Saddle Squares in Random Two Person Zero Sum Games with Finitely Many Strategies

by

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ABSTRACT

By the von Neumann min-max theorem, a two person zero sum game with finitely many pure strategies has a unique value for each player (summing to zero) and each player has a non-empty set of optimal mixed strategies. If the payoffs are independent, identically distributed (iid) uniform $(0,1)$ random variables, then with probability one, both players have unique optimal mixed strategies utilizing the same number of pure strategies with positive probability (Jonasson 2004). The pure strategies with positive probability in the unique optimal mixed strategies are called saddle squares. In 1957, Goldman evaluated the probability of a saddle point (a 1 by 1 saddle square), which was rediscovered by many authors including Thorp (1979). Thorp gave two proofs of the probability of a saddle point, one using combinatorics and one using a beta integral. In 1965, Falk and Thrall investigated the integrals required for the probabilities of a 2 by 2 saddle square for $2 \times n$ and $m \times 2$ games with iid uniform $(0,1)$ payoffs, but they were not able to evaluate the integrals.

This dissertation generalizes Thorp's beta integral proof of Goldman's probability of a saddle point, establishing an integral formula for the probability that a $m \times n$ game with iid uniform $(0,1)$ payoffs has a k by k saddle square $(k \leq m,n)$. Additionally, the probabilities of a 2 by 2 and a 3 by 3 saddle square for a 3×3 game with iid uniform$(0,1)$ payoffs are found. For these, the 14 integrals observed by Falk and Thrall are dissected into 38 disjoint domains, and the integrals are evaluated using the basic properties of the dilogarithm function. The final results for the probabilities of a 2 by 2 and a 3 by 3 saddle square in a 3×3 game are linear combinations of 1, π^2, and $\ln(2)$ with rational coefficients.

DEDICATION

To my loving wife whose patience, support,

and motivational assistance

made this dissertaion possible.

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INTRODUCTION

1.1 Overview of Game Theory

With their book "Theory of Games and Economic Behavior" [MvN1980], first published in 1944, John von Neumann and Oskar Morgenstern laid the foundation for the study of optimal mixed strategies, value and utility theory, and bargaining and coalition formation for games of strategy with $p \geq 2$ players. For a play of the game, each player simultaneously announces a pure strategy chosen from their set of pure strategies; each player then receives a payoff determined by the pure strategies chosen by all of the players. If the sum of the payoffs is always zero, then we say that the game is a zero sum game. A nonzero sum game can be embedded in a larger zero sum game by adding a fictitious player who has one pure strategy and whose payoff is minus the sum of the payoffs of the players of the original game. See APPENDIX A for examples of zero sum games of strategy with finitely many pure strategies.

The celebrated min-max theorem, first proved by von Neumann [vN1928], states that for a two person zero sum game with finitely many pure strategies, there is a unique value for each player and these two values sum to zero. A mixed strategy for a player is an assignment of probabilities, which sum to one, to the player's set of pure strategies. Each player has a non-empty set of optimal mixed strategies which maximize their expectation given that their opponent knows their mixed strategy. If either player uses any optimal mixed strategy, then their expectation is greater than or equal to their value of the game regardless of their opponent's mixed strategy. Dantzig [Dan1963] developed the simplex method (see APPENDIX B) which is often used by government planners, business operators, and many others to solve linear optimization problems which include applying the von Neumann min-max theorem to large economic games with two interests.

John Nash [Nas1951] and Lloyd S. Shapley [Sha2004] gave two important theorems which partially extend the von Neumann min-max theorem to zero sum games with $p \geq 3$ players. Nash [Nas1951] used Brower's fixed point theorem [Bro1910] to show the existence of an equilibrium point: a mixed strategy for each player such that no player can increase their expectation by using a different mixed strategy. The players may have different expectations for different Nash equilibrium points. Shapley [Sha2004] gave an axiomatic development of a unique value for each player determined by the characteristic function of a game: a superadditive real valued function on the collection of all subsets of the players. Each player may have no mixed strategy which ensures that their expectation is greater than or equal to their Shapley value even though all of the mixed strategies of their opponents are known. See APPENDIX A for a zero sum three player coin matching game which illustrates these limitations of the Nash equilibrium point theorem and Shapley's unique value theorem.

While a great part of the research in the last sixty years has focused on coalition formation, bargaining, utility theory and other modern concepts, a few researchers have focused on the structure of the solutions of von Neumann and Nash for games of strategy with random payoffs.

A. J. Goldman [Gol1957] gave the probability that an $m \times n$ matrix, where $m, n \geq 1$ and whose entries are independent, identically distributed (iid) random variables (rvs) from a continuous distribution, has a saddle point: an entry which is the minimum of its row and the maximum of its column. If the random $m \times n$ matrix is the payoff matrix for a two person zero sum game (see [PZ1996]), then Goldman's theorem gives the probability that both players have a unique optimal pure strategy, which we refer to as a 1 by 1 saddle square.

R. M. Thrall and J. E. Falk [FT1965], Edward O. Thorp [Tho1979] and G. P.

Papavassilopoulos [Pap1995] rediscovered Goldman's theorem and its simple proof. Thorp also gave an alternative proof of Goldman's theorem using the gamma function and the beta integral [Tho1979].

William G. Faris and Robert S. Maier [FM1987] consider random two person zero sum games in which both players have n pure strategies and the payoffs are iid rvs from a continuous distribution with certain other properties. Faris and Maier observe that Goldman's saddle point probability tends to 0 as n tends to infinity. Additionally, Faris and Maier show that if the payoffs have a normal distribution, then the probability that all pure strategies occur with positive probability in an optimal mixed strategy, which we call an n by n saddle square, is bounded above by 2^{1-n}. Faris and Maier also investigated the distribution of the number of pure strategies which occur with positive probability in an optimal mixed strategy when n is large and the payoffs have a Gaussian or uniform distribution.

Johan Jonasson [Jon2004] extended the results of Faris and Maier [FM1987] and proved an important theorem: if the payoffs in a two person zero sum game with finitely many strategies are iid rvs from a continuous distribution, then with probability 1 (wp1) both players have a unique optimal mixed strategy, and the number of pure strategies which occur with positive probability in both optimal mixed strategies are the same. Jonasson proved that for an $n \times n$ two person zero sum game with iid payoffs from a standard normal distribution, the expected number of pure strategies occuring with positive probability in both players' unique optimal mixed strategies is $n/2 + o(n)$.

In the Spring 2005 semester, my advisor, Kevin W. J. Kadell, gave a MAT/STP 598 course in game theory. He demonstrated that Jonasson's theorem [Jon2004] follows by examining the final tableau obtained by applying the simplex method to find optimal strategies for both players. We use the term "saddle square" to describe the unique optimal mixed strategies which occur wp1 by Jonasson's theorem, which are the object of study in this dissertation.

A number of researchers have shown that, unlike random two person zero sum games, the limit of the probability of a 1 by 1 saddle square in two person nonzero sum games is $1 - 1/e$, as the number of pure strategies for both players tends to infinity. Goldman, K. Goldberg and M. Newman [GGN1968] first proved this in 1968. Melvin Dresher [Dre1970] extended the result of Goldman, Goldberg, and Newman to all nonzero sum games with $p \geq 2$ players. G. P. Papavassilopoulos [Pap1995] rediscovered the results of Goldman, Goldberg, Newman and Dresher and added the probability of a pure equilibrium solution in a random p person nonzero sum game with the number of strategies fixed for each player tends to 0 as p tends to infinity.

This dissertation studies the structure of optimal mixed strategies for two person zero sum games with finitely many pure strategies with a random payoff matrix whose entries are iid uniform (0,1) rvs. We study the number of pure strategies which occur with positive probability in the optimal mixed strategies given by the von Neumann min-max theorem. We generalize Thorp's [Tho1979] second proof of Goldman's theorem [Gol1957], finding the probability that a random uniform $(0,1)$ payoff matrix A with $m,n \geq 2$ has a 2 by 2 saddle square. This means that both players have one optimal mixed strategy which uses exactly two of their pure strategies.

1.2 The von Neumann Min-Max Theorem

The set of $m \times n$ matrices with real entries is given by

$$M_{m,n}(S) = \left\{ (a_{i,j})_{i \in [m],\, j \in [n]} \;\middle|\; \forall i \in [m],\, \forall j \in [n],\, a_{i,j} \in S \right\}, \qquad m,n \geq 1, \quad S \subseteq \mathbb{R},$$

where $\forall$ is an abbreviation of for all, $[k] = \{1, \ldots, k\}$ for a positive integer k, and $\mathbb{R}$ is the set of real numbers.

We follow the standard setup [PZ1996] of a two person zero sum game with finitely many pure strategies, represented by the payoff matrix $A \in M_{m,n}(\mathbb{R})$. Both players simultaneously announce their choices of pure strategies $i \in [m],\ j \in [n]$, then player I recieves a payoff of $a_{i,j}$ units, and player II recieves a payoff of $-a_{i,j}$ units. If $a_{i,j} > 0$ then player I receives $a_{i,j}$ units from player II. If $a_{i,j} < 0$ then player II receives $-a_{i,j}$ units from player I. If $a_{i,j} = 0$ then there is no payoff to either player.

The mixed strategy spaces for players I and II are given by

$$P = \Big\{ \widetilde{p} = (p_1, \dots, p_m) \ \Big| \ 0 \le p_1, \dots, p_m,\ \sum_{i \in [m]} p_i = 1 \Big\},$$
$$Q = \Big\{ \widetilde{q} = (q_1, \dots, q_n) \ \Big| \ 0 \le q_1, \dots, q_n,\ \sum_{j \in [n]} q_j = 1 \Big\}.$$

Note that p_i and q_j are the probabilities that player I chooses pure strategy $i \in [m]$ and player II chooses pure strategy $j \in [n]$, respectively. For any two mixed strategies $\widetilde{p} \in P$ and $\widetilde{q} \in Q$, the expected payoff is

$$\widetilde{p} A \widetilde{q}^{\mathsf{T}} = \sum_{i \in [m]} \sum_{j \in [n]} p_i a_{i,j} q_j, \qquad \widetilde{p} \in P, \quad \widetilde{q} \in Q.$$

It is a standard result of real analysis (see H. L. Royden [Roy1988]) that a continuous function, such as $(\widetilde{p}, \widetilde{q}) \longrightarrow \widetilde{p} A \widetilde{q}^{\mathsf{T}}$, on a compact set

$$P \times Q = \big\{ (\widetilde{p}, \widetilde{q}) \ \big| \ \widetilde{p} \in P,\ \widetilde{q} \in Q \big\},$$

with the product topology, is bounded above and below and attains its maximum and minimum values. Hence we define

$$\mathrm{v}_{\mathrm{maxmin}}(A) = \max_{\widetilde{p} \in P} \Big(\min_{\widetilde{q} \in Q} \widetilde{p} A \widetilde{q}^{\mathsf{T}} \Big) = \max_{\widetilde{p} \in P} \Big(\min_{j \in [n]} \sum_{i \in [m]} p_i a_{i,j} \Big), \tag{1.2.1}$$

$$P^{\mathrm{opt}}(A) = \Big\{ \widetilde{p} \in P \ \Big| \ \forall \widetilde{q} \in Q,\ \widetilde{p} A \widetilde{q}^{\mathsf{T}} \ge \mathrm{v}_{\mathrm{maxmin}}(A) \Big\}, \tag{1.2.2}$$

and note

$$P^{\mathrm{opt}}(A) \neq \varnothing, \qquad A \in M_{m,n}(\mathbb{R}). \tag{1.2.3}$$

Similarly, we define

$$\mathrm{v}_{\mathrm{minmax}}(A) = \min_{\widetilde{q} \in Q} \Big(\max_{\widetilde{p} \in P} \widetilde{p} A \widetilde{q}^{\mathsf{T}} \Big) = \min_{\widetilde{q} \in Q} \Big(\max_{i \in [m]} \sum_{j \in [n]} a_{i,j} q_j \Big), \tag{1.2.4}$$

$$Q^{\mathrm{opt}}(A) = \left\{ \widetilde{q} \in Q \;\middle|\; \forall \widetilde{p} \in P,\; \widetilde{p} A \widetilde{q}^{\mathsf{T}} \leq \mathrm{v}_{\mathrm{minmax}}(A) \right\}, \tag{1.2.5}$$

and also note

$$Q^{\mathrm{opt}}(A) \neq \varnothing, \qquad A \in M_{m,n}(\mathbb{R}). \tag{1.2.6}$$

Observe by (1.2.1) and (1.2.4) that

$$\mathrm{v}_{\mathrm{maxmin}}(A) \leq \mathrm{v}_{\mathrm{minmax}}(A).$$

The essential part of the von Neumann min-max theorem is given by the following theorem.

Theorem 1.2.1. von Neumann's min-max theorem [vN1928, MvN1980]. *If $A \in M_{m,n}(\mathbb{R})$ is the payoff matrix for a two person zero sum game, then*

$$\mathrm{v}_{\mathrm{maxmin}}(A) = \mathrm{v}_{\mathrm{minmax}}(A).$$

By Theorem 1.2.1,

$$\begin{gathered} \forall \widetilde{p} \in P^{\mathrm{opt}}(A) \text{ and } \forall \widetilde{q} \in Q^{\mathrm{opt}}(A), \\ \mathrm{v}(A) = \mathrm{v}_{\mathrm{maxmin}}(A) = \mathrm{v}_{\mathrm{minmax}}(A) = \sum_{i \in [m]} \sum_{j \in [n]} p_i \, a_{i,j} \, q_j. \end{gathered} \tag{1.2.7}$$

1.3 Goldman's Theorem

We define

$A \sim UN_{m,n}(0,1)$ iff $A \in M_{m,n}(0,1)$ and the entries of A are iid un(0,1) rvs,

where $\mathrm{un}(0,1)$ denotes the uniform distribution on the interval $(0,1)$. Similarly for comparisons in Chapter 4 we define

$A \sim EX_{m,n}(1)$ iff $A \in M_{m,n}(0,\infty)$ and the entries of A are iid ex(1) rvs, where $\mathrm{ex}(1)$ denotes the exponential distribution with single parameter 1, and

$$A \sim N_{m,n}(0,1) \text{ iff } A \in M_{m,n}(-\infty,\infty) \text{ and the entries of } A \text{ are iid n(0,1) rvs,}$$

where $\mathrm{n}(0,1)$ denotes the standard normal distribution.

Definition 1.3.1. The entry $a_{i,j}$ of $A \in M_{m,n}(\mathbb{R})$ where $i \in [m],\ j \in [n],\ m,n \geq 1$, is a saddle point of A iff

$$a_{1,j},\ \ldots,\ a_{i-1,j}, a_{i+1,j},\ \ldots,\ a_{m,j} \leq a_{i,j} \leq a_{i,1},\ \ldots,\ a_{i,j-1},\ a_{i,j+1},\ \ldots,\ a_{i,n}. \tag{1.3.1}$$

A. J. Goldman [Gol1957] established the following theorem which gives the probability that $A \sim UN_{m,n}(0,1)$ has a saddle point.

Theorem 1.3.2. Goldman's theorem [Gol1957].

$$\begin{aligned} S_1(m,n) &= \mathrm{Prob}\Big(A \text{ has a saddle point} \ \Big|\ A \sim UN_{m,n}(0,1)\Big) \\ &= \frac{m!n!}{(m+n-1)!}, \qquad m,n \geq 1. \end{aligned} \tag{1.3.2}$$

1.4 Jonasson's Theorem

Johan Jonasson [Jon2004] established the following theorem which characterizes $P^{\mathrm{opt}}(A)$ and $Q^{\mathrm{opt}}(A)$ over random two person zero sum games with finitely many pure strategies.

Theorem 1.4.1. Jonasson's theorem [Jon2004]. *If $A \sim UN_{m,n}(0,1)$ is the payoff matrix of a two person zero sum game, then wp1 both players have a unique optimal mixed strategy*

$$P^{\mathrm{opt}} = \left\{\widetilde{p}^{\mathrm{Opt}}\right\},\ Q^{\mathrm{opt}} = \left\{\widetilde{q}^{\mathrm{Opt}}\right\}, \tag{1.4.1}$$

where both $\widetilde{p}^{\mathrm{Opt}}$ and $\widetilde{q}^{\mathrm{Opt}}$ require, with positive probability, the same number of pure strategies

$$\left|\left\{i \in [m] \ \middle|\ p_i^{\mathrm{Opt}} > 0\right\}\right| = \left|\ \left\{j \in [n] \ \middle|\ q_j^{\mathrm{Opt}} > 0\right\}\right|. \tag{1.4.2}$$

We define

$$M^*_{m,n}(S) = \Big\{ A \in M_{m,n}(S) \;\Big|\; \text{the conclusion (1.4.1), (1.4.2) of Jonasson's Theorem 1.4.1 holds} \Big\}.$$

Following Theorem 1.3.2 and Theorem 1.4.1 we establish the following definitions.

Definition 1.4.2. $A \in M_{m,n}(\mathbb{R})$ has a $k \times k$ saddle square iff

$$P^{\text{opt}} = \Big\{ \widetilde{p}^{\text{Opt}} \Big\}, \quad Q^{\text{opt}} = \Big\{ \widetilde{q}^{\text{Opt}} \Big\}, \quad \text{and}$$

$$\Big| \Big\{ i \in [m] \;\Big|\; p_i^{\text{Opt}} > 0 \Big\} \Big| = \Big| \Big\{ j \in [n] \;\Big|\; q_j^{\text{Opt}} > 0 \Big\} \Big| = k, \quad \text{for} \quad 1 \le k \le m,n.$$

Definition 1.4.3.

$$S_k(m,n) = \text{Prob}\Big(A \text{ has a } k \text{ by } k \text{ saddle square} \;\Big|\; A \sim UN_{m,n}(0,1) \Big), \quad 1 \le k \le m,n.$$

Observe that Goldman's Theorem 1.3.2 can be written as

$$\begin{aligned} S_1(m,n) = \text{Prob}\Big(& \exists i \in [m] \text{ with } P^{\text{opt}} = \big\{ (0_1, \ldots, 0_{i-1}, 1, 0_{i+1}, \ldots, 0_m) \big\}, \\ & \exists j \in [n] \text{ with } Q^{\text{opt}} = \big\{ (0_1, \ldots, 0_{j-1}, 1, 0_{j+1}, \ldots, 0_n) \big\}, \\ & \Big| A \sim UN_{m,n}(0,1) \Big) \\ = \frac{m!n!}{(m+n-1)!}, & \qquad m,n \ge 1. \end{aligned}$$

This dissertation concerns the value of the saddle square probabilities $S_k(m,n)$ when $A \sim UN_{m,n}(0,1)$. Specifically we will compute $S_2(2,n)$, $S_2(m,2)$, $S_2(3,3)$ and $S_3(3,3)$. In Chapter 2 we present two proofs of Goldman's Theorem 1.3.2, the first based on basic combinatorial analysis and the second using the beta integral and gamma function. Chapter 3 generalizes the second proof of Goldman's Theorem 1.3.2, expressing $S_k(m,n)$ as a function of the expected value of probabilities $\Theta(A)$ and $\Phi(A)$. Following the establishment of $S_k(m,n)$ in Chapter 4 we derive algebraic formulas for

$\Theta(A)$ and $\Phi(A)$ for $k = 2$ and present the integral form of $S_2(m,n)$ for $A \sim UN_{2,2}(0,1)$. We conclude Chapter 4 finding expressions for $\Theta(A)$ and $\Phi(A)$ for $k = 2$ when the entries of matrix A are iid rvs from two alternative distributions, exponential (1) ($A \sim EX_{2,2}(1)$) and normal (0,1) ($A \sim N_{2,2}(0,1)$), demonstrating with a large simulation the dependence of $\Theta(A)\Phi(A)$ on the choice of the underlying distribution. Chapter 5 presents the preliminaries neccessary for computing the integral form of $S_2(3,3)$. The partial details of the integral computation are provided in Chapters 6, 7, and 8. We conclude the dissertation with the computational results in Chapter 9.

Chapter 2

SADDLE POINTS

2.1 Proof of Goldman's Theorem

Recall the following from Chapter 1:

Definition 1.3.1

The entry $a_{i,j}$ of $A \in M_{m,n}(\mathbb{R})$ where $i \in [m]$, $j \in [n]$, $m,n \geq 1$, is a saddle point of A iff

$$a_{1,j},\ \ldots,\ a_{i-1,j},\ a_{i+1,j},\ \ldots,\ a_{m,j} \leq a_{i,j} \leq a_{i,1},\ \ldots,\ a_{i,j-1},\ a_{i,j+1},\ \ldots,\ a_{i,n},$$

Definition 1.4.3, setting $k = 1$

$$S_1(m,n) = \text{Prob}\Big(A \text{ has a saddle point} \,\Big|\, A \sim UN_{m,n}(0,1)\Big) \quad 1 \leq m,n,$$

Goldman's Theorem 1.3.2

$$\begin{aligned} S_1(m,n) &= \text{Prob}\Big(A \text{ has a saddle point} \,\Big|\, A \sim UN_{m,n}(0,1)\Big) \\ &= \frac{m!n!}{(m+n-1)!}, \qquad 1 \leq m,n. \end{aligned}$$

Goldman [Gol1957] gave the following proof of his Theorem 1.3.2.

Proof. Assume throughout the proof that $A \in M_{m,n}$ is an m by n random matrix, $A \sim UN_{m,n}(0,1)$, $m,n \geq 1$. All events are conditioned on $A \sim UN_{m,n}(0,1)$.

Observe by (1.3.1) each entry of A is equally likely to be a saddle point. By the von Neumann min-max Theorem 1.2.1, all of the saddle points of A are equal. Since the entries of A are distinct wp1, A has at most one saddle point, and the probability of the event (1.3.1) is unchanged if strict inequalities are used. Hence,

$$\begin{aligned} S_1(m,n) &= mn\,\text{Prob}\Big(a_{1,1} \text{ is a saddle point of } A\Big) \\ &= mn\,\text{Prob}\Big(a_{2,1},\ldots,a_{m,1} < a_{1,1} < a_{1,2},\ldots,a_{1,n}\Big). \end{aligned} \tag{2.1.1}$$

Let

$$\text{LEG} = \left\{a_{2,1}, \ldots, a_{m,1}\right\},\ \text{CORNER} = \left\{a_{1,1}\right\},\ \text{ARM} = \left\{a_{1,2}, \ldots, a_{1,n}\right\}, \tag{2.1.2}$$

$$\text{HOOK} = \text{LEG} \cup \text{CORNER} \cup \text{ARM}.$$

Observe that (2.1.1) can be written as

$$S_1(m,n) = mn\,\text{Prob}\Big(\mathscr{LCA}\Big), \tag{2.1.3}$$

where $\mathscr{LCA}$ is the event

$$\mathscr{LCA} : \text{LEG} < \text{CORNER} < \text{ARM}, \tag{2.1.4}$$

with the set inequality

$$U < V \quad \text{iff} \quad \forall u \in U, \quad \forall v \in V, \quad u < v, \qquad U,\, V \subseteq \mathbb{R}.$$

Since finitely many iid rvs from a continuous distribution are distinct wp1,

$$|\text{HOOK}| = m+n-1,$$

and using order statistics (see David [Dav1981]),

$$\text{HOOK} = \{hk_{(1)}, \ldots, hk_{(m+n-1)}\}, \qquad 0 < hk_{(1)} < \ldots < hk_{(m+n-1)} < 1.$$

Observe

$$\mathscr{LCA} \quad \text{iff} \quad \mathscr{L} \text{ and } \mathscr{C} \text{ and } \mathscr{A},$$

where $\mathscr{L}$, $\mathscr{C}$ and $\mathscr{A}$ are the events

$$\begin{aligned} \mathscr{L} &: \text{LEG} = \left\{hk_{(1)}, \ldots, hk_{(m-1)}\right\}, \\ \mathscr{C} &: \text{CORNER} = \left\{hk_{(m)}\right\}, \\ \mathscr{A} &: \text{ARM} = \left\{hk_{(m+1)}, \ldots, hk_{(m+n-1)}\right\}. \end{aligned} \tag{2.1.5}$$

Note the events $\mathscr{LCA}$, $\mathscr{L}$, $\mathscr{C}$, and $\mathscr{A}$ are independent of the set HOOK. Hence,

$$\text{Prob}\Big(\mathscr{C}\Big) = \frac{1}{(m+n-1)},$$

$$\text{Prob}\Big(\mathscr{L}\;\Big|\;\mathscr{C}\Big) = \frac{1}{\binom{m+n-2}{m-1}} = \frac{(m+n-2)!}{(m-1)!(n-1)!},$$

$$\text{Prob}\Big(\mathscr{A}\;\Big|\;\mathscr{L}\text{ and }\mathscr{C}\Big) = 1,$$

which yields

$$\begin{aligned}\text{Prob}\Big(\mathscr{L}\mathscr{C}\mathscr{A}\Big) &= \text{Prob}\Big(\mathscr{C}\Big)\,\text{Prob}\Big(\mathscr{L}\;\Big|\;\mathscr{C}\Big)\,\text{Prob}\Big(\mathscr{A}\;\Big|\;\mathscr{L}\text{ and }\mathscr{C}\Big)\\ &= \frac{1}{(m+n-1)}\frac{(m-1)!(n-1)!}{(m+n-2)!}\\ &= \frac{(m-1)!(n-1)!}{(m+n-1)!}.\end{aligned} \tag{2.1.6}$$

Substituting (2.1.6) into (2.1.3) provides the required result of Theorem 1.3.2. □

Note the probabilities of events (2.1.5) are independent of the distribution of the entries of A, thus this proof suffices for random games, whose entries are iid rvs from any continuous distribution.

2.2 Thorp's Alternative Proof of Goldman's Theorem

Thorp [Tho1979] rediscovered Goldman's Theorem 1.3.2 and its simple proof above, and presented the following reformulation (idea credited to S. Karamardian) using the gamma function and the beta integral.

Proof. Again, it is assumed throughout the proof that $A \sim UN_{m,n}(0,1)$ where $m,n \geq 1$, and all events are conditioned on $A \sim UN_{m,n}(0,1)$. Define

$$\Theta(t) = \text{Prob}\big(t < u \;\big|\; u \sim \text{un}(0,1)\big) = 1-t, \qquad 0<t<1,$$

$$\Phi(t) = \text{Prob}\big(u < t \;\big|\; u \sim \text{un}(0,1)\big) = t, \qquad 0<t<1.$$

Recall the sets LEG, CORNER, ARM, and HOOK (2.1.2) and events $\mathscr{L}\mathscr{C}\mathscr{A}$ (2.1.4), $\mathscr{L}$, $\mathscr{C}$, and $\mathscr{A}$ (2.1.5). Again note that the events $\mathscr{L}\mathscr{C}\mathscr{A}$, $\mathscr{L}$, $\mathscr{C}$, and $\mathscr{A}$ are independent of the set HOOK. Since

$$\text{Prob}\Big(\mathscr{C}\;\Big|\;\mathscr{L}\text{ and }\mathscr{A}\Big)=1,$$

we have

$$\mathscr{L}\mathscr{C}\mathscr{A}\quad\text{iff}\quad\mathscr{L}\text{ and }\mathscr{A}. \tag{2.2.1}$$

Hence,

$$\begin{aligned}\text{Prob}\Big(\mathscr{L}\;\Big|\;a_{1,1}=t\Big)&=\text{Prob}\Big(a_{2,1},\ldots,a_{m,1}<t\;\Big|\;a_{1,1}=t\Big)\\&=\text{Prob}\Big(a_{2,1}<t\text{ and }\ldots\text{ and }a_{m,1}<t\;\Big|\;a_{1,1}=t\Big).\end{aligned}$$

Since $a_{i,j},\ i\in[m],\ \ j\in[n],$ are iid un(0,1) rvs,

$$\begin{aligned}&\text{Prob}\Big(a_{2,1}<t\text{ and }\ldots\text{ and }a_{m,1}<t\;\Big|\;a_{1,1}=t\Big)\\&\quad=\text{Prob}\Big(a_{2,1}<t\;\Big|\;a_{1,1}=t\Big)\cdots\text{Prob}\Big(a_{m,1}<t\;\Big|\;a_{1,1}=t\Big)\\&\quad=\Phi(t)^{m-1}\\&\quad=t^{m-1},\qquad 0<t<1.\end{aligned} \tag{2.2.2}$$

Similarly,

$$\begin{aligned}\text{Prob}\Big(\mathscr{A}\;\Big|\;a_{1,1}=t\Big)&=\text{Prob}\Big(t<a_{1,2},\ldots,a_{1,n}\;\Big|\;a_{1,1}=t\Big)\\&=\text{Prob}\Big(t<a_{1,2}\text{ and }\ldots\text{ and }t<a_{1,n}\;\Big|\;a_{1,1}=t\Big)\\&=\text{Prob}\Big(t<a_{1,2}\;\Big|\;a_{1,1}=t\Big)\cdots\text{Prob}\Big(t<a_{1,n}\;\Big|\;a_{1,1}=t\Big)\\&=\Theta(t)^{n-1}\\&=(1-t)^{n-1},\qquad 0<t<1.\end{aligned} \tag{2.2.3}$$

Observe that

$$\mathscr{L}\;\Big|\;a_{1,1}=t\text{ and }\mathscr{A}\;\Big|\;a_{1,1}=t\text{ are independent},\qquad 0<t<1. \tag{2.2.4}$$

Combining our results (2.2.1), (2.2.2), (2.2.3), and (2.2.4) yields

$$\begin{aligned}\text{Prob}\Big(\mathscr{L}\mathscr{C}\mathscr{A}\;\Big|\;a_{1,1}=t\Big)&=\text{Prob}\Big(\mathscr{L}\text{ and }\mathscr{A}\;\Big|\;a_{1,1}=t\Big)\\&=\text{Prob}\Big(\mathscr{L}\;\Big|\;a_{1,1}=t\Big)\text{Prob}\Big(\mathscr{A}\;\Big|\;a_{1,1}=t\Big)\\&=t^{m-1}(1-t)^{n-1},\qquad 0<t<1.\end{aligned} \tag{2.2.5}$$

Substituting (2.2.5) into (2.1.3)

$$S_1(m,n) = mn \int_{t=0}^{1} t^{m-1}(1-t)^{n-1}\,dt. \tag{2.2.6}$$

The gamma function and the beta integral [AAR1999] are given by

$$\Gamma(x) = \int_{t=0}^{\infty} t^{x-1}e^{-t}\,dt, \qquad Re(x) > 0,$$

$$B(x,y) = \int_{t=0}^{1} t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y-1)}, \qquad Re(x) > 0, Re(y) > 0.$$

Note the functional equation

$$\Gamma(x+1) = x\Gamma(x). \tag{2.2.7}$$

Using the functional equation (2.2.7), the fact that $\Gamma(1) = 1$, and the convention $0! = 1$,

$$\Gamma(n) = (n-1)!, \qquad n \geq 0.$$

Thus (2.2.6) becomes

$$\begin{aligned} S_1(m,n) &= mnB(m,n) \\ &= mn\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n-1)} \\ &= \frac{mn\,(m-1)!\,(n-1)!}{(m+n-1)\,(m+n-2)!} \\ &= \frac{m!n!}{(m+n-1)!}, \qquad m,n \geq 1, \end{aligned}$$

as required by Theorem 1.3.2. □

Chapter 3

SADDLE SQUARES

Denote the submatrix

$$A_{k,\ell} = \begin{pmatrix} a_{1,1} & \dots & a_{1,\ell} \\ \vdots & & \vdots \\ a_{k,1} & \dots & a_{k,\ell} \end{pmatrix}, \qquad A \in M_{m,n}(\mathbb{R}), \quad 1 \le k \le m, \quad 1 \le \ell \le n.$$

Observe by Jonasson's Theorem 1.4.1 that

$$\text{if } A \sim UN_{m,n}(0,1), \text{ then wp1 } A \in M^*_{m,n}(0,1).$$

Let

$$r(A) = \left| \left\{ i \in [m] \;\middle|\; p_i^{\text{Opt}} > 0 \right\} \right| = \left| \left\{ j \in [n] \;\middle|\; q_j^{\text{Opt}} > 0 \right\} \right|, \qquad A \in M^*_{m,n}(0,1),$$

where $\widetilde{p}^{\text{Opt}}$ and $\widetilde{q}^{\text{Opt}}$ are the unique optimal strategies of the players.

Following Thorp's alternative proof of Goldman's Theorem 1.3.2 using the gamma function and beta integral [Tho1979], we establish the following representation of $S_k(m,n)$.

$$\begin{aligned} S_k(m,n) &= \binom{m}{k}\binom{n}{k} \text{Prob}\Big(r(A_{k,k}) = k \text{ and } A_{k,k} \text{ remains a } k \text{ by } k \text{ saddle square} \\ &\qquad \Big| \, A \sim UN_{m,n}(0,1) \Big) \\ &= \binom{m}{k}\binom{n}{k} \text{Prob}\Big(r(A_{k,k}) = k \,\Big|\, A \sim UN_{m,n}(0,1) \Big) \text{Prob}\Big(A_{k,k} \text{ remains a} \\ &\qquad k \text{ by } k \text{ saddle square} \,\Big|\, r(A_{k,k}) = k,\, A \sim UN_{m,n}(0,1) \Big) \\ &= \binom{m}{k}\binom{n}{k} S_k(k,k) \text{Prob}\Big(A_{k,k} \text{ remains a } k \text{ by } k \text{ saddle square} \\ &\qquad \Big|\, r(A_{k,k}) = k,\, A \sim UN_{m,n}(0,1) \Big), \qquad 1 \le k \le m,n. \end{aligned} \tag{3.0.1}$$

Given $A \sim UN_{m,n}(0,1)$ and $r(A_{k,k}) = k$, the following three events are equivalent

$$
\begin{aligned}
&E : A_{k,k} \text{ remains a } k \text{ by } k \text{ saddle square},\\
&F : \text{player I rejects his/her pure strategies } k+1,\dots,m, \text{ and}\\
&\quad\;\; \text{player II rejects his/her pure strategies } k+1,\dots,n, \qquad (3.0.2)\\
&G : (a_{i,1},\dots,a_{i,k})\widetilde{q}^{\text{opt}}(A_{k,k})^{\mathsf{T}} < \mathrm{v}(A_{k,k}), \qquad k+1 \le i \le m, \text{ and}\\
&\quad\;\; \widetilde{p}^{\text{opt}}(A_{k,k})(a_{1,j},\dots,a_{k,j})^{\mathsf{T}} > \mathrm{v}(A_{k,k}), \qquad k+1 \le j \le n.
\end{aligned}
$$

Definition 3.0.1.

$$
\begin{aligned}
\Theta(A) &= \mathrm{Prob}\left(\sum_{i=1}^{k} p_i^{\text{Opt}}(A)u_i > \mathrm{v}(A) \;\middle|\; u_1,\dots,u_k \text{ iid un}(0,1)\right),\\
\Phi(A) &= \mathrm{Prob}\left(\sum_{i=1}^{k} q_i^{\text{Opt}}(A)u_i < \mathrm{v}(A) \;\middle|\; u_1,\dots,u_k \text{ iid un}(0,1)\right), \qquad (3.0.3)\\
&\qquad\qquad A \in M^*_{k,k}(0,1).
\end{aligned}
$$

Using the independence of events in G (3.0.2), we reformulate

$$
\begin{aligned}
&\mathrm{Prob}\Big(A_{k,k} \text{ remains a } k \text{ by } k \text{ saddle square} \;\Big|\; A \sim UN_{m,n}(0,1),\ r(A_{k,k}) = k\Big)\\
&= \mathrm{Prob}\Big((a_{k+1,1},\dots,a_{k+1,k})\widetilde{q}^{\text{opt}}(A_{k,k})^{\mathsf{T}} < \mathrm{v}(A_{k,k}) \;\Big|\; A \sim UN_{m,n}(0,1),\ r(A_{k,k}) = k\Big)\cdots\\
&\quad \mathrm{Prob}\Big((a_{m,1},\dots,a_{m,k})\widetilde{q}^{\text{opt}}(A_{k,k})^{\mathsf{T}} < \mathrm{v}(A_{k,k}) \;\Big|\; A \sim UN_{m,n}(0,1),\ r(A_{k,k}) = k\Big)\\
&\times \mathrm{Prob}\Big(\widetilde{p}^{\text{opt}}(A_{k,k})^{\mathsf{T}}(a_{1,k+1},\dots,a_{k,k+1}) > \mathrm{v}(A_{k,k}) \;\Big|\; A \sim UN_{m,n}(0,1),\ r(A_{k,k}) = k\Big)\cdots\\
&\quad \mathrm{Prob}\Big(\widetilde{p}^{\text{opt}}(A_{k,k})^{\mathsf{T}}(a_{1,n},\dots,a_{k,n}) > \mathrm{v}(A_{k,k}) \;\Big|\; A \sim UN_{m,n}(0,1),\ r(A_{k,k}) = k\Big)\\
&= \Theta^{n-k}\big(A_{k,k} \;\big|\; A \sim UN_{m,n}(0,1),\ r(A_{k,k}) = k\big)\\
&\quad \times \Phi^{m-k}\big(A_{k,k} \;\big|\; A \sim UN_{m,n}(0,1),\ r(A_{k,k}) = k\big). \qquad (3.0.4)
\end{aligned}
$$

Combining (3.0.1) and (3.0.4) we obtain

$$
\begin{aligned}
S_k(m,n) &= \binom{m}{k}\binom{n}{k} S_k(k,k)\Theta^{n-k}\big(A_{k,k} \;\big|\; A \sim UN_{m,n}(0,1),\ r(A_{k,k}) = k\big)\\
&\quad \times \Phi^{m-k}\big(A_{k,k} \;\big|\; A \sim UN_{m,n}(0,1),\ r(A_{k,k}) = k\big), \qquad 1 \le k \le m,n. \qquad (3.0.5)
\end{aligned}
$$

Note $\Theta(A)$ and $\Phi(A)$, when conditioned on $A \sim UN_{k,k}(0,1),\ r(A)=k$ as in (3.0.5), are random quantities. Thus, we rewrite $S_k(m,n)$ in terms of the $\mathrm{E}\Big(\Theta^{n-k}(A)\Phi^{m-k}(A)\Big)$, obtaining the first representation

$$S_k(m,n) = \binom{m}{k}\binom{n}{k} S_k(k,k)\mathrm{E}\Big(\Theta^{n-k}(A)\Phi^{m-k}(A)\ \Big|\ A \sim UN_{m,n},\ r(A)=k\Big), \qquad 1 \le k \le m,n. \tag{3.0.6}$$

Expressing $\mathrm{E}\Big(\Theta^{n-k}(A)\Phi^{m-k}(A)\ \Big|\ A \sim UN_{m,n},\ r(A)=k\Big)$ as a k^2 - fold integral

$$\frac{1}{S_k(k,k)} \underset{\substack{A\in M^*_{k,k}(0,1)\\ r(A)=k}}{\int\cdots\int} \Theta^{n-k}(A)\Phi^{m-k}(A)d\mu(A), \tag{3.0.7}$$

where the element of probability, $d\mu(A)$, is the k^2 dimensional Lebesgue measure (see [Roy1988]), and $\frac{1}{S_k(k,k)}$ is the normalization factor, we obtain the second representation

$$S_k(m,n) = \binom{m}{k}\binom{n}{k} \underset{\substack{A\in M^*_{k,k}(0,1)\\ r(A)=k}}{\int\cdots\int} \Theta^{n-k}(A)\Phi^{m-k}(A)d\mu(A). \tag{3.0.8}$$

Chapter 4

$S_2(M,N)$

4.1 Partial Results

The following lemma evaluates $S_2(2,n),\ \ n \geq 2$, and $S_2(m,2),\ \ m \geq 2$.

Lemma 4.1.1. [FT1965]

$$S_2(2,n) = \frac{(n-1)}{(n+1)}, \qquad n \geq 2,$$

$$S_2(m,2) = \frac{(m-1)}{(m+1)}, \qquad m \geq 2.$$

Proof. Let $A \sim UN_{2,n}(0,1)$. By Theorem 1.4.1, we have $1 \leq r(A) \leq 2$ and hence

$$1 = S_1(2,n) + S_2(2,n). \tag{4.1.1}$$

Solving (4.1.1) for $S_2(2,n)$ and using Theorem 1.3.2, we obtain

$$S_2(2,n) = 1 - S_1(2,n) = 1 - \frac{2}{(n+1)} = \frac{(n-1)}{(n+1)}, \qquad n \geq 2.$$

Similarly, we obtain

$$S_2(m,2) = 1 - S_1(m,2) = 1 - \frac{2}{(m+1)} = \frac{(m-1)}{(m+1)}, \qquad m \geq 2,$$

as required by Lemma 4.1.1. □

We use Lemma 4.1.1 to find the marginal distributions of $\Theta(A)$ and $\Phi(A)$, where $A \sim UN_{2,2}(0,1)$ and $r(A) = 2$. Using (3.0.6) with $k = 2$ and $m = 2$

$$S_2(2,n) = \binom{2}{2}\binom{n}{2} S_2(2,2) \mathrm{E}\Big(\Theta^{n-2}(A)\Big) = 1 - \frac{2}{(n+1)} = \frac{(n-1)}{(n+1)}.$$

Thus

$$\mathrm{E}\Big(\Theta^{n-2}(A)\Big) = \frac{(n-1)}{(n+1)}\frac{3}{\binom{n}{2}}$$
$$= \frac{(n-1)}{(n+1)}\frac{3}{\left(\frac{n(n-1)}{2}\right)} \tag{4.1.2}$$
$$= \frac{6}{n(n+1)},$$

Replacing n by $n+2$ and rearranging (4.1.2) we find

$$\mathrm{E}\Big(\Theta^{n}(A)\Big) = \frac{6}{(n+2)(n+3)}, \qquad n \geq 0.$$

Similarly,

$$\mathrm{E}\Big(\Phi^{m}(A)\Big) = \frac{6}{(m+2)(m+3)}, \qquad m \geq 0.$$

Let $X \sim \text{beta}(2,2)$. The density function of X

$$f_X(t) = 6t(1-t), \qquad 0 < t < 1,$$

is obtained by normalizing the beta integral (2.2) with $m = n = 2$. We have the moments

$$\mathrm{E}(X^n) = 6\int_{t=0}^{1} t^{n+1}(1-t)\,dt$$
$$= 6\frac{\Gamma(n+2)\Gamma(2)}{\Gamma(n+4)}$$
$$= \frac{6}{(n+2)(n+3)}, \qquad n \geq 0.$$

By Hausdorff's theorem [Hau1921], we see for $A \sim UN_{2,n}(0,1)$ and $r(A) = 2$

$$\Theta(A) \sim \text{beta}(2,2) \tag{4.1.3}$$

and

$$\Phi(A) \sim \text{beta}(2,2). \tag{4.1.4}$$

Using the marginal distributions of $\Theta(A)$ and $\Phi(A)$ for $A \sim UN_{2,n}(0,1)$, $r(A) = 2$, we find bounds for $S_2(3,3)$. From (4.1.3) and (4.1.4) we have, conditioning on

$A \sim UN_{2,2}(0,1)$ and $r(A) = 2$,

$$\begin{aligned} \mathrm{E}\Big(\Theta(A)\Big) = \mathrm{E}\Big(\Phi(A)\Big) &= \frac{1}{2}, \\ \mathrm{E}\Big(\Theta^2(A)\Big) = \mathrm{E}\Big(\Phi^2(A)\Big) &= \frac{3}{10}, \\ \mathrm{E}\Big(\big(\Theta(A)+\Phi(A)\big)^2\Big) = 1 + \mathrm{Var}\Big(\Theta(A)+\Phi(A)\Big) &> 1. \end{aligned} \tag{4.1.5}$$

Furthermore,

$$\begin{aligned} \mathrm{E}\Big(\big(\Theta(A)+\Phi(A)\big)^2\Big) &= E\Big(\Theta^2(A) + 2\Theta(A)\Phi(A) + \Phi^2(A)\Big) \\ &= E\Big(\Theta^2(A)\Big) + 2E\Big(\Theta(A)\Phi(A)\Big) + E\Big(\Theta^2(A)\Big) \\ &= \frac{3}{10} + 2E\Big(\Theta(A)\Phi(A)\Big) + \frac{3}{10} \\ &= \frac{3}{5} + 2E\Big(\Theta(A)\Phi(A)\Big). \end{aligned} \tag{4.1.6}$$

(4.1.5) and (4.1.6) together imply

$$\frac{1}{5} < E\Big(\Theta(A)\Phi(A)\Big). \tag{4.1.7}$$

Substituting (4.1.7) into (3.0.6) with $k = 2$ and $m = n = 3$, we find

$$\frac{3}{5} < S_2(3,3). \tag{4.1.8}$$

By Theorem 1.4.1,

$$S_1(3,3) + S_2(3,3) + S_3(3,3) = 1. \tag{4.1.9}$$

We solve for an upper bound for $S_2(3,3)$.

$$S_2(3,3) = 1 - S_1(3,3) - S_3(3,3) = 1 - \frac{3}{10} - S_3(3,3) < \frac{7}{10}. \tag{4.1.10}$$

Combining (4.1.8) and (4.1.10) yields bounds

$$\frac{3}{5} < S_2(3,3) < \frac{7}{10}. \tag{4.1.11}$$

Additionally, for purposes in Section 4.5, we compute an upper bound for $\mathrm{Var}\Big(\Theta(A)\Phi(A)\Big)$ conditioning on $A \sim UN_{2,2}(0,1)$ and $r(A) = 2$. From (4.1.7) we have

$$\frac{1}{25} < \bigg(E\Big(\Theta(A)\Phi(A)\Big)\bigg)^2. \tag{4.1.12}$$

Similar to (4.1.9),

$$S_1(4,4) + S_2(4,4) + S_3(4,4) + S_4(4,4) = 1.$$

Solving for $S_2(4,4)$ and using Theorem 1.3.2

$$S_2(4,4) = 1 - S_1(4,4) - S_3(4,4) - S_4(4,4) = 1 - \frac{4}{35} - S_3(4,4) - S_4(4,4) < 31/35,$$

using (3.0.6) with $k = 2$ $m = n = 4$, we have

$$\begin{aligned} \binom{4}{2}\binom{4}{2}\frac{1}{3}E\Big(\Theta(A)^2\Phi(A)^2\Big) &< 31/35, \\ E\Big(\Theta(A)^2\Phi(A)^2\Big) &< \frac{31}{420}. \end{aligned} \tag{4.1.13}$$

Combining (4.1.12) and (4.1.13), we find

$$\operatorname{Var}\Big(\Theta(A)\Phi(A)\Big) = E\Big(\Theta(A)^2\Phi(A)^2\Big) - \Big(E\Big(\Theta(A)\Phi(A)\Big)\Big)^2 < \frac{71}{2100}. \tag{4.1.14}$$

4.2 Saddle Point Conditions

For convenience let $a = a_{1,1}$, $b = a_{1,2}$, $c = a_{2,1}$, and $d = a_{2,2}$ for $A \in M^*_{2,2}(0,1)$:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{4.2.1}$$

Following from Definition 1.3.1 and the proof of Theorem 1.3.2 we see that

$$\begin{pmatrix} a \text{ is a saddle point iff } c < a < b \\ b \text{ is a saddle point iff } d < b < a \\ c \text{ is a saddle point iff } a < c < d \\ d \text{ is a saddle point iff } b < d < c \end{pmatrix}. \tag{4.2.2}$$

Using (4.2.1) and (4.2.2) we obtain

$$r(A) = 2 \text{ iff } \{a,d\} < \{b,c\} \text{ or } \{b,c\} < \{a,d\}.$$

Thus, the exsistence of a saddle point is completely determined by the order statistics of the entries of A. That is, $A \sim UN_{m,n}(0,1)$ has a saddle point iff the matrix of the ranks of the entries of A has a saddle point. Unfortunately, Maple 13 simulation quickly reveals that this fails to give $S_k(m,n)$ when $k \geq 2$ and $m,n \geq 3$.

We let α, β, γ, and δ represent the order statistics of a,b,c, and d by

$$\{a,b,c,d\} = \{\alpha,\beta,\gamma,\delta\}, \quad \alpha \leq \beta \leq \gamma \leq \delta.$$

Given $S = \{\alpha,\beta,\gamma,\delta\}$ with $0 < \alpha < \beta < \gamma < \delta < 1$, there are 4! equally likely possibilities for $A \sim UN_{m,n}(0,1)$ given by

$$\text{OneByOne} = \begin{pmatrix} \begin{pmatrix} \beta & \gamma \\ \alpha & \delta \end{pmatrix}, & \begin{pmatrix} \beta & \delta \\ \alpha & \gamma \end{pmatrix}, & \begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix}, & \begin{pmatrix} \gamma & \delta \\ \beta & \alpha \end{pmatrix} \\ \begin{pmatrix} \gamma & \beta \\ \delta & \alpha \end{pmatrix}, & \begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix}, & \begin{pmatrix} \delta & \gamma \\ \alpha & \beta \end{pmatrix}, & \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix} \\ \begin{pmatrix} \alpha & \delta \\ \beta & \gamma \end{pmatrix}, & \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, & \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, & \begin{pmatrix} \beta & \alpha \\ \gamma & \delta \end{pmatrix} \\ \begin{pmatrix} \delta & \alpha \\ \gamma & \beta \end{pmatrix}, & \begin{pmatrix} \gamma & \alpha \\ \delta & \beta \end{pmatrix}, & \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}, & \begin{pmatrix} \beta & \alpha \\ \delta & \gamma \end{pmatrix} \end{pmatrix}, \tag{4.2.3}$$

$$\text{TwoByTwo} = \begin{pmatrix} \begin{pmatrix} \alpha & \delta \\ \gamma & \beta \end{pmatrix}, & \begin{pmatrix} \delta & \alpha \\ \beta & \gamma \end{pmatrix}, & \begin{pmatrix} \beta & \gamma \\ \delta & \alpha \end{pmatrix}, & \begin{pmatrix} \gamma & \beta \\ \alpha & \delta \end{pmatrix} \\ \begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix}, & \begin{pmatrix} \delta & \beta \\ \alpha & \gamma \end{pmatrix}, & \begin{pmatrix} \beta & \delta \\ \gamma & \alpha \end{pmatrix}, & \begin{pmatrix} \gamma & \alpha \\ \beta & \delta \end{pmatrix} \end{pmatrix}. \tag{4.2.4}$$

4.3 $S_2(m,n)$

Setting $k=2$ in (3.0.8),

$$S_2(m,n) = \binom{m}{2}\binom{n}{2} \iiiint\limits_{\substack{A\in M^*_{2,2}(0,1)\\ r(A)=2}} \Theta^{n-2}(A)\Phi^{m-2}(A)d\mu(A). \tag{4.3.1}$$

where

$$d\mu(A) = d\alpha\, d\beta\, d\gamma\, d\delta.$$

For the multiple integral in (4.3.1) we observe geometrically (Figure 4.1) the following cases for $\Theta(A)$ and $\Phi(A)$, given $A \sim UN_{2,2}(0,1)$ and $r(A)=2$.

Definition 4.3.1.

$$\text{casa}(A) = \begin{pmatrix} 1 & \iff & p_1^{\text{Opt}} > \text{v}(A) \text{ and } p_2^{\text{Opt}} > \text{v}(A) \\ 2 & \iff & p_1^{\text{Opt}} > \text{v}(A) \text{ and } p_2^{\text{Opt}} < \text{v}(A) \\ 3 & \iff & p_1^{\text{Opt}} < \text{v}(A) \text{ and } p_2^{\text{Opt}} > \text{v}(A) \\ 4 & \iff & p_1^{\text{Opt}} < \text{v}(A) \text{ and } p_2^{\text{Opt}} < \text{v}(A) \end{pmatrix}, \tag{4.3.2}$$

$$\text{maison}(A) = \begin{pmatrix} 1 & \iff & q_1^{\text{Opt}} > \text{v}(A), \text{ and } q_2^{\text{Opt}} > \text{v}(A) \\ 2 & \iff & q_1^{\text{Opt}} > \text{v}(A), \text{ and } q_2^{\text{Opt}} < \text{v}(A) \\ 3 & \iff & q_1^{\text{Opt}} < \text{v}(A), \text{ and } q_2^{\text{Opt}} > \text{v}(A) \\ 4 & \iff & q_1^{\text{Opt}} < \text{v}(A), \text{ and } q_2^{\text{Opt}} < \text{v}(A) \end{pmatrix}. \tag{4.3.3}$$

Of the 4^2 cases given above, only 14 are possible since $\text{casa}(A)=1$ and $\text{maison}(A)=4$ gives contradiction

$$p_1^{\text{Opt}} + p_2^{\text{Opt}} > q_1^{\text{Opt}} + q_2^{\text{Opt}} \quad \bigwedge \quad p_1^{\text{Opt}} + p_2^{\text{Opt}} = q_1^{\text{Opt}} + q_2^{\text{Opt}} = 1,$$

and $\text{casa}(A)=4$ and $\text{maison}(A)=1$ also gives contradiction

$$p_1^{\text{Opt}} + p_2^{\text{Opt}} < q_1^{\text{Opt}} + q_2^{\text{Opt}} \quad \bigwedge \quad p_1^{\text{Opt}} + p_2^{\text{Opt}} = q_1^{\text{Opt}} + q_2^{\text{Opt}} = 1.$$

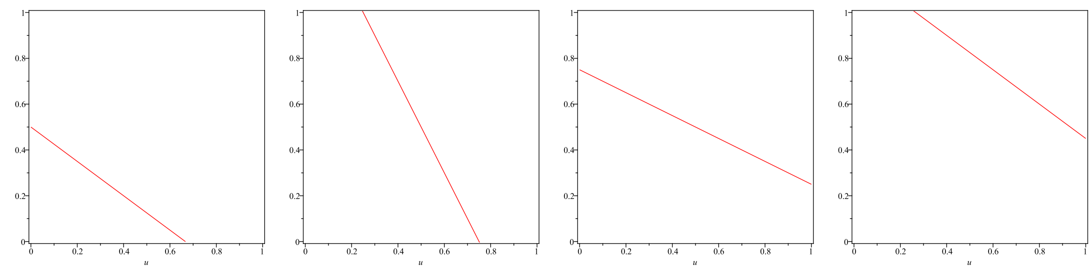

Figure 4.1: For $A \sim UN_{2,2}(0,1)$, $\Theta(A)$ is the area above $p_1^{\text{Opt}} u_1 + p_2^{\text{Opt}} u_2 = \text{v}(A)$ and $\Phi(A)$ is the area below $q_1^{\text{Opt}} u_1 + q_2^{\text{Opt}} u_2 = \text{v}(A)$.

For each representative in (4.2.4), using formulas (B.0.3), (B.0.4), (B.0.5), we can express p_1^{Opt}, p_2^{Opt}, q_1^{Opt}, q_2^{Opt}, and $\text{v}(A)$ in terms of $\alpha, \beta, \gamma, \delta$. Further examination of the expressions with $0 < \alpha < \beta < \gamma < \delta < 1$ reveals only 6 cases of the 14 possible for each representative. Furthermore, observe by the inherent symmetry the set $\{p_1^{\text{Opt}}, p_2^{\text{Opt}}, q_1^{\text{Opt}}, q_2^{\text{Opt}}\}$ together and $\text{v}(A)$ are invarient over the set (4.2.4) of equally likely representatives. More specifically, swapping rows of A will switch p_1^{Opt} and p_2^{Opt}, swapping columns of A will switch q_1^{Opt} and q_2^{Opt}, and swapping rows for columns in A will switch $\{p_1^{\text{Opt}}, p_2^{\text{Opt}}\}$ for $\{q_1^{\text{Opt}}, q_2^{\text{Opt}}\}$. Hence it is sufficient to consider a single representative of (4.2.4) to compute integral (4.3.1).

For the remainder of this Chapter we will consider the following representative $A \in$ TwoByTwo (4.2.4)

$$A = \begin{pmatrix} \alpha & \delta \\ \gamma & \beta \end{pmatrix}, \qquad 0 < \alpha < \beta < \gamma < \delta < 1. \tag{4.3.4}$$

Using formulas (B.0.3), (B.0.4), and (B.0.5) with A in (4.3.4) we have

$$\begin{aligned} p_1^{\text{Opt}} &= \frac{\gamma - \beta}{\delta - \alpha + \gamma - \beta}, & p_2^{\text{Opt}} &= \frac{\delta - \alpha}{\delta - \alpha + \gamma - \beta}, \\ q_1^{\text{Opt}} &= \frac{\delta - \beta}{\delta - \alpha + \gamma - \beta}, & q_2^{\text{Opt}} &= \frac{\gamma - \alpha}{\delta - \alpha + \gamma - \beta}, \\ \text{v}(A) &= \frac{\gamma\delta - \alpha\beta}{\delta - \alpha + \gamma - \beta}. \end{aligned} \tag{4.3.5}$$

Substituting (4.3.5) into Definition 4.3.1 and simplifying the system of inequalities, we find

$$\text{casa}(A) = \text{maison}(A) = 1 \iff \gamma\delta - \alpha\beta < \gamma - \beta \tag{4.3.6}$$

$$\text{casa}(A) = 3,\ \text{maison}(A) = 1 \iff \gamma - \beta < \gamma\delta - \alpha\beta < \min(\delta - \beta, \gamma - \alpha) \tag{4.3.7}$$

$$\text{casa}(A) = 3,\ \text{maison}(A) = 2 \iff \gamma - \alpha < \gamma\delta - \alpha\beta < \delta - \beta \tag{4.3.8}$$

$$\text{casa}(A) = 3,\ \text{maison}(A) = 3 \iff \delta - \beta < \gamma\delta - \alpha\beta < \gamma - \alpha \tag{4.3.9}$$

$$\text{casa}(A) = 3,\ \text{maison}(A) = 4 \iff \max(\delta - \beta, \gamma - \alpha) < \gamma\delta - \alpha\beta < \delta - \alpha \tag{4.3.10}$$

$$\text{casa}(A) = \text{maison}(A) = 4 \iff \delta - \alpha < \gamma\delta - \alpha\beta \tag{4.3.11}$$

Recall the definitions of $\Theta(A)$ and $\Phi(A)$ in (3.0.3). Setting $k = 2$,

$$\begin{aligned} \Theta(A) &= \text{Prob}\left(p_1^{\text{Opt}}(A)u_1 + p_2^{\text{Opt}}(A)u_2 > \text{v}(A) \;\middle|\; u_1, u_2 \text{ iid un}(0,1) \right), \\ \Phi(A) &= \text{Prob}\left(q_1^{\text{Opt}}(A)u_1 + q_2^{\text{Opt}}(A)u_2 < \text{v}(A) \;\middle|\; u_1, u_2 \text{ iid un}(0,1) \right), \end{aligned} \tag{4.3.12}$$

Using simple geometry, (4.3.12), and expressions (4.3.5) we obtain the following formulas for $\Theta(A)$ and $\Phi(A)$ required for the 6 cases of (4.3.6) to (4.3.11).

$$\begin{aligned} \Theta_1(A) &= 1 - \frac{(\gamma\delta - \alpha\beta)^2}{2(\gamma - \beta)(\delta - \alpha)} \\ \Theta_3(A) &= 1 - \frac{2(\gamma\delta - \alpha\beta) - \gamma + \beta}{2\,(\delta - \alpha)} \\ \Theta_4(A) &= \frac{(\gamma - \beta + \delta - \alpha - \gamma\delta + \alpha\beta)^2}{2(\gamma - \beta)(\delta - \alpha)} \\ \Phi_1(A) &= \frac{(\gamma\delta - \alpha\beta)^2}{2(\gamma - \alpha)(\delta - \beta)} \\ \Phi_2(A) &= \frac{2(\gamma\delta - \alpha\beta) - \gamma + \alpha}{2\,(\delta - \beta)} \\ \Phi_3(A) &= \frac{2(\gamma\delta - \alpha\beta) - \delta + \beta}{2(\gamma - \alpha)} \\ \Phi_4(A) &= 1 - \frac{(\gamma - \alpha + \delta - \beta - \gamma\delta + \alpha\beta)^2}{2(\gamma - \alpha)(\delta - \beta)}. \end{aligned} \tag{4.3.13}$$

4.4 Alternative Distributions

In this section, we compute formulas for $\Theta(A)$ and $\Phi(A)$ when $A \sim EX_{2,2}(1)$ and $A \sim N_{2,2}(0,1)$. Although our main focus is $S_2(m,n)$ for $A \sim UN_{2,2}(0,1)$, we investigate the dependence of $\Theta(A)\Phi(A)$ on the chosen distribution of the entries of A.

Let $A \sim EX_{2,2}(1)$ and assume $A \in$ TwoByTwo where

$$A = \begin{pmatrix} \alpha & \delta \\ \gamma & \beta \end{pmatrix}, \qquad 0 < \alpha < \beta < \gamma < \delta < 1.$$

Following (4.3.12)

$$\begin{aligned} \Theta(A) &= \text{Prob}\left(p_1^{\text{Opt}}(A)u_1 + p_2^{\text{Opt}}(A)u_2 > \text{v}(A) \;\middle|\; u_1, u_2 \text{ iid ex}(1) \right), \\ \Phi(A) &= \text{Prob}\left(q_1^{\text{Opt}}(A)u_1 + q_2^{\text{Opt}}(A)u_2 < \text{v}(A) \;\middle|\; u_1, u_2 \text{ iid ex}(1) \right), \end{aligned} \tag{4.4.1}$$

we define the following boundary points for $\Theta(A)$ and $\Phi(A)$

$$\begin{aligned} u &= \frac{v(A)}{p_2^{\text{Opt}}(A)} = \frac{\alpha\delta - \gamma\beta}{\alpha - \beta}, \\ w &= \frac{v(A)}{p_1^{\text{Opt}}(A)} = \frac{\alpha\delta - \gamma\beta}{\delta - \gamma}, \\ s &= \frac{v(A)}{q_2^{\text{Opt}}(A)} = \frac{\alpha\delta - \gamma\beta}{\alpha - \gamma}, \\ t &= \frac{v(A)}{q_1^{\text{Opt}}(A)} = \frac{\alpha\delta - \gamma\beta}{\delta - \beta}. \end{aligned} \tag{4.4.2}$$

With the joint exponential probability density function with parameters $(1,1)$

$$f_{EX}(x,y) = e^{-x}e^{-y} \qquad 0 < x, \quad 0 < y,$$

and (4.4.1) we obtain

$$\begin{aligned}\Theta(A) &= 1 - \int_{x=0}^{w} \int_{y=0}^{u-\frac{u}{w}x} e^{-x}e^{-y}\, dy\, dx \\ &= 1 - \int_{x=0}^{w} e^{x}\left(1 - e^{\frac{u}{w}x-u}\right) dx \\ &= 1 + \frac{e^{(\frac{u}{w}-1)w-u}}{(\frac{u}{w}-1)} - \frac{e^{-u}}{(\frac{u}{w}-1)} + e^{-w} - 1 \\ &= \frac{u}{u-w}e^{-w} - \frac{w}{u-w}e^{-u},\end{aligned} \tag{4.4.3}$$

and

$$\begin{aligned}\Phi(A) &= \int_{x=0}^{t} \int_{y=0}^{s-\frac{s}{t}x} e^{-x}e^{-y}\, dy\, dx \\ &= \int_{x=0}^{t} e^{x}\left(1 - e^{\frac{s}{t}x-s}\right) dx \\ &= -\frac{e^{(\frac{s}{t}-1)t-s}}{(\frac{s}{t}-1)} + \frac{e^{-s}}{(\frac{s}{t}-1)} - e^{-t} + 1 \\ &= \frac{t}{s-t}e^{-s} - \frac{s}{s-t}e^{-t} + 1.\end{aligned} \tag{4.4.4}$$

Again, assuming $A \in \text{TwoByTwo}$ as (4.3.4) and letting $A \sim N_{2,2}(0,1)$, $\Theta(A)$ and $\Phi(A)$ of (4.3.12) become

$$\begin{aligned}\Theta(A) &= \text{Prob}\left(p_1^{\text{Opt}}(A)u_1 + p_2^{\text{Opt}}(A)u_2 > \text{v}(A) \;\middle|\; u_1, u_2 \text{ iid n}(0,1)\right), \\ \Phi(A) &= \text{Prob}\left(q_1^{\text{Opt}}(A)u_1 + q_2^{\text{Opt}}(A)u_2 < \text{v}(A) \;\middle|\; u_1, u_2 \text{ iid n}(0,1)\right).\end{aligned} \tag{4.4.5}$$

Considering the joint standard normal probability density function

$$f_N(x,y) = \frac{e^{\frac{-x^2}{2}}e^{\frac{-y^2}{2}}}{2\pi} \qquad -\infty < x < \infty, \quad -\infty < y < \infty,$$

we have the following two cases.

$v(A) > 0:$

Using the same boundary points of (4.4.2) we find orthogonal distances to lines $ux + wy = uw$ and $sx + ty = st$,

$$\begin{aligned} h_\Theta &= \frac{uw}{\sqrt{u^2+w^2}}, \\ h_\Phi &= \frac{st}{\sqrt{s^2+t^2}}. \end{aligned} \tag{4.4.6}$$

Hence by the symmetry of the joint standard normal distribution

$$\begin{aligned} \Theta(A) &= 1 - \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{h_\Theta} \frac{e^{\frac{-x^2}{2}} e^{\frac{-y^2}{2}}}{2\pi} \, dy \, dx \\ &= 1 - \int_{x=-\infty}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \Big(F_N(h_\Theta) \Big) \, dx \\ &= 1 - F_N(h_\Theta) \int_{x=-\infty}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \, dx \\ &= 1 - F_N(h_\Theta), \end{aligned} \tag{4.4.7}$$

and

$$\begin{aligned} \Phi(A) &= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{h_\Phi} \frac{e^{\frac{-x^2}{2}} e^{\frac{-y^2}{2}}}{2\pi} \, dy \, dx \\ &= \int_{x=-\infty}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \Big(F_N(h_\Phi) \Big) \, dx \\ &= F_N(h_\Phi) \int_{x=-\infty}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \, dx \\ &= F_N(h_\Phi). \end{aligned} \tag{4.4.8}$$

where $F_N(x)$ is the standard normal cumulative distribution function.

$v(A) < 0$:

Following from (4.4.6), (4.4.7), and (4.4.8)

$$\Theta(A) = 1 - F_N(-h_\Theta) \tag{4.4.9}$$

and

$$\Phi(A) = F_N(-h_\Phi). \tag{4.4.10}$$

4.5 Confidence Intervals

Monte-Carlo simulations using Maple 13 were performed with payoff matrix entries distributed as iid un(0,1), ex(1), and n(0,1). Note from Chapters 1 and 2 we see Theorem 1.3.2 and Theorem 1.4.1 hold as long as the entries of A are iid rvs from some continuous distribution. Thus Lemma 4.1.1, the subsequent results (4.1.3) and (4.1.4), and consequently the bounds for $S_2(3,3)$ (4.1.11) and $\mathrm{Var}\Big(\Theta(A)\Phi(A)\Big)$ (4.1.14) remain for $\Theta(A)$ and $\Phi(A)$ defined in (4.4.1) and (4.4.5). Hence we construct large sample normal approximation 99% confidence intervals for $E\big(\Theta(A)\Phi(A)\big)$ by [Leh1999]

$$\left(\overline{\Theta(A)\Phi(A)} - z_{0.005}\frac{S}{\sqrt{n}}\,,\ \overline{\Theta(A)\Phi(A)} + z_{0.005}\frac{S}{\sqrt{n}}\right),$$

where

$$\overline{\Theta(A)\Phi(A)} = \frac{1}{n}\sum^{n}\Theta(A)\Phi(A),$$

$$S = \sqrt{\frac{1}{n-1}\sum^{n}\Big(\Theta(A)\Phi(A) - \overline{\Theta(A)\Phi(A)}\Big)^2},$$

$$X \sim N(0,1), \qquad \mathrm{Prob}\big(X < z_{0.005}\big) = 0.995.$$

Large samples of 10^8 observations were used to compute the following 99% confidence intervals

$$A \sim UN_{2,2}(0,1),\ \Theta(A)\Phi(A):\quad (0.2006010062,\ 0.2006295257), \tag{4.5.1}$$

$$A \sim EX_{2,2}(1),\ \Theta(A)\Phi(A):\quad (0.2002027093,\ 0.2002312857), \tag{4.5.2}$$

$$A \sim N_{2,2}(0,1),\ \Theta(A)\Phi(A):\quad (0.2002259383,\ 0.2002538277). \tag{4.5.3}$$

Note the interval for $A \sim UN_{2,2}(0,1)$ is disjoint of the other two. We are 99% confident that $E\big(\Theta(A)\Phi(A)\big)$ is dependent on the distribution of the entries of A. Additional comparisons can be made with the following plots (Figures 4.2 to 4.4) of the support boundary of the joint distribution of $\Theta(A)$ and $\Phi(A)$ for each of the underlying distributions.

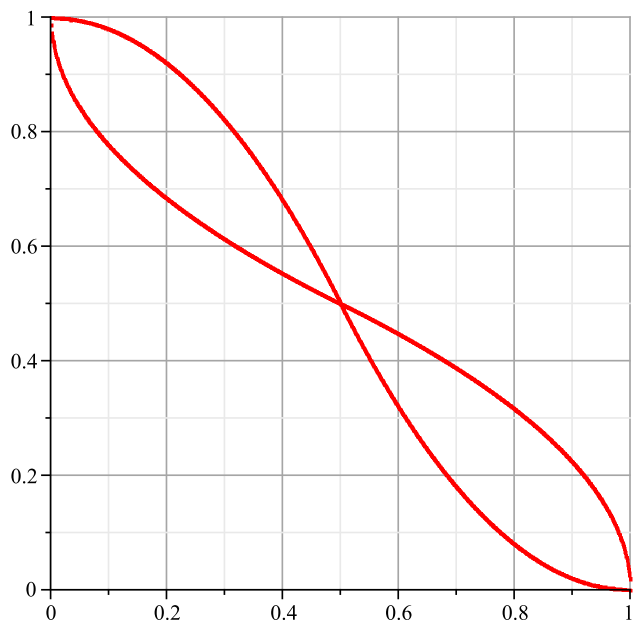

Figure 4.2: Support boundary for joint distribution of $\Theta(A)$ and $\Phi(A)$ for $A \sim UN_{2,2}(0,1)$.

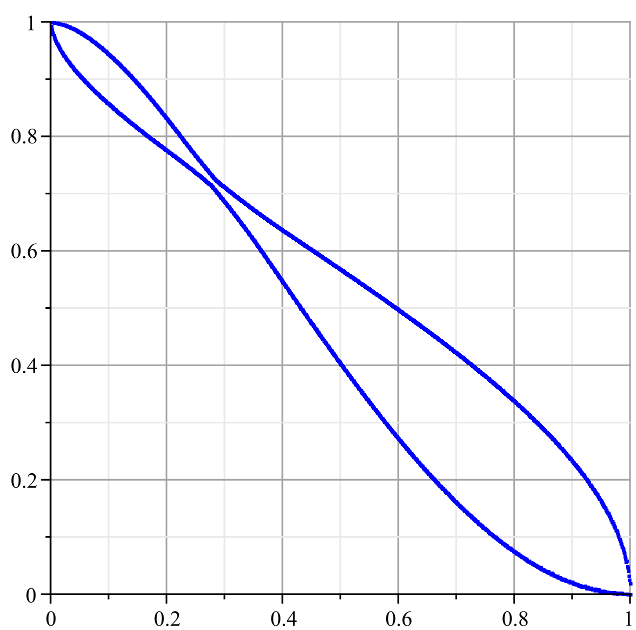

Figure 4.3: Support boundary for joint distribution of $\Theta(A)$ and $\Phi(A)$ for $A \sim EX_{2,2}(1)$.

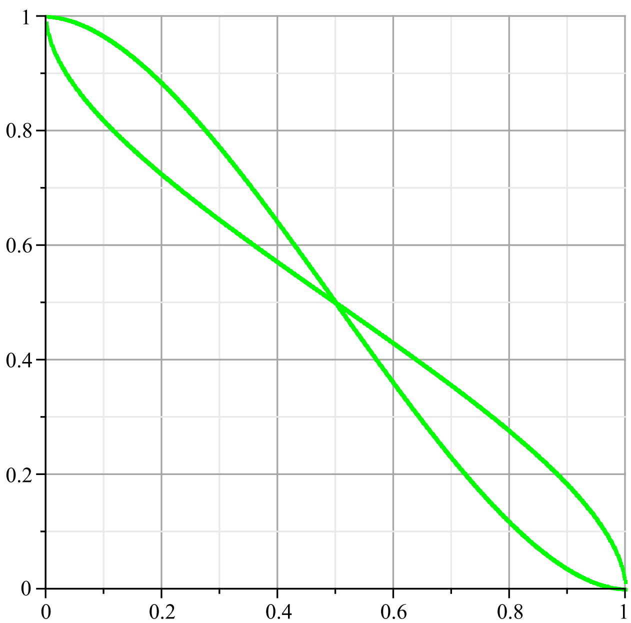

Figure 4.4: Support boundary for joint distribution of $\Theta(A)$ and $\Phi(A)$ for $A \sim N_{2,2}(0,1)$.

Chapter 5

COMPUTATIONAL PRELIMINARIES

5.1 A Transformation

We begin with defining the following integrals with payoff matrix representative $A \in$ TwoByTwo (4.3.4) for $A \sim UN_{2,2}(0,1)$ and the six regions (4.3.6) to (4.3.11),

$$\mathscr{A}(m,n) = \iiiint\limits_{\substack{A=\begin{pmatrix}\alpha & \delta\\ \gamma & \beta\end{pmatrix}\\ 0<\alpha<\beta<\gamma<\delta<1\\ \text{casa}(A)=\text{maison}(A)=1}} \Theta_1(A)^m \Phi_1(A)^n \, d\alpha\, d\beta\, d\gamma\, d\delta, \tag{5.1.1}$$

$$\mathscr{B}(m,n) = \iiiint\limits_{\substack{A=\begin{pmatrix}\alpha & \delta\\ \gamma & \beta\end{pmatrix}\\ 0<\alpha<\beta<\gamma<\delta<1\\ \text{casa}(A)=3,\ \text{maison}(A)=1}} \Theta_3(A)^m \Phi_1(A)^n \, d\alpha\, d\beta\, d\gamma\, d\delta, \tag{5.1.2}$$

$$\mathscr{C}(m,n) = \iiiint\limits_{\substack{A=\begin{pmatrix}\alpha & \delta\\ \gamma & \beta\end{pmatrix}\\ 0<\alpha<\beta<\gamma<\delta<1\\ \text{casa}(A)=3,\ \text{maison}(A)=2}} \Theta_3(A)^m \Phi_2(A)^n \, d\alpha\, d\beta\, d\gamma\, d\delta, \tag{5.1.3}$$

$$\mathscr{D}(m,n) = \iiiint\limits_{\substack{A=\begin{pmatrix}\alpha & \delta\\ \gamma & \beta\end{pmatrix}\\ 0<\alpha<\beta<\gamma<\delta<1\\ \text{casa}(A)=3,\ \text{maison}(A)=3}} \Theta_3(A)^m \Phi_3(A)^n \, d\alpha\, d\beta\, d\gamma\, d\delta, \tag{5.1.4}$$

$$\mathscr{E}(m,n) = \iiiint\limits_{\substack{A=\begin{pmatrix}\alpha & \delta\\ \gamma & \beta\end{pmatrix}\\ 0<\alpha<\beta<\gamma<\delta<1\\ \text{casa}(A)=3,\ \text{maison}(A)=4}} \Theta_3(A)^m \Phi_4(A)^n \, d\alpha\, d\beta\, d\gamma\, d\delta, \tag{5.1.5}$$

$$\mathscr{F}(m,n) = \iiiint\limits_{\substack{A=\begin{pmatrix}\alpha & \delta\\ \gamma & \beta\end{pmatrix}\\ 0<\alpha<\beta<\gamma<\delta<1\\ \text{casa}(A)=4,\ \text{maison}(A)=4}} \Theta_4(A)^m \Phi_4(A)^n \, d\alpha\, d\beta\, d\gamma\, d\delta. \tag{5.1.6}$$

Consider the bijective transformation

$$T : (a,b,c,d,) \longrightarrow (e,f,g,h) \quad \text{by}$$

$$e = 1-d,$$

$$f = 1-c,$$

$$g = 1-b,$$

$$h = 1-a.$$

Transformation T preserves uniform$(0,1)$ iid rvs. If

$$a,b,c,d \overset{\text{iid}}{\sim} un(0,1)$$

then

$$h,g,f,e \overset{\text{iid}}{\sim} un(0,1).$$

Furthermore

$$a < b < c < d \Longrightarrow e < f < g < h.$$

Recall the six cases (4.3.6) to (4.3.11) of the representative payoff matrix in TwoByTwo (4.3.4). Substituting $\alpha = 1-\theta$, $\beta = 1-\eta$, $\gamma = 1-\zeta$, and $\delta = 1-\varepsilon$ we find

$$0 < \alpha < \beta < \gamma < \delta < 1 \overset{T}{\Longrightarrow} 0 < \varepsilon < \zeta < \eta < \theta < 1, \tag{5.1.7}$$

$$\gamma\delta - \alpha\beta < \gamma - \beta \overset{T}{\Longrightarrow} \theta - \varepsilon < \eta\theta - \varepsilon\zeta,$$
$$\gamma - \beta < \gamma\delta - \alpha\beta < \min(\delta-\beta, \gamma-\alpha) \overset{T}{\Longrightarrow} \max(\theta-\zeta, \eta-\varepsilon) < \eta\theta - \varepsilon\zeta < \theta - \varepsilon,$$
$$\gamma - \alpha < \gamma\delta - \alpha\beta < \delta - \beta \overset{T}{\Longrightarrow} \eta - \varepsilon < \eta\theta - \varepsilon\zeta < \theta - \zeta. \tag{5.1.8}$$

Moreover,

$$
\begin{aligned}
\Theta_1(A) &= 1-\frac{(\gamma\delta-\alpha\beta)^2}{2(\gamma-\beta)(\delta-\alpha)} \overset{T}{\Longrightarrow} 1-\frac{(\theta-\zeta-(\eta\theta-\varepsilon\zeta)+\eta-\varepsilon)^2}{2(\eta-\zeta)(\theta-\varepsilon)} = 1-\Theta_4(A),\\
\Theta_3(A) &= 1-\frac{2(\gamma\delta-\alpha\beta)-\gamma+\beta}{2\,(\delta-\alpha)} \overset{T}{\Longrightarrow} \frac{(\eta\theta-\varepsilon\zeta)}{(\theta-\varepsilon)}-\frac{(\eta-\zeta)}{2(\theta-\varepsilon)} = 1-\Theta_3(A),\\
\Phi_2(A) &= \frac{2(\gamma\delta-\alpha\beta)-\gamma+\alpha}{2(\delta-\beta)} \overset{T}{\Longrightarrow} 1-\frac{2(\eta\theta-\varepsilon\zeta)-\theta+\zeta}{2(\eta-\varepsilon)} = 1-\Phi_3(A),\\
\Phi_1(A) &= \frac{(\gamma\delta-\alpha\beta)^2}{2(\gamma-\alpha)(\delta-\beta)} \overset{T}{\Longrightarrow} \frac{(\theta-\zeta-(\eta\theta-\varepsilon\zeta)+\eta-\varepsilon)^2}{2(\eta-\varepsilon)(\theta-\zeta)} = \Phi_4(A).
\end{aligned}
\tag{5.1.9}
$$

The Jacobian determinant of T is

$$
\begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} = 1. \tag{5.1.10}
$$

Observe by setting $m=n=1$ in (5.1.1) we have, following (5.1.8), (5.1.9), (5.1.10), a change of variables by T

$$
\begin{aligned}
\mathscr{A}(1,1) &= \iiiint\limits_{\substack{A=\begin{pmatrix}\alpha & \delta\\ \gamma & \beta\end{pmatrix}\\ 0<\alpha<\beta<\gamma<\delta<1\\ \text{Casa}(A)=\text{Maison}(A)=1}} \Theta_1(A)\Phi_1(A)\, d\alpha\, d\beta\, d\gamma\, d\delta \\
&= \iiiint\limits_{\substack{A=\begin{pmatrix}\alpha & \delta\\ \gamma & \beta\end{pmatrix}\\ 0<\alpha<\beta<\gamma<\delta<1\\ \text{Casa}(A)=\text{Maison}(A)=4}} \Big(1-\Theta_4(A)\Big)\Big(1-\Phi_4(A)\Big)\, d\alpha\, d\beta\, d\gamma\, d\delta.
\end{aligned}
$$

Expanding the product $(1-\Theta_4(A))(1-\Phi_4(A))$ and solving we obtain

$$\begin{aligned}
\mathscr{F}(1,1) &= \iiiint\limits_{\substack{A=\begin{pmatrix}\alpha & \delta\\ \gamma & \beta\end{pmatrix}\\ 0<\alpha<\beta<\gamma<\delta<1\\ \mathrm{Casa}(A)=\mathrm{Maison}(A)=4}} \Theta_4(A)\Phi_4(A)\, d\alpha\, d\beta\, d\gamma\, d\delta \\
&= \iiiint\limits_{\substack{A=\begin{pmatrix}\alpha & \delta\\ \gamma & \beta\end{pmatrix}\\ 0<\alpha<\beta<\gamma<\delta<1\\ \mathrm{Casa}(A)=\mathrm{Maison}(A)=1}} \Theta_1(A)\Phi_1(A)\, d\alpha\, d\beta\, d\gamma\, d\delta \\
&\qquad - \iiiint\limits_{\substack{A=\begin{pmatrix}\alpha & \delta\\ \gamma & \beta\end{pmatrix}\\ 0<\alpha<\beta<\gamma<\delta<1\\ \mathrm{Casa}(A)=\mathrm{Maison}(A)=1}} \Theta_1(A)\, d\alpha\, d\beta\, d\gamma\, d\delta \\
&\qquad - \iiiint\limits_{\substack{A=\begin{pmatrix}\alpha & \delta\\ \gamma & \beta\end{pmatrix}\\ 0<\alpha<\beta<\gamma<\delta<1\\ \mathrm{Casa}(A)=\mathrm{Maison}(A)=1}} \Phi_1(A)\, d\alpha\, d\beta\, d\gamma\, d\delta \\
&\qquad + \iiiint\limits_{\substack{A=\begin{pmatrix}\alpha & \delta\\ \gamma & \beta\end{pmatrix}\\ 0<\alpha<\beta<\gamma<\delta<1\\ \mathrm{Casa}(A)=\mathrm{Maison}(A)=1}} d\alpha\, d\beta\, d\gamma\, d\delta \\
&= \mathscr{A}(1,1)-\mathscr{A}(1,0)-\mathscr{A}(0,1)+\mathscr{A}(0,0).
\end{aligned} \tag{5.1.11}$$

Expressions for $\mathscr{D}(1,1)$ and $\mathscr{E}(1,1)$ follow similarly.

$$\begin{aligned}
\mathscr{D}(1,1) &= \mathscr{C}(1,1)-\mathscr{C}(1,0)-\mathscr{C}(0,1)+\mathscr{C}(0,0),\\
\mathscr{E}(1,1) &= \mathscr{B}(1,1)-\mathscr{B}(1,0)-\mathscr{B}(0,1)+\mathscr{B}(0,0).
\end{aligned} \tag{5.1.12}$$

We now turn our focus to computing $S_2(3,3)$. From (4.3.1) with $m=n=3$ we have the integral

$$\iiiint\limits_{\substack{A\in M^*_{2,2}(0,1)\\ r(A)=2}} \Theta(A)\Phi(A)\, d\alpha\, d\beta\, d\gamma\, d\delta. \tag{5.1.13}$$

Following from Chapter 4, representative (4.3.4), definitions (5.1.1) to (5.1.6), and results (5.1.11) and (5.1.12) we obtain

$$\begin{aligned}
\iiiint\limits_{\substack{A\in M^*_{2,2}(0,1)\\ r(A)=2}} \Theta(A)\Phi(A)\, d\alpha\, d\beta\, d\gamma\, d\delta &= 8 \iiiint\limits_{\substack{A=\begin{pmatrix}\alpha & \delta\\ \gamma & \beta\end{pmatrix}\\ 0<\alpha<\beta<\gamma<\delta<1}} \Theta(A)^m \Phi(A)^n\, d\gamma\, d\beta\, d\delta\, d\alpha \\
&= 8\Big(\mathscr{A}(1,1)+\mathscr{B}(1,1)+\mathscr{C}(1,1) \\
&\quad +\mathscr{D}(1,1)+\mathscr{E}(1,1)+\mathscr{F}(1,1)\Big) \\
&= 8\Big(2\mathscr{A}(1,1)-\mathscr{A}(1,0)-\mathscr{A}(0,1)+\mathscr{A}(0,0) \\
&\quad +2\mathscr{B}(1,1)-\mathscr{B}(1,0)-\mathscr{B}(0,1)+\mathscr{B}(0,0) \\
&\quad +2\mathscr{C}(1,1)-\mathscr{C}(1,0)-\mathscr{C}(0,1)+\mathscr{C}(0,0)\Big).
\end{aligned} \tag{5.1.14}$$

5.2 Integration Limits

In order to compute the integtrals of (5.1.14), we need to express the regions defined in cases (4.3.6), (4.3.7), and (4.3.8) as limits of integration. Starting with casa$(A) =$ maison$(A) = 1$ we have

$$\gamma\delta - \alpha\beta < \gamma - \beta \quad \text{with} \;\; 0 < \alpha < \beta < \gamma < \delta < 1. \tag{5.2.1}$$

To avoid quadratic solutions we choose the order of integration, from interior to exterior, to be γ, β, δ, α. Solving the inequalites (5.2.1) for γ,

$$\frac{\beta(1-\alpha)}{1-\delta} < \gamma, \quad \beta < \gamma, \quad \gamma < \delta.$$

Since

$$\beta < \frac{\beta(1-\alpha)}{1-\delta},$$

then we find limits for γ

$$\frac{\beta(1-\alpha)}{1-\delta} < \gamma < \delta. \tag{5.2.2}$$

Continuing to β, the remaining inequalties from (5.2.2) and (5.2.3) are

$$\beta < \frac{\beta(1-\alpha)}{1-\delta} < \delta, \quad \alpha < \beta < \delta.$$

Solving for β yields

$$\beta < \frac{\delta(1-\delta)}{1-\alpha}, \quad \alpha < \beta, \quad \beta < \delta.$$

Again it is clear that

$$\frac{\delta(1-\delta)}{1-\alpha} < \delta,$$

and hence we find the limits for β

$$\alpha < \beta < \frac{\delta(1-\delta)}{1-\alpha}. \tag{5.2.3}$$

For δ the remaining inequalities are

$$\alpha < \frac{\delta(1-\delta)}{1-\alpha} < \delta, \quad \delta < 1.$$

Solving we find

$$\alpha(1-\alpha) < \delta(1-\delta), \quad \alpha < \delta, \quad \delta < 1. \tag{5.2.4}$$

Examining (5.2.4) reveals limits for δ

$$\alpha < \delta < 1-\alpha, \tag{5.2.5}$$

and α

$$0 < \alpha < \frac{1}{2}. \tag{5.2.6}$$

Combining (5.2.2), (5.2.3), (5.2.5), and (5.2.6) we see (5.2.1) is equivalent to

$$\begin{array}{c} \frac{\beta(1-\alpha)}{1-\delta} < \gamma < \delta \\ \alpha < \beta < \frac{\delta(1-\delta)}{1-\alpha} \\ \alpha < \delta < 1-\alpha \\ 0 < \alpha < \frac{1}{2} \end{array}. \tag{5.2.7}$$

For $\text{casa}(A) = 3$, $\text{maison}(A) = 1$, the system of inequalties becomes substantially more complicated. We outline one iteration of the systematic technique used to find the limits of integration. Initial investigation indicates the integration order γ, β, α, δ, is best. From (4.3.8) we have

$$\gamma-\beta<\gamma\delta-\alpha\beta<\min(\delta-\beta,\gamma-\alpha) \text{ with } 0<\alpha<\beta<\gamma<\delta<1.$$

To begin solving we first recognize the following two cases

$$\gamma-\beta<\gamma\delta-\alpha\beta<\delta-\beta, \quad \delta-\beta<\gamma-\alpha \text{ with } 0<\alpha<\beta<\gamma<\delta<1, \quad (5.2.8)$$

$$\gamma-\beta<\gamma\delta-\alpha\beta<\gamma-\alpha, \quad \gamma-\alpha<\delta-\beta \text{ with } 0<\alpha<\beta<\gamma<\delta<1. \quad (5.2.9)$$

Solving (5.2.8) for γ we find

$$\gamma<\frac{\beta(1-\alpha)}{1-\delta}, \quad \gamma<\frac{\delta-\beta+\alpha\beta}{\delta}, \quad \delta-\beta+\alpha<\gamma, \quad \beta<\gamma, \quad \gamma<\delta. \quad (5.2.10)$$

Considering all possible orderings in (5.2.10), there initially appears to be $2!\cdot 3!$ distinct limits for γ. Note the contradiction

$$\left(\delta<\frac{\beta(1-\alpha)}{1-\delta} \text{ and } \delta<\frac{\delta-\beta+\alpha\beta}{\delta}\right) \text{ or } \left(\frac{\beta(1-\alpha)}{1-\delta}<\delta \text{ and } \frac{\delta-\beta+\alpha\beta}{\delta}<\delta\right)$$
$$\implies \quad \Big(\delta(1-\delta)<\beta(1-\alpha) \text{ and } \beta(1-\alpha)<\delta(1-\delta)\Big).$$

Hence we have the following 4 orderings for (5.2.10), verified by simulation:

$$\beta<\delta-\beta+\alpha<\gamma<\frac{\beta(1-\alpha)}{1-\delta}<\delta<\frac{\delta-\beta+\alpha\beta}{\delta}, \quad (5.2.11)$$

$$\beta<\delta-\beta+\alpha<\gamma<\frac{\delta-\beta+\alpha\beta}{\delta}<\delta<\frac{\beta(1-\alpha)}{1-\delta}, \quad (5.2.12)$$

$$\delta-\beta+\alpha<\beta<\gamma<\frac{\beta(1-\alpha)}{1-\delta}<\delta<\frac{\delta-\beta+\alpha\beta}{\delta}, \quad (5.2.13)$$

$$\delta-\beta+\alpha<\beta<\gamma<\frac{\delta-\beta+\alpha\beta}{\delta}<\delta<\frac{\beta(1-\alpha)}{1-\delta}, \quad (5.2.14)$$

giving the following limits for γ

$$\delta-\beta+\alpha<\gamma<\frac{\beta(1-\alpha)}{1-\delta},$$
$$\delta-\beta+\alpha<\gamma<\frac{\delta-\beta+\alpha\beta}{\delta},$$
$$\beta<\gamma<\frac{\beta(1-\alpha)}{1-\delta},$$
$$\beta<\gamma<\frac{\delta-\beta+\alpha\beta}{\delta}.$$

We then proceed to β repeating the process for each system (5.2.11) to (5.2.14) with $\alpha < \beta < \delta$.

Completion of the process outlined above yields limits for (5.2.8)

$$
\begin{array}{ccc}
\begin{array}{c}
\delta-\beta+\alpha<\gamma<\frac{\beta(1-\alpha)}{1-\delta} \\
\frac{(\delta+\alpha)(1-\delta)}{2-\alpha+\delta}<\beta<\frac{\delta+\alpha}{2} \\
0<\alpha<1-2\delta \\
\frac{1}{3}<\delta<\frac{1}{2}
\end{array}, &
\begin{array}{c}
\beta<\gamma<\frac{\beta(1-\alpha)}{1-\delta} \\
\frac{\delta+\alpha}{2}<\beta<\frac{\delta(1-\delta)}{1-\alpha} \\
0<\alpha<1-2\delta \\
\frac{1}{3}<\delta<\frac{1}{2}
\end{array}, &
\begin{array}{c}
\beta<\gamma<\frac{\delta-\beta+\alpha\beta}{\delta} \\
\frac{\delta(1-\delta)}{1-\alpha}<\beta<\frac{\delta}{1-\alpha+\delta} \\
0<\alpha<1-2\delta \\
\frac{1}{3}<\delta<\frac{1}{2}
\end{array}, \\
\begin{array}{c}
\delta-\beta+\alpha<\gamma<\frac{\beta(1-\alpha)}{1-\delta} \\
\frac{(\delta+\alpha)(1-\delta)}{2-\alpha+\delta}<\beta<\frac{\delta+\alpha}{2} \\
0<\alpha<\delta \\
0<\delta<\frac{1}{3}
\end{array}, &
\begin{array}{c}
\beta<\gamma<\frac{\beta(1-\alpha)}{1-\delta} \\
\frac{\delta+\alpha}{2}<\beta<\frac{\delta(1-\delta)}{1-\alpha} \\
0<\alpha<\delta \\
0<\delta<\frac{1}{3}
\end{array}, &
\begin{array}{c}
\beta<\gamma<\frac{\delta-\beta+\alpha\beta}{\delta} \\
\frac{\delta(1-\delta)}{1-\alpha}<\beta<\frac{\delta}{1-\alpha+\delta} \\
0<\alpha<\delta \\
0<\delta<\frac{1}{3}
\end{array}, \\
\begin{array}{c}
\delta-\beta+\alpha<\gamma<\frac{\beta(1-\alpha)}{1-\delta} \\
\frac{(\delta+\alpha)(1-\delta)}{2-\alpha+\delta}<\beta<\frac{\delta(1-\delta)}{1-\alpha} \\
1-2\delta<\alpha<\delta \\
\frac{1}{3}<\delta<\frac{1}{2}
\end{array}, &
\begin{array}{c}
\delta-\beta+\alpha<\gamma<\frac{\delta-\beta+\alpha\beta}{\delta} \\
\frac{\delta(1-\delta)}{1-\alpha}<\beta<\frac{\delta+\alpha}{2} \\
1-2\delta<\alpha<\delta \\
\frac{1}{3}<\delta<\frac{1}{2}
\end{array}, &
\begin{array}{c}
\beta<\gamma<\frac{\delta-\beta+\alpha\beta}{\delta} \\
\frac{\delta+\alpha}{2}<\beta<\frac{\delta}{1-\alpha+\delta} \\
1-2\delta<\alpha<\delta \\
\frac{1}{3}<\delta<\frac{1}{2}
\end{array}, \\
\begin{array}{c}
\delta-\beta+\alpha<\gamma<\frac{\beta(1-\alpha)}{1-\delta} \\
\frac{(\delta+\alpha)(1-\delta)}{2-\alpha+\delta}<\beta<\frac{\delta(1-\delta)}{1-\alpha} \\
0<\alpha<1-\delta \\
\frac{1}{2}<\delta<1
\end{array}, &
\begin{array}{c}
\delta-\beta+\alpha<\gamma<\frac{\delta-\beta+\alpha\beta}{\delta} \\
\frac{\delta(1-\delta)}{1-\alpha}<\beta<\frac{\delta+\alpha}{2} \\
0<\alpha<1-\delta \\
\frac{1}{2}<\delta<1
\end{array}, &
\begin{array}{c}
\beta<\gamma<\frac{\delta-\beta+\alpha\beta}{\delta} \\
\frac{\delta+\alpha}{2}<\beta<\frac{\delta}{1-\alpha+\delta} \\
0<\alpha<1-\delta \\
\frac{1}{2}<\delta<1
\end{array}.
\end{array}
\tag{5.2.15}
$$

Similarly for (5.2.9) we find limits

$$\begin{array}{ccc}
\begin{array}{c}\frac{\alpha(1-\beta)}{1-\delta}<\gamma<\frac{\beta(1-\alpha)}{1-\delta}\\ \alpha<\beta<\frac{\alpha}{1-\delta+\alpha}\\ 0<\alpha<\frac{(1-\delta)^2}{2-\delta}\\ \frac{2-\sqrt{2}}{2}<\delta<1\end{array}, &
\begin{array}{c}\beta<\gamma<\frac{\beta(1-\alpha)}{1-\delta}\\ \frac{\alpha}{1-\delta+\alpha}<\beta<\frac{(\delta+\alpha)(1-\delta)}{2-\alpha+\delta}\\ 0<\alpha<\frac{(1-\delta)^2}{2-\delta}\\ \frac{2-\sqrt{2}}{2}<\delta<1\end{array}, &
\begin{array}{c}\beta<\gamma<\delta-\beta+\alpha\\ \frac{(\delta+\alpha)(1-\delta)}{2-\alpha+\delta}<\beta<\frac{\delta+\alpha}{2}\\ 0<\alpha<\frac{(1-\delta)^2}{2-\delta}\\ \frac{2-\sqrt{2}}{2}<\delta<1\end{array}, \\
\begin{array}{c}\frac{\alpha(1-\beta)}{1-\delta}<\gamma<\frac{\beta(1-\alpha)}{1-\delta}\\ \alpha<\beta<\frac{\alpha}{1-\delta+\alpha}\\ 0<\alpha<\delta\\ 0<\delta<\frac{2-\sqrt{2}}{2}\end{array}, &
\begin{array}{c}\beta<\gamma<\frac{\beta(1-\alpha)}{1-\delta}\\ \frac{\alpha}{1-\delta+\alpha}<\beta<\frac{(\delta+\alpha)(1-\delta)}{2-\alpha+\delta}\\ 0<\alpha<\delta\\ 0<\delta<\frac{2-\sqrt{2}}{2}\end{array}, &
\begin{array}{c}\beta<\gamma<\delta-\beta+\alpha\\ \frac{(\delta+\alpha)(1-\delta)}{2-\alpha+\delta}<\beta<\frac{\delta+\alpha}{2}\\ 0<\alpha<\delta\\ 0<\delta<\frac{2-\sqrt{2}}{2}\end{array}, \\
\begin{array}{c}\frac{\alpha(1-\beta)}{1-\delta}<\gamma<\frac{\beta(1-\alpha)}{1-\delta}\\ \alpha<\beta<\frac{(\delta+\alpha)(1-\delta)}{2-\alpha+\delta}\\ \frac{(1-\delta)^2}{2-\delta}<\alpha<\delta\\ \frac{2-\sqrt{2}}{2}<\delta<\frac{3-\sqrt{5}}{2}\end{array}, &
\begin{array}{c}\frac{\alpha(1-\beta)}{1-\delta}<\gamma<\delta-\beta+\alpha\\ \frac{(\delta+\alpha)(1-\delta)}{2-\alpha+\delta}<\beta<\frac{\alpha}{1-\delta+\alpha}\\ \frac{(1-\delta)^2}{2-\delta}<\alpha<\delta\\ \frac{2-\sqrt{2}}{2}<\delta<\frac{3-\sqrt{5}}{2}\end{array}, &
\begin{array}{c}\beta<\gamma<\delta-\beta+\alpha\\ \frac{\alpha}{1-\delta+\alpha}<\beta<\frac{\delta+\alpha}{2}\\ \frac{(1-\delta)^2}{2-\delta}<\alpha<\delta\\ \frac{2-\sqrt{2}}{2}<\delta<\frac{3-\sqrt{5}}{2}\end{array}, \\
\begin{array}{c}\frac{\alpha(1-\beta)}{1-\delta}<\gamma<\frac{\beta(1-\alpha)}{1-\delta}\\ \alpha<\beta<\frac{(\delta+\alpha)(1-\delta)}{2-\alpha+\delta}\\ \frac{(1-\delta)^2}{2-\delta}<\alpha<(1-\delta)^2\\ \frac{3-\sqrt{5}}{2}<\delta<1\end{array}, &
\begin{array}{c}\frac{\alpha(1-\beta)}{1-\delta}<\gamma<\delta-\beta+\alpha\\ \frac{(\delta+\alpha)(1-\delta)}{2-\alpha+\delta}<\beta<\frac{\alpha}{1-\delta+\alpha}\\ \frac{(1-\delta)^2}{2-\delta}<\alpha<(1-\delta)^2\\ \frac{3-\sqrt{5}}{2}<\delta<1\end{array}, &
\begin{array}{c}\beta<\gamma<\delta-\beta+\alpha\\ \frac{\alpha}{1-\delta+\alpha}<\beta<\frac{\delta+\alpha}{2}\\ \frac{(1-\delta)^2}{2-\delta}<\alpha<(1-\delta)^2\\ \frac{3-\sqrt{5}}{2}<\delta<1\end{array}, \\
\begin{array}{c}\frac{\alpha(1-\beta)}{1-\delta}<\gamma<\frac{\beta(1-\alpha)}{1-\delta}\\ \alpha<\beta<\frac{(\delta+\alpha)(1-\delta)}{2-\alpha+\delta}\\ (1-\delta)^2<\alpha<\delta\\ \frac{3-\sqrt{5}}{2}<\delta<\frac{1}{2}\end{array}, &
\begin{array}{c}\frac{\alpha(1-\beta)}{1-\delta}<\gamma<\delta-\beta+\alpha\\ \frac{(\delta+\alpha)(1-\delta)}{2-\alpha+\delta}<\beta<\frac{\alpha}{1-\delta+\alpha}\\ (1-\delta)^2<\alpha<\delta\\ \frac{3-\sqrt{5}}{2}<\delta<\frac{1}{2}\end{array}, &
\begin{array}{c}\beta<\gamma<\delta-\beta+\alpha\\ \frac{\alpha}{1-\delta+\alpha}<\beta<\frac{\delta+\alpha}{2}\\ (1-\delta)^2<\alpha<\delta\\ \frac{3-\sqrt{5}}{2}<\delta<\frac{1}{2}\end{array}, \\
\begin{array}{c}\frac{\alpha(1-\beta)}{1-\delta}<\gamma<\frac{\beta(1-\alpha)}{1-\delta}\\ \alpha<\beta<\frac{(\delta+\alpha)(1-\delta)}{2-\alpha+\delta}\\ (1-\delta)^2<\alpha<1-\delta\\ \frac{1}{2}<\delta<1\end{array}, &
\begin{array}{c}\frac{\alpha(1-\beta)}{1-\delta}<\gamma<\delta-\beta+\alpha\\ \frac{(\delta+\alpha)(1-\delta)}{2-\alpha+\delta}<\beta<\frac{\alpha}{1-\delta+\alpha}\\ (1-\delta)^2<\alpha<1-\delta\\ \frac{1}{2}<\delta<1\end{array}, &
\begin{array}{c}\beta<\gamma<\delta-\beta+\alpha\\ \frac{\alpha}{1-\delta+\alpha}<\beta<\frac{\delta+\alpha}{2}\\ (1-\delta)^2<\alpha<1-\delta\\ \frac{1}{2}<\delta<1\end{array}.
\end{array} \tag{5.2.16}$$

Applying the solution process again to casa$(A)=3$, maison$(A)=2$, with $0<\alpha<\beta<\gamma<\delta<1$ and following order of integration γ, β, δ, α, region (4.3.8) becomes

$$\begin{array}{c}\beta<\gamma<\frac{\alpha(1-\beta)}{1-\delta}\\ \alpha<\beta<\frac{\alpha}{1-\delta+\alpha}\\ \alpha<\delta<1-\alpha\\ 0<\alpha<\frac{1}{2}\end{array},\quad \begin{array}{c}\beta<\gamma<\frac{\delta-\beta+\alpha\beta}{\delta}\\ \alpha<\beta<\frac{\delta}{1-\alpha+\delta}\\ 1-\alpha<\delta<\sqrt{1-\alpha}\\ 0<\alpha<\frac{1}{2}\end{array},\quad \begin{array}{c}\beta<\gamma<\frac{\delta-\beta+\alpha\beta}{\delta}\\ \alpha<\beta<\frac{\delta}{1-\alpha+\delta}\\ 1-\frac{\alpha}{2}<\delta<1\\ 0<\alpha<\frac{2}{3}\end{array},$$

$$\begin{array}{c}\beta<\gamma<\frac{\delta-\beta+\alpha\beta}{\delta}\\ \alpha<\beta<\frac{\delta}{1-\alpha+\delta}\\ \sqrt{1-\alpha}<\delta<1-\frac{\alpha}{2}\\ 0<\alpha<\frac{\sqrt{5}-1}{2}\end{array},\quad \begin{array}{c}\beta<\gamma<\frac{\delta-\beta+\alpha\beta}{\delta}\\ \alpha<\beta<\frac{\delta}{1-\alpha+\delta}\\ \alpha<\delta<1\\ \frac{2}{3}<\alpha<1\end{array},\quad \begin{array}{c}\beta<\gamma<\frac{\delta-\beta+\alpha\beta}{\delta}\\ \alpha<\beta<\frac{\delta}{1-\alpha+\delta}\\ \alpha<\delta<1-\frac{\alpha}{2}\\ \frac{\sqrt{5}-1}{2}<\alpha<\frac{2}{3}\end{array}, \qquad (5.2.17)$$

$$\begin{array}{c}\beta<\gamma<\frac{\delta-\beta+\alpha\beta}{\delta}\\ \alpha<\beta<\frac{\delta}{1-\alpha+\delta}\\ \alpha<\delta<\sqrt{1-\alpha}\\ \frac{1}{2}<\alpha<\frac{\sqrt{5}-1}{2}\end{array}.$$

5.3 Polylogarithms

Integration of (5.1.1), (5.1.2), (5.1.3), with $m,n\leq 1$, requires the use of the dilogrithm function. The polylogartihm functions are defined by

$$Li_s(z)=\sum_{n\geq 1}\frac{z^n}{n^s},\qquad z\in\mathbb{C},\ |z|<1.$$

We have the well known Taylor series

$$-ln(1-z)=\sum_{n\geq 1}\frac{z^n}{n},\qquad z\in\mathbb{C},\ |z|<1.$$

Using the principal branch of the complex logarithm function, we have the analytic continuation

$$Li_1(z)=-\ln(1-z),\qquad z\in\mathbb{C}-[1,\infty)$$

of $Li_1(z)$ to $\mathbb{C}-[1,\infty)$, which is the complex plane cut from one to infinity. Observe that the recurrence relation

$$Li_s(z) = \int\limits_{t=0}^{z} \frac{Li_{s-1}(t)}{t}\,dt, \qquad z\in\mathbb{C}-(1,\infty), \qquad 2\leq s,$$

where the integral may be taken over any path which is contained in $\mathbb{C}-(1,\infty)$, provides an analytic continuation of $Li_s(z)$ to $\mathbb{C}-(1,\infty)$ which is the complex plane cut from one to infinity with one included.

For our purposes we require some special properties of the dilogrithm $Li_2(z)$ funtion over the real numbers

$$Li_2(x) = \int\limits_{t=0}^{x} \frac{-ln(1-t)}{t}\,dt, \qquad x\leq 1. \tag{5.3.1}$$

Following from [Lew1981] we have the following relations

$$\begin{aligned}
Li_2\left(\frac{-1}{x}\right)+Li_2(-x) &= -\frac{\pi^2}{6}-\frac{\ln(x)^2}{2}, \qquad 0<x,\\
Li_2(x)+Li_2(1-x) &= \frac{\pi^2}{6}-\ln(x)ln(1-x), \qquad 0<x<1,\\
Li_2(x)+Li_2\left(\frac{-x}{1-x}\right) &= -\frac{ln(1-x)^2}{2}, \qquad x<1,\\
Li_2(x^2) &= \frac{Li_2(x)}{2}+\frac{Li_2(-x)}{2}, \qquad -1<x<1.
\end{aligned} \tag{5.3.2}$$

We also require the use of some particular know values of the dilogarithm

$$\begin{aligned}
Li_2(0) &= 0,\\
Li_2\left(\frac{1}{2}\right) &= \frac{\pi^2}{12}-\frac{\ln(2)^2}{2},\\
Li_2(1) &= \frac{\pi^2}{6},\\
Li_2(-1) &= -\frac{\pi^2}{12}.
\end{aligned} \tag{5.3.3}$$

5.4 Integral Computation

We are ready to proceed in finding $S_2(3,3)$ by computing integrals $\mathscr{A}(0,0)$, $\mathscr{A}(1,0)$, $\mathscr{A}(0,1)$, $\mathscr{A}(1,1)$, $\mathscr{B}(0,0)$, $\mathscr{B}(1,0)$, $\mathscr{B}(0,1)$, $\mathscr{B}(1,1)$, $\mathscr{C}(0,0)$, $\mathscr{C}(1,0)$, $\mathscr{C}(0,1)$, and $\mathscr{C}(1,1)$ given by (5.1.1), (5.1.2), and (5.1.3). Note, while invariably long and tedious, all functions have real valued antiderivates (utilizing the dilogartihm for $x \leq 1$), and the integration is carried out by the fundamental theroem of calculus. Maple 13 integration and simplification routines were utilitzed extensively to mitigate errors and speed the computation. Also note the author found it neccessary to write several special routines, as the standard Maple 13 routines were often unsuccessful.

In the following chapters we present an outline of the computations for each of the integrals and state the remaining results. Chapter 6 covers the computation of $\mathscr{A}(m,n)$ for $m,n \leq 1$. Chapter 7 contains $\mathscr{B}(m,n)$ for $m,n \leq 1$. Chapter 8 contains the final integral computation necessary for $S_2(3,3)$, $\mathscr{C}(m,n)$ for $m,n \leq 1$.

Chapter 6

CASA= 1, MAISON= 1

6.1 Setup

Using limits (5.2.7), integral (5.1.1) becomes

$$\mathscr{A}(m,n) = \int_{\alpha=0}^{1/2} \int_{\delta=\alpha}^{1-\alpha} \int_{\beta=\alpha}^{\frac{\delta(1-\delta)}{1-\alpha}} \int_{\gamma=\frac{\beta(1-\alpha)}{1-\delta}}^{\delta} \Theta_1(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \, d\delta \, d\alpha. \tag{6.1.1}$$

6.2 $\mathscr{A}(0,0)$

Substituting $m = n = 0$ in (6.1.1),

$$\mathscr{A}(0,0) = \int_{\alpha=0}^{1/2} \int_{\delta=\alpha}^{1-\alpha} \int_{\beta=\alpha}^{\frac{\delta(1-\delta)}{1-\alpha}} \int_{\gamma=\frac{\beta(1-\alpha)}{1-\delta}}^{\delta} d\gamma \, d\beta \, d\delta \, d\alpha.$$

Integrating the inner integral with respect to γ, we obtain

$$\mathscr{A}(0,0) = \int_{\alpha=0}^{1/2} \int_{\delta=\alpha}^{1-\alpha} \int_{\beta=\alpha}^{\frac{\delta(1-\delta)}{1-\alpha}} \left(\delta - \frac{\beta(1-\alpha)}{1-\delta} \right) d\beta \, d\delta \, d\alpha.$$

Integrating with respect to β, simplifying, and taking partial fractions with respect to δ yields

$$\mathscr{A}(0,0) = \int_{\alpha=0}^{1/2} \int_{\delta=\alpha}^{1-\alpha} \left(\frac{\delta^2(1-\delta)}{2(1-\alpha)} - \delta\alpha + \frac{\alpha^2(1-\alpha)}{2(1-\delta)} \right) d\delta \, d\alpha.$$

Integrating with respect to δ and taking partial fractions with respect to α we have

$$\mathscr{A}(0,0) = \int_{\alpha=0}^{1/2} \left(-\frac{(1-\alpha)^3}{8} - \frac{(1-\alpha)^2\alpha}{2} + \frac{(1-\alpha)^2}{6} + \left(\frac{\alpha^3}{2} - \frac{\alpha^2}{2} \right) \ln(\alpha) \right.$$
$$\left. + \frac{\alpha^4}{8(1-\alpha)} + \frac{\alpha^3}{2} - \frac{\alpha^3}{6(1-\alpha)} - \left(\frac{\alpha^3}{2} - \frac{\alpha^2}{2} \right) \ln(1-\alpha) \right) d\alpha.$$

Lastly, we integrate with respect to α and simplify, obtaining,

$$\mathscr{A}(0,0) = \frac{1}{288}. \tag{6.2.1}$$

6.3 $\mathscr{A}(1,0)$

Substituting $m = 1$ and $n = 0$, and formula (4.3.13) for $\Theta_1(A)$ in (6.1.1) yields

$$\mathscr{A}(1,0) = \int_{\alpha=0}^{1/2} \int_{\delta=\alpha}^{1-\alpha} \int_{\beta=\alpha}^{\frac{\delta(1-\delta)}{1-\alpha}} \int_{\gamma=\frac{\beta(1-\alpha)}{1-\delta}}^{\delta} \left(1 - \frac{(\gamma\delta - \alpha\beta)^2}{2(\gamma-\beta)(\delta-\alpha)}\right) d\gamma\, d\beta\, d\delta\, d\alpha.$$

Integrating the inner integral with respect to γ and rearranging the result in terms of powers of β we obtain

$$\begin{aligned}\mathscr{A}(1,0) = \int_{\alpha=0}^{1/2} \int_{\delta=\alpha}^{1-\alpha} \int_{\beta=\alpha}^{\frac{\delta(1-\delta)}{1-\alpha}} \Bigg(& -\frac{\left(\delta^3 - 4\delta + 4\alpha\right)\delta}{4(\delta-\alpha)} + \frac{\beta^2(\delta-\alpha)}{2}\ln(\delta-\alpha) \\ & - \frac{\left(-\delta^4 + \delta^3 + 2\delta^3\alpha - 2\alpha\delta^2 + 2\delta - 2\alpha\delta + 2\alpha^2 - 2\alpha\right)\beta}{2(\delta-\alpha)(1-\delta)} \\ & + \frac{(1-\alpha)\delta\left(-4\alpha + 3\alpha\delta + 3\delta - 2\delta^2\right)\beta^2}{4(\delta-\alpha)(1-\delta)^2} \\ & - \frac{(\delta-\alpha)\beta^2}{2}\ln(\delta-\beta) + \frac{\beta^2}{2}(\delta-\alpha)\ln(\beta) \\ & - \frac{\beta^2(\delta-\alpha)}{2}\ln(1-\delta)\Bigg)\, d\beta\, d\delta\, d\alpha.\end{aligned}$$

Integrating with respect to β, simplifying, and taking partial fractions with respect to δ yields

$$\mathscr{A}(1,0) = \int_{\alpha=0}^{1/2} \int_{\delta=\alpha}^{1-\alpha} \Bigg(-\frac{(3\alpha-2)}{12} - \alpha\delta - \frac{\left(\alpha^3-\alpha^2-1\right)\delta^2}{2(1-\alpha)} + \frac{\left(2\alpha^2-6+3\alpha\right)\delta^3}{12(1-\alpha)^2}$$
$$- \frac{\alpha(-4+3\alpha)\delta^4}{12(1-\alpha)^2} + \frac{\alpha^2(2\alpha-3)(\alpha-2)}{12(1-\delta)} - \frac{\alpha^3(1-\alpha)}{12(1-\delta)^2}$$
$$+ \frac{\delta^3(\delta-\alpha)}{6}\ln(\delta) - \frac{\alpha^3(\delta-\alpha)}{6}\ln(\alpha) + \frac{\alpha^3(\delta-\alpha)}{6}\ln(1-\delta)$$
$$- \frac{\delta^3(\delta-\alpha)}{6}\ln(1-\alpha)\Bigg)\, d\delta\, d\alpha.$$

Integrating with respect to δ and taking partial fractions with respect to α we have

$$\mathscr{A}(1,0) = \int_{\alpha=0}^{1/2} \Bigg(\frac{17}{200} - \frac{35\alpha}{96} + \frac{29\alpha^2}{40} + \frac{3\alpha^3}{80} + \frac{3\alpha^4}{40} - \frac{4\alpha^5}{75} - \frac{13}{240(1-\alpha)}$$
$$+ \frac{1}{240(1-\alpha)^2} - \frac{\alpha^2\left(3\alpha^3-20\alpha+20\right)}{40}\ln(\alpha)$$
$$+ \frac{\alpha^2\left(3\alpha^3-20\alpha+20\right)}{40}\ln(1-\alpha)\Bigg)\, d\alpha.$$

We integrate with respect to α and simplify, obtaining,

$$\mathscr{A}(1,0) = \frac{1}{360}. \tag{6.3.1}$$

6.4 $\mathscr{A}(0,1)$

Substituting $m = 0$ and $n = 1$, and formula (4.3.13) for $\Phi_1(A)$ in (6.1.1) yields

$$\mathscr{A}(0,1) = \int_{\alpha=0}^{1/2} \int_{\delta=\alpha}^{1-\alpha} \int_{\beta=\alpha}^{\frac{\delta(1-\delta)}{1-\alpha}} \int_{\gamma=\frac{\beta(1-\alpha)}{1-\delta}}^{\delta} \left(\frac{(\gamma\delta-\alpha\beta)^2}{2(\gamma-\alpha)(\delta-\beta)} \right) d\gamma\, d\beta\, d\delta\, d\alpha.$$

Integrating the inner integral with respect to γ and taking partial fractions of the result with respect to β,

$$\mathscr{A}(0,1) = \int_{\alpha=0}^{1/2}\int_{\delta=\alpha}^{1-\alpha}\int_{\beta=\alpha}^{\frac{\delta(1-\delta)}{1-\alpha}} \left(-\frac{\delta^2\left(8\alpha\delta - 4\alpha\delta^2 - 2\alpha^2 - 2\alpha + \alpha^2\delta - \delta\right)}{4(1-\delta)^2} \right.$$
$$+ \frac{(1-\alpha)(-4\alpha+3\alpha\delta+\delta)\delta\beta}{4(1-\delta)^2} - \frac{\delta^3(-\delta+2)(\delta-\alpha)^2}{4(\delta-\beta)(1-\delta)^2}$$
$$- \frac{(\delta-\beta)\alpha^2}{2}\ln(\beta - \alpha\beta - \alpha + \alpha\delta)$$
$$+ \frac{\alpha^2(\delta-\beta)}{2}\ln(\delta-\alpha)$$
$$\left. + \frac{(\delta-\beta)\alpha^2}{2}\ln(1-\delta) \right) d\beta\, d\delta\, d\alpha.$$

Integrating with respect to β, simplifying, and taking partial fractions with respect to δ yields

$$\mathscr{A}(0,1) = \int_{\alpha=0}^{1/2}\int_{\delta=\alpha}^{1-\alpha} \left(-\frac{4\alpha^5 - 8\alpha^4 + 5\alpha^3 - 5\alpha^2 + 2}{8(1-\alpha)} + \frac{\left(-2\alpha^3 + \alpha^4 + 2\alpha - \alpha^2 - 1\right)\delta}{4(1-\alpha)^2} \right.$$
$$- \frac{(2\alpha-1)\left(4\alpha^3 - 7\alpha^2 + 2\alpha - 2\right)\delta^2}{8(1-\alpha)^2} + \frac{(2\alpha-1)\delta^3}{4(1-\alpha)}$$
$$+ \frac{\left(4\alpha^2 + 1 - 6\alpha\right)\delta^4}{8(1-\alpha)^2} + \frac{(1-\alpha)\left(\alpha^3 + 3\alpha + 1\right)}{4(1-\delta)}$$
$$- \frac{\alpha(\alpha+2)(1-\alpha)^2}{8(1-\delta)^2} - \frac{(-2+\delta)(\delta-\alpha)^2\delta^3}{4(1-\delta)^2}\ln(\delta)$$
$$- \frac{(\alpha-2)(\delta-\alpha)^2\alpha^3}{4(1-\alpha)^2}\ln(\alpha) + \frac{(\alpha-2)(\delta-\alpha)^2\alpha^3}{4(1-\alpha)^2}\ln(1-\delta)$$
$$\left. + \frac{(-2+\delta)(\delta-\alpha)^2\delta^3}{4(1-\delta)^2}\ln(1-\alpha) \right) d\delta\, d\alpha.$$

Integrating with respect to δ and taking partial fractions with respect to α, we have,

$$\mathscr{A}(0,1) = \int_{\alpha=0}^{1/2} \Bigg(-\frac{2947}{1800} + \frac{425\alpha}{96} - \frac{799\alpha^2}{360} - \frac{11\alpha^3}{240} - \frac{11\alpha^4}{120} - \frac{4\alpha^5}{75} + \frac{1}{720(1-\alpha)}$$
$$- \frac{1}{360(1-\alpha)^2} + \frac{\alpha\left(11\alpha^4 - 20\alpha^2 + 240\alpha - 240\right)}{120} \ln(\alpha)$$
$$- \frac{(1-\alpha)(\alpha-2)}{2} Li_2(1-\alpha) + \frac{(1-\alpha)(\alpha-2)}{2} Li_2(\alpha)$$
$$- \frac{\alpha\left(11\alpha^4 - 20\alpha^2 + 240\alpha - 240\right)}{120} \ln(1-\alpha)$$
$$+ \frac{(1-\alpha)(\alpha-2)}{2} \ln(1-\alpha)\ln(\alpha)$$
$$- \frac{(1-\alpha)(\alpha-2)}{2} \ln(1-\alpha)^2 \Bigg) \, d\alpha.$$

We integrate with respect to α, evaluate dilogarithms $Li_2(0)$, $Li_2\left(\frac{1}{2}\right)$, $Li_2(1)$ by (5.3.3) and simplify, obtaining

$$\mathscr{A}(0,1) = -\frac{493}{720} + \frac{5\pi^2}{72}. \tag{6.4.1}$$

6.5 $\mathscr{A}(1,1)$

Substituting $m = n = 1$ and formulas for $\Theta_1(A)$ and $\Phi_1(A)$ (4.3.13) into (6.1.1) yields

$$\mathscr{A}(1,1) = \int_{\alpha=0}^{1/2} \int_{\delta=\alpha}^{1-\alpha} \int_{\beta=\alpha}^{\frac{\delta(1-\delta)}{1-\alpha}} \int_{\gamma=\frac{\beta(1-\alpha)}{1-\delta}}^{\delta} \left(\left(1 - \frac{(\gamma\delta - \alpha\beta)^2}{2(\gamma-\beta)(\delta-\alpha)} \right) \right.$$
$$\left. \left(\frac{(\gamma\delta - \alpha\beta)^2}{2(\gamma-\alpha)(\delta-\beta)} \right) \right) d\gamma \, d\beta \, d\delta \, d\alpha.$$

Integrating the inner integral with respect to γ and taking partial fractions of the result with respect to β,

$$\begin{aligned}\mathscr{A}(1,1) = \int_{\alpha=0}^{1/2}\int_{\delta=\alpha}^{1-\alpha}\int_{\beta=\alpha}^{\frac{\delta(1-\delta)}{1-\alpha}} \Bigg(& \frac{\delta^2}{24(\delta-\alpha)(1-\delta)^3}\Big(-6\delta^3+6\delta^2+6\delta\alpha+69\alpha\delta^3 \\
& -6\alpha^2\delta^3-96\alpha^2\delta^2+18\delta\alpha^3-27\alpha^3\delta^3-12\alpha^3 \\
& +12\alpha^3\delta^2-2\delta^4+72\delta\alpha^2+39\delta^4\alpha+63\alpha^2\delta^4 \\
& -27\alpha^2\delta^5+11\alpha^3\delta^4-90\alpha\delta^5+36\delta^6\alpha-12\alpha^2 \\
& -54\alpha\delta^2-12\delta^5+21\delta^6-9\delta^7\Big) \\
& +\frac{\delta\beta}{24(\delta-\alpha)(1-\delta)^3}\Big(6\delta^2-5\delta^4-6\delta^3+24\alpha^2-30\delta\alpha \\
& -9\alpha\delta^3+141\alpha^2\delta^4-12\delta\alpha^2-102\alpha^2\delta^3-6\alpha^3\delta^2 \\
& +5\alpha^3\delta^4+42\delta\alpha^3+42\alpha\delta^2+45\delta^4\alpha-3\delta^5-24\alpha^3 \\
& +12\delta^6-6\delta^7-72\alpha\delta^5-51\alpha^2\delta^5+30\delta^6\alpha-6\alpha^2\delta^2 \\
& -15\alpha^3\delta^3\Big) \\
& +\frac{\delta^2(\alpha-1)\beta^2}{24(\delta-\alpha)(1-\delta)^3}\Big(36\alpha^2+53\alpha\delta^2-36\delta\alpha-60\delta\alpha^2 \\
& -21\alpha\delta^3+11\delta^2+26\alpha^2\delta^2-15\delta^3+6\delta^4\Big) \\
& +\frac{(\delta-\alpha)^2\delta^3\left(-12-27\delta^3+11\delta^4+12\delta^2+18\delta\right)}{24(\delta-\beta)(1-\delta)^3} \\
& +\frac{(\delta-\alpha)^3\beta^4}{4(\beta-\alpha)(\delta-\beta)}\ln(\beta) \\
& +\frac{1}{4(\delta-\alpha)(\delta-\beta)(\beta-\alpha)}\Big(\beta^4\delta^4+\alpha^4\delta^4-4\delta\alpha^4\beta \\
& -4\beta^4\delta^3\alpha-4\alpha^4\delta^3\beta-2\alpha^3\delta^3+2\alpha^2\delta^3\beta+2\alpha^4\beta^2 \\
& +2\alpha^3\beta\delta^2-4\delta^2\alpha^2\beta^2+2\delta\beta^3\alpha^2+2\alpha^3\beta^2\delta \\
& +6\beta^4\delta^2\alpha^2+6\alpha^4\delta^2\beta^2+2\alpha^4\delta^2-4\alpha^3\delta\beta^4 \\
& -2\alpha^3\beta^3-4\delta\alpha^4\beta^3+2\alpha^4\beta^4\Big)\ln(\delta-\alpha)\end{aligned}$$

$$
\begin{aligned}
&- \frac{(\delta-\alpha)^3\beta^4}{4(\beta-\alpha)(\delta-\beta)}\ln(\delta-\beta) \\
&+ \frac{1}{4(\delta-\alpha)(\delta-\beta)}\Big(-\alpha^2\beta\delta^4 - \alpha\beta^2\delta^4 + 4\alpha^2\beta^2\delta^3 \\
&\quad -2\alpha^3\beta^2 - \alpha^3\delta^4 + 2\alpha^2\delta^3 + \beta^3\delta^4 + 4\alpha^3\delta\beta \\
&\quad -6\alpha^3\delta^2\beta^2 + 4\alpha^3\delta^3\beta - 6\beta^3\delta^2\alpha^2 + 2\delta\alpha^2\beta^2 \\
&\quad +4\delta\alpha^3\beta^3 - 4\alpha^2\beta\delta^2 - 2\alpha^3\delta^2 + 4\beta^3\delta^3\alpha\Big)\ln(1-\delta) \\
&- \frac{(\delta-\beta)\alpha^2}{4(\beta-\alpha)(\delta-\alpha)}\Big(\alpha^2\beta^2 - 2\delta\alpha^2\beta + 2\delta\beta - 2\alpha\beta + 2\alpha^2 \\
&\quad +\alpha^2\delta^2 - 2\delta\alpha\Big)\ln(\beta-\alpha\beta-\alpha+\delta\alpha)\Bigg)\,d\beta\,d\delta\,d\alpha.
\end{aligned}
$$

Integrating with respect to β, simplifying, and taking partial fractions with respect to δ yields

$$
\begin{aligned}
\mathscr{A}(1,1) = \int_{\alpha=0}^{1/2}\int_{\delta=\alpha}^{1-\alpha} \Bigg(&- \frac{1}{144(1-\alpha)^2}\Big(49\alpha^8 - 86\alpha^7 - 86\alpha^6 + 310\alpha^5 - 296\alpha^4 \\
&\quad + 256\alpha^3 - 117\alpha^2 - 90\alpha + 72\Big) \\
&+ \frac{\delta}{72(1-\alpha)^3}\Big(-63\alpha^2 - 43\alpha^5 + 84\alpha - 54\alpha^3 + 7\alpha^8 - 6\alpha^7 - 30 \\
&\quad + 114\alpha^4 - 3\alpha^6\Big) \\
&- \frac{\delta^2}{144(1-\alpha)^3}\Big(-105\alpha^6 + 49\alpha^7 + 48 - 69\alpha^5 - 387\alpha^3 - 162\alpha \\
&\quad + 383\alpha^4 + 261\alpha^2\Big) \\
&- \frac{(\alpha+1)\left(14\alpha^5 - 35\alpha^4 - 7\alpha^3 + 81\alpha^2 - 72\alpha + 18\right)\delta^3}{72(1-\alpha)^3} \\
&- \frac{\left(-416\alpha^2 + 49\alpha^5 - 177\alpha^4 - 30 + 348\alpha^3 + 228\alpha\right)\delta^4}{144(1-\alpha)^3} \\
&+ \frac{\left(7\alpha^4 - 15\alpha^3 + 15\alpha^2 - 17\alpha + 12\right)\delta^5}{72(1-\alpha)^3} - \frac{49\delta^6}{144}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{12\alpha^5-28\alpha^4+21\alpha^3-81\alpha^2+54\alpha+42}{72\,(1-\delta)}\\
&-\frac{(1-\alpha)\left(8\alpha^4-15\alpha^3-21\alpha^2+78\alpha+12\right)}{144\,(1-\delta)^2}\\
&+\frac{\alpha\left(2\alpha^2+3\alpha+6\right)(1-\alpha)^2}{72\,(1-\delta)^3}\\
&+\frac{(\delta-\alpha)^2\,\alpha}{24\,(1-\alpha)^3}\Big(-3\delta^2\alpha-2\delta\alpha^2+9\delta^2\alpha^2-12\alpha^2-6\delta^3+29\alpha^3\\
&\quad-21\alpha^4+18\alpha\delta^3-18\alpha^2\delta^3-9\alpha^3\delta^2+6\alpha^3\delta^3+3\alpha^4\delta^2\\
&\quad+6\delta\alpha^3-6\delta\alpha^4+6\alpha^5+2\alpha^5\delta\Big)\ln(1-\delta)\\
&-\frac{(\delta-\alpha)^2\,\alpha}{24\,(1-\alpha)^3}\Big(-3\delta^2\alpha-2\delta\alpha^2+9\delta^2\alpha^2-12\alpha^2-6\delta^3+29\alpha^3\\
&\quad-21\alpha^4+18\alpha\delta^3-18\alpha^2\delta^3-9\alpha^3\delta^2+6\alpha^3\delta^3+3\alpha^4\delta^2\\
&\quad+6\delta\alpha^3-6\delta\alpha^4+6\alpha^5+2\alpha^5\delta\Big)\ln(\alpha)\\
&+\frac{(\delta-\alpha)^2\,\delta}{24\,(1-\delta)^3}\Big(-12\delta^2-2\delta^2\alpha-3\delta\alpha^2+9\delta^2\alpha^2+29\delta^3-21\delta^4\\
&\quad-6\alpha^3-6\delta^4\alpha+6\alpha\delta^3+3\alpha^2\delta^4-9\alpha^2\delta^3-18\alpha^3\delta^2\\
&\quad+6\alpha^3\delta^3+18\delta\alpha^3+6\delta^5+2\delta^5\alpha\Big)\ln(1-\alpha)\\
&-\frac{(\delta-\alpha)^2\,\delta}{24\,(1-\delta)^3}\Big(-12\delta^2-2\delta^2\alpha-3\delta\alpha^2+9\delta^2\alpha^2+29\delta^3-21\delta^4\\
&\quad-6\alpha^3-6\delta^4\alpha+6\alpha\delta^3+3\alpha^2\delta^4-9\alpha^2\delta^3-18\alpha^3\delta^2\\
&\quad+6\alpha^3\delta^3+18\delta\alpha^3+6\delta^5+2\delta^5\alpha\Big)\ln(\delta)\\
&+\frac{(\delta-\alpha)^2\,\delta^4}{8}\ln(1-\alpha)^2-\frac{(\delta-\alpha)^2\,\delta^4}{4}\ln(1-\delta)\ln(1-\alpha)\\
&-\frac{(\delta-\alpha)^2\,\delta^4}{8}\ln(\delta)^2+\frac{(\delta-\alpha)^2\,\delta^4}{4}\ln(\delta)\ln(1-\delta)\\
&-\frac{(\delta-\alpha)^2\,\delta^4}{4}\,Li_2\left(1-\frac{\alpha}{\delta}\right)+\frac{(\delta-\alpha)^2\,\delta^4}{4}\,Li_2\left(\frac{\delta-\alpha}{1-\alpha}\right)\\
&+\frac{\alpha^4\,(\delta-\alpha)^2}{4}\,Li_2\left(\frac{1-\alpha-\delta}{1-\alpha}\right)
\end{aligned}
$$

$$+\frac{\alpha^4(\delta-\alpha)^2}{4}\,Li_2\left(1-\frac{1-\delta}{\alpha}\right)$$

$$-\frac{\alpha^4(\delta-\alpha)^2}{4}\,Li_2\left(1-\frac{\delta(1-\delta)}{\alpha(1-\alpha)}\right)\Bigg)\,d\delta\,d\alpha.$$

Integrating with respect to δ and rewriting the result by taking partial fractions with respect to α, we have

$$\begin{aligned}\mathscr{A}(1,1)=\int_{\alpha=0}^{1/2}\Bigg(&-\frac{847}{300}+\frac{218501\alpha}{30240}-\frac{3991\alpha^2}{1260}-\frac{67\alpha^3}{2016}-\frac{19\alpha^4}{252}-\frac{253\alpha^5}{8400}+\frac{\alpha^6}{24}\\&+\frac{17}{1440(1-\alpha)}-\frac{17}{4320(1-\alpha)^2}+\frac{1}{2160(1-\alpha)^3}\\&+\frac{\alpha}{2520}\Big(105\alpha^6+12\alpha^5-225\alpha^4+4\alpha^3+528\alpha^2\\&\qquad-7440\alpha+8640\Big)\ln(1-\alpha)\\&-\frac{\alpha}{2520}\Big(105\alpha^6+12\alpha^5-225\alpha^4+4\alpha^3\\&\qquad+528\alpha^2-7440\alpha+8640\Big)\ln(\alpha)\\&+\left(-\frac{7\alpha^2}{10}+\frac{7\alpha}{3}-\frac{12}{7}\right)Li_2(\alpha)+\left(\frac{7\alpha^2}{10}-\frac{7\alpha}{3}+\frac{12}{7}\right)Li_2(1-\alpha)\\&+\frac{\alpha^7}{420}\ln(\alpha)^2+\left(\frac{\alpha^7}{420}+\frac{7\alpha^2}{10}-\frac{7\alpha}{3}+\frac{12}{7}\right)\ln(1-\alpha)^2\\&\qquad+\left(-\frac{\alpha^7}{210}-\frac{7\alpha^2}{10}+\frac{7\alpha}{3}-\frac{12}{7}\right)\ln(1-\alpha)\ln(\alpha)\Bigg)\,d\alpha.\end{aligned}$$

We integrate with respect to α, evaluate $Li_2(0)$, $Li_2\left(\frac{1}{2}\right)$, and $Li_2(1)$ by (5.3.3) and simplify, obtaining

$$\mathscr{A}(1,1)=-\frac{11099}{8640}+\frac{25\pi^2}{192}. \tag{6.5.1}$$

Chapter 7

CASA= 3, MAISON= 1

7.1 Setup

Using limits (5.2.15) and (5.2.16), and grouping integrals sharing common limits for α and δ, we rewrite $\mathscr{B}(m,n)$ (5.1.2) as a sum of 10 integral sets

$$\begin{aligned}\mathscr{B}(m,n) =&\mathscr{B}_1(m,n)+\mathscr{B}_2(m,n)+\mathscr{B}_3(m,n)+\mathscr{B}_4(m,n)+\mathscr{B}_5(m,n)\\&+\mathscr{B}_6(m,n)+\mathscr{B}_7(m,n)+\mathscr{B}_8(m,n)+\mathscr{B}_9(m,n)+\mathscr{B}_{10}(m,n),\end{aligned}$$

with,

$$\begin{aligned}\mathscr{B}_1(m,n) = \int_{\delta=\frac{1}{3}}^{\frac{1}{2}}\int_{\alpha=0}^{1-2\delta}\Bigg(&\int_{\beta=\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}}^{\frac{\delta+\alpha}{2}}\int_{\gamma=\delta-\beta+\alpha}^{\frac{\beta(1-\alpha)}{1-\delta}}\Theta_3(A)^m\Phi_1(A)^n\,d\gamma\,d\beta\\&+\int_{\beta=\frac{\delta+\alpha}{2}}^{\frac{\delta(1-\delta)}{1-\alpha}}\int_{\gamma=\beta}^{\frac{\beta(1-\alpha)}{1-\delta}}\Theta_3(A)^m\Phi_1(A)^n\,d\gamma\,d\beta\\&+\int_{\beta=\frac{\delta(1-\delta)}{1-\alpha}}^{\frac{\delta}{1-\alpha+\delta}}\int_{\gamma=\beta}^{\frac{\delta-\beta+\alpha\beta}{\delta}}\Theta_3(A)^m\Phi_1(A)^n\,d\gamma\,d\beta\Bigg)\,d\alpha\,d\delta,\\
\mathscr{B}_2(m,n) = \int_{\delta=0}^{\frac{1}{3}}\int_{\alpha=0}^{\delta}\Bigg(&\int_{\beta=\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}}^{\frac{\delta+\alpha}{2}}\int_{\gamma=\delta-\beta+\alpha}^{\frac{\beta(1-\alpha)}{1-\delta}}\Theta_3(A)^m\Phi_1(A)^n\,d\gamma\,d\beta\\&+\int_{\beta=\frac{\delta+\alpha}{2}}^{\frac{\delta(1-\delta)}{1-\alpha}}\int_{\gamma=\beta}^{\frac{\beta(1-\alpha)}{1-\delta}}\Theta_3(A)^m\Phi_1(A)^n\,d\gamma\,d\beta\\&+\int_{\beta=\frac{\delta(1-\delta)}{1-\alpha}}^{\frac{\delta}{1-\alpha+\delta}}\int_{\gamma=\beta}^{\frac{\delta-\beta+\alpha\beta}{\delta}}\Theta_3(A)^m\Phi_1(A)^n\,d\gamma\,d\beta\Bigg)\,d\alpha\,d\delta,\end{aligned}\tag{7.1.1}$$

$$
\begin{aligned}
\mathscr{B}_3(m,n) = \int_{\delta=\frac{1}{3}}^{\frac{1}{2}} \int_{\alpha=1-2\delta}^{\delta} \Bigg(& \int_{\beta=\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}}^{\frac{\delta(1-\delta)}{1-\alpha}} \int_{\gamma=\delta-\beta+\alpha}^{\frac{\beta(1-\alpha)}{1-\delta}} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \\
& + \int_{\beta=\frac{\delta(1-\delta)}{1-\alpha}}^{\frac{\delta+\alpha}{2}} \int_{\gamma=\delta-\beta+\alpha}^{\frac{\delta-\beta+\alpha\beta}{\delta}} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \\
& + \int_{\beta=\frac{\delta+\alpha}{2}}^{\frac{\delta}{1-\alpha+\delta}} \int_{\gamma=\beta}^{\frac{\delta-\beta+\alpha\beta}{\delta}} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \Bigg) \, d\alpha \, d\delta, \\
\mathscr{B}_4(m,n) = \int_{\delta=\frac{1}{2}}^{1} \int_{\alpha=0}^{1-\delta} \Bigg(& \int_{\beta=\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}}^{\frac{\delta(1-\delta)}{1-\alpha}} \int_{\gamma=\delta-\beta+\alpha}^{\frac{\beta(1-\alpha)}{1-\delta}} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \\
& + \int_{\beta=\frac{\delta(1-\delta)}{1-\alpha}}^{\frac{\delta+\alpha}{2}} \int_{\gamma=\delta-\beta+\alpha}^{\frac{\delta-\beta+\alpha\beta}{\delta}} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \\
& + \int_{\beta=\frac{\delta+\alpha}{2}}^{\frac{\delta}{1-\alpha+\delta}} \int_{\gamma=\beta}^{\frac{\delta-\beta+\alpha\beta}{\delta}} \Theta_3(A)^m \Phi_1(A)^n d\gamma \, d\beta \Bigg) \, d\alpha \, d\delta, \\
\mathscr{B}_5(m,n) = \int_{\delta=\frac{2-\sqrt{2}}{2}}^{1} \int_{\alpha=0}^{\frac{(1-\delta)^2}{2-\delta}} \Bigg(& \int_{\beta=\alpha}^{\frac{\alpha}{1-\delta+\alpha}} \int_{\gamma=\frac{\alpha(1-\beta)}{1-\delta}}^{\frac{\beta(1-\alpha)}{1-\delta}} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \\
& + \int_{\beta=\frac{\alpha}{1-\delta+\alpha}}^{\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}} \int_{\gamma=\beta}^{\frac{\beta(1-\alpha)}{1-\delta}} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \\
& + \int_{\beta=\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}}^{\frac{\delta+\alpha}{2}} \int_{\gamma=\beta}^{\delta-\beta+\alpha} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \Bigg) \, d\alpha \, d\delta,
\end{aligned}
\tag{7.1.1}
$$

$$
\begin{aligned}
\mathscr{B}_6(m,n) = \int_{\delta=0}^{\frac{2-\sqrt{2}}{2}} \int_{\alpha=0}^{\delta} \Bigg(& \int_{\beta=\alpha}^{\frac{\alpha}{1-\delta+\alpha}} \int_{\gamma=\frac{\alpha(1-\beta)}{1-\delta}}^{\frac{\beta(1-\alpha)}{1-\delta}} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \\
& + \int_{\beta=\frac{\alpha}{1-\delta+\alpha}}^{\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}} \int_{\gamma=\beta}^{\frac{\beta(1-\alpha)}{1-\delta}} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \\
& + \int_{\beta=\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}}^{\frac{\delta+\alpha}{2}} \int_{\gamma=\beta}^{\delta-\beta+\alpha} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \Bigg) \, d\alpha \, d\delta, \\
\mathscr{B}_7(m,n) = \int_{\delta=\frac{2-\sqrt{2}}{2}}^{\frac{3-\sqrt{5}}{2}} \int_{\alpha=\frac{(1-\delta)^2}{2-\delta}}^{\delta} \Bigg(& \int_{\beta=\alpha}^{\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}} \int_{\gamma=\frac{\alpha(1-\beta)}{1-\delta}}^{\frac{\beta(1-\alpha)}{1-\delta}} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \\
& + \int_{\beta=\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}}^{\frac{\alpha}{1-\delta+\alpha}} \int_{\gamma=\frac{\alpha(1-\beta)}{1-\delta}}^{\delta-\beta+\alpha} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \\
& + \int_{\beta=\frac{\alpha}{1-\delta+\alpha}}^{\frac{\delta+\alpha}{2}} \int_{\gamma=\beta}^{\delta-\beta+\alpha} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \Bigg) \, d\alpha \, d\delta, \\
\mathscr{B}_8(m,n) = \int_{\delta=\frac{3-\sqrt{5}}{2}}^{1} \int_{\alpha=\frac{(1-\delta)^2}{2-\delta}}^{(1-\delta)^2} \Bigg(& \int_{\beta=\alpha}^{\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}} \int_{\gamma=\frac{\alpha(1-\beta)}{1-\delta}}^{\frac{\beta(1-\alpha)}{1-\delta}} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \\
& + \int_{\beta=\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}}^{\frac{\alpha}{1-\delta+\alpha}} \int_{\gamma=\frac{\alpha(1-\beta)}{1-\delta}}^{\delta-\beta+\alpha} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \\
& + \int_{\beta=\frac{\alpha}{1-\delta+\alpha}}^{\frac{\delta+\alpha}{2}} \int_{\gamma=\beta}^{\delta-\beta+\alpha} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \Bigg) \, d\alpha \, d\delta,
\end{aligned}
\tag{7.1.1}
$$

$$\mathscr{B}_9(m,n) = \int_{\delta=\frac{3-\sqrt{5}}{2}}^{\frac{1}{2}} \int_{\alpha=(1-\delta)^2}^{\delta} \left(\int_{\beta=\alpha}^{\frac{\alpha}{1-\delta+\alpha}} \int_{\gamma=\frac{\alpha(1-\beta)}{1-\delta}}^{\frac{\beta(1-\alpha)}{1-\delta}} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \right.$$
$$+ \int_{\beta=\frac{\alpha}{1-\delta+\alpha}}^{\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}} \int_{\gamma=\beta}^{\frac{\beta(1-\alpha)}{1-\delta}} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta$$
$$\left. + \int_{\beta=\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}}^{\frac{\delta+\alpha}{2}} \int_{\gamma=\beta}^{\delta-\beta+\alpha} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \right) d\alpha \, d\delta,$$

$$\mathscr{B}_{10}(m,n) = \int_{\delta=\frac{1}{2}}^{1} \int_{\alpha=(1-\delta)^2}^{1-\delta} \left(\int_{\beta=\alpha}^{\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}} \int_{\gamma=\frac{\alpha(1-\beta)}{1-\delta}}^{\frac{\beta(1-\alpha)}{1-\delta}} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \right.$$
$$+ \int_{\beta=\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}}^{\frac{\alpha}{1-\delta+\alpha}} \int_{\gamma=\frac{\alpha(1-\beta)}{1-\delta}}^{\delta-\beta+\alpha} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta$$
$$\left. + \int_{\beta=\frac{\alpha}{1-\delta+\alpha}}^{\frac{\delta+\alpha}{2}} \int_{\gamma=\beta}^{\delta-\beta+\alpha} \Theta_3(A)^m \Phi_1(A)^n \, d\gamma \, d\beta \right) d\alpha \, d\delta.$$

(7.1.1)

7.2 $\mathscr{B}(0,0)$

Substituting $m = n = 0$ in $\mathscr{B}_1(m,n)$ of (7.1.1) yields

$$\mathscr{B}_1(0,0) = \int_{\delta=\frac{1}{3}}^{\frac{1}{2}} \int_{\alpha=0}^{1-2\delta} \left(\int_{\beta=\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}}^{\frac{\delta+\alpha}{2}} \int_{\gamma=\delta-\beta+\alpha}^{\frac{\beta(1-\alpha)}{1-\delta}} d\gamma \, d\beta + \int_{\beta=\frac{\delta+\alpha}{2}}^{\frac{\delta(1-\delta)}{1-\alpha}} \int_{\gamma=\beta}^{\frac{\beta(1-\alpha)}{1-\delta}} d\gamma \, d\beta \right.$$
$$\left. + \int_{\beta=\frac{\delta(1-\delta)}{1-\alpha}}^{\frac{\delta}{1-\alpha+\delta}} \int_{\gamma=\beta}^{\frac{\delta-\beta+\alpha\beta}{\delta}} d\gamma \, d\beta \right) d\alpha \, d\delta.$$

Starting integration of the three inner integrals with respect to γ, we obtain

$$\mathscr{B}_1(0,0) = \int_{\delta=\frac{1}{3}}^{\frac{1}{2}} \int_{\alpha=0}^{1-2\delta} \left(\int_{\beta=\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}}^{\frac{\delta+\alpha}{2}} \left(-\delta-\alpha+\frac{(2-\alpha-\delta)\,\beta}{1-\delta} \right) d\beta + \int_{\beta=\frac{\delta+\alpha}{2}}^{\frac{\delta(1-\delta)}{1-\alpha}} \left(\frac{\beta\,(\delta-\alpha)}{1-\delta} \right) d\beta + \int_{\beta=\frac{\delta(1-\delta)}{1-\alpha}}^{\frac{\delta}{1-\alpha+\delta}} \left(1-\frac{(1-\alpha+\delta)\,\beta}{\delta} \right) d\beta \right) d\alpha\, d\delta.$$

Integrating the three inner integrals with respect to β, combining, simplifying, and taking partial fractions with respect to α yields

$$\mathscr{B}_1(0,0) = \int_{\delta=\frac{1}{3}}^{\frac{1}{2}} \int_{\alpha=0}^{1-2\delta} \left(-1+\frac{\delta^2}{4}+\frac{\delta}{2}-\frac{\alpha}{2}-\frac{\alpha^2}{4}-\frac{\delta\,(1-\delta)}{2\,(1-\alpha)} + \frac{\delta}{2\,(1-\alpha+\delta)}+\frac{2\,(1-\delta)}{2-\alpha-\delta} \right) d\alpha\, d\delta.$$

Integrating with respect to α and taking partial fractions with respect to δ, we find

$$\mathscr{B}_1(0,0) = \int_{\delta=\frac{1}{3}}^{\frac{1}{2}} \left(-\frac{4}{3}+4\delta-\frac{11\delta^2}{4}+\frac{\delta^3}{6}+2\,(1-\delta)\ln(2-\delta)-\frac{\delta^2}{2}\ln(\delta) + \frac{\delta\,(1-\delta)}{2}\ln(2)-\frac{\delta}{2}\ln(3)+\left(\frac{5\delta}{2}-2\right)\ln(1+\delta) \right) d\delta.$$

Finishing by integrating with respect to δ and simplifying, we obtain

$$\mathscr{B}_1(0,0) = \frac{15745}{31104}-\frac{1045}{162}\ln(3)+\frac{1747}{162}\ln(2)-\frac{5}{9}\ln(5)\,. \qquad (7.2.1)$$

Similarly we carry out the four-fold integration of remaining integral groupings of (7.1.1) with $m = n = 0$, yielding

$$\mathscr{B}_2(0,0) = \frac{623}{1944} + \frac{5}{9}\ln(5) - \frac{58}{81}\ln(3) - \frac{50}{81}\ln(2), \tag{7.2.2}$$

$$\mathscr{B}_3(0,0) = -\frac{337}{864} - \frac{187}{16}\ln(2) + \frac{371}{48}\ln(3), \tag{7.2.3}$$

$$\mathscr{B}_4(0,0) = \frac{431}{1152} + \frac{17}{48}\ln(2) - \frac{9}{16}\ln(3), \tag{7.2.4}$$

$$\mathscr{B}_5(0,0) = \frac{19\sqrt{2}}{18} - \frac{67}{96} + \frac{43}{48}\ln(2) - \frac{77}{48}\ln\left(1+\sqrt{2}\right), \tag{7.2.5}$$

$$\mathscr{B}_6(0,0) = -\frac{91}{288} + \frac{5\sqrt{2}}{144} + \frac{49}{48}\ln(2) - \frac{1}{2}\ln\left(1+\sqrt{2}\right), \tag{7.2.6}$$

$$\begin{aligned}\mathscr{B}_7(0,0) = &\left(\frac{3\sqrt{5}}{4} - \frac{7}{6}\right)\ln(2) + \left(-\frac{3\sqrt{5}}{16} + \frac{15}{32}\right)\ln(5) + \frac{101}{48}\ln\left(1+\sqrt{2}\right)\\ &+ \left(-\frac{\sqrt{5}}{4} - \frac{5}{4}\right)\ln\left(1+\sqrt{5}\right) - \frac{5}{144} - \frac{157\sqrt{2}}{144} + \frac{91\sqrt{5}}{144},\end{aligned} \tag{7.2.7}$$

$$\begin{aligned}\mathscr{B}_8(0,0) = &\left(-\frac{35}{12} - \frac{3\sqrt{5}}{4}\right)\ln(2) + \left(\frac{3\sqrt{5}}{16} - \frac{15}{32}\right)\ln(5)\\ &+ \left(\frac{3\sqrt{5}}{4} + \frac{35}{12}\right)\ln\left(1+\sqrt{5}\right) + \frac{2165}{2016} - \frac{321\sqrt{5}}{224},\end{aligned} \tag{7.2.8}$$

$$\begin{aligned}\mathscr{B}_9(0,0) = &\left(\frac{131}{24} - \sqrt{5}\right)\ln(2) - \frac{5}{4}\ln(5) + \left(\frac{\sqrt{5}}{2} - \frac{5}{3}\right)\ln\left(1+\sqrt{5}\right) - \frac{16931}{10752}\\ &+ \frac{1615\sqrt{5}}{2016} + \frac{3}{16}\ln(3),\end{aligned} \tag{7.2.9}$$

$$\mathscr{B}_{10}(0,0) = \left(\sqrt{5} - \frac{19}{8}\right)\ln(2) + \frac{9883}{10752} - \sqrt{5}\ln\left(1+\sqrt{5}\right) + \frac{5}{4}\ln(5) - \frac{3}{16}\ln(3). \tag{7.2.10}$$

Combining (7.2.1) to (7.2.10) and simplifying, we obtain

$$\mathscr{B}(0,0) = \frac{13}{72} - \frac{1}{4}\ln(2). \tag{7.2.11}$$

7.3 $\mathscr{B}(1,0)$

Substituting $m=1$ and $n=0$ in $\mathscr{B}_1(m,n)$ of (7.1.1), and formula (4.3.13) for $\Theta_3(A)$ yields

$$\mathscr{B}_1(1,0) = \int_{\delta=\frac{1}{3}}^{\frac{1}{2}} \int_{\alpha=0}^{1-2\delta} \left(\int_{\beta=\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}}^{\frac{\delta+\alpha}{2}} \int_{\gamma=\delta-\beta+\alpha}^{\frac{\beta(1-\alpha)}{1-\delta}} \left(1-\frac{2(\gamma\delta-\alpha\beta)-\gamma+\beta}{2(\delta-\alpha)}\right) d\gamma\, d\beta \right.$$
$$+ \int_{\beta=\frac{\delta+\alpha}{2}}^{\frac{\delta(1-\delta)}{1-\alpha}} \int_{\gamma=\beta}^{\frac{\beta(1-\alpha)}{1-\delta}} \left(1-\frac{2(\gamma\delta-\alpha\beta)-\gamma+\beta}{2(\delta-\alpha)}\right) d\gamma\, d\beta$$
$$\left. + \int_{\beta=\frac{\delta(1-\delta)}{1-\alpha}}^{\frac{\delta}{1-\alpha+\delta}} \int_{\gamma=\beta}^{\frac{\delta-\beta+\alpha\beta}{\delta}} \left(1-\frac{2(\gamma\delta-\alpha\beta)-\gamma+\beta}{2(\delta-\alpha)}\right) d\gamma\, d\beta \right) d\alpha\, d\delta.$$

Starting with the integration of the 3 inner double integrals with respect to γ and rewriting the result in terms of powers of β, we obtain

$$\mathscr{B}_1(1,0) = \int_{\delta=\frac{1}{3}}^{\frac{1}{2}} \int_{\alpha=0}^{1-2\delta} \left(\int_{\beta=\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}}^{\frac{\delta+\alpha}{2}} \left(\frac{(\delta+\alpha)\left(-5\delta+3\alpha+2\delta^2+2\delta\alpha\right)}{4\delta-4\alpha} \right.\right.$$
$$+ \frac{\left(3\delta-3\delta\alpha-\alpha-3\delta^2+2\delta^2\alpha+\alpha^2\delta+\delta^3\right)\beta}{(\delta-\alpha)(1-\delta)}$$
$$\left. - \frac{\left(-3\alpha+2\delta\alpha+2\delta^2-3\delta+2\right)(2-\alpha-\delta)\beta^2}{4(\delta-\alpha)(1-\delta)^2} \right) d\beta$$
$$+ \int_{\beta=\frac{\delta+\alpha}{2}}^{\frac{\delta(1-\delta)}{1-\alpha}} \left(\frac{\beta(\delta-\alpha)}{1-\delta} + \frac{(-3+2\delta)(\delta-\alpha)\beta^2}{4(1-\delta)^2} \right) d\beta$$
$$+ \int_{\beta=\frac{\delta(1-\delta)}{1-\alpha}}^{\frac{\delta}{1-\alpha+\delta}} \left(-\frac{\left(\delta-4\delta\alpha-3\alpha+2\alpha^2+1+2\delta^2\right)\beta}{2(\delta-\alpha)\delta} \right.$$
$$- \frac{\left(\alpha+2\delta\alpha+\delta-2\delta^2-1\right)(1-\alpha+\delta)\beta^2}{4(\delta-\alpha)\delta^2}$$
$$\left.\left. + \frac{-2\delta+4\alpha-1}{4\delta-4\alpha} \right) d\beta \right) d\alpha\, d\delta.$$

Integrating the 3 inner integrals with respect to β, combining, simplifying, and taking partial fractions with respect to α yields

$$\begin{aligned}\mathscr{B}_1(1,0) = \int_{\delta=\frac{1}{3}}^{\frac{1}{2}} \int_{\alpha=0}^{1-2\delta} \Bigg(&-\frac{\delta^3}{24} + \frac{5\delta^2}{24} + \frac{\delta}{2} - \frac{7}{6} + \left(\frac{12-\delta^2}{24}\right)\alpha + \left(\frac{\delta-5}{24}\right)\alpha^2 \\ &+ \frac{\alpha^3}{24} + \frac{\delta(-5+4\delta)}{12(1-\alpha)} + \frac{\delta^2(1-\delta)}{12(1-\alpha)^2} - \frac{7\delta-8}{6(2-\alpha-\delta)} \\ &- \frac{2(1-\delta)}{3(2-\alpha-\delta)^2} + \frac{\delta(5+2\delta)}{12-12\alpha+12\delta} - \frac{\delta^2}{6(1-\alpha+\delta)^2}\Bigg)\, d\alpha\, d\delta.\end{aligned}$$

Integrating with respect to α and taking partial fractions with respect to δ, we find

$$\begin{aligned}\mathscr{B}_1(1,0) = \int_{\delta=\frac{1}{3}}^{\frac{1}{2}} \Bigg(&-\frac{89}{288} + \frac{13\delta}{3} - \frac{125\delta^2}{48} + \frac{7\delta^3}{72} + \frac{\delta^4}{18} - \frac{2}{(6-3\delta)} - \frac{7}{6(1+\delta)} \\ &+ \left(\frac{\delta^2}{6} + \frac{11\delta}{4} - \frac{8}{3}\right)\ln(1+\delta) + \left(\frac{8-7\delta}{3}\right)\ln(2-\delta) - \frac{\delta^2}{2}\ln(\delta) \\ &- \frac{\delta(-5+4\delta)}{12}\ln(2) - \frac{\delta(5+2\delta)}{12}\ln(3)\Bigg)\, d\delta.\end{aligned}$$

We finish by integrating with respect to δ and simplify, obtaining

$$\mathscr{B}_1(1,0) = \frac{4344773}{5598720} - \frac{4987}{486}\ln(3) + \frac{913}{54}\ln(2) - \frac{41}{54}\ln(5). \qquad (7.3.1)$$

In a similar fashion we carry out the four-fold integration of remaining integral groupings of (7.1.1) with $m = 1$ and $n = 0$, yielding

$$\mathscr{B}_2(1,0) = \frac{138529}{349920} - \frac{1}{54}\ln(2) - \frac{355}{243}\ln(3) + \frac{41}{54}\ln(5), \qquad (7.3.2)$$

$$\mathscr{B}_3(1,0) = -\frac{2331689}{3732480} + \frac{3697}{288}\ln(3) - \frac{175}{9}\ln(2), \qquad (7.3.3)$$

$$\mathscr{B}_4(1,0) = \frac{11525}{27648} + \frac{7}{6}\ln(2) - \frac{107}{96}\ln(3), \qquad (7.3.4)$$

$$\mathscr{B}_5(1,0) = \left(\frac{29}{96} + \frac{25}{144}\sqrt{2}\right)\ln(2) + \left(-\frac{73}{288} - \frac{\sqrt{2}}{6}\right)\ln\left(1+\sqrt{2}\right) - \frac{41}{128} + \frac{1825\sqrt{2}}{6912}, \qquad (7.3.5)$$

$$\mathscr{B}_6(1,0) = \left(\frac{469}{288} - \frac{25\sqrt{2}}{144}\right)\ln(2) + \left(-\frac{11}{12} + \frac{\sqrt{2}}{6}\right)\ln\left(1+\sqrt{2}\right)$$
$$+ \frac{5827\sqrt{2}}{34560} - \frac{10313}{17280}, \tag{7.3.6}$$
$$\mathscr{B}_7(1,0) = \left(-\frac{47}{36} + \frac{41\sqrt{5}}{72}\right)\ln(2) + \left(\frac{115}{192} - \frac{49\sqrt{5}}{288}\right)\ln(5) + \frac{589}{720} - \frac{623\sqrt{2}}{1440}$$
$$+ \left(-\frac{11\sqrt{5}}{72} - \frac{17}{24}\right)\ln\left(\sqrt{5}+1\right) + \frac{337}{288}\ln\left(1+\sqrt{2}\right) - \frac{43\sqrt{5}}{288}, \tag{7.3.7}$$
$$\mathscr{B}_8(1,0) = \left(-\frac{41\sqrt{5}}{72} - \frac{83}{24}\right)\ln(2) + \left(-\frac{115}{192} + \frac{49\sqrt{5}}{288}\right)\ln(5)$$
$$+ \left(\frac{349\sqrt{5}}{360} + \frac{67}{24}\right)\ln\left(\sqrt{5}+1\right) - \frac{933\sqrt{5}}{896} + \frac{16453}{120960}, \tag{7.3.8}$$
$$\mathscr{B}_9(1,0) = \left(-\frac{49\sqrt{5}}{30} + \frac{379}{48}\right)\ln(2) + \left(-\frac{25}{12} + \frac{49\sqrt{5}}{60}\right)\ln\left(\sqrt{5}+1\right) - \frac{134067}{57344}$$
$$+ \frac{9601\sqrt{5}}{8064} - \frac{43}{24}\ln(5) - \frac{7}{96}\ln(3), \tag{7.3.9}$$
$$\mathscr{B}_{10}(1,0) = \left(\frac{49\sqrt{5}}{30} - \frac{527}{144}\right)\ln(2) + \frac{10313957}{7741440} - \frac{49\sqrt{5}}{30}\ln\left(\sqrt{5}+1\right) + \frac{43}{24}\ln(5)$$
$$+ \frac{7}{96}\ln(3). \tag{7.3.10}$$

Combining (7.3.1) to (7.3.10), and simplifying the logarithms, we obtain

$$\mathscr{B}(1,0) = \frac{1}{72}\ln(2) - \frac{1}{216}. \tag{7.3.11}$$

7.4 $\mathscr{B}(0,1)$

Substituting $m=0$ and $n=1$ in $\mathscr{B}_1(m,n)$ of (7.1.1), and formula (4.3.13) for $\Phi_1(A)$ yields

$$\mathscr{B}_1(0,1)=\int_{\delta=\frac{1}{3}}^{\frac{1}{2}}\int_{\alpha=0}^{1-2\delta}\left(\int_{\beta=\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}}^{\frac{\delta+\alpha}{2}}\int_{\gamma=\delta-\beta+\alpha}^{\frac{\beta(1-\alpha)}{1-\delta}}\left(\frac{(\gamma\delta-\alpha\beta)^2}{2(\delta-\beta)(\gamma-\alpha)}\right)d\gamma\,d\beta\right.$$
$$+\int_{\beta=\frac{\delta+\alpha}{2}}^{\frac{\delta(1-\delta)}{1-\alpha}}\int_{\gamma=\beta}^{\frac{\beta(1-\alpha)}{1-\delta}}\left(\frac{(\gamma\delta-\alpha\beta)^2}{2(\delta-\beta)(\gamma-\alpha)}\right)d\gamma\,d\beta$$
$$\left.+\int_{\beta=\frac{\delta(1-\delta)}{1-\alpha}}^{\frac{\delta}{1-\alpha+\delta}}\int_{\gamma=\beta}^{\frac{\delta-\beta+\alpha\beta}{\delta}}\left(\frac{(\gamma\delta-\alpha\beta)^2}{2(\delta-\beta)(\gamma-\alpha)}\right)d\gamma\,d\beta\right)d\alpha\,d\delta.$$

Again, starting with the integration of the 3 inner double integrals with respect to γ and taking partial fractions with respect to β of each result, we obtain

$$\mathscr{B}_1(0,1)=\int_{\delta=\frac{1}{3}}^{\frac{1}{2}}\int_{\alpha=0}^{1-2\delta}\left(\int_{\beta=\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}}^{\frac{\delta+\alpha}{2}}\left(-\frac{\delta}{4(1-\delta)^2}\Big(-2\delta^3+2\delta\alpha+\delta^4+3\alpha^2\delta^2+4\alpha^2\right.\right.$$
$$-6\alpha^2\delta+4\alpha\delta^3+2\delta^2-8\delta^2\alpha\Big)$$
$$-\frac{(2-\alpha-\delta)\left(3\delta\alpha-4\alpha+\delta^2\right)\delta\beta}{4(1-\delta)^2}$$
$$+\frac{\delta^2(\delta-\alpha)^2}{4(1-\delta)^2(\delta-\beta)}-\frac{\alpha^2(\delta-\beta)}{2}\ln(\delta-\beta)$$
$$-\frac{\alpha^2(\delta-\beta)}{2}\ln(1-\delta)$$
$$\left.+\frac{\alpha^2}{2}(\delta-\beta)\ln(\beta-\alpha\beta-\alpha+\delta\alpha)\right)d\beta$$

$$+\int\limits_{\beta=\frac{\delta+\alpha}{2}}^{\frac{\delta(1-\delta)}{1-\alpha}}\Bigg(-\frac{\delta^2(2-\delta)(\delta-\alpha)^2}{4(1-\delta)^2}-\frac{\alpha^2(\delta-\beta)}{2}\ln(\beta-\alpha)$$

$$-\frac{\left(3\delta\alpha-4\alpha+2\delta-\delta^2\right)\delta(\delta-\alpha)\beta}{4(1-\delta)^2}$$

$$+\frac{\delta^3(2-\delta)(\delta-\alpha)^2}{4(1-\delta)^2(\delta-\beta)}-\frac{\alpha^2(\delta-\beta)}{2}\ln(1-\delta)$$

$$+\frac{\alpha^2(\delta-\beta)}{2}\ln(\beta-\alpha\beta-\alpha+\delta\alpha)\Bigg)\,d\beta$$

$$+\int\limits_{\beta=\frac{\delta(1-\delta)}{1-\alpha}}^{\frac{\delta}{1-\alpha+\delta}}\Bigg(\frac{\delta\left(2\alpha+1-2\delta\alpha+\alpha^2+\delta^2\right)}{4}$$

$$-\frac{(3\alpha+1-\delta)(1-\alpha+\delta)\beta}{4}$$

$$-\frac{\delta^2(\delta-\alpha)^2}{4(\delta-\beta)}+\frac{\alpha^2(\delta-\beta)}{2}\ln(\delta-\beta)$$

$$-\frac{\alpha^2(\delta-\beta)}{2}\ln(\delta)-\frac{\alpha^2(\delta-\beta)}{2}\ln(\beta-\alpha)$$

$$+\frac{\alpha^2(\delta-\beta)}{2}\ln(1-\alpha)\Bigg)\,d\beta\Bigg)\,d\alpha\,d\delta.$$

Integrating the 3 inner integrals with respect to β, combining, simplifying, and taking partial fractions with respect to α yields

$$\mathscr{B}_1(0,1)=\int\limits_{\delta=\frac{1}{3}}^{\frac{1}{2}}\int\limits_{\alpha=0}^{1-2\delta}\Bigg(-\frac{\delta^4}{8}+\frac{\delta^3}{4}-\frac{\delta^2}{4}+\frac{7\delta}{4}-\frac{7}{4}-\frac{\left(\delta^4+3\delta^2+3-3\delta^3-5\delta\right)\alpha}{4(1-\delta)}$$

$$-\frac{\left(1+\delta^2-\delta\right)\alpha^2}{4(1-\delta)}-\frac{\alpha^3\delta}{4}+\frac{\alpha^4}{8}-\frac{\delta}{4(1-\alpha+\delta)}+\frac{4(1-\delta)}{2-\alpha-\delta}$$

$$-\frac{(1+4\delta)(1-\delta)}{4(1-\alpha)}+\frac{(1-\delta)^2\delta}{4(1-\alpha)^2}+\frac{\delta^3(-2+\delta)(\delta-\alpha)^2\ln(\delta)}{4(1-\delta)^2}$$

$$-\frac{(\delta+\alpha)(\delta-\alpha)^3}{4}\ln(1-\alpha+\delta)+\frac{(\delta+\alpha)(\delta-\alpha)^3}{4}\ln(2)$$

$$+\frac{(-\alpha+2\delta\alpha-\delta)(\delta-\alpha)^3}{4(1-\alpha)^2(1-\delta)^2}\ln(2-\alpha-\delta)$$

$$-\frac{(\delta+\delta\alpha-\alpha)(-\delta+\delta\alpha-\alpha)(\delta-\alpha)^2}{4(1-\delta)^2}\ln(1-\alpha)\Bigg)\,d\alpha\,d\delta.$$

Integrating with respect to α and then taking partial fractions with respect to δ, we find

$$\begin{aligned}
\mathscr{B}_1(0,1) = \int_{\delta=\frac{1}{3}}^{\frac{1}{2}} \Bigg(& -\frac{379}{90} + \frac{249\delta}{20} - \frac{959\delta^2}{90} - \frac{947\delta^3}{720} + \frac{69\delta^4}{20} - \frac{391\delta^5}{180} + \frac{1}{3(1-\delta)} \\
& + \frac{\left(-163\delta^3 + 80 - 268\delta + 26\delta^4 + 324\delta^2\right)}{12(1-\delta)^2} \ln(2-\delta) \\
& - \frac{(1-\delta)(2-\delta)}{2} Li_2\left(\frac{-2\delta}{1-\delta}\right) \\
& + \frac{(1-\delta)(2-\delta)}{2} \ln(\delta)\ln(1-\delta) \\
& - \frac{1}{120(1-\delta)^2\delta}\Big(-18\delta^7 + 9\delta^8 + 9\delta^6 + 754\delta - 590\delta^5 - 2093\delta^2 \\
& \qquad + 1674\delta^3 + 190\delta^4 - 15\Big)\ln(1+\delta) \\
& + \frac{(1-\delta)(2-\delta)}{2} \ln(2)\ln(1-\delta) \\
& - \frac{\delta\left(20 - 120\delta + 270\delta^2 - 270\delta^3 + 81\delta^4\right)}{40} \ln(3) \\
& + \frac{1}{120(1-\delta)^2}\Big(-5910\delta^4 - 3418\delta^6 + 764\delta^7 + 5994\delta^5 \\
& \qquad + 267\delta - 1356\delta^2 - 6 + 3585\delta^3\Big)\ln(2) \\
& + \frac{\delta\left(120 + 9\delta^6 + 370\delta^2 - 360\delta - 18\delta^5 + 9\delta^4 - 140\delta^3\right)\ln(\delta)}{120(1-\delta)^2} \\
& \qquad + \frac{(1-\delta)(2-\delta)}{2} Li_2\left(\frac{-1}{1-\delta}\right)\Bigg)\, d\delta.
\end{aligned}$$

Integrating with respect to δ and simplifying, we obtain

$$\begin{aligned}
\mathscr{B}_1(0,1) = & \frac{13}{81}\ln(3)^2 - \frac{83}{324} Li_2\left(-\frac{3}{2}\right) + \frac{17}{24} Li_2\left(-\frac{1}{2}\right) - \frac{5}{6} Li_2\left(-\frac{2}{3}\right) \\
& + \frac{1}{8} Li_2\left(-\frac{1}{3}\right) + \frac{19}{4} Li_2\left(\frac{1}{3}\right) - \frac{19}{4} Li_2\left(\frac{1}{4}\right) - \frac{955}{648}\ln(5) \\
& + \frac{187\pi^2}{3888} - \frac{109}{162}\ln(2)^2 + \frac{83}{162}\ln(2)\ln(3) - \frac{268481}{38880}\ln(3) \\
& \qquad + \frac{26344831}{2099520}\ln(2) + \frac{8401859}{10077696} + \frac{5}{6} Li_2(-2).
\end{aligned} \tag{7.4.1}$$

Similarly we carry out the four-fold integration of remaining integral groupings of (7.1.1) with $m=0$ and $n=1$, yielding

$$\begin{aligned}\mathscr{B}_2(0,1) = &-\frac{13}{81}\,Li_2\left(-\frac{3}{2}\right)+\frac{5}{12}\,Li_2\left(-\frac{2}{3}\right)-\frac{218791}{524880}\ln(2)-\frac{47843}{19440}\ln(3)\\ &-\frac{187\pi^2}{3888}+\frac{2620379}{3149280}+\frac{955}{648}\ln(5)-\frac{13}{81}\ln(2)^2+\frac{26}{81}\ln(2)\ln(3)\\ &-\frac{239}{648}\ln(3)^2-\frac{5}{12}\,Li_2(-2),\end{aligned} \tag{7.4.2}$$

$$\begin{aligned}\mathscr{B}_3(0,1) = &\frac{5}{24}\ln(3)^2-\frac{421781}{829440}+\frac{1}{8}\,Li_2\left(-\frac{1}{2}\right)-\frac{1}{8}\,Li_2\left(-\frac{1}{3}\right)-\frac{13}{3}\,Li_2\left(\frac{1}{3}\right)\\ &+\frac{13}{3}\,Li_2\left(\frac{1}{4}\right)-\frac{\pi^2}{144}-\frac{7}{24}\ln(2)^2+\frac{314513}{27648}\ln(3)-\frac{20637181}{1244160}\ln(2)\\ &-\frac{1}{2}\,Li_2(-2)+\frac{5}{12}\,Li_2(-3),\end{aligned} \tag{7.4.3}$$

$$\begin{aligned}\mathscr{B}_4(0,1) = &\frac{284683}{276480}+\frac{16555}{9216}\ln(2)+\frac{\pi^2}{24}-\frac{10287}{5120}\ln(3)-\frac{5}{12}\,Li_2\left(\frac{1}{4}\right)\\ &-\frac{1}{8}\ln(2)^2+\frac{1}{12}\,Li_2(-2)+\frac{5}{12}\,Li_2\left(-\frac{1}{2}\right),\end{aligned} \tag{7.4.4}$$

$$\begin{aligned}\mathscr{B}_5(0,1) = &-\frac{\pi^2}{288}-\frac{760241}{34560}+\frac{413}{12}\,Li_2\left(-\frac{\sqrt{2}}{2}\right)-\frac{7219\sqrt{2}}{960}\ln(2)-\frac{7\sqrt{5}}{10}\ln(2)^2\\ &-\frac{\sqrt{2}\pi^2}{96}-\frac{\sqrt{2}}{192}\ln(2)^2-\frac{379}{192}\ln(2)^2+\frac{59461}{5760}\ln(2)+\frac{125173\sqrt{2}}{4320}\\ &-\frac{14\sqrt{5}}{15}\ln\left(1+\sqrt{2}\right)\ln\left(-1+\sqrt{5}+\sqrt{2}\right)+\frac{7\sqrt{5}}{15}\ln(2)\ln\left(\sqrt{5}+1\right)\\ &+\left(-\frac{13723}{640}+\frac{623}{48}\ln(2)+\frac{7\sqrt{5}}{30}\ln(2)+\frac{57217\sqrt{2}}{2880}-\frac{\sqrt{2}}{48}\ln(2)\right.\\ &\qquad\left.-\frac{14\sqrt{5}}{15}\ln\left(\sqrt{5}+2\right)\right)\ln\left(1+\sqrt{2}\right)\\ &+\left(-\frac{\sqrt{2}}{48}-\frac{769}{48}+\frac{7\sqrt{5}}{15}\right)\ln\left(1+\sqrt{2}\right)^2+\left(-\frac{\sqrt{2}}{24}-\frac{23}{24}\right)Li_2\left(-\sqrt{2}\right)\\ &+\left(-\frac{2}{3}-\frac{7\sqrt{5}}{15}\right)Li_2\left(\frac{1}{\sqrt{5}+2}\right)\\ &+\left(-\frac{2}{3}+\frac{7\sqrt{5}}{15}\right)Li_2\left(\frac{-1}{-2+\sqrt{5}}\right)\end{aligned} \tag{7.4.5}$$

$$+\left(\frac{7\sqrt{5}}{15}+\frac{2}{3}\right) Li_2\left(\frac{1+\sqrt{2}}{\sqrt{5}+2}\right)$$
$$+\left(\frac{7\sqrt{5}}{15}+\frac{2}{3}\right) Li_2\left(\frac{\sqrt{2}+\sqrt{5}}{\sqrt{5}+1}\right)$$
$$+\left(\frac{2}{3}-\frac{7\sqrt{5}}{15}\right) Li_2\left(\frac{-1-\sqrt{2}}{-2+\sqrt{5}}\right)$$
$$+\left(\frac{2}{3}-\frac{7\sqrt{5}}{15}\right) Li_2\left(\frac{-\sqrt{2}}{-1+\sqrt{5}}\right)+\frac{7\sqrt{5}}{15}\ln(2)\ln\left(-1+\sqrt{5}+\sqrt{2}\right)$$
$$+\left(-\frac{5}{24}-\frac{\sqrt{2}}{24}\right) Li_2\left(-1-\sqrt{2}\right), \tag{7.4.5}$$

$$\begin{aligned}\mathscr{B}_6(0,1) =&\frac{4091\sqrt{2}}{4320}+\frac{41\sqrt{2}}{2880}\ln(2)+\frac{1069}{384}\ln(2)-\frac{\pi^2}{288}-\frac{5}{12}Li_2\left(-\frac{\sqrt{2}}{2}\right)\\ &+\frac{\sqrt{2}\pi^2}{96}-\frac{14107}{6912}+\left(-\frac{7469}{5760}-\frac{5\sqrt{2}}{64}\right)\ln\left(1+\sqrt{2}\right)\\ &+\left(-\frac{\sqrt{2}}{24}-\frac{1}{8}\right) Li_2\left(\frac{\sqrt{2}}{2}\right),\end{aligned} \tag{7.4.6}$$

$$\begin{aligned}\mathscr{B}_7(0,1) =&\left(\frac{\sqrt{2}}{48}\ln(2)-\frac{623}{48}\ln(2)-\frac{1781\sqrt{2}}{90}+\frac{4093}{180}-\frac{7\sqrt{5}}{30}\ln(2)\right)\ln\left(1+\sqrt{2}\right)\\ &+\left(-\frac{2}{3}+\frac{7\sqrt{5}}{15}\right) Li_2\left(\frac{-\sqrt{2}}{-1+\sqrt{5}}\right)\\ &+\left(-\frac{2}{3}-\frac{7\sqrt{5}}{15}\right) Li_2\left(\frac{1+\sqrt{2}}{\sqrt{5}+2}\right)+\left(-\frac{2}{3}-\frac{7\sqrt{5}}{15}\right) Li_2\left(\frac{\sqrt{2}}{\sqrt{5}+1}\right)\\ &+\left(\frac{769}{48}-\frac{7\sqrt{5}}{15}+\frac{\sqrt{2}}{48}\right)\ln\left(1+\sqrt{2}\right)^2\\ &+\left(-\frac{1}{3}\ln(5)-\frac{7\sqrt{5}}{30}\ln(5)\right)\ln\left(\sqrt{5}+2\right)\\ &+\left(-\frac{77431}{1440}-\frac{1}{4}\ln(5)+\frac{1333}{12}\ln(2)+\frac{45181\sqrt{5}}{1440}\right)\ln\left(\sqrt{5}+1\right)\end{aligned} \tag{7.4.7}$$

$$
\begin{aligned}
&+\left(-\frac{1567}{1440}+\frac{7\sqrt{5}}{15}\ln(2)+\frac{2}{3}\ln(2)+\frac{1}{3}\ln(5)\right.\\
&\qquad\qquad\left.+\frac{659\sqrt{5}}{1440}-\frac{7\sqrt{5}}{30}\ln(5)\right)\ln\left(3+\sqrt{5}\right)\\
&+\left(-\frac{1177}{24}+\frac{7\sqrt{5}}{15}\right)\ln\left(\sqrt{5}+1\right)^2+\left(-\frac{2}{3}+\frac{7\sqrt{5}}{15}\right)\,Li_2\left(\frac{-1-\sqrt{2}}{-2+\sqrt{5}}\right)\\
&+\left(\frac{2}{3}-\frac{7}{15}\sqrt{5}\right)\,Li_2\left(\frac{2\left(-1+\sqrt{5}\right)}{-2+\sqrt{5}}\right)\\
&+\frac{39329\sqrt{5}}{8640}+34\,Li_2\left(\frac{1-\sqrt{5}}{2}\right)+\frac{5}{24}\,Li_2\left(-1-\sqrt{2}\right)\\
&+\frac{2}{3}\ln(5)\ln(2)-\frac{29}{24}\,Li_2\left(1-\sqrt{5}\right)+\frac{1351}{180}\ln(2)\sqrt{2}\\
&-\frac{16343\sqrt{5}}{720}\ln(2)+\left(-\frac{3}{8}+\frac{\sqrt{5}}{24}\right)\,Li_2\left(3-\sqrt{5}\right)-\frac{1441\sqrt{5}}{192}\ln(5)\\
&+\frac{52217}{1728}+\frac{2}{3}\,Li_2\left(\frac{\sqrt{5}}{\sqrt{5}+2}\right)+\left(-\frac{2}{3}-\frac{7\sqrt{5}}{15}\right)\ln\left(\sqrt{5}+1\right)\ln\left(3+\sqrt{5}\right)\\
&-\frac{1}{2}\,Li_2\left(-\sqrt{5}\right)-\frac{17}{24}\,Li_2\left(\frac{-2+\sqrt{5}}{2}\right)+\frac{7\sqrt{5}}{30}\ln(2)^2+\frac{\sqrt{2}}{192}\ln(2)^2\\
&+\frac{7\sqrt{5}}{15}\,Li_2\left(5-2\sqrt{5}\right)-\frac{61\sqrt{5}}{120}\,Li_2\left(\frac{-2+\sqrt{5}}{2}\right)+\frac{173\sqrt{5}\pi^2}{1440}\\
&+\frac{\sqrt{2}}{24}\,Li_2\left(-1-\sqrt{2}\right)+\frac{\sqrt{5}}{24}\,Li_2\left(1-\sqrt{5}\right)+\left(\frac{1}{8}+\frac{\sqrt{2}}{24}\right)\,Li_2\left(\frac{\sqrt{2}}{2}\right)\\
&-\frac{2693\sqrt{2}}{90}-34\,Li_2\left(\frac{-\sqrt{2}}{2}\right)+\frac{7\pi^2}{288}-\frac{11533}{192}\ln(2)^2+\frac{12203}{720}\ln(2)\\
&+\frac{3515}{192}\ln(5)+\frac{14\sqrt{5}}{15}\ln\left(1+\sqrt{2}\right)\ln\left(3+\sqrt{5}+\sqrt{2}\sqrt{5}+2\sqrt{2}\right)\\
&-\frac{7\sqrt{5}}{15}\ln(2)\ln\left(4+\sqrt{2}\sqrt{5}+\sqrt{2}\right)+\frac{5}{6}\,Li_2\left(-\sqrt{2}\right),
\end{aligned}
\tag{7.4.7}
$$

$$
\begin{aligned}
\mathscr{B}_8(0,1) =& -\frac{1}{8}\, Li_2\left(\frac{3-\sqrt{5}}{\sqrt{5}+1}\right) - \frac{835}{24}\, Li_2\left(\frac{1-\sqrt{5}}{2}\right) - \frac{4}{3}\ln(5)\ln(2) \\
& - \frac{11811527419}{1676505600} + \frac{7\sqrt{5}}{15}\ln(5)\ln(2) + \frac{7\sqrt{5}}{15}\ln(5)\ln\left(\sqrt{5}+2\right) \\
& + \frac{1571291\sqrt{5}}{66528}\ln(2) + \left(\frac{7\sqrt{5}}{15}+\frac{3}{2}\right)\ln\left(\sqrt{5}+1\right)\ln\left(3+\sqrt{5}\right) \\
& + \frac{1441\sqrt{5}}{192}\ln(5) - \frac{1}{16}\ln\left(3+\sqrt{5}\right)^2 + \frac{1}{2}\, Li_2\left(-\sqrt{5}\right) \\
& + \left(-\frac{20564213\sqrt{5}}{665280} + \frac{7}{12}\ln(5) - \frac{1367}{12}\ln(2) + \frac{3601279}{60480}\right. \\
& \qquad \left. - \frac{7\sqrt{5}}{30}\ln(5)\right)\ln\left(\sqrt{5}+1\right) \\
& + \left(-\frac{11}{8}\ln(2) - \frac{7\sqrt{5}}{15}\ln(2)\right)\ln\left(3+\sqrt{5}\right) \\
& + \left(\frac{797}{16} - \frac{7\sqrt{5}}{15}\right)\ln\left(\sqrt{5}+1\right)^2 + \left(-\frac{2}{3} - \frac{7\sqrt{5}}{15}\right) Li_2\left(\frac{\sqrt{5}}{\sqrt{5}+2}\right) \\
& + \left(\frac{9}{8} + \frac{61\sqrt{5}}{120}\right) Li_2\left(\frac{-2+\sqrt{5}}{2}\right) + \left(\frac{7\sqrt{5}}{15} + \frac{2}{3}\right) Li_2\left(\frac{1}{\sqrt{5}+2}\right) \\
& + \left(-\frac{2}{3} + \frac{7\sqrt{5}}{15}\right) Li_2\left(\frac{-\sqrt{5}}{-2+\sqrt{5}}\right) + \left(\frac{2}{3} - \frac{7\sqrt{5}}{15}\right) Li_2\left(\frac{-1}{-2+\sqrt{5}}\right) \\
& - \frac{\pi^2}{32} + \frac{1537}{24}\ln(2)^2 - \frac{428059}{11880}\ln(2) - \frac{3515}{192}\ln(5) - \frac{\sqrt{5}}{24}\, Li_2\left(1-\sqrt{5}\right) \\
& - \frac{173\sqrt{5}\pi^2}{1440} + \frac{7\sqrt{5}}{15}\ln(2)^2 + \left(-\frac{\sqrt{5}}{24} + \frac{3}{8}\right) Li_2\left(\frac{-2+2\sqrt{5}}{\sqrt{5}+1}\right),
\end{aligned}
\tag{7.4.8}
$$

$$
\begin{aligned}
\mathscr{B}_9(0,1) =& \frac{1}{8}\, Li_2\left(\frac{3-\sqrt{5}}{\sqrt{5}+1}\right) + \frac{1}{3}\, Li_2(-3) + \frac{19}{24}\, Li_2\left(\frac{1-\sqrt{5}}{2}\right) \\
& + \frac{5}{12}\, Li_2\left(\frac{-2+\sqrt{5}}{2}\right) + \frac{11}{24}\ln(5)\ln(2) - \frac{23}{24}\, Li_2\left(\frac{1}{\sqrt{5}+1}\right) \\
& - \frac{23}{24}\, Li_2\left(\frac{-1}{-1+\sqrt{5}}\right) + \frac{709109671\sqrt{5}}{335301120} - \frac{848917770971}{214592716800}
\end{aligned}
\tag{7.4.9}
$$

$$
\begin{aligned}
&-\frac{1421719\sqrt{5}}{332640}\ln(2)+\frac{1}{8}\ln\left(3+\sqrt{5}\right)^2-\frac{1}{3}Li_2\left(\frac{-3}{2}\right)\\
&+\left(-\frac{349177}{60480}+\frac{9}{4}\ln(2)+\frac{1922051\sqrt{5}}{665280}\right)\ln\left(\sqrt{5}+1\right)\\
&-\frac{19}{24}Li_2\left(\frac{-1}{2}\right)-\frac{11\pi^2}{96}+\left(\frac{1567}{1440}-\frac{659}{1440}\sqrt{5}\right)\ln\left(3+\sqrt{5}\right)\\
&-\frac{13}{6}\ln(2)^2+\frac{844498367}{56770560}\ln(2)-\frac{1188933}{6307840}\ln(3)-\frac{1825}{576}\ln(5)\\
&-\frac{37}{48}\ln\left(\sqrt{5}+1\right)^2+\frac{11}{8}\ln(2)\ln(3)-\frac{11}{24}\ln(3)\ln(5)+\frac{5}{6}Li_2\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)\\
&-\frac{1}{4}\ln\left(\sqrt{5}+1\right)\ln\left(3+\sqrt{5}\right),
\end{aligned} \tag{7.4.9}
$$

$$
\begin{aligned}
\mathscr{B}_{10}(0,1)=&\frac{1}{16}\ln\left(3+\sqrt{5}\right)^2-\frac{1}{3}Li_2(-3)+\frac{694321877083}{214592716800}-\frac{11}{24}\ln(5)\ln(2)\\
&+\frac{23}{24}Li_2\left(\frac{-1}{-1+\sqrt{5}}\right)+\frac{23}{24}Li_2\left(\frac{1}{\sqrt{5}+1}\right)+\frac{161\sqrt{5}}{48}\ln(2)+\frac{1825}{576}\ln(5)\\
&-\frac{161\sqrt{5}}{48}\ln\left(\sqrt{5}+1\right)-\frac{1}{8}\ln(2)\ln\left(3+\sqrt{5}\right)+\frac{1}{3}Li_2\left(\frac{-3}{2}\right)-\frac{\pi^2}{96}\\
&+\frac{19}{24}Li_2\left(\frac{-1}{2}\right)+\frac{7}{8}\ln(2)^2-\frac{1041382717}{170311680}\ln(2)+\frac{1188933}{6307840}\ln(3)\\
&-\frac{11}{8}\ln(2)\ln(3)+\frac{11}{24}\ln(3)\ln(5).
\end{aligned} \tag{7.4.10}
$$

Combining (7.4.1) to (7.4.10), simplifying and combining logarithms, and using dilogarithm identities (5.3.2) and values (5.3.3), we obtain

$$
\mathscr{B}(0,1)=\frac{173}{288}+\frac{1}{8}\ln(2)-\frac{5\pi^2}{72}. \tag{7.4.11}
$$

7.5 $\mathscr{B}(1,1)$

Substituting $m=n=1$ in $\mathscr{B}_1(m,n)$ of (7.1.1), and expressions $\Theta_3(A)$ and $\Phi_1(A)$ from (4.3.13) yields

$$\begin{aligned}\mathscr{B}_1(1,1) = \int_{\delta=\frac{1}{3}}^{\frac{1}{2}} \int_{\alpha=0}^{1-2\delta} \Bigg(& \int_{\beta=\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}}^{\frac{\delta+\alpha}{2}} \int_{\gamma=\delta-\beta+\alpha}^{\frac{\beta(1-\alpha)}{1-\delta}} \left(1-\frac{2\gamma\delta-2\alpha\beta-\gamma+\beta}{2(\delta-\alpha)}\right) \\ & \qquad \left(\frac{(\gamma\delta-\alpha\beta)^2}{2(\delta-\beta)(\gamma-\alpha)}\right) d\gamma\, d\beta \\ & + \int_{\beta=\frac{\delta+\alpha}{2}}^{\frac{\delta(1-\delta)}{1-\alpha}} \int_{\gamma=\beta}^{\frac{\beta(1-\alpha)}{1-\delta}} \left(1-\frac{2\gamma\delta-2\alpha\beta-\gamma+\beta}{2(\delta-\alpha)}\right) \\ & \qquad \left(\frac{(\gamma\delta-\alpha\beta)^2}{2(\delta-\beta)(\gamma-\alpha)}\right) d\gamma\, d\beta \\ & + \int_{\beta=\frac{\delta(1-\delta)}{1-\alpha}}^{\frac{\delta}{1-\alpha+\delta}} \int_{\gamma=\beta}^{\frac{\delta-\beta+\alpha\beta}{\delta}} \left(1-\frac{2\gamma\delta-2\alpha\beta-\gamma+\beta}{2(\delta-\alpha)}\right) \\ & \qquad \left(\frac{(\gamma\delta-\alpha\beta)^2}{2(\delta-\beta)(\gamma-\alpha)}\right) d\gamma\, d\beta \Bigg) d\alpha\, d\delta.\end{aligned}$$

Starting integration of the 3 inner double integrals with respect to γ and taking partial fractions with respect to β of each result, we obtain

$$\begin{aligned}
\mathscr{B}_1(1,1) = \int_{\delta=\frac{1}{3}}^{\frac{1}{2}} \int_{\alpha=0}^{1-2\delta} \Bigg(\int_{\beta=\frac{(\delta+\alpha)(1-\delta)}{2-\alpha-\delta}}^{\frac{\delta+\alpha}{2}} & \Big(-\frac{\delta}{24\,(\delta-\alpha)\,(1-\delta)^3} \Big(-81\alpha\delta^5 - 20\delta^6 + 12\alpha\delta^2 \\
& + 27\alpha^2\delta - 93\delta^2\alpha^2 + 6\alpha^3\delta + 45\alpha^3\delta^2 \\
& + 13\delta^3 - 35\delta^4 - 12\alpha^3 + 144\alpha^2\delta^3 + 4\delta^7 \\
& - 120\alpha^2\delta^4 - 59\alpha^3\delta^3 + 22\alpha^3\delta^4 - 78\delta^3\alpha \\
& + 135\alpha\,\delta^4 + 36\delta^5 + 36\alpha^2\delta^5 + 18\delta^6\alpha \Big) \\
& + \frac{\beta}{24\,(\delta-\alpha)\,(1-\delta)^3} \Big(36\delta^6\alpha + 8\delta^7 + 72\alpha\delta^2 \\
& - 216\delta^3\alpha - 108\delta^2\alpha^2 + 303\alpha^2\delta^3 + 8\delta^3 \\
& + 99\alpha^3\delta^2 - 115\alpha^3\delta^3 - 40\delta^4 + 6\alpha^3 + 63\delta^5 \\
& - 6\alpha^2\delta - 36\alpha^3\delta - 255\alpha^2\delta^4 + 72\alpha^2\delta^5 \\
& + 44\alpha^3\delta^4 + 255\alpha\delta^4 - 153\alpha\delta^5 - 37\delta^6 \Big) \\
& + \frac{(2-\alpha-\delta)\,\beta^2}{24\,(\delta-\alpha)\,(1-\delta)^3} \Big(42\alpha^2\delta - 6\alpha^2 + 29\alpha\delta^2 \\
& - 35\delta^3\alpha + 14\alpha\delta^4 + 22\alpha^2\delta^3 - 56\delta^2\alpha^2 \\
& + 9\delta^3 - 12\alpha\delta + 4\delta^5 - 9\delta^4 - 2\delta^2 \Big) \\
& - \frac{(-5+7\delta)\,(\delta-\alpha)^2\,\delta^2}{24\,(\delta-\beta)\,(1-\delta)^3} \\
& + \frac{\alpha^2\,(\delta-\beta)}{4\,(\delta-\alpha)} \Big(-\beta + 2\alpha\beta - 2\alpha\delta \\
& + 2\delta - \alpha \Big) \ln(\beta - \alpha\beta - \alpha + \alpha\delta) \\
& - \frac{\alpha^2\,(\delta-\beta)}{4\,(\delta-\alpha)} \Big(-\beta + 2\alpha\beta - 2\alpha\delta \\
& + 2\delta - \alpha \Big) \ln(\delta-\beta)
\end{aligned}$$

$$- \frac{\alpha^2(\delta-\beta)}{4(\delta-\alpha)}\Big(-\beta+2\alpha\beta-2\alpha\delta$$

$$+2\delta-\alpha\Big)\ln(1-\delta)\Big)\,d\beta$$

$$+\int_{\beta=\frac{\delta+\alpha}{2}}^{\frac{\delta(1-\delta)}{1-\alpha}}\left(\frac{\delta^2\left(-12+4\delta^3-17\delta^2+27\delta\right)(\delta-\alpha)^2}{24(1-\delta)^3}\right.$$

$$-\frac{\delta\beta}{24(1-\delta)^3}\Big(-36\alpha\delta+12\delta^2+18\alpha^2-27\alpha^2\delta$$

$$-27\delta^3+78\alpha\delta^2-46\delta^3\alpha+8\alpha\delta^4-\delta^2\alpha^2$$

$$+8\alpha^2\delta^3-4\delta^5+17\delta^4\Big)$$

$$+\frac{\beta^2}{24(1-\delta)^3}\Big(42\alpha^2\delta-6\alpha^2+9\delta^3-27\alpha\delta^2$$

$$-11\delta^4+37\delta^3\alpha-56\delta^2\alpha^2+22\alpha^2\delta^3$$

$$-14\alpha\delta^4+4\delta^5\Big)$$

$$-\frac{\left(-12+4\delta^3-17\delta^2+27\delta\right)(\delta-\alpha)^2\delta^3}{24(\delta-\beta)(1-\delta)^3}$$

$$+\frac{\alpha^2(\delta-\beta)}{4(\delta-\alpha)}\Big(-\beta+2\alpha\beta-2\alpha\delta$$

$$+2\delta-\alpha\Big)\ln(\beta-\alpha\beta-\alpha+\alpha\delta)$$

$$-\frac{\alpha^2(\delta-\beta)}{4(\delta-\alpha)}\Big(-\beta+2\alpha\beta-2\alpha\delta$$

$$+2\delta-\alpha\Big)\ln(\beta-\alpha)$$

$$-\frac{\alpha^2(\delta-\beta)}{4(\delta-\alpha)}\Big(-\beta+2\alpha\beta-2\alpha\delta$$

$$\left.+2\delta-\alpha\Big)\ln(1-\delta)\right)d\beta$$

$$+\int_{\beta=\frac{\delta(1-\delta)}{1-\alpha}}^{\frac{\delta}{1-\alpha+\delta}}\left(\frac{\delta}{24(\delta-\alpha)}\Big(-3\alpha-6\alpha^2+2-15\alpha\delta^2-12\delta^2\alpha^2\right.$$

$$+2\delta+4\alpha^3\delta+3\alpha^2\delta+6\alpha\delta-5\alpha^3+5\delta^3$$

$$+12\delta^3\alpha-4\delta^4\Big)$$

$$
\begin{aligned}
&- \frac{\beta}{24\,(\delta-\alpha)}\Big(-3\alpha - 6\alpha^2 + 4 + 8\alpha^3\delta + 6\alpha\delta \\
&\qquad - 5\delta^3 + 27\alpha\delta^2 - 57\alpha^2\delta + 11\alpha^3 - 12\delta^3\alpha \\
&\qquad\qquad + \delta + 4\delta^4\Big) \\
&- \frac{(1-\alpha+\delta)\,\beta^2}{24\,(\delta-\alpha)\,\delta}\Big(22\alpha^2\delta - 2\alpha^2 - 3\delta^2 + 3\delta - 2 \\
&\qquad + 5\alpha\delta - 14\alpha\delta^2 - 2\alpha + 4\delta^3\Big) \\
&+ \frac{(-5+4\delta)\,(\delta-\alpha)^2\,\delta^2}{24\,(\delta-\beta)} \\
&+ \frac{\alpha^2\,(\delta-\beta)}{4\,(\delta-\alpha)}\Big(-\beta + 2\alpha\beta - 2\alpha\delta \\
&\qquad + 2\delta - \alpha\Big)\ln(1-\alpha) \\
&+ \frac{\alpha^2\,(\delta-\beta)}{4\,(\delta-\alpha)}\Big(-\beta + 2\alpha\beta - 2\alpha\delta \\
&\qquad + 2\delta - \alpha\Big)\ln(\delta-\beta) \\
&- \frac{\alpha^2\,(\delta-\beta)}{4\,(\delta-\alpha)}\Big(-\beta + 2\alpha\beta - 2\alpha\delta \\
&\qquad + 2\delta - \alpha\Big)\ln(\beta-\alpha) \\
&- \frac{\alpha^2\,(\delta-\beta)}{4\,(\delta-\alpha)}\Big(-\beta + 2\alpha\beta - 2\alpha\delta \\
&\qquad + 2\delta - \alpha\Big)\ln(\delta)\Big)\,d\beta\Big)\,d\alpha\,d\delta.
\end{aligned}
$$

Integrating the 3 inner integrals with respect to β, combining and simplifying the results, and taking partial fractions with respect to α yields

$$\begin{aligned}
\mathscr{B}_1(1,1) = \int_{\delta=\frac{1}{3}}^{\frac{1}{2}} \int_{\alpha=0}^{1-2\delta} \Bigg(& -\frac{5\delta^6 - 18\delta^5 + 27\delta^4 - 24\delta^3 + 124\delta^2 - 266\delta + 148}{48\,(1-\delta)} \\
& - \frac{\left(-52\delta^3 + 57\delta^4 - 132\delta - 28\delta^5 + 98\delta^2 + 5\delta^6 + 56\right)\alpha}{48\,(1-\delta)^2} \\
& - \frac{\left(-14\delta^4 - 11\delta + 12\delta^3 + 5\delta^5 + 8 - 2\delta^2\right)\alpha^2}{24\,(1-\delta)^2} \\
& + \frac{\delta\,(5\delta-9)\,\alpha^3}{24} + \left(\frac{5}{48}\delta + \frac{5}{48}\right)\alpha^4 - \frac{5\alpha^5}{48} \\
& - \frac{(1-\delta)\left(6\delta^2 - 15\delta - 2\right)}{24\,(1-\alpha)^2} - \frac{(\delta+2)\,(1-\delta)^2\,\delta}{24\,(1-\alpha)^3} \\
& - \frac{(1-2\delta)\,\delta}{24\,(1-\alpha+\delta)} + \frac{\delta^2}{12\,(1-\alpha+\delta)^2} - 2\frac{11\delta-12}{6-3\alpha-3\delta} \\
& - \frac{\left(4\alpha^2 - 5\alpha + 4\alpha\delta + 4\delta^2 - 5\delta\right)(\delta-\alpha)^3}{24}\ln(2) \\
& - \frac{(\delta-\alpha)^2}{24\,(1-\delta)^3}\Big(15\delta^2\alpha^2 - 5\alpha^2\delta^3 + 7\delta^3 - 5\delta^2 - 12\alpha^3\delta^2 \\
& \qquad + 4\alpha^3\delta^3 + 5\alpha^2 - 4\alpha^3 - 15\alpha^2\delta + 12\alpha^3\delta\Big)\ln(1-\alpha) \\
& - \frac{4\,(1-\delta)}{3\,(2-\alpha-\delta)^2} + \frac{34\delta^2 - 29\delta - 16}{24\,(1-\alpha)} \\
& + \frac{\left(4\alpha^2 - 5\alpha + 4\alpha\delta + 4\delta^2 - 5\delta\right)(\delta-\alpha)^3}{24}\ln(1-\alpha+\delta) \\
& + \frac{(\delta-\alpha)^3}{24\,(1-\alpha)^3\,(1-\delta)^3}\Big(7\alpha^2 - 21\alpha^2\delta + 16\delta^2\alpha^2 + 22\alpha\delta \\
& \qquad - 5\alpha - 21\alpha\delta^2 + 7\delta^2 - 5\delta\Big)\ln(2-\alpha-\delta) \\
& + \frac{\left(-12 + 4\delta^3 - 17\delta^2 + 27\delta\right)(\delta-\alpha)^2\,\delta^3\ln(\delta)}{24\,(1-\delta)^3}\Bigg)\,d\alpha\,d\delta.
\end{aligned}$$

Integrating with respect to α and then taking partial fractions with respect to δ, we find

$$\begin{aligned}
\mathscr{B}_1(1,1) = \int_{\delta=\frac{1}{3}}^{\frac{1}{2}} \Bigg(& -\frac{7319}{1728} + \frac{6547\delta}{360} - \frac{141031\delta^2}{8640} + \frac{207\delta^3}{160} - \frac{787\delta^4}{1440} + \frac{2051\delta^5}{2160} \\
& - \frac{359\delta^6}{540} + \frac{1}{96\delta} - \frac{4}{-3\delta+6} - \frac{11}{4(1-\delta)} + \frac{10}{9(1-\delta)} - \frac{1}{9(1-\delta)^2} \\
& - \frac{1}{720(1-\delta)^3}\Big(7632\delta^9 - 37436\delta^8 + 76146\delta^7 - 84172\delta^6 \\
& + 51810\delta^5 - 12843\delta^4 - 5231\delta^3 + 5301\delta^2 - 1377\delta + 10\Big)\ln(2) \\
& + \left(\frac{2\delta^2}{3} - \frac{9\delta}{4} + \frac{5}{3}\right) Li_2\left(\frac{-1}{1-\delta}\right) \\
& + \left(\frac{2\delta^2}{3} - \frac{9\delta}{4} + \frac{5}{3}\right)\ln(\delta)\ln(1-\delta) \\
& + \left(\frac{2\delta^2}{3} - \frac{9\delta}{4} + \frac{5}{3}\right)\ln(2)\ln(1-\delta) \\
& + \left(\frac{-2\delta^2}{3} + \frac{9\delta}{4} - \frac{5}{3}\right) Li_2\left(\frac{-2\delta}{1-\delta}\right) \\
& - \frac{1}{72(1-\delta)^3}\Big(-800 - 1593\delta^4 + 3360\delta + 4327\delta^3 - 5498\delta^2 \\
& + 202\delta^5\Big)\ln(2-\delta) \\
& - \frac{1}{1440(1-\delta)^3\delta^2}\Big(76\delta^{11} - 318\delta^{10} + 498\delta^9 - 346\delta^8 + 9470\delta^7 \\
& - 8364\delta^6 - 36803\delta^5 + 79913\delta^4 - 60156\delta^3 + 16860\delta^2 \\
& - 525\delta + 15\Big)\ln(1+\delta) \\
& - \frac{1}{240}\delta\left(20 - 450\delta^2 - 40\delta + 1890\delta^3 - 2835\delta^4 + 1458\delta^5\right)\ln(3) \\
& + \frac{\delta}{720(1-\delta)^3}\Big(38\delta^8 - 159\delta^7 + 249\delta^6 - 173\delta^5 + 1225\delta^4 \\
& - 4470\delta^3 + 6730\delta^2 - 4620\delta + 1200\Big)\ln(\delta)\Bigg)\, d\delta.
\end{aligned}$$

Integrating with respect to δ and simplifying, we obtain

$$\begin{aligned}\mathscr{B}_1(1,1) =& -\frac{79}{9}\,Li_2\left(\frac{1}{4}\right)+\frac{31}{72}\,Li_2\left(\frac{-1}{2}\right)+\frac{175002521}{6531840}\ln(2)+\frac{2117\pi^2}{23328}\\ &+\frac{79}{243}(\ln(2))^2+\frac{79}{243}(\ln(3))^2-\frac{158}{243}\ln(3)\ln(2)-\frac{55}{72}\,Li_2(-3)\\ &+\frac{7014381113}{4232632320}+\frac{55}{36}\,Li_2(-2)-\frac{19313}{7776}\ln(5)-\frac{101700169}{6531840}\ln(3)\\ &+\frac{79}{243}\,Li_2\left(\frac{-3}{2}\right)+\frac{79}{9}\,Li_2\left(\frac{1}{3}\right)+\frac{1}{3}\,Li_2\left(\frac{-1}{3}\right)-\frac{55}{72}\,Li_2\left(\frac{-2}{3}\right).\end{aligned} \tag{7.5.1}$$

We carry out the four-fold integration of remaining integral groupings of (7.1.1) with $m=1$ and $n=1$, yielding results

$$\begin{aligned}\mathscr{B}_2(1,1) =& \frac{218035697}{132269760}-\frac{2117\pi^2}{23328}-\frac{86747}{90720}\ln(2)-\frac{79}{243}\,Li_2\left(\frac{-3}{2}\right)+\frac{19313}{7776}\ln(5)\\ &-\frac{79}{243}\ln(2)^2+\frac{158}{243}\ln(3)\ln(2)-\frac{2749}{3888}\ln(3)^2-\frac{3419053}{816480}\ln(3)\\ &+\frac{55}{72}\,Li_2\left(\frac{-2}{3}\right)-\frac{55}{72}\,Li_2(-2),\end{aligned} \tag{7.5.2}$$

$$\begin{aligned}\mathscr{B}_3(1,1) =& -\frac{1748156341}{52254720}\ln(2)-\frac{101791}{102060}-\frac{53\pi^2}{3456}-\frac{163}{288}\ln(2)^2+\frac{133072957}{5806080}\ln(3)\\ &-\frac{91}{96}\,Li_2(-2)+\frac{55}{144}\ln(3)^2+\frac{79}{9}\,Li_2\left(\frac{1}{4}\right)+\frac{1}{3}\,Li_2\left(\frac{-1}{2}\right)\\ &+\frac{55}{72}\,Li_2(-3)-\frac{79}{9}\,Li_2\left(\frac{1}{3}\right)-\frac{1}{3}\,Li_2\left(\frac{-1}{3}\right),\end{aligned} \tag{7.5.3}$$

$$\begin{aligned}\mathscr{B}_4(1,1) =& \frac{951659}{387072}\ln(2)-\frac{167\pi^2}{3456}+\frac{11013967}{5806080}+\frac{53}{288}\,Li_2(-2)+\frac{163}{288}\ln(2)^2\\ &-\frac{55}{72}\,Li_2\left(\frac{-1}{2}\right)-\frac{226661}{71680}\ln(3),\end{aligned} \tag{7.5.4}$$

$$
\begin{aligned}
\mathscr{B}_5(1,1) =& \frac{15736163}{241920}\ln(2) - \frac{29192929}{1451520} + \frac{14767}{144}\,Li_2\left(\frac{-\sqrt{2}}{2}\right) - \frac{97}{72}\,Li_2\left(\frac{1}{\sqrt{5}+2}\right) \\
&+ \left(-\frac{1651}{900}\sqrt{5}\ln\left(\sqrt{5}+2\right) + \frac{6509}{108}\sqrt{2} - \frac{778759}{16128} + \frac{\sqrt{5}}{40} + \frac{2797}{144}\ln(2)\right. \\
&\qquad \left. - \frac{1651\sqrt{5}}{900}\ln\left(-1+\sqrt{5}+\sqrt{2}\right) + \frac{1651\sqrt{5}}{3600}\ln(2)\right)\ln\left(1+\sqrt{2}\right) \\
&+ \left(\frac{1651\sqrt{5}}{1800} - \frac{6979}{72}\right)\ln\left(1+\sqrt{2}\right)^2 + \left(-\frac{241}{144} - \frac{7}{72}\sqrt{2}\right)\,Li_2\left(-\sqrt{2}\right) \\
&+ \left(-\frac{\sqrt{5}}{20} + \frac{1651\sqrt{5}}{1800}\ln(2)\right)\ln\left(-1+\sqrt{5}+\sqrt{2}\right) - \frac{35\sqrt{2}\pi^2}{1728} \\
&+ \left(-\frac{1651\sqrt{5}}{1800} + \frac{97}{72}\right)\,Li_2\left(\frac{-1-\sqrt{2}}{-2+\sqrt{5}}\right) - \frac{162937\sqrt{2}}{4032}\ln(2) \\
&+ \left(\frac{1651}{1800}\sqrt{5} + \frac{97}{72}\right)\,Li_2\left(\frac{1+\sqrt{2}}{\sqrt{5}+2}\right) + \left(\frac{1651\sqrt{5}}{1800} + \frac{97}{72}\right)\,Li_2\left(\frac{\sqrt{2}}{\sqrt{5}+1}\right) \\
&+ \left(-\frac{1651}{1800}\sqrt{5} + \frac{97}{72}\right)\,Li_2\left(\frac{-\sqrt{2}}{-1+\sqrt{5}}\right) + \frac{41056459\sqrt{2}}{1451520} - \frac{7\sqrt{2}}{288}\ln(2)^2 \\
&- \frac{7}{72}\,Li_2\left(\frac{-\sqrt{2}}{2}\right)\sqrt{2} - \frac{1651\sqrt{5}}{1200}\ln(2)^2 - \frac{387}{4}\,Li_2\left(-1-\sqrt{2}\right) \\
&- \frac{1651}{1800}\,Li_2\left(\frac{1}{\sqrt{5}+2}\right)\sqrt{5} + \frac{3}{40}\sqrt{5}\ln(2) - \frac{27919\pi^2}{3456} - \frac{1039}{32}\ln(2)^2 \\
&+ \left(-\frac{\sqrt{5}}{20} + \frac{1651\sqrt{5}}{1800}\ln(2)\right)\ln\left(\sqrt{5}+1\right) \\
&+ \left(\frac{1651\sqrt{5}}{1800} - \frac{97}{72}\right)\,Li_2\left(\frac{-1}{-2+\sqrt{5}}\right)
\end{aligned}
\tag{7.5.5}
$$

$$
\begin{aligned}
\mathscr{B}_6(1,1) =& \left(-\frac{11}{48} - \frac{7\sqrt{2}}{72}\right)\,Li_2\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{11}{48} + \frac{7}{72}\sqrt{2}\right)\,Li_2\left(\frac{-\sqrt{2}}{2}\right) + \frac{7\sqrt{2}\pi^2}{288} \\
&+ \left(-\frac{652087}{241920} + \frac{139\sqrt{2}}{630}\right)\ln\left(1+\sqrt{2}\right) + \frac{2225141\sqrt{2}}{1451520} + \frac{457703}{80640}\ln(2) \\
&- \frac{1032683}{290304} - \frac{11\pi^2}{1728} - \frac{55}{72}\,Li_2\left(\frac{-\sqrt{2}}{2}\right) - \frac{28081\sqrt{2}}{60480}\ln(2),
\end{aligned}
\tag{7.5.6}
$$

$$
\begin{aligned}
\mathscr{B}_7(1,1) =& \frac{79489}{3024}\ln(2) + \frac{28873}{1152}\ln(5) + \left(\frac{27839}{288} - \frac{1651\sqrt{5}}{1800} + \frac{7\sqrt{2}}{144}\right)\ln\left(1+\sqrt{2}\right)^2 \\
&+ \left(\frac{1651\sqrt{5}}{1800} - \frac{97}{72}\right) Li_2\left(-\frac{1+\sqrt{2}}{-2+\sqrt{5}}\right) - \frac{304241\sqrt{5}}{51840} + \frac{7\sqrt{2}}{576}\ln(2)^2 \\
&+ \left(\frac{1651\sqrt{5}}{1800} - \frac{97}{72}\right) Li_2\left(-\frac{\sqrt{2}}{-1+\sqrt{5}}\right) + \frac{19243373}{362880} + \frac{1651\sqrt{5}}{3600}\ln(2)^2 \\
&+ \left(-\frac{1651\sqrt{5}}{1800} + \frac{97}{72}\right) Li_2\left(-\frac{\sqrt{5}}{-2+\sqrt{5}}\right) - \frac{45085\sqrt{2}}{1512} + \frac{10031\sqrt{5}\pi^2}{43200} \\
&+ \left(\frac{5\sqrt{5}}{144} - \frac{289}{144}\right) Li_2\left(-\sqrt{5}+1\right) + \left(\frac{7}{72}\sqrt{2} + \frac{13855}{144}\right) Li_2\left(\sqrt{2}+2\right) \\
&+ \left(-\frac{387}{8}\ln(5) + \frac{107335\sqrt{5}}{12096} - \frac{3747641}{60480} + \frac{22769}{72}\ln(2)\right)\ln\left(\sqrt{5}+1\right) \\
&+ \left(-\frac{1651\sqrt{5}}{3600}\ln(5) - \frac{97}{144}\ln(5)\right)\ln\left(\sqrt{5}+2\right) \\
&+ \left(\frac{1651\sqrt{5}}{1800} + \frac{97}{72}\right) Li_2\left(\frac{\sqrt{5}}{\sqrt{5}+2}\right) + \left(\frac{11}{48} + \frac{7\sqrt{2}}{72}\right) Li_2\left(1 - 1/2\sqrt{2}\right) \\
&+ \left(-\frac{5671}{288}\ln(2) + \frac{385421}{7560} + \frac{7\sqrt{2}}{144}\ln(2) - \frac{228649\sqrt{2}}{3780} + \frac{\sqrt{5}}{40}\right. \\
&\left. + \frac{1651\sqrt{5}}{900}\ln\left(3+\sqrt{5}+2\sqrt{2}+\sqrt{5}\sqrt{2}\right) - \frac{1651\sqrt{5}}{3600}\ln(2)\right)\ln\left(1+\sqrt{2}\right) \\
&+ \left(-\frac{14453}{144} + \frac{1651\sqrt{5}}{1800}\right)\ln\left(\sqrt{5}+1\right)^2 - \frac{\sqrt{5}}{20}\ln\left(5 - \sqrt{5}\sqrt{2} + \sqrt{5}\right) \\
&+ \left(\frac{97}{144}\ln(5) + \frac{97}{72}\ln(2) - \frac{1651\sqrt{5}}{3600}\ln(5) + \frac{23791\sqrt{5}}{60480}\right. \\
&\qquad\qquad \left. + \frac{1651\sqrt{5}}{1800}\ln(2) - \frac{61319}{60480}\right)\ln\left(3+\sqrt{5}\right) \\
&+ \left(-\frac{97}{72} - \frac{1651\sqrt{5}}{1800}\right) Li_2\left(\frac{\sqrt{2}}{\sqrt{5}+1}\right) + \left(\frac{5\sqrt{5}}{144} - \frac{9}{16}\right) Li_2\left(\frac{3-\sqrt{5}}{\sqrt{5}+1}\right) \\
&+ \left(-\frac{97}{72} - \frac{1651\sqrt{5}}{1800}\right) Li_2\left(\frac{1+\sqrt{2}}{\sqrt{5}+2}\right) + \frac{405}{4} Li_2\left(\frac{1-\sqrt{5}}{2}\right) \\
&+ \left(-\frac{223}{144} - \frac{3427\sqrt{5}}{3600}\right) Li_2\left(\frac{\sqrt{5}-1}{2}\right) - \frac{405}{4} Li_2\left(\frac{-\sqrt{2}}{2}\right)
\end{aligned}
\tag{7.5.7}
$$

$$+\frac{309017\sqrt{2}}{7560}\ln(2)-\frac{2585\sqrt{5}}{54}\ln(2)+\frac{13}{9}\,Li_2\left(-\sqrt{2}\right)$$
$$+\left(-\frac{97}{72}-\frac{1651\sqrt{5}}{1800}\right)\ln\left(\sqrt{5}+1\right)\ln\left(3+\sqrt{5}\right)$$
$$+\frac{97}{72}\ln(5)\ln(2)-\frac{387}{4}\,Li_2\left(-\sqrt{5}\right)+\frac{125387}{8064}\ln(5)\sqrt{5}$$
$$-\frac{1651\sqrt{5}}{1800}\ln(2)\ln\left(4+\sqrt{2}+\sqrt{5}\sqrt{2}\right),$$

(7.5.7)

$$\mathscr{B}_8(1,1)=\left(-\frac{1651\sqrt{5}}{1800}+\frac{97}{72}\right)\,Li_2\left(\frac{-1}{-2+\sqrt{5}}\right)+\frac{1281499361\sqrt{5}}{25945920}\ln(2)$$
$$+\left(-\frac{1651\sqrt{5}}{1800}\ln(2)-\frac{79}{32}\ln(2)\right)\ln\left(3+\sqrt{5}\right)-\frac{28873}{1152}\ln(5)$$
$$+\left(\frac{1651\sqrt{5}}{1800}+\frac{97}{72}\right)\,Li_2\left(\frac{1}{\sqrt{5}+2}\right)+\frac{1651\sqrt{5}}{1800}\ln(5)\ln\left(\sqrt{5}+2\right)$$
$$-\frac{29557}{288}\,Li_2\left(\frac{1-\sqrt{5}}{2}\right)-\frac{200383387}{1853280}\ln(2)-\frac{3936561848567}{130767436800}$$
$$+\frac{387}{4}\,Li_2\left(-\sqrt{5}\right)+\left(\frac{1651\sqrt{5}}{1800}+\frac{67}{24}\right)\ln\left(\sqrt{5}+1\right)\ln\left(3+\sqrt{5}\right)$$
$$+\frac{1651\sqrt{5}}{1800}\ln(5)\ln(2)-\frac{31}{192}\ln\left(3+\sqrt{5}\right)^2+\frac{1651\sqrt{5}}{1800}\ln(2)^2$$
$$-\frac{10031\sqrt{5}\pi^2}{43200}+\frac{571\pi^2}{72}+\frac{63139}{288}\ln(2)^2+\frac{5735960129\sqrt{5}}{5230697472}$$
$$+\left(\frac{7063}{144}\ln(5)-\frac{361461337\sqrt{5}}{51891840}+\frac{25973231}{362880}-\frac{46235}{144}\ln(2)\right.$$
$$\left.-\frac{1651\sqrt{5}}{3600}\ln(5)\right)\ln\left(\sqrt{5}+1\right)$$
$$+\left(-\frac{5\sqrt{5}}{144}+\frac{9}{16}\right)\,Li_2\left(\frac{2(\sqrt{5}-1)}{\sqrt{5}+1}\right)-\frac{31}{96}\,Li_2\left(-\frac{\sqrt{5}-3}{\sqrt{5}+1}\right)$$
$$+\left(\frac{19523}{192}-\frac{1651\sqrt{5}}{1800}\right)\ln\left(\sqrt{5}+1\right)^2-\frac{625927\sqrt{5}}{40320}\ln(5)$$
$$+\left(\frac{1651\sqrt{5}}{1800}-\frac{97}{72}\right)\,Li_2\left(\frac{-\sqrt{5}}{-2+\sqrt{5}}\right)-\frac{97}{36}\ln(5)\ln(2)$$

(7.5.8)

$$+\left(\frac{3427\sqrt{5}}{3600}+\frac{107}{48}\right) Li_2\left(\frac{1-\sqrt{5}}{2}\right)$$
$$+\left(\frac{289}{144}-\frac{5\sqrt{5}}{144}\right) Li_2\left(-\sqrt{5}+1\right)$$
$$+\left(-\frac{97}{72}-\frac{1651\sqrt{5}}{1800}\right) Li_2\left(\frac{\sqrt{5}}{\sqrt{5}+2}\right),$$

(7.5.8)

$$\mathscr{B}_9(1,1) = \frac{29310078449}{1107025920}\ln(2)+\frac{55}{72}\,Li_2\left(\frac{-1+\sqrt{5}}{2}\right)-\frac{509}{288}\,Li_2\left(\frac{1}{\sqrt{5}-1}\right)$$
$$+\frac{1358057}{7028736}\ln(3)-\frac{167}{288}\,Li_2\left(\frac{-3}{2}\right)-\frac{76961}{13824}\ln(5)+\frac{221}{96}\ln(3)\ln(2)$$
$$+\left(\frac{86885893\sqrt{5}}{51891840}+\frac{97}{24}\ln(2)-\frac{697477}{72576}\right)\ln\left(\sqrt{5}+1\right)-\frac{27\pi^2}{128}$$
$$+\left(\frac{61319}{60480}+\frac{149923\sqrt{5}}{120960}\right)\ln\left(3+\sqrt{5}\right)-\frac{31}{48}\ln\left(\sqrt{5}+1\right)\ln\left(3+\sqrt{5}\right)$$
$$+\frac{397}{288}\,Li_2\left(\frac{1-\sqrt{5}}{2}\right)-\frac{757}{576}\ln\left(\sqrt{5}+1\right)^2+\frac{31}{96}\ln\left(3+\sqrt{5}\right)^2$$
$$-\frac{348241177\sqrt{5}}{51891840}\ln(2)-\frac{76391928061061}{8369115955200}-\frac{509}{288}\,Li_2\left(\frac{1}{\sqrt{5}+1}\right)$$
$$-\frac{2117}{576}\ln(2)^2+\frac{31}{96}\,Li_2\left(\frac{3-\sqrt{5}}{\sqrt{5}+1}\right)+\frac{124811000819\sqrt{5}}{26153487360}$$
$$-\frac{221}{288}\ln(3)\ln(5)+\frac{13}{9}\,Li_2\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)+\frac{167}{288}\,Li_2(-3)$$
$$-\frac{397}{288}\,Li_2\left(\frac{-1}{2}\right)+\frac{221}{288}\ln(5)\ln(2),$$

(7.5.9)

$$
\begin{aligned}
\mathscr{B}_{10}(1,1) = &-\frac{34035755219}{3321077760}\ln(2) - \frac{1358057}{7028736}\ln(3) + \frac{167}{288}\,Li_2\left(\frac{-3}{2}\right) + \frac{76961}{13824}\ln(5) \\
&+ \frac{58214748198469}{8369115955200} - \frac{221}{96}\ln(3)\ln(2) - \frac{31}{96}\ln(2)\ln\left(3+\sqrt{5}\right) \\
&+ \frac{509}{288}\,Li_2\left(\frac{1}{\sqrt{5}+1}\right) + \frac{31}{192}\ln\left(3+\sqrt{5}\right)^2 - \frac{31\pi^2}{1152} + \frac{275}{192}\ln(2)^2 \\
&- \frac{3263\sqrt{5}}{480}\ln\left(\sqrt{5}+1\right) + \frac{221}{288}\ln(3)\ln(5) + \frac{3263\sqrt{5}}{480}\ln(2) \\
&- \frac{167}{288}\,Li_2(-3) + \frac{509}{288}\,Li_2\left(\frac{-1}{\sqrt{5}-1}\right) + \frac{397}{288}\,Li_2\left(\frac{-1}{2}\right) \\
&- \frac{221}{288}\ln(5)\ln(2)\,.
\end{aligned}
\tag{7.5.10}
$$

Combining (7.5.1) to (7.5.10), simplifying by combining logarithms and dilogarithms, and using relations (5.3.2) and dilogrithm values (5.3.3), we obtain

$$
\mathscr{B}(1,1) = \frac{26009}{20160} - \frac{47}{1008}\ln(2) - \frac{55\pi^2}{432}. \tag{7.5.11}
$$

Chapter 8

CASA= 3, MAISON= 2

8.1 Setup

Using limits (5.2.17), we rewrite integral (5.1.3) as

$$\begin{aligned}\mathscr{C}(m,n) =&\mathscr{C}_1(m,n)+\mathscr{C}_2(m,n)+\mathscr{C}_3(m,n)+\mathscr{C}_4(m,n)+\mathscr{C}_5(m,n)\\ &+\mathscr{C}_6(m,n)+\mathscr{C}_7(m,n),\end{aligned}$$

with,

$$\begin{aligned}
\mathscr{C}_1(m,n) &= \int_{\alpha=0}^{\frac{1}{2}} \int_{\delta=\alpha}^{1-\alpha} \int_{\beta=\alpha}^{\frac{\alpha}{1-\delta+\alpha}} \int_{\gamma=\beta}^{\frac{\alpha(1-\beta)}{1-\delta}} \Theta_3(A)^m \Phi_2(A)^n \, d\gamma \, d\beta \, d\delta \, d\alpha,\\
\mathscr{C}_2(m,n) &= \int_{\alpha=0}^{\frac{1}{2}} \int_{\delta=1-\alpha}^{\sqrt{1-\alpha}} \int_{\beta=\alpha}^{\frac{\delta}{1-\alpha+\delta}} \int_{\gamma=\beta}^{\frac{\delta-\beta+\alpha\beta}{\delta}} \Theta_3(A)^m \Phi_2(A)^n \, d\gamma \, d\beta \, d\delta \, d\alpha,\\
\mathscr{C}_3(m,n) &= \int_{\alpha=0}^{\frac{2}{3}} \int_{\delta=1-\frac{\alpha}{2}}^{1} \int_{\beta=\alpha}^{\frac{\delta}{1-\alpha+\delta}} \int_{\gamma=\beta}^{\frac{\delta-\beta+\alpha\beta}{\delta}} \Theta_3(A)^m \Phi_2(A)^n \, d\gamma \, d\beta \, d\delta \, d\alpha,\\
\mathscr{C}_4(m,n) &= \int_{\alpha=0}^{\frac{-1+\sqrt{5}}{2}} \int_{\delta=\sqrt{1-\alpha}}^{1-\frac{\alpha}{2}} \int_{\beta=\alpha}^{\frac{\delta}{1-\alpha+\delta}} \int_{\gamma=\beta}^{\frac{\delta-\beta+\alpha\beta}{\delta}} \Theta_3(A)^m \Phi_2(A)^n \, d\gamma \, d\beta \, d\delta \, d\alpha,\\
\mathscr{C}_5(m,n) &= \int_{\alpha=\frac{2}{3}}^{1} \int_{\delta=\alpha}^{1} \int_{\beta=\alpha}^{\frac{\delta}{1-\alpha+\delta}} \int_{\gamma=\beta}^{\frac{\delta-\beta+\alpha\beta}{\delta}} \Theta_3(A)^m \Phi_2(A)^n \, d\gamma \, d\beta \, d\delta \, d\alpha,\\
\mathscr{C}_6(m,n) &= \int_{\alpha=\frac{-1+\sqrt{5}}{2}}^{\frac{2}{3}} \int_{\delta=\alpha}^{1-\frac{\alpha}{2}} \int_{\beta=\alpha}^{\frac{\delta}{1-\alpha+\delta}} \int_{\gamma=\beta}^{\frac{\delta-\beta+\alpha\beta}{\delta}} \Theta_3(A)^m \Phi_2(A)^n \, d\gamma \, d\beta \, d\delta \, d\alpha,\\
\mathscr{C}_7(m,n) &= \int_{\alpha=\frac{1}{2}}^{\frac{-1+\sqrt{5}}{2}} \int_{\delta=\alpha}^{\sqrt{1-\alpha}} \int_{\beta=\alpha}^{\frac{\delta}{1-\alpha+\delta}} \int_{\gamma=\beta}^{\frac{\delta-\beta+\alpha\beta}{\delta}} \Theta_3(A)^m \Phi_2(A)^n \, d\gamma \, d\beta \, d\delta \, d\alpha.
\end{aligned} \tag{8.1.1}$$

8.2 $\mathscr{C}(0,0)$

Substituting $m = n = 0$ in $\mathscr{C}_1(m,n)$ of (8.1.1) yields

$$\mathscr{C}_1(0,0) = \int_{\alpha=0}^{\frac{1}{2}} \int_{\delta=\alpha}^{1-\alpha} \int_{\beta=\alpha}^{\frac{\alpha}{1-\delta+\alpha}} \int_{\gamma=\beta}^{\frac{\alpha(1-\beta)}{1-\delta}} d\gamma\, d\beta\, d\delta\, d\alpha.$$

Integrating the inner integral with respect to γ and rearranging the result in terms of powers of β, we obtain

$$\mathscr{C}_1(0,0) = \int_{\alpha=0}^{\frac{1}{2}} \int_{\delta=\alpha}^{1-\alpha} \int_{\beta=\alpha}^{\frac{\alpha}{1-\delta+\alpha}} \left(\frac{\alpha}{1-\delta} - \frac{(1-\delta+\alpha)}{1-\delta}\beta\right) d\beta\, d\delta\, d\alpha.$$

Integrating with respect to β, simplifying, and taking partial fractions with respect to δ yields

$$\mathscr{C}_1(0,0) = \int_{\alpha=0}^{\frac{1}{2}} \int_{\delta=\alpha}^{1-\alpha} \left(\frac{\alpha^2}{2} + \frac{\alpha\,(1-\alpha)^2}{2(1-\delta)} - \frac{\alpha}{2(1-\delta+\alpha)}\right) d\delta\, d\alpha.$$

Integrating with respect to δ and simplifying, we find

$$\mathscr{C}_1(0,0) = \int_{\alpha=0}^{\frac{1}{2}} \left(\frac{\alpha^2}{2} - \alpha^3 + \frac{\alpha^2\,(2-\alpha)}{2}\ln(\alpha) + \frac{\alpha}{2}\ln(1-\alpha) + \frac{\alpha\,(1-\alpha)^2}{2}\ln(2)\right) d\alpha.$$

Integrating with respect to α and simplifying, we obtain

$$\mathscr{C}_1(0,0) = -\frac{5}{192} + \frac{1}{24}\ln(2)\,. \tag{8.2.1}$$

Similarly we carry out the four-fold integration of remaining integrals of (8.1.1) with $m = n = 0$, yielding,

$$\mathscr{C}_2(0,0) = \frac{23\sqrt{2}}{168} - \frac{14983}{64512} + \frac{325}{768}\ln(2) + \frac{1}{16}\ln\left(1+\sqrt{2}\right) - \frac{1}{4}\ln\left(\sqrt{2}+2\right), \tag{8.2.2}$$

$$\mathscr{C}_3(0,0) = -\frac{89}{243} - \frac{74}{81}\ln(2) + \frac{74}{81}\ln(3)\,, \tag{8.2.3}$$

$$\begin{aligned}
\mathscr{C}_4(0,0) = & \frac{14089\sqrt{5}}{8064} + \frac{1}{4}\ln\left(\sqrt{5}+1\right) + \left(\frac{13}{32} - \frac{17\sqrt{5}}{96}\right)\ln\left(3-\sqrt{5}\right) \\
& - \left(\frac{5-17\sqrt{5}}{48}\right)\ln\left(5-\sqrt{5}\right) + \left(\frac{7\sqrt{5}}{32} - \frac{227}{96}\right)\ln(2) \\
& + \left(\frac{61}{72} - \frac{3\sqrt{5}}{8}\right)\ln\left(11-3\sqrt{5}\right) - \frac{29105}{8064},
\end{aligned} \tag{8.2.4}$$

$$\mathscr{C}_5(0,0) = -\frac{419}{7776} + \frac{38}{81}\ln(2) - \frac{20}{81}\ln(3), \tag{8.2.5}$$

$$\begin{aligned}
\mathscr{C}_6(0,0) = & \left(\frac{421}{144} - \frac{19\sqrt{5}}{48}\right)\ln(2) - \frac{2}{3}\ln(3) + \left(\frac{5}{48} - \frac{17\sqrt{5}}{48}\right)\ln\left(5-\sqrt{5}\right) \\
& - \left(\frac{61}{72} - \frac{3\sqrt{5}}{8}\right)\ln\left(11-3\sqrt{5}\right) - \left(\frac{37}{48} - \frac{17\sqrt{5}}{48}\right)\ln\left(-1+\sqrt{5}\right) \\
& + \frac{1289}{288} - \frac{191\sqrt{5}}{96},
\end{aligned} \tag{8.2.6}$$

$$\begin{aligned}
\mathscr{C}_7(0,0) = & -\frac{22657}{64512} - \frac{23\sqrt{2}}{168} + \frac{1955\sqrt{5}}{8064} - \frac{1}{16}\ln\left(1+\sqrt{2}\right) - \frac{1}{4}\ln\left(\sqrt{5}+1\right) \\
& + \left(\frac{17\sqrt{5}}{96} - \frac{253}{768}\right)\ln(2) - \left(\frac{13}{32} - \frac{17\sqrt{5}}{96}\right)\ln\left(3-\sqrt{5}\right) + \frac{1}{4}\ln\left(\sqrt{2}+2\right) \\
& + \left(\frac{37}{48} - \frac{17\sqrt{5}}{48}\right)\ln\left(-1+\sqrt{5}\right).
\end{aligned} \tag{8.2.7}$$

Combining results (8.2.1) to (8.2.7) and simplifying yields

$$\mathscr{C}(0,0) = -\frac{47}{288} + \frac{1}{4}\ln(2). \tag{8.2.8}$$

8.3 $\mathscr{C}(1,0)$

Substituting $m = 1$ and $n = 0$ in $\mathscr{C}_1(m,n)$ of (8.1.1), and formula (4.3.13) for $\Theta_3(A)$ yields

$$\mathscr{C}_1(1,0) = \int_{\alpha=0}^{\frac{1}{2}} \int_{\delta=\alpha}^{1-\alpha} \int_{\beta=\alpha}^{\frac{\alpha}{1-\delta+\alpha}} \int_{\gamma=\beta}^{\frac{\alpha(1-\beta)}{1-\delta}} \left(1 - \frac{2(\gamma\delta-\alpha\beta)-\gamma+\beta}{2(\delta-\alpha)}\right) d\gamma\, d\beta\, d\delta\, d\alpha.$$

Integrating the inner integral with respect to γ and rearranging the result in terms of powers of β, we obtain

$$\mathscr{C}_1(1,0) = \int_{\alpha=0}^{\frac{1}{2}} \int_{\delta=\alpha}^{1-\alpha} \int_{\beta=\alpha}^{\frac{\alpha}{1-\delta+\alpha}} \Bigg(\frac{\alpha\left(4\delta-4\delta^2-3\alpha+2\alpha\delta\right)}{4(\delta-\alpha)(1-\delta)^2} - \frac{\left(5\alpha\delta-4\alpha\delta^2-3\alpha^2+2\alpha^2\delta-\alpha+2\delta-4\delta^2+2\delta^3\right)\beta}{2(\delta-\alpha)(1-\delta)^2} + \frac{(1-\delta+\alpha)\beta^2}{4(\delta-\alpha)(1-\delta)^2}\Big(-3\alpha+2\alpha\delta-2\delta^2 + 1+\delta\Big)\Bigg)\, d\beta\, d\delta\, d\alpha.$$

Integrating with respect to β, simplifying, and taking partial fractions with respect to δ yields

$$\mathscr{C}_1(1,0) = \int_{\alpha=0}^{\frac{1}{2}} \int_{\delta=\alpha}^{1-\alpha} \Bigg(\frac{\alpha^2(3-\alpha)}{6} + \frac{\alpha(1-\alpha)\left(2\alpha^2-5\alpha+5\right)}{12(1-\delta)} - \frac{(1-\alpha)^2\alpha^2}{12(1-\delta)^2} - \frac{\alpha(2\alpha+5)}{12(1-\delta+\alpha)} + \frac{\alpha^2}{6(1-\delta+\alpha)^2}\Bigg)\, d\delta\, d\alpha.$$

Integrating with respect to δ and simplifying,

$$\mathscr{C}_1(1,0) = \int_{\alpha=0}^{\frac{1}{2}} \Bigg(\frac{7\alpha^2}{12} - \frac{4\alpha^3}{3} + \frac{\alpha^4}{3} + \frac{\alpha^2\left(2\alpha^2+12-7\alpha\right)}{12}\ln(\alpha) + \frac{\alpha(1-\alpha)\left(2\alpha^2-5\alpha+5\right)}{12}\ln(1-\alpha) + \frac{\alpha(2\alpha+5)}{12}\ln(2)\Bigg)\, d\alpha.$$

We finish by integrating with respect to α and simplify, finding,

$$\mathscr{C}_1(1,0) = -\frac{481}{17280} + \frac{31}{720}\ln(2). \tag{8.3.1}$$

Again we state the results of the remaining integrals of (8.1.1) with $m = 1$ and

$n = 0$:

$$\mathscr{C}_2(1,0) = -\frac{3008897}{9676800} + \frac{3641}{7680}\ln(2) - \frac{91}{288}\ln\left(1+\sqrt{2}\right) + \frac{1247\sqrt{2}}{6720}, \tag{8.3.2}$$

$$\mathscr{C}_3(1,0) = -\frac{27641}{72900} - \frac{7181}{7290}\ln(2) + \frac{7061}{7290}\ln(3), \tag{8.3.3}$$

$$\begin{aligned}\mathscr{C}_4(1,0) = &\left(-\frac{12353}{25920} - \frac{29\sqrt{5}}{576}\right)\ln(2) + \left(-\frac{143}{576} + \frac{155\sqrt{5}}{576}\right)\ln(5) \\ &+ \left(\frac{235}{288} - \frac{155\sqrt{5}}{288}\right)\ln\left(\sqrt{5}+1\right) + \left(-\frac{155\sqrt{5}}{576} + \frac{193}{320}\right)\ln\left(3-\sqrt{5}\right) \\ &+ \left(-\frac{1451}{3888} + \frac{23\sqrt{5}}{144}\right)\ln\left(11-3\sqrt{5}\right) + \frac{667453\sqrt{5}}{725760} - \frac{6306857}{3628800},\end{aligned} \tag{8.3.4}$$

$$\mathscr{C}_5(1,0) = -\frac{29831}{388800} + \frac{1891}{3645}\ln(2) - \frac{1877}{7290}\ln(3), \tag{8.3.5}$$

$$\begin{aligned}\mathscr{C}_6(1,0) = &\left(-\frac{9665}{7776} + \frac{247\sqrt{5}}{288}\right)\ln(2) + \left(\frac{143}{576} - \frac{155\sqrt{5}}{576}\right)\ln(5) + \frac{32}{45}\ln\left(\sqrt{5}+1\right) \\ &+ \left(\frac{1451}{3888} - \frac{23}{144\sqrt{5}}\right)\ln\left(11-3\sqrt{5}\right) - \frac{13241\sqrt{5}}{12960} + \frac{53851}{23328} - \frac{32}{45}\ln(3),\end{aligned} \tag{8.3.6}$$

$$\begin{aligned}\mathscr{C}_7(1,0) = &\left(-\frac{155\sqrt{5}}{192} + \frac{5061}{2560}\right)\ln(2) + \left(-\frac{733}{480} + \frac{155\sqrt{5}}{288}\right)\ln\left(\sqrt{5}+1\right) \\ &+ \left(-\frac{193}{320} + \frac{155\sqrt{5}}{576}\right)\ln\left(3-\sqrt{5}\right) + \frac{8227\sqrt{5}}{80640} + \frac{91}{288}\ln\left(1+\sqrt{2}\right) \\ &- \frac{1247\sqrt{2}}{6720} + \frac{159049}{9676800}.\end{aligned} \tag{8.3.7}$$

Combining results (8.3.1) to (8.3.7) and simplifying yields

$$\mathscr{C}(1,0) = \frac{37}{120}\ln(2) - \frac{8977}{43200}. \tag{8.3.8}$$

8.4 $\mathscr{C}(0,1)$

Substituting $m=0$ and $n=1$ in $\mathscr{C}_1(m,n)$ of (8.1.1), and formula (4.3.13) for $\Phi(A)$ yields

$$\mathscr{C}_1(0,1)=\int_{\alpha=0}^{\frac{1}{2}}\int_{\delta=\alpha}^{1-\alpha}\int_{\beta=\alpha}^{\frac{\alpha}{1-\delta+\alpha}}\int_{\gamma=\beta}^{\frac{\alpha(1-\beta)}{1-\delta}}\left(\frac{2\left(\gamma\delta-\alpha\beta\right)-\gamma+\alpha}{2\left(\delta-\beta\right)}\right)d\gamma\,d\beta\,d\delta\,d\alpha.$$

Integrating the inner integral with respect to γ and taking partial fractions with respect to β of the result, we have

$$\begin{aligned}\mathscr{C}_1(0,1)=\int_{\alpha=0}^{\frac{1}{2}}\int_{\delta=\alpha}^{1-\alpha}\int_{\beta=\alpha}^{\frac{\alpha}{1-\delta+\alpha}}\Bigg(&\frac{1}{4\left(1-\delta\right)^2}\Big(-5\alpha^2\delta+10\alpha\delta^2+4\alpha^2-8\alpha\delta+2\alpha\\&+2\delta^2\alpha^2-4\delta^3\alpha+4\delta^2-5\delta^3+2\delta^4-\delta\Big)\\&+\frac{\left(1-\delta+\alpha\right)\left(-3\alpha+2\alpha\delta+3\delta-2\delta^2-1\right)\beta}{4\left(1-\delta\right)^2}\\&-\frac{\left(2\delta-1\right)\left(\delta-\alpha\right)^2}{4\left(\delta-\beta\right)}\Bigg)\,d\beta\,d\delta\,d\alpha.\end{aligned}$$

Integrating with respect to β, simplifying, and taking partial fractions with respect to δ yields

$$\begin{aligned}\mathscr{C}_1(0,1)=\int_{\alpha=0}^{\frac{1}{2}}\int_{\delta=\alpha}^{1-\alpha}\Bigg(&-\frac{\left(2-\alpha\right)\alpha}{8}+\frac{\alpha\left(3\alpha-1\right)\delta}{4}-\frac{\alpha\delta^2}{2}-\frac{\alpha^3\left(1-\alpha\right)}{4\left(1-\delta\right)}\\&+\frac{\left(1-\alpha\right)^2\alpha^2}{8\left(1-\delta\right)^2}+\frac{\alpha}{4\left(1-\delta+\alpha\right)}+\frac{\left(2\delta-1\right)\left(\delta-\alpha\right)^2}{4}\ln\left(1-\delta\right)\\&-\frac{\left(2\delta-1\right)\left(\delta-\alpha\right)^2}{4}\ln\left(1-\delta+\alpha\right)\Bigg)\,d\delta\,d\alpha.\end{aligned}$$

Integrating with respect to δ and simplifying, we find

$$\mathscr{C}_1(0,1) = \int_{\alpha=0}^{\frac{1}{2}} \Bigg(-\frac{\alpha}{12} + \frac{7\alpha^2}{16} - \frac{5\alpha^3}{8} + \frac{\alpha^4}{6} + \frac{\alpha^3 (4-5\alpha) \ln(\alpha)}{24} - \frac{(1-\alpha)\left(5\alpha^3+\alpha^2+\alpha-1\right)}{24} \ln(1-\alpha) - \frac{\alpha(2\alpha-1)\left(4\alpha^2-6\alpha+3\right)}{12} \ln(2) \Bigg)\, d\alpha.$$

We complete $\mathscr{C}_1(0,1)$ by integrating with respect to α and simplifying, obtaining

$$\mathscr{C}_1(0,1) = \frac{7}{1920} - \frac{1}{240} \ln(2). \tag{8.4.1}$$

Similarly, we carry out the four-fold integration of remaining integrals of (8.1.1) with $m = 0$ and $n = 1$ giving the following results:

$$\mathscr{C}_2(0,1) = \left(-\frac{10007}{26880} - \frac{121\sqrt{2}}{3360}\right) \ln\left(1+\sqrt{2}\right) + \frac{138067\sqrt{2}}{564480} + \frac{5587}{8960} \ln(2) - \frac{148747}{352800} + \frac{121\sqrt{2}}{6720} \ln(2), \tag{8.4.2}$$

$$\mathscr{C}_3(0,1) = -\frac{2081}{32400} - \frac{1909}{7290} \ln(2) + \frac{274}{1215} \ln(3), \tag{8.4.3}$$

$$\mathscr{C}_4(0,1) = \left(-\frac{80137}{20160} + \frac{935\sqrt{5}}{672}\right) \ln(2) + \left(-\frac{23}{256} + \frac{5\sqrt{5}}{384}\right) \ln(5) + \left(\frac{1763}{2688} - \frac{5\sqrt{5}}{192}\right) \ln\left(\sqrt{5}+1\right) + \left(-\frac{437}{1680} + \frac{5\sqrt{5}}{42}\right) \ln\left(3-\sqrt{5}\right) + \left(\frac{28793}{17280} - \frac{145\sqrt{5}}{192}\right) \ln\left(11-3\sqrt{5}\right) + \frac{1325\sqrt{5}}{1568} - \frac{437573}{235200}, \tag{8.4.4}$$

$$\mathscr{C}_5(0,1) = -\frac{11047}{129600} + \frac{5257}{7290} \ln(2) - \frac{917}{2430} \ln(3), \tag{8.4.5}$$

$$\mathscr{C}_6(0,1) = \left(-\frac{175\sqrt{5}}{96} + \frac{33781}{8640}\right)\ln(2) + \left(-\frac{5\sqrt{5}}{384} + \frac{23}{256}\right)\ln(5)$$
$$+ \left(\frac{65\sqrt{5}}{192} - \frac{349}{384}\right)\ln\left(\sqrt{5}+1\right) + \left(-\frac{28793}{17280} + \frac{145\sqrt{5}}{192}\right)\ln\left(11-3\sqrt{5}\right)$$
$$- \frac{3419\sqrt{5}}{2880} + \frac{41}{270}\ln(3) + \frac{2863}{1080}, \tag{8.4.6}$$
$$\mathscr{C}_7(0,1) = \left(-\frac{121\sqrt{2}}{6720} + \frac{145\sqrt{5}}{336} - \frac{18953}{26880}\right)\ln(2) + \left(\frac{10007}{26880} + \frac{121\sqrt{2}}{3360}\right)\ln\left(1+\sqrt{2}\right)$$
$$+ \left(-\frac{5\sqrt{5}}{16} + \frac{85}{336}\right)\ln\left(\sqrt{5}+1\right) + \left(\frac{437}{1680} - \frac{5\sqrt{5}}{42}\right)\ln\left(3-\sqrt{5}\right)$$
$$+ \frac{48281\sqrt{5}}{141120} - \frac{610639}{1411200} - \frac{138067\sqrt{2}}{564480}. \tag{8.4.7}$$

Combining the results (8.4.1) to (8.4.7) and simplifying yields

$$\mathscr{C}(0,1) = -\frac{3019}{14400} + \frac{37}{120}\ln(2). \tag{8.4.8}$$

8.5 $\mathscr{C}(1,1)$

Substituting $m = n = 1$ and formulas for $\Theta_3(A)$ and $\Phi_2(A)$ from (4.3.13) in $\mathscr{C}_1(m,n)$ of (8.1.1) yields

$$\mathscr{C}_1(1,1) = \int_{\alpha=0}^{\frac{1}{2}} \int_{\delta=\alpha}^{1-\alpha} \int_{\beta=\alpha}^{\frac{\alpha}{1-\delta+\alpha}} \int_{\gamma=\beta}^{\frac{\alpha(1-\beta)}{1-\delta}} \left(\left(1 - \frac{2(\gamma\delta-\alpha\beta)-\gamma+\beta}{2(\delta-\alpha)}\right)\right.$$
$$\left.\left(\frac{2(\gamma\delta-\alpha\beta)-\gamma+\alpha}{2(\delta-\beta)}\right)\right) d\gamma\, d\beta\, d\delta\, d\alpha.$$

Integrating inner integral of $\mathscr{C}_1(1,1)$ with respect to γ and rearranging the result in terms of powers of β yields

$$\begin{aligned}\mathscr{C}_1(1,1) = \int_{\alpha=0}^{\frac{1}{2}}\int_{\delta=\alpha}^{1-\alpha}\int_{\beta=\alpha}^{\frac{\alpha}{1-\delta+\alpha}} \Bigg(&-\frac{1}{24(\delta-\alpha)(1-\delta)^3}\Big(-15\delta\alpha+9\alpha^2+5\delta^2+87\alpha\delta^2 \\ &-81\alpha^2\delta-63\alpha^3\delta+195\alpha^2\delta^2+71\alpha^3\delta^2-213\alpha^2\delta^3 \\ &+24\alpha^3+65\delta^4-71\delta^5-29\delta^3-195\alpha\delta^3+213\delta^4\alpha \\ &-38\alpha^3\delta^3+114\alpha^2\delta^4-114\delta^5\alpha+8\alpha^3\delta^4-24\alpha^2\delta^5 \\ &+24\delta^6\alpha+38\delta^6-8\delta^7\Big) \\ &-\frac{\beta}{24(\delta-\alpha)(1-\delta)^3}\Big(-3\alpha+5\delta+51\delta\alpha-24\alpha^2-159\alpha\delta^2 \\ &+123\alpha^2\delta+59\alpha^3\delta-29\delta^2-38\alpha^3\delta^2 \\ &+114\alpha^2\delta^3-33\alpha^3-71\delta^4+38\delta^5+65\delta^3 \\ &+201\alpha\delta^3-114\delta^4\alpha+8\alpha^3\delta^3-24\alpha^2\delta^4 \\ &+24\delta^5\alpha-8\delta^6-189\alpha^2\delta^2\Big) \\ &-\frac{(1-\delta+\alpha)\beta^2}{24(\delta-\alpha)(1-\delta)^3}\Big(14\alpha^2+8\alpha^2\delta^2-20\alpha^2\delta-16\alpha\delta^3 \\ &+\alpha+38\alpha\delta^2-18\delta^3+11\delta^2+8\delta^4-1\Big) \\ &+\frac{(2\delta-1)(4\delta-5)(\delta-\alpha)^2}{24(\delta-\beta)}\Bigg)\, d\beta\, d\delta\, d\alpha.\end{aligned}$$

Continuing by integrating with respect to β,simplifying the result, and taking partial fractions with respect to δ, we obtain

$$\mathscr{C}_1(1,1) = \int_{\alpha=0}^{\frac{1}{2}} \int_{\delta=\alpha}^{1-\alpha} \Bigg(-\frac{\left(-15\alpha+4\alpha^2+6+8\alpha^3\right)\alpha}{144} + \frac{\left(39\alpha-3+8\alpha^2\right)\alpha\delta}{72} - \frac{(1+2\alpha)\,\alpha\delta^2}{4} + \frac{\alpha\delta^3}{3} - \frac{\alpha^3\left(17+8\alpha^2-23\alpha\right)}{72\,(1-\delta)} + \frac{\alpha^2(1-\alpha)\left(8\alpha^2-15\alpha+15\right)}{144\,(1-\delta)^2} - \frac{\alpha^3(1-\alpha)^2}{36\,(1-\delta)^3} + \frac{(-2\alpha+1)\,\alpha}{24\,(1-\delta+\alpha)} - \frac{\alpha^2}{12\,(1-\delta+\alpha)^2} + \frac{(2\delta-1)(4\delta-5)(\delta-\alpha)^2}{24}\ln(1-\delta+\alpha) - \frac{(2\delta-1)(4\delta-5)(\delta-\alpha)^2}{24}\ln(1-\delta) \Bigg)\, d\delta\, d\alpha.$$

Integrating with respect to δ and taking partial fractions of the result with respect to α yields

$$\mathscr{C}_1(1,1) = \int_{\alpha=0}^{\frac{1}{2}} \Bigg(\frac{\alpha}{60} - \frac{17\alpha^2}{160} + \frac{113\alpha^3}{240} - \frac{17\alpha^4}{20} + \frac{2\alpha^5}{5} + \frac{\alpha^3\left(40-65\alpha+24\alpha^2\right)}{240}\ln(\alpha) + \left(-\frac{\alpha}{72} - \frac{\alpha^3}{6} - \frac{\alpha^2}{36} + \frac{13\alpha^4}{48} - \frac{\alpha^5}{10} + \frac{7}{720} \right)\ln(1-\alpha) - \frac{\alpha\left(-15+30\alpha+80\alpha^2-200\alpha^3+128\alpha^4\right)}{360}\ln(2) \Bigg)\, d\alpha.$$

Integrating with respect to α, we obtain

$$\mathscr{C}_1(1,1) = \frac{901}{103680} - \frac{17}{1440}\ln(2). \tag{8.5.1}$$

The four-fold integration of $\mathscr{C}_2(1,1)$ to $\mathscr{C}_7(1,1)$ is carried out similar to

$\mathscr{C}_1(1,1)$. We present following results

$$\begin{aligned}
\mathscr{C}_2(1,1) =& -\frac{157829681}{228614400}+\frac{37089499\sqrt{2}}{91445760}+\left(\frac{1580279}{1451520}+\frac{6737\sqrt{2}}{362880}\right)\ln(2) \\
&+\left(-\frac{1003169}{1451520}-\frac{6737\sqrt{2}}{181440}\right)\ln\left(1+\sqrt{2}\right), \qquad (8.5.2)\\
\mathscr{C}_3(1,1) =& -\frac{4806343}{47239200}-\frac{37346}{98415}\ln(2)+\frac{131059}{393660}\ln(3), \qquad (8.5.3)\\
\mathscr{C}_4(1,1) =& \frac{3624511}{4233600}-\frac{1947439\sqrt{5}}{5715360}-\left(\frac{54799}{34020}-\frac{18509\sqrt{5}}{90720}\right)\ln(2) \\
&+\left(\frac{8623\sqrt{5}}{34560}-\frac{35189}{58320}\right)\ln\left(11-3\sqrt{5}\right)+\frac{6707}{9072}\ln\left(\sqrt{5}+1\right) \\
&+\left(\frac{10219\sqrt{5}}{90720}-\frac{11519}{45360}\right)\ln\left(3-\sqrt{5}\right)+\left(\frac{353}{864}-\frac{2819\sqrt{5}}{13824}\right)\ln(5) \\
&+\left(\frac{353}{432}-\frac{2819\sqrt{5}}{6912}\right)\ln\left(-1+\sqrt{5}\right), \qquad (8.5.4)\\
\mathscr{C}_5(1,1) =& -\frac{23824751}{188956800}+\frac{333983}{393660}\ln(2)-\frac{82769}{196830}\ln(3), \qquad (8.5.5)
\end{aligned}$$

$$\begin{aligned}
\mathscr{C}_6(1,1) =& -\frac{6113}{31104}+\frac{3293\sqrt{5}}{38880}-\left(\frac{7117}{19440}-\frac{\sqrt{5}}{432}\right)\ln(2)+\frac{1277}{14580}\ln(3) \\
&+\left(\frac{2819\sqrt{5}}{13824}-\frac{353}{864}\right)\ln(5)+\left(\frac{3071\sqrt{5}}{34560}-\frac{1}{9}\right)\ln\left(\sqrt{5}-1\right) \qquad (8.5.6)\\
&-\left(\frac{8623\sqrt{5}}{34560}-\frac{35189}{58320}\right)\ln\left(11-3\sqrt{5}\right), \\
\mathscr{C}_7(1,1) =& -\frac{2299327}{57153600}-\frac{37089499\sqrt{2}}{91445760}+\frac{182921\sqrt{5}}{714420}-\frac{6707}{9072}\ln\left(\sqrt{5}+1\right) \\
&+\left(\frac{412627}{483840}-\frac{6737}{362880}\sqrt{2}-\frac{18719\sqrt{5}}{90720}\right)\ln(2) \\
&+\left(\frac{1003169}{1451520}+\frac{6737\sqrt{2}}{181440}\right)\ln\left(1+\sqrt{2}\right) \qquad (8.5.7)\\
&+\left(\frac{689\sqrt{5}}{2160}-\frac{305}{432}\right)\ln\left(\sqrt{5}-1\right)+\left(\frac{11519}{45360}-\frac{10219\sqrt{5}}{90720}\right)\ln\left(3-\sqrt{5}\right).
\end{aligned}$$

Combining $\mathscr{C}_1(1,1)$ (8.5.1) to $\mathscr{C}_7(1,1)$ (8.5.7) and simplifying logarithms, we find

$$\mathscr{C}(1,1) = -\frac{25069}{86400}+\frac{911}{2160}\ln(2). \qquad (8.5.8)$$

Chapter 9

RESULTS

We summarize the results of the dissertation. Recall Definition 1.4.3

$$S_k(m,n) = \text{Prob}\Big(A \text{ has a } k\times k \text{ saddle square} \,\Big|\, A \sim UN_{m,n}(0,1)\Big),$$
$$1 \le k \le m,n.$$

Goldman's Theorem 1.3.2 states that for $A \sim UN_{m,n}(0,1)$

$$S_1(m,n) = \frac{m!n!}{(m+n-1)!}, \qquad m,n \ge 1.$$

Following the definitions of $\text{v}_{\text{minmax}}(A)$ (1.2.4) and $\text{v}_{\text{minmax}}(A)$ (1.2.1) and Thorp's second proof of Goldman's theorem, we establish a formula for the probability that $A \sim UN_{m,n}(0,1)$ has a k by k saddle square (3.0.6)

$$S_k(m,n) = \binom{m}{k}\binom{n}{k} S_k(k,k)\,\text{E}\Big(\Theta^{n-k}(A)\Phi^{m-k}(A) \,\Big|\, A \sim UN_{m,n},\ r(A) = k\Big),$$
$$1 \le k \le m,n,$$

based on the expected value of the probabilities (Definition 3.0.1)

$$\Theta(A) = \text{Prob}\left(\sum_{i=1}^{m} p_i^{\text{Opt}}(A)u_i > \text{v}(A) \,\middle|\, u_1,\ldots,u_m \text{ iid un}(0,1)\right),$$
$$\Phi(A) = \text{Prob}\left(\sum_{i=1}^{n} q_i^{\text{Opt}}(A)u_i < \text{v}(A) \,\middle|\, u_1,\ldots,u_n \text{ iid un}(0,1)\right),$$
$$A \in M^*_{m,n}(0,1).$$

We present in Lemma 4.1.1 the probabilities that $A \sim UN_{2,n}(0,1)$ and $A \sim UN_{m,2}(0,1)$ have a 2 by 2 saddle square (observed by Falk and Thrall)

$$S_2(2,n) = \frac{(n-1)}{(n+1)}, \qquad n \ge 2,$$

$$S_2(m,2) = \frac{(m-1)}{(m+1)}, \qquad m \ge 2.$$

As a consequence of Lemma 4.1.1, we find the marginal distributions of $\Theta(A)$ and $\Phi(A)$ when $A \sim UN_{2,2}(0,1)$ and $r(A) = 2$, (4.1.3) and (4.1.4),

$$\Theta(A) \sim \text{beta}(2,2)$$

and

$$\Phi(A) \sim \text{beta}(2,2).$$

Chapters 5 to 8 are devoted to finding our main result $S_2(3,3)$. Following from the integral form of $S_2(m,n)$ (4.3.1) with $m = n = 3$, we have

$$\begin{aligned} S_2(3,3) &= \binom{3}{2}\binom{3}{2} \iiiint\limits_{\substack{A\in M^*_{2,2}(0,1)\\ r(A)=2}} \Theta^{3-2}(A)\Phi^{3-2}(A)\, d\alpha\, d\beta\, d\gamma\, d\delta, \\ &= 9 \iiiint\limits_{\substack{A\in M^*_{2,2}(0,1)\\ r(A)=2}} \Theta(A)\Phi(A)\, d\alpha\, d\beta\, d\gamma\, d\delta. \end{aligned} \tag{9.0.1}$$

In Chapter 5 we define integrals $\mathscr{A}(m,n)$ (5.1.1), $\mathscr{B}(m,n)$ (5.1.2), and $\mathscr{C}(m,n)$ (5.1.3), and rewrite $S_2(3,3)$ (9.0.1) following (5.1.14)

$$\begin{aligned} 9 \iiiint\limits_{\substack{A\in M^*_{2,2}(0,1)\\ r(A)=2}} \Theta(A)\Phi(A)\, d\alpha\, d\beta\, d\gamma\, d\delta &= 72 \iiiint\limits_{\substack{A=\begin{pmatrix}\alpha & \delta\\ \gamma & \beta\end{pmatrix}\\ 0<\alpha<\beta<\gamma<\delta<1}} \Theta(A)\Phi(A)\, d\alpha\, d\beta\, d\gamma\, d\delta \\ &= 72\Big(2\mathscr{A}(1,1) - \mathscr{A}(1,0) - \mathscr{A}(0,1) + \mathscr{A}(0,0) \\ &\quad + 2\mathscr{B}(1,1) - \mathscr{B}(1,0) - \mathscr{B}(0,1) + \mathscr{B}(0,0) \\ &\quad + 2\mathscr{C}(1,1) - \mathscr{C}(1,0) - \mathscr{C}(0,1) + \mathscr{C}(0,0)\Big) \end{aligned} \tag{9.0.2}$$

From the computation outlined in Chapters 6 to 8 we have the following results

$$\mathscr{A}(0,0) = \tfrac{1}{288}, \tag{6.2.1}$$

$$\mathscr{A}(1,0) = \tfrac{1}{360}, \tag{6.3.1}$$

$$\mathscr{A}(0,1) = -\tfrac{493}{720} + \tfrac{5\pi^2}{72}, \tag{6.4.1}$$

$$\mathscr{A}(1,1) = -\tfrac{11099}{8640} + \tfrac{25\pi^2}{192}, \tag{6.5.1}$$

$$\mathscr{B}(0,0) = \tfrac{13}{72} - \tfrac{1}{4}\ln(2), \tag{7.2.11}$$

$$\mathscr{B}(1,0) = -\tfrac{1}{216} + \tfrac{1}{72}\ln(2), \tag{7.3.11}$$

$$\mathscr{B}(0,1) = \tfrac{173}{288} - \tfrac{5\pi^2}{72} + \tfrac{1}{8}\ln(2), \tag{7.4.11}$$

$$\mathscr{B}(1,1) = \tfrac{26009}{20160} - \tfrac{55\pi^2}{432} - \tfrac{47}{1008}\ln(2), \tag{7.5.11}$$

$$\mathscr{C}(0,0) = -\tfrac{47}{288} + \tfrac{1}{4}\ln(2), \tag{8.2.8}$$

$$\mathscr{C}(1,0) = -\tfrac{8977}{43200} + \tfrac{37}{120}\ln(2), \tag{8.3.8}$$

$$\mathscr{C}(0,1) = -\tfrac{3019}{14400} + \tfrac{37}{120}\ln(2), \tag{8.4.8}$$

$$\mathscr{C}(1,1) = -\tfrac{25069}{86400} + \tfrac{911}{2160}\ln(2). \tag{8.5.8}$$

Substituting the above results $\mathscr{A}(0,0)$ to $\mathscr{C}(1,1)$ into (9.0.2), we obtain our final results of the dissertation

$$S_2(3,3) = -\frac{909}{280} + \frac{5\pi^2}{12} - \frac{8}{21}\ln(2), \tag{9.0.3}$$

and by Jonasson's Theorem 1.4.1,

$$\begin{aligned} S_3(3,3) &= 1 - S_1(3,3) - S_2(3,3) \\ &= 1 - \frac{3}{10} - \left(\frac{909}{280} + \frac{5\pi^2}{12} - \frac{8}{21}\ln(2)\right) \\ &= \frac{221}{56} - \frac{5\pi^2}{12} + \frac{8}{21}\ln(2). \end{aligned} \tag{9.0.4}$$

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APPENDIX A

N PERSON ZERO SUM GAME EXAMPLES

A.1 The Game of Rock, Paper, Scissors

A classic example of a two person zero sum game with three strategies is rock, paper, and scissors: both players simultaneously show rock, paper or scissors and there is no payoff unless the choices are different, in which case rock beats scissors, scissors beats paper, paper beats rock. The player with the winning strategy receives a payoff of one unit from the loser. This is a fair game with expected value zero to each player and unique optimal mixed strategy of showing rock, paper and scissors each with probability $1/3$. If either player chooses any other mixed strategy, then the other player can counter with a mixed strategy which gives them a positive expectation. In the game of rock, paper, scissors, both of the unique optimal mixed strategies require all three pure strategies, which we call a 3 by 3 saddle square.

$$\begin{array}{cc} & \text{P II} \\ & \begin{array}{ccc} \text{rock} & \text{paper} & \text{scissors} \end{array} \\ \text{P I} \quad \begin{array}{r} \text{rock} \\ \text{paper} \\ \text{scissors} \end{array} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array} \tag{A.1.1}$$

A.2 The Three Person Coin Matching Game

A standard example of a three person zero sum game involves coin matching: all three players simultaneously show heads or tails and there is no payoff unless two players show heads (respectively tails) and their opponent shows tails (respectively heads), in which case the two players showing the same face both give one unit to the third player. Since the game is symmetric and the Shapley value is unique, the Shapley value for each player is zero. This is obtained if each player flips a fair coin to decide whether to show heads or tails, which is a Nash equilibrium point. If one player always shows heads and another player always shows tails, then the third player will lose one unit regardless of their mixed strategy. If the third player flips a fair coin to decide to show heads or tails while the other two players always show heads and always show tails, then this is another Nash equilibrium point for the game.

APPENDIX B

LINEAR PROGRAMMING AND THE SIMPLEX METHOD

Observe that the computation of $\mathrm{v}_{\mathrm{maxmin}}(A)$ (1.2.1) requires one to

$$\text{maximize} \ \ \min_j \sum_i p_i a_{i,j} \qquad \text{subject to} \begin{cases} \sum_i p_i = 1, \\ 0 \le p_i. \end{cases}$$

Using the slack variable x, this becomes the linear programming problem

$$\text{maximize} \ \ x \qquad \text{subject to} \begin{cases} x \le \sum_i p_i a_{i,1}, \\ \quad \vdots \\ x \le \sum_i p_i a_{i,n}, \\ \sum_i p_i = 1, \\ 0 \le p_i. \end{cases} \tag{B.0.1}$$

Similarly, computing $\mathrm{v}_{\mathrm{minmax}}(A)$ (1.2.4) requires one to

$$\text{minimize} \ \ \max_i \sum_j a_{i,j} q_j \qquad \text{subject to} \begin{cases} \sum_j q_j = 1, \\ 0 \le q_j. \end{cases}$$

Using the slack variable y this becomes the linear programming problem

$$\text{minimize} \ \ y \qquad \text{subject to} \begin{cases} y \ge \sum_j a_{1,j} q_j, \\ \quad \vdots \\ y \ge \sum_j a_{m,j} q_j, \\ \sum_j q_j = 1, \\ 0 \le q_j. \end{cases} \tag{B.0.2}$$

Note that linear program (B.0.2) is the dual to (B.0.1), and by Dantzig's simplex algorithm [Dan1963], we can find the optimal strategies of each player and the value of the game.

In the case $A \in M_{2,2}(\mathbb{R})$ the linear program (B.0.1) becomes

$$\text{maximize } x \qquad \text{subject to} \begin{cases} x \le p_1\, a_{1,1} + p_2\, a_{2,1}, \\ x \le p_1\, a_{1,2} + p_2\, a_{2,2}, \\ p_1 + p_2 = 1, \\ 0 \le p_1, \\ 0 \le p_2, \end{cases}$$

and linear program for (B.0.2) becomes

$$\text{minimize } y \qquad \text{subject to} \begin{cases} y \ge a_{1,1}\, q_1 + a_{1,2}\, q_2, \\ y \ge a_{2,1}\, q_1 + a_{2,2}\, q_2, \\ q_1 + q_2 = 1, \\ 0 \le q_1, \\ 0 \le q_2. \end{cases}$$

If we assume A, with $m = n = 2$, does not have a saddle point ($r(A) = 2$), then x is maximized at the intersection of lines $p_1\,(a_{1,1} - a_{1,2}) + p_2\,(a_{2,1} - a_{2,2}) = 0$ and $p_1 + p_2 = 1$, and y is minimized at the intersections of lines $(a_{1,1} - a_{2,1})\,q_1 + (a_{1,2} - a_{2,2})\,q_2 = 0$ and $q_1 + q_2 = 1$, giving optimal strategies and game value algebraically:

$$\widetilde{p}^{\text{opt}}(A) = \left(\frac{a_{2,2} - a_{2,1}}{a_{1,1} - a_{1,2} + a_{2,2} - a_{2,1}}, \frac{a_{1,1} - a_{1,2}}{a_{1,1} - a_{1,2} + a_{2,2} - a_{2,1}} \right), \tag{B.0.3}$$

$$\widetilde{q}^{\text{opt}}(A) = \left(\frac{a_{2,2} - a_{1,2}}{a_{1,1} - a_{1,2} + a_{2,2} - a_{2,1}}, \frac{a_{1,1} - a_{2,1}}{a_{1,1} - a_{1,2} + a_{2,2} - a_{2,1}} \right), \tag{B.0.4}$$

$$\mathrm{v}(A) = \frac{a_{1,1}\, a_{2,2} - a_{1,2}\, a_{2,1}}{a_{1,1} - a_{1,2} + a_{2,2} - a_{2,1}}. \tag{B.0.5}$$