Three Essays in Economics

by

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ABSTRACT

This dissertation presents three essays in economics. Firstly, I study the problem of allocating an indivisible good between two agents under incomplete information. I provide a characterization of mechanisms that maximize the sum of the expected utilities of the agents among all feasible strategy-proof mechanisms: Any optimal mechanism must be a convex combination of two fixed price mechanisms and two option mechanisms. Secondly, I study the problem of allocating a non-excludable public good between two agents under incomplete information. An equal-cost sharing mechanism which maximizes the sum of the expected utilities of the agents among all feasible strategy-proof mechanisms is proved to be optimal. Under the equal-cost sharing mechanism, when the built cost is low, the public good is provided whenever one of the agents is willing to fund it at half cost; when the cost is high, the public good is provided only if both agents are willing to fund it. Thirdly, I analyze the problem of matching two heterogeneous populations. If the payoff from a match exhibits complementarities, it is well known that absent any friction positive assortative matching is optimal. Coarse matching refers to a situation in which the populations into a finite number of classes, then randomly matched within these classes. The focus of this essay is the performance of coarse matching schemes with a finite number of classes. The main results of this essay are the following ones. First, assuming a multiplicative match payoff function, I derive a lower bound on the performance of n-class coarse matching under mild conditions on the distributions of agents' characteristics. Second, I prove that this result generalizes to a large class of match payoff functions. Third, I show that these results are applicable to a broad class of applications, including a monopoly pricing problem with incomplete information, as well as to a cost-sharing problem with incomplete information. In these problems, standard models predict that opti-

mal contracts sort types completely. The third result implies that a monopolist can capture a large fraction of the second-best profits by offering pooling contracts with a small number of qualities.

DEDICATION

I dedicate my dissertation to my family and many friends. A special feeling of gratitude to my loving parents, Yanxing Shao and Daihong Shi made all of this possible, for their endless encouragement and support.

I also dedicate this dissertation to my wife, Na Wang. There is no doubt that without her continued support I could not have completed this process.

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CHAPTER 1

OPTIMAL ALLOCATION OF AN INDIVISIBLE GOOD

1.1 Introduction

In this paper we consider the problem of allocating an indivisible good among two agents when agents' valuations of the good are private information. A typical problem of such is the bilateral bargaining model in which a seller and a buyer negotiate with each other as to if and how to trade a particular good. This problem has generated a large literature since the pioneering work of Myerson and Satterthwaite (1982). Another more practical example is how to allocate a more desirable office among two interested employees. The intense research interest in this type of problem is derived from a fundamental dilemma due to Green and Laffont (1977): when agents' valuations are private information, it is impossible to find a costless method that always gives the good to the agent with the higher valuation.

Several methods are commonly used in practice to solve such allocation problems: Lotteries, seniority ranking or other type of queuing, or even auctions, are just few of the well-known examples. Many of such methods are quite effective in soliciting agents' revelation of true valuations. Yet some methods may often assign the good to the agent with the lower valuation (lotteries), others may incur negative cash outflows from the agents (auctions). At the more theoretical level, two particular classes of methods have received researchers' attentions. The first class consists of all Vickery-Clark-Groves pivotal mechanisms (Vickery, 1961, Clarke, 1971, Groves, 1973) that extend the conventional English auction scheme. The second class consists of all fixed-price mechanisms (Hagerty and Rogerson, 1987), in which the good is assigned to one agent (the seller) unless both agents are willing to trade the good at a predetermined price. It is well-known that these two classes of methods have their own strength and weakness, yet no one has ever carried out any formal comparisons of these two methods, not to mention comparisons of more general methods.

We shall conduct a systemic investigation of various allocation methods from the optimal mechanism design perspective. We are focused on methods that are both robust and practical. Specifically, we require that all methods be immune to individual manipulation so that it must be a dominant strategy for each agent to reveal their true valuations. We also require that all methods be feasible so that there would be no need of injection of money from outside. Our task is to find among all such methods those that maximize the sum of the utilities of both agents. We show that fixed price mechanisms are indeed optimal. In addition, this exercise leads us to another class of mechanisms, called the option mechanisms. In an option mechanism, one agent is the temporary holder of the good and the other agent is the recipient of a call option that allows him to purchase the good from the first agent at a predetermined price. The good changes hands as long as the option recipient is willing to buy the good from the temporary holder at the predetermined price. In comparison, under the fixed price mechanism, the good changes hands only when both agents agree to the trade at a predetermined price. We can show that option mechanisms are also optimal. In fact, the main result of our paper is that any optimal mechanisms must be a lottery of fixed price mechanisms and option mechanisms.

Our result makes a significant contribution to the literature of mechanism design. While the optimal mechanism approach has been standard for the study of Bayesian mechanisms, it has rarely been applied by anyone to study strategy-proof mechanisms[1]. Our result is one of the very few that have identified the structure of

[1] The fact that there few known results with strategy-proof mechanisms is not

optimal strategy-proof mechanisms in a canonical allocation model[2]. It is also interesting that we have found a new role for options. In the traditional finance literature options are either used as instruments for risk management for investors or as means to provide incentives for managers. In our model, however, they are used as a part of a mechanism to maximize the sum of agents' utilities.

1.2 The Main Result

Consider a model with two agents and one indivisible good. The good is a private good so that it can be consumed by one agent only. Each agent has a quasi-linear utility function for the good and the money transfer,

$$v_i(x_i, t_i; \theta_i) = \theta_i x_i + t_i.$$

Here the parameter $\theta_i \in [0,1]$ is agent i's valuation of the good, or i's type and $x_i \in [0,1]$ is the probability that agent i receives the good. The value of θ_i is known to agent i only.

When the values of θ_1 and θ_2 are commonly known, the efficient allocation is to give the good to the agent with the higher θ_i. However, when θ_1 and θ_2 are privately known only, it is not always possible to identify and execute the efficient

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the only motivation behind our work. The predicted outcomes of a strategy-proof mechanism are deemed reliable since all agents have unambiguous optimal actions regardless others' actions. The predicted outcomes of a Bayesian mechanism are accepted only under strong informational and behavioral assumptions. For example, one must assume that the distribution of agents' types is common knowledge among all agents and is also known to the designer of the mechanism. We will not debate on the relative merits of Bayesian mechanisms, strategy-proof mechanisms and other alternatives here. Interested readers can find them in other papers on this issue (Chung and Ely, 2004, d'Aspremont and Gerard-Varet (1979), Bergemann and Morris, 2005, Jehiel et al, 2006).

[2] Miller considers a model that is similar to ours in a recent working paper (Miller 2007, see also Athey and Miller 2007). After conducting a series of simulations, he arrives at the conclusion that the fixed price mechanism maximizes the sum of the agents' utilities as a conjecture. Moreover, there is no mentioning of the option mechanism and the characterization of all optimal mechanisms.

allocation. We consider here direct mechanisms that ask agents to report their types and use their reported types to determine the allocation as well as transfers to the agents. This attention on direct mechanisms is not excessively restrictive since, by the revelation principle, our analysis immediately extends to all direct mechanisms in which both agents have dominant strategies at every type profile. Otherwise, we allow nearly all direct mechanisms. In particular, we allow all mechanisms that allocate the good to agents randomly.

Formally, a direct mechanism M consists of four integrable functions on $[0,1]\times[0,1]$: $x_1(\theta_1,\theta_2)$, $x_2(\theta_1,\theta_2)$, $t_1(\theta_1,\theta_2)$, $t_2(\theta_1,\theta_2)$. Since $x_1(\theta_1,\theta_2)$ and $x_2(\theta_1,\theta_2)$ are respectively the probabilities agent 1 and agent 2 receive the good, they must satisfy

$$x_i(\theta_1,\theta_2)\in[0,1]\text{, and } x_1(\theta_1,\theta_2)+x_2(\theta_1,\theta_2)\leq 1,\ \forall\theta_1,\theta_2.$$

The functions $t_1(\theta_1,\theta_2)$ and $t_2(\theta_1,\theta_2)$ are transfers to the agents.

To ensure that agents have incentives to report their true types, we require that all mechanisms under consideration be strategy-proof, i.e.,

$$\begin{aligned}\theta_1 x_1(\theta_1,\theta_2)+t_1(\theta_1,\theta_2)&\geq\theta_1 x_1(\tilde{\theta}_1,\theta_2)+t_1(\tilde{\theta}_1,\theta_2),\ \forall\theta_1,\tilde{\theta}_1,\theta_2\\ \theta_2 x_2(\theta_1,\theta_2)+t_2(\theta_1,\theta_2)&\geq\theta_2 x_1(\theta_1,\tilde{\theta}_2)+t_1(\theta_1,\tilde{\theta}_2),\ \forall\theta_1,\theta_2,\tilde{\theta}_2.\end{aligned}\tag{IC}$$

We also require that all mechanisms be feasible so that it does not need outside money,

$$t_1(\theta_1,\theta_2)+t_2(\theta_1,\theta_2)\leq 0,\ \forall\theta_1,\theta_2.\tag{F}$$

The majority of work on bilateral bargaining further assumes budget-balanced-ness, i.e.,

$$t_1(\theta_1,\theta_2)+t_2(\theta_1,\theta_2)=0,\ \forall\theta_1,\theta_2.\tag{BB}$$

We don't want to impose budget-balanced-ness in our work. Although money burning seems inefficient ex post, it might conceivably increase the ex ante efficiency of a mechanism since it could provide incentives more effectively. In fact, all Vickery-Groves-Clarke mechanisms that implement the efficient allocation of the good do not satisfy budget-balanced-ness. If we were to impose budget-balanced-ness, we would have excluded a large class of mechanisms that are popular in the literature of mechanism design from consideration and the end result would be much weaker.

So far the basic structure our model looks virtually the same as the auction model with private values (Myerson, 1980) and the bilateral bargaining model (Myerson and Satterthwaite, 1983, Hagerty and Rogerson, 1987). The main distinction between our model and the others is the difference in objectives. In the auction literature the objective is maximal revenue extraction by an outsider from the agents, whereas in the bilateral bargaining model the objective is maximal revenue extraction by one agent (the seller) from the other (the buyer). In contrast, we adopt the utilitarian viewpoint here and our objective is to find mechanisms that maximize the sum of utilities of both agents.

Given any feasible strategy-proof mechanism $M = \{x_1(\theta_1, \theta_2),\ x_2(\theta_1, \theta_2), t_1(\theta_1 \theta_2), t_2(\theta_1, \theta_2)\}$, the sum of agents' utilities is at each (θ_1, θ_2) is

$$U_1(\theta_1, \theta_2) + U_2(\theta_1, \theta_2) = \theta_1 x_1(\theta_1, \theta_2) + \theta_2 x_2(\theta_1, \theta_2) + t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2).$$

Since the mechanism M satisfies feasibility,

$$U_1(\theta_1, \theta_2) + U_2(\theta_1, \theta_2) \leq Max\{\theta_1, \theta_2\},\ \forall \theta_1, \theta_2.$$

If we could find a feasible strategy-proof mechanism M for which

$$U_1(\theta_1, \theta_2) + U_2(\theta_1, \theta_2) = Max\{\theta_1, \theta_2\},\ \forall \theta_1, \theta_2,$$

then M would be the optimal mechanism. However, this is impossible the classical result by Green and Laffont (1977) shows. On the other hand, for every $(\tilde{\theta}_1, \tilde{\theta}_2)$, it is easy to find a feasible strategy-proof mechanism $M_{(\tilde{\theta}_1,\tilde{\theta}_2)}$ for which

$$U_1(\tilde{\theta}_1,\tilde{\theta}_2)+U_2(\tilde{\theta}_1,\tilde{\theta}_2)=Max\left\{\tilde{\theta}_1,\tilde{\theta}_2\right\},$$

(just the trivial mechanism that always give the good to the agent with the higher $\tilde{\theta}_i$). Hence, we cannot find a first-best mechanism that (weakly) dominates all others at all type profiles. As a result, a meaningful criterion of optimality should be based on some average measurement.

In a Bayesian model one must specify a distribution function P of agents' types and one would naturally use P to calculate the average. We have not specified a distribution of agents' types yet since in our model agents choose their best actions using the simple idea of dominance strategies. In the absence of a distribution of types as the primitive of the model, we may use some discretion in choosing a probability distribution to calculate the average. In his classical article on utilitarianism (1955), Harsanyi argues that one may assume the uniform distribution on the unknown when one is behind the "veil of ignorance." Hence, we will use in this paper the uniform distribution on agents' types as the basis for to calculate the average[3].

Let $\mathscr{F}$ denote the class of all feasible strategy-proof mechanisms. For each $M\in\mathscr{F}$, the total utilities of M is given by:

$$TU(M)=\int_0^1\int_0^1(\theta_1x_1(\theta_1,\theta_2)+\theta_2x_2(\theta_1,\theta_2)+t_1(\theta_1,\theta_2)+t_2(\theta_1,\theta_2))d\theta_1d\theta_2.$$

[3]While this is a restrictive assumption, we want to emphasize that this assumption in our dominance-strategy-based model is not nearly as strong as the same assumption in a Bayesian model. First, agents' optimal actions are independent of this distribution assumption. Second, as a consequence, the set of all mechanisms under consideration is independent of this distribution assumption. We are simply using some probability distribution, from the viewpoint of the designer of the mechanism, to evaluate the efficiency of various mechanisms. The use of the uniform distribution reflects the fact that we are not really sure about the true distribution of agents' types, which is one of the main reasons why we are studying strategy-proof mechanisms in the first place.

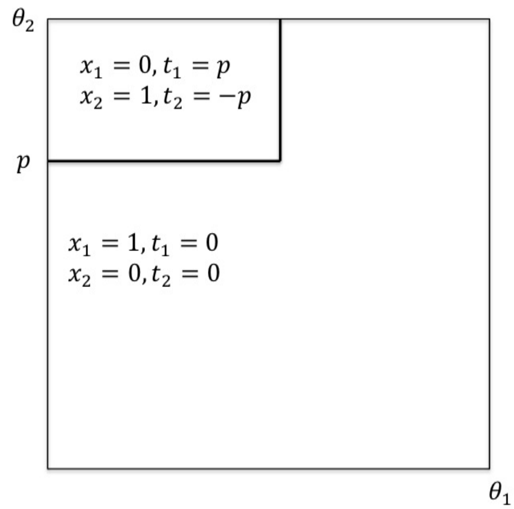

Figure 1: The Fixed Price Mechanism

Our goal is to characterize all mechanisms $M^* \in \mathscr{F}$ that yield the highest average total utilities, that is,

$$TU(M^*) = Max_{M \in \mathscr{F}} TU(M).$$

Let us calculate $TU(M)$ for some well-known mechanisms.

First, the canonical pivotal mechanism (or the second price auction mechanism) M_{SP} has the total utility $TU(M_{SP}) = \frac{1}{3}$, which is not very high. It is not even the best one among all V-C-G mechanisms. In a separate paper we find the best V-C-G mechanism M_{BVCG} with $TU(M_{BVCG}) = \frac{7}{12}$ (Shao and Zhou, 2007).

Hagerty and Rogerson (1987) consider fixed price mechanisms: assuming that agent one is the seller and agent two the buyer, a trade will take place at some fixed price p if and only both the seller and the buyer agree. Formally, the fixed price mechanism with price p is defined as follows (see Figure 1):

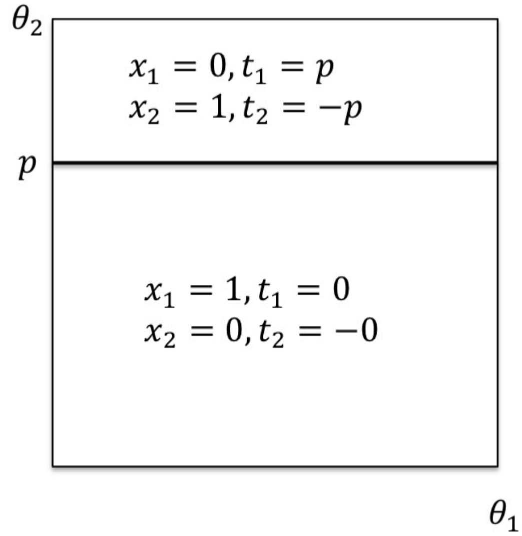

Figure 2: The Option Mechanism

$$\begin{cases} x_1(\theta_1,\theta_2) = 0, \\ t_1(\theta_1,\theta_2) = p, \\ x_2(\theta_1,\theta_2) = 1, \\ t_2(\theta_1,\theta_2) = -p, \end{cases} \text{ when } \theta_1 \leq p \text{ and } \theta_2 \geq p; \text{ and } \begin{cases} x_1(\theta_1,\theta_2) = 1, \\ t_1(\theta_1,\theta_2) = 0, \\ x_2(\theta_1,\theta_2) = 0, \\ t_2(\theta_1,\theta_2) = 0, \end{cases} \text{ otherwise.}$$

Among all fixed price mechanism, the mechanism with the price $p = \frac{1}{2}$ yields the highest total utility $TU(M_F) = \frac{5}{8}$. (The same holds for the fixed price mechanism in which agent two is the designated seller.)

In this paper we also consider another mechanism, called the option mechanism, which is related to, but different from, the fixed price mechanism. It gives the good to agent one conditionally and, at the same time, issues a call option to agent two that allows him to buy the good from agent one at a fixed exercise price of p. Obviously, agent two will exercise the option if and only if $\theta_2 > p$. Formally, it is defined as follows (see Figure 2):

$$\begin{cases} x_1(\theta_1,\theta_2)=0, \\ t_1(\theta_1,\theta_2)=p, \\ x_2(\theta_1\theta_2)=1, \\ t_2(\theta_1,\theta_2)=-p, \end{cases} \text{when } \theta_2 \geq p\text{; and } \begin{cases} x_1(\theta_1,\theta_2)=1, \\ t_1(\theta_1,\theta_2)=0, \\ x_2(\theta_1,\theta_2)=0, \\ t_2(\theta_1,\theta_2)=0, \end{cases} \text{otherwise}$$

Among all option mechanisms, the mechanism with the option price $p=\frac{1}{2}$ yields the highest total utility $TU(M_O)=\frac{5}{8}$. (The same holds for the option mechanism in which agent two is the conditional owner of the good and agent one is awarded the option.)

It is interesting that the best fixed price mechanism and the best option mechanism yield the same level of total utilities. These two mechanisms differ only in the region $\theta_1 \geq \frac{1}{2}$ and $\theta_2 \geq \frac{1}{2}$ where both agents' types are greater than or equal to $\frac{1}{2}$. The fixed price mechanism favors agent one by giving the whole region to agent one, whereas the option mechanism favors agent two. The total utilities are the same since agents' types are distributed symmetrically.

The main finding of our paper is that both the fixed price mechanisms and the option mechanisms (with $p=\frac{1}{2}$) are optimal. Moreover, all optimal mechanisms are convex combinations of these four mechanisms.

Theorem Every optimal mechanism is a convex combination of the two fixed-price mechanisms and the two option mechanisms.

Proof We will divide the proof into two parts. In Part I we show $Max_{M\in\mathscr{F}}TU(M)=\frac{5}{8}$. In Part II we demonstrate that any mechanism that satisfies $TU(M)=\frac{5}{8}$ must be a convex combination of the four mechanisms.

Part I We begin with the structure of a generic mechanism $M\in\mathscr{F}$. First, since M is strategy-proof, $x_1(\theta_1,\theta_2)$ is non-decreasing in θ_1 and $x_2(\theta_1,\theta_2)$ is non-decreasing in θ_2. For every pair of such $x_1(\theta_1,\theta_2)$ and $x_2(\theta_1,\theta_2)$, we can find a continuum of pairs of $t_1(\theta_1,\theta_2)$ and $t_2(\theta_1,\theta_2)$ such that these four functions define a feasible strategy-proof mechanism. The canonical transfers are the generalized pivotal-taxes:

$$t_1^p(\theta_1,\theta_2) = -\theta_1 x_1(\theta_1,\theta_2) + \int_0^{\theta_1} x_1(\alpha,\theta_2)d\alpha, \text{ and}$$
$$t_2^p(\theta_1,\theta_2) = -\theta_2 x_2(\theta_1,\theta_2) + \int_0^{\theta_2} x_2(\theta_1,\beta)d\beta.$$

By definition, the pivotal-taxes are non-positive so that they define a feasible mechanism. However, they represent outflows of money from agents so the resulting mechanism is not efficient. To improve the efficiency of the mechanism, we consider redistribution of the pivotal transfers between two agents while keeping the incentive property of the mechanism intact. To achieve this goal, we add some function of θ_2 only —$h_1(\theta_2)$— to $t_1^p(\theta_1,\theta_2)$ and some function of θ_1 only —$h_2(\theta_1)$— to $t_2^p(\theta_1,\theta_2)$:

$$t_1(\theta_1,\theta_2) = -\theta_1 x_1(\theta_1,\theta_2) + \int_0^{\theta_1} x_1(\alpha,\theta_2)d\alpha + h_1(\theta_2), \text{ and}$$
$$t_2(\theta_1,\theta_2) = -\theta_2 x_2(\theta_1,\theta_2) + \int_0^{\theta_2} x_2(\theta_1,\beta)d\beta + h_2(\theta_1).$$

Obviously, the new mechanism is still strategy-proof. In fact, this is actually the only way to maintain the incentives. We may consider these functions $h_1(\theta_2)$ and $h_2(\theta_1)$as rebates to the agents —$h_1(\theta_2)$ is the amount of money agent 1 receives when agent 2's type is θ_2 and $h_2(\theta_1)$ is the amount of money agent 2 receives when agent 1's type is θ_1. The total amounts of money that can be redistributed between the agents are limited from above by the feasibility condition:

$$h_1(\theta_2) + h_2(\theta_1) \leq \theta_1 x_1(\theta_1,\theta_2) - \int_0^{\theta_1} x_1(\alpha,\theta_2)d\alpha - \int_0^{\theta_2} x_2(\theta_1,\beta)d\beta. \qquad \text{(F)}$$

Now let us estimate the upper-bound of

$$TU(M) = \int_0^1\int_0^1 (\theta_1 x_1(\theta_1,\theta_2) + \theta_2 x_2(\theta_1,\theta_2) + t_1(\theta_1,\theta_2) + t_2(\theta_1,\theta_2))d\theta_1 d\theta_2.$$

Since $\theta_1 x_1(\theta_1,\theta_2)+t_1(\theta_1,\theta_2)=\int_0^{\theta_1} x_1(a,\theta_2)da+h_1(\theta_2)$, we have

$$\begin{aligned}
&\int_0^1\int_0^1 (\theta_1 x_1(\theta_1,\theta_2)+t_1(\theta_1,\theta_2))d\theta_1 d\theta_2\\
&=\int_0^1\int_0^1\left(\int_0^{\theta_1} x_1(\tau,\theta_2)d\tau\right)d\theta_1 d\theta_2+\int_0^1 h_1(\theta_2)d\theta_2\\
&=\int_0^1\int_0^1\left(\int_\tau^1 x_1(\tau,\theta_2)d\theta_1\right)d\tau d\theta_2+\int_0^1 h_1(\theta_2)d\theta_2\\
&=\int_0^1\int_0^1 (1-\tau)x_1(\tau,\theta_2)d\tau d\theta_2+\int_0^1 h_1(\theta_2)d\theta_2\\
&=\int_0^1\int_0^1 (1-\theta_1)x_1(\theta_1,\theta_2)d\theta_1 d\theta_2+\int_0^1 h_1(\theta_2)d\theta_2.
\end{aligned}$$

Thus,

$$\begin{aligned}
&TU(M)\\
&=\int_0^1\int_0^1 (\theta_1 x_1(\theta_1,\theta_2)+\theta_2 x_2(\theta_1,\theta_2)+t_1(\theta_1,\theta_2)+t_2(\theta_1,\theta_2))\ d\theta_1 d\theta_2\\
&=\int_0^1\int_0^1 ((1-\theta_1)x_1(\theta_1,\theta_2)+(1-\theta_2)x_2(\theta_1,\theta_2))\ d\theta_1 d\theta_2\\
&+\int_0^1 h_1(\theta_2)d\theta_2+\int_0^1 h_2(\theta_1)d\theta_1\\
&=\int_0^1\int_0^1 (x_1(\theta_1,\theta_2)+x_2(\theta_1,\theta_2))\ d\theta_1 d\theta_2+\int_0^1 h_1(\theta_2)d\theta_2+\int_0^1 h_2(\theta_1)d\theta_1\\
&-\int_0^1\int_0^1 (\theta_1 x_1(\theta_1,\theta_2)+\theta_2 x_2(\theta_1,\theta_2))\ d\theta_1 d\theta_2
\end{aligned}$$

$$\leq \int_0^1 \int_0^1 (x_1(\theta_1,\theta_2) + x_2(\theta_1,\theta_2))\, d\theta_1 d\theta_2 + \int_0^1 h_1(\theta_2) d\theta_2 + \int_0^1 h_2(\theta_1) d\theta_1$$

$$- \int_0^1 \int_0^1 (\theta_1 x_1(\theta_1,\theta_2) + \theta_2 x_2(\theta_1,\theta_2) + t_1(\theta_1,\theta_2) + t_2(\theta_1,\theta_2))\, d\theta_1 d\theta_2$$

$$= \int_0^1 \int_0^1 (x_1(\theta_1,\theta_2) + x_2(\theta_1,\theta_2))\, d\theta_1 d\theta_2 + \int_0^1 h_1(\theta_2) d\theta_2$$

$$+ \int_0^1 h_2(\theta_1) d\theta_1 - TU(M).$$

Hence,

$$TU(M) \leq \frac{1}{2} \left(\int_0^1 \int_0^1 (x_1(\theta_1,\theta_2) + x_2(\theta_1,\theta_2))\, d\theta_1 d\theta_2 \right) + \frac{1}{2} \left(\int_0^1 h_1(\theta_2) d\theta_2 + \int_0^1 h_2(\theta_1) d\theta_1 \right). \qquad \text{(E)}$$

The equation holds in (E) if and only if $\int_0^1 \int_0^1 (t_1(\theta_1,\theta_2) + t_2(\theta_1,\theta_2))\, d\theta_1 d\theta_2 = 0$. The next Lemma gives us an estimate of the second term on the right hand side of (E):

Lemma Let A be the area in that is below the minor diagonal $\theta_1 + \theta_2 = 1$:

$$\int_0^1 h_1(\theta_2)\, d\theta_2 + \int_0^1 h_2(\theta_1)\, d\theta_1 \leq \frac{3}{4} - \int_A (x_1(\theta_1,\theta_2) + x_2(\theta_1,\theta_2)) d\theta_1 d\theta_2.$$

Proof of Lemma Let us consider the feasibility inequality on the minor diagonal $\theta_1 = \theta$, and $\theta_2 = 1 - \theta$:

$$h_1(\theta) + h_2(1-\theta)$$
$$\leq \theta x_1(\theta, 1-\theta) - \int_0^\theta x_1(\alpha, 1-\theta)\, d\alpha + (1-\theta) x_2(\theta, 1-\theta) - \int_0^{1-\theta} x_2(\theta,\beta)\, d\beta.$$

We integrate the above inequality over $\theta_1 \in [0,1]$,

$$\begin{aligned}
&\int_0^1 h_1(\theta)\,d\theta + \int_0^1 h_2(1-\theta)\,d\theta \\
&\le \int_0^1 \theta x_1(\theta,1-\theta)d\theta - \int_0^1\int_0^\theta x_1(\alpha,1-\theta)\,d\alpha d\theta + \int_0^1 (1-\theta)x_2(\theta,1-\theta)d\theta \\
&- \int_0^1\int_0^{1-\theta} x_2(\theta,\beta)\,d\beta d\theta \\
&= \int_0^1 (\theta x_1(\theta,1-\theta) + (1-\theta)x_2(\theta,1-\theta))\,d\theta - \int_0^1\int_0^\theta x_1(\alpha,1-\theta)\,d\alpha d\theta \\
&- \int_0^1\int_0^{1-\theta} x_2(\theta,\beta)\,d\beta d\theta.
\end{aligned}$$

The first term is easy to estimate: Since

$$\max_{x_1+x_2\le 1} \{\theta x_1(\theta,1-\theta) + (1-\theta)x_2(\theta,1-\theta)\} = \begin{cases} 1-\theta, & \theta \le \frac{1}{2} \\ \theta, & \theta \ge \frac{1}{2} \end{cases},$$

we have

$$\int_0^1 (\theta x_1(\theta,1-\theta) + (1-\theta)x_2(\theta,1-\theta))\,d\theta \le \int_0^{\frac{1}{2}} (1-\theta)d\theta + \int_{\frac{1}{2}}^1 \theta d\theta = \frac{3}{4}.$$

Through the change of variables, we can express the double integrals in the second and the third term as integrals over the area A,

$$\begin{aligned}
&\int_0^1\int_0^\theta x_1(\alpha,1-\theta)\,d\alpha d\theta + \int_0^1\int_0^{1-\theta} x_2(\theta,\beta)\,d\beta d\theta \\
&= \int_0^1\int_0^{1-\tilde{\theta}} x_1\left(\alpha,\tilde{\theta}\right)d\alpha d\tilde{\theta} + \int_0^1\int_0^{1-\theta} x_2(\theta,\beta)\,d\beta d\theta \\
&= \int_A (x_1(\theta_1,\theta_2) + x_2(\theta_1,\theta_2))d\theta_1 d\theta_2.
\end{aligned}$$

Putting these two inequalities together, we prove the lemma. Finally, we can apply the lemma to (E) to obtain the desired estimate

$$\begin{aligned}
TU(M) &\leq \frac{1}{2}\left(\int_0^1\int_0^1 (x_1(\theta_1,\theta_2)+x_2(\theta_1,\theta_2))d\theta_1 d\theta_2\right) \\
&+\frac{1}{2}\left(\int_0^1 h_1(\theta_2)d\theta_2+\int_0^1 h_2(\theta_1)d\theta_1\right) \\
&\leq \frac{1}{2}\left(\int_0^1\int_0^1 (x_1(\theta_1,\theta_2)+x_2(\theta_1,\theta_2))d\theta_1 d\theta_2\right) \\
&+\frac{1}{2}\left(\frac{3}{4}-\int_A (x_1(\theta_1,\theta_2)+x_2(\theta_1,\theta_2))d\theta_1 d\theta_2\right) \\
&=\frac{1}{2}\left(\int_{[0,1]\times[0,1]\setminus A} (x_1(\theta_1,\theta_2)+x_2(\theta_1,\theta_2))d\theta_1 d\theta_2\right)+\frac{3}{8} \\
&\leq \frac{1}{2}\times\frac{1}{2}+\frac{3}{8}=\frac{5}{8}.
\end{aligned}$$

Since $TU(M)=\frac{5}{8}$ for both the fixed price mechanisms and the option mechanisms, these mechanisms are all optimal.

Part II We now show that any mechanism that satisfies $TU(M)=\frac{5}{8}$ must be a convex combination of the fixed price mechanisms and the option mechanisms. We can see from the proof above that any mechanism M satisfies $TU(M)=\frac{5}{8}$ must also satisfy

$$\begin{aligned}
&h_1(\theta_2)+h_2(\theta_1) \\
=&\theta_1 x_1(\theta_1,\theta_2)-\int_0^{\theta_1} x_1(\alpha,\theta_2)\,d\alpha+\theta_2 x_2(\theta_1,\theta_2)-\int_0^{\theta_2} x_2(\theta_1,\beta)\,d\beta, \text{ a.e.} \qquad \text{(B1)}
\end{aligned}$$

$$x_1(\theta_1,\theta_2)+x_2(\theta_1,\theta_2)=1 \text{ (a.e.) on } [0,1]\times[0,1]\setminus A. \qquad \text{(B2)}$$

To avoid unnecessary repetitions, we will drop the qualifier (a.e.) from the proof whenever we invoke (B1) and (B2)[4]. We now divide $[0,1]\times[0,1]$ into four small squares of equal size and study M on each of them separately.

Part II-1 Consider first the upper-left square $(0,\frac{1}{2}]\times(\frac{1}{2},1]$. On this area, agent 2's type is always higher than agent 1's type. It is intuitive that the good should and could be given to agent 2, i.e., $x_2(\theta_1,\theta_2)=1$ on $\left(0,\frac{1}{2}\right]\times\left(\frac{1}{2},1\right]$. Let us present a formal proof. We begin with the upper half of $\left(0,\frac{1}{2}\right]\times\left(\frac{1}{2},1\right]$ and prove

$$x_2(\theta_1,\theta_2)=1 \text{ for } (\theta_1,\theta_2) \text{ with } \theta_1<\frac{1}{2} \text{ and } \theta_1+\theta_2>1.$$

Suppose, on the contrary, $x_2(\theta_1^*,\theta_2^*)\le 1-\delta$ with $\delta>0$ for some (θ_1^*,θ_2^*) with $\theta_1^*<\frac{1}{2}$ and $\theta_1^*+\theta_2^*>1$. Since $x_1(\theta_1,\theta_2)$ is non-decreasing in θ_1 and $x_2(\theta_1,\theta_2)$ is non-increasing in θ_2 (since $x_2(\theta_1,\theta_2)=1-x_1(\theta_1,\theta_2)$ on this region), we must have

$$x_2(\theta_1,\theta_2)\le 1-\delta \text{ for all } (\theta_1,\theta_2) \text{ with } \theta_1\ge\theta_1^* \text{ and } \theta_2\le\theta_2^* \text{ (see Figure3).}$$

In particular, this inequality holds on the minor diagonal $\theta_1=\theta,$ and $\theta_2=1-\theta$ when $\theta_1^*\le\theta_1\le\frac{1}{2}$. Repeating a part of the proof of Lemma, we have

$$\begin{aligned}&\int_0^1 h_1(\theta)\,d\theta+\int_0^1 h_2(1-\theta)\,d\theta\\ \le&\int_0^1 \theta x_1(\theta,1-\theta)d\theta-\int_0^1\int_0^\theta x_1(\alpha,1-\theta)\,d\alpha d\theta+\int_0^1(1-\theta)x_2(\theta,1-\theta)d\theta\\ &-\int_0^1\int_0^{1-\theta} x_2(\theta,\beta)\,d\beta d\theta\end{aligned}$$

[4] The proof remains valid if we have to be more rigorous. At some steps we need to use the Fubini theorem to justify our argument.

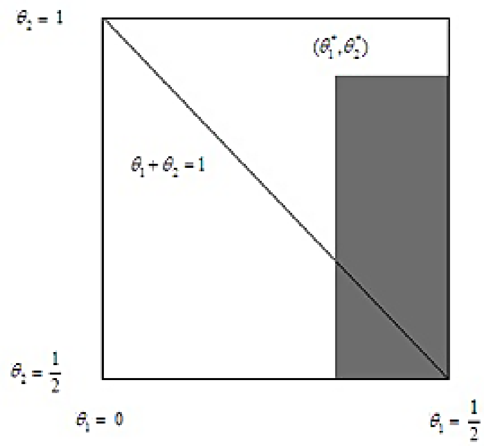

Figure 3

$$= \int_0^{\theta_1^*} (\theta x_1(\theta, 1-\theta) + (1-\theta)x_2(\theta, 1-\theta))\, d\theta - \int_A (x_1(\theta_1, \theta_2) + x_2(\theta_1, \theta_2)) d\theta_1 d\theta_2$$

$$+ \int_{\theta_1^*}^{\frac{1}{2}} (\theta x_1(\theta, 1-\theta) + (1-\theta)x_2(\theta, 1-\theta))\, d\theta$$

$$+ \int_{\frac{1}{2}}^{1} (\theta x_1(\theta, 1-\theta) + (1-\theta)x_2(\theta, 1-\theta))\, d\theta$$

$$\leq \int_0^{\theta_1^*} (1-\theta) d\theta + \int_{\theta_1^*}^{\frac{1}{2}} (\theta\delta + (1-\theta)(1-\delta))\, d\theta$$

$$+ \int_{\frac{1}{2}}^{1} \theta d\theta - \int_A (x_1(\theta_1, \theta_2) + x_2(\theta_1, \theta_2)) d\theta_1 d\theta_2$$

$$< \frac{3}{4} - \int_A (x_1(\theta_1, \theta_2) + x_2(\theta_1, \theta_2)) d\theta_1 d\theta_2.$$

When we plug this into the estimation of $TU(M)$, we have

$$TU(M) < \frac{5}{8}.$$

This contradicts the assumption that M is optimal. Thus

$$x_2(\theta_1, \theta_2) = 1 \text{ for } (\theta_1, \theta_2) \text{ with } \theta_1 < \frac{1}{2} \text{ and } \theta_1 + \theta_2 > 1.$$

Now we show that $x_2(\theta_1, \theta_2) = 1$ also holds for the other half of $[0, \frac{1}{2}] \times (\frac{1}{2}, 1]$ in which $\theta_2 > \frac{1}{2}$ and $\theta_1 + \theta_2 \leq 1$.

First, for any $\theta_2' > \theta_2 > \frac{1}{2}$, we can find some θ_1 with $\theta_1 < \frac{1}{2}$ and $\theta_1 + \theta_2 > 1$. Then

$$h_1(\theta_2) + h_2(\theta_1) = \theta_2 - \int_0^{\theta_2} x_2(\theta_1, \beta)\, d\beta \text{ and}$$

$$h_1(\theta'_2) + h_2(\theta_1) = \theta_2' - \int_0^{\theta_2'} x_2(\theta_1, \beta)\, d\beta.$$

Taking the difference of the two, we have

$$h_1(\theta'_2) - h_1(\theta_2) = \theta_2' - \theta_2 - \int_{\theta_2}^{\theta_2'} x_2(\theta_1, \beta)\, d\beta = 0.$$

Hence, $h_1(\theta_2)$ is a constant h_{12}^* on $\theta_2 \in \left(\frac{1}{2}, 1\right]$. A similar argument also shows that $h_2(\theta_1)$ is a constant h_{21}^* on $\theta_1 \in \left(0, \frac{1}{2}\right]$. Consider any (θ_1, θ_2) with $\theta_2 > \frac{1}{2}$ and $\theta_1 + \theta_2 \leq 1$. Since x_1 is zero to the right of the diagonal, x_1 is also zero on this area for x_1 is non-increasing in θ_1. Hence

$$h_1(\theta_2) + h_2(\theta_1) = \theta_2 x_2(\theta_1, \theta_2) - \int_0^{\theta_2} x_2(\theta_1, \beta)\, d\beta.$$

Choosing any $\theta_2' > 1 - \theta_1 \geq \theta_2$, we have

$$h_1(\theta'_2) + h_2(\theta_1) = \theta_2' - \int_0^{\theta_2'} x_2(\theta_1, \beta)\, d\beta.$$

Subtracting one from the other yields

$$\theta_2' - \theta_2 x_2(\theta_1, \theta_2) = \int_0^{\theta_2'} x_2(\theta_1, \beta) d\beta - \int_0^{\theta_2} x_2(\theta_1, \beta) d\beta = \int_{\theta_2}^{\theta_2'} x_2(\theta_1, \beta) d\beta \leq \theta_2' - \theta_2.$$

This shows $x_2(\theta_1, \theta_2) = 1$. Hence,

$$x_2(\theta_1, \theta_2) = 1 \text{ on } \left(0, \frac{1}{2}\right] \times \left(\frac{1}{2}, 1\right]. \tag{II-1}$$

Part II-2 When we work with the lower-right square $\left(\frac{1}{2},1\right] \times \left(0,\frac{1}{2}\right]$, we can show that $h_1(\theta_2)$ is a constant h_{11}^* on $\theta_2 \in \left(0,\frac{1}{2}\right]$ and that $h_2(\theta_1)$ is a constant h_{22}^* on $\theta_1 \in \left(\frac{1}{2},1\right]$, and

$$x_1(\theta_1,\theta_2) = 1 \text{ on } \left(\frac{1}{2},1\right] \times \left(0,\frac{1}{2}\right]. \tag{II-2}$$

Part II-3 Now consider the upper-right square $\left(\frac{1}{2},1\right] \times \left(\frac{1}{2},1\right]$. (B2) implies

$$h_{12}^* + h_{22}^* = \theta_1 x_1(\theta_1,\theta_2) - \int_0^{\theta_1} x_1(\alpha,\theta_2)\,d\alpha + \theta_2 x_2(\theta_1,\theta_2) - \int_0^{\theta_2} x_2(\theta_1,\beta)\,d\beta.$$

Since $x_1(\alpha,\theta_2) = 0$ for $\alpha < \frac{1}{2}$, and $x_2(\theta_1,\beta) = 0$ for $\beta < \frac{1}{2}$,

$$h_{12}^* + h_{22}^* = \theta_1 x_1(\theta_1,\theta_2) - \int_{\frac{1}{2}}^{\theta_1} x_1(\alpha,\theta_2)\,d\alpha + \theta_2 x_2(\theta_1,\theta_2) - \int_{\frac{1}{2}}^{\theta_2} x_2(\theta_1,\beta)\,d\beta.$$

If we let both $\theta_1 \to \frac{1}{2}$ and $\theta_2 \to \frac{1}{2}$, we obtain $h_{12}^* + h_{22}^* = \frac{1}{2}$. Now plug it back into the equation above,

$$\frac{1}{2} = \theta_1 x_1(\theta_1,\theta_2) - \int_{\frac{1}{2}}^{\theta_1} x_1(\alpha,\theta_2)\,d\alpha + \theta_2 x_2(\theta_1,\theta_2) - \int_{\frac{1}{2}}^{\theta_2} x_2(\theta_1,\beta)\,d\beta, \text{ or}$$

$$\int_{\frac{1}{2}}^{\theta_1} x_1(\alpha,\theta_2)\,d\alpha + \int_{\frac{1}{2}}^{\theta_2} x_2(\theta_1,\beta)\,d\beta = \left(\theta_1 - \frac{1}{2}\right) x_1(\theta_1,\theta_2) + \left(\theta_2 - \frac{1}{2}\right) \theta_2 x_2(\theta_1,\theta_2).$$

Since $x_1(\theta_1,\theta_2)$ is non-decreasing in θ_1 and $x_2(\theta_1,\theta_2)$ is non-decreasing in θ_2, the equation above implies

$$x_1(\alpha,\theta_2) = x_1(\theta_1,\theta_2), \text{ for all } \alpha \in \left[\frac{1}{2},\theta_1\right), \text{ and}$$
$$x_2(\theta_1,\beta) = x_2(\theta_1,\theta_2), \text{ for all } \beta \in \left[\frac{1}{2},\theta_2\right).$$

It is easy to verify that this holds for all $(\theta_1,\theta_2) \in \left(\frac{1}{2},1\right] \times \left(\frac{1}{2},1\right]$ if and only if there are two constants c_1 and c_2 with $c_1 + c_2 = 1$ such that

$$x_1(\theta_1,\theta_2) = c_1 \text{ and } x_2(\theta_1,\theta_2) = c_2 \text{ on } \left(\frac{1}{2},1\right] \times \left(\frac{1}{2},1\right]. \tag{II-3}$$

Part II-4 Lastly, we consider the lower-left square $\left(0,\frac{1}{2}\right]\times\left(0,\frac{1}{2}\right]$. We know

$$h_{11}^{*}+h_{21}^{*}=\theta_1 x_1(\theta_1,\theta_2)-\int_0^{\theta_1} x_1(\alpha,\theta_2)\,d\alpha+\theta_2 x_2(\theta_1,\theta_2)-\int_0^{\theta_2} x_2(\theta_1,\beta)\,d\beta.$$

When let both $\theta_1\to\frac{1}{2}$ and $\theta_2\to\frac{1}{2}$, we obtain $h_{11}^{*}+h_{21}^{*}=0$. Hence,

$$\theta_1 x_1(\theta_1,\theta_2)+\theta_2 x_2(\theta_1,\theta_2)=\int_0^{\theta_1} x_1(\alpha,\theta_2)\,d\alpha+\int_0^{\theta_2} x_2(\theta_1,\beta)\,d\beta.$$

So we can conclude that there are two constants d_1 and d_2 such that

$$x_1(\theta_1,\theta_2)=d_1 \text{ and } x_2(\theta_1,\theta_2)=d_2 \text{ on } \left(\frac{1}{2},1\right]\times\left(\frac{1}{2},1\right]. \tag{II-4}$$

To find out more about d_1 and d_2, we put together we have shown about h's and x's. First, on the upper-left square, we have

$$h_{11}^{*}+h_{21}^{*}=\theta_2 x_2(\theta_1,\theta_2)-\int_0^{\theta_2} x_2(\theta_1,\beta)\,d\beta=\theta_2-\left(\theta_2-\frac{1}{2}\right)-\int_0^{\frac{1}{2}} d_2 d\beta=\frac{1}{2}(1-d_2).$$

Similarly, on the lower-right square

$$h_{21}^{*}+h_{12}^{*}=\frac{1}{2}(1-d_1).$$

On the other hand, we already know

$$h_{12}^{*}+h_{22}^{*}=\frac{1}{2} \text{ and } h_{11}^{*}+h_{21}^{*}=0.$$

These four equations together lead to $d_1+d_2=1$.

Hence, any optimal mechanism can be characterized by two parameters $c\in[0,1]$ and $d\in[0,1]$ with the allocation probabilities given by the table in Figure 4:

There are four different combinations of extreme values of c and d:

(i) $c=0$ and $d=0$: this corresponds to the fixed price mechanism M_{00} in which

$x_1 = 1$ $x_2 = 0$	$x_1 = c$ $x_2 = 1 - c$
$x_1 = d$ $x_2 = 1 - d$	$x_1 = 0$ $x_2 = 1$

Figure 4

agent two is the seller;

(ii) $c = 1$ and $d = 1$: this corresponds to the fixed price mechanism M_{11} in which agent one is the seller;

(iii) $c = 0$ and $d = 1$: this corresponds to the option mechanism M_{01} in which agent one is given the good and agent two is given the option; and

(iv) $c = 1$ and $d = 0$: this corresponds to the option mechanism M_{10} in which agent two is given the good and agent one is given the option.

Any other optimal mechanism M is just a convex combination of these four mechanisms:

$$M = c \circ M_{11} + (d - c) \circ M_{01} + (1 - c - d) \circ M_{00}, \text{ for } c \leq d; \text{ or}$$

$$M = d \circ M_{11} + (c - d) \circ M_{10} + (1 - c - d) \circ M_{00}, \text{ for } c > d.$$

This provides the characterization optimal mechanisms among all feasible strategy-proof mechanisms.

Q.E.D.

1.3 Discussions

Before closing, we discuss more about our result in comparison with other known results in the literature and explore some possible extensions of our result for future research.

Every feasible strategy-proof mechanism consists of two parts: the first part is the rule of assigning the good to the agents, and the second part transfers that are necessary to force the agents to report their types truthfully. The loss of efficiency may come both sources: either the good is not assigned to the agent with the higher valuation, or the total transfer lead to a money outflow. A very natural and important question is: What is the optimal trade-off between these two types of inefficiency? However, this issue has never been addressed formally by others in the previous literature. Most papers in the literature either focus on V-C-G mechanisms or on trading mechanisms in which money changes hands between two agents. These two classes of mechanisms are mutually exclusive by definition: all V-C-G mechanisms are immune from the first type of inefficiency, and most trading mechanisms — including the fixed-price mechanisms — are immune from the second type of inefficiency. Hence, it is impossible to discuss the potential trade-off within between the two types of inefficiency in either model. In order to address this issue, we must adopt a more general setting that includes both V-C-G mechanisms and trading mechanisms as subclasses of admissible mechanisms. We have done it successfully in this paper. The finding is somewhat surprising: although it might be expected that an optimal mechanism entails a compromise of both types of inefficiency, our result indicates the second type of inefficiency seems more damaging and it must be completely absent at any optimal mechanism.

Our model also differs from the model of Hagerty and Rogerson, and other models in bilateral bargaining, in another dimension for we do not impose individual rationality on mechanisms under consideration in our model. Since this enlarges the class of admissible mechanisms, this makes our result even stronger. In addition, we are able to discover the two optimal option mechanisms, which have never been studied before largely because they do not satisfy individual rationality when one agent the designated seller and the other agent the designated buyer[5]. In many allocation problems where neither agent originally owns the good, such as the office assignment problem, the option mechanisms are better alternatives than the fixed-price mechanisms as they are more equitable ex ante. This advantage is not huge as it disappears once lotteries are admissible.

Once we break away from the bilateral bargaining model, it is then natural to consider a model in which an indivisible good (or even multiple units of the good) are allocated among more than two agents. We still do not have any formal results in such a model yet. Admittedly, it will be quite difficult to obtain a complete characterization of all optimal mechanisms. However, it is reasonable to believe that we can still derive some partial results. While we are unsure how to extend the fixed-price mechanism, we have found a generalization of the option mechanism. We assign the good to an agent, say agent one, conditionally and direct him to run a second-price auction of the good with the other $n-1$ agents with a reservation price α. This mechanism always balances the budget since money just changes hands from one agent to another. It should also be reasonably efficient, depending on the value of α. The best value of α can be found by maximizing the sum of the expected utilities of all agents (the transfers are absent

[5]Nevertheless, that the option mechanisms do satisfy the weak individual rationality condition that no agent has negative utility at any profile.

since the budget is always balanced):

$$\alpha^* = \arg\max \left(\frac{1}{2}\alpha^{n-1} + \int_{\alpha}^{1} y d(y^{n-1}) \right).$$

Hence, $\alpha^* = \frac{1}{2}$. Note that this optimal reservation value is the same as that in Myerson's mechanism in which the expected revenue of the seller is maximized. It would be remarkable that the maximum efficiency in allocating an indivisible good among n agents can be achieved when we give the good to one of the agents and direct him to conduct a revenue maximizing auction with the other $n-1$ agents as buyers. Of course, this is just a conjecture at this point, and further research is needed to yield a formal answer.

Finally, we return to an issue we already mentioned when we set up the basic model. Although we have made some justifications for our use of the uniform distribution to calculate the expected utilities of the agents, one may still ask what will happen to our main finding if different distributions are used. Although it is clear that optimal mechanisms will change, we have not been able to derive a full characterization of all optimal mechanisms for a general distribution function of agents' types. In a separate paper (Shao and Zhou, 2008), we undertake a more modest task. Instead of including all feasible strategy-proof mechanisms, we consider only the V-C-G mechanisms and the fixed-price mechanisms (or the option mechanisms), two classes of mechanisms that are most prominent in the literature. Assuming that the distributions of agents' types are independent and symmetric, we manage to find separately the best mechanism among all V-C-G mechanisms and the best mechanism among all fixed-priced mechanisms. Then we compare these two mechanisms to see which one is better. For some distributions, the best V-C-G mechanism actually outperforms the best fixed-price mechanism. However, in two important cases when the distribution function of types is either concave or convex, we show that the best fixed-price mechanism beats the best V-C-G mechanism. We should point out that the model of quasi-linear preferences with a general distribution of types is mathematically equivalent to the model of more general preferences with a uniform distribution of types. Copic and Ponsati (2004) have reported some results for the latter model in the bilateral bargaining framework. While their work has made some progresses in dealing with non-linearity of preferences, it still shares the similar weakness that Hagerty and Rogerson's work exhibits. For instance, it assumes budget-balanced-ness so it excludes the V-C-G mechanisms as well as many other potentially mechanisms from consideration. Hence, it cannot even compare the efficiency of the fixed-priced mechanisms and the V-C-G mechanisms. This being said, their work is already a rather complicated mathematical exercise. We certainly cannot underestimate the difficulty we shall face when we try to find optimal feasible strategy-proof mechanisms when we assume a general distribution of agents' types.

CHAPTER 2

OPTIMAL EQUAL-COST SHARING SCHEME FOR THE ALLOCATION OF NON-EXCLUDABLE PUBLIC GOODS

2.1 Introduction

Public goods have non-rival and non-excludable properties. The consumption of the good by one individual does not exclude the amount available for others. By correctly perceiving the negligible influence on the aggregate level provided one has, individuals would take advantage of the public good without contributing much. The direct impact reflected on the market is, the level of the public good provided is usually far from being sufficient, efficient allocation would not be generated under the gain-seeking strategic behaviors. Samuelson (1954) has shown, the competitive market system is not appropriate to cover the allocation of public goods. Different from purely private goods economy, increasing the size of the population does not ameliorate incentive to ”misbehave”, it is even more serious (Roberts 1976). The degree of insufficiency also depends on the distribution of income. The under provision could be mitigated with the increasing difference of income distribution (Olson 1982).

The recognized failure of the decentralized allocation encourages enormous analysis of alternative mechanisms and the evaluation against efficient yardstick. One natural solution is to introduce a central bureau, or called a government, to coordinate decisions over the level of a public good provided and shares of cost. Both production level and cost sharing are related to information of individuals’ preferences. To induce individuals report their privately known information, feasible mechanisms should satisfy certain incentive compatibility conditions. One branch of the literature proceeds with scheme design in the framework of general equilibrium models. Groves and Ledyard (1977) presented a class of optimal government rules, with which every competitive allocation relative to a government scheme is Pareto optimal. Moreover, truth-telling marginal willingness to pay for the public good is Nash equilibrium, yet not yielding dominant-strategy equilibria. As an alternative to Groves-Ledyard mechanism, a simpler incentive compatible scheme was provided by Mark Walker (1981), and achieves Pareto optimal as well.

The other branch carries out incentive compatible mechanisms in the partial equilibrium models with transferable utility. Groves (1973) developed a general scheme to solve the conflict between efficiency and incentive in the context of a team decision model. Furthermore, truth-telling is a dominant strategy for each agent. It was later applied to a production problem with a group of firms using public inputs, in which the coordinating agent determines the optimal quantity of the public inputs according to the information each firm sends (Groves and Loeb 1975). The efficiency and strong incentive properties make Groves scheme glowing in the bunch of mechanisms. However, as Green and Laffont (1977) pointed out, Groves scheme generally incurs negative aggregate transfers. Redefine the measure of efficiency by adding aggregate transfer to the sum of agents' utilities, Groves scheme is no longer satisfactory. The negative result was also obtained independently in Walker's model with restricted individual preference (1980). Both results implied that dominant strategy property can generally be obtained only by sacrificing Pareto optimality. Green, Kohlberg and Laffont (1975) showed this loss can be kept small by subjecting a small randomly selected sample of the population to a particular Groves scheme. With retention of optimality, the natural response is to use other implementation concepts including undominated perfect Nash equilibrium (Bagnoli and Lipman 1989), subgame perfect equilibrium (Jackson and Moulin 1991), and slack the strong incentive compatibility, for instance, Bayesian incentive compatibility. D'Aspremont and Gerard-Varet (1979) proposed a mechanism that is Bayesian incentive compatible, budget-balanced and optimal, but typical criticisms aim at the fact that the outcome relies too much on the strong common knowledge assumption, and eventually leads to the unreliable predicted outcome.

The objective in this paper is to investigate various allocation schemes of a non-excludable public good between two agents using the optimal mechanism design approach. Agents' valuations of the public good are privately known. For a robust predicted result, we require that all individuals have dominant strategies to report their true valuations. We also require that the allocation schemes are feasible so that there is no fund from outside. The goal is to find the optimal mechanisms that maximize the sum of agents' utility and aggregate transfer among all possible allocation schemes. Our main result shows that the optimal mechanism is an equal-cost sharing scheme. It suggests that the potentially large aggregate transfer could indeed overturn the optimality of the commonly used Groves scheme.

The paper is organized as follows. The model is described in section 2. In section 3, we characterize all feasible strategy-proof schemes. In section 4, the optimality of the proposed scheme is proved.

2.2 Model

The planner is concerned with the construction of a public project. The choice set of the planner contains only two alternatives $d = \{0, 1\}$. The decision $d = 1$ represents the agreement to build a public project–a bridge, a park, or a street lamp etc. at a cost of c ($0 \leq c < 2$), and $d = 0$ represents the decision not to build.

There are two agents, indexed by $i = 1, 2$. Agents' types θ_1 and θ_2 are private information, independently and uniformly distributed over $[0, 1]$. Each agent's utility function is quasi-linear:

$$v_i(d, t_i; \theta_i) = \theta_i d + t_i$$

where t_i is the transfer paid for the construction of the public good to the planner.

In order to solve the free-rider problem, the planner provides a mechanism under which agents have dominant strategies to report their true valuations.

Definition 1 A scheme is $\{d(\theta_1, \theta_2); t_1(\theta_1, \theta_2), t_2(\theta_1, \theta_2)\}$ where $d : [0,1] \times [0,1] \to \{0,1\}$, $t_i : [0,1] \times [0,1] \to R$.

Definition 2 A scheme is said to be strategy-proof, if for all $\theta_1, \theta_1^{'}, \theta_2 \in [0,1]$, $\{d(\theta_1, \theta_2); t_1(\theta_1, \theta_2), t_2(\theta_1, \theta_2)\}$ satisfies,

$$\theta_1 d(\theta_1, \theta_2) + t_1(\theta_1, \theta_2) \geq \theta_1 d(\theta_1^{'}, \theta_2) + t_1(\theta_1^{'}, \theta_2)$$
$$\theta_2 d(\theta_1, \theta_2) + t_2(\theta_1, \theta_2) \geq \theta_2 d(\theta_1, \theta_2^{'}) + t_1(\theta_1, \theta_2^{'}).$$

Definition 3 A scheme is feasible, if for all $(\theta_1, \theta_2) \in [0,1] \times [0,1]$,

$$t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2) + cd(\theta_1, \theta_2) \leq 0.$$

The objective function of the planner is

$$EC(M) = \int_0^1 \int_0^1 [(\theta_1 + \theta_2) d(\theta_1, \theta_2) + t_1(\theta_1 + \theta_2) + t_2(\theta_1 + \theta_2)] d\theta_1 d\theta_2.$$

Remark 1 No mechanism outperforms all others at every profile of types. Hence, we assume the government just chooses a scheme to maximize the expected value of the sum of agents' utilities and transfers, rather than pointwise optimization.

Definition 4 A scheme is optimal, if $\{d^*(\theta_1, \theta_2); t_1^*(\theta_1, \theta_2), t_2^*(\theta_1, \theta_2)\}$

$$\max_{\{d(\cdot); t_1(\cdot), t_2(\cdot)\}} \int_0^1 \int_0^1 [(\theta_1 + \theta_2) d(\theta_1, \theta_2) + t_1(\theta_1 + \theta_2) + t_2(\theta_1 + \theta_2)] d\theta_1 d\theta_2$$

subject to strategy-proofness and feasibility.

2.3 The characterization of feasible and strategy-proof scheme

Mimic Myerson's analysis (1981), we first characterize the feasible and strategy-proof schemes in this part, then introduce one specific scheme in the class of strategy-proof and feasible mechanisms —the equal–cost sharing scheme that attracts the most interest.

By strategy-proofness, we know $d(\theta_1,\theta_2)$ is non-decreasing in $\theta_1$and θ_2. Define $\phi_1(\theta_2)=\inf\{\theta_1\,|d(\theta_1,\theta_2)=1\}$ and $\phi_2(\theta_1)=\inf\{\theta_2\,|d(\theta_1,\theta_2)=1\}$. Graphically, $\phi_2(\theta_1)$ and $\phi_2(\theta_1)$ are the curves carving the whole area $[0,1]\times[0,1]$ into two parts, one on which $d(\theta_1,\theta_2)=1$, i.e. the public good is built, and the other on which $d(\theta_1,\theta_2)=0$. Furthermore, functions $\phi_2(\theta_1)$ and $\phi_2(\theta_1)$ exhibit non-increasing because of the non-decreasing property of $d(\theta_1,\theta_2)$. The strategy-proof mechanism can be described by $\{\phi_2(\theta_1),\phi_1(\theta_2),h_1(\theta_2),h_2(\theta_1)\}$

$$d(\theta_1,\theta_2)=\begin{cases}1\ if\ \theta_2>\phi_2(\theta_1)\\ 0\ if\ \theta_2<\phi_2(\theta_1)\end{cases}$$

$$t_1(\theta_1,\theta_2)=\begin{cases}h_1(\theta_2)-\phi_1(\theta_2)\ if\ \theta_1>\phi_1(\theta_2)\\ h_1(\theta_2)\ if\ \theta_1<\phi_1(\theta_2)\end{cases}$$

$$t_2(\theta_1,\theta_2)=\begin{cases}h_2(\theta_1)-\phi_2(\theta_1)\ if\ \theta_2<\phi_2(\theta_1)\\ h_2(\theta_2)\ if\ \theta_2<\phi_2(\theta_1)\end{cases}$$

where $h_1(\theta_2)$, $h_2(\theta_1)$ are arbitrary functions on $[0,1]$.

Remark 2 We do not specify whether public goods should be built or not and how to distribute the cost between agents when $\theta_2=\phi(\theta_1)$. However, it does not affect the estimation of the upper bound of $EC(M)$ later. So no condition is actually imposed on the issue of breaking the tie.

Besides strategy-proofness, $\{\phi_2(\theta_1), \phi_1(\theta_2), h_1(\theta_2), h_2(\theta_1)\}$ should also satisfy the following feasibility conditions,

$$h_1(\theta_2) + h_2(\theta_1) - \phi_1(\theta_2) - \phi_2(\theta_1) + c \leq 0 \; if \; \theta_2 \geq \phi_2(\theta_1)$$

$$h_1(\theta_2) + h_2(\theta_1) \leq 0 \; if \; \theta_2 < \phi_2(\theta_1).$$

2.4 The optimal scheme

Theorem The optimal scheme for the allocation of a public goods is an equal-cost sharing scheme.

Proof This argument proceeds separately in two cases, these ares, $2 > c \geq 1$ and $1 > c > 0$. For each case, we show, for all strategy-proof feasible schemes, the expected value of the allocation of a non-excludable public good has an upper bound, which is exactly the expected value achieved under an equal-cost sharing scheme. Hence the equal-cost sharing scheme is optimal.

From previous description:

$$t_1(\theta_1, \theta_2) = v_1(\theta_1, \theta_2) - \theta_1 d(\theta_1, \theta_2) = \int_0^{\theta_1} d(s, \theta_2) ds + h_1(\theta_2) - \theta_1 d(\theta_1, \theta_2).$$

$$t_2(\theta_1, \theta_2) = v_2(\theta_1, \theta_2) - \theta_2 d(\theta_1, \theta_2) = \int_0^{\theta_2} d(\theta_1, s) ds + h_2(\theta_1) - \theta_2 d(\theta_1, \theta_2).$$

$$\begin{aligned} EC(M) &= \int_0^1 \int_0^1 [(\theta_1 + \theta_2) d(\theta_1, \theta_2) + t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2)] d\theta_1 d\theta_2 \\ &= \int_0^1 \int_0^1 [(\theta_1 + \theta_2) d(\theta_1, \theta_2) + \int_0^{\theta_1} d(s, \theta_2) ds + h_1(\theta_2) - \theta_1 d(\theta_1, \theta_2) \\ &+ \int_0^{\theta_2} d(\theta_1, s) ds + h_2(\theta_1) - \theta_2 d(\theta_1, \theta_2)] d\theta_1 d\theta_2 \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \int_0^1 [\int_0^{\theta_1} d(s,\theta_2)ds + h_1(\theta_2) + \int_0^{\theta_2} d(\theta_1,s)ds + h_2(\theta_1)]d\theta_1 d\theta_2 \\
&= \int_0^1 \int_0^1 [\int_s^1 d(s,\theta_2)d\theta_1]dsd\theta_2 + \int_0^1 \int_0^1 [\int_s^1 d(\theta_1,s)d\theta_2]dsd\theta_1 \\
&\quad + \int_0^1 \int_0^1 [h_1(\theta_2) + h_2(\theta_1)]d\theta_1 d\theta_2 \\
&\quad = \int_0^1 \int_0^1 (1-\theta_1)d(\theta_1,\theta_2)d\theta_1 d\theta_2 + \int_0^1 \int_0^1 (1-\theta_2)d(\theta_1,\theta_2)d\theta_1 d\theta_2 \\
&\quad + \int_0^1 \int_0^1 [h_1(\theta_2) + h_2(\theta_1)]d\theta_1 d\theta_2 \\
&\quad = \int_0^1 \int_0^1 (2-\theta_1-\theta_2)d(\theta_1,\theta_2)d\theta_1 d\theta_2 + \int_0^1 h_1(\theta_2)d\theta_2 + \int_0^1 h_2(\theta_1)d\theta_1.
\end{aligned}
$$

Both sides are subtracted by $\int_0^1 \int_0^1 [t_1(\theta_1,\theta_2) + t_2(\theta_1,\theta_2)]d\theta_1 d\theta_2$, we have

$$
\begin{aligned}
&EC(M) - \int_0^1 \int_0^1 [t_1(\theta_1,\theta_2) + t_2(\theta_1,\theta_2)]d\theta_1 d\theta_2 \\
&= \int_0^1 \int_0^1 2d(\theta_1,\theta_2)d\theta_1 d\theta_2 - EC(M) + \int_0^1 h_1(\theta_2)d\theta_2 + \int_0^1 h_2(\theta_1)d\theta_1.
\end{aligned}
$$

Simplify the above equation, and we get

$$
\begin{aligned}
EC(M) &= \int_0^1 \int_0^1 d(\theta_1,\theta_2)d\theta_1 d\theta_2 + \frac{1}{2}\int_0^1 \int_0^1 [t_1(\theta_1,\theta_2) + t_2(\theta_1,\theta_2)]d\theta_1 d\theta_2 \\
&\quad + \frac{1}{2}\int_0^1 h_1(\theta_2)d\theta_2 + \frac{1}{2}\int_0^1 h_2(\theta_1)d\theta_1.
\end{aligned}
$$

Denote the area on which $d(\theta_1,\theta_2)=1$ as A and the rest, $d(\theta_1,\theta_2)=0$, as B.

$$EC(M)=\iint_A d\theta_1 d\theta_2+\frac{1}{2}\iint_A [t_1(\theta_1+\theta_2)+t_2(\theta_1+\theta_2)]\,d\theta_1 d\theta_2$$
$$+\frac{1}{2}\iint_B [t_1(\theta_1+\theta_2)+t_2(\theta_1+\theta_2)]\,d\theta_1 d\theta_2+\frac{1}{2}\int_0^1 h_1(\theta_2)\,d\theta_2+\int_0^1 h_2(\theta_1)\,d\theta_1$$
$$\leq \iint_A d\theta_1 d\theta_2+\frac{1}{2}\int_0^1 h_1(\theta_2)\,d\theta_2+\int_0^1 h_2(\theta_1)\,d\theta_1.$$

The last inequality holds due to the feasibility conditions.

Case I If $2>c\geq 1$:

First define the equal-cost sharing scheme M_E^1 in this case as:

$$d(\theta_1,\theta_2)=\begin{cases}1\ if\ \theta_1>\frac{c}{2}\text{ and }\theta_2>\frac{c}{2}\\ 0\ otherwise\end{cases}$$

$$t_1(\theta_1,\theta_2)=t_2(\theta_1,\theta_2)=\begin{cases}-\frac{c}{2}\ if\ \theta_1>\frac{c}{2}\text{ and }\theta_2>\frac{c}{2}\\ 0\ otherwise\end{cases}.$$

Clearly, the equal-cost sharing scheme M_E^1 is strategy-proof and feasible.

Moreover, $EC(M_E^1)=\frac{(2-c)^3}{8}$. In order to find the appropriate upper bound of $EC(M)$, we divide the set of $\phi_1(\theta_2)$, $\phi_2(\theta_1)$ into several subsets. Those subsets cover all possibilities. We then show that, for each possibility, $EC(M)$ has the same upper bound $\frac{(2-c)^3}{8}$.

I-a Suppose that $(\frac{c}{2},\frac{c}{2})\in A$.

Let $\theta^*=\inf\{\theta\,|(\theta,\theta)\in A\}$, and obviously $\theta^*\leq\frac{c}{2}$. From Figure 5,

we see

$$Area(A)=(1-\theta^*)^2+\int_{\theta^*}^1(\theta^*-\phi_1(\theta_2))d\theta_2+\int_{\theta^*}^1(\theta^*-\phi_2(\theta_1))d\theta_1.$$

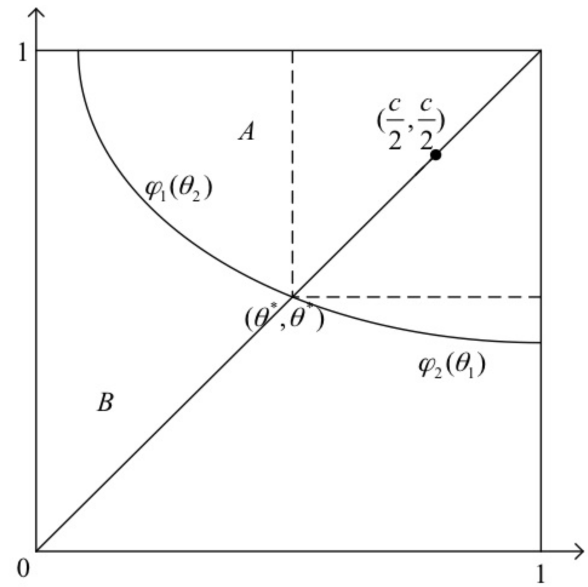

Figure 5

Next, we try to estimate $\int_0^1 h_1(\theta_2)d\theta_2 + \int_0^1 h_2(\theta_1)d\theta_1$. We may assume WLOG that

$$\sup_{\theta_2\in[0,\theta^*]} h_1(\theta_2) \leq 0 \text{ and } \sup_{\theta_1\in[0,\theta^*]} h_2(\theta_1) \leq 0.$$

Hence,

$$\int_0^{\theta^*} h_1(\theta_2)d\theta_2 \leq 0 \text{ and } \int_0^{\theta^*} h_2(\theta_1)d\theta_1 \leq 0.$$

On the other hand, on the square $[\theta^*,1]\times[\theta^*,1]$, feasibility implies

$$h_1(\theta_2)+h_2(\theta_1) \leq \phi_1(\theta_2)+\phi_2(\theta_1)-c.$$

As we integrate the inequality above over $[\theta^*,1]\times[\theta^*,1]$, we get the following:

$$\int_{\theta^*}^1 h_1(\theta_2)d\theta_2 + \int_{\theta^*}^1 h_2(\theta_1)d\theta_1 \leq \int_{\theta^*}^1 \phi_1(\theta_2)d\theta_2 + \int_{\theta^*}^1 \phi_2(\theta_1)d\theta_1 - c(1-\theta^*).$$

Thus,

$$\int_0^1 h_1(\theta_2)d\theta_2 + \int_0^1 h_2(\theta_1)d\theta_1 \leq \int_{\theta^*}^1 \phi_1(\theta_2)d\theta_2 + \int_{\theta^*}^1 \phi_2(\theta_1)d\theta_1 - c(1-\theta^*).$$

Hence,

$$
\begin{aligned}
EC(M) &\leq \left(1-\frac{c}{2}\right)\iint\limits_{A} d\theta_1 d\theta_2 + \frac{1}{2}\int_{0}^{1} h_1(\theta_2)\, d\theta_2 + \int_{0}^{1} h_2(\theta_1)\, d\theta_1 \\
&= (1-\frac{c}{2})Area(A) + \frac{1}{2}\int_{0}^{1} h_1(\theta_2)d\theta_2 + \frac{1}{2}\int_{0}^{1} h_2(\theta_1)d\theta_1 \\
&\leq (1-\frac{c}{2})[(1-\theta^*)^2 + \int_{\theta^*}^{1}(\theta^* - \phi_1(\theta_2))d\theta_2 + \int_{\theta^*}^{1}(\theta^* - \phi_2(\theta_1))d\theta_1] \\
&+ \frac{1}{2}\int_{\theta^*}^{1}\phi_1(\theta_2)d\theta_2 + \frac{1}{2}\int_{\theta^*}^{1}\phi_2(\theta_1)d\theta_1 - \frac{c}{2}(1-\theta^*)
\end{aligned}
$$

$$
\begin{aligned}
&= (1-\frac{c}{2})(1-\theta^*)^2 + 2(1-\frac{c}{2})\theta^*(1-\theta^*) + (\frac{c}{2}-\frac{1}{2})\int_{\theta^*}^{1}\phi_1(\theta_2)d\theta_2 \\
&+ (\frac{c}{2}-\frac{1}{2})\int_{\theta^*}^{1}\phi_2(\theta_1)d\theta_1 - \frac{c}{2}(1-\theta^*) \\
&\leq (1-\frac{c}{2})(1-\theta^*)(1+\theta^*) + (\frac{c}{2}-\frac{1}{2})(1-\theta^*)c - \frac{c}{2}(1-\theta^*) \\
&= (1-\frac{c}{2})(1-\theta^*)[(1+\theta^*)-c] \\
&= (1-\frac{c}{2})(1-\theta^*)(1+\theta^*) + (\frac{c}{2}-1)(1-\theta^*)c.
\end{aligned}
$$

The last inequality holds since

$$\phi_1(\theta^*) + \phi_2(\theta^*) = \theta^* + \theta^* \leq \frac{c}{2} + \frac{c}{2} = c.$$

Observe that $(1-\frac{c}{2})(1-\theta^*)(1+\theta^*-c)$ achieves maximum value when $1-\theta^* = 1+\theta^*-c$ since $1-\theta^*+1+\theta^*-c = 2-c$ is a constant. Thus $\theta^* = \frac{c}{2}$. Hence $EC(M) \leq \frac{(2-c)^3}{8}$.

II-b $(\frac{c}{2},\frac{c}{2}) \notin A$, but there exists a $(\hat{\theta},\hat{\theta}) \in A$, such that $\phi_2(\hat{\theta}) + \phi_1(\hat{\theta}) = c$. Denote $y = \phi_2(\hat{\theta})$ and $x = \phi_1(\hat{\theta})$. We first estimate the upper bound of

$\int_0^1 h_1(\theta_2)d\theta_2 + \int_0^1 h_2(\theta_1)d\theta_1$. We may assume WLOG that

$$\sup_{\theta_2 \in [0,y]} h_1(\theta_2) = \sup_{\theta_1 \in [0,x]} h_2(\theta_1) = -a \leq 0.$$

The feasibility condition on $[x,\hat{\theta}] \times [0,y]$, $h_1(\theta_2) + h_2(\theta_1) \leq 0$, implies

$$\sup_{\theta_1 \in [x,\hat{\theta}]} h_2(\theta_1) \leq a.$$

Similarly, we also have

$$\sup_{\theta_2 \in [y,\hat{\theta}]} h_1(\theta_2) \leq a.$$

Hence

$$\begin{aligned}
&\int_0^{\hat{\theta}}\int_0^{\hat{\theta}} (h_1(\theta_2) + h_2(\theta_1))d\theta_1 d\theta_2 \\
&= \iint_C (h_1(\theta_2) + h_2(\theta_1))d\theta_1 d\theta_2 + \iint_D (h_1(\theta_2) + h_2(\theta_1))d\theta_1 d\theta_2 \\
&+ \iint_E (h_1(\theta_2) + h_2(\theta_1))d\theta_1 d\theta_2 + \iint_F (h_1(\theta_2) + h_2(\theta_1))d\theta_1 d\theta_2 \\
&\leq \iint_C (h_1(\theta_2) + h_2(\theta_1))d\theta_1 d\theta_2 + \iint_F (h_1(\theta_2) + h_2(\theta_1))d\theta_1 d\theta_2 \\
&\leq -2axy + 2a(\hat{\theta} - x)(\hat{\theta} - y) \\
&\leq -2axy + 2a(1-x)(1-y) \\
&= 2a(-xy + 1 - x - y + xy) \\
&= 2a(1-x-y) \\
&= 2a(1-c) \leq 0.
\end{aligned}$$

The last inequality holds due to $2 > c \geq 1$, and area C, D, E, F shown in the Figure 6. Then,

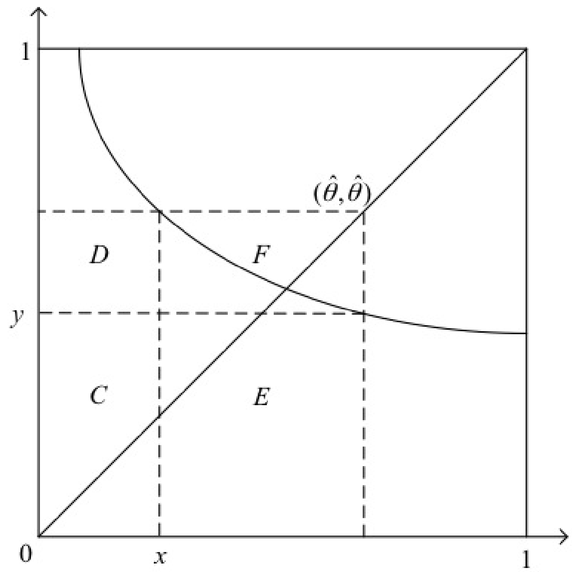

Figure 6

$$\int_0^1 h_1(\theta_2)d\theta_2 + \int_0^1 h_2(\theta_1)d\theta_1 \leq \int_{\hat{\theta}}^1 h_1(\theta_2)d\theta_2 + \int_{\hat{\theta}}^1 h_2(\theta_1)d\theta_1$$

$$\leq \int_{\hat{\theta}}^1 \phi_2(\theta_1)d\theta_1 + \int_{\hat{\theta}}^1 \phi_1(\theta_2)d\theta_2 - c(1-\hat{\theta}).$$

The last part is obtained by integrating the feasibility constraint over $[\hat{\theta},1]\times[\hat{\theta},1]$.

Therefore,

$$EC(M) \leq (1-\frac{c}{2})Area(A) + \int_0^1 h_1(\theta_2)d\theta_2 + \int_0^1 h_2(\theta_1)d\theta_1$$

$$\leq (1-\frac{c}{2})Area(A) + \int_{\hat{\theta}}^1 \phi_2(\theta_1)d\theta_1 + \int_{\hat{\theta}}^1 \phi_1(\theta_2)d\theta_2 - c(1-\hat{\theta}).$$

Construct the new functions $\hat{\phi}_2(\theta_1)$ and $\hat{\phi}_1(\theta_2)$ as follows,

$$\hat{\phi}_2(\theta_1) = \begin{cases} \phi_2(\hat{\theta})1 \; if \; \theta_1 \in [\phi_1(\hat{\theta}), \hat{\theta}] \\ \phi_2(\theta_1) \; otherwise \end{cases}$$

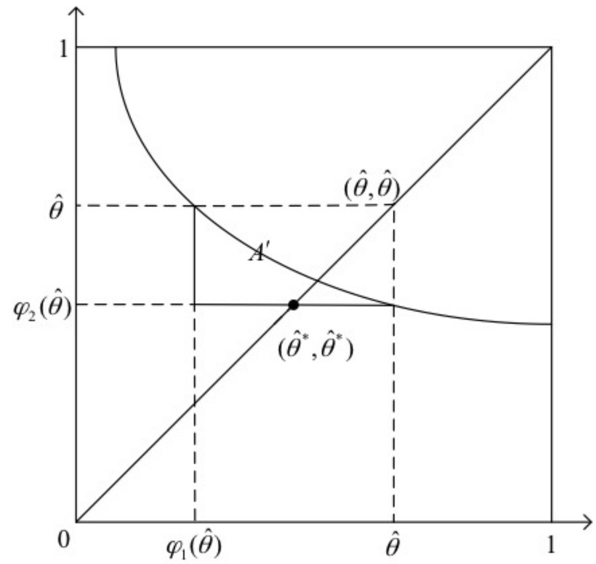

Figure 7

$$\hat{\phi}_1(\theta_2) = \begin{cases} \phi_1(\hat{\theta})1 \; if \; \theta_2 \in [\phi_2(\hat{\theta}), \hat{\theta}] \\ \phi_1(\theta_2) \; otherwise \end{cases}.$$

Denote the area that is shaped by $\hat{\phi}_2(\theta_1)$, $\hat{\varphi}_1(\theta_2)$, $x = 1$ and $y = 1$ as A', which is shown in Figure 7. $\hat{\theta}^*$ is the intersection of $\theta_2 = \theta_1$ and $\hat{\phi}_2(\theta_1)$ or $\hat{\varphi}_1(\theta_2)$. Obviously, $\hat{\theta}^* = \max\{\phi_1(\hat{\theta}), \phi_2(\hat{\theta})\}$.

$$\int_{\hat{\theta}^*}^{1} \hat{\phi}_2(\theta_1) d\theta_1 + \int_{\hat{\theta}^*}^{1} \hat{\phi}_1(\theta_2) d\theta_2 - c(1 - \hat{\theta}^*)$$

$$= \int_{\hat{\theta}^*}^{\hat{\theta}} \phi_2(\hat{\theta}) d\theta_1 + \int_{\hat{\theta}^*}^{\hat{\theta}} \phi_1(\hat{\theta}) d\theta_2 - c(\hat{\theta} - \hat{\theta}^*) + \int_{\hat{\theta}}^{1} \phi_2(\theta_1) d\theta_1 + \int_{\hat{\theta}}^{1} \phi_1(\theta_2) d\theta_2 - c(1 - \hat{\theta})$$

$$= [\phi_2(\hat{\theta}) + \phi_1(\hat{\theta}) - c](\hat{\theta} - \hat{\theta}^*) + \int_{\hat{\theta}}^{1} \phi_2(\theta_1) d\theta_1 + \int_{\hat{\theta}}^{1} \phi_1(\theta_2) d\theta_2 - c(1 - \hat{\theta})$$

$$= \int_{\hat{\theta}}^{1} \phi_2(\theta_1) d\theta_1 + \int_{\hat{\theta}}^{1} \phi_1(\theta_2) d\theta_2 - c(1 - \hat{\theta}).$$

The last equality holds since $\phi_2(\hat{\theta})+\phi_1(\hat{\theta})=c$. The upper bound then is

$$
\begin{aligned}
EC(M) &\leq (1-\frac{c}{2})Area(A)+\frac{1}{2}\int_0^1 h_1(\theta_2)d\theta_2+\frac{1}{2}\int_0^1 h_2(\theta_1)d\theta_1 \\
&\leq (1-\frac{c}{2})Area(A)+\frac{1}{2}\int_{\hat{\theta}}^1 \phi_1(\theta_2)d\theta_2+\frac{1}{2}\int_{\hat{\theta}}^1 \phi_2(\theta_1)d\theta_1-\frac{c}{2}(1-\hat{\theta}) \\
&\leq (1-\frac{c}{2})Area(A')+\frac{1}{2}\int_{\hat{\theta}^*}^1 \hat{\phi}_1(\theta_2)d\theta_2+\frac{1}{2}\int_{\hat{\theta}^*}^1 \hat{\phi}_2(\theta_1)d\theta_1-\frac{c}{2}(1-\hat{\theta}^*) \\
&= (1-\frac{c}{2})[(1-\hat{\theta}^*)^2+\int_{\hat{\theta}^*}^1 (\hat{\theta}^*-\hat{\phi}_1(\theta_2))d\theta_2+\int_{\hat{\theta}^*}^1 (\hat{\theta}^*-\hat{\phi}_2(\theta_1))d\theta_1] \\
&+\frac{1}{2}\int_{\hat{\theta}^*}^1 \hat{\phi}_2(\theta_1)d\theta_1+\frac{1}{2}\int_{\hat{\theta}^*}^1 \hat{\phi}_1(\theta_2)d\theta_2-\frac{c}{2}(1-\hat{\theta}^*) \\
&= (1-\frac{c}{2})(1-\hat{\theta}^*)^2+2(1-\frac{c}{2})\hat{\theta}^*(1-\hat{\theta}^*)+(\frac{c}{2}-\frac{1}{2})\int_{\hat{\theta}^*}^1 \hat{\phi}_1(\theta_2)d\theta_2 \\
&+(\frac{c}{2}-\frac{1}{2})\int_{\hat{\theta}^*}^1 \hat{\phi}_2(\theta_1)d\theta_1-\frac{c}{2}(1-\hat{\theta}^*)
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\frac{c}{2})(1-\hat{\theta}^*)(1+\hat{\theta}^*)+(\frac{c}{2}-\frac{1}{2})\int_{\hat{\theta}^*}^1 \hat{\phi}_1(\hat{\theta}^*)d\theta_2+(\frac{c}{2}-\frac{1}{2})\int_{\hat{\theta}^*}^1 \hat{\phi}_2(\hat{\theta}^*)d\theta_1 \\
&-\frac{c}{2}(1-\hat{\theta}^*) \\
&= (1-\frac{c}{2})(1-\hat{\theta}^*)(1+\hat{\theta}^*)+(\frac{c}{2}-\frac{1}{2})(1-\hat{\theta}^*)c-\frac{c}{2}(1-\hat{\theta}^*) \\
&= (1-\frac{c}{2})(1-\hat{\theta}^*)(1+\hat{\theta}^*-c).
\end{aligned}
$$

As in previous case, EC(M) reaches maximum $\frac{(2-c)^3}{8}$ when $\hat{\theta}^*=\frac{c}{2}$.

I-c $(\frac{c}{2},\frac{c}{2})\notin A$, but there does not exist a $(\hat{\theta},\hat{\theta})\in A$, where $\phi_2(\hat{\theta})+\phi_1(\hat{\theta})=c$. The characteristics of $\phi_2(\theta)$ and $\phi_1(\theta)$ in this case imply that, for all $\theta\in[\theta^*,1]$, $\phi_2(\theta)+\phi_1(\theta)-c>0$, where $\theta^*=\inf\{\theta\,|(\theta,\theta)\in A\}$. Particularly, $\phi_2(1)+\phi_1(1)-c=\varepsilon>0$.

Construct a new feasible mechanism determined by

$\{\tilde{\phi}_2(\theta_1), \tilde{\phi}_1(\theta_2), \tilde{h}_1(\theta_2), \tilde{h}_2(\theta_1)\}$, where $\tilde{\phi}_2(\theta_1) = \phi_2(1) - \frac{\varepsilon}{2}$, for $\theta_1 \in [\phi_1(1) - \frac{\varepsilon}{2}, 1]$; $\tilde{\phi}_1(\theta_2) = \phi_1(1) - \frac{\varepsilon}{2}$, for $\theta_2 \in [\phi_2(1) - \frac{\varepsilon}{2}, 1]$; $\tilde{h}_1(\theta_2) = 0$, for $\theta_2 \in [0,1]$ and $\tilde{h}_2(\theta_1) = 0$, for $\theta_1 \in [0,1]$.

Notice that $\tilde{\phi}_2(1) + \tilde{\phi}_1(1) - c = 0$. Denote the area under this new mechanism on which $d(\theta_1, \theta_2) = 1$ as A". It is clearly to see $A \subset A$" by our construction. So we have

$$\begin{aligned} 8EC(M) &\leq \frac{1}{2}(2-c)\iint\limits_{A} d\theta_1 d\theta_2 + \int_0^1 h_1(\theta_2)d\theta_2 + \int_0^1 h_2(\theta_1)d\theta_1 \\ &\leq \frac{1}{2}(2-c)\iint\limits_{A"} d\theta_1 d\theta_2 + \int_0^1 \tilde{h}_1(\theta_2)d\theta_2 + \int_0^1 \tilde{h}_2(\theta_1)d\theta_1. \end{aligned}$$

The new constructed mechanism is the one belonging to case I-b, using the proved result there that the upper bound is $\frac{(2-c)^3}{8}$, we get the same upper bound here.

Case II If $1 > c > 0$:

Define the equal-cost sharing scheme M_E^2 in this case as:

$$d(\theta_1, \theta_2) = \begin{cases} 1 \ if \ \theta_1 > \frac{c}{2} or \theta_2 > \frac{c}{2} \\ 0 \ otherwise \end{cases}$$

$$t_1(\theta_1, \theta_2) = t_2(\theta_1, \theta_2) = \begin{cases} -\frac{c}{2} \ if \ \theta_1 > \frac{c}{2} or \theta_2 > \frac{c}{2} \\ 0 \ otherwise \end{cases}.$$

Here, M_E^2 is strategy-proof and feasible. Moreover, $EC(M_E^1) = \frac{1}{8}c^3 - c + 1$. We next show that in the class of all strategy-proof and feasible mechanisms $EC(M)$ has the upper bound $\frac{1}{8}c^3 - c + 1$. Let $x = \varphi_1(\frac{c}{2})$ and $y = \varphi_2(\frac{c}{2})$. WLOG, we assume that $x \leq y$. First, find the upper bound of $\int_0^1 \int_0^1 [h_1(\theta_2) + h_2(\theta_1)] d\theta_1 d\theta_2$, given each $\varphi_2(\theta_1)$ and $\varphi_1(\theta_2)$.

Define the following function

$$\lambda(\theta_1,\theta_2)=\begin{cases}\frac{1-c+y}{(1-\frac{c}{2})^2}\ if\ (\theta_1,\theta_2)\in[\frac{c}{2},1]^2\\ \frac{1}{(1-\frac{c}{2})}\ if\ (\theta_1,\theta_2)\in[y,\frac{c}{2}]\times[\frac{c}{2},1]\cup[\frac{c}{2},1]\times[y,\frac{c}{2}]\\ \frac{1}{y}\ if\ (\theta_1,\theta_2)\in[0,y]\times[0,y]\\ 0\ otherwise\end{cases}.$$

Note that $\int\limits_0^1 \lambda(\theta_1,\theta_2)d\theta_1 = 1$ for all θ_2 and $\int\limits_0^1 \lambda(\theta_1,\theta_2)d\theta_2 = 1$ for all θ_1. Hence

$$\int_0^1\int_0^1 [h_1(\theta_2)+h_2(\theta_1)]d\theta_1 d\theta_2 = \int_0^1\int_0^1 \lambda(\theta_1,\theta_2)[h_1(\theta_2)+h_2(\theta_1)]d\theta_1 d\theta_2.$$

From the feasibility condition, we have

$$\begin{aligned}&\int_0^1\int_0^1 \lambda(\theta_1,\theta_2)[h_1(\theta_2)+h_2(\theta_1)]d\theta_1 d\theta_2\\ &=\int_{\frac{c}{2}}^1 \varphi_2(\theta_1)d\theta_1+\int_{\frac{c}{2}}^1 \varphi_1(\theta_2)d\theta_2-c(1-y)+\int_y^{\frac{c}{2}} \varphi_2(\theta_1)d\theta_1+\int_y^{\frac{c}{2}} \varphi_1(\theta_2)d\theta_2.\end{aligned}$$

Plug this into $EC(M)$, we have

$$
\begin{aligned}
EC(M) &\leq \left(1-\frac{c}{2}\right)Area(A)+\frac{1}{2}\int_0^1\int_0^1[h_1(\theta_2)+h_2(\theta_1)]d\theta_1 d\theta_2\\
&\leq \left(1-\frac{c}{2}\right)\left(1-\frac{c}{2}\right)^2+\left(1-\frac{c}{2}\right)\int_{\frac{c}{2}}^1\left(\frac{c}{2}-\varphi_2(\theta_1)\right)d\theta_1\\
&+\left(1-\frac{c}{2}\right)\int_{\frac{c}{2}}^1\left(\frac{c}{2}-\varphi_1(\theta_2)\right)d\theta_2+\left(1-\frac{c}{2}\right)\int_y^{\frac{c}{2}}\left(\frac{c}{2}-\varphi_1(\theta_2)\right)d\theta_2\\
&+\frac{1}{2}\int_{\frac{c}{2}}^1\varphi_2(\theta_1)d\theta_1+\frac{1}{2}\int_{\frac{c}{2}}^1\varphi_1(\theta_2)d\theta_2-\frac{1}{2}c(1-y)\\
&+\frac{1}{2}\int_y^{\frac{c}{2}}\varphi_2(\theta_1)d\theta_1+\frac{1}{2}\int_y^{\frac{c}{2}}\varphi_1(\theta_2)d\theta_2\\
&=\left(1-\frac{c}{2}\right)^3+c\left(1-\frac{c}{2}\right)^2+\left(\frac{c}{2}-\frac{1}{2}\right)\int_{\frac{c}{2}}^1\varphi_2(\theta_1)d\theta_1\\
&+\left(\frac{c}{2}-\frac{1}{2}\right)\int_{\frac{c}{2}}^1\varphi_1(\theta_2)d\theta_2+\frac{c}{2}\left(1-\frac{c}{2}\right)\left(\frac{c}{2}-y\right)\\
&+\left(\frac{c}{2}-\frac{1}{2}\right)\int_y^{\frac{c}{2}}\varphi_1(\theta_2)d\theta_2+\frac{1}{2}\int_y^{\frac{c}{2}}\varphi_2(\theta_1)d\theta_1-\frac{1}{2}c(1-y)\\
&=\left(1-\frac{c}{2}\right)^3+c\left(1-\frac{c}{2}\right)^2+\frac{c}{2}\left(1-\frac{c}{2}\right)\left(\frac{c}{2}-y\right)\\
&+\left(\frac{c}{2}-\frac{1}{2}\right)\int_y^{\frac{c}{2}}\varphi_1(\theta_2)d\theta_2+\frac{1}{2}\int_y^{\frac{c}{2}}\varphi_2(\theta_1)d\theta_1-\frac{1}{2}c(1-y)\\
&\leq\left(1-\frac{c}{2}\right)^3+c\left(1-\frac{c}{2}\right)^2+\frac{c}{2}\left(1-\frac{c}{2}\right)\left(\frac{c}{2}-y\right)\\
&+\left(\frac{c}{2}-\frac{1}{2}\right)\left(\int_x^{\frac{c}{2}}\varphi_2(\theta_1)d\theta_1+\frac{c}{2}(x-y)\right)+\frac{1}{2}\int_y^{\frac{c}{2}}\varphi_2(\theta_1)d\theta_1-\frac{1}{2}c(1-y)\\
&=\left(1-\frac{c}{2}\right)^3+c\left(1-\frac{c}{2}\right)^2+\frac{c}{2}\left(1-\frac{c}{2}\right)\left(\frac{c}{2}-y\right)+\left(\frac{c}{2}-\frac{1}{2}\right)\int_x^y\varphi_2(\theta_1)d\theta_1\\
&+\left(\frac{c}{2}-\frac{1}{2}\right)\frac{c}{2}(x-y)+\frac{c}{2}\int_y^{\frac{c}{2}}\varphi_2(\theta_1)d\theta_1-\frac{1}{2}c(1-y)\\
&\leq\left(1-\frac{c}{2}\right)^3+c\left(1-\frac{c}{2}\right)^2+\frac{c}{2}\left(1-\frac{c}{2}\right)\left(\frac{c}{2}-y\right)+\left(\frac{c}{2}-\frac{1}{2}\right)(y-x)\varphi_2(y)\\
&+\left(\frac{c}{2}-\frac{1}{2}\right)\frac{c}{2}(x-y)+\frac{c}{2}\left(\frac{c}{2}-y\right)\varphi_2(y)-\frac{1}{2}c(1-y).
\end{aligned}
$$

Let $\varphi_2(y)=z$, note that $0\leq x\leq y\leq z\leq\frac{c}{2}$, this is also shown in Figure 8.

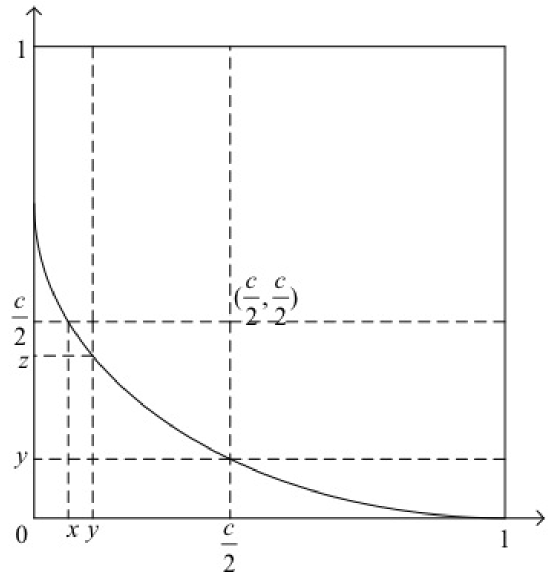

Figure 8

Define

$$L(x,y,z) = \left(1-\frac{c}{2}\right)^3 + c\left(1-\frac{c}{2}\right)^2 + \frac{c}{2}\left(1-\frac{c}{2}\right)\left(\frac{c}{2}-y\right) + \left(\frac{c}{2}-\frac{1}{2}\right)(y-x)z$$
$$+\left(\frac{c}{2}-\frac{1}{2}\right)\frac{c}{2}(x-y) + \frac{c}{2}\left(\frac{c}{2}-y\right)z - \frac{1}{2}c(1-y).$$

For all $0 \leq x \leq y \leq z \leq \frac{c}{2}$, $L(x,y,z) \leq L(0, \frac{c}{2}, \frac{c}{2}) = \frac{1}{8}c^3 - c + 1$. Combing all the cases discussed above, we can conclude that equal-cost sharing scheme is the optimal scheme for the allocation of non-excludable public good.

Q.E.D.

2.5 Concluding Remarks

Although Groves scheme exhibits seemly prominent mitigation of conflict between allocation efficiency and incentive compatibility, the unbalanced budget is a deadly blemish when the overall efficiency being evaluated includes the aggregate transfer. Shao and Zhou (2008) challenges the wide acceptance of Groves scheme by showing that all mechanisms including Groves mechanisms are weakly inferior to a particular class of mechanisms with budget-balance-ness when allocating an indivisible private good between two agents under incomplete information. In this paper, we prove the optimal scheme is equal-cost sharing, rather than Groves scheme, in the allocation of a public good. Only a non-excludable public good is considered here. However, for many non-rival goods, it is also feasible to exclude consumers from usage. An efficient mechanism for public goods with use exclusion is provided by Peter (2004), and he claimed that a fixed fee mechanism is almost optimal.

One interesting follow-up research is to find the optimal scheme for the allocation of public goods when there are more than two agents. The proof we provided in this paper are mostly based on the geometric properties in R^2. Things appear much more complicated when the type space is extended to $[-1,1]^n$. Our conjecture here is the optimality of the equal-cost sharing could survive through such extension, but we may turn to an alternative method to show it.

CHAPTER 3

GENERALIZED COARSE MATCHING

3.1 Introduction

Consider a canonical matching problem: there are two heterogeneous populations of equal size. Agents within each population have the same preferences regarding the other population. For instance, one population may consist of men and a second of women. Within each population, agents differ by ability, beauty, education, etc., and they each prefer agents from the other population with higher ability, beauty, education, etc. If two agents (e.g., a man and woman) match, their payoffs depend on the characteristics of both partners.

It is well known that if a payoff function exhibits complementarities, it is optimal to match the two populations in a positive assortative fashion. (See Becker (1973).) That is, men with the best (worst) characteristics are matched with the women with the best (worst) characteristics. This method of matching is clearly better than randomly assigning men to women. Indeed, the efficiency gain of positive assortative matching over random matching can be significant.

This paper focuses on an intermediate method of pairing two populations, namely, *coarse matching*. In its simplest version, each population is split into two categories of agents—those with high characteristics and those with low characteristics. (The number of agents with high characteristics is the same in the two populations.) The set of men with high attributes is randomly matched with the set of women with high attributes, and similarly with the sets of men and women with low attributes.

More generally, coarse matching proceeds by partitioning the population into n classes and then randomly matching individuals within each class. This is a phenomenon we observe in practice. For instance, dating services classify agents according to a small number of attributes that in fact partition them into coarse classes. Similar examples can be found in labor markets.

This paper asks: how much of total surplus is captured by coarse matching (henceforth, CM) when compared with total surplus generated by positive assortative matching (henceforth, PAM)?[1] Before coming to the answer derived, let us explain the importance of the question.

Observe that PAM requires that a social planner know the actual type of each member of each population. For instance, a matchmaker must know the exact characteristics of each man and woman in given populations. When the populations are large, this is a strong informational requirement. On the other hand, CM only requires that the planner have coarse information, which may be relatively easier to acquire. For instance, the matchmaker may have—or may easily acquire—information on the income bracket of each man and woman in the population (e.g., whether they earn less than \$50,000, more than \$200,000 or between \$50,000 and \$200,000).

In practice, there is a cost of acquiring information on agents' types. This cost is not modeled in typical analyses. We would think that the cost of acquiring perfect information is significantly greater than the cost of acquiring coarse information. Given these costs, should the social planner implement PAM, or should he settle for CM? The answer depends on whether the efficiency gain of using PAM over CM is small vs. large. If it is small, it may not justify the cost of acquiring this additional information.[2]

[1]In this paper, by PAM we mean perfect sorting, although CM also involves positive assortative matching in a very coarse sense.

[2]This is not the only justification for CM. For instance, McAfee (2002) argues that "The use of a continuum of priorities is not feasible in many circumstances-

This question is not new. It was first addressed by McAfee (2002) in a seminal paper. He considered coarse matching, where each population was partitioned into *exactly* two classes. He showed that when the match payoff function takes a certain (multiplicative) form and distributions of attributes satisfy certain hazard rate conditions, then CM can capture at least half of the total surplus generated by PAM.

McAfee (2002) raised an important question—but one that he did not answer completely. For instance, he implicitly assumed that the coarse information available indicates whether a particular member of the population is above or below the mean. This may not be the type of information available to the planner. Or, more interestingly, the planner may have the ability to choose the type of coarse matching he desires. For instance, instead of the matchmaker acquiring information on whether a man's income is above the mean, he can acquire information on whether his income is above a (different) threshold, or whether it lies in one of three income brackets, etc. In other words, an important open problem is how to analyze the performance of n-class CM when the planner does not constrain himself to a particular way of partitioning the populations.

This paper tackles this problem and investigates the performance of n-class CM (i.e., the fraction of the total surplus it obtains) for a very general class of distributions. The main result shows that for each n, there is a way to construct an n-class CM so that there is a "meaningful" lower bound on its performance. We

using many priorities makes the scheme unwieldy to administer and opaque to consumers. Moreover, if the priority prices are determined by bidding, as is natural, the auction process will be complex and expensive to operate when there are many service classes." For the cost, Hoppe, Moldovanu, and Ozdenoren (2010) suggests that "These costs may take the form of: communication, complexity (or menu), and evaluation costs for the intermediary (who needs more detailed information about the environment in order to implement a fine scheme), and for the agents (who need precise information about their own and others' attributes in order to optimally respond to a fin scheme), or higher production costs for firms offering different qualities."

show that the efficiency loss from using n-class CM instead of PAM is bounded by an expression that is proportional to $1/n^2$, where the constant of proportionality depends on the distributions of agents' characteristics but not on n. The paper provides a method to compute this constant of proportionality, and it argues, by way of example, that this constant—and hence the lower bound—is easy to compute. It also shows by example that this lower bound is tight.

As in McAfee (2002), our main result assumes that the match payoff is multiplicative in agents' types. We show, however, that it generalizes to match payoff functions that are supermodular in agents' characteristics, so long as a condition that involves both the payoff function and the distributions of attributes is satisfied. Therefore, this paper substantially enlarges the class of matching problems in which the performance of CM can be assessed.

An important step of the argument is that we choose how to divide two given populations into n-class, so that the lower bound obtains. This is quite different than the exercise in McAfee (2002) who assumes that the 2-class must be divided based on the mean. Indeed, the structure of the mathematical argument is also quite distinct from McAfee (2002). To trace a clearer parallel, Theorem 3.3.2 below shows that McAfee's argument extends to 2^n-classes. (This is intuitively clear but not mathematically obvious.) The key is that, in this case, we can divide the partitions into smaller partitions and still preserve the monotone hazard rate requirements. It is not apparent, however, that the same applies to a partition of arbitrary n classes, a problem that we are able to address successfully in our n-class CM analysis.[3]

[3] A key trick in McAfee (2002) is to split the two populations at the mean value, starting from random matching. We show that this extends to 2^n classes by repeatedly splitting each class at the conditional mean associated with random matching for that class. But in the case of an arbitrary partition there is no natural substitute for the role that the conditional mean plays, precluding an extension to n classes.

This raises the question: are there situations of applied interest, in which the planner can effectively choose which coarse information to obtain? One difficulty is that members of one or both populations may have incentives to misreport and so not give the planner necessary information.

Nonetheless, we argue that the planner may indeed be able to obtain such information. To make this argument, we consider a textbook problem of monopoly pricing under incomplete information about a buyer's valuation (type). This is not typically thought of as a matching problem. But we show that, mathematically, it can be thought of as a matching problem. It is well known that the optimal pricing scheme involves a continuum of price-quantity pairs (or price-quality pairs), each one tailored to a particular possible value of a buyer's type. This is akin to PAM. In practice, such a finely tuned pricing scheme may be impractical or too costly to implement. Thus, an important issue for a firm is to assess how much profit is sacrificed by using 'simpler' pricing schemes that pool intervals of types (e.g., involving only a small number of qualities or quantities offered). This corresponds to CM.

It is not straightforward to apply CM results here to evaluate the profit loss by using pooling contracts. Unlike the original canonical matching model where incentive compatibility and participation conditions are absent, in this problem, the payoffs of the optimal contract and pooling contracts are twisted differently due to different numbers of incentive compatibility and participation conditions faced by the monopolist. Therefore, the connection between pooling contracts and the optimal contract is not as simple as the connection between n-class CM and PAM.

As another application, we analyze CM in a cost-sharing problem. In this circumstance, a principal wants to procure a product from a firm. The cost of production for the firm is randomly distributed. The firm can reduce the initial cost by exerting effort and bearing some disutility. Both initial cost and effort level are unobservable to the principal. The principal can only observe realized cost. The principal's goal is to maximize cost reduction from the initial cost. This problem involves both incomplete information of the firm's initial cost and a hidden action of the firm's choice of effort levels. When facing a firm with initial cost drawn from an interval, the optimal contract induces the firm to exert a different effort level for each possible value of initial cost. Similar to the monopoly pricing problem, this application is also amenable to a reinterpretation as a matching problem. Once we account for incentive compatibility and participation, we apply CM results to give a lower bound regarding the amount of cost reduction accomplished by pooling contracts.

RELATED LITERATURE. Wilson (1989) considers a model where a monopolist sells goods with limited supply to a continuum of consumers with different valuations. He analyzed the efficiency gain of using a priority pricing schedule. The priority pricing schedule involves each consumer receiving a good out of the available supply in different priorities and paying different prices. This schemes forms n priority groups of consumers based on the consumers' valuations. Wilson (1989) shows that the efficiency loss of using n priorities converges to zero at a rate of $O\left(1/n^2\right)$. However, knowing the convergence rate is not quite informative for understanding the performance of pricing schedules with a given number of priority groups, especially when this number is small. McAfee (2002) considers 2-class coarse matching. He proves that 2-class coarse matching can achieve at least half of the efficiency gain under certain conditions. In addition, he also provides a way to map Wilson (1989) into his matching model. Therefore, 2-class coarse matching result can be used to evaluate the efficiency gain of using 2 priorities in Wilson (1989). By contrast, our results allow any number of classes. In our paper the efficiency loss is bounded above by a number proportional to $1/n^2$ Damiano and Li (2007) and Hoppe, Moldovanu, and Ozdenoren (2010) study a matching model with two-sided private information. In Damiano and Li (2007) the goal of the matchmaker is to maximize its revenue. They provide necessary and sufficient conditions under which the matchmaker achieves maximum revenue with efficient sorting pattern (PAM). In our paper the matchmaker is a social planner whose objective is to maximize total efficiency from matching. Hoppe, Moldovanu, and Ozdenoren (2010) also restricts attention to 2-class coarse matching when establishing lower bounds on total efficiency, the revenue of the matchmaker, and the welfare of the agents. Rogerson (2003) and Chu and Sappington (2007) consider a procurement model with one-sided private information and moral hazard under very special distributional assumptions. The purpose of their papers is to establish lower bounds of the performance of the optimal 2-level contracts in terms of cost. Although the model they consider is quite different from the matching model in McAfee (2002), Rogerson (2003) points out that there seems to be a common mathematical structure behind these models. Our paper shows that the coarse matching results can be applied to this procurement model and establishes a lower bound on the performance of n-level contracts under much more general distributional assumptions. Hence, this paper also sheds some light on the common mathematical structure behind these models.

Our paper develops as follows: In Section 2, we introduce the formal model, notations and assumptions. In Section 3, we first briefly discuss the result of McAfee (2002). Then we show that under strictly weaker conditions his result could be extended to 2^n-class coarse matching. Section 4 contains our main theorem. Section 5 consists of three applications. Also see Appendix for details of proofs and algebra of examples.

3.2 The Model

A two-sided market consists of two populations of agents of equal size. For clarity, we call the agents of each population men and women, each of whom, are characterized by types. Denote the types as x and y respectively. Agents' types are randomly distributed over $[0, \tau]$ where $\tau < \infty$ according to distribution functions $F(x)$ and $G(y)$. Throughout the paper, we assume that corresponding density functions $f(x)$ and $g(y)$ are continuous and positive over $(0, \tau)$. We also assume that $F(x)$ and $G(y)$ have finite variances.

Each agent is assumed to be matched with one agent from the other category, namely, one man may only marry one woman. The payoff for matched agents with type x and type y equals $x \cdot y$. The couple shares this payoff in proportion, e.g., they split it equally.

It is known in the literature (e.g., Becker (1973)) that if a match payoff function exhibits complementarity in agents' types, the optimal allocation involves PAM i.e., the highest-type man matches the highest-type woman and the second highest-type man matches the second highest-type woman and so on.

To interpret what follows, it is instructive to think there is a matchmaker who can manipulate a certain type of man to match a certain type of women. The goal of the matchmaker is to maximize the total payoff from matching. However, to achieve PAM, the matchmaker has to know an inordinate amount of information, i.e., the true type of each agent. Alternatively, the matchmaker can create several locations, or sub-markets, for men and women with types belong to a certain interval to meet at one sub-market. Within each sub-market, agents match randomly. This matching scheme requires much less information than PAM since the matchmaker only needs to know the interval to which an agent's type belongs instead of the exact type. In Section 5.1, we discuss in detail how to implement such a matching scheme and why it requires less information relative to PAM.

The matching rule $\phi(x)$ for PAM is a function defined by $G(\phi(x)) = F(x)$, which specifies with whom an agent of type x is matched. The total payoff from PAM is

$$u_\infty = \int_0^\tau x\phi(x)\, dF(x).$$

By contrast, the total payoff from random matching is

$$u_1 = \int_0^\tau x dF(x) \int_0^\tau y dG(y).$$

Now suppose the matchmaker creates n sub-markets where men and women meet. Within each sub-market, agents match randomly. We denote such a matching scheme as n-class CM. For each sub-market, one to one matching requires the same mass on both sides. Hence, given distributions $F(\cdot)$, $G(\cdot)$ and a type range contained in the sub-market, e.g., $[a,b]$ and $[c,d]$ for each category respectively, we have $F(b) - F(a) = G(c) - G(d)$. The match payoff from this sub-market is

$$\int_a^b x \frac{f(x)}{F(b) - F(a)} dx \int_c^d y \frac{g(y)}{G(c) - G(d)} dy,$$

where $\phi(a) = c$ and $\phi(b) = d$. Given cutoff points $\{x_i\}_{i=1}^{n-1}$, the total payoff for n-class CM is

$$u_n = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) \int_{x_{i-1}}^{x_i} \frac{xf(x)}{F(x_i) - F(x_{i-1})} dx \int_{\phi(x_{i-1})}^{\phi(x_i)} \frac{yg(y)}{G(\phi(x_i)) - G(\phi(x_{i-1}))} dy,$$

where $x_0 = y_0 = 0$ and $x_n = y_n = \tau$. If n goes to infinity, CM becomes PAM. If $n = 1$, CM is random matching. Using a change of variables, u_n could also be written as

$$u_n = \sum_{i=1}^{n} (c_i - c_{i-1}) \int_{c_{i-1}}^{c_i} \frac{F^{-1}(x)}{c_i - c_{i-1}} dx \int_{c_{i-1}}^{c_i} \frac{G^{-1}(y)}{c_i - c_{i-1}} dy,$$

where $c_i = F(x_i)$. Throughout the paper, the cutoff points refer to three sets of points $\{x_i\}_{i=1}^{n-1}$, $\{y_i\}_{i=1}^{n-1}$ and $\{c_i\}_{i=1}^{n-1}$ with $y_i = \phi(x_i)$ and $c_i = F(x_i)$. Given distributions F and G, any set of cutoff points could imply the other two sets. Whenever we use u_n, we implicitly assume that there exists cutoff points $\{x_i\}_{i=1}^{n-1}$ such that the match payoff is u_n.

The measure we use to evaluate the efficiency gain of an n-class CM is

$$\frac{u_n - u_1}{u_\infty - u_1}. \tag{3.1}$$

The denominator is the total surplus of PAM over random matching. The numerator is the surplus of an n-class matching over random matching. The payoff difference between PAM and an n-class CM, $u_\infty - u_n$, can be written as

$$\begin{aligned} u_\infty - u_n = \sum (c_i - c_{i-1}) \bigg[& \int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} F^{-1}(z)\, G^{-1}(z)\, dz \\ & - \int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} F^{-1}(z)\, dz \int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} G^{-1}(z)\, dz \bigg]. \end{aligned}$$

Note that

$\int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} F^{-1}(z)\, G^{-1}(z)\, dz - \int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} F^{-1}(z)\, dz \int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} G^{-1}(z)\, dz$ is the surplus of PAM over random matching of one particular sub-market. Therefore, the payoff difference $u_\infty - u_n$ is a weighted sum of surpluses of each sub-market.

Under current payoff function xy, any non-negative and bounded type space can be normalized to $[0,1]$ without changing the value of (1) (see Appendix). Therefore, henceforth, we assume that the type space is $[0,1]$. We can immediately see that the efficiency gain is zero when $n = 1$, and the efficiency gain is one when n goes to infinity. Intuitively, when n becomes larger, the CM scheme becomes finer and the efficiency gain larger. Our goal is to establish a lower bound for the efficiency gain for CM with any n classes. Such bound should be an increasing function of n.

3.3 An Extension of McAfee (2002)

Before we present our main theorem, it is instructive to discuss the result of McAfee (2002). McAfee's 2-class CM result together with our extension of his result regarding 2^n-class CM give a hint of the form of the efficiency gain for CM with n classes. Once the main theorem for n-class CM is presented in the next section we can also compare it to results in this section.

Theorem 3.3.1 *(McAfee 2002) If for distribution functions F and G*

1. *$F(x)/f(x)$ and $G(y)/g(y)$ are both increasing,*
2. *$[1-F(x)]/f(x)$ and $[1-G(y)]/g(y)$ are decreasing,*

   with $x_1 = E(X)$ or $\phi(x_1) = E(Y)$. Then

$$\frac{u_2 - u_1}{u_\infty - u_1} \geq \frac{1}{2}.$$

The main tool used in the proof of the above theorem is Chebyshev's inequality. One of the difficulties McAfee mentioned that prevents him from further extending Theorem 3.3.1 to CM with more classes is that there are no alternatives to conditions 1 and 2. However, we show in Appendix that such alternative conditions indeed exist when considering 2^n classes that are directly implied by conditions 1 and 2 from Theorem 3.3.1. Hence, the above theorem can be generalized to 2^n-class CM.

Theorem 3.3.2 *If for distribution functions F and G*

1. *$F(x)/f(x)$ and $G(y)/g(y)$ are both increasing, and*

2. $[1-F(x)]/f(x)$ *and* $[1-G(y)]/g(y)$ *are decreasing,*

with $x_{2^i\cdot j}=\int_{x_{2^i\cdot(j-1)}}^{x_{2^i\cdot(j+1)}}\frac{z}{F\left(x_{2^i\cdot(j+1)}\right)-F\left(x_{2^i\cdot(j-1)}\right)}dF(z)$ *or*

$\phi\left(x_{2^i\cdot j}\right)=\int_{x_{2^i\cdot(j-1)}}^{x_{2^i\cdot(j+1)}}\frac{z}{G\left(\phi\left(x_{2^i\cdot(j+1)}\right)\right)-G\left(\phi\left(x_{2^i\cdot(j-1)}\right)\right)}dG(z)$ *with*

$j\,(\mathrm{mod}\ 2)\neq 0$*, then*

$$\frac{u_{2^n}-u_1}{u_\infty-u_1}\geq 1-\frac{1}{2^n}.$$

In this theorem, the index of the cutoff point is represented as $2^i\cdot j$. In order to ensure the uniqueness of the representation, we require that j be an odd number.

SKETCH OF THE PROOF. We prove Theorem 3.3.2 by induction. The idea is that by doubling the number of classes the loss of efficiency gain is reduced by half. Theorem 3.3.1 actually illustrates this point by doubling the number of classes from $n=1$, random matching, to $n=2$, 2-class CM. By assuming that the result holds for n-class CM, we want to show that the efficiency loss from $2n$-class CM is no more than half of the efficiency loss of given n-class CM. In particular, we want to show $2\left(u_{2n}-u_n\right)\geq u_\infty-u_n$. As in the discussion of Section 2, the right hand side of this inequality is a weighted sum of surpluses of PAM over random matching of all sub-markets. Loosely speaking, we could apply Theorem 3.3.1 to each sub-market, and then sum them together to obtain $2\left(u_{2n}-u_n\right)\geq u_\infty-u_n$.

One key step of the proof is to determine cutoff points. It is trivial to conclude that if all cutoff points collapse to one end point, 0 or 1, the n-class CM scheme becomes random matching. In the proof, to double the number of classes, we split each interval by the conditional mean of such an interval. To generalize the result for matching with any n classes, we need to divide one interval into several sub-intervals. The trouble is that there is no analog rule to choose cutoff points to divide a given interval into more than two sub-intervals. If it exists, such a rule would give the same cutoff points, the conditional mean, if we divide the interval into two parts.

WEAKER CONDITIONS. Conditions in Theorem 3.3.1 are usually referred to as hazard rate conditions. These conditions are essential to Chebyshev's inequality. They are sufficient but not necessary. It is possible that the result of Theorem 3.3.1 still holds while one of these conditions is violated.

Example 3.3.1 *Consider* $F(x) = G(x) = 1 - (1-x)^{\frac{1}{4}}$*. Since* $(F(x)/f(x))'$ *is negative when x is close to one, condition 1 in Theorem 3.3.1 is violated. Since* $[(1-F(x))/f(x)]' = -4 < 0$*, condition 2 in Theorem 3.3.1 is satisfied. Using the cutoff point* $c = E[x] = \frac{4}{5}$*, we calculate* $(u_2 - u_1)/(u_\infty - u_1)$ *directly, which yields* $(u_2 - u_1)/(u_\infty - u_1) - \frac{1}{2} = 0.22676 > 0$.

In fact, hazard rate conditions in Theorem 3.3.1 can be relaxed to incorporate the above example. By replacing Chebyshev's inequality in the proof of Theorem 3.3.1 with Lemma 2 (see Appendix), we have the following theorem.

Theorem 3.3.3 *If for distributions F and G*

1. $E\left[\frac{F(x)}{f(x)} \mid x < t\right] \leq E\left[\frac{F(x)}{f(x)} \mid x < x_1\right]$ *and*
   $E\left[\frac{G(y)}{g(y)} \mid y < \phi(t)\right] \leq E\left[\frac{G(y)}{g(y)} \mid y < \phi(x_1)\right]$ *for any* $t \leq x_1$
2. $E\left[\frac{1-F(x)}{f(x)} \mid x > t\right] \geq E\left[\frac{1-F(x)}{f(x)} \mid x > x_1\right]$ *and*
   $E\left[\frac{1-G(y)}{g(y)} \mid y > \phi(t)\right] \geq E\left[\frac{1-G(y)}{g(y)} \mid y > \phi(x_1)\right]$ *for any* $t > x_1$
   with $x_1 = E(X)$ *or* $\phi(x_1) = E(Y)$*. Then*
   $$\frac{u_2 - u_1}{u_\infty - u_1} \geq \frac{1}{2}.$$

The conditions in Theorem 3.3.3 are still sufficient but weaker than conditions in Theorem 3.3.1. It is not difficult to see that conditions 1 and 2 in Theorem 3.3.3 are implied by hazard rate conditions in Theorem 3.3.1. The next example shows that the conditions in Theorem 3.3.3 are indeed weaker.

Example 3.3.2 *Return to Example 3.3.1. We need to verify conditions in Theorem 3.3.3. Condition 2 is implied by the fact that* $[(1-F(x))/f(x)]' < 0$. *For condition 1: we can show that* $E[F(x)/f(x) \mid x < t] \leq E\left[F(x)/f(x) \mid x < \frac{4}{5}\right]$ *when* $t \leq \frac{4}{5}$.*Therefore, Theorem 3.3.3 can be applied.*

Similarly, we could also apply Lemma 2 to derive weaker conditions for Theorem 3.3.2 to hold. But as n increases, the number of conditions to be checked grows quickly.

3.4 Main Results

The previous section provides a result regarding 2^n-class CM (as much as can be derived using McAfee's argument), which relies on the fixed cutoffs at the mean and certain hazard rate conditions. In this section, we develop a different line of attack, which provides results for n-class CM. A conjecture for n-class CM from Theorem 3.3.2 is

$$\frac{u_n - u_1}{u_\infty - u_1} \geq 1 - \frac{1}{n}. \tag{3.2}$$

In fact, our main theorem states

$$\frac{u_n - u_1}{u_\infty - u_1} \geq 1 - \frac{\beta}{n^2}, \tag{3.3}$$

where β is independent of n and it is only a function of the distributions.

To show that inequality (3) holds is equivalent to showing

$$\frac{u_\infty - u_n}{u_\infty - u_1} \leq \frac{\beta}{n^2}. \tag{3.4}$$

The total surplus of PAM, $u_\infty - u_1$, can also be written as

$$\begin{aligned} u_\infty - u_1 &= \int_0^1 (x-\mu_x)(\phi(x)-\mu_y)\,dF(x) \\ &= COV(x,\phi(x)). \end{aligned}$$

The last equality is due to the fact that x and $\phi(x)$ are defined over the same probability space. Then it follows the definition of covariance:

$$\begin{aligned} COV(x,y) &= \int\int (x-\mu_x)(y-\mu_y)\,dF(x,y) \\ &= \int (x-\mu_x)(\phi(x)-\mu_y)\,dF(x). \end{aligned}$$

If both women and men have the same distribution, the covariance $COV(x,\phi(x))$ degenerates to the variance. The reason to write $u_\infty - u_1$ as $COV(x,\phi(x))$ is to highlight that such value only depends on distributions F and G.

Main Theorem

Theorem 3.4.1 is our main theorem, providing the lower bound of efficiency gain of CM with any n classes. The lower bound is $1-\left(\beta/n^2\right)$. The value of β picks the minimum of β_i corresponding to an n-class CM scheme associated with a different vector of cutoff points. Each β_i is represented in the form of p-norm. p-norm $\|f\|_p^{[a,b]}$ is defined as $\|f\|_p^{[a,b]} = \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}}$ for $p \geq 1$ and $\|f\|_\infty^{[a,b]} = ess\sup_{[a,b]}\{|f|\}$. Given our assumption about densities, $f(x)$ and $g(y)$ are continuous and positive over $(0,1)$. Hence, $\|f\|_p^{[0,1]}$ may go to infinity only when $f(x)$ goes to infinity at 0 or 1. Similarly, $\|1/f\|_p^{[0,1]}$ may go to infinity only when $f(x)$ goes to zero.

Theorem 3.4.1 *Given F and G, for each n, there exists a vector of cutoff points $(x_1,\ldots,x_{n-1})$ such that*

$$\frac{u_n - u_1}{u_\infty - u_1} \geq 1 - \frac{\beta}{n^2},$$

where $\beta = \min\{\beta_1, \beta_2, \beta_3\}$

1. $\beta_1 = \dfrac{1}{4COV(x, \phi(x))}$

2. $\beta_2 = \min\left\{ \min_{p,q} \dfrac{\left\|F^{-1\prime}\right\|_p^{[0,1]} \left\|G^{-1\prime}\right\|_q^{[0,1]}}{8COV(x,\phi(x))}, \dfrac{\left\|F^{-1\prime}\right\|_\infty^{[0,1]} \left\|G^{-1\prime}\right\|_\infty^{[0,1]}}{12COV(x,\phi(x))} \right\},$

   where $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p, q \in [1, \infty]$

3. $\beta_3 = \min\left\{\beta_3^a, \beta_3^b\right\},$

   where $\beta_3^a = \min_{p_1,q_1} \dfrac{\|f\|_{p_1}^{[0,1]}}{2\sqrt{2}\left[(q_1+1)(2q_1+1)\right]^{\frac{1}{2q_1}} COV(x,\phi(x))},$

   $\beta_3^b = \min_{p_2,q_2} \dfrac{\|g\|_{p_2}^{[0,1]}}{2\sqrt{2}\left[(q_2+1)(2q_2+1)\right]^{\frac{1}{2q_2}} COV(x,\phi(x))},$

   $\frac{1}{p_j} + \frac{1}{q_j} = 1$ *and* $p_j, q_j \in [1, \infty],\ j = 1, 2.$

Remark 3.4.1 *The cutoff points* $\{x_i\}_{i=1}^{n-1}$ *associated with each condition are pinned down as follows:*

1. *If* $\beta = \beta_1$, $\{x_i\}_{i=1}^{n-1}$ *should satisfy* $(x_i - x_{i-1})(\phi(x_i) - \phi(x_{i-1})) = 1/n^2$ *with* $i = 1$ *to whenever possible up to* $i \leq n-1$.

2. *If* $\beta = \beta_2$, $\{x_i\}_{i=1}^{n-1}$ *should satisfy* $F(x_i) - F(x_{i-1}) = 1/n$, *where* $i \leq n$.

3. *a) If* $\beta = \beta_3^a$, $\{x_i\}_{i=1}^{n-1}$ *should satisfy*
   $(x_i - x_{i-1})^{1+\frac{1}{q_1}} (\phi(x_i) - \phi(x_{i-1})) = (1/n)^{2+\frac{1}{q_1}}$ *with* $i = 1$ *to whenever possible up to* $i \leq n-1$.

   b) If $\beta = \beta_3^b$, $\{x_i\}_{i=1}^{n-1}$ *should satisfy*
   $(x_i - x_{i-1})(\phi(x_i) - \phi(x_{i-1}))^{1+\frac{1}{q_2}} = (1/n)^{2+\frac{1}{q_2}}$ *with* $i = 1$ *to whenever possible up to* $i \leq n-1$.

SKETCH OF THE PROOF. We sketch the proof of Theorem 3.4.1 here (see Appendix for a complete proof). One key tool used to prove the theorem is Gruss inequality. Unlike Chebyshev's inequality, which only implies that the payoff of PAM is greater than the payoff of random matching, Gruss inequality gives the upper bound of difference in payoffs. We prove this theorem by showing that inequality (4) holds. Since $u_\infty - u_n$ is a weighted sum of surpluses of all sub-markets, we prove that the surplus of each sub-market is bounded above by a number proportional to $1/n^2$ by using Gruss-type inequalities. Then the weighted sum yields the upper bound for $u_\infty - u_n$, which can be represented as B/n^2, where B is a fixed value. By letting $\beta = B/COV\left(x, \phi\left(x\right)\right)$, we have $\left(u_\infty - u_n\right)/\left(u_\infty - u_1\right) \leq \beta/n^2$. Due to the different versions of Gruss inequality we use, the value of B could be different as could be the value of β. Hence, we choose the minimum among all possible values of β.

Given the number of classes n, the optimal n-class CM yields the highest payoff by choosing cutoffs optimally. Such optimal n-class CM usually involves solving for a quite complex programming problem and is not informative once n has changed. Cutoff points $\{x_i\}_{i=1}^{n-1}$ from the above theorem need not be chosen optimally nor be the same as cutoff points in Theorem 3.3.2 when restricted to 2^m classes. The lower bound is tight if cutoffs from Theorem 3.4.1 coincide with cutoffs of optimal CM.

The choice of cutoff points is important for Theorem 3.4.1 to hold. To be more specific, it is critical for us to represent the upper bound of $u_\infty - u_n$ as a function of $1/n^2$. The reason is that the size of each sub-market is determined by cutoff points. The larger the size of the sub-market, the larger the efficiency loss due to the mismatch within such a sub-market. If all cutoff points concentrate together, the efficiency gain from CM is close to the efficiency gain from random matching. Therefore, we need to spread cutoff points to keep a proper size of each sub-market so that efficiency loss can be bounded by a number proportional to $1/n^2$. The reason we can do this relies on the assumption that the type space is bounded. As a consequence, the weighted sum of those efficiency losses is bounded by a function of $1/n^2$.

Three forms of β_i are given to ensure the tightness of the lower bound. Generally speaking, β_1 performs relatively well if distributions are symmetric. β_2 utilizes information of $1/f$ and $1/g$. Hence, if densities could achieve very large value or go to infinity, β_2 performs better than other two forms. The third form β_3 uses f and g directly. Hence, if densities go to some small value, β_3 would be better.

Benefits of Theorem 3.4.1 include that 1) in order to evaluate the performance of n-class CM, people do not need to know anything about cutoff points to derive the lower bound for efficiency gain; 2) the value of β only depends on distributions; 3) the lower bound is tight, which is shown in Example 3.4.1; 4) instead of solving for the optimal n-level CM scheme, cutoff points from Remark 3.4.1 provide a much easier way to find an n-level CM scheme with efficiency gain no less than the lower bound.

Note that although given certain distributions, β_2, β_3 may go to infinity, β_1 is always finite. Therefore, β is always a finite number. This fact guarantees that the lower bound is meaningful at least when n is large enough. Comparing the lower bound derived in Theorem 3.4.1 to inequality (2), it is obvious that Theorem 3.4.1 is better if $\beta \leq n$ in the sense that the lower bound is larger. Since β is fixed over n, there exists an n^* such that Theorem 3.4.1 is better for all $n > n^*$. This fact gives us a way to compare Theorem 3.4.1 and Theorem 3.3.2 when we restrict ourselves to a non-negative and bounded type space with 2^m classes. When m is large enough, the lower bound from Theorem 3.4.1 is tighter. Especially if β is less than 2, Theorem 3.4.1 is better from $m = 1$. For this reason, we are particularly interested in comparing β to 2 later on. Moreover, Theorem 3.4.1 holds with much more general conditions on distribution functions.

To apply Theorem 3.4.1, we need to go through all possible values of all β_is to pick the minimum, which often requires a large amount of calculations. The value of one particular β_i is often good enough. By sacrificing tightness slightly, focusing on one particular β_i with a particular value of p_i decreases the amount of calculations dramatically. Furthermore, if the distributions behave "nicely" enough, we can derive the lower bound in a much easier way. The following corollary is developed to illustrate this point. If densities are bounded from zero, then instead of utilizing detailed information about the density, the corollary only involves those bounds. Therefore, the calculation of the lower bound is simplified. Certainly, the lower bounds could potentially be improved if we employed the much more complex Theorem 3.4.1.

Corollary 3.4.1 *The lower bound of efficiency gain* $\dfrac{u_n - u_1}{u_\infty - u_1}$ *is:*

1. $1 - \dfrac{1}{4an^2}$ *if* $COV(x, \phi(x)) \geq a$,
2. $1 - \dfrac{\sqrt{3}AB^{\frac{3}{2}}}{n^2}$ *if densities satisfy* $f(x) \leq A, g(x) \leq B$,
3. $1 - \dfrac{a^2b^2}{n^2}$ *if densities satisfy* $f(x) \geq 1/a$ *and* $g(x) \geq 1/b$,
4. $1 - \dfrac{1}{n^2}\dfrac{AB}{ab}$ *if densities* $a \leq f(x) \leq A$ *and* $b \leq g(y) \leq B$,

Lower bound 1 of above corollary is simply a direct implication from β_1 in Theorem 3.4.1. In particular, if $f(x) = g(x)$, $COV(x, \phi(x))$ becomes the variance σ_x^2. Then the larger the variance, the larger the lower bound. Roughly speaking, this result indicates that the higher the variation, the less the efficiency loss. Lower bound 2-4 provide large lower bounds as long as densities don't vary dramatically.

Now we apply the above corollary to the following simple example.

Example 3.4.1 *Consider $F(x) = G(x) = x$. Density equals 1. By Corollary 3.4.1, we have $(u_n - u_1)/(u_\infty - u_1) \geq 1 - 1/n^2$. In fact, the lower bound is tight. The highest efficiency gain can that be achieved by an n-level CM scheme is $1 - 1/n^2$.*

Corollary 3.4.1 provides an easier way to derive lower bounds by employing less information on distributions. The tradeoff then is the sacrifice of tightness of the lower bound. In Example 3.4.2, we apply part of Theorem 3.4.1, which gives a better bound than the above corollary.

Example 3.4.2 *Suppose $F(x) = G(x) = \frac{1}{\sqrt{e}-1} e^{\frac{1}{2}x} - \frac{1}{\sqrt{e}-1}$ with $x \in [0,1]$. Then $f(x) = \frac{1}{2}\frac{1}{\sqrt{e}-1} e^{\frac{1}{2}x} \in \left[\frac{1}{2}\frac{1}{\sqrt{e}-1}, \frac{1}{2}\frac{1}{\sqrt{e}-1} e^{\frac{1}{2}}\right]$ and $F^{-1}(x) = 2\log((\sqrt{e}-1)x+1)$. It is easy to check that none of the bounds derived from Corollaries 4.1 are less than 2. We now derive the bound using β_2 with $p = q = \infty$. We have $\beta_2 = 1.7045 < 2$. Hence, $(u_n - u_1)/(u_\infty - u_1) > 1 - 2/n^2$.*

The value of β_1 in Theorem 3.4.1 can be improved if the inverse functions of F and G are totally (completely) monotonic or absolutely monotonic. A function $f(x)$ is totally (completely) monotonic if $(-1)^n f^{(n)}(x) \geq 0$ for all $n = 0, 1, 2, \ldots$. A function $f(x)$ is absolutely monotonic if it has nonnegative derivatives of all orders.

Corollary 3.4.2 *For any F and G,*

$$\beta_1 = \frac{1}{12COV\left(x, \phi\left(x\right)\right)} \text{ if } F^{-1} \text{ and } G^{-1} \text{ are totally (completely) monotonic or}$$
$$\beta_1 = \frac{4}{45COV\left(x, \phi\left(x\right)\right)} \text{ if } F^{-1} \text{ and } G^{-1} \text{ are absolutely monotonic.}$$

Example 3.4.3 *For distributions $F(x) = G(x) = x^{\frac{1}{\alpha}}$ defined over $[0,1]$ with integer α greater than 1, the inverse function $F^{-1}(x) = x^{\alpha}$ is absolutely monotonic. The value of $COV(x,x) = \frac{\alpha^2}{(\alpha+1)^2(2\alpha+1)} \geq \frac{2}{45}$ when $\alpha = 1,..,8$. By Corollary 3.4.1, $\beta_1 = \frac{4}{45COV(x,x)} \leq 2$ when $\alpha = 1,..,8$.*

The value of β from Theorem 3.4.1 depends on distributions. If β is too large, the lower bound is meaningless for CM with small n. As we discussed above, the value of β depends on the symmetry of distributions. In the following example we consider distributions $F(x) = G(x) = x^{\frac{1}{\alpha}}$, where $\alpha \in (0,\infty)$, and show that $\beta \leq 3$ for any α. Therefore, the lower bound implied by Theorem 3.4.1 is useful. The reason we are interested in this class of distributions is that as α changes this class covers both symmetric and asymmetric distributions. If $\alpha = 1$, the distributions are uniform and symmetric. If α is closed to 0 or ∞, the distributions are highly asymmetric. We believe this example suggests that in general β from Theorem 3.4.1 cannot be too large. The following example also illustrates the advantage of Theorem 3.4.1 over Theorem 3.3.1 and 3.3.2.

Example 3.4.4 *Consider a class of distributions $F(x) = G(x) = x^{\frac{1}{\alpha}}$ over $[0,1]$ where $\alpha \in (0,\infty)$.*

(i) From Theorem 3.3.2, condition 1 and 2 are satisfied only if $\alpha \leq 1$. Therefore, for $\alpha \in (0,1]$ we have

$$\frac{u_{2^m} - u_1}{u_\infty - u_1} \geq 1 - \frac{1}{2^m}.$$

(ii) In this case, simply applying Theorem 3.4.1 won't give a tight lower bound over all α. Special treatment is needed. We show in Appendix that by adapting the proof of Theorem 3.4.1, we can derive the following result:

$$\frac{u_2 - u_1}{u_\infty - u_1} \geq \frac{1}{2} \qquad \textit{and} \qquad \frac{u_n - u_1}{u_\infty - u_1} \geq 1 - \frac{3}{n^2}$$

for any $\alpha \in (0, \infty)$ and $n \geq 3$.

Compared to the lower bound derived from part (i), part (ii) provides a tighter bound when restricted to $\alpha \leq 1$ and the number of classes equals 2^m. Furthermore, it provides a bound for any $\alpha \in (0, \infty)$ and any number of classes.

MEASURE OF PERFORMANCE. Ratio u_n/u_∞ is also used as a measure of the performance of CM in the literature. As we show in Appendix, when the match payoff function is xy, ratio $(u_n - u_1) / (u_\infty - u_1)$ is convenient since its value is not affected by shifting the type space while u_n/u_∞ is. For a general match payoff function $u(x, y)$, ratio $(u_n - u_1) / (u_\infty - u_1)$ may lose such advantage due to the curvature of $u(x, y)$. Depending on interest, it is sometimes also more sensible to use u_n/u_∞ as a measure of the performance of CM. In fact, Theorem 3.4.1 is enough to establish a lower bound for u_n/u_∞. Because $\frac{u_n - u_1}{u_\infty - u_1} \geq 1 - \beta/n^2$ implies that $\frac{u_n}{u_\infty} \geq 1 - \beta/n^2$. If we replace $COV(x, \phi(x))$ in β with u_∞, a tighter lower bound for u_n/u_∞ is obtained. Note that u_∞ only depends on distributions F and G.

General match payoff Function

We derive all previous results by assuming that the match payoff function is simply $x \cdot y$. However, those results can be easily generalized to matching with payoff function $m(x)n(y)$, where $m(\cdot)$ and $n(\cdot)$ are continuous monotonic functions. Treat $F(x(m))$ and $G(y(n))$ as new distribution functions w.r.t. m and n, where

$x(m)$ and $y(n)$ are inverse function of $m(x)$ and $n(y)$. Then the payoff function becomes mn. Hence all previous results hold if the payoff function is separable.

In general, there is no reason why the payoff function should be separable. But results regarding general payoff functions are rarely seen in related literature. We show below that the main result extends if a condition imposed jointly on the payoff function and on the densities holds

Formally, we consider matching with a general payoff function $u(x,y)$, which is complementary in x and y. Due to the curvature of $u(x,y)$, we are no longer able to assume that the type space is $[0,1]$ without loss of generality. Here, we assume the type space $[a,b]$ is nonnegative and bounded.

We have the following similar notations for match payoff of random matching and PAM:

$$u_1 = \int_0^1 \int_0^1 u\left(F^{-1}(\alpha), G^{-1}(\beta)\right) d\alpha d\beta$$
$$u_\infty = \int_0^1 u\left(F^{-1}(\alpha), G^{-1}(\alpha)\right) d\alpha.$$

Theorem 3.4.2 *For c.d.f. F,G defined over a non-negative bounded type space $[a,b]$ with $\frac{\partial^2}{\partial x \partial y} u\left(F^{-1}(x), G^{-1}(y)\right) \in \left[\underline{A}, \overline{A}\right]$, we have*

$$\frac{u_n - u_1}{u_\infty - u_1} \geq 1 - \frac{\beta}{n^2},$$

where $\beta = \overline{A}/\underline{A}$. The cutoff points are $\{c_i\}_{i=1}^n$ such that $c_i = i/n$.

The condition required in Theorem 3.4.2 $\frac{\partial^2}{\partial x \partial y} u\left(F^{-1}(x), G^{-1}(y)\right)$ equals $u_{21}/(fg)$. The value of u_{21} characterizes the degree of complementarity of the match payoff function in matched agents' types. This condition indicates that a lower bound exists if the degree of complementarity together with the reciprocal of densities are bounded away from zero and infinity. The lower bound is large if $u_{21}/(fg)$ doesn't vary significantly over the type space.

To illustrate Theorem 3.4.2 consider the following example:

Example 3.4.5 *Suppose $u(x,y) = e^{\frac{1}{4}xy}$ and x and y are uniformly distributed over $[0,1]$. Then $u\left(F^{-1}(x), G^{-1}(y)\right) = e^{\frac{1}{4}xy}$. It's easy to check that $u_{21}F^{-1\prime}G^{-1\prime} = \frac{1}{16}e^{\frac{1}{4}xy}(xy+4) \in \left[\frac{1}{4}, \frac{5}{16}e^{\frac{1}{4}}\right]$ and $\beta = \overline{A}/\underline{A} = 1.605$. By Theorem 3.4.2, we have*

$$\frac{u_n - u_1}{u_\infty - u_1} \geq 1 - \frac{1.605}{n^2}.$$

In Appendix, we provide two more technical conditions for the result in Theorem 3.4.2 to hold.

3.5 Applications

All of our results so far are derived under a canonical matching model assuming the matchmaker can at least obtain coarse information of agents' types. As we mention in the introduction, in practice, agents may have incentive to misreport their types. For example, in the monopolistic pricing model, a monopolist could produce products with various qualities and sell them to consumers with different valuations that are privately known. It is well known that to maximize its profit, the monopolist can design an optimal contract to provide agents incentives to report honestly. Under such an optimal contract, the consumer with the highest valuation chooses the product with the highest quality and so on. From the practical point of view, seldom does a firm adopt such an optimal contract by offering a product with different quality for each consumer. Instead, a firm usually offers a simple contract with products possessing only several different qualities that target different groups of consumers. In the same spirit of coarse matching, we are interested in the performance of such ”simple contracts”.

We start this section with a matching model with private information. We illustrate how to implement coarse matching when agents’ types are only known to themselves. We then turn our attention to contracting models. The first to be examined is a monopolistic pricing model, which is a pure adverse selection model. The second is a cost-sharing model, which has both private information and hidden actions.

Coarse Matching with Private Information

In this application, we discuss how to implement coarse matching with private information. Consider the previous marriage model with individual’s type privately known to themselves. Assume that the match couple shares the payoff xy equally. The setting is quite similar to Damiano and Li (2007). However, instead of revenue, as a social planner, the matchmaker cares about the total payoff from matching which is a sum over all payoffs of individuals and transfers between individuals and the matchmaker.

Given the number of classes adopted by the matchmaker, our CM results provide a lower bound of the efficiency gain of n-class CM with perfect information. Next we show that the same lower bound of the efficiency gain can be derived for n-class coarse matching. To implement a particular n-class CM scheme given distributions of individuals’ types, the matchmaker creates n sub-markets for individuals with types belonging to different intervals. Individuals within each sub-market match randomly. The cutoffs of types of each sub-market is public announced by the matchmaker. Hence, any individual has an expectation of payoffs from participating different sub-markets. Denote cutoffs men and women associated with ith sub-market as $\{x_{i-1}, x_i\}$ and $\{y_{i-1}, y_i\}$ respectively with $y_i = \phi(x_i)$ and $i = 1, ..., n$. Also denote the participation fees for men and women for ith sub-market are p_i^m and p_i^w respectively. The matchmaker could induce individuals to choose right sub-markets by setting participation fees properly. A fee schedule $\{p_i^m, p_i^n\}_{i=1}^n$ is incentive compatible for men with type $x \in [x_{i-1}, x_i]$ and women with type $y \in [\phi(x_{i-1}), \phi(x_i)]$ if

$$\int_{y_{i-1}}^{y_i} \frac{xy}{G(y_i) - G(y_{i-1})} dG(y) - p_i^m \geq \int_{y_{j-1}}^{y_j} \frac{xy}{G(y_j) - G(y_{j-1})} dG(y) - p_j^m$$
$$\int_{x_{i-1}}^{x_i} \frac{xy}{F(x_i) - F(x_{i-1})} dF(x) - p_i^w \geq \int_{x_{j-1}}^{x_j} \frac{xy}{F(x_j) - F(x_{j-1})} dF(x) - p_j^w.$$

Such fee schedule indeed exists. For illustration, suppose a matchmaker adopts a 2-class CM. To implement, the matchmaker creates two locations in which men and women to meet, one for individuals with types above thresholds x^* and y^* (high type) and the other one for individuals with types below x^* and y^* (low type). Now men and women who want to participate the high type location will be charged fees p_m and p_w respectively. It is free to participate in the low type location. Knowing they will match some individuals with high (low) type by participating in high (low) type locations, they will pay the following fees

$$p_m = \frac{1}{2} \int_{y^*}^{1} \frac{x^* y}{1 - G(y^*)} dG(y) - \frac{1}{2} \int_{0}^{y^*} \frac{x^* y}{G(y^*)} dG(y)$$
$$p_w = \frac{1}{2} \int_{x^*}^{1} \frac{x y^*}{1 - F(x^*)} dF(x) - \frac{1}{2} \int_{0}^{x^*} \frac{x y^*}{F(x^*)} dF(x),$$

which are incentive compatible. All men and women with types above (below) x^* and y^* respectively will participate in high (low) type locations. Since transfers between individuals and the matchmaker don't affect overall payoffs from CM, 2-class CM can be implemented. Alternatively, we could interpret those individuals who are willing to pay fees as being premium members, with the matchmaker only introducing premium members to premium members. It is the same procedure to implement CM with any number of classes, even PAM, by charging fees continuously for each type. The reasons why a matchmaker may prefer CM to PAM may be 1) there is a cost for creating more locations for individuals to meet or alternatively, it is costly to introduce a new tier of membership; 2) the welfare concern of individuals means that to implement PAM the matchmaker has to take a large share of total payoff away from individuals.

In sum, the efficiency gain of n-class CM with private information of individuals' types can be bounded below by $1-\beta/n^2$ from Theorem 3.4.1.

Monopolistic Pricing

In this application, we consider a textbook monopoly pricing model. Under standard assumptions, the optimal contract offered by the monopolist induces a consumer with a higher valuation to buy a product with a higher quality and pay a higher price continuously. This sorting result can be thought of as PAM between a consumer's type and product's quality. It is natural to interpret CM with n classes in this setting as an n-level contract consisting of products with n different qualities for n groups of consumers, which induces each group of consumers with valuations within an interval to buy products with the same quality and pay for the same price. Given the number of different qualities the monopolist could produce, the monopolist could change the size of each group of consumers by choosing cutoffs of consumers' valuations to maximize its profit. The natural question to ask is how to apply our previous CM result to evaluate the performance of optimal n-level contracts relative to the optimal contracts.

Unlike our original canonical matching model where the incentive compatibility (*IC* henceforth) and participation (*IR* henceforth) conditions are absent, in this model, the payoff of the optimal contract and optimal n-level contract are twisted differently away from the product of consumer's valuation and product's quality due to different numbers of *IC* and *IR* conditions faced by the monopolist. Therefore, the connection between n-level contracts and the optimal contract is not as simple as the connection between n-class CM and PAM. It is not straightforward to apply CM results here to obtain the profit gain (analog to the efficiency gain) of the optimal n-level contract. To overcome the difficulty caused by *IC* and *IR* conditions, we first treat the payoff of the optimal contract as PAM then apply CM results mechanically to this payoff, which implies the lower bound of n-class CM. Note, n-class CM derived here is based on the optimal contract's payoff, which is generated endogenously taking *IC* conditions into account. It does not correspond to any n-level contract directly. To link n-class CM and n-level contract, in the second step, we construct an n-level (feasible) contract that satisfies *IC* and *IR* conditions. The profit gain of such a feasible contract can be bounded below by the lower bound of n-class CM derived in the first step. Since the optimal n-level contract should have a (weakly) higher profit than any feasible n-level contract, the lower bound derived in the first step, therefore, is the lower bound of the profit gain for the optimal n-level contract.

PROBLEM FORMULATION. A monopolist seller could choose to produce a product with quality a at cost $c \cdot a$. The buyer's utility from such a good is assumed to be $\theta u(a)$, where θ is this buyer's type privately known to herself and $u(a)$ is an increasing concave function of quality a. Type θ is randomly distributed over $[0,1]$ with c.d.f. $\hat{F}(\theta)$. The monopolist maximizes its profit by solving the following problem:

$$\begin{aligned}
&\max_{a(\theta),t(\theta)} \int_0^1 [t(\theta) - ca(\theta)]\, d\hat{F}(\theta) \\
\text{s.t } &\theta u(a(\theta)) - t(\theta) \geq \theta u\left(a\left(\theta'\right)\right) - t\left(\theta'\right) \text{ for any } \theta, \theta' \in [0,1] \quad (IC) \\
&\theta u(a(\theta)) - t(\theta) \geq 0 \text{ for any } \theta \in [0,1] \qquad\qquad (IR).
\end{aligned}$$

THE OPTIMAL CONTRACT. It is standard to assume that $\left(1-\hat{F}(\theta)\right)/\hat{f}(\theta)$ is a decreasing function. Then by using standard procedures, we can solve for the optimal contract. Denote the optimal contract as $a^*(\theta)$, which specifies the quality of the product a buyer gets if his valuation is θ. The monopolist's profit is

$$\pi_\infty = \underbrace{\int_{\underline{\theta}}^1 \left(\theta - \frac{1-\hat{F}(\theta)}{\hat{f}(\theta)}\right) u(a^*(\theta))\, d\hat{F}(\theta)}_{\text{revenue}} - \underbrace{\int_{\underline{\theta}}^1 ca^*(\theta)\, d\hat{F}(\theta)}_{\text{cost}},$$

where $\underline{\theta}$ satisfies $\underline{\theta} - \left(1-\hat{F}(\underline{\theta})\right)/\hat{f}(\underline{\theta}) = 0$.

COARSE MATCHING. Let $\varphi(\theta) = \left[\theta - \left(1-\hat{F}(\theta)\right)\right]/\hat{f}(\theta)$. Denote revenue under the optimal contract as R_∞ which can be written as

$$R_\infty = \int_{\underline{\theta}}^1 \varphi(\theta)\, u(a^*(\theta))\, d\hat{F}(\theta).$$

Since both $\varphi(\theta)$ and $u(a^*(\theta))$ are increasing, R_∞ can be thought of as PAM between "agents" with types $\varphi(\theta)$ and $u(a^*(\theta))$. Denote the inverse function of $\varphi(\theta)$ as $\varphi^{-1}(x)$. Then revenue R_∞ can be written as $\int_0^1 xu\left(a^*\left(\varphi^{-1}(x)\right)\right) d\hat{F}\left(\varphi^{-1}(x)\right)$. We then apply Theorem 3.4.1 directly to R_∞ to derive a lower bound for $(R_n - R_1)/(R_\infty - R_1)$. Here R_n is defined as

$$\begin{aligned}
R_n = \sum \Bigg(&\frac{1}{F\left(\varphi^{-1}(x_i)\right) - F\left(\varphi^{-1}(x_{i-1})\right)} \int_{x_{i-1}}^{x_i} x d\hat{F}\left(\varphi^{-1}(x)\right) \\
&\times \int_{x_{i-1}}^{x_i} u\left(a^*\left(\varphi^{-1}(x)\right)\right) d\hat{F}\left(\varphi^{-1}(x)\right) \Bigg).
\end{aligned}$$

By letting $\pi_n = R_n - \int_{\underline{\theta}}^{1} ca^*(\theta)\, d\hat{F}(\theta)$, we have equality $(R_n - R_1)/(R_\infty - R_1) = (\pi_n - \pi_1)/(\pi_\infty - \pi_1)$. Note that R_n and π_n do not have any economic meaning in this setting thus far. Both R_n and π_n are numbers implied by Theorem 3.4.1.

CONNECTION TO n-LEVEL CONTRACTS. Our goal is to establish a lower bound for the profit gain of the optimal n-level contract. Denote the profit under the optimal n-level contract as π_n^*. We want to utilize the lower bound of $(R_n - R_1)/(R_\infty - R_1)$ derived above to establish a lower bound for $(\pi_n^* - \pi_1)/(\pi_\infty - \pi_1)$. This can be achieved if we can show that π_n^* is not less than π_n.

Let π_n be the profit from an n-level contract satisfying both *IC* and *IR* conditions. By definition, the profit of the optimal n-level contract π_n^* is no less than π_n. In order to do a comparison, we need show that there indeed exists an n-level contract that satisfies *IC* and *IR* conditions under which the monopolist has profit equal to π_n. We show this by construction. Consider an n-level stochastic contract offered by the monopolist such that a consumer with type $\theta \in [\theta_{i-1}, \theta_i]$ gets a product with quality randomly distributed according to $a_i(x) = a^*(x)$ where x is randomly distributed over $[\theta_{i-1}, \theta_i]$ by $\hat{F}(x)$ and $\{\theta_i\}_{i=1}^n$ are cutoff points associated with R_n. By offering products with stochastic qualities within certain intervals and charging fix prices, the monopolist could induce consumers to buy products targeted to their types. We claim such a contract satisfies *IC* and *IR* conditions (hence implementable), and it has profit equal to π_n (see Appendix). In fact, this n-level contract also has revenue equal to R_n.

RESULT. To apply Theorem 3.4.1, define $F(x) = \hat{F}\left(\varphi^{-1}(x)\right)$, $G\left(u\left(a^*\left(\varphi^{-1}(x)\right)\right)\right) = F(x)$ and $\phi(x) = u\left(a^*\left(\varphi^{-1}(x)\right)\right)$. Let $C = \phi(1) - \phi(0)$. Now distributions $F(\cdot)$ and $G(\cdot)$ are defined over $[0,1]$ and $[\phi(0), \phi(1)]$. By adapting the proof of Theorem 3.4.1 $(\pi_n^* - \pi_1)/(\pi_\infty - \pi_1) \geq 1 - C\left(\beta/n^2\right)$ where β is the same as in Theorem 3.4.1. However, unlike in the canonical matching model, $(\pi_n^* - \pi_1)/(\pi_\infty - \pi_1)$ may not be a natural measure for the profit gain of the optimal n-level contract. On the contrary, measure π_n^*/π_∞ captures the fraction of the profit that can be generated from the optimal n-level contract relative to the optimal contract. By the logic of the proof of Theorem 3.4.1, with a little modification, we have the following result regarding π_n^*/π_∞.

Proposition 3.5.1 *The profit of the optimal n-level contract π_n^* satisfies*

$$\frac{\pi_n^*}{\pi_\infty} \geq 1 - \frac{\hat{\beta}}{n^2},$$

where $\hat{\beta} = \min\left\{\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3\right\}$

1. $\hat{\beta}_1 = \dfrac{C}{4\pi_\infty}$

2. $\hat{\beta}_2 =$
$$C \min\left\{\min_{p,q} \frac{\left\|F^{-1\prime}\right\|_p^{[0,1]} \left\|G^{-1\prime}\right\|_q^{[\phi(0),\phi(1)]}}{8\pi_\infty}, \frac{\left\|F^{-1\prime}\right\|_\infty^{[0,1]} \left\|G^{-1\prime}\right\|_\infty^{[\phi(0),\phi(1)]}}{12\pi_\infty}\right\},$$
where $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p, q \in [1, \infty]$

3. $\hat{\beta}_3 = \min\left\{\hat{\beta}_3^a, \hat{\beta}_3^b\right\},$

where $\hat{\beta}_3^a = \min\limits_{p_1,q_1} \dfrac{C\,\|f\|_{p_1}^{[0,1]}}{2\sqrt{2}\left[(q_1+1)(2q_1+1)\right]^{\frac{1}{2q_1}}\pi_\infty},$

$\hat{\beta}_3^b = \min\limits_{p_2,q_2} \dfrac{C\,\|g\|_{p_2}^{[\phi(0),\phi(1)]}}{2\sqrt{2}\left[(q_2+1)(2q_2+1)\right]^{\frac{1}{2q_2}}\pi_\infty},$

$\frac{1}{p_j} + \frac{1}{q_j} = 1$ *and* $p_j, q_j \in [1, \infty]$, $j = 1, 2$.

To illustrate this proposition, we apply it to the following example.

Example 3.5.1 *Consumer's utility function is* $\theta u(a) = \theta a^{\frac{1}{2}}$. *The c.d.f. of valuation is* $\hat{F}(\theta) = \theta^{\alpha}$ *over* $[0,1]$. *Then the optimal contract is* $a(\theta) = \varphi^2(\theta)/4c^2$, *where* $\varphi(\theta) = \left(\theta - \left(1 - \hat{F}(\theta)\right)/\hat{f}(\theta)\right)$. *The revenue under the optimal contract is* $R_\infty = \frac{1}{2c}\int_{\underline{\theta}}^{1} \varphi^2(\theta)\, dF(\theta)$. *Hence,* $F(x) = G(x) = \hat{F}\left(\varphi^{-1}(x)\right)$. *We have* $R_\infty = \frac{1}{2c}\int_0^1 x^2 dF(x)$. *For illustration, we simply choose* $\hat{\beta}_3^a$ *with* $p_1 = \infty$ *from the previous proposition*

$$\beta_3^a = \frac{\sqrt{3}\,\|f\|_\infty^{[0,1]}}{12\pi_\infty}.$$

We can show that $\beta_3 \leq 2$, *for any* $\alpha \in [1,20]$. *Therefore, for any* $\alpha \in [1,20]$,

$$\frac{\pi_n^*}{\pi_\infty} \geq 1 - \frac{2}{n^2}.$$

WITHOUT THE OPTIMAL CONTRACT. To derive $\hat{\beta}$ for the profit gain of an optimal n-level contract, we need to know the entire function of the optimal contract $a^*(\theta)$. However, the closed form of $a^*(\theta)$ is usually difficult or impossible to derive. Now the question is, how do we know the gain of profit recovered by an optimal n-level contract if we don't have the optimal contract? In this part, we consider an alternative to solving the optimal contract.

By observing $\hat{\beta}$ in the above proposition, both C and π_∞ are related to optimal contract $a^*(\theta)$. However, $C = u(a^*(1)) - u(a^*(\underline{\theta}))$, which depends on the value of $a^*(\theta)$ at only two points, $a^*(1)$ and $a^*(\underline{\theta})$. Furthermore, from the conditions that the optimal contract need to satisfy, we could imply that $a^*(\underline{\theta}) = 0$ and $u'(a^*(1)) = c$. Hence, once the form of $u(\cdot)$ is known, the value of $a^*(1)$ can be solved immediately. By contrast, the value of π_∞ is determined by the entire function $a^*(\theta)$. Denote the value of $\hat{\beta}$ by replacing π_∞ with π_2^* as $\tilde{\beta}$, where π_2^* the profit of the optimal 2-level contract. Since the optimal 2-level contract only involves two *IC* conditions, it is easy to derive π_2^*. Furthermore, $\pi_2^* \leq \pi_\infty$ implies that $\tilde{\beta} \geq \hat{\beta}$. Therefore, $\pi_n^*/\pi_\infty \geq 1 - \tilde{\beta}/n^2$.

To illustrate the above discussion, we consider the same example as Example 3.5.1 with $\alpha = 1$.

Example 3.5.2 *For Example 3.5.1 with $\alpha = 1$, It is easy to solve for an optimal 2-level contract that has profit $\pi_2^* = \frac{1}{25c}$. Based on previous discussion, for simplicity, let $\tilde{\beta}$ be $\hat{\beta}_1$ with π_∞ replacing by π_2^*. Hence, $\tilde{\beta} = u(a^*(1)) / (4\pi_2^*)$. That is $u(a^*(1)) = 1/(2c)$. Therefore, $\tilde{\beta} = 25/8$.*

Hence,

$$\frac{\pi_n^*}{\pi_\infty} \geq 1 - \frac{\tilde{\beta}}{n^2}.$$

Cost-Sharing

In the cost-sharing model, a principal wants to procure a product from a firm. The cost of production for the firm is randomly distributed. The firm could reduce the initial cost by exerting effort and bearing some disutility. Both initial cost and effort are unobservable to the principal. The principal only observes realized cost. Compared to the monopolistic pricing model, this model includes both adverse selection and hidden actions. Laffont and Tirole (1986) studied this model in a very general setting. They completely solved the problem for the optimal contract containing a continuum of items. However, Rogerson (2003) argued that such a contract was not widely used due to its complexity. He then studied this model within a very restrictive environment and showed that a ”simple contract” performs relatively well in terms of cost. Chu and Sappington (2007) followed this idea. They analyzed the same cost-sharing model as Rogerson (2003) under a slightly more general class of distributions. They provided a different ”simple contract” and showed that such contract could often outperform the one used in Rogerson (2003) under different distributions.

The ”simple contract” in these two papers refers to a 2-level contract, which is a combination of two of the following three contracts: fixed price (FP) contract, cost reimbursement (CR) contract and linear cost sharing (LCS) contract. Under a FP contract, the principal pays a fixed price for the product regardless the realized cost. Under a CR contract, the principal pays the realized cost to the firm. Under an LCS contract, the principal pays a lump-sum payment and shares a fixed fraction of the realized cost. In Rogerson (2003), the ”simple contract” is a 2-level contract which has FP and CR. In Chu and Sappington (2007), the ”simple contract” is a 2-level contract containing LCS and CR.

We consider the same cost-sharing model as Rogerson (2003) and Chu and Sappington (2007) with much more general distributions. Unlike the previous two papers, we want to evaluate the performance of n-level contracts. To apply CM results to this model, we have the same difficulty as in the monopolistic pricing case due to feasibility (IC and IR) conditions required for n-level contracts. PAM is generated endogenously under the optimal contract. The CM scheme, in general, is not a feasible contract. By the same logic as in monopoly pricing, to apply our CM results, we construct a ”simple contract” that can be bounded by the lower bound derived from CM result. Moreover, such contract is feasible. The ”simple contract” to be analyzed is an n-level linear cost sharing cost reimbursement (LCSCR) contract. Such a contract contains n items with $n-1$ different LCS items and one CR item.

To derive the lower bound of the efficiency gain for the optimal n-level LCSCR contract, we first formulate this problem and solve for the optimal contract enabling us to see the sorting feature in the principal’s value function. Due to sorting, we can apply our CM results and establish a lower bound. In the end, we show that this lower bound is indeed a lower bound of the efficiency gain of a feasible n-level LCSCR contract.

MODEL DESCRIPTION. A risk-neutral principal wants to buy a unit of product from a firm. The firm's initial cost of production is a random variable following a c.d.f. $F(x)$ over $[0,1]$. The firm could exert effort y to reduce the initial cost, and the disutility of effort is $\frac{1}{4k}y^2$. Such an effort could not be observed by the principal. The principal could only observe the realized cost, which is $c = x - y$. Since the firm's initial cost is private information, the contract should satisfy both *IC* and *IR* conditions. The goal of the principal is to offer a feasible contract that minimize expected transfer to the firm.

THE OPTIMAL CONTRACT. Following the standard procedure developed in Laffont and Tirole (1986), we have $y(x) = 0$ if $F(x)/f(x) \geq 2k$ and $y(x) = 2k - (F(x)/f(x))$ if $F(x)/f(x) \leq 2k$ (see Appendix). Function $F(x)/f(x)$ is usually assumed to be an increasing function. Hence, the effort $y(x)$ is decreasing in x and bounded below by zero.

Denote $x^* = \min\{\{x,1\} \mid [F(x)/f(x)] = 2k.\}$. The expected transfer to the firm is

$$\mu_x - \underbrace{k\int_0^{x^*}\left[\frac{1}{2k}\frac{F(x)}{f(x)} - 1\right]^2 dF(x)}_{\text{cost reduction}}.$$

Under a CR contract, the principal pays the firm for the observed cost. It is easy to check that a CR contract is feasible, and under this contract the firm exerts no effort and the expected cost for the principal is μ_x. Hence, the second term $k\int_0^{x^*}[1 - F(x)/2kf(x)]^2$
$dF(x)$ is the cost reduction from the CR contract. The objective of the principal is equivalent to finding a contract that maximizes cost reduction. Under any contract, the expected transfer to the firm can be represented as μ_x subtracting an amount of cost reduction. The performance of a "simple contract" then can be measured by

the amount of cost reduction.

COARSE MATCHING. Denote cost reduction under the optimal contract as u_∞. That is $u_\infty = k\int_0^{x^*}[1-F(x)/2kf(x)]^2\,dF(x)$. Note that u_∞ can be viewed as PAM between "agents" with two identical types $F(x)/2kf(x)-1$. As in monopoly pricing model, we could apply CM results to u_∞ and provide a lower bound of $(u_{n-1}-u_1)/(u_\infty-u_1)$. Here,

$$u_{n-1} = k\sum_{i=1}^{n-1}\frac{1}{F(x_i)-F(x_{i-1})}\left(\int_{x_{i-1}}^{x_i}[F(x)/2kf(x)-1]\,dF(x)\right)^2,$$

where $x_0 = 0$, $x_{n-1} = x^*$ and $\{x_i\}$ are determined by the main theorem. Note u_{n-1} again is the a value implied by the CM result and has no meaning in this model thus far.

CONNECTION TO THE n-LEVEL CONTRACT. In order to use the lower bound of $(u_{n-1}-u_1)/(u_\infty-u_1)$ to evaluate the performance of the optimal n-level contract, we need to show that there exists a feasible n-level contract which has the amount of cost reduction equals to u_{n-1}. The n-level contract we analyze is an n-level LCSCR contract, where the first $n-1$ items are $n-1$ different LCS contracts for the firm with type less than x^* and the last is a CR contract for the firm with type greater than x^*. The reason such an n-level contract has cost reduction equal to u_{n-1} is because the CR part of an n-level LCSCR contract provides no cost reduction.

We prove the existence by construction. The principal offers an n-level LCSCR contract that has LCS contract $\{T_i,\alpha_i\}$ if $i \le n-1$ and CR contract if $i = n$. $\{\alpha_i\}_{i=1}^{n-1}$ are chosen to satisfy

$$(1-\alpha_i)^2 = \frac{1}{F(x_i)-F(x_{i-1})}\left(\int_{x_{i-1}}^{x_i}\left[1-\frac{1}{2k}\frac{F(x)}{f(x)}\right]dF(x)\right)^2,$$

where $\{x_i\}_{i=1}^{n-1}$ are the same as the cutoff point for u_{n-1}. Lump-sum transfers $\{T_i\}_{i=1}^{n-1}$ can be solved recursively so that IC and IR conditions can be satisfied.

Therefore, the n-level LCSCR contract we constructed is feasible. Moreover, the cost reduction of this n-level contract is the same as u_{n-1}. Hence, the value of u_{n-1} can be treated as the cost reduction of some n-level LCSCR contract. Note that such an n-level LCSCR contract needs not be optimal.

THE OPTIMAL n-LEVEL LCSCR CONTRACT. So far we have established the lower bound for $(u_{n-1} - u_1) / (u_\infty - u_1)$ and showed that there exists a feasible n-level LCSCR under which the cost reduction is u_{n-1}. By definition, the optimal n-level LCSCR contract should provide a weakly higher cost reduction denoted as u^*_{n-1}. The lower bound of $(u_{n-1} - u_1) / (u_\infty - u_1)$ then is a lower bound of $\left(u^*_{n-1} - u_1\right) / (u_\infty - u_1)$.

Similarly to monopoly pricing model, u^*_{n-1}/u_∞ here is a more sensible measure, which captures the fraction of cost reduction can be achieved by the optimal n-level LCSCR contract of the optimal contract. We could derive an analog result to Proposition 3.5.1. For simplicity, by using the proof of β_1 in Theorem 3.4.1, we show the following result.

Proposition 3.5.2 *If $\frac{F(x)}{f(x)}$ is increasing, the cost reduction of the optimal n-level LCSCR contract u^*_{n-1} satisfies $u^*_{n-1}/u_\infty \geq 1 - \beta / (n-1)^2$, where $\beta = \dfrac{k}{4u_\infty}$.*

Rogerson (2003) shows that with uniform distribution an optimal two-level FPCR contract has $(u^*_1/u_\infty) \geq 0.75$. Chu and Sappington (2007) shows that under a c.d.f. $F(x) = \left(\frac{x-\underline{x}}{\overline{x}-\underline{x}}\right)^{\alpha}$, where $\alpha \in [0, \infty)$ an optimal 2-level LCSCR has $u^*_1/u_\infty \geq 2/e \approx 0.73$.

By restricting ourselves to distributions used in above two papers, our results may not outperform their lower bounds. Because our results are derived for much more general environments and for contracts with any number of levels. On the contrary, under specific forms of distributions, the optimal two-level contracts can be solved. Hence, tighter lower bounds can be provided.

3.6 Concluding Remarks

We have analyzed the efficiency gain of n-class CM of two-sided markets with heterogeneous agents relative to the optimal matching scheme that requires PAM. We revisited McAfee (2002) and showed that his result extends in an intuitive yet non-obvious way to 2^n classes. But the same method cannot be applied to further extend the result to CM with n classes. We then developed a completely different method, one that involves a novel use of some powerful inequalities. Our first main result provides a lower bound on the efficiency gain of CM for suitable n-class partitions of the populations. Furthermore, the distributions required are very mild, as they only need to have finite variance and continuous densities over a non-negative bounded interval. The second main result of the paper provides an extension to more general payoff functions that are supermodular in the agents' characteristics, so long as a condition that involves both the payoff function and the distributions of attributes is satisfied. In the end, we showed that there are several problems seemingly unrelated to matching in which our results can be fruitfully applied. We illustrated this by adapting the coarse matching results to models like monopoly pricing problem and cost-sharing problem so as to evaluate the performance of pooling contracts that are easy implemented relative to optimal contracts. Compared to the results from Rogerson (2003) and Chu and Sappington (2007) regarding the cost-sharing problem, our results hold for contracts with any n levels under much more general conditions of distributions. This application also sheds some light on the comments made in Rogerson (2003) about the common mathematical structure behind the matching and cost-sharing models.

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APPENDIX A

PROOF AND OMITTED STEPS OF CHAPTER 3

Normalization of Type Space

In this part, we show that if the match payoff is xy, any non-negative and bounded type space $[a,b]$ can be normalized to $[0,1]$ without affecting the value of $(u_n - u_1)/(u_\infty - u_1)$. It's easy to see that the length of the type space can be assumed to be one without loss of generality.

Define

$$u_\infty^a = \int_a^{a+1} x\tilde{\phi}(x)\,d\tilde{F}(x),\; u_1^a = \int_a^{a+1} xd\tilde{F}(x)\int_a^{a+1} yd\tilde{G}(y)$$
$$u_n^a = \sum_{i=1}^{n} \frac{1}{F(x_i) - F(x_{i-1})}\int_{x_{i-1}}^{x_i} xd\tilde{F}(x)\int_{\phi(x_{i-1})}^{\phi(x_i)} yd\tilde{G}(y).$$

Here, for any $x, y \in [a, a+1]$, $\tilde{F}(x) = F(x-a)$ and $\tilde{G}(y) = G(y-a)$ and $G(\phi(x-a)) = \tilde{G}\left(\tilde{\phi}(x)\right)$. Therefore, $\tilde{\phi}(x) = \phi(x-a) + a$.

Lemma 1 $\dfrac{u_\infty^a - u_n^a}{u_\infty^a - u_1^a} = \dfrac{u_\infty^0 - u_n^0}{u_\infty^0 - u_1^0}$.

Proof. For $x, y \in [a+1, b+1]$, $\tilde{F}(x) = F(x-1)$, $\tilde{G}(y) = G(y-1)$, $F(x) = G(\phi(x))$ and $G(\phi(x-1)) = \tilde{G}\left(\tilde{\phi}(x)\right)$. Hence, $\tilde{\phi}(x) = \phi(x-1) + 1$.

To prove the lemma, it is enough to prove the following equality:

$$\int_a^b x\phi(x)\,dF(x) - \frac{1}{F(b) - F(a)}\int_a^b xdF(x)\int_a^b ydG(y)$$
$$= \int_{a+1}^{b+1} x\tilde{\phi}(x)\,d\tilde{F}(x) - \frac{1}{\tilde{F}(b+1) - \tilde{F}(a+1)}\int_{a+1}^{b+1} xd\tilde{F}(x)\int_{a+1}^{b+1} yd\tilde{G}(y),$$

where $b - a \leq 1$.

To show it,

$$\int_{a+1}^{b+1} x\tilde{\phi}(x)\,d\tilde{F}(x) - \frac{1}{\tilde{F}(b+1)-\tilde{F}(a+1)}\int_{a+1}^{b+1} xd\tilde{F}(x)\int_{a+1}^{b+1} xd\tilde{G}(x)$$
$$= \int_a^b (y+1)\,\tilde{\phi}(y+1)\,dF(x) - \frac{1}{F(b)-F(a)}\int_a^b (y+1)\,dF(y)\int_a^b (y+1)\,dG(y)$$
$$= \int_a^b (y+1)(\phi(y)+1)\,dF(x) - \frac{1}{F(b)-F(a)}\int_a^b (y+1)\,dF(y)\int_a^b (y+1)\,dG(y)$$
$$= \int_a^b x\phi(x)\,dF(x) + \int_a^b \phi(x)\,dF(x) + \int_a^b xdF(x) + \int_a^b 1dF(x)$$
$$- \frac{1}{b-a}\int_a^b xdF(x)\int_a^b xdG(x) - \int_a^b xdG(x) - \int_a^b xdF(x) - \int_a^b 1dF(x)$$
$$= \int_a^b x\phi(x)\,dF(x) - \frac{1}{b-a}\int_a^b xdF(x)\int_a^b xdG(x).$$

If the type space is an arbitrary interval $[a,b]$, we can redefine $\tilde{F}(x) = F\left(\frac{x-a}{b-a}\right)$ and $\tilde{G}(y) = G\left(\frac{y-a}{b-a}\right)$. Then the same argument goes through. ■

From the above proof, one could immediately see that ratio

$$\frac{u_1^a}{u_\infty^a} = \frac{\int_0^1 \left(F^{-1}(z)+a\right)dz \int_0^1 \left(G^{-1}(z)+a\right)dz}{\int_0^1 \left(F^{-1}(z)+a\right)\left(G^{-1}(z)+a\right)dz}$$

is not immune to the shift of domain. In fact, by letting a go to infinity, u_1^a/u_∞^a goes to 1.

Preliminary Results

The following results are collected from Mitrinovic et al., 1993.

Lemma 2 (Steffensen's Inequality) *Let $g,h:[a,b]\to\mathbb{R}$ and $F:[a,b]\to[0,1]$ be a distribution function. Suppose that g is monotonically increasing. Define $H_F:(a,b]\to\mathbb{R}$, $H_F(t) = \int_a^t h(s)\,dF(s)\,/\int_a^t dF(s)$. If $H_F(t)\le H_F(b)$ for all $t\in(a,b]$, then*

$$\int_a^b g(s)\,h(s)\,dF(s) \ge \int_a^b g(s)\,dF(s)\int_a^b h(s)\,dF(s).$$

Define

$$T\left(f,g;p\right)=\frac{1}{\int_a^b p\left(x\right)dx}\left(\int_a^b p\left(x\right)f\left(x\right)g\left(x\right)dx\right.$$
$$\left.-\frac{1}{\int_a^b p\left(x\right)dx}\int_a^b p\left(x\right)f\left(x\right)dx\int_a^b p\left(x\right)g\left(x\right)dx\right).$$

In particular, if $p\left(x\right)=\frac{1}{b-a}$, $T\left(f,g\right)=T\left(f,g;p\right)$.

Lemma 3 (Korkine's identity)

$$T\left(f,g;p\right)=\frac{1}{2}\int_a^b\int_a^b p\left(t\right)p\left(s\right)\left(f\left(t\right)-f\left(s\right)\right)\left(g\left(t\right)-g\left(s\right)\right)dtds.$$

Lemma 4

$$T\left(f,g\right)=f'\left(\varepsilon\right)g'\left(\eta\right)T\left(x,x\right)\qquad\varepsilon,\eta\in\left[a,b\right].$$

Lemma 5 *Assume* $\varphi\leq f\left(x\right)\leq\phi$ *and* $\gamma\leq g\left(x\right)\leq\Gamma$*, then*

$$T\left(f,g\right)\leq\frac{1}{4}\left(\phi-\varphi\right)\left(\Gamma-\gamma\right).$$

Lemma 6 *If* $f\left(\cdot\right),g\left(\cdot\right)$ *are totally(completely) monotonic,*

$$T\left(f,g\right)\leq\frac{1}{12}\left(f\left(b\right)-f\left(a\right)\right)\left(g\left(b\right)-g\left(a\right)\right).$$

If $f\left(\cdot\right),g\left(\cdot\right)$ are absolutely monotonic,

$$T\left(f,g\right)\leq\frac{4}{45}\left(f\left(b\right)-f\left(a\right)\right)\left(g\left(b\right)-g\left(a\right)\right).$$

Define $K\left(f,p,q\right)=\int_a^b q\left(x\right)f\left(x,x\right)dx-\int_a^b\int_a^b f\left(x,y\right)p\left(x,y\right)dxdy.$

Lemma 7 $K\left(f,p,q\right)=f_{21}\left(\varepsilon,\eta\right)K\left(\left(x-a\right)\left(y-a\right),p,q\right)$

Lemma 8

$$\frac{1}{b-a}\int_a^b u\left(F^{-1}(\alpha),G^{-1}(\alpha)\right)d\alpha-\left(\frac{1}{b-a}\right)^2\int_a^b\int_a^b u\left(F^{-1}(\alpha),G^{-1}(\beta)\right)d\alpha d\beta$$
$$\leq\frac{1}{4}\left(u\left(F^{-1}(b),G^{-1}(b)\right)+u\left(F^{-1}(a),G^{-1}(a)\right)\right.$$
$$\left.-u\left(F^{-1}(a),G^{-1}(b)\right)-u\left(F^{-1}(b),G^{-1}(a)\right)\right)$$
$$=\frac{1}{4}\int_a^b\int_a^b u_{21}F^{-1\prime}G^{-1\prime}d\alpha d\beta$$

The following result is from *Elezovic, Marangunic and Pecaric 2007.*

Lemma 9 *If $f(\cdot)$ and $g(\cdot)$ are absolutely continuous, for $\alpha,\beta>1,\frac{1}{\alpha}+\frac{1}{\beta}\leq 1$ we have*

$$|T(f,g)|\leq\frac{1}{12}\left(\frac{3}{2}\right)^{\frac{1}{\alpha}+\frac{1}{\beta}}(b-a)^{2-\frac{1}{\alpha}-\frac{1}{\beta}}\left\|f'\right\|_\alpha\left\|g'\right\|_\beta.$$

The following result is from *Barnett, Cerone, and Dragomir etc. 2001.*

Lemma 10 *Given p.d.f function $f(x)$ defined over $[a,b]$, if standard deviation σ exists, then*

$$\sigma\leq\begin{cases}\frac{\sqrt{3}(b-a)^2}{6}\|f\|_\infty, & \textit{provided } f\in L_\infty[a,b],\ ;\\ \frac{\sqrt{2}(b-a)^{1+\frac{1}{q}}}{2[(q+1)(2q+1)]^{\frac{1}{2q}}}\|f\|_p, & \textit{provided } f\in L_p[a,b],\\ & \textit{and } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{2}(b-a), & \textit{provided } p=1.\end{cases}.$$

An Extension of McAfee (2002): Proof of Theorem 3.3.2 and 3.3.3

The proof of Theorem 3.3.2 requires the following lemma.

Lemma 11 *If conditions (1) and (2) of Theorem 3.3.1 hold,*

1. *$(z-c)F^{-1\prime}(z)$ and $(z-c)G^{-1\prime}(z)$ are increasing for z in $(c,1]$, and*

2. *$(c-z)\,F^{-1'}(z)$ and $(c-z)\,G^{-1'}(z)$ are decreasing for z in $[0,c)$.*

Proof. Function $(z-c)\,F^{-1'}(z)$ is increasing if and only if $(F(x)-c)/f(x)$ is increasing. From condition (1) of Theorem 3.3.1, $F(x)/f(x)$ is increasing. The derivative is

$$\frac{d}{dx}\frac{F(x)}{f(x)} = \frac{f^2(x) - F(x)\,f'(x)}{f^2(x)} > 0.$$

The derivative of $(F(x)-c)/f(x)$ is

$$\frac{d}{dx}\frac{F(x)-c}{f(x)} = \frac{f^2(x) - (F(x)-c)\,f'(x)}{f^2(x)}.$$

If $f'(x) < 0$, $[(F(x)-c)/f(x)]' > 0$ due to $(F(x)-c) > 0$. If $f'(x) > 0$, we have $f^2(x) - (F(x)-c)\,f'(x) > f^2(x) - F(x)\,f'(x)$. That is $[(F(x)-c)/f(x)]' > 0$. Similarly, we can prove $(z-c)\,G^{-1'}(z)$ is increasing.

For the second statement:

Function $(c-z)\,F^{-1'}(z)$ is decreasing if and only if $(c-F(x))/f(x)$ is decreasing. From condition (2) of Theorem 3.3.1, $(1-F(x))/f(x)$ is decreasing. The derivative is

$$\frac{d}{dx}\frac{1-F(x)}{f(x)} = \frac{-f^2(x) - (1-F(x))\,f'(x)}{f^2(x)} < 0.$$

The derivative of $(c-F(x))/f(x)$ is

$$\frac{d}{dx}\frac{c-F(x)}{f(x)} = \frac{-f^2(x) - (c-F(x))\,f'(x)}{f^2(x)}.$$

Note that $c-F(x) > 0$ for $z \in [0,c)$. If $f'(x) > 0$, $[(c-F(x))/f(x)]' < 0$. If $f'(x) < 0$, $-(c-F(x))\,f'(x) < -(1-F(x))\,f'(x)$. Therefore,

$$\frac{d}{dx}\frac{c-F(x)}{f(x)} = \frac{-f^2(x) - (c-F(x))\,f'(x)}{f^2(x)} < \frac{-f^2(x) - (1-F(x))\,f'(x)}{f^2(x)} < 0. \quad \blacksquare$$

Proof of Theorem 3.3.2. Instead of proving the theorem directly, we show that

$$\frac{u_{2^n}-u_{2^{n-1}}}{u_\infty-u_{2^{n-1}}}\geq\frac{1}{2}$$

holds for any given n by induction. Theorem 3.3.1 shows that the above inequality holds when $n=1$. By induction, assume it is true for $n-1$. We need to show that

$$\frac{u_{2^n}-u_{2^{n-1}}}{u_\infty-u_{2^{n-1}}}\geq\frac{1}{2}.$$

First rewrite $u_{2^{n-1}}$ as

$$\begin{aligned}u_{2^{n-1}}=\sum_{i=1}^{2^{n-1}}\left(F\left(x_i\right)-F\left(x_{i-1}\right)\right)&\left(\int_{x_{i-1}}^{x_i}x\frac{f\left(x\right)}{F\left(x_i\right)-F\left(x_{i-1}\right)}dx\right.\\&\left.\times\int_{\phi\left(x_{i-1}\right)}^{\phi\left(x_i\right)}y\frac{g\left(y\right)}{G\left(\phi\left(x_i\right)\right)-G\left(\phi\left(x_{i-1}\right)\right)}dy\right)\end{aligned}$$

with fixed cutoff points $\{x_i\}_{i=0}^{2^{n-1}}$.

To obtain 2^n classes with cutoff point $\{x_i\}_{i=0}^{2^{n-1}}\cup\{z_i\}_{i=0}^{2^{n-1}}$, we insert 2^{n-1} cutoff points such that $z_i\in[x_{i-1},x_i]$ for $i=1,...,2^{n-1}$. The payoff for this 2^n-class CM is

$$\begin{aligned}u_{2^n}=\sum_{i=1}^{2^{n-1}}\left(F\left(z_i\right)-F\left(x_{i-1}\right)\right)&\left(\int_{x_{i-1}}^{z_i}x\frac{f\left(x\right)}{F\left(x_i\right)-F\left(x_{i-1}\right)}dx\right.\\&\left.\times\int_{\phi\left(x_{i-1}\right)}^{\phi\left(z_i\right)}y\frac{g\left(y\right)}{G\left(\phi\left(x_i\right)\right)-G\left(\phi\left(x_{i-1}\right)\right)}dy\right)\\&+\sum_{i=2}^{2^{n-1}}\left(F\left(x_i\right)-F\left(z_{i-1}\right)\right)\left(\int_{z_{i-1}}^{x_i}x\frac{f\left(x\right)}{F\left(x_i\right)-F\left(z_{i-1}\right)}dx\right.\\&\left.\times\int_{\phi\left(z_{i-1}\right)}^{\phi\left(x_i\right)}y\frac{g\left(y\right)}{G\left(\phi\left(x_i\right)\right)-G\left(\phi\left(z_{i-1}\right)\right)}dy\right).\end{aligned}$$

Let $c_i=F\left(x_i\right)=G\left(\phi\left(x_i\right)\right)$ and $d_i=F\left(z_i\right)=G\left(\phi\left(z_i\right)\right)$. Rewrite above equations to obtain:

$$\begin{aligned}u_{2^{n-1}}&=\sum_{i=1}^{2^{n-1}}\left(c_i-c_{i-1}\right)\int_{c_{i-1}}^{c_i}\frac{F^{-1}\left(z\right)}{c_i-c_{i-1}}dz\int_{c_{i-1}}^{c_i}\frac{G^{-1}\left(z\right)}{c_i-c_{i-1}}dz\\u_{2^n}&=\sum_{i=1}^{2^{n-1}}\left(d_i-c_{i-1}\right)\int_{c_{i-1}}^{d_i}\frac{F^{-1}\left(z\right)}{d_i-c_{i-1}}dz\int_{c_{i-1}}^{d_i}\frac{G^{-1}\left(z\right)}{d_i-c_{i-1}}dz\\&+\sum_{i=1}^{2^{n-1}}\left(c_i-d_i\right)\int_{d_i}^{c_i}\frac{F^{-1}\left(z\right)}{c_i-d_i}dz\int_{d_i}^{c_{i+1}}\frac{G^{-1}\left(z\right)}{c_i-d_i}dz.\end{aligned}$$

Therefore,

$$
\begin{aligned}
& u_{2^n} - u_{2^{n-1}} \\
&= \sum_{i=1}^{2^{n-1}} \frac{1}{d_i - c_{i-1}} \left(\int_{c_{i-1}}^{d_i} \left(F^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz \right) dz \right. \\
&\times \left. \int_{c_{i-1}}^{d_i} \left(G^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz \right) dz \right) \\
&+ \sum_{i=1}^{2^{n-1}} \frac{1}{c_i - d_i} \left(\int_{d_i}^{c_i} \left(F^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz \right) dz \right. \\
&\times \left. \int_{d_i}^{c_i} \left(G^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz \right) dz \right).
\end{aligned}
$$

Rewrite $u_\infty - u_{2^{n-1}}$ as

$$
\begin{aligned}
& u_\infty - u_{2^{n-1}} \\
&= \int_0^1 F^{-1}(z) G^{-1}(z) dz - \sum_{i=1}^{2^{n-1}} (c_i - c_{i-1}) \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz \\
&= \sum_{i=1}^{2^{n-1}} \int_{c_{i-1}}^{c_i} \left(F^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz \right) \left(G^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz \right) dz.
\end{aligned}
$$

Focus on one of those terms, we have

$$
\begin{aligned}
& \int_{c_{i-1}}^{c_i} \left(F^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz \right) \left(G^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz \right) dz \\
&= \int_{c_{i-1}}^{d_i} \left(F^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz \right) \left(G^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz \right) dz \\
&+ \int_{d_i}^{c_i} \left(F^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz \right) \left(G^{-1}(z) - \int_{c_i}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz \right) dz.
\end{aligned}
$$

Integrate by parts and collect terms to obtain

$$
\begin{aligned}
&= \left(c_i - c_{i-1}\right)\left(F^{-1}\left(d_i\right) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz\right)\left(G^{-1}\left(d_i\right) - \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz\right) \\
&- \int_{c_{i-1}}^{d_i} \left(z - c_{i-1}\right) F^{-1'}(z) \left(G^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz\right) dz \\
&- \int_{c_{i-1}}^{d_i} \left(z - c_{i-1}\right) G^{-1'}(z) \left(F^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz\right) dz \\
&+ \int_{d_i}^{c_i} \left(c_i - z\right) F^{-1'}(z) \left(G^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz\right) dz \\
&+ \int_{d_i}^{c_i} \left(c_i - z\right) G^{-1'}(z) \left(F^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz\right) dz.
\end{aligned}
$$

Given the conditions required in Theorem 3.3.2 and Lemma 12,

$$
\begin{aligned}
&\leq \left(c_i - c_{i-1}\right)\left(F^{-1}\left(d_i\right) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz\right)\left(G^{-1}\left(d_i\right) - \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz\right) \\
&- \frac{1}{d_i - c_{i-1}} \int_{c_{i-1}}^{d_i} \left(z - c_{i-1}\right) F^{-1'}(z)\, dz \int_{c_{i-1}}^{d_i} \left(G^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz\right) dz \\
&- \frac{1}{d_i - c_{i-1}} \int_{c_{i-1}}^{d_i} \left(z - c_{i-1}\right) G^{-1'}(z)\, dz \int_{c_{i-1}}^{d_i} \left(F^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz\right) dz \\
&+ \frac{1}{c_i - d_i} \int_{d_i}^{c_i} \left(c_i - z\right) F^{-1'}(z)\, dz \int_{d_i}^{c_i} \left(G^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz\right) dz \\
&+ \frac{1}{c_i - d_i} \int_{d_i}^{c_i} \left(c_i - z\right) G^{-1'}(z)\, dz \int_{d_i}^{c_i} \left(F^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz\right) dz.
\end{aligned}
$$

Integrate by parts and collect terms:

$$
\begin{aligned}
&= \left(c_i - c_{i-1}\right)\left(F^{-1}\left(d_i\right) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz\right)\left(G^{-1}\left(d_i\right) - \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz\right) \\
&+ \frac{2}{d_i - c_{i-1}} \left(\int_{c_{i-1}}^{d_i} \left(F^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz\right) dz\right. \\
&\times \left.\int_{c_{i-1}}^{d_i} \left(G^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz\right) dz\right) \\
&+ \frac{2}{c_i - d_i} \left(\int_{d_i}^{c_i} \left(G^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz\right) dz\right. \\
&\times \left.\int_{d_i}^{c_i} \left(F^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz\right) dz\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&u_\infty - u_{2^{n-1}} \\
&= \sum_{i=1}^{2^{n-1}} \int_{c_{i-1}}^{c_i} \left(F^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz \right) \left(G^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz \right) dz \\
&\leq \sum_{i=1}^{2^{n-1}} (c_i - c_{i-1}) \left(F^{-1}(d_i) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz \right) \left(G^{-1}(d_i) - \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz \right) \\
&+ \sum_{i=1}^{2^{n-1}} \frac{2}{d_i - c_{i-1}} \left(\int_{c_{i-1}}^{d_i} \left(F^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz \right) dz \right. \\
&\left. \times \int_{c_{i-1}}^{d_i} \left(G^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz \right) dz \right) \\
&+ \sum_{i=1}^{2^{n-1}} \frac{2}{c_i - d_i} \left(\int_{d_i}^{c_i} \left(G^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz \right) dz \right. \\
&\left. \times \int_{d_i}^{c_i} \left(F^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz \right) dz \right) \\
&= \sum_{i=1}^{2^{n-1}} (c_i - c_{i-1}) \left(F^{-1}(d_i) - \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz \right) \left(G^{-1}(d_i) - \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz \right) \\
&+ 2\left(u_{2^n} - u_{2^{n-1}}\right).
\end{aligned}
$$

Let $F^{-1}(d_i) = \int_{c_{i-1}}^{c_i} \frac{F^{-1}(z)}{c_i - c_{i-1}} dz$ or $G^{-1}(d_i) = \int_{c_{i-1}}^{c_i} \frac{G^{-1}(z)}{c_i - c_{i-1}} dz$. Hence

$$\frac{u_{2^n} - u_{2^{n-1}}}{u_\infty - u_{2^{n-1}}} \geq \frac{1}{2}.$$

This immediately implies that $\frac{u_{2^n} - u_1}{u_\infty - u_1} \geq 1 - \frac{1}{2^n}$. ■

Proof of Theorem 3.3.3. From the proof of Theorem 3.3.1 in McAfee (2002), conditions (1) and (2) in Theorem 3.3.1 could ensure the validity of Chebyshev's inequality. Since Lemma 2 gives exactly the same inequality but under a weaker condition, we only need to verify that the conditions in Lemma 2 are satisfied given conditions (1) and (2) from Theorem 3.3.3. In the proof of Theorem 3.3.1 in

McAfee (2002), Chebyshev's inequality is applied to the following four terms:

$$\int_0^c zF^{-1\prime}(z)\left(G^{-1}(z)-\mu_y\right)dz, \int_0^c zG^{-1\prime}(z)\left(F^{-1}(z)-\mu_x\right)dz$$
$$\int_c^1 (1-z)F^{-1\prime}(z)\left(G^{-1}(z)-\mu_y\right)dz \text{ and } \int_c^1 (1-z)G^{-1\prime}(z)\left(F^{-1}(z)-\mu_x\right)dz.$$

It is enough for us to consider

$$\int_0^c zF^{-1\prime}(z)\left(G^{-1}(z)-\mu_y\right)dz \text{ and } \int_c^1 (1-z)F^{-1\prime}(z)\left(G^{-1}(z)-\mu_y\right)dz.$$

Define $H_F(t)=\frac{1}{t}\int_0^t zF^{-1\prime}(z)\,dz$. Since $G^{-1}(z)-\mu_y$ is increasing in z, for Steffensen's inequality to hold, we need to show that

$$\frac{1}{t}\int_0^t zF^{-1\prime}(z)\,dz \leq \frac{1}{c}\int_0^c zF^{-1\prime}(z)\,dz \text{ for all } t\in(0,c]$$

which is equivalent to

$$\frac{1}{t}\int_0^{F^{-1}(t)} F(x)\,dx \leq \frac{1}{c}\int_0^{F^{-1}(c)} F(x)\,dx.$$

Similarly, we need to show

$$\frac{1}{1-t}\int_0^{1-t} zF^{-1\prime}(1-z)\,dz \geq \frac{1}{1-c}\int_0^{1-c} zF^{-1\prime}(1-z)\,dz$$

Rewrite $\int_t^1 (1-z)F^{-1\prime}(z)\,dz$ as $\int_0^{1-t} zF^{-1\prime}(1-z)\,dz$. This is equivalent to

$$\frac{1}{1-t}\int_t^1 (1-z)F^{-1\prime}(z)\,dz \geq \frac{1}{1-c}\int_c^1 (1-z)F^{-1\prime}(z)\,dz \text{ for all } t\in(c,1].$$

Therefore, conditions (1) and (2) in Theorem 3.3.3 are enough to guarantee that Steffensen's inequality to holds. Replace Chebyshev's inequality by Steffensen's inequality in the proof of Theorem 3.3.1 in McAfee (2002), and Theorem 3.3.3 holds immediately. ■

Generalized CM: Proof of Theorem 3.4.1

Proof of Theorem 3.4.1. To show that inequality (4) holds, we first show that numerator $u_\infty - u_n \leq \left(1/n^2\right) B_i$. Next, by letting $\beta_i = B_i / COV\left(x, \phi\left(x\right)\right)$, we have $\left(u_\infty - u_n\right) / \left(u_\infty - u_1\right) \leq \beta_i / n^2$, which implies $\left(u_n - u_1\right) / \left(u_\infty - u_1\right) \leq 1 - \beta_i / n^2$. The following three circumstances derive B_i by using different versions of the Gruss inequality.

Part 1:

$$\begin{aligned}
u_\infty - u_n &= \sum_{i=1}^{n} \left(c_i - c_{i-1}\right) [\int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} F^{-1}\left(z\right) G^{-1}\left(z\right) dz \\
&\quad - \int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} F^{-1}\left(z\right) dz \int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} G^{-1}\left(z\right) dz] \\
&\leq \frac{1}{4} \sum_{i=1}^{n} \left(c_i - c_{i-1}\right) \left(F^{-1}\left(c_i\right) - F^{-1}\left(c_{i-1}\right)\right) \left(G^{-1}\left(c_i\right) - G^{-1}\left(c_{i-1}\right)\right) \\
&\leq \frac{1}{4} \frac{1}{n^2}.
\end{aligned}$$

The first inequality is due to Lemma 5. To show the second inequality, we choose c_i s.t. $\left(F^{-1}\left(c_i\right) - F^{-1}\left(c_{i-1}\right)\right) \left(G^{-1}\left(c_i\right) - G^{-1}\left(c_{i-1}\right)\right) = \frac{1}{n^2}$ starting from $i = 1$ up to some i until $\left(F^{-1}\left(c_i\right) - F^{-1}\left(c_{i-1}\right)\right) \left(G^{-1}\left(c_i\right) - G^{-1}\left(c_{i-1}\right)\right) \leq \frac{1}{n^2}$. WLOG, we could assume that for all $i \leq n-1$,
$\left(F^{-1}\left(c_i\right) - F^{-1}\left(c_{i-1}\right)\right) \left(G^{-1}\left(c_i\right) - G^{-1}\left(c_{i-1}\right)\right) = \frac{1}{n^2}$. Then the last term is $\left(1 - \sum_{i=1}^{n-1} a_i\right) \left(1 - \sum_{i=1}^{n-1} b_i\right)$, where $a_i = \left(F^{-1}\left(c_i\right) - F^{-1}\left(c_{i-1}\right)\right)$ and $b_i = \left(G^{-1}\left(c_i\right) - G^{-1}\left(c_{i-1}\right)\right)$. It is not difficult to show that $\left(1 - \sum_{i=1}^{n-1} a_i\right) \left(1 - \sum_{i=1}^{n-1} b_i\right) \leq \frac{1}{n^2}$.

Part 2: By Lemma 9

$$\begin{aligned}
u_\infty - u_n &= \sum_{i=1}^{n} \left(c_i - c_{i-1}\right) \left(\int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} F^{-1}\left(z\right) G^{-1}\left(z\right) dz \right. \\
&\quad \left. - \int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} F^{-1}\left(z\right) dz \int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} G^{-1}\left(z\right) dz \right) \\
&\leq \frac{1}{12} \left(\frac{3}{2}\right)^{\frac{1}{p}+\frac{1}{q}} \sum_{i=1}^{n} \left(c_i - c_{i-1}\right)^{3-\frac{1}{p}-\frac{1}{q}} \left\| F^{-1\prime}\left(z\right) \right\|_p^{\left[c_i, c_{i-1}\right]} \left\| G^{-1\prime}\left(z\right) \right\|_q^{\left[c_i, c_{i-1}\right]}
\end{aligned}$$

Let $c_i - c_{i-1} = \frac{1}{n}$ and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\leq \frac{1}{8n^2} \left\| F^{-1\prime}(z) \right\|_p^{[0,1]} \left\| G^{-1\prime}(z) \right\|_q^{[0,1]}.$$

The above inequality is due to the Holder's inequality. That is

$$\begin{aligned}
&\sum_{i=1}^{n} \left\| F^{-1\prime}(z) \right\|_p^{[c_i,c_{i-1}]} \left\| G^{-1\prime}(z) \right\|_q^{[c_i,c_{i-1}]} \\
=&\sum_{i=1}^{n} \left(\int_{c_{i-1}}^{c_i} \left(F^{-1\prime}(x) \right)^p dx \right)^{\frac{1}{p}} \left(\int_{c_{i-1}}^{c_i} \left(G^{-1\prime}(x) \right)^q dx \right)^{\frac{1}{q}} \\
\leq& \left(\sum_{i=1}^{n} \left[\left(\int_{c_{i-1}}^{c_i} \left(F^{-1\prime}(x) \right)^p dx \right)^{\frac{1}{p}} \right]^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \left[\left(\int_{c_{i-1}}^{c_i} \left(G^{-1\prime}(x) \right)^q dx \right)^{\frac{1}{q}} \right]^q \right)^{\frac{1}{q}} \\
=& \left(\int_0^1 \left(F^{-1\prime}(x) \right)^p dx \right)^{\frac{1}{p}} \left(\int_0^1 \left(G^{-1\prime}(x) \right)^q dx \right)^{\frac{1}{q}} \\
=& \left\| F^{-1\prime}(z) \right\|_p^{[0,1]} \left\| G^{-1\prime}(z) \right\|_q^{[0,1]}.
\end{aligned}$$

If $p = q = \infty$,

$$\begin{aligned}
&\frac{1}{12} \left(\frac{3}{2} \right)^{\frac{1}{p}+\frac{1}{q}} \sum_{i=1}^{n} (c_i - c_{i-1})^{3-\frac{1}{p}-\frac{1}{q}} \left\| F^{-1\prime}(z) \right\|_p^{[c_i,c_{i-1}]} \left\| G^{-1\prime}(z) \right\|_q^{[c_i,c_{i-1}]} \\
&\leq \frac{1}{12n^3} \sum_{i=1}^{n} \left\| F^{-1\prime}(z) \right\|_\infty^{[0,1]} \left\| G^{-1\prime}(z) \right\|_\infty^{[0,1]} \\
&= \frac{1}{12n^2} \left\| F^{-1\prime}(z) \right\|_\infty^{[0,1]} \left\| G^{-1\prime}(z) \right\|_\infty^{[0,1]}.
\end{aligned}$$

Hence,

$$u_\infty - u_n \leq \frac{1}{12n^2} \left\| F^{-1\prime}(z) \right\|_\infty^{[0,1]} \left\| G^{-1\prime}(z) \right\|_\infty^{[0,1]}$$

Part 3: By Holder's inequality,

$$\begin{aligned}
u_\infty - u_n \leq& \sum_{i=1}^{n} (c_i - c_{i-1}) \left(\frac{1}{c_i - c_{i-1}} \int_{c_{i-1}}^{c_i} \left(F^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} F^{-1}(z)\, dz \right)^2 dz \right)^{\frac{1}{2}} \\
&\times \left(\frac{1}{c_i - c_{i-1}} \int_{c_{i-1}}^{c_i} \left(G^{-1}(z) - \int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} G^{-1}(z)\, dz \right)^2 dz \right)^{\frac{1}{2}}
\end{aligned}$$

According to Lemma 10,

$$\left(\frac{1}{c_i-c_{i-1}}\int_{c_{i-1}}^{c_i}\left(F^{-1}(z)-\int_{c_{i-1}}^{c_i}\frac{1}{c_i-c_{i-1}}F^{-1}(z)\,dz\right)^2 dz\right)^{\frac{1}{2}}$$
$$\le \frac{1}{c_i-c_{i-1}}\frac{\sqrt{2}\left(F^{-1}(c_i)-F^{-1}(c_i-1)\right)^{1+\frac{1}{q}}}{2\left[(q+1)(2q+1)\right]^{\frac{1}{2q}}}\|f\|_p^{\left[F^{-1}(c_{i-1}),F^{-1}(c_i)\right]}.$$

Also by Lemma 10 with $p=1$,

$$\left(\frac{1}{c_i-c_{i-1}}\int_{c_{i-1}}^{c_i}\left(G^{-1}(z)-\int_{c_{i-1}}^{c_i}\frac{1}{c_i-c_{i-1}}G^{-1}(z)\,dz\right)^2 dz\right)^{\frac{1}{2}}$$
$$\le \frac{1}{2}\left(G^{-1}(c_i)-G^{-1}(c_i-1)\right).$$

Let $\left(F^{-1}(c_i)-F^{-1}(c_i-1)\right)^{1+\frac{1}{q}}\left(G^{-1}(c_i)-G^{-1}(c_i-1)\right)=\left(\frac{1}{n}\right)^{2+\frac{1}{q}}$. With an analog argument as in Part 1, we could have

$$\begin{aligned}
&u_\infty-u_n\\
&\le \sum_{i=1}^{n}\frac{\sqrt{2}}{4\left[(q+1)(2q+1)\right]^{\frac{1}{2q}}n^{2+\frac{1}{q}}}\left(\int_{\frac{i-1}{n}}^{\frac{i}{n}}f^p(\theta)\,d\theta\right)^{\frac{1}{p}}\\
&\le \frac{\sqrt{2}}{4\left[(q+1)(2q+1)\right]^{\frac{1}{2q}}n^{2+\frac{1}{q}}}\left(\sum_{i=1}^{n}\left(\left(\int_{\frac{i-1}{n}}^{\frac{i}{n}}f^p(\theta)\,d\theta\right)^{\frac{1}{p}}\right)^{p}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}1\right)^{\frac{1}{q}}\\
&=\frac{\sqrt{2}}{4\left[(q+1)(2q+1)\right]^{\frac{1}{2q}}n^{2+\frac{1}{q}}}\|f\|_p^{[0,1]}\times n^{\frac{1}{q}}\\
&=\frac{\sqrt{2}}{4\left[(q+1)(2q+1)\right]^{\frac{1}{2q}}n^{2}}\|f\|_p^{[0,1]}.
\end{aligned}$$

The second inequality is due to Holder's inequality.

Similarly, let $\left(F^{-1}(c_i)-F^{-1}(c_i-1)\right)\left(G^{-1}(c_i)-G^{-1}(c_i-1)\right)^{1+\frac{1}{q}}=\left(\frac{1}{n}\right)^{2+\frac{1}{q}}$.

Then we also have

$$u_\infty-u_n\le\frac{\sqrt{2}}{4\left[(q+1)(2q+1)\right]^{\frac{1}{2q}}n^{2}}\|g\|_p^{[0,1]}.$$

Since $F^{-1\prime}(\cdot)$ is a continuous function, $\left\|F^{-1\prime}\right\|_p^{[0,1]}\left\|G^{-1\prime}\right\|_q^{[0,1]}$ is continuous in p. From $(1/p)+(1/q)=1$, it is enough for us to focus on p. We also know that when p goes to infinity the value of $\left\|F^{-1\prime}\right\|_p^{[0,1]}$ increases and $\left\|G^{-1\prime}\right\|_q^{[0,1]}$ goes to $\left\|G^{-1\prime}\right\|_1^{[0,1]}$, which is finite. Hence, the value of $\left\|F^{-1\prime}\right\|_p^{[0,1]}\left\|G^{-1\prime}\right\|_q^{[0,1]}$ is either finite or goes to infinity when p goes to infinity. Similarly, the value of $\left\|F^{-1\prime}\right\|_p^{[0,1]}\left\|G^{-1\prime}\right\|_q^{[0,1]}$ is either finite or goes to infinity when p approaches 1. Therefore, the minimum value of $\left\|F^{-1\prime}\right\|_p^{[0,1]}\left\|G^{-1\prime}\right\|_q^{[0,1]}$ exists. For a similar reason, the minimum of $\|f\|_p^{[0,1]}/\left(4\left[(q+1)(2q+1)\right]^{\frac{1}{2q}}\right)$ and $\|g\|_p^{[0,1]}/\left(4\left[(q+1)(2q+1)\right]^{\frac{1}{2q}}\right)$ over p also exist. Therefore, β can be obtained.

■

Proof of Corollary 3.4.1. 2): By Lemma 4, we have
$COV\left(x,\phi\left(x\right)\right)=\frac{1}{12}F^{-1\prime}\left(\varepsilon\right)G^{-1\prime}\left(\eta\right)$
$=1/\left(12fg\right)\geq 1/\left(12AB\right)$. β_3 in Theorem 3.4.1 is less than
$\left(\sqrt{3}A\right)/\left(12COV\left(x,\phi\left(x\right)\right)\right)$ if $p=\infty$. Similarly β_3 could also be
$\left(\sqrt{3}B\right)/\left(12COV\left(x,\phi\left(x\right)\right)\right)$. Hence,
$\beta_3\leq\left(\sqrt{3}\sqrt{AB}\right)/\left(12COV\left(x,\phi\left(x\right)\right)\right)=\sqrt{3}AB^{\frac{3}{2}}$.

3): By Lemma 3, we have $COV\left(x,\phi\left(x\right)\right)=\frac{1}{2}\int_0^1\int_0^1 f\left(x\right)g\left(y\right)\left(x-y\right)^2dxdy\geq\frac{1}{12ab}$.
Using β_2 of Theorem 3.4.1 with $p=q=\infty$,

$$\beta_2=\frac{\left\|F^{-1\prime}\right\|_\infty^{[0,1]}\left\|G^{-1\prime}\right\|_\infty^{[0,1]}}{12COV\left(x,\phi\left(x\right)\right)}=\frac{ab}{12COV\left(x,\phi\left(x\right)\right)}.$$

That is, $\beta_2\leq a^2b^2$.

4): By Lemma 4 $u_\infty-u_n=\sum\frac{1}{12n^2}F^{-1\prime}\left(\varepsilon_i\right)G^{-1\prime}\left(\eta_i\right)\leq\frac{1}{12n^2ab}$. By Lemma 4 $COV\left(x,\phi\left(x\right)\right)=\frac{1}{12}F^{-1\prime}\left(\varepsilon\right)G^{-1\prime}\left(\eta\right)\geq\frac{1}{12AB}$. It is obvious that the result holds.

■

CM with General Match Payoff: Proof of Theorem 3.4.2

The following theorem is a stronger version of Theorem 3.4.2.

Theorem 3.4.2′ For any c.d.f. $F(\theta), G(\theta)$ defined over a non-negative bounded type space $[a,b]$ with $u_{21}F^{-1\prime}G^{-1\prime} \geq \underline{A}$, we have

$$\frac{u_n - u_1}{u_\infty - u_1} \geq 1 - \frac{\tilde{\beta}}{n^2},$$

where $\tilde{\beta} = \min\left\{\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3\right\}$

1. $\tilde{\beta}_1 = \frac{\overline{A}}{\underline{A}}$ if $u_{21}F^{-1\prime}G^{-1\prime} \leq \overline{A}$,
2. $\beta_2 = \frac{3}{\underline{A}} \int_a^b u_{21}\left(F^{-1}(\alpha), G^{-1}(\alpha)\right) F^{-1\prime}(\alpha) G^{-1\prime}(\alpha)\, d\alpha$

   if $\frac{\partial^4}{\partial\alpha^2\partial\beta^2} u\left(F^{-1}(\alpha), G^{-1}(\beta)\right) \geq 0$,
3. $\beta_3 = \frac{3}{\underline{A}}\left(u(b,b) + u(a,a) - u(a,b) - u(b,a)\right)$

   if $\frac{\partial^4}{\partial\alpha^2\partial\beta^2} u\left(F^{-1}(\alpha), G^{-1}(\beta)\right) \leq 0$.

Proof. Part 1: By Lemma 7,

$$\begin{aligned} u_\infty - u_n &= \sum (c_i - c_{i-1}) \left(\int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} u\left(F^{-1}(\alpha), G^{-1}(\alpha)\right) d\alpha \right. \\ &\quad \left. - \left(\frac{1}{c_i - c_{i-1}}\right)^2 \int_{c_{i-1}}^{c_i} \int_{c_{i-1}}^{c_i} u\left(F^{-1}(\alpha), G^{-1}(\beta)\right) d\alpha d\beta \right) \\ &= \sum (c_i - c_{i-1}) u_{21}(\varepsilon_i, \eta_i) F^{-1\prime}(\varepsilon_i) G^{-1\prime}(\eta_i) \left[\frac{1}{12}(c_i - c_{i-1})^2\right] \end{aligned}$$

By letting $c_i - c_{i-1} = \frac{1}{n}$ together with $u_{21}F^{-1\prime}G^{-1\prime} \in \left[\underline{A}, \overline{A}\right]$,

$$u_\infty - u_n \leq \frac{1}{12n^2}\overline{A}.$$

Also

$$u_\infty - u_1 = \frac{1}{12} u_{21}(\varepsilon, \eta) F^{-1\prime}(\varepsilon) G^{-1\prime}(\eta) \geq \frac{1}{12}\underline{A}.$$

Hence, by letting $\tilde{\beta}_1 = \overline{A}/\underline{A}$, we have

$$\frac{u_n - u_1}{u_\infty - u_1} \geq 1 - \frac{\tilde{\beta}_1}{n^2}.$$

Part 2: By Lemma 8,

$$
\begin{aligned}
u_\infty - u_n = {} & \sum (c_i - c_{i-1}) \left(\int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} u\left(F^{-1}(\alpha), G^{-1}(\alpha)\right) d\alpha \right. \\
& \left. - \left(\frac{1}{c_i - c_{i-1}} \right)^2 \int_{c_{i-1}}^{c_i} \int_{c_{i-1}}^{c_i} u\left(F^{-1}(\alpha), G^{-1}(\beta)\right) d\alpha d\beta \right) \\
\leq {} & \frac{1}{4} \sum (c_i - c_{i-1}) \int_{c_{i-1}}^{c_i} \int_{c_{i-1}}^{c_i} u_{21}\left(F^{-1}(\alpha), G^{-1}(\beta)\right) F^{-1\prime}(\alpha) G^{-1\prime}(\beta) d\alpha d\beta.
\end{aligned}
$$

If $\frac{\partial^4}{\partial\alpha^2\partial\beta^2} u\left(F^{-1}(\alpha), G^{-1}(\beta)\right) \geq 0$,

$$
\begin{aligned}
& \int_{c_{i-1}}^{c_i} \int_{c_{i-1}}^{c_i} u_{21}\left(F^{-1}(\alpha), F^{-1}(\beta)\right) F^{-1\prime}(\alpha) G^{-1\prime}(\beta) d\alpha d\beta \\
& \leq (c_i - c_{i-1}) \int_{c_{i-1}}^{c_i} u_{21}\left(F^{-1}(\alpha), G^{-1}(\alpha)\right) F^{-1\prime}(\alpha) G^{-1\prime}(\alpha) d\alpha.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \frac{1}{4} \sum (c_i - c_{i-1}) \int_{c_{i-1}}^{c_i} \int_{c_{i-1}}^{c_i} u_{21} F^{-1\prime} G^{-1\prime} d\alpha d\beta \\
& \leq \frac{1}{4} \sum (c_i - c_{i-1})^2 \int_{c_{i-1}}^{c_i} u_{21}\left(F^{-1}(\alpha), G^{-1}(\alpha)\right) F^{-1\prime}(\alpha) G^{-1\prime}(\alpha) d\alpha \\
& = \frac{1}{4n^2} \int_0^1 u_{21}\left(F^{-1}(\alpha), G^{-1}(\alpha)\right) F^{-1\prime}(\alpha) G^{-1\prime}(\alpha) d\alpha.
\end{aligned}
$$

The last equality holds by letting $c_i - c_{i-1} = \frac{1}{n}$. Similarly as in part 1, we have

$$
\beta_2 = \frac{3}{\underline{A}} \int_0^1 u_{21}\left(F^{-1}(\alpha), G^{-1}(\alpha)\right) F^{-1\prime}(\alpha) G^{-1\prime}(\alpha) d\alpha.
$$

Part 3: Using Lemma 8,

$$
\begin{aligned}
& \frac{1}{4} \sum (c_i - c_{i-1}) \int_{c_{i-1}}^{c_i} \int_{c_{i-1}}^{c_i} u_{21}\left(F^{-1}(\alpha), G^{-1}(\beta)\right) F^{-1\prime}(\alpha) G^{-1\prime}(\beta) d\alpha d\beta \\
& \leq \frac{1}{4n^2} \int_0^1 \int_0^1 u_{21} F^{-1\prime}(\alpha) G^{-1\prime}(\beta) d\alpha d\beta \\
& = \frac{1}{4n^2} \left(u(b,b) + u(a,a) - u(a,b) - u(b,a)\right).
\end{aligned}
$$

The inequality is proved below. From here, by letting

$$
\beta_3 = \frac{3}{\underline{A}} \left(u(b,b) + u(a,a) - u(a,b) - u(b,a)\right)
$$

the result holds. Cutoff points $\{c_i\}$ are chosen such that $c_i = \frac{i}{n}$.

We will illustrate the inequality by showing the case $n = 3$ with $[a,b] = [0,1]$. It is the same procedure to prove for any n and any domain $[a,b]$. Denote $U(\alpha,\beta) = u\left(F^{-1}(\alpha), G^{-1}(\alpha)\right)$.

$$\sum (c_i - c_{i-1}) \int_{c_{i-1}}^{c_i} \int_{c_{i-1}}^{c_i} u_{21} F^{-1\prime} G^{-1\prime} d\alpha d\beta$$
$$= \frac{1}{3} \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} U_{21}(\alpha,\beta)\, d\alpha d\beta + \frac{1}{3} \int_{\frac{1}{3}}^{\frac{2}{3}} \int_{\frac{1}{3}}^{\frac{2}{3}} U_{21}(\alpha,\beta)\, d\alpha d\beta + \frac{1}{3} \int_{\frac{2}{3}}^{1} \int_{\frac{2}{3}}^{1} U_{21}(\alpha,\beta)\, d\alpha d\beta$$

Since $\frac{\partial^4}{\partial\alpha^2 \partial\beta^2} u\left(F^{-1}(\alpha), F^{-1}(\beta)\right) \leq 0$ implies that U_{21} is substitute in α and β, we claim that the following inequality holds

$$\int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} U_{21}(\alpha,\beta)\, d\alpha d\beta + \int_{\frac{1}{3}}^{\frac{2}{3}} \int_{\frac{1}{3}}^{\frac{2}{3}} U_{21}(\alpha,\beta)\, d\alpha d\beta$$
$$\leq \int_0^{\frac{1}{3}} \int_{\frac{1}{3}}^{\frac{2}{3}} U_{21}(\alpha,\beta)\, d\alpha d\beta + \int_{\frac{1}{3}}^{\frac{2}{3}} \int_0^{\frac{1}{3}} U_{21}(\alpha,\beta)\, d\alpha d\beta$$

To see this, rewrite the following terms:

$$\int_0^{\frac{1}{3}} \int_{\frac{1}{3}}^{\frac{2}{3}} U_{21}(\alpha,\beta)\, d\alpha d\beta = \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} U_{21}\left(\alpha, \beta + \frac{1}{3}\right) d\alpha d\beta$$
$$\int_{\frac{1}{3}}^{\frac{2}{3}} \int_0^{\frac{1}{3}} U_{21}(\alpha,\beta)\, d\alpha d\beta = \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} U_{21}\left(\alpha + \frac{1}{3}, \beta\right) d\alpha d\beta$$
$$\int_{\frac{1}{3}}^{\frac{2}{3}} \int_{\frac{1}{3}}^{\frac{2}{3}} U_{21}(\alpha,\beta)\, d\alpha d\beta = \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} U_{21}\left(\alpha + \frac{1}{3}, \beta + \frac{1}{3}\right) d\alpha d\beta$$

Then $\frac{\partial^4}{\partial\alpha^2 \partial\beta^2} u\left(F^{-1}(\alpha), F^{-1}(\beta)\right) < 0$ implies

$$U_{21}\left(\alpha + \frac{1}{3}, \beta + \frac{1}{3}\right) + U_{21}(\alpha,\beta)$$
$$\leq U_{21}\left(\alpha + \frac{1}{3}, \beta\right) + U_{21}\left(\alpha, \beta + \frac{1}{3}\right).$$

From this,

$$
\begin{aligned}
&\int_0^{\frac{2}{3}}\int_0^{\frac{2}{3}} U_{21}(\alpha,\beta)\,d\alpha d\beta \\
&= \int_0^{\frac{1}{3}}\int_0^{\frac{1}{3}} U_{21}(\alpha,\beta)\,d\alpha d\beta + \int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}} U_{21}(\alpha,\beta)\,d\alpha d\beta \\
&+ \int_0^{\frac{1}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}} U_{21}(\alpha,\beta)\,d\alpha d\beta + \int_{\frac{1}{3}}^{\frac{2}{3}}\int_0^{\frac{1}{3}} U_{21}(\alpha,\beta)\,d\alpha d\beta \\
&\geq 2\int_0^{\frac{1}{3}}\int_0^{\frac{1}{3}} U_{21}(\alpha,\beta)\,d\alpha d\beta + 2\int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}} U_{21}(\alpha,\beta)\,d\alpha d\beta
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}} U_{21}(\alpha,\beta)\,d\alpha d\beta + \int_{\frac{2}{3}}^{1}\int_{\frac{2}{3}}^{1} U_{21}(\alpha,\beta)\,d\alpha d\beta \\
&\leq \int_{\frac{2}{3}}^{1}\int_{\frac{1}{3}}^{\frac{2}{3}} U_{21}(\alpha,\beta)\,d\alpha d\beta + \int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{2}{3}}^{1} U_{21}(\alpha,\beta)\,d\alpha d\beta \\
&\int_0^{\frac{1}{3}}\int_0^{\frac{1}{3}} U_{21}(\alpha,\beta)\,d\alpha d\beta + \int_{\frac{2}{3}}^{1}\int_{\frac{2}{3}}^{1} U_{21}(\alpha,\beta)\,d\alpha d\beta \\
&\leq \int_{\frac{2}{3}}^{1}\int_0^{\frac{1}{3}} U_{21}(\alpha,\beta)\,d\alpha d\beta + \int_0^{\frac{1}{3}}\int_{\frac{2}{3}}^{1} U_{21}(\alpha,\beta)\,d\alpha d\beta.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&3\left(\int_0^{\frac{1}{3}}\int_0^{\frac{1}{3}} U_{21}(\alpha,\beta)\,d\alpha d\beta + \int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}} U_{21}(\alpha,\beta)\,d\alpha d\beta + \int_{\frac{2}{3}}^{1}\int_{\frac{2}{3}}^{1} U_{21}(\alpha,\beta)\,d\alpha d\beta\right) \\
&\leq \int_0^{\frac{1}{3}}\int_0^{\frac{1}{3}} U_{21}(\alpha,\beta)\,d\alpha d\beta + \int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}} U_{21}(\alpha,\beta)\,d\alpha d\beta + \int_{\frac{2}{3}}^{1}\int_{\frac{2}{3}}^{1} U_{21}(\alpha,\beta)\,d\alpha d\beta \\
&+ \int_{\frac{2}{3}}^{1}\int_{\frac{1}{3}}^{\frac{2}{3}} U_{21}(\alpha,\beta)\,d\alpha d\beta + \int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{2}{3}}^{1} U_{21}(\alpha,\beta)\,d\alpha d\beta + \int_{\frac{2}{3}}^{1}\int_0^{\frac{1}{3}} U_{21}(\alpha,\beta)\,d\alpha d\beta \\
&+ \int_0^{\frac{1}{3}}\int_{\frac{2}{3}}^{1} U_{21}(\alpha,\beta)\,d\alpha d\beta + \int_0^{\frac{1}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}} U_{21}(\alpha,\beta)\,d\alpha d\beta + \int_{\frac{1}{3}}^{\frac{2}{3}}\int_0^{\frac{1}{3}} U_{21}(\alpha,\beta)\,d\alpha d\beta \\
&= \int_0^{1}\int_0^{1} U_{21}(\alpha,\beta)\,d\alpha d\beta.
\end{aligned}
$$

■

Omitted Algebra from Example 3.2 and 4.4

Example 3.1 and 3.2. Consider $F(x) = G(x) = 1 - (1-x)^{\frac{1}{4}}$ with p.d.f.

$f(x) = g(x) = \frac{1}{4}(1-x)^{-\frac{3}{4}}$.

The cutoff point $c = \mu = \frac{4}{5}$.

$\frac{d}{dx}\left(\frac{1-F(x)}{f(x)}\right) = -4 < 0$. Hence, the second condition is satisfied.

$\frac{F(x)}{f(x)} = \frac{1-(1-x)^{\frac{1}{4}}}{\frac{1}{4}(1-x)^{-\frac{3}{4}}}$. $\frac{d}{dx}\left(\frac{F(x)}{f(x)}\right) = \frac{1}{(1-x)^{\frac{1}{4}}}\left(4(1-x)^{\frac{1}{4}} - 3\right)$, which is not always positive.

In particular, $\frac{d}{dx}\left(\frac{F(x)}{f(x)}\right)|_{x=\mu} = -0.48605 < 0$

$E\left[\frac{F(x)}{f(x)} \mid x < \frac{4}{5}\right] = \frac{1}{25}\sqrt{5} + \frac{1}{25}\sqrt[4]{5} + \frac{1}{25}5^{\frac{3}{4}} + \frac{1}{25}$and

$E\left[\frac{F(x)}{f(x)} \mid x < t\right] = -\frac{1}{\sqrt[4]{1-t}-1}\left(t - \frac{4}{5}t\sqrt[4]{1-t} + \frac{4}{5}\sqrt[4]{1-t} - \frac{4}{5}\right)$ increasing in t when $t \leq \mu$.

Hence, the result still holds. In fact, $\frac{u_2-u_0}{u_\infty-u_0} - \frac{1}{2} = 0.22676$. ■

Example 3.4.4. Consider the same distributions $F(x) = G(x) = x^{\frac{1}{\alpha}}$ over $[0,1]$ with densities $f(x) = g(x) = \frac{1}{\alpha}x^{\frac{1}{\alpha}-1}$ and $\alpha \in (0,\infty)$. The inverse function of c.d.f. is $F^{-1}(x) = x^{\alpha}$.

(i) Note that $(F(x)/f(x))' = \alpha > 0$ and
$[(1-F(x))/f(x)]' = -x^{-\frac{1}{\alpha}}\left(x^{\frac{1}{\alpha}}\alpha - \alpha + 1\right)$, which implies that $F(x)$ is condition

2 of Theorem 3.3.2 over $[0,1]$ only when $\alpha \leq 1$. Therefore, for $\alpha \leq 1$ we have

$$\frac{u_{2^n} - u_1}{u_\infty - u_1} \geq 1 - \frac{1}{2^n}.$$

(ii) Now we will use Theorem 3.4.1 and corollaries derived above to generalize this result. Since both sides have the same distribution function,

$$COV(x,x) = \sigma_x^2 = \int_0^1 x^{2\alpha}dx - \left(\int_0^1 x^{\alpha}dx\right)^2 = \frac{1}{2\alpha+1} - \left(\frac{1}{\alpha+1}\right)^2 = \frac{\alpha^2}{(\alpha+1)^2(2\alpha+1)}.$$

For $\alpha \geq 1$, the density is $f(x) = \frac{1}{\alpha} x^{\frac{1}{\alpha}-1}$. If we use Theorem 3.4.1 directly, the problem is that $f(x)$ goes to infinity when x approaches zero. Hence, β_2 involving $\|f\|_p^{[0,1]}$ is not tight enough to give a meaningful lower bound. For β_3, which involves $\left\|F^{-1\prime}(z)\right\|_p^{[0,1]}$, $F^{-1\prime}(z) = \alpha x^{\alpha-1}$ becomes so large as α increases. Therefore, when α is large enough, β_3 is not tight enough. Also when α is large, the distribution is highly asymmetric, which makes β_1 is not tight. If α is close to zero, we have the same problem. Hence, Theorem 3.4.1 can not provide a meaningful lower bound when α is either too large or too small. To provide a meaningful lower bound, we'll use a combination of both $\|f\|_p$ and $\left\|F^{-1\prime}(z)\right\|_p$. To do this, we split the type space into two intervals by a properly chosen point c^*. Then we use $\left\|F^{-1\prime}(z)\right\|_p^{[0,c]}$ and $\|f\|_p^{\left[F^{-1}(c),1\right]}$ respectively on each interval.

Given n, Let θ^* and corresponding $c^* = F(\theta^*)$ be the splitting point. We consider n to be an even number and an odd number separately. If $n = 2m$ with $m \in \mathbb{N}$,

$$
\begin{aligned}
& u_\infty - u_n \\
&= \sum_{i=0}^{m} (c_i - c_{i-1}) \left(\int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} F^{-1}(z) G^{-1}(z)\, dz \right. \\
&\left. - \int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} F^{-1}(z)\, dz \int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} G^{-1}(z)\, dz \right) \\
&+ \sum_{i=m+1}^{n} (c_i - c_{i-1}) \left(\int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} F^{-1}(z) G^{-1}(z)\, dz \right. \\
&\left. - \int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} F^{-1}(z)\, dz \int_{c_{i-1}}^{c_i} \frac{1}{c_i - c_{i-1}} G^{-1}(z)\, dz \right) \\
&\leq \frac{c^{*2}}{12(m)^2} \left\|F^{-1\prime}(z)\right\|_\infty^{[0,c^*]} \left\|G^{-1\prime}(z)\right\|_\infty^{[0,c^*]} + \frac{\sqrt{3}(1-\theta^*)^3}{12(m)^2} \|f\|_\infty^{[\theta^*,1]}.
\end{aligned}
$$

Let $\theta^* = \left(\frac{1}{2}\right)^{\frac{1}{\sqrt{\alpha}}}$ and $c^* = F(\theta^*) = \left(\frac{1}{2}\right)^{\sqrt{\alpha}}$. Note that the total number of classes is $2m$. In the spirit of the proof of Theorem 3.4.1 we have

$$
\beta_1^* = \frac{c^{*2} \left\|F^{-1\prime}(z)\right\|_\infty^{[0,c^*]} \left\|G^{-1\prime}(z)\right\|_\infty^{[0,c^*]} + \sqrt{3}(1-\theta^*)^3 \|f\|_\infty^{[\theta^*,1]}}{3\sigma_x^2}.
$$

Denote $B=\beta_1^*\sigma_x^2$. In order to compare β_1^* and 2, we compare

$$
\begin{aligned}
& COV\,(x,x)-2B \\
&= \frac{\alpha^2}{(\alpha+1)^2(2\alpha+1)}-\frac{1}{6}\left(\left(\frac{1}{2}\right)^{\sqrt{\alpha}}\right)^2\alpha^2\left(\left(\frac{1}{2}\right)^{\sqrt{\alpha}}\right)^{2\alpha-2} \\
&-\frac{\sqrt{3}}{6}\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\sqrt{\alpha}}}\right)^3\frac{1}{\alpha}\left(\frac{1}{2}\right)^{\frac{1}{\sqrt{\alpha}}\left(\frac{1}{\alpha}-1\right)} \\
&\geq 0.
\end{aligned}
$$

The last inequality holds for $\alpha\geq 1$. Therefore, $\beta_1^*\leq 2$ if $n=2m$ and $\alpha\geq 1$.

If $n=2m+1$ with $m\in\mathbb{N}$,

$$
\begin{aligned}
u_\infty-u_n &= \sum(c_i-c_{i-1})\left(\int_{c_{i-1}}^{c_i}\frac{1}{c_i-c_{i-1}}F^{-1}(z)\,G^{-1}(z)\,dz\right. \\
&\left.-\int_{c_{i-1}}^{c_i}\frac{1}{c_i-c_{i-1}}F^{-1}(z)\,dz\int_{c_{i-1}}^{c_i}\frac{1}{c_i-c_{i-1}}G^{-1}(z)\,dz\right) \\
&= \sum_{i=0}^{m-1}(c_i-c_{i-1})\left(\int_{c_{i-1}}^{c_i}\frac{1}{c_i-c_{i-1}}F^{-1}(z)\,G^{-1}(z)\,dz\right. \\
&\left.-\int_{c_{i-1}}^{c_i}\frac{1}{c_i-c_{i-1}}F^{-1}(z)\,dz\int_{c_{i-1}}^{c_i}\frac{1}{c_i-c_{i-1}}G^{-1}(z)\,dz\right) \\
&+\sum_{i=m}^{n}(c_i-c_{i-1})\left(\int_{c_{i-1}}^{c_i}\frac{1}{c_i-c_{i-1}}F^{-1}(z)\,G^{-1}(z)\,dz\right. \\
&\left.-\int_{c_{i-1}}^{c_i}\frac{1}{c_i-c_{i-1}}F^{-1}(z)\,dz\int_{c_{i-1}}^{c_i}\frac{1}{c_i-c_{i-1}}G^{-1}(z)\,dz\right) \\
&\leq \frac{c^{*2}}{12\,(m)^2}\left\|F^{-1\prime}(z)\right\|_\infty^{[0,c^*]}\left\|G^{-1\prime}(z)\right\|_\infty^{[0,c^*]}+\frac{\sqrt{3}\,(1-\theta^*)^3}{12\,(m+1)^2}\left\|f\right\|_\infty^{[\theta^*,1]}.
\end{aligned}
$$

Let $\theta^*=\left(\frac{1}{2}\right)^{\frac{1}{\sqrt{\alpha}}}$ and $c^*=F(\theta^*)=\left(\frac{1}{2}\right)^{\sqrt{\alpha}}$. Similarly,

$$
\begin{aligned}
& n^2\frac{c^{*2}}{12\,(m)^2}\left\|F^{-1\prime}(z)\right\|_\infty^{[0,c^*]}\left\|G^{-1\prime}(z)\right\|_\infty^{[0,c^*]}+n^2\frac{\sqrt{3}\,(1-\theta^*)^3}{12\,(m+1)^2}\left\|f\right\|_\infty^{[\theta^*,1]} \\
&= (2m+1)^2\frac{c^{*2}}{12\,(m)^2}\left\|F^{-1\prime}(z)\right\|_\infty^{[0,c^*]}\left\|G^{-1\prime}(z)\right\|_\infty^{[0,c^*]}+(2m+1)^2\frac{\sqrt{3}\,(1-\theta^*)^3}{12\,(m+1)^2}\left\|f\right\|_\infty^{[\theta^*,1]} \\
&\leq \frac{3c^{*2}}{4}\left\|F^{-1\prime}(z)\right\|_\infty^{[0,c^*]}\left\|G^{-1\prime}(z)\right\|_\infty^{[0,c^*]}+\frac{\sqrt{3}\,(1-\theta^*)^3}{4}\left\|f\right\|_\infty^{[\theta^*,1]}.
\end{aligned}
$$

Let

$$\beta_2^* = \frac{3c^{*2}}{4\sigma_x^2} \left\| F^{-1\prime}(z) \right\|_\infty^{[0,c^*]} \left\| G^{-1\prime}(z) \right\|_\infty^{[0,c^*]} + \frac{\sqrt{3}\,(1-\theta^*)^3}{4\sigma_x^2} \|f\|_\infty^{[\theta^*,1]}.$$

Since $n = 2m+1$, we start by comparing β_2 to 3.

Denote $B' = \beta_2^* \sigma_x^2$. Then

$$\begin{aligned}\sigma_x^2 - \frac{1}{3}B' &= \frac{\alpha^2}{(\alpha+1)^2(2\alpha+1)} - \frac{1}{4}\left(\left(\frac{1}{2}\right)^{\sqrt{\alpha}}\right)^2 \alpha^2 \left(\left(\frac{1}{2}\right)^{\sqrt{\alpha}}\right)^{2\alpha-2} \\ &- \frac{\sqrt{3}}{12}\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\sqrt{\alpha}}}\right)^3 \frac{1}{\alpha}\left(\frac{1}{2}\right)^{\frac{1}{\sqrt{\alpha}}\left(\frac{1}{\alpha}-1\right)} \\ &\geq 0.\end{aligned}$$

The inequality holds for $\alpha \geq 1$. Hence, $\beta_2^* \leq 3$ for $n = 2m+1$ and $\alpha \geq 1$.

By far, we have shown that for even number classes $\beta_1^* \leq 2$, for odd number classes we have $\beta_1^* \leq 3$.

Similarly, we can have the same result for $\alpha < 1$, by choosing $\|f\|_\infty^{[0,\theta^*]}$ and $\left\| F^{-1\prime}(z) \right\|_\infty^{[c^*,1]}$ instead. For $n = 2m$ we have $\beta_1^* \leq 2$ due to

$$\frac{\alpha^2}{(\alpha+1)^2(2\alpha+1)} \geq \frac{1}{6}\left(1-\left(\frac{1}{2}\right)^{\sqrt{\alpha}}\right)^2 \alpha^2 \left(\left(\frac{1}{2}\right)^{\sqrt{\alpha}}\right)^{2\alpha-2} + \frac{\sqrt{3}}{6}\left(\frac{1}{2}\right)^{\frac{3}{\sqrt{\alpha}}} \frac{1}{\alpha}\left(\frac{1}{2}\right)^{\frac{1}{\sqrt{\alpha}}\left(\frac{1}{\alpha}-1\right)}.$$

For $n = 2m+1$, we have $\beta_2^* \leq 3$ due to

$$\frac{\alpha^2}{(\alpha+1)^2(2\alpha+1)} \geq \frac{1}{4}\left(1-\left(\frac{1}{2}\right)^{\sqrt{\alpha}}\right)^2 \alpha^2 \left(\left(\frac{1}{2}\right)^{\sqrt{\alpha}}\right)^{2\alpha-2} + \frac{\sqrt{3}}{12}\left(\frac{1}{2}\right)^{\frac{3}{\sqrt{\alpha}}} \frac{1}{\alpha}\left(\frac{1}{2}\right)^{\frac{1}{\sqrt{\alpha}}\left(\frac{1}{\alpha}-1\right)}.$$

In sum,

$$\frac{u_2-u_1}{u_\infty-u_1} \geq \frac{1}{2} \qquad \text{or} \qquad \frac{u_n-u_1}{u_\infty-u_1} \geq 1-\frac{3}{n^2}$$

for any $\alpha \in (0,\infty)$ and $n \geq 3$. ■

Monopolistic Pricing: Omitted Steps and Proof of Proposition 3.5.1

OPTIMAL CONTRACT. Incentive compatibility condition together with integration by parts implies

$$t(\theta) = t(0) + \theta u(a(\theta)) - \int_0^\theta u(a(\theta))\, d\hat{F}(\theta).$$

Substituting back into the objective function we get:

$$\int_0^1 \left[\left(\theta - \frac{1-\hat{F}(\theta)}{\hat{f}(\theta)} \right) u(a(\theta)) - ca(\theta) \right] d\hat{F}(\theta).$$

It is standard in the literature to assume that $\left(1-\hat{F}(\theta)\right)/\hat{f}(\theta)$ is increasing in θ.Let $\underline{\theta}$ satisfy $\underline{\theta} - \frac{1-\hat{F}(\underline{\theta})}{\hat{f}(\underline{\theta})} = 0$. The optimal contract then has $a(\theta) = t(\theta) = 0$ for $\theta < \underline{\theta}$. The optimal contract also needs to satisfy

$$\left(\theta - \frac{1-\hat{F}(\theta)}{\hat{f}(\theta)} \right) u'(a(\theta)) = c.$$

This condition together with $\left(1-\hat{F}(\theta)\right)/\hat{f}(\theta)$ is an increasing function, implying that the optimal contract is monotonic in agent's type.

Denote the optimal contract as $a^*(\theta)$. The maximized profit obtained by the monopolist is

$$\pi_\infty = \int_{\underline{\theta}}^1 \left[\left(\theta - \frac{1-\hat{F}(\theta)}{\hat{f}(\theta)} \right) u(a^*(\theta)) - ca^*(\theta) \right] d\hat{F}(\theta).$$

EXISTENCE OF n-LEVEL CONTRACT. It would be enough to show the existence of a 3-level contract. The same procedure extends it for arbitrary n.

For any 3-level contract $\{(a_1,t_1),(a_2,t_2),(a_3,t_3)\}$ that satisfies *IC* and *IR* conditions, given *IR* conditions, we need $\underline{\theta}u(a_1) - t_1 \geq 0$, which is binding to maximize the profit. Hence, $t_1 = \underline{\theta}u(a_1)$. From *IC* conditions:

$$\theta u(a_1) - t_1 \geq \theta u(a_2) - t_2 \text{ and } \theta u(a_1) - t_1 \geq \theta u(a_3) - t_3 \text{ if } \theta \leq \theta_1.$$

By the first inequality, $\theta_1 = \dfrac{t_2 - t_1}{u(a_2) - u(a_1)}$. Similarly, $\theta_2 = \dfrac{t_3 - t_2}{u(a_3) - u(a_2)}$.

We need to verify that $\theta u(a_1) - t_1 \geq \theta u(a_3) - t_3$ if $\theta \leq \theta_1$. Since $\theta u(a_1) - t_1 \geq \theta u(a_2) - t_2$ if $\theta \leq \theta_1$, $\theta u(a_1) - t_1 \geq \theta u(a_3) - t_3$ if $\theta u(a_2) - t_2 \geq \theta u(a_3) - t_3$, which is $\theta \leq \dfrac{t_3 - t_2}{u(a_3) - u(a_2)} = \theta_2$. To verify $\theta u(a_3) - t_3 \geq \theta u(a_1) - t_1$ if $\theta \geq \theta_3$, similarly, we only need to verify $\theta u(a_2) - t_2 \geq \theta u(a_1) - t_1$. If $\theta u(a_2) - t_2 \geq \theta u(a_1) - t_1$, all *IC* conditions hold. This is equivalent to $\theta \geq \dfrac{t_2 - t_1}{u(a_2) - u(a_1)} = \theta_1$.

Now we construct a 3-level stochastic contract which has the same profit as π_3, the valued yielded by 3-class CM. Let $a_1(x) = a^*(x)$ if $x \in [\underline{\theta}, \theta_1]$, $a_2(x) = a^*(x)$ if $x \in [\theta_1, \theta_2]$ and $a_3(x) = a^*(x)$ if $x \in [\theta_2, 1]$. The expected utility for an agent with type $\theta \in [\underline{\theta}, \theta_1]$ is

$$\theta Eu(a_1) = \frac{\theta}{\left[\hat{F}(\theta_1) - \hat{F}(\underline{\theta})\right]} \int_{\underline{\theta}}^{\theta_1} u(a^*(\theta))\, d\hat{F}(\theta).$$

The expected utility for an agent with type $\theta \in [\theta_1, \theta_2]$ is

$$\theta Eu(a_2) = \frac{\theta}{\left[\hat{F}(\theta_2) - \hat{F}(\theta_1)\right]} \int_{\theta_1}^{\theta_2} u(a^*(\theta))\, d\hat{F}(\theta).$$

The expected utility for an agent with type $\theta \in [\theta_2, 1]$ is

$$\theta Eu(a_3) = \frac{\theta}{\left[1 - \hat{F}(\theta_2)\right]} \int_{\theta_2}^{1} u(a^*(\theta))\, d\hat{F}(\theta).$$

The transfers correspondingly are

$$t_1 = \underline{\theta} Eu(a_1), \;\; t_2 = \theta_1 (Eu(a_2) - Eu(a_1)) + \underline{\theta} Eu(a_1)$$
$$t_3 = \theta_2 (Eu(a_3) - Eu(a_2)) + \theta_1 (Eu(a_2) - Eu(a_1)) + \underline{\theta} Eu(a_1).$$

By the above argument, such a contract satisfies *IC* and *IR* conditions.

By offering such a contract, the profit is

$$\begin{aligned}
\tilde{\pi}_3 &= \underline{\theta} Eu(a_1)\left[\hat{F}(\theta_1)-\hat{F}(\underline{\theta})\right] \\
&+ \left(\theta_1\left(Eu(a_2)-Eu(a_1)\right)+\underline{\theta} Eu(a_1)\right)\left[\hat{F}(\theta_2)-\hat{F}(\theta_1)\right] \\
&+ \left(\theta_2\left(Eu(a_3)-Eu(a_2)\right)+\theta_1\left(Eu(a_2)-Eu(a_1)\right)+\underline{\theta} Eu(a_1)\right)\left[1-\hat{F}(\theta_2)\right] \\
&- c\int_{\underline{\theta}}^{1} a^*(\theta)\, d\hat{F}(\theta) \\
&= \underline{\theta} Eu(a_1)\left[1-\hat{F}(\underline{\theta})\right]+\theta_1\left(Eu(a_2)-Eu(a_1)\right)\left[1-\hat{F}(\theta_1)\right] \\
&+ \theta_2\left(Eu(a_3)-Eu(a_2)\right)\left[1-\hat{F}(\theta_2)\right]-c\int_{\underline{\theta}}^{1} a^*(\theta)\, d\hat{F}(\theta),
\end{aligned}$$

while the 3-class CM yields

$$\pi_3 = \frac{1}{\left[\hat{F}(\theta_1)-\hat{F}(\underline{\theta})\right]}\int_{\underline{\theta}}^{\theta_1}\phi(\theta)\, d\hat{F}(\theta)\int_{\underline{\theta}}^{\theta_1} u(a^*(\theta))\, d\hat{F}(\theta)$$

$$\begin{aligned}
&+ \frac{1}{\left[\hat{F}(\theta_2)-\hat{F}(\theta_1)\right]}\int_{\theta_1}^{\theta_2}\phi(\theta)\, d\hat{F}(\theta)\int_{\theta_1}^{\theta_2} u(a^*(\theta))\, d\hat{F}(\theta) \\
&+ \frac{1}{\left[1-\hat{F}(\theta_2)\right]}\int_{\theta_2}^{1}\phi(\theta)\, d\hat{F}(\theta)\int_{\theta_2}^{1} u(a^*(\theta))\, d\hat{F}(\theta)-c\int_{\underline{\theta}}^{1} a^*(\theta)\, d\hat{F}(\theta) \\
&= \underline{\theta} Eu(a_1)\left[\hat{F}(\theta_1)-\hat{F}(\underline{\theta})\right]+(\underline{\theta}-\theta_1) Eu(a_1)\left[1-\hat{F}(\theta_1)\right] \\
&+ \theta_1 Eu(a_2)\left[\hat{F}(\theta_2)-\hat{F}(\theta_1)\right]+(\theta_1-\theta_2) Eu(a_2)\left[1-\hat{F}(\theta_2)\right] \\
&+ \theta_2 Eu(a_3)\left[1-\hat{F}(\theta_2)\right]-c\int_{\underline{\theta}}^{1} a^*(\theta)\, d\hat{F}(\theta) \\
&= \underline{\theta} Eu(a_1)\left[1-\hat{F}(\underline{\theta})\right]+\theta_1\left(Eu(a_2)-Eu(a_1)\right)\left[1-\hat{F}(\theta_1)\right] \\
&+ \theta_2\left(Eu(a_3)-Eu(a_2)\right)\left[1-\hat{F}(\theta_2)\right]-c\int_{\underline{\theta}}^{1} a^*(\theta)\, d\hat{F}(\theta).
\end{aligned}$$

Hence, $\tilde{\pi}_3 = \pi_3$.

Proof of Proposition 3.5.1. Applying the analog argument in the proof of Theorem 3.4.1 and taking the fact that G is defined over $[\phi(0), \phi(1)]$ into account, we show that $\pi_\infty - \pi_n^* \leq C\left(\hat{B}_i/n^2\right)$. Then by letting $\hat{\beta}_i = \hat{B}_i/\pi_\infty$, we get the result.

■

Example 3.5.1. Consider $u(a) = a^{\frac{1}{2}}$ and $\hat{F}(\theta) = \theta^\alpha$. From $\varphi(\theta)u'(a) = c$ we can derive that $a = \varphi^2/4c^2$. From the assumption that the distribution satisfies IFR, $\alpha \geq 1$, $R_\infty = \int_{\underline{\theta}}^1 \left(\theta - \frac{1-\hat{F}(\theta)}{\hat{f}(\theta)}\right) u(a(\theta))\, dF(\theta) = \frac{1}{2c}\int_{\underline{\theta}}^1 \varphi^2 d\hat{F}(\theta)$. Without loss of generality, let $c = \frac{1}{2}$.

$$R_\infty = \int_{\underline{\theta}}^1 \varphi(\theta)^2 d\hat{F}(\theta) = \int_{\left(\frac{1}{\alpha+1}\right)^{\frac{1}{\alpha}}}^1 \left(\theta - \frac{1-\theta^\alpha}{\alpha\theta^{\alpha-1}}\right)^2 \alpha\theta^{\alpha-1} d\theta$$
$$= \frac{\alpha}{\alpha^2-4}\left(\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}} + \alpha\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}} - 1\right),$$
$$\pi_\infty = R_\infty - \frac{1}{2}\int_{\underline{\theta}}^1 \varphi(\theta)^2 dF(\theta) = \frac{1}{2}R_\infty$$

and

$$\frac{\partial\varphi}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\theta - \frac{1-\theta^\alpha}{\alpha\theta^{\alpha-1}}\right) = \frac{1}{\theta^\alpha\alpha}(\alpha + \theta^\alpha\alpha + \theta^\alpha - 1).$$
$$\frac{\partial\theta}{\partial\varphi} = \frac{1}{\frac{1}{\theta^\alpha\alpha}(\alpha + \theta^\alpha\alpha + \theta^\alpha - 1)}.$$

By a change of variable, $\int_{\underline{\theta}}^1 \varphi(\theta)\, dF(\theta) = \int_0^1 \varphi d\hat{F}(\theta(\varphi))$. Here $\theta(\varphi)$ is the inverse function of $\varphi(\theta)$. Denote $F(\varphi) = \hat{F}(\theta(\varphi))$. We have $f(\varphi) = \hat{f}(\theta(\varphi))\,\theta'(\varphi) = \alpha\theta^{\alpha-1}\frac{\partial\theta}{\partial\varphi}$.

$$\text{That is } f(\varphi) = \alpha\theta^{\alpha-1}\frac{1}{\frac{1}{\theta^\alpha\alpha}(\alpha + \theta^\alpha\alpha + \theta^\alpha - 1)}$$
$$= \theta^{2\alpha-1}\frac{\alpha^2}{\alpha + \theta^\alpha\alpha + \theta^\alpha - 1}, \text{ where } \theta \in \left[\left(\frac{1}{\alpha+1}\right)^{\frac{1}{\alpha}}, 1\right].$$

Taking the derivative of $f(\phi)$, we have

$$\frac{d}{d\theta}\left(\theta^{2\alpha-1}\frac{\alpha^2}{\alpha + \theta^\alpha\alpha + \theta^\alpha - 1}\right)$$
$$= \theta^{2\alpha-2}\alpha^2\frac{\alpha-1}{(\alpha + \theta^\alpha + \theta^\alpha\alpha - 1)^2}(2\alpha + \theta^\alpha + \theta^\alpha\alpha - 1) > 0,$$

because $\theta^{\alpha}(1+\alpha) \geq 1$. Hence, $\theta^{2\alpha-1}\frac{\alpha^2}{\alpha+\theta^{\alpha}\alpha+\theta^{\alpha}-1}$ is increasing in θ, and, $\|f\|_{\infty}^{[\theta,1]} = \alpha/2$. Use β_3 from Proposition 3.5.1, $\beta_3 = \frac{\sqrt{3}\|f\|_{\infty}^{[\theta,1]}}{12\pi_{\infty}}$. If $\alpha \in [1,20]$,

$$\pi_{\infty} - \frac{\sqrt{3}}{24}\|f\|_{\infty}^{[\theta,1]} = \frac{\alpha}{\alpha^2-4}\left(\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}} + \alpha\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}} - 1\right) - \frac{\sqrt{3}}{48}\alpha$$
$$\geq 0 .$$

Hence, we have $\beta_3 \leq 2$. Therefore, for any $\alpha \in [1,20]$, $\frac{\pi_n^*}{\pi_{\infty}} \geq 1 - \frac{2}{n^2}$. ■

Example 3.5.2. Now we solve for the optimal 2-level contract:

$\max \theta_1 a_1^{\frac{1}{2}}(\theta_2-\theta_1) + \left[\theta_2\left(a_2^{\frac{1}{2}} - a_1^{\frac{1}{2}}\right) + \theta_1 a_1^{\frac{1}{2}}\right][1-\theta_2] - ca_1(\theta_2-\theta_1) - ca_2[1-\theta_2]$.

FOC:

$\theta_1 : a_1^{\frac{1}{2}}(1-2\theta_1) + ca_1 = 0$

$\theta_2 : \left(a_2^{\frac{1}{2}} - a_1^{\frac{1}{2}}\right)[1-2\theta_2] - ca_1 + ca_2 = 0$

$a_1 : \frac{1}{2}\theta_1 a_1^{-\frac{1}{2}}(\theta_2-\theta_1) + \frac{1}{2}a_1^{-\frac{1}{2}}[\theta_1-\theta_2][1-\theta_2] - c(\theta_2-\theta_1) = 0$

$a_2 : \frac{1}{2}\theta_2 a_2^{-\frac{1}{2}}[1-\theta_2] - c[1-\theta_2] = 0.$

Solving those equations we get:

$\theta_1 = \frac{3}{5},\ \theta_2 = \frac{4}{5},\ ca_1^{\frac{1}{2}} = \frac{1}{5},\ ca_2^{\frac{1}{2}} = \frac{2}{5}.$

The profit is $\pi_2^* = \frac{1}{25c}$.Hence, $\tilde{\beta} = u(a^*(1))/(4\pi_2^*)$. We've already shown that $a^*(\theta) = \phi(\theta)/2c$. Therefore, $\tilde{\beta} = 25/8$. Therefore, we have $\frac{\pi_n^*}{\pi_{\infty}} \geq 1 - \frac{\tilde{\beta}}{n^2}$. ■

Cost-Sharing: Omitted Steps and Proof of Proposition 3.5.2

PROBLEM FORMULATION. The principal offers a contract to minimize expected transfer $T(x)$ to the firm, which is:

$$\min_{T(x),y(x)} \int_0^1 T(x)\,dF(x)$$
$$\text{s.t. } T(x) - (x - y(x|x)) - \frac{1}{4k}y^2(x|x)$$
$$\geq T(\tilde{x}) - (x - y(\tilde{x}|x)) - \frac{1}{4k}y^2(\tilde{x}|x) \quad \text{and}$$
$$T(x) - (x - y(x)) - \frac{1}{4k}y^2(x|x) \geq 0 \text{ for all } x \in [0,1],$$

where $x - y(\tilde{x}|x) = \tilde{x} - y(\tilde{x})$.

THE OPTIMAL CONTRACT. Define $u(x) = T(x) - (x - y(x|x)) - \frac{1}{4k}y^2(x|x)$.

From the envelope theorem for the agent,

$$u'(x) = -\frac{dC(y(\tilde{x}|x))}{dx}|_{\tilde{x}=x} = -C'(y(x)) \leq 0.$$

Hence, $u(1) = 0$.

$$u(x) = \int_x^1 C'(y(\tilde{x}))\,d\tilde{x}.$$

This implies that

$$\int_0^1 u(x)\,dF(x) = \int_0^1 C'(y(x))\frac{F(x)}{f(x)}dF(x) \text{ and}$$
$$\int_0^1 T(x)\,dF(x) = \int_0^1 \left[x - y(x) + C(y(x)) + C'(y)\frac{F(x)}{f(x)}\right]dF(x).$$

The derivative w.r.t. $y(x)$ is

$$-1 + C'(y(x)) + C''(y(x))\frac{F(x)}{f(x)}.$$

Since $C(y) = \frac{1}{4k}y^2$, the derivative becomes

$$-1 + \frac{1}{2k}y(x) + \frac{1}{2k}\frac{F(x)}{f(x)}.$$

If $F(x)/f(x) \geq 2k$, to minimize cost $y(x) = 0$. If $F(x)/f(x) \leq 2k$, then $y(x) = 2k - F(x)/f(x)$. Since $F(x)/f(x)$ is an increasing function, $y(x)$ is decreasing in x and bounded below by zero.

Denote $x^* = \min\{\{x,1\} \,|\, [F(x)/f(x)] = 2k.\}$. The expected transfer to the firm is

$$\int_0^1 \left[x - y^*(x) + \frac{1}{4k} y^*(x)^2 + \frac{1}{2k} y^*(x) \frac{F(x)}{f(x)}\right] dF(x)$$
$$= \mu_x - \int_0^1 y^*(x) \left[1 - \frac{1}{4k} y^*(x) - \frac{1}{2k} \frac{F(x)}{f(x)}\right] dF(x)$$
$$= \mu_x - \int_0^{x^*} \left[2k - \frac{F(x)}{f(x)}\right] \left[\frac{1}{2} - \frac{1}{4k} \frac{F(x)}{f(x)}\right] dF(x)$$
$$= \mu_x - k \int_0^{x^*} \left[1 - \frac{1}{2k} \frac{F(x)}{f(x)}\right]^2 dF(x).$$

EXISTENCE OF n-level LCSCR CONTRACT WITH COST REDUCTION EQUALS u_{n-1}.

For illustration purpose, we show the existence of a 3-level LCSCR contract that has cost reduction equals to u_2. The same procedure shows it for arbitrary n.

The principal offers a firm a contract containing $\{T_1, \alpha_1\}, \{T_2, \alpha_2\}$ if the firm has a type belonging to $[0, x_1^*]$ and $[x_1^*, x^*]$ respectively and a cost reimbursement contract if the firm has a type greater than x^*. The firm with $x < x^*$ needs to solve

$$\max_y (1 - \alpha_i) y - \frac{1}{4k} y^2.$$

Then $y_i = 2k(1 - \alpha_i)$.

The firm's profit under such contract is

$$T_i + \alpha_i (x - y(x)) - C(y(x)) - (x - y(x))$$
$$= T_i - [1 - \alpha_i][x - 2k(1 - \alpha_i)] - \frac{1}{4k} [2k(1 - \alpha_i)]^2.$$

The firm with $x > x^*$ exerts no effort and the profit is zero.

When $x = x^*$, the profit of the firm is zero. Hence,

$$T_2 - [1 - \alpha_2][x^* - 2k(1 - \alpha_2)] - \frac{1}{4k} [2k(1 - \alpha_2)]^2 = 0.$$

This implies

$$T_2 = (1-\alpha_2)\, x^* - k\,(1-\alpha_2)^2 .$$

A firm with type x_1^* should be indifferent to choosing between the two contracts.

Therefore,

That is

$$T_1 = (1-\alpha_2)\, x^* + (\alpha_2-\alpha_1)\, x_1^* - k\,(1-\alpha_1)^2 .$$

The principal's expected cost is

$$\int_0^{x_1^*} T_1 + \alpha_1 \left[x - y(x)\right] dF(x) + \int_{x_1^*}^{x^*} T_2 + \alpha_2 \left[x - y(x)\right] dF(x)$$
$$= \int_0^{x_1^*} \left[(1-\alpha_2)\, x^* + (\alpha_2-\alpha_1)\, x_1^* + \alpha_1 x - k\left(1-\alpha_1^2\right)\right] dF(x)$$
$$+ \int_{x_1^*}^{x^*} \left[\alpha_2 x + (1-\alpha_2)\, x^* - k\left(1-\alpha_2^2\right)\right] dF(x) .$$

The principal minimizes expected cost over α_1 and α_2, which yields first order conditions

$$\int_0^{x_1^*} \left[x - x_1^*\right] dF(x) = -2k\alpha_1 \int_0^{x_1^*} dF(x) \qquad \text{and}$$
$$\int_{x_1^*}^{x^*} \left[x - x^*\right] dF(x) = -2k\alpha_2 \int_{x_1^*}^{x^*} dF(x) + \int_0^{x_1^*} \left[x^* - x_1^*\right] dF(x) .$$

The cost reduction associate with such contract is

$$\int_0^{x_1^*} \left[-(1-\alpha_2)\, x^* - (\alpha_2-\alpha_1)\, x_1^* + (1-\alpha_1)\, x + k\left(1-\alpha_1^2\right)\right] dF(x)$$
$$+ \int_{x_1^*}^{x^*} \left[(1-\alpha_2)\,(x - x^*) + k\left(1-\alpha_2^2\right)\right] dF(x)$$
$$= \int_0^{x_1^*} \left[(1-\alpha_1)\,(x - x_1^*) + (1-\alpha_2)\,(x_1^* - x^*) + k\left(1-\alpha_1^2\right)\right] dF(x)$$
$$+ \int_{x_1^*}^{x^*} \left[(1-\alpha_2)\,(x - x^*) + k\left(1-\alpha_2^2\right)\right] dF(x)$$

$$
\begin{aligned}
&= -2k\alpha_1 (1-\alpha_1) \int_0^{x_1^*} dF(x) + \int_0^{x_1^*} \left((1-\alpha_2)(x_1^* - x^*) + k\left(1-\alpha_1^2\right)\right) dF(x) \\
&- 2k\alpha_2 (1-\alpha_2) \int_{x_1^*}^{x^*} dF(x) + \int_{x_1^*}^{x^*} \left[k\left(1-\alpha_2^2\right)\right] dF(x) + (1-\alpha_2) \int_0^{x_1^*} [x^* - x_1^*] dF(x) \\
&= -2k\alpha_1 (1-\alpha_1) \int_0^{x_1^*} dF(x) + \int_0^{x_1^*} k\left(1-\alpha_1^2\right) dF(x) \\
&- 2k\alpha_2 (1-\alpha_2) \int_{x_1^*}^{x^*} dF(x) + \int_{x_1^*}^{x^*} \left[k\left(1-\alpha_2^2\right)\right] dF(x) \\
&= \int_0^{x_1^*} k(1-\alpha_1)^2 dF(x) + \int_{x_1^*}^{x^*} k(1-\alpha_2)^2 dF(x).
\end{aligned}
$$

Since

$$
\begin{aligned}
u_2 = &\frac{k}{F\left(x_1^*\right)} \left(\int_0^{x_1^*} \left[1 - \frac{1}{2k}\frac{F(x)}{f(x)}\right] dF(x)\right)^2 \\
&+ \frac{k}{F(x^*) - F\left(x_1^*\right)} \left(\int_{x_1^*}^{x^*} \left[1 - \frac{1}{2k}\frac{F(x)}{f(x)}\right] dF(x)\right)^2,
\end{aligned}
$$

by letting

$$
\begin{aligned}
1-\alpha_1 &= \frac{1}{F\left(x_1^*\right)} \int_0^{x_1^*} \left[1 - \frac{1}{2k}\frac{F(x)}{f(x)}\right] dF(x) \\
1-\alpha_2 &= \frac{1}{F(x^*) - F\left(x_1^*\right)} \int_{x_1^*}^{x^*} \left[1 - \frac{1}{2k}\frac{F(x)}{f(x)}\right] dF(x),
\end{aligned}
$$

such a 3-level LCSCR contract has cost reduction equals u_2.

Proof of Proposition 3.5.2. By the proof of Theorem 3.4.1, we have

$$
u_\infty - u_{n-1} \leq \frac{kF(x^*)}{4(n-1)^2} \leq \frac{k}{4(n-1)^2}.
$$

Then

$$
\frac{u_\infty - u_{n-1}}{u_\infty} \leq \frac{1}{4(n-1)^2}\frac{k}{u_\infty}.
$$

This implies

$$
\frac{u_{n-1}}{u_\infty} \geq 1 - \frac{\beta}{(n-1)^2},
$$

where $\beta = \dfrac{k}{4u_\infty}$. ■