, which exists since $\delta(V_1, U_0 \cup U_1) \ge |U_0 \cup U_1| - 1$ and $|U_0| \ge s-1$. Since $\deg(u_0, V_2') > s$, we can move v_1 along with u_0. So we may suppose that there exists some $u_2 \in U_2$ such that $\deg(u_2, V_1) \ge s$. Let $V_2' := \{v \in V_2 : \deg(v, U_1) \le s-1\}$. If say $|V_2 \setminus V_2'| > \frac{n}{8}$, then since $\Delta(U_1, V_2) \le 2\alpha^{1/3}k_2 s$ we could move b_2 vertices from V_2 and we would be done. So we may suppose that $|V_2'| > \frac{n}{4}$. Note that we have

$$\delta(V_1, U_0) + \deg(V_2', U_0) \ge n + 3s - 4 - (|U_1| + s - 1 + |U_2| + s - 1) = |U_0| + s - 2. \quad (6.43)$$

Let $v_1 \in V_1$ such that $\deg(v_1, U_2) = s-1$ and let $L := N(v_1) \cap U_2$. Let $\tilde{V}_2 := V_2' \cap N(L)$ and note that $|\tilde{V}_2| > \frac{n}{8}$. For all $\tilde{v} \in \tilde{V}_2$ we have $\deg(\tilde{v}, N(v_1) \cap U_0) \ge s-2$ by (6.43). Since $|\tilde{V}_2| > |N(v_1) \cap U_0|$, there exists $u_0 \in N(v_1) \cap U_0$ such that $\deg(u_0, \tilde{V}_2) \ge s-1$. This completes a copy of $K_{s,s}$

which allows us to move v_1.

Case 3.2.2.2. $|U_1| > \ell_1 s$. Let $a_1 := |U_1| - \ell_1 s$. Recall $\ell_2 = m - \ell_1$, $b_1 = |V_1| - \ell_1 s$, $a_2 = |U_2| - (\ell_2 s - s)$, and $b_2 := |V_2| - (\ell_2 s - s)$. We have

$$\delta(V_1, U_2)+\delta(U_2, V_1) \geq n+3s-5-(\ell_1 s+s-a_2)-(\ell_2 s-b_1) = 2s-5+a_2+b_1 \quad (6.44)$$

and

$$\delta(V_2, U_1)+\delta(U_1, V_2) \geq n+3s-5-(\ell_2 s-a_1)-(\ell_1 s+s-b_2) = 2s-5+a_1+b_2 \quad (6.45)$$

Case 3.2.2.2.1. For some $i \in \{1, 2\}$ we have $\delta(V_i, U_{3-1}) \geq s$ or $\delta(U_{3-i}, V_i) \geq s$. Without loss of generality (all cases are similar, but not exactly the same), suppose $\delta(V_2, U_1) \geq s$. This implies by Lemma 6.4.4(iii) that there is a set of a_1 vertex disjoint s-stars from U_1 to V_2 and a set of b_2 vertex disjoint s-stars from V_2 to U_1. So if we can move a_2 vertices from U_2 or b_1 vertices from V_1, then we say that we are done. If $\delta(V_1, U_2) \geq s$ or $\delta(U_2, V_1) \geq s$, then we can apply Lemma 6.4.4(i) or (iii) and we are done, so suppose not. This implies $2 \leq a_2 + b_1 \leq 3$ by (6.44). Furthermore, if $a_2 + b_1 = 3$, then $\delta(V_1, U_2) + \delta(U_2, V_1) \geq 2s - 2$ and we may suppose $\delta(V_1, U_2) = s - 1$ and $\delta(U_2, V_1) = s - 1$. Let $U_2' := \{u \in U_2 : \deg(u, V_1) \leq s - 1\}$ and $V_1' := \{v \in V_1 : \deg(v, U_2) \leq s - 1\}$.

Since $2 \leq a_2 + b_1 \leq 3$, either $a_2 = 1$ or $b_1 = 1$. Without loss of generality suppose $a_2 = 1$ and thus $1 \leq b_1 \leq 2$. If there is a vertex $u_2 \in U_2$ such that $\deg(u_2, V_1) \geq s$, then we can move u_2 and we are done, so suppose $\Delta(U_2, V_1) \leq s - 1$. For all $u \in U_2$ and $v \in V_1'$ we have $n + 3s - 5 \leq \deg(u) + \deg(v) \leq \ell_1 s + s - 1 + s - 1 + \ell_2 s - b_1 + s - 1 \leq n + 3s - 4$ and thus $\delta(U_2, V_0 \cup V_2) \geq |V_0 \cup V_2| - 1$ and $\delta(V_1', U_0 \cup U_1) \geq |U_0 \cup U_1| - 1$. If $b_1 = 1$, then we may suppose $\Delta(V_1, U_2) \leq s - 1$ or else we are done. In this case $V_1' = V_1$. If $b_1 = 2$, then $\delta(V_1, U_2) \geq s - 1$. If there are two vertex disjoint s-stars from V_1 to U_2, then we are done since $b_1 \leq 2$. This implies that $|V_1'| \geq |V_1| - 2s\alpha^{1/3}k_2 s$. So in either case there exists a vertex $u_2 \in U_2$ such that $\deg(u_2, V_1') = s-1$. Since $\delta(V_2, U_1) \geq s$, there is a set of s vertex disjoint s-stars from $N(u_2) \cap V_2$ to U_1. Finally since $\delta(V_2', U_0 \cup U_1) \geq |U_0 \cup U_1| - 1$, the leaf set of one of the s-stars from V_2 to U_1 will form a $K_{s-1,s-1}$ with $s-1$ vertices in $N(u_2) \cap V_1'$ and $s-1$ vertices in U_1. Then we move $b_2 - 1$ more vertices from V_2.

Case 3.2.2.2.2. For all $i \in \{1,2\}$ we have $\delta(V_i, U_{3-i}) \leq s-1$ and $\delta(U_{3-i}, V_i) \leq s-1$. So by (6.44) and (6.45), we may suppose $2 \leq a_1 + b_2 \leq 3$ and $2 \leq a_2 + b_1 \leq 3$. We have

$$\delta(V_2, U_1) \geq k_2 s + 2s - 5 - r - (\ell_2 s - a_1) = (k_2 - \ell_2)s + 2s - 5 - r + a_1 \geq (k_2 - \ell_2)s + s - 4 + a_1. \tag{6.46}$$

If $\ell_1 > k_1$, then $k_2 > \ell_2$ and $\delta(V_2, U_1) \geq s$ by (6.46). So suppose $\ell_1 = k_1$ and thus $\ell_2 = k_2$. We also have

$$\delta(V_1, U_2) \geq k_2 s + 2s - 5 - r - (k_1 s + s - a_2) = (k_2 - k_1)s + s - 5 - r + a_2. \tag{6.47}$$

If $k_2 \geq k_1 + 2$, then $\delta(V_1, U_2) \geq s$ by (6.47). So suppose $k_2 \leq k_1 + 1$. If $k_2 = k_1$, then $r \leq \frac{s-6}{2}$ by Claim 6.4.1 and thus (6.46) gives $\delta(V_2, U_1) \geq 2s - 5 - \frac{s-6}{2} + a_1 \geq s$. So suppose $k_2 = k_1 + 1$ which implies $r \leq s - 3$ by Claim 6.4.1. If $r \leq s-4$, then (6.46) implies $\delta(V_2, U_1) \geq s - 1 + a_1 \geq s$. So suppose $r = s-3$. Finally if either $a_1 \geq 2$ or $a_2 \geq 2$, then (6.46) or (6.47) implies $\delta(V_1, U_2) \geq s$ or $\delta(V_2, U_1) \geq s$. So suppose $a_1 = 1 = a_2$ and thus $\delta(V_1, U_2) = s - 1 = \delta(V_2, U_1)$. For $i = 1, 2$, let $V_i' := \{v \in V_i : \deg(v, U_{3-i}) \leq s-1\}$. For all $v \in V_i$, $\deg(v, U_0 \cup U_i) \geq k_2 s + 2s - 5 - r - (s-1) = k_2 s - 1 = |U_0 \cup U_i|$, thus $G[V_i', U_0 \cup U_i]$ is complete.

First suppose $b_1 = 2 = b_2$. Since $|V_1| > |U_2|$ and $|V_2| > |U_1|$, there are vertices $u_1 \in U_1$ and $u_2 \in U_2$ such that $\deg(u_1, V_2) \geq s$ and $\deg(u_2, V_1) \geq s$. If

$|V_i \setminus V_i'| > 2s\alpha^{1/3}k_2 s$ for some i, then we would be done by moving two vertices from $V_i \setminus V_i'$ and moving u_i from U_i for some $i = 1, 2$. So we may assume that $|V_i'| \geq |V_i| - s\alpha^{1/3}n$ for $= 1, 2$. Since $\delta(V_1', U_2) \geq s - 1$ and $|V_1'| \geq |V_1| - s\alpha^{1/3}n$, there exists $u_2 \in U_2$ such that $\deg(u_2, V_1') \geq s - 2$ and there exists $u_1 \in U_1$ such that $\deg(u_1, V_2') \geq 2$. Now since $G[V_1', U_0 \cup U_1]$ and $G[V_2', U_0 \cup U_2]$ are complete, we have a copy of $K_{s,s}$ with $s - 2$ vertices in V_1', 2 vertices in V_2', $s - 2$ vertices in U_0, 1 vertex in U_1 and 1 vertex in U_2. Then we move the remaining $s - 4$ vertices from V_0 to V_1

Now suppose $b_i = 2$ and $b_{3-i} = 1$ for some i. Without loss of generality, suppose $b_1 = 1$ and $b_2 = 2$. Since $|V_2| > |U_1|$, there is a vertex $u_1 \in U_1$ such that $\deg(u_1, V_2) \geq s$. So we would be done unless $\Delta(V_1, U_2) \leq s - 1$ and thus $V_1' = V_1$. Let $u_2, u_2' \in U_2$ be the centers of two vertex disjoint $(s-1)$-stars from U_2 to V_1. Then since $\delta(V_2, U_1) \geq s - 1$ we can choose two vertex disjoint $(s-1)$-stars from $(N(u_2) \cap N(u_2')) \cap V_2$ to U_1. Then since $G[V_1, U_0 \cup U_1]$ is complete we are done.

Finally suppose $b_1 = 1 = b_2$. If there exists $v_2 \in V_2$ (without loss of generality) such that $\deg(v, U_1) \geq s$, then there is a vertex $u_1 \in U_1$ such that $\deg(u_1, V_2) \geq s$. So we would be done unless $\Delta(V_1, U_2) \leq s - 1$ and $\Delta(U_2, V_1) \leq s - 1$. Thus $G[V_1, U_0 \cup U_1]$ is complete. Let $u_2, u_2' \in U_2$ be the centers of two vertex disjoint $(s-1)$-stars from U_2 to V_1. Then since $\delta(V_2, U_1) \geq s - 1$ we can choose two vertex disjoint $(s-1)$-stars from $N(u_2) \cap N(u_2') \cap V_2$ to U_1. Then since $G[V_1, U_0 \cup U_1]$ is complete we are done. Otherwise $\Delta(V_i, U_{3-i}) \leq s - 1$ for $i = 1, 2$ in which case $G[V_i, U_0 \cup U_i]$ is complete for $i = 1, 2$. Let $u_1 \in U_1$ such that $\deg(u_1, V_2) \geq s - 1$ and let $v_1 u_2 \in E(V_1, U_2)$. Since $G[V_1, U_0 \cup U_1]$ and $G[V_2, U_0 \cup U_2]$ are complete, we have a copy of $K_{s,s}$ with $s - 1$ vertices in V_2, 1 vertex in V_1, $s - 2$ vertices in U_0, 1 vertex in U_1, and 1 vertex in U_2. Then we move the remaining $s - 2$ vertices from V_0 to V_2.

6.5 Examples when δ_U is small

6.5.1 A probabilistic example

We prove Theorem 6.1.15. We ignore floors and ceilings since they are not vital to our calculations.

Proof. Given a positive integer s, let $c := s^{1/3}$, $d := 2c$, $a := s^c$, and $b := \frac{s}{d}a = \frac{s^{c+1}}{d}$. Let s be large enough so that $s^{2s^{2/3}}\left(\frac{(3d)^d}{s^{(c-1)s}}\right)^s < \frac{1}{2}$. Let A, B be sets such that $|A| = a$ and $|B| = b$. Consider the random bipartite graph by adding the pair from $A \times B$ with probability $p := \frac{3d}{s}$ (all choices made independently). Then for $u \in A$, $\mathbb{E}(\deg(u)) = pb = 3s^c$ and for $v \in B$, $\mathbb{E}(\deg(v)) = pa = 3ds^{c-1}$. The probability that there exists $u \in A$ with $\deg(u) < 2s^c$ or $v \in B$ with $\deg(v) < 2ds^{c-1}$ is less than 1/2 by a standard application of Chernoff's bound. In addition, the probability that there exists $K_{d,s}$ with d vertices in A is at most

$$\binom{a}{d}\binom{b}{s}p^{ds} < a^d b^s p^{ds} = s^{cd}\frac{s^{(c+1)s}}{d^s}\frac{(3d)^{ds}}{s^{ds}} = \frac{s^{cd}}{s^{(d-(c+1))s}}\left(\frac{(3d)^d}{d}\right)^s$$
$$\leq s^{2s^{2/3}}\left(\frac{(3d)^d}{s^{(c-1)s}}\right)^s < \frac{1}{2}.$$

Consequently there exists a graph H on $A \cup B$ such that

- $\deg(u) \geq 2s^c$ for every $u \in A$, $\deg(v) \geq 2ds^{c-1}$ for $v \in B$ and
- H has no $K_{d,s}$ with d vertices in A.

Let G be obtained from H by adding a set A' of $n - a$ vertices to A and a set B' of $n - b$ vertices to B with n large as usual. We add all edges between A' and $B \cup B'$. The sum of degrees in G is at least $2s^c + (n - s^c) = n + s^c$.

Suppose that G can be tiled with $K_{s,s}$. Since $G[A, B']$ is empty, any copy of $K_{s,s}$ touching A must have s vertices in B. Also, any copy touching A must have at most $d-1$ vertices from A, since H has no $K_{d,s}$ with d vertices in A. So the number of copies touching A is at least $\frac{a}{d-1}$. However, this implies that $s\frac{a}{d-1} \leq |B| = \frac{s}{d}a$, a contradiction. $\square$

6.5.2 Concrete examples

We do not provide a general class of counterexamples in this section, however we provide two specific cases of graphs with $\delta_U = O(1)$ and $\delta_U + \delta_V \geq n + 2s - 2\lceil\sqrt{s}\rceil + c(s)$ which cannot be tiled with $K_{s,s}$.

Let $s = 5$. First note that $n + 2s - 2\lceil\sqrt{s}\rceil + c(s) = n + 5$. We will show that there exists a graph with $\delta_U + \delta_V = n + 5$ which cannot be tiled with $K_{5,5}$. Let $G[U, V]$ be a balanced bipartite graph with the following properties. Let $|U| = |V| = 5m =: n$. Partition U as $U = U_1 \cup U_2$ where $|U_1| = 3$, $|U_2| = n - 3$ and V as $V = V_1 \cup V_2$ where $|V_1| = 4$ and $|V_2| = n - 4$. Let $G[U_i, V_i]$ be complete for $i = 1, 2$. Let $G[V_1, U_2]$ be complete. Finally suppose $U_1 = \{a, b, c\}$ and let $N(a) \cap V_2 = \{a_1, a_2, a_3, a_4\}$, $N(b) \cap V_2 = \{b_1, b_2, b_3, b_4\}$, and $N(c) \cap V_2 = \{c_1, c_2, c_3, c_4\}$ where $a_4 = b_1$, $b_4 = c_1$, $c_4 = a_1$, and $a_2, a_3, b_2, b_3, c_2, c_3$ are distinct (see Figure 6.5.2). Note that $\delta_U = 8$, $\delta_V = n - 3$ and thus $\delta_U + \delta_V = n + 5 = n + 2s - 2\lceil\sqrt{s}\rceil + c(s)$. Suppose G can be tiled with $K_{5,5}$. Since $|N(a, b, c)| = 4$, it is not the case that a, b, c all belong to one copy. So either a, b, and c are in distinct copies, or say b and c belong to the same copy. First suppose that a, b, and c are in distinct copies and let A, B and C be copies of $K_{5,5}$ such that $a \in A$, $b \in B$, and $c \in C$. Let $\alpha := |V(A) \cap V_1|$, $\beta := |V(B) \cap V_1|$, and $\gamma := |V(C) \cap V_1|$. Since $|V_1| = 4$, we have $\alpha + \beta + \gamma \leq 4$. Also since $|(N(a) \cup N(b) \cup N(c)) \cap V_2| = 9$, we have $5 - \alpha + 5 - \beta + 5 - \gamma \leq 9$ which implies $6 \leq \alpha + \beta + \gamma$, a contradiction. So suppose that b and c belong to the same copy. But since $|N(b, c) \cap V_2| = 1$, we have $|N(b, c) \cap V_1| = 4$. But since $|N(a) \cap V_2| = 4$, it is not possible for a to belong to a disjoint copy of $K_{5,5}$.

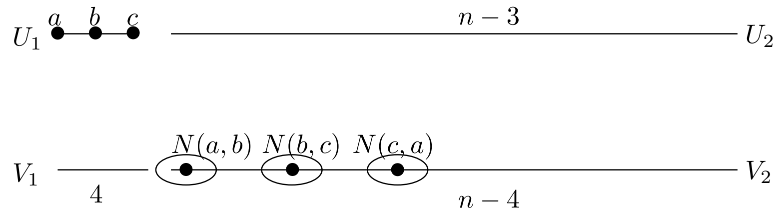


Figure 6.4: $s = 5$

Let $s = 10$. First note that $n + 2s - 2\lceil\sqrt{s}\rceil + c(s) = n + 13$. We will show that there exists a graph with $\delta_U + \delta_V = n + 15$ which cannot be tiled with $K_{10,10}$. Let $G[U, V]$ be a balanced bipartite graph with the following properties. Let $|U| = |V| = 10m =: n$. Partition U as $U = U_1 \cup U_2$ where $|U_1| = 4$, $|U_2| = n - 4$ and V as $V = V_1 \cup V_2$ where $|V_1| = 9$ and $|V_2| = n - 9$. Let $G[U_i, V_i]$ be complete for $i = 1, 2$. Let $G[V_1, U_2]$ be complete. Finally suppose $U_1 = \{a, b, c, d\}$ and let $N(a) \cap V_2 = \{a_1, \ldots, a_{10}\}$, $N(b) \cap V_2 = \{b_1, \ldots, b_{10}\}$, $N(c) \cap V_2 = \{c_1, \ldots, c_{10}\}$, and $N(d) = \{d_1, \ldots, d_{10}\}$ where $\{a_7, a_8, a_9, a_{10}\} = \{b_1, b_2, b_3, b_4\}$, $\{b_7, b_8, b_9, b_{10}\} = \{c_1, c_2, c_3, c_4\}$, $\{c_7, c_8, c_9, c_{10}\} = \{d_1, d_2, d_3, d_4\}$, $\{d_7, d_8, d_9, d_{10}\} = \{a_1, a_2, a_3, a_4\}$ and $a_5, a_6, b_5, b_6, c_5, c_6, d_5, d_6$ are distinct (see Figure 6.5.2). Note that $\delta_U = 19$, $\delta_V = n - 4$ and thus $\delta_U + \delta_V = n + 15 = n + 2s - 2\lceil\sqrt{s}\rceil + c(s)$. Suppose G can be tiled with $K_{10,10}$. Since $|N(x, y, z)| = 9$, for any $x, y, z \in \{a, b, c, d\}$ it is not the case that any three of a, b, c, d all belong to one copy. A similar analysis as given in the $s = 5$ case will lead to a contradiction here.

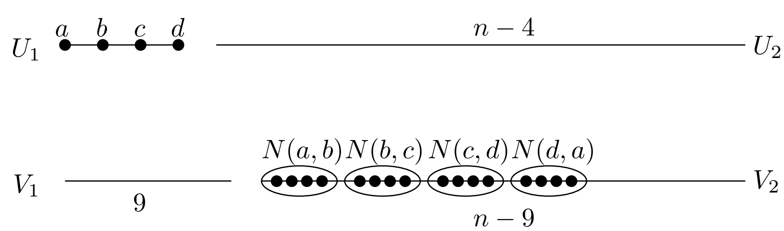


Figure 6.5: $s = 10$

6.6 Conclusion

In Theorem 6.1.8 and Theorem 6.1.13 we show that if $\delta(G)$ is $\Omega(n)$, then $\delta_U + \delta_V \geq n + 3s - 5$ suffices to tile G with $K_{s,s}$. The only example we have which shows $n + 3s - 5$ is best possible has the property that $\delta_U = \delta_V$. When $\delta_V > \delta_U$ we have examples which show that we can't do better than $n + 3s - 7$. This raises the question of whether $n + 3s - 6$ suffices when $\delta_V > \delta_U$.

In Theorem 6.1.15, we show that there exist balanced bipartite graphs on $2n$ vertices with $\delta_U + \delta_V \geq n + s^{s^{1/3}}$ which cannot be tiled with $K_{s,s}$. An interesting problem would be to determine the largest possible value of $\delta_U + \delta_V$ such that $G[U, V]$ cannot be tiled with $K_{s,s}$. We note that if $G[U, V]$ is a graph with $\delta_U + \delta_V \geq (1 + \epsilon)n$, then $\delta_U \geq \epsilon n$ and thus we can apply Theorem 6.1.8 or Theorem 6.1.13 to obtain a tiling of G.

Finally, while we don't address the case of tiling with $K_{s,t}$ here, we point out that it is easy to prove an analog of Theorem 6.1.13 for $K_{s,t}$. In fact, even if we only assume $\delta_U + \delta_V \geq n$, we can tile G with $K_{s,t}$: the proof of Theorem 6.1.13 is easy when there exists ℓ such that $|U_1| \leq \ell s$ and $|V_0 \cup V_1| \geq \ell s$ by Claim 6.4.7, so we just remove copies of $K_{s,t}$ from $G[U_1, V_1]$, each with t vertices in U_1, until the desired property holds and then we can finish the tiling as we do here.

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