Global Behavior Of Finite Energy Solutions To The Focusing Nonlinear Schrödinger Equation In d Dimension

by

Cristi Darley Guevara

A Dissertation Presented in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

Approved April 2011 by the
Graduate Supervisory Committee:

Svetlana Roudenko, Chair
Carlos Castillo-Chavez
Donald Jones
Alex Mahalov
Sergei Suslov

ARIZONA STATE UNIVERSITY

May 2011

ABSTRACT

Nonlinear dispersive equations model nonlinear waves in a wide range of physical and mathematics contexts. They reinforce or dissipate effects of linear dispersion and nonlinear interactions, and thus, may be of a focusing or defocusing nature. The nonlinear Schrödinger equation or NLS is an example of such equations. It appears as a model in hydrodynamics, nonlinear optics, quantum condensates, heat pulses in solids and various other nonlinear instability phenomena. In mathematics, one of the interests is to look at the wave interaction: waves propagation with different speeds and/or different directions produces either small perturbations comparable with linear behavior, or creates solitary waves, or even leads to singular solutions.

This dissertation studies the global behavior of finite energy solutions to the d-dimensional focusing NLS equation, $i\partial_t u + \Delta u + |u|^{p-1}u = 0$, with initial data $u_0 \in H^1$, $x \in \mathbb{R}^d$; the nonlinearity power p and the dimension d are chosen so that the scaling index $s = \frac{d}{2} - \frac{2}{p-1}$ is between 0 and 1, thus, the NLS is mass-supercritical ($s > 0$) and energy-subcritical ($s < 1$).

For solutions with $\mathcal{ME}[u_0] < 1$ ($\mathcal{ME}[u_0]$ stands for an invariant and conserved quantity in terms of the mass and energy of u_0), a sharp threshold for scattering and blowup is given. Namely, if the renormalized gradient $\mathcal{G}_u$ of a solution u to NLS is initially less than 1, i.e., $\mathcal{G}_u(0) < 1$, then the solution exists globally in time and scatters in H^1 (approaches some linear Schrödinger evolution as $t \to \pm\infty$); if the renormalized gradient $\mathcal{G}_u(0) > 1$, then the solution exhibits a blowup behavior, that is, either a finite time blowup occurs, or there is a divergence of H^1 norm in infinite time.

This work generalizes the results for the 3d cubic NLS obtained in a series of papers by Holmer-Roudenko and Duyckaerts-Holmer-Roudenko with the key ingredients, the concentration compactness and localized variance, developed in

the context of the energy-critical NLS and Nonlinear Wave equations by Kenig and Merle.

One of the difficulties is fractional powers of nonlinearities which are overcome by considering Besov-Strichartz estimates and various fractional differentiation rules.

To my parents Jorge and Blanca.

My brothers Jorge and Javier

My friend and husband Omar

And my little Son Fabio

ACKNOWLEDGEMENTS

I would like to thank my advisor, Professor Svetlana Roudenko, for her expertise, for introducing me to the dispersive equations and to the problem, for her continuous advice, support and guidance. I appreciate her unbelievable patience.

I would like to thank Professor Carlos Kenig, Gustavo Ponce, and Felipe Linares for the fruitful discussions on the subject and helpful suggestions.

I would like to thank Professor Justin Holmer for his thorough and careful review.

I would like to thank Professor Carlos Castillo-Chavez for the support all the way through the program.

I would also like to thank my committee members, Professors Carlos Castillo-Chavez, Sergey Suslov, Alex Mahalov, and Don Jones for their support.

I could not have completed my doctoral program without the financial support from a number of sources. The research in this dissertation and my graduate studies were partially funded by National Science Foundation (NSF - Grant DMS - 0808081 and NSF - Grant DUE-0633033; PI Roudenko), the Alfred P. Sloan Foundation (nominated by Dr. Carlos Castillo-Chavez), More Graduate Education at Mountain State Alliance (MGE@MSA), and School of Mathematical and Statistical Sciences (formelly Department of Mathematics) at Arizona State University.

I would like to acknowledge those around me that have supported, inspired and motivated me since the first day of my graduate school. Thank you, Alejandra and Dori.

A special thank you to my parents: Jorge and Blanca, my brothers: Jorge and Javier, lovely friend and husband Omar and my little son Fabio for joining me throughout this entire journey.

TABLE OF CONTENTS

LIST OF FIGURES

Figure Page

Chapter 1

INTRODUCTION

In the past twenty years, the field of nonlinear dispersive PDEs has dramatically grown and attracted the interest of Harmonic Analysis, Geometry and PDE audiences. Most of the problems, originate within physics in subjects such as general relativity, quantum mechanics, quantum condensates, water waves, hydrodynamics, nonlinear optics, nonlinear acoustics, nonlinear elasticity, heat pulses in solids and various other nonlinear instability phenomena.

In mathematics, the interest comes from understanding the wave interaction and measuring dispersion since the waves do not obey to the superposition principle, as in the linear theory. Consequently, new ideas and techniques such as the use of dispersive or L^p and Strichartz estimates for linear dispersive equations, the vector fields method for the linear and nonlinear wave equation, estimates for bilinear and multilinear wave interactions and the use of wave packet methods, among others. The harmonic analysis ideas have become important for understanding the structure of nonlinearities, and even created a two way interaction between harmonic analysis and the analysis of dispersive equations. In addition, geometry plays an important role, for instance, the geometric properties of the target spaces (manifolds) for the solutions may determine certain characteristics of the equations, or involve obstacles or create compactly supported metric perturbations.

There are a large number of dispersive PDEs, the simplest ones include nonlinear Schrödinger equation or NLS, nonlinear wave equation or NLW, Korteweg de Vries or KdV, some more advanced ones are Benjamin-Ono, Boussinesq equations, and there are also systems like Zakharov system. A large part of research is centered in understanding and developing techniques and principles to analyze the existence of solutions and long term behavior of solutions at either high or low regularities.

In what follows, H^s and $\dot{H}^s$ stand for (inhomogeneous or homogeneous) Sobolev spaces, exact definition is given in Section 1.4.

1.1 Background

In this work, we study the global behavior of solutions to the d-dimensional focusing critical NLS equation with finite energy initial data (i.e., $u_0 \in H^1(\mathbb{R}^d)$). We consider the Cauchy problem for the nonlinear Schrödinger equation, denoted by $\mathrm{NLS}_p^{\pm}(\mathbb{R}^d)$,

$$\begin{cases} i\partial_t u + \Delta u + \mu |u|^{p-1} u = 0 \\ u(x,0) = u_0(x) \in H^1(\mathbb{R}^d), \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is a complex-valued function in space-time $\mathbb{R}_x^d \times \mathbb{R}_t$, $p \geq 1$ and $\mu = \pm 1$. The value $\mu = -1$ denotes the *defocusing*[1] NLS equation or $\mathrm{NLS}_p^{-}(\mathbb{R}^d)$, and $\mu = +1$ yields the *focusing*[2] NLS equation or $\mathrm{NLS}_p^{+}(\mathbb{R}^d)$.

Definition 1.1 (Solution)**.** Let $I \subseteq \mathbb{R}$ such that $0 \in I$. A function $u : \mathbb{R}^d \times I \to \mathbb{C}$ is a *strong* solution to $\mathrm{NLS}_p^{\pm}(\mathbb{R}^d)$ if and only if it belongs to $C(I, H^1(\mathbb{R}^d))$ and for all $t \in I$ satisfies the integral equation

$$u(t) = e^{it\Delta} u_0 + i\mu \int_0^t e^{i(t-\tau)\Delta} \left(|u|^{p-1} u(\tau) \right) d\tau. \tag{1.2}$$

A function $u : \mathbb{R}^d \times I \to \mathbb{C}$ is a *weak* solution to $\mathrm{NLS}_p^{\pm}(\mathbb{R}^d)$ if and only if it belongs to $L^\infty(I, H^1(\mathbb{R}^d))$ and for all $t \in I$ satisfies the integral equation (1.2).

[1] Intuitively, an equation is defocusing if the nonlinearity dissipates the solution when it is concentrated.

[2] The nonlinearity reinforces the solution when it is large and diminishes it when it is small.

For a fixed $\lambda \in (0,\infty)$, the rescaled function $u_\lambda(x,t) := \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t)$ is a solution of $\mathrm{NLS}_p^{\pm}(\mathbb{R}^d)$ in (1.1) if and only if $u(x,t)$ is. This scaling property gives rise to scale-invariant norms.

The scale-invariant Lebesgue norm is $L^{q_c}(\mathbb{R}^d)$ with $q_c := \frac{d(p-1)}{2}$, i.e., $\|u_\lambda\|_{L^{q_c}(\mathbb{R}^d)} = \|u\|_{L^{q_c}(\mathbb{R}^d)}$, where

$$\|u\|_{L^{q_c}}^{q_c} = \int_{(\mathbb{R}^d)} |u(x,t)|^{q_c} dx,$$

and the scale-invariant Sobolev norm is $\dot{H}^{s_c}(\mathbb{R}^d)$ with $s_c := \frac{d}{2} - \frac{2}{p-1}$, i.e., $\|u_\lambda\|_{\dot{H}^{s_c}(\mathbb{R}^d)} = \|u\|_{\dot{H}^{s_c}(\mathbb{R}^d)}$, where

$$\|u\|_{\dot{H}^{s_c}}^2 = \int_{(\mathbb{R}^d)} |\xi|^{2s_c} |\hat{u}(\xi,t)|^2 d\xi.$$

If $s_c = 0$, this means that $p = \frac{4}{d} + 1$, the problem is known as the *mass-critical* or L^2-*critical* and examples of this are $\mathrm{NLS}_5^{+}(\mathbb{R}^1)$ and $\mathrm{NLS}_3^{+}(\mathbb{R}^2)$; when $s_c = 1$, this means that $p = \frac{d+2}{d-2}$, it is called *energy-critical* or $\dot{H}^1-$*critical*, the $\mathrm{NLS}_5^{+}(\mathbb{R}^3)$ and $\mathrm{NLS}_4^{+}(\mathbb{R}^4)$ equations belong to this class.

The *mass-supercritical and energy-subcritical* problem is when $0 < s_c < 1$, that is,

$$\begin{cases} p > 5 & d = 1 \\ p > 3 & d = 2 \\ \frac{4+d}{d} < p < \frac{d+2}{d-2} & d \geq 3, \end{cases}$$

examples in this category are $\mathrm{NLS}_5^{+}(\mathbb{R}^2)$, $\mathrm{NLS}_3^{+}(\mathbb{R}^3)$, $\mathrm{NLS}_{\frac{5}{3}}^{+}(\mathbb{R}^7)$ and $\mathrm{NLS}_{\frac{13}{9}}^{+}(\mathbb{R}^{10})$.

Finally, the *energy-supercritical* equation is when $s_c > 1$, so $p = \frac{d+2}{d-2}$, and examples of this type are $\mathrm{NLS}_5^{+}(\mathbb{R}^4)$ and $\mathrm{NLS}_7^{+}(\mathbb{R}^3)$.

Definition 1.2 (Wellposedness)**.** The problem $\mathrm{NLS}_p^{\pm}(\mathbb{R}^d)$ is *locally wellposed* in $H^1(\mathbb{R}^d)$ if for any $u_0 \in H^1(\mathbb{R}^d)$ there exist a time $T > 0$ and an open ball B in $H^1(\mathbb{R}^d)$ containing u_0, and a subset X of $C([-T,T], H^1(\mathbb{R}^d))$, such that for each $u_0 \in B$ there exists a unique solution $u \in X$ to the integral equation (1.2), and furthermore, the map $u_0 \mapsto u$ is continuous from B to X. If T can be taken arbitrarily large ($T = +\infty$), the problem is *globally wellposed.*

Definition 1.3 (Interval of existence)**.** The maximal interval of existence in time of solutions to $\mathrm{NLS}_p^{\pm}(\mathbb{R}^d)$ is denoted by the interval (T_*, T^*). We say a solution is global in forward time if $T^* = +\infty$. Similarly, if $T_* = -\infty$, the solution is global in backward time. If we say solution is global, it means $(T_*, T^*) = \mathbb{R}$.

Definition 1.4 (Scattering in H^s)**.** A global solution $u(t)$ to $\mathrm{NLS}_p^{\pm}(\mathbb{R}^d)$ *scatters in* $H^s(\mathbb{R}^d)$ as $t \to +\infty$ if there exists $\psi^+ \in H^s(\mathbb{R}^d)$ such that

$$\lim_{t\to+\infty} \|u(t) - e^{it\Delta}\psi^+\|_{H^s(\mathbb{R}^d)} = 0. \tag{1.3}$$

Similarly, we can define scattering in $H^s(\mathbb{R}^d)$ for $t \to -\infty$.

A standard question about the initial value problem (1.1) is whether it has a solution which is locally (globally) wellposed. Ginibre-Velo [Ginibre and Velo, 1979a, Ginibre and Velo, 1979b] showed that the initial-value problem $\mathrm{NLS}_p^{\pm}(\mathbb{R}^d)$ with initial data $u(x,0) = u_0(x) \in H^1$, where $1 \le p < \frac{d+2}{d-2}$ for the defocusing case and $1 \le p < \frac{d+4}{d}$ in the focusing case, is locally well-posed in $H^s(\mathbb{R}^d)$ with $s \le 1$. Further, Cazenave-Weissler [Cazenave and Weissler, 1990] showed that for small initial data in $\dot{H}^s(\mathbb{R}^d)$, with $0 \le s < \frac{d}{2}$ and $0 < p \le \frac{d+2}{d-2}$, there exists a unique solution to $\mathrm{NLS}_p^{\pm}(\mathbb{R}^d)$ defined for all times.

On their maximal interval of existence, solutions to $\mathrm{NLS}_p^{\pm}(\mathbb{R}^d)$, have three conserved quantities: mass $M[u](t) = M[u_0]$, energy $E[u](t) = E[u_0]$ and momentum $P[u](t) = P[u_0]$, where

$$\begin{aligned}
M[u](t) &= \int_{\mathbb{R}^d} |u(x,t)|^2 dx,\\
E[u](t) &= \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(x,t)|^2 dx - \frac{\mu}{p+1}\int_{\mathbb{R}^d} |u(x,t)|^{p+1} dx,\\
P[u](t) &= \mathrm{Im}\int_{\mathbb{R}^d} \bar{u}(x,t)\nabla u(x,t) dx.
\end{aligned}$$

Moreover, since

$$\|u_\lambda\|_{L^2(\mathbb{R}^d)} = \lambda^{-s_c}\|u\|_{L^2(\mathbb{R}^d)}, \quad \|\nabla u_\lambda\|_{L^2(\mathbb{R}^d)} = \lambda^{-s_c+1}\|\nabla u\|_{L^2(\mathbb{R}^d)},$$

and

$$\|u_\lambda\|^{p+1}_{L^{p+1}(\mathbb{R}^d)} = \lambda^{2(-s_c+1)}\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^d)},$$

the below quantities are scaling invariant [Holmer and Roudenko, 2007]

$$\|u\|^{1-s_c}_{L^2(\mathbb{R}^d)}\|\nabla u\|^{s_c}_{L^2(\mathbb{R}^d)}, \quad \text{and} \quad E[u]^{s_c}M[u]^{1-s_c}.$$

Considering the history of mathematical developments of NLS, we start with the defocusing NLS equation $\mathrm{NLS}_p^-(\mathbb{R}^d)$. Bourgain in 1999 [Bourgain, 1999], for the energy critical NLS (i.e., $s_c = 1$) with initial radial data in $H^1(\mathbb{R}^3)$, established scattering in H^s with $s \geq 1$ for radial functions using the "induction on energy"[3] argument, in dimensions $d = 3, 4$. Grillakis [Grillakis, 2000] showed preservation of smoothness in H^1 with spherically symmetric initial data in 3 dimensions. Tao [Tao, 2005] extended Bourgain's result for radial data, to dimension 5 and higher with initial data in $\dot{H}^1$. Ginibre-Velo [Ginibre and Velo, 1985] established scattering in H^1 for solutions to the energy critical NLS in 3d ($\mathrm{NLS}_5^-(\mathbb{R}^3)$) with initial data in H^1 using Morawetz inequality[4]. Colliander-Keel-Staffilani-Takaoka-Tao [Colliander et al., 2008] simplified Ginibre-Velo argument using interaction Morawetz estimate and used the induction analysis in both momentum and configuration spaces. The interaction Morawetz estimate removes the localization at the origin (as it is observed in the usual Morawetz estimate) making it

[3] This technique allows to focus on the "minimal energy blowup solutions" which are localized both in space and frequency.

[4] Morawetz inequalities are monotonicity formula for nonlinear Schrodinger and wave equations, where the monotone quantity is usually generated by integrating the momentum density against a bounded vector field such as the outgoing spatial normal $\frac{x_i}{|x|}$. They are particularly useful for obtaining scattering results in Sobolev spaces such as in the energy class.

possible to handle the nonradial contributions of solutions. A further simplification of this proof is done by Killip-Visan [Killip and Visan, 2011]. Ryckman-Visan [Ryckman and Visan, 2007] extended scattering to $\text{NLS}_3^-(\mathbb{R}^4)$ with $u_0 \in \dot{H}^1(\mathbb{R}^4)$, and Visan [Visan, 2007] $\text{NLS}^-_{\frac{d+2}{d-2}}(\mathbb{R}^d)$ and $u_0 \in \dot{H}^1(\mathbb{R}^d)$ for $d \geq 5$.

Recent further works in the case $s_c = 0$ of $\text{NLS}_p^{\pm}(\mathbb{R}^d)$, i.e., $p = \frac{4}{d} + 1$, are by: Killip-Tao-Visan [Killip et al., 2009], Tao-Visan-Zhang [Tao et al., 2008] and Killip-Visan-Zhang [Killip et al., 2008], where they study scattering of globally existing solutions in the defocusing case (and also in the focusing under the threshold $M[u] < M[Q]$) in dimensions $d \geq 3$ for large spherically symmetric $L^2(\mathbb{R}^d)$ initial data. The recent work of Dodson has resolved the scattering question for the mass-critical NLS with L^2 initial data (see [Dodson, 2009, Dodson, 2010a, Dodson, 2010b]).

The focusing case has a different story. The local wellposedness is similar to the defocusing case, however, the global behavior of solutions in the focusing case is a largely open question. Some of the challenging cases here are the mass-critical ($s_c = 0$) and the energy-critical ($s_c = 1$) NLS equations when the initial data is also taken in L^2 or $\dot{H}^1$, correspondingly. For L^2-critical NLS equation with initial data $u_0 \in H^1(\mathbb{R}^d)$, Weinstein in [Weinstein, 1982] established a sharp threshold for global existence, namely, the condition $\|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)}$, where Q is the ground state solution[5], guarantees a global existence of evolution $u_0 \rightsquigarrow u(t)$. Solutions at the threshold mass, i.e., when $\|u_0\|_{L^2(\mathbb{R}^d)} = \|Q\|_{L^2(\mathbb{R}^d)}$, may blowup in finite time, such solutions are called the minimal mass blowup solutions. Merle in [Merle, 1993] characterized the minimal mass blowup H^1 solutions showing that all such solutions are pseudo-conformal transformations of the ground state (up

[5]See Section 2.4

to H^1 symmetries), that is,

$$u_T(x,t) = \frac{e^{i/(T-t)} e^{i|x|^2/(T-t)}}{T-t} Q\left(\frac{x}{T-t}\right).$$

In the energy-critical case $s_c = 1$, the known results are as follows. Kenig-Merle [Kenig and Merle, 2006] studied global behavior of solutions for the energy-critical $\mathrm{NLS}_p^+(\mathbb{R}^d)$ with $p = \frac{d+2}{d-2}$, and initial data in $\dot{H}^1(\mathbb{R}^d)$ in dimensions$d = 3, 4$, and 5. They showed that under a certain energy threshold (namely, $E[u_0] < E[W]$, where W is the positive solution of $\Delta W + W^p = 0$, decaying at ∞), it is possible to characterize global existence versus finite blowup depending on the size of the L^2–norm of gradient, and also prove scattering for globally existing solutions. To obtain the last property, they applied the concentration-compactness and rigidity technique. The concentration-compactness method appears in the context of wave equation in Gérard [Gérard, 1996] and NLS in Keraani [Keraani, 2001], and dates back to works of P-L. Lions [Lions, 1984] and Brezis-Coron [Brezis and Coron, 1985]. The rigidity argument (estimates on a localized variance) is the technique of F. Merle from mid 1980's. Killip-Visan [Killip and Visan, 2010] generalized the above result of Kenig-Merle [Kenig and Merle, 2006] for dimension $d = 5$ and higher.

The mass-supercritical and energy-subcritical case ($0 < s_c < 1$) is discussed in detail in the next section, and the energy-supercritical case ($s_c > 1$) is largely open.

1.2 The mass-supercritical and energy-subcritical problem

Another interesting critical focusing NLS problem is the mass- supercritical and energy-subcritical NLS ($0 < s_c < 1$), that is, the Cauchy problem (1.1) with $\mu = +1$ ($\mathrm{NLS}_p^+(\mathbb{R}^d)$) and $\begin{cases} p > 5 & d = 1 \\ p > 3 & d = 2 \\ \frac{4+d}{d} < p < \frac{d+2}{d-2} & d \geq 3 \end{cases}.$

Recall the invariant norm $\dot{H}^{s_c}(\mathbb{R}^d)$ with $s_c = \frac{d}{2} - \frac{2}{p-1}$, $s_c \in (0,1)$.

In physics, the 2d or 3d cubic NLS (a variant of Gross-Pitaevskii equation with zero potential) is the most relevant equation of this range ($0 < s_c < 1$) and it appears in modeling of several physical phenomena (see, for example, [Sulem and Sulem, 1999]). The $\mathrm{NLS}_3(\mathbb{R}^2)$ appears as a model in nonlinear optics, *Laser propagation in a Kerr medium* [Sulem and Sulem, 1999]. The equation $\mathrm{NLS}_3^{\pm}(\mathbb{R}^3)$ appears as a model for the Bose-Einstein condensate in condensed matter physics [Dalfovo et al., 1999] and together with nonlinear wave equation yields Zakharov system in plasma physics, *Langmuir turbulence in a weakly magnetized plasma*, [Zakharov, 1972].

The 3d cubic NLS equation with H^1 data has been studied in a series of papers [Holmer and Roudenko, 2008], [Duyckaerts et al., 2008], [Duyckaerts and Roudenko, 2010], [Holmer and Roudenko, 2010c] and [Holmer et al., 2010]. The authors obtained a sharp scattering threshold for radial initial data in [Holmer and Roudenko, 2008], then extension of these result to the nonradial data was obtained in [Duyckaerts et al., 2008]. This results hold under a so called *mass-energy threshold*

$$M[u]E[u] < M[Q]E[Q],$$

where Q is the ground state solution (see description Section in 2.4). Behavior of solutions and characterization of all solutions at the *mass-energy threshold* $M[u]E[u] = M[Q]E[Q]$ was done in [Duyckaerts and Roudenko, 2010] using spectral techniques. Furthermore, for infinite variance nonradial solutions Holmer-Roudenko in [Holmer and Roudenko, 2010c] introduced a first application of concentration-compactness and rigidity arguments to prove the existence of a "weak blowup"[6]. In addition, Holmer-Platte-Roudenko [Holmer et al., 2010]

[6]See Remark 1.7 and Chapter 4 for exact formulation and discussion.

consider (both theoretically and numerically) solutions to the 3d cubic NLS above the *mass-energy threshold* and give new blowup criteria in that region. They also predict the asymptotic behavior of solutions for different classes of initial data (Gaussian, super-Gaussian, off-centered Gaussian, and oscillatory Gaussian) and provide several conjectures in relation to the threshold for scattering.

In the spirit of [Duyckaerts et al., 2008], [Holmer and Roudenko, 2008], [Holmer and Roudenko, 2010c], Carreon-Guevara [Carreon and Guevara, 2011] study the long-term behavior of solutions for the 2d quintic NLS equation with H^1 initial data, which corresponds to the mass-supercritical and energy-subcritical NLS with $s_c = \frac{1}{2}$. Mainly, for the initial value problem $\mathrm{NLS}_5^+(\mathbb{R}^2)$ scattering and blowup was proven, including the existence of a "weak blowup". This equation is interesting since first of all, it has a higher power of nonlinearity (higher than cubic), secondly, recently a nontrivial blowup result (on a standing ring) was exhibited by Raphaël in [Raphaël, 2006], there are further extensions of [Raphaël, 2006] to higher dimensions and different nonlinearities in [Raphaël and Szeftel, 2009], also [Holmer and Roudenko, 2010b], and [Zwiers, 2010]; an H^1 control on the outside of the blowup core is shown in [Holmer and Roudenko, 2010a], which improves the result of [Raphaël and Szeftel, 2009].

As it was mentioned before, the key argument to obtain scattering and "weak blowup" is the concentration compactness technique together with a rigidity theorem. Note that for $2 < q < \frac{2d}{d-2}$ the embedding $H^1(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ is not compact[7]; however, a profile decomposition allows to manage this lack of compactness and to produce a "critical element". Then a localization principle proves scattering or weak blowup, depending on the initial assumptions.

[7]In fact, given any $f \in H^1(\mathbb{R}^d)$, the sequence $f_n(x) = f(x - x_n)$, where the sequence $x_n \to \infty$ in $\mathbb{R}^d$, is uniformly bounded in $H^1(\mathbb{R}^d)$, but has no convergent sequence on L^q.

To conclude this section, we point out that the concentration compactness-ridigity method can be used for various types of PDEs, not necessary dispersive ones. For example, a recent work of Kenig-Koch [Kenig and Koch, 2009] presents an alternative to study regularity of solutions to the Navier-Stokes equations in a critical space [Escauriaza et al., 2003]; they proved that mild solutions which remain bounded in L^3 for all times do not become singular in finite time using the concentration compactness and rigidity theorem.

1.3 Overview of the results

Throughout this document, unless otherwise specified, we will always assume that $0 < s < 1$ and $s = \frac{d}{2} - \frac{2}{p-1}$, $\alpha := \frac{\sqrt{d(p-1)}}{2}$, and $\beta := 1 - \frac{(d-2)(p-1)}{4}$, and let

$$u_{_Q}(x,t) := e^{i\beta t} Q(\alpha x). \tag{1.4}$$

Then $u_{_Q}(x,t)$ solves the equation (1.1), provided Q solves[8]

$$-\beta\, Q + \alpha^2\, \Delta Q + Q^p = 0, \quad Q = Q(x), \quad x \in \mathbb{R}^d. \tag{1.5}$$

The theory of nonlinear elliptic equations (Berestycki-Lions [Berestycki and Lions, 1983a, Berestycki and Lions, 1983b]) shows that (1.5) has an infinite number of solutions in $H^1(\mathbb{R}^d)$, but a unique solution of minimal L^2-norm, which we denote by $Q(x)$. It is positive, radial, exponentially decaying (for example, [Tao, 2006, Appendix B]) and is called the *ground state* solution.

As it was mentioned in Section 1.1 the quantities

$$\|u\|^{1-s_c}_{L^2(\mathbb{R}^d)} \|\nabla u\|^{s_c}_{L^2(\mathbb{R}^d)} \quad \text{and} \quad E[u]^{s_c} M[u]^{1-s_c}$$

[8]Here, in the equation (1.5) and definition of Q, we use the notation from Weinstein [Weinstein, 1982]. Rescaling $Q(x) \mapsto \beta^{\frac{1}{p-1}} Q\Big(\sqrt{\frac{\beta}{\alpha}}x\Big)$ will solve a more common version of the nonlinear elliptic equation $-Q + \Delta Q + Q^p = 0$.

are scaled invariant, therefore, we introduce the following notation:

- the renormalized gradient $$\mathcal{G}_u(t) := \frac{\|u\|^{1-s}_{L^2(\mathbb{R}^d)}\|\nabla u(t)\|^s_{L^2(\mathbb{R}^d)}}{\|u_Q\|^{1-s}_{L^2(\mathbb{R}^d)}\|\nabla u_Q\|^s_{L^2(\mathbb{R}^d)}}, \quad (1.6)$$
- the renormalized momentum $$\mathcal{P}[u] := \frac{P[u]^s\|u\|^{1-2s}_{L^2(\mathbb{R}^d)}}{\|u_Q\|^{1-s}_{L^2(\mathbb{R}^d)}\|\nabla u_Q\|^s_{L^2(\mathbb{R}^d)}}, \quad (1.7)$$
- the renormalized Mass-Energy $$\mathcal{ME}[u] := \frac{M[u]^{1-s}E[u]^s}{M[u_Q]^{1-s}E[u_Q]^s}. \quad (1.8)$$

Remark 1.5 (Negative energy). Note that the renormalized mass-energy $\mathcal{ME}[u] < 1$ defined in (1.8) is not defined when $E[u] < 0$ and s is fractional. However, if $E[u] < 0$, the blowup from the dichotomy in Theorem 1.6 Part II (a) applies. (It follows from the standard convexity blow up argument and the work of Glangetas-Merle [Glangetas and Merle, 1995]). Thus, we only consider positive energy in what follows.

The main result of this dissertation is

Theorem 1.6. *Consider* $NLS_p^+(\mathbb{R}^d)$ *with* $u_0 \in H^1(\mathbb{R}^d)$ *with* $d \geq 1$ *and let* $u(t)$ *be the corresponding solution in its maximal time interval of existence* (T_*, T^*)*, and* $s := s_c \in (0,1)$*. Let* $u_Q(x,t)$ *be as in (1.4), and assume*

$$(\mathcal{ME}[u])^{\frac{1}{s}} - \frac{d}{2s}(\mathcal{P}[u])^{\frac{2}{s}} < 1. \quad (1.9)$$

I. If

$$[\mathcal{G}_u(0)]^{\frac{2}{s}} - (\mathcal{P}[u])^{\frac{2}{s}} < 1, \quad (1.10)$$

then

(a) $[\mathcal{G}_u(t)]^{\frac{2}{s}} - (\mathcal{P}[u])^{\frac{2}{s}} < 1$ *for all* $t \in \mathbb{R}$*, and thus, the solution is global in time (i.e.,* $T_* = -\infty$*,* $T^* = +\infty$*) and*

(b) u *scatters in* $H^1(\mathbb{R}^d)$*, i.e., there exists* $\phi_\pm \in H^1(\mathbb{R}^d)$ *such that*

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta}\phi_\pm\|_{H^1(\mathbb{R}^d)} = 0.$$

II. If

$$[\mathcal{G}_u(0)]^{\frac{2}{s}} - (\mathcal{P}[u])^{\frac{2}{s}} > 1, \tag{1.11}$$

then $[\mathcal{G}_u(t)]^{\frac{2}{s}} - (\mathcal{P}[u])^{\frac{2}{s}} > 1$ *for all* $t \in (T_*, T^*)$ *and*

(a) if u_0 *is radial (for* $d \geq 3$ *and in* $d = 2,\ 3 < p \leq 5$*) or* u_0 *is of finite variance, i.e.,* $|x|u_0 \in L^2(\mathbb{R}^d)$*, then the solution blows up in finite time (i.e.,* $T^* < +\infty,\ T_* > -\infty$*).*

(b) If u_0 *is non-radial and of infinite variance, then either the solution blows up in finite time (i.e.,* $T^* < +\infty,\ T_* > -\infty$*) or there exists a sequence of times* $t_n \to +\infty$ *(or* $t_n \to -\infty$*) such that* $\|\nabla u(t_n)\|_{L^2(\mathbb{R}^d)} \to \infty$.

Remark 1.7 (Weak blowup). We say there is a "weak blowup" if under the $\mathcal{ME}[u] < 1$, $u(t)$ exists globally for all positive time and there exists a sequence of times $t_n \to +\infty$ such that $\|\nabla u(t_n)\|_{L^2} \to \infty$. In other words, L^2 norm of the gradient diverges in infinite time.

1.4 Notation

Throughout this dissertation we write $X \lesssim Y$ whenever there exists some constant C independent of the parameters, so that $X \leq CY$. The abbreviation $\mathcal{O}(X)$ denotes a finite linear combination of terms that "look like" X, but possibly with some factors replaced by their complex conjugates. We use the 'Japanese bracket' convention: $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$ and $\langle \nabla \rangle := (1 - \Delta)^{\frac{1}{2}}$, where the derivative operator ∇ refers only to the space variable.

Define the Fourier transform on $\mathbb{R}^d$

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) dx,$$

and the inverse Fourier transform on $\mathbb{R}^d$ is given by

$$f(x) := \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \hat{f}(\xi) d\xi.$$

We regularly refer to the *space-time* norms

$$\|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} = \|u\|_{L_t^q L_x^r} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} |u(x,t)|^r dx \right)^{\frac{q}{r}} dt \right)^{\frac{1}{q}}$$

with the corresponding changes when either $q = \infty$ or $r = \infty$.

We work with the *fractional differentiation operators* D^s defined by

$$\widehat{D^s f}(\xi) := |\xi|^s \hat{f}(\xi).$$

The *inhomogeneous Sobolev* norm $H^s(\mathbb{R}^d)$ is defined by (when s is an integer)

$$\|f\|_{H^s(\mathbb{R}^d)} = \|f\|_{H^s} := \sum_{|\alpha|=0}^{s} \|\partial_x^\alpha f\|_{L^2(\mathbb{R}^d)},$$

when s is any real number as,

$$\|f\|_{H^s(\mathbb{R}^d)} = \|f\|_{H^s} := \Big(\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 (1+|\xi|^2)^s d\xi \Big)^{\frac{1}{2}},$$

and the *homogeneous Sobolev* norm $\dot{H}^s(\mathbb{R}^d)$ is defined as

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} = \|f\|_{\dot{H}^s} := \Big(\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 |\xi|^{2s} d\xi \Big)^{\frac{1}{2}}.$$

Let $e^{it\Delta} f$ be the free Schrödinger propagator, i.e., a solution of the linear Schödinger equation $iu_t + \Delta u = 0$ with $u(0,x) = f(x)$. In physical space, this is given by

$$e^{it\Delta} f(x) = \frac{1}{(4\pi i t)^{d/2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} f(y) dy,$$

and in frequency space

$$\widehat{e^{it\Delta} f}(\xi) = e^{-4\pi^2 it|\xi|^2} \hat{f}(\xi).$$

In particular, the propagator preserves the homogeneous Sobolev norms and obeys the dispersive inequality

$$\|e^{it\Delta} f\|_{L_x^\infty(\mathbb{R}^d)} \lesssim |t|^{-\frac{d}{2}} \|f\|_{L_x^1(\mathbb{R}^d)} \tag{1.12}$$

for all times $t \neq 0$, for example, see Cazenave [Cazenave, 2003, Proposition 2.2.3].

We employ some *Littlewood-Paley operators* theory. Specifically, let $\varphi \in C^\infty_{comp}(\mathbb{R}^d)$ be such that

$$\varphi(\xi) = \begin{cases} 1 & |\xi| \leq 1 \\ 0 & |\xi| \geq 2. \end{cases}$$

For each dyadic number $N \in 2^{\mathbb{Z}}$ and a Schwartz function f, define the Littlewood-Paley operators

$$\widehat{P_{\leq N} f}(\xi) := \varphi\Big(\frac{\xi}{N}\Big)\hat{f}(\xi), \qquad \widehat{P_{>N} f}(\xi) := \bigg(1 - \varphi\Big(\frac{\xi}{N}\Big)\bigg)\hat{f}(\xi),$$

$$\widehat{P_N f}(\xi) := \bigg(\varphi\Big(\frac{\xi}{N}\Big) - \varphi\Big(\frac{2\xi}{N}\Big)\bigg)\hat{f}(\xi).$$

Thus, $\widehat{P_N f}$, $\widehat{P_{\leq N} f}$, and $\widehat{P_{>N} f}$ are smooth out projections to the regions $|\xi| \sim N$, $|\xi| \leq 2N$, and $|\xi| > N$, respectively. Note that for all Schwartz functions f, and $M \in \mathbb{Z}$

$$\widehat{P_{\leq N} f}(\xi) = \sum_{M \leq N} \widehat{P_M f}, \qquad \widehat{P_{>N} f}(\xi) = \sum_{M > N} \widehat{P_M f}, \qquad f = \sum_M P_M f.$$

Similarly, $P_{<N}$, $P_{\geq N}$, and $P_{M<\cdot\leq N} := P_{\leq N} - P_{\leq M}$ can be defined whenever M and N are dyadic numbers with $M < N$.

Note that Littlewood-Paley operators commute with derivative operators, the free propagator, and complex conjugation. They are self-adjoint and bounded on every L^p and $\dot{H}^s_x$ space for $1 \leq p < \infty$ and $s \geq 0$. They also obey the following Sobolev and Bernstein estimates:

$$\begin{aligned}
\big\|P_{\geq N} f\big\|_{L^p_x} &\lesssim_{p,s,n} N^{-s}\big\|D^s P_{\geq N} f\big\|_{L^p_x} = N^{-s}\big\|P_{\geq N} D^s f\big\|_{L^p_x}, \\
\big\|P_{\leq N} D^s f\big\|_{L^p_x} = \big\|D^s P_{\leq N} f\big\|_{L^p_x} &\lesssim_{p,s,n} N^s\big\|P_{\leq N} f\big\|_{L^p_x}, \\
\big\|P_N D^{\pm s} f\big\|_{L^p_x} = \big\|D^{\pm s} P_N f\big\|_{L^p_x} &\sim_{p,s,n} N^{\pm s}\big\|P_N f\big\|_{L^p_x}, \\
\big\|P_{\leq N} f\big\|_{L^q_x} &\lesssim_{p,q,n} N^{\frac{n}{p}-\frac{n}{q}}\big\|P_{\leq N} f\big\|_{L^p_x}, \\
\big\|P_N f\big\|_{L^q_x} &\lesssim_{p,q,n} N^{\frac{n}{p}-\frac{n}{q}}\big\|P_N f\big\|_{L^p_x},
\end{aligned}$$

whenever $s \geq 0$ and $1 \leq p \leq q < \infty$, for further discussion see Tao [Tao, 2006].

Let $S'(\mathbb{R}^d)$ be the space of tempered distributions on $\mathbb{R}^d$. For $1 \leq p, q \leq \infty$ and $\sigma > \frac{d}{p}$, define the *inhomogeneous Besov space* $\beta^\sigma_{p,q}(\mathbb{R}^d) = \{u \in S'(\mathbb{R}^d) : \|u\|_{B^\sigma_{p,q}(\mathbb{R}^d)} < \infty\}$, where

$$\|u\|_{\beta^\sigma_{p,q}(\mathbb{R}^d)} := \|P_{\leq N}u\|_{L^p_x} + \left(\sum_{j=1}^{\infty} \left(2^{j\sigma} \|P_{2^j}u\|_{L^p_x} \right)^q \right)^{\frac{1}{q}}$$
$$= \|P_{\leq N}u\|_{L^p} + \Big(\sum_{N \in 2^{\mathbb{Z}}} \big(N^\sigma \|P_N u\|_{L^p_x} \big)^q \Big)^{\frac{1}{q}},$$

and the *homogeneous Besov space* $\dot{\beta}^\sigma_{p,q}(\mathbb{R}^d) = \{u \in S'(\mathbb{R}^d) : \|u\|_{\dot{\beta}^\sigma_{p,q}(\mathbb{R}^d)} < \infty\}$, where

$$\|u\|_{\dot{\beta}^\sigma_{p,q}(\mathbb{R}^d)} := \Big(\sum_{N \in 2^{\mathbb{Z}}} \big(N^\sigma \|P_N u\|_{L^p_x} \big)^q \Big)^{\frac{1}{q}}.$$

Note that most of the L^p, H^s, $\dot{H}^s$, $\beta^\sigma_{p,q}$ and $\dot{\beta}^\sigma_{p,q}$ norms are defined on $\mathbb{R}^d$, thus, we will omit the symbol $\mathbb{R}^d$ unless we need a specific space dimension.

Chapter 2

PRELIMINARIES

In this chapter, we review the Strichartz estimates (e.g., see Cazenave [Cazenave, 2003], Keel-Tao [Keel and Tao, 1998], Foschi [Foschi, 2005]), fractional calculus tools and local theory; these are the instruments to treat the nonlinearity $F(u) = |u|^{p-1}u$. In addition, we survey the ground state properties and the reduction to the zero momentum which allows us to restate Theorem 1.6 into a simpler version.

2.1 Fractional calculus tools

For Lemmas 2.1, 2.3, 2.2, assume $p, p_i \in (1,\infty)$, $\frac{1}{p} = \frac{1}{p_i} + \frac{1}{p_{i+1}}$, with $i = 1,2,3$.

Lemma 2.1 (Chain rule [Kenig et al., 1993]). *Suppose $F \in C^1(\mathbb{C})$. Let $\sigma \in (0,1)$, then*

$$\|D^\sigma F(u)\|_{L^p} \lesssim \|F'(u)\|_{L^{p_1}} \|D^\sigma f\|_{L^{p_2}}.$$

Lemma 2.2 (Leibniz rule [Kenig et al., 1993]). *Let $\sigma \in (0,1)$, then*

$$\|D^\sigma (fg)\|_{L^p} \lesssim \Big(\|f\|_{L^{p_1}} \|D^\sigma g\|_{L^{p_2}} + \|g\|_{L^{p_3}} \|D^\sigma f\|_{L^{p_4}} \Big).$$

Lemma 2.3 (Chain rule for Hölder-continuous functions [Visan, 2007]). *Let F be a Hölder-continuous function of order $0 < \rho < 1$, then for every $0 < \sigma < \rho$, and $\frac{\sigma}{\rho} < \nu < 1$ we have*

$$\big\|D^\sigma F(u)\big\|_{L^p} \lesssim \big\||u|^{\rho - \frac{\sigma}{\nu}}\big\|_{L^{p_1}} \big\|D^\nu u\big\|^{\frac{\sigma}{\nu}}_{L^{\frac{\sigma}{\nu}p_2}},$$

provided $(1 - \frac{\sigma}{\rho\nu})p_1 > 1$.

2.2 Strichartz type estimates

We say the pair (q,r) is $\dot{H}^s-$Strichartz admissible if

$$\frac{2}{q}+\frac{d}{r}=\frac{d}{2}-s, \qquad \text{with} \quad 2\leq q,r\leq\infty \quad \text{and} \quad (q,r,d)\neq(2,\infty,2);$$

and the pair (q,r) is $\frac{d}{2}-$acceptable if

$$1\leq q,r\leq\infty, \qquad \frac{1}{q}<d\Big(\frac{1}{2}-\frac{1}{r}\Big), \quad \text{or} \quad (q,r)=(\infty,2).$$

As usual we denote by q' and r' the Hölder conjugates of q and r, respectively (i.e., $\frac{1}{r}+\frac{1}{r'}=1$).

2.2.1 Strichartz estimates

The Strichartz estimates (e.g., see Cazenave [Cazenave, 2003], [Keel and Tao, 1998], Foschi [Foschi, 2005]) are

$$\left\|e^{it\triangle}\phi\right\|_{L_t^qL_x^r}\lesssim\|\phi\|_{L^2}, \qquad \left\|\int e^{-i\tau\triangle}f(\tau)d\tau\right\|_{L^2}\lesssim\|\phi\|_{L_t^{q'}L_x^{r'}}, \tag{2.1}$$

$$\left\|\int_{\tau<t}e^{i(t-\tau)\triangle}f(\tau)d\tau\right\|_{L_t^qL_x^r}\lesssim\|f\|_{L_t^{q'}L_x^{r'}}, \tag{2.2}$$

where (q,r) is an $\dot{H}^s-$Strichartz admissible pair. The retarded estimate (2.2) have a wider range of admissibility and holds when the pair (q,r) is $\frac{d}{2}-$acceptable [Kato, 1994].

In order to include the appropriate (for our goals) admissible pairs for the (2.2), define the *Strichartz space* $S(\dot{H}^s)=S(\dot{H}^s(\mathbb{R}^d\times I))$ as the closure of all test functions under the norm $\|\cdot\|_{S(\dot{H}^s)}$ with

$$\|u\|_{S(\dot H^s)} = \begin{cases} \sup\left\{ \|u\|_{L_t^q L_x^r} \;\middle|\; \begin{array}{c} (q,r)\ \dot H^s \text{ admissible with} \\ \left(\frac{2}{1-s}\right)^+ \le q \le \infty, \quad \frac{2d}{d-2s} \le r \le \left(\frac{2d}{d-2}\right)^- \end{array} \right\} & \text{if } d \ge 3 \\ \sup\left\{ \|u\|_{L_t^q L_x^r} \;\middle|\; \begin{array}{c} (q,r)\ \dot H^s \text{ admissible with} \\ \left(\frac{2}{1-s}\right)^+ \le q \le \infty, \quad \frac{2}{1-s} \le r \le \left(\left(\frac{2}{1-s}\right)^+\right)' \end{array} \right\} & \text{if } d = 2 \\ \sup\left\{ \|u\|_{L_t^q L_x^r} \;\middle|\; \begin{array}{c} (q,r)\ \dot H^s \text{ admissible with} \\ \frac{4}{1-2s} \le q \le \infty, \quad \frac{2}{1-2s} \le r \le \infty \end{array} \right\} & \text{if } d = 1. \end{cases}$$

Here, $(a^+)'$ is defined as $(a^+)' := \frac{a^+ \cdot a}{a^+ - a}$, so that $\frac{1}{a} = \frac{1}{(a^+)'} + \frac{1}{a^+}$ for any positive real value a, with a^+ being a fixed number slightly larger than a. Likewise, a^- is a fixed number slightly smaller than a.

Remark 2.4. Note that $\frac{2d}{d-2s} < \left(\frac{2d}{d-2}\right)^- < \frac{2d}{d-2}$, if $d \ge 3$. Additionally, when $d = 2$ and $s \ne \frac{1}{2}$, the quantity $r = \frac{2d}{d-2s}$ might be very large, but $\frac{2d}{d-2s} < \left(\left(\frac{2}{1-s}\right)^+\right)'$.

Similarly, define the *dual Strichartz space* $S'(\dot H^{-s}) = S'(\dot H^{-s}(\mathbb{R}^d \times I))$ as the closure of all test functions under the norm $\|\cdot\|_{S'(\dot H^{-s})}$ with

$$\|u\|_{S'(\dot H^{-s})} = \begin{cases} \inf\left\{ \|u\|_{L_t^{q'} L_x^{r'}} \;\middle|\; \begin{array}{c} (q,r)\ \dot H^{-s} \text{ admissible with} \\ \left(\frac{2}{1+s}\right)^+ \le q \le \left(\frac{1}{s}\right)^-, \ \left(\frac{2d}{d-2s}\right)^+ \le r \le \left(\frac{2d}{d-2}\right)^- \end{array} \right\} & \text{if } d \ge 3 \\ \inf\left\{ \|u\|_{L_t^{q'} L_x^{r'}} \;\middle|\; \begin{array}{c} (q,r)\ \dot H^{-s} \text{ admissible with} \\ \left(\frac{2}{1+s}\right)^+ \le q \le \left(\frac{1}{s}\right)^-, \ \left(\frac{2}{1-s}\right)^+ \le r \le \left(\left(\frac{2}{1+s}\right)^+\right)' \end{array} \right\} & \text{if } d = 2 \\ \inf\left\{ \|u\|_{L_t^{q'} L_x^{r'}} \;\middle|\; \begin{array}{c} (q,r)\ \dot H^{-s} \text{ admissible with} \\ \frac{2}{1+2s} \le q \le \left(\frac{1}{s}\right)^-, \ \left(\frac{2}{1-s}\right)^+ \le r \le \infty \end{array} \right\} & \text{if } d = 1. \end{cases}$$

Remark 2.5. Note that $S(L^2) = S(\dot H^0)$ and $S'(L^2) = S'(\dot H^{-0})$. In this dissertation, if (q,r) is $\dot H^{-0}$ admissible we say a pair (q', r') is L^2 *dual admissible*.

Under the above definitions, the Strichartz estimates (2.1) become

$$\|e^{it\Delta}\phi\|_{S(L^2)} \leq c\|\phi\|_{L^2} \quad \text{and} \quad \Big\| \int_{s<t} e^{i(t-s)\Delta} f(s) ds \Big\|_{S(L^2)} \leq c\|f\|_{S'(L^2)} \tag{2.3}$$

and in this paper, we refer to them as *the (standard) Strichartz* estimates.

Combining (2.3) with the Sobolev embedding $W_x^{s,r}(\mathbb{R}^d) \hookrightarrow L_x^{\frac{nr}{n-sr}}(\mathbb{R}^d)$ for $s < \frac{n}{r}$ and interpolating yields *the Sobolev Strichartz* estimates

$$\|e^{it\Delta}\phi\|_{S(\dot{H}^s)} \leq c\|\phi\|_{\dot{H}^s} \quad \text{and} \quad \Big\| \int_0^t e^{i(t-s)\Delta} f(s) ds \Big\|_{S(\dot{H}^s)} \leq c\|D^s f\|_{S'(L^2)}, \tag{2.4}$$

and in similar fashion (2.2) leads to the *Kato's Strichartz* estimate [Kato, 1987, Foschi, 2005]

$$\Big\| \int_0^t e^{i(t-s)\Delta} f(s) ds \Big\|_{S(\dot{H}^s)} \leq c\|f\|_{S'(\dot{H}^{-s})}. \tag{2.5}$$

Kato's Strichartz estimate along with the Sobolev embedding imply the inhomogeneous estimate (second estimate in (2.4)) and it is the key estimate in the long term perturbation argument (Proposition 2.17).

2.2.2 Besov Strichartz estimates

We will also address a question of non-integer nonlinearities for $\mathrm{NLS}_p^+(\mathbb{R}^d)$. Thus, the following remark is due

Remark 2.6. The complex derivative of the nonlinearity $F(u) = |u|^{p-1}u$ is $F_z(z) = \frac{p+1}{2}|z|^{p-1}$ and $F_{\bar{z}}(z) = \frac{p-1}{2}|z|^{p-1}\frac{z}{\bar{z}}$. They are Hölder-continuous functions of order p, and for any $u, v \in \mathbb{C}$, we have

$$F(u) - F(v) = \int_0^1 \Big[F_z(v + t(u-v))(u-v) + F_{\bar{z}}(v + t(u-v))\overline{(u-v)} \Big] dt, \tag{2.6}$$

thus,

$$|F(u) - F(v)| \lesssim |u-v|\big(|u|^{p-1} + |v|^{p-1}\big). \tag{2.7}$$

Hence, the nonlinearity $F(u)$ satisfies

(a) $F \in C^2(\mathbb{C})$, if $2 \le d < 5$, or $d = 5$ when $\frac{1}{2} < s_c < 1$,

(b) $F \in C^1(\mathbb{C})$, if $d \ge 6$, or $d = 5$ when $0 < s_c \le \frac{1}{2}$.

When estimating the fractional derivatives of (2.6), in the case (b), there is a lack of smoothness. This issue is resolved by using the Besov Spaces.

Define the *Besov Strichartz* space $\dot{\beta}^{\sigma}_{S(\dot{H}^s)} = \dot{\beta}^{\sigma}_{S(\dot{H}^s)}(\mathbb{R}^d \times I)$ as the closure of all test functions under the semi-norm $\|\cdot\|_{\dot{\beta}^{\sigma}_{S(\dot{H}^s)}}$ with

$$\|u\|_{\dot{\beta}^{\sigma}_{S(\dot{H}^s)}} = \begin{cases} \sup\left\{ \|u\|_{L^q_t \dot{\beta}^{\sigma}_{r,2}} \;\middle|\; \begin{array}{c} (q,r)\ \dot{H}^s \text{ admissible with} \\ \left(\frac{2}{1-s}\right)^+ \le q \le \infty, \quad \frac{2d}{d-2s} \le r \le \left(\frac{2d}{d-2}\right)^- \end{array} \right\} & \text{if } d \ge 3 \\ \sup\left\{ \|u\|_{L^q_t \dot{\beta}^{\sigma}_{r,2}} \;\middle|\; \begin{array}{c} (q,r)\ \dot{H}^s \text{ admissible with} \\ \left(\frac{2}{1-s}\right)^+ \le q \le \infty, \quad \frac{2}{1-s} \le r \le \left(\left(\frac{2}{1-s}\right)^+\right)' \end{array} \right\} & \text{if } d = 2. \\ \sup\left\{ \|u\|_{L^q_t \dot{\beta}^{\sigma}_{r,2}} \;\middle|\; \begin{array}{c} (q,r)\ \dot{H}^s \text{ admissible with} \\ \frac{4}{1-2s} \le q \le \infty, \quad \frac{2}{1-2s} \le r \le \infty \end{array} \right\} & \text{if } d = 1. \end{cases}$$

Similary, define the *dual Besov Strichartz* space $\dot{\beta}^{\sigma}_{S'(\dot{H}^{-s})} = \dot{\beta}^{\sigma}_{S'(\dot{H}^{-s})}(\mathbb{R}^d \times I)$ as the closure of all test functions under the semi-norm $\|\cdot\|_{\dot{\beta}^{\sigma}_{S'(\dot{H}^{-s})}}$ with

$$\|u\|_{\dot{\beta}^{\sigma}_{S'(\dot{H}^{-s})}} = \begin{cases} \inf\left\{ \|u\|_{L^{q'}_t \dot{\beta}^{\sigma}_{r',2}} \;\middle|\; \begin{array}{c} (q,r)\ \dot{H}^{-s} \text{ admissible with} \\ \left(\frac{2}{1+s}\right)^+ \le q \le \left(\frac{1}{s}\right)^-,\ \left(\frac{2d}{d-2s}\right)^+ \le r \le \left(\frac{2d}{d-2}\right)^- \end{array} \right\} & \text{if } d \ge 3 \\ \inf\left\{ \|u\|_{L^{q'}_t \dot{\beta}^{\sigma}_{r',2}} \;\middle|\; \begin{array}{c} (q,r)\ \dot{H}^{-s} \text{ admissible with} \\ \left(\frac{2}{1+s}\right)^+ \le q \le \left(\frac{1}{s}\right)^-,\ \left(\frac{2}{1-s}\right)^+ \le r \le \left(\left(\frac{2}{1+s}\right)^+\right)' \end{array} \right\} & \text{if } d = 2 \\ \inf\left\{ \|u\|_{L^{q'}_t \dot{\beta}^{\sigma}_{r',2}} \;\middle|\; \begin{array}{c} (q,r)\ \dot{H}^{-s} \text{ admissible with} \\ \frac{2}{1+2s} \le q \le \left(\frac{1}{s}\right)^-,\ \left(\frac{2}{1-s}\right)^+ \le r \le \infty \end{array} \right\} & \text{if } d = 1. \end{cases}$$

Lemma 2.7. *If* $u \in \dot{\beta}^{\sigma}_{S(\dot{H}^s)}$ *and* $\sigma \geq 0,\ s \in \mathbb{R},$ *then*

$$\left\|D^{\sigma} u\right\|_{S(\dot{H}^s)} \lesssim \|u\|_{\dot{\beta}^{\sigma}_{S(\dot{H}^s)}}.$$

Proof. Let (q, r) be $\dot{H}^s$ admissible pair, then

$$\begin{aligned}\left\|D^{\sigma} u\right\|_{L^q_t L^r_x} &\lesssim \left\|\Big(\sum_{N \in 2^{\mathbb{Z}}} \left|P_N D^{\sigma} u\right|^2\Big)^{\frac{1}{2}}\right\|_{L^q_t L^r_x} \lesssim \left\|\Big(\sum_{N \in 2^{\mathbb{Z}}} \|P_N D^{\sigma} u\|^2_{L^r_x}\Big)^{\frac{1}{2}}\right\|_{L^q_t} \\ &\approx \left\|\Big(\sum_{N \in 2^{\mathbb{Z}}} N^{2\sigma} \|P_N u\|^2_{L^r_x}\Big)^{\frac{1}{2}}\right\|_{L^q_t} \lesssim \|u\|_{\dot{\beta}^{\sigma}_{L^q_t \dot{\beta}^{\sigma}_{r,2}}}.\end{aligned}$$

Taking sup over all (q, r) $\dot{H}^s-$admissible pairs yields the claim. □

Lemma 2.8 (Embedding)**.** *For any compact time interval* I*, assume* $0 \leq \sigma < \rho$*,* $1 \leq r, r_1, q \leq \infty$*. Then*

$$\|D^{\sigma} u\|_{L^q_t L^r_x} \lesssim \|D^{\rho} u\|_{L^q_t L^{r_1}_x}, \tag{2.8}$$

where $r_1 = \frac{rd}{(\rho - \sigma) r + d}$ *and* $q_1 = q_2$*.*

Proof. The Sobolev embedding $\dot{W}^{\rho, r_1}_x(\mathbb{R}^d) \hookrightarrow \dot{W}^{\sigma, r}_x(\mathbb{R}^d)$ yields the inequality (2.8). □

Remark 2.9. If q', r' and r'_1 are the Hölder's conjugates of r, q and r_1, respectively, then we have

$$\|D^{\rho} u\|_{L^{q'}_t L^{r'_1}_x} \lesssim \|D^{\sigma} u\|_{L^{q'}_t L^{r'}_x}.$$

Lemma 2.10 (Linear Besov-Strichartz)**.** *Let* $u \in \dot{\beta}^{\sigma}_{S(L^2)}$ *be a solution to the forced Schrödinger equation*

$$iu_t + \Delta u = \sum_{m=1}^{M} F_m \tag{2.9}$$

for some functions $F_1, \ldots, F_M$ *and* $\sigma = 0$ *or* $\sigma = s$*. Then on* $\mathbb{R}^d \times I$ *we have*

$$\|u\|_{\dot{\beta}^{\sigma}_{S(\dot{H}^s)}} \lesssim \|u_0\|_{\dot{H}^{\sigma}} + \sum_{m=1}^{M} \|F_m\|_{\dot{\beta}^{\sigma}_{S'(L^2)}}. \tag{2.10}$$

Proof. It suffices to prove the statement for $M = 1$, since combining Duhamel's formula (1.2) and the triangle inequality yield the proof for $M \geq 1$. Furthermore, it is enough to prove for $\sigma = 0$ because applying D^s to both sides of the equation (2.9), and observing that D^s and Littlewood-Paley operators commute with $i\partial_t + \Delta$ give that for all dyadic N

$$i\partial_t P_N u + \Delta P_N u = P_N F_1.$$

Note that the standard Strichartz estimates (2.4) yield

$$\|P_N u\|_{S(\dot{H}^s)} \lesssim \|P_N u(t_0)\|_{L^2_x} + \|P_N F_1\|_{S'(L^2)}, \tag{2.11}$$

squaring (2.11), summing over all dyadic N, and combining with the Littlewood-Paley inequality, the claim is obtained. □

Lemma 2.11 (Inhomogeneous Besov Strichartz estimate)**.** *If* $F \in \dot{\beta}^{\sigma}_{S'(\dot{H}^{-s})}$, *then*

$$\Big\| \int_0^t e^{i(t-\tau)\Delta} F(\tau) d\tau \Big\|_{\dot{\beta}^{\sigma}_{S(\dot{H}^s)}} \lesssim \|F\|_{\dot{\beta}^{\sigma}_{S'(\dot{H}^{-s})}}. \tag{2.12}$$

Proof. The dispersive inequality (1.12) and interpolation with the L^2_x norm whenever $t \neq \tau$ yield

$$\big\|e^{i(t-\tau)\Delta} F(\tau)\big\|_{L^r_x} \lesssim |t-\tau|^{-d(\frac{1}{r'}-\frac{1}{2})} \big\|F(\tau)\big\|_{L^{r'}_x}.$$

In particular, if (q,r) $\dot{H}^s$ admissible, integration on $\mathbb{R}^d \times I$ combined with Minkowski's inequality imply

$$\begin{aligned} \Big\| \int_{t_0}^t e^{i(t-\tau)\Delta} F(\tau) d\tau \Big\|_{L^q_t L^r_x} &\lesssim \Big\| \int_{t_0}^t \big\|e^{i(t-\tau)\Delta} F(\tau)\big\|_{L^r_x} d\tau \Big\|_{L^q_t} \\ &\lesssim \Big\| \int_{t_0}^t \big\|F(\tau)\big\|_{L^{r'}_x} * |t|^{d(\frac{1}{r}-\frac{1}{2})} \Big\|_{L^q_t} \lesssim \big\|F\big\|_{L^{q'}_t L^{r'}_x}. \end{aligned}$$

Thus, Littlewood-Paley theory gives

$$\Big\|P_N \int_{t_0}^t e^{i(t-\tau)\Delta} F(\tau) d\tau \Big\|_{L^q_t L^r_x} \lesssim \big\|P_N F(\tau)\big\|_{L^{q'}_t L^{r'}_x}.$$

Therefore, (2.12) is obtained by multiplying both sides of the above estimate by N^{σ}, squaring and summing over all dyadic $N's$. □

Lemma 2.12 (Interpolation inequalities for Besov spaces [Triebel, 1978]). *Let* $1 \le p_i, q_i \le \infty$ *and* $u \in \beta^{\sigma_i}_{p_i,q_i}(\mathbb{R}^d)$, *where* $i = 1, 2, 3$. *Then*

$$\|u\|_{\beta^{\sigma_1}_{p_1,q_1}(\mathbb{R}^d)} = \|u\|^{1-\theta}_{\beta^{\sigma_2}_{p_2,q_2}(\mathbb{R}^d)}\|u\|^{\theta}_{\beta^{\sigma_3}_{p_3,q_3}(\mathbb{R}^d)}$$

provided that

$$\sigma_1 = (1-\theta)\sigma_2 + \theta\sigma_3, \qquad \frac{1}{p_1} = \frac{1-\theta}{p_2} + \frac{\theta}{p_3} \qquad \text{and} \qquad \frac{1}{q_1} = \frac{1-\theta}{q_2} + \frac{\theta}{q_3}.$$

2.3 Local Theory

In this section the global existence and scattering in $H^1(\mathbb{R}^d)$ for small data in $\dot{H}^s$ (Propositions 2.13 and 2.21), and a long perturbation argument (Proposition 2.17) are examined. The proofs lie on paraproduct[9] techniques and Besov spaces which allow us to treat the lack of smoothness of the nonlinearity $F(u) = |u|^{p-1}u$ (see Remark 2.6).

Proposition 2.13 (Small data). *Suppose* $\|u_0\|_{\dot{H}^s} \lesssim A$. *There exists* $\delta_{sd} = \delta_{sd}(A) > 0$ *such that if* $\|e^{it\triangle}u_0\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \lesssim \delta_{sd}$, *then u(t) solving the* $NLS^+_p(\mathbb{R}^d)$ *is global in* $\dot{H}^s(\mathbb{R}^d)$ *and*

$$\|u\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \lesssim 2\|e^{it\triangle}u_0\|_{\dot{\beta}^0_{S(\dot{H}^s)}}, \qquad \|u\|_{\dot{\beta}^s_{S(L^2)}} \lesssim 2c\|u_0\|_{\dot{H}^s}.$$

Proof. Using a fixed point argument in a ball B, the existence of solutions to (1.1) and continuous dependence on the initial data is proven as follows.

Let

$$B = \Big\{\|u\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \lesssim 2\|e^{it\triangle}u_0\|_{\dot{\beta}^0_{S(\dot{H}^s)}}, \quad \|u\|_{\dot{\beta}^s_{S(L^2)}} \lesssim 2c\|u_0\|_{\dot{H}^s}\Big\}.$$

[9] Bilinear, non-commutative operator that satisfies product reconstruction and linearization formulas (up to smooth errors), a Hölder-type inequality, and a Leibniz-type rule.

Assume $F(u) = |u|^{p-1}u$ and the map $u \mapsto \Phi_{u_0}(u)$ defined via

$$\Phi_{u_0}(u) := e^{it\triangle}u_0 + i\int_0^t e^{i(t-\tau)\triangle}F(u(\tau))d\tau.$$

Combining the triangle inequality and the Linear Besov Strichartz estimates (2.10) and the fact that $F(u) \in C^1$, we obtain

$$\|\Phi_{u_0}(u)\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \lesssim \|e^{it\triangle}u_0\|_{\dot{\beta}^0_{S(\dot{H}^s)}} + \|F(u)\|_{\dot{\beta}^s_{S'(L^2)}},$$
$$\|\Phi_{u_0}(u)\|_{\dot{\beta}^s_{S(L^2)}} \lesssim \|u_0\|_{\dot{\beta}^s_{S(L^2)}} + \|F(u)\|_{\dot{\beta}^s_{S'(L^2)}}.$$

For each dyadic number $N \in 2^{\mathbb{Z}}$, the fractional chain rule (Lemma 2.1) and Hölder's inequality lead to

$$\begin{aligned}\|D^sF(u)\|_{S'(L^2)} &\lesssim \|D^s(|u|^{p-1}u)\|_{L_t^{\frac{d}{2s}}L_x^{\frac{2d^2(p-1)}{d^2(p-1)+16}}}\\ &\lesssim \|u\|^{p-1}_{L_t^{\frac{dp}{2s}}L_x^{\frac{d^2p(p-1)}{2(d+4)}}}\|D^su\|_{L_t^{\frac{dp}{2s}}L_x^{\frac{2d^2p}{d^2p-8s}}} \lesssim \|u\|^{p-1}_{S(\dot{H}^s)}\|D^su\|_{S(L^2)},\end{aligned}$$

thus, Littlewood-Paley theory yields

$$\||u|^{p-1}u\|_{\dot{\beta}^s_{S'(L^2)}} \lesssim \|u\|^{p-1}_{\dot{\beta}^0_{S(\dot{H}^s)}}\|u\|_{\dot{\beta}^s_{S(L^2)}}. \tag{2.13}$$

Therefore,

$$\|\Phi_{u_0}(u)\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \lesssim \|e^{it\triangle}u_0\|_{\dot{\beta}^0_{S(\dot{H}^s)}} + \|u\|^{p-1}_{\dot{\beta}^0_{S(\dot{H}^s)}}\|u\|_{\dot{\beta}^s_{S(L^2)}},$$
$$\|\Phi_{u_0}(u)\|_{\dot{\beta}^s_{S(L^2)}} \lesssim \|u_0\|_{\dot{\beta}^s_{S(L^2)}} + \|u\|^{p-1}_{\dot{\beta}^0_{S(\dot{H}^s)}}\|u\|_{\dot{\beta}^s_{S(L^2)}}$$

and choosing $\delta_1 = \min\left\{\frac{1}{2^pc_1^{p-1}A^{p-2}},\ \sqrt[p-1]{\frac{1}{2^pc_2^{p-1}A}}\right\}$ leads to $\Phi_{u_0}(u) \in B$.

To complete the proof, we need to show that the map $u \mapsto \Phi_{u_0}(u)$ is a contraction. Take $u, v \in B$, and note that triangle inequality and Besov Strichartz estimates yield

$$\begin{aligned}\|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{\dot{\beta}^0_{S(\dot{H}^s)}} &\lesssim \left\|\int_0^t e^{i(t-\tau)\triangle}\Big(F\big(u(\tau)\big) - F\big(v(\tau)\big)\Big)d\tau\right\|_{\dot{\beta}^0_{S(\dot{H}^s)}}\\ &\lesssim \|D^s\big(F(u) - F(v)\big)\|_{\dot{\beta}^0_{S'(L^2)}} \approx \|F(u) - F(v)\|_{\dot{\beta}^s_{S'(L^2)}},\end{aligned}$$

and

$$\begin{aligned}\|\Phi_{u_0}(u)-\Phi_{u_0}(v)\|_{\dot\beta^s_{S(\dot H^s)}} &\approx \|D^s(\Phi_{u_0}(u)-\Phi_{u_0}(v))\|_{\dot\beta^0_{S(L^2)}}\\ &\lesssim \Big\|\int_0^t e^{i(t-\tau)\triangle}D^s\Big(F\big(u(\tau)\big)-F\big(v(\tau)\big)\Big)d\tau\Big\|_{\dot\beta^0_{S(L^2)}}\\ &\lesssim \|D^s\big(F(u)-F(v)\big)\|_{\dot\beta^0_{S'(L^2)}} \approx \|F(u)-F(v)\|_{\dot\beta^s_{S'(L^2)}}.\end{aligned}$$

For each dyadic number $N\in 2^{\mathbb{Z}}$, we estimate $\|D^s\big(F(u)-F(v)\big)\|_{S'(L^2)}$. Recall that we are considering the mass-supercritical energy-subcritical NLS, i.e., $0<s<1$ and $p=1+\frac{4}{d-2s}$. Due to the lack of smoothness of the nonlinearity (Remark 2.6), we consider two (complementary) cases:

(a) The function $F(u)$ is **at least** in $C^2(\mathbb{C})$.

(b) The nonlinearity $F(u)$ is **at most** in $C^1(\mathbb{C})$.

In the rest of the proof we examine these cases separately, and after the proof we give specific examples to illustrate our approach.

Case (a). $F(u)$ is **at least** in $C^2(\mathbb{C})$: this case occurs when $1\le d\le 4+2s$, i.e., dimensions 2, 3, and 4 for $0<s<1$, or dimension 5 when $\frac{1}{2}\le s<1$. Combining (2.7), chain rule (Lemma 2.1) and Hölder's inequality, gives

$$\begin{aligned}\|D^s\big(F(u)-F(v)\big)\|_{S'(L^2)} &\lesssim \|D^s(|u|^{p-1}u-|v|^{p-1}v)\|_{L_t^{\frac{d}{2s}}L_x^{\frac{2d^2(p-1)}{d^2(p-1)+16}}}\\ &\lesssim \|D^s|u-v|\|_{L_t^{\frac{dp}{2s}}L_x^{\frac{2d^2p}{d^2p-8s}}}\Big(\|u\|^{p-1}_{L_t^{\frac{dp}{2s}}L_x^{\frac{d^2p(p-1)}{2(d+4)}}}+\|v\|^{p-1}_{L_t^{\frac{dp}{2s}}L_x^{\frac{d^2p(p-1)}{2(d+4)}}}\Big)\\ &\lesssim \|D^s|u-v|\|_{S(L^2)}\Big(\|u\|^{p-1}_{S(\dot H^s)}+\|v\|^{p-1}_{S(\dot H^s)}\Big).\end{aligned}$$

Here, we used the Hölder split

$$\frac{2d^2(p-1)}{d^2(p-1)+16}=\frac{d^2p-8s}{2d^2p}+(p-1)\frac{2(d+4)}{d^2p(p-1)} \tag{2.14}$$

together with the fact that the pair $\left(\frac{d}{2s},\frac{2d^2(p-1)}{d^2(p-1)+16}\right)$ is L^2 dual admissible, the pair $\left(\frac{dp}{2s},\frac{2d^2p}{d^2p-8s}\right)$ is L^2 admissible and the pair $\left(\frac{dp}{2s},\frac{d^2p(p-1)}{2(d+4)}\right)$ is $\dot H^s$ admissible.

Therefore, $\|F(u)-F(v)\|_{\dot{\beta}^s_{S'(L^2)}} \lesssim \|u-v\|_{\dot{\beta}^s_{S(L^2)}}\Big(\|u\|^{p-1}_{\dot{\beta}^0_{S(\dot{H}^s)}} + \|v\|^{p-1}_{\dot{\beta}^0_{S(\dot{H}^s)}}\Big)$. Letting $\delta_2 = \min\left\{\sqrt[p-1]{\frac{1}{2^pC}}, \frac{1}{2^pA^p-2C}\right\}$ implies that Φ_{u_0} is a contraction.

Case (b). $F(u)$ is **at most** in $C^1(\mathbb{C})$: this corresponds to dimensions higher than $4+2s$, i.e., $d=5$ with $0<s<\frac{1}{2}$ or $d\geq 6$ with $0<s<1$. Let $w=u-v$, therefore (2.6) and the triangle inequality imply

$$\|D^s\big(F(u)-F(v)\big)\|_{S'(L^2)} \lesssim \|D^s(|u|^{p-1}u-|v|^{p-1}v)\|_{L_t^{\frac{d}{2s}}L_x^{\frac{2d^2(p-1)}{d^2(p-1)+16}}}$$

$$\lesssim \|D^sF_z(v+w)w\|_{L_t^{\frac{d}{2s}}L_x^{\frac{2d^2(p-1)}{d^2(p-1)+16}}} + \|D^sF_{\bar{z}}(v+\bar{w})\bar{w}\|_{L_t^{\frac{d}{2s}}L_x^{\frac{2d^2(p-1)}{d^2(p-1)+16}}}. \tag{2.15}$$

To estimate (2.15), we consider the subcases $s\leq p-1$ and $s>p-1$.

(i) If dimensions $4+2s<d\leq\frac{4+2s^2}{s}$, then $s\leq p-1$, thus,

$$\|D^sF_z(u)w\|_{L_t^{\frac{d}{2s}}L_x^{\frac{2d^2(p-1)}{d^2(p-1)+16}}} \lesssim \|D^{\frac{s(p-1)}{2}}F_z(u)w\|_{L_t^{\frac{d}{2s}}L_x^{\frac{4d^2(p-1)}{(d+4)(d-dp+8)+d^2p(p-1)}}} \tag{2.16}$$

$$\lesssim \|D^{\frac{s(p-1)}{2}}F_z(u)\|_{L_t^{\frac{dp}{2s(p-1)}}L_x^{\frac{8d^2p}{(p-1)^2((d^2-3ds+2s^2)(d+4)+8s^2)}}}\|w\|_{L_t^{\frac{dp}{2s}}L_x^{\frac{d^2p(p-1)}{2(d+4)}}} \tag{2.17}$$

$$+\|u\|^{p-1}_{L_t^{\frac{d(p-1)}{s}}L_x^{\frac{d^2(d-1)}{2(d-s)}}}\|D^{\frac{s(p-1)}{2}}w\|_{L_t^{\frac{d}{s}}L_x^{\frac{d^2(d-1)}{2d+s^2(p-1)^2}}} \tag{2.18}$$

$$\lesssim \|u\|^{\frac{p-1}{2}}_{L_t^{\frac{dp}{2s}}L_x^{\frac{d^2p(p-1)}{2(d+4)}}}\|D^su\|^{\frac{p-1}{2}}_{L_t^{\frac{dp}{2s}}L_x^{\frac{2d^2p}{d^2p-8s}}}\|D^sw\|_{L_t^{\frac{dp}{2s}}L_x^{\frac{2d^2p}{d^2p-8s}}} \tag{2.19}$$

$$+\|u\|^{p-1}_{L_t^{\frac{d(p-1)}{s}}L_x^{\frac{d^2(d-1)}{2(d-s)}}}\|D^sw\|_{L_t^{\frac{d}{s}}L_x^{\frac{2d^2}{d^2-4s}}} \tag{2.20}$$

$$\lesssim \|D^sw\|_{S(L^2)}\Big(\|u\|^{\frac{p-1}{2}}_{S(\dot{H}^s)}\|D^su\|^{\frac{p-1}{2}}_{S(L^2)} + \|u\|^{p-1}_{S(\dot{H}^s)}\Big),$$

here, Remark 2.9 yields (2.16) since $\frac{4d^2(p-1)}{(d+4)(d-dp+8)+d^2p(p-1)}$, $\frac{2d^2(p-1)}{d^2(p-1)+16}$ are Hölder conjugates and $\frac{s(p-1)}{2}<s$. Leibniz rule gives (2.17) and (2.18). Then applying chain rule for Hölder-continuous functions (Lemma 2.3) with $\rho:=p-1$, $\sigma:=\frac{s(p-1)}{2}$ and $\nu:=s$ to (2.17), we obtain (2.19). Noticing that $L_x^{\frac{2d^2}{d^2-4s}} \hookrightarrow L_x^{\frac{d^2(d-1)}{2d+s^2(p-1)^2}}$, Lemma 2.8 implies (2.20). The last line comes from the fact that the pairs

$\left(\frac{dp}{2s}, \frac{d^2p(p-1)}{2(d+4)}\right)$, $\left(\frac{d(p-1)}{s}, \frac{d^2(d-1)}{2(d-s)}\right)$ are $\dot{H}^s$ admissible, and the pairs $\left(\frac{dp}{2s}, \frac{2d^2p}{d^2p-8s}\right)$, $\left(\frac{d}{s}, \frac{2d^2}{d^2-4s}\right)$ are L^2 admissible. In a similar fashion, we obtain the estimate for the conjugate

$$\|D^s F_{\bar{z}}(v+\bar{w})\bar{w}\|_{L_t^{\frac{d}{2s}} L_x^{\frac{2d^2(p-1)}{d^2(p-1)+16}}} \lesssim \|D^s w\|_{S(L^2)} \big(\|u\|_{S(\dot{H}^s)}^{\frac{p-1}{2}} \|D^s u\|_{S(L^2)}^{\frac{p-1}{2}} + \|u\|_{S(\dot{H}^s)}^{p-1}\big).$$

Thus, Littlewood-Paley theory implies that

$$\|F(u)-F(v)\|_{\dot{\beta}^s_{S'(L^2)}} \lesssim 2\|u-v\|_{\dot{\beta}^s_{S(L^2)}} \left(\|u\|_{\dot{\beta}^0_{S(\dot{H}^s)}}^{\frac{p-1}{2}} \|u\|_{\dot{\beta}^s_{S(L^2)}}^{\frac{p-1}{2}} + \|u\|_{\dot{\beta}^0_{S(\dot{H}^s)}}^{p-1}\right),$$

and letting $\delta_3 \leq \sqrt[\frac{p-1}{2}]{\frac{1}{2^{(p+2)}CA^{\frac{p-1}{2}}}}$ gives that Φ_{u_0} is a contraction.

(ii)

If the dimensions $d > \frac{4+2s^2}{s}$, then $s > p-1$. Therefore, we make an estimate for $\|D^s F_z(u)w\|_{L_t^{\frac{d}{2s}} L_x^{\frac{2d^2(p-1)}{d^2(p-1)+16}}}$, as follows

$$\|D^s F_z(u)w\|_{L_t^{\frac{d}{2s}} L_x^{\frac{2d^2(p-1)}{d^2(p-1)+16}}} \lesssim \|D^{(p-1)^2} F_z(u)w\|_{L_t^{\frac{d}{2s}} L_x^{\frac{2(d+4)+d(p-1)^3}{d^2(p-1)}}} \tag{2.21}$$

$$\lesssim \|D^{(p-1)^2} F_z(u)\|_{L_t^{\frac{dp}{2s(p-1)}} L_x^{\frac{d^2p}{2(d+4)+dp(p-1)^2}}} \|w\|_{L_t^{\frac{dp}{2s}} L_x^{\frac{d^2p(p-1)}{2(d+4)}}} \tag{2.22}$$

$$+ \|u\|_{L_t^{\frac{d(p-1)}{s}} L_x^{\frac{d^2(p-1)}{2(d-s)}}}^{p-1} \|D^{(p-1)^2} w\|_{L_t^{\frac{d}{s}} L_x^{\frac{2d^2}{d^2+2d(p-1)^2-2s(d+2)}}} \tag{2.23}$$

$$\lesssim \|u\|_{L_t^{\frac{dp}{2s}} L_x^{\frac{d^2p(p-1)}{2(d+4)}}}^{\frac{(p-1)(1+s-p)}{s}} \|D^s u\|_{L_t^{\frac{dp}{2s}} L_x^{\frac{2d^2p}{d^2p-8s}}}^{\frac{(p-1)^2}{s}} \|D^s w\|_{L_t^{\frac{dp}{2s}} L_x^{\frac{2d^2p}{d^2p-8s}}} \tag{2.24}$$

$$+ \|u\|_{L_t^{\frac{d(p-1)}{s}} L_x^{\frac{d^2(p-1)}{2(d-s)}}}^{p-1} \|D^s w\|_{L_t^{\frac{d}{s}} L_x^{\frac{2d^2}{d^2-4s}}} \tag{2.25}$$

$$\lesssim \|D^s w\|_{S(L^2)} \left(\|u\|_{S(\dot{H}^s)}^{\frac{(p-1)(1+s-p)}{s}} \|D^s u\|_{S(L^2)}^{\frac{(p-1)^2}{s}} + \|u\|_{S(\dot{H}^s)}^{p-1}\right),$$

as before in (i), Remark 2.9 yields (2.21) since $\frac{2(d+4)+d(p-1)^3}{d^2(p-1)}$, $\frac{2d^2(p-1)}{d^2(p-1)+16}$ are Hölder conjugates and $(p-1)^2 < s$. Leibniz rule gives (2.22) and (2.23). To obtain (2.24), we use chain rule for Hölder-continuous functions (Lemma 2.3) with $\rho := (p-1)^2$ and $\nu := s$ to (2.17). The line (2.20) follows from Lemma 2.8, and finally, since the pairs $\left(\frac{dp}{2s}, \frac{d^2p(p-1)}{2(d+4)}\right)$, $\left(\frac{d(p-1)}{s}, \frac{d^2(d-1)}{2(d-s)}\right)$ are $\dot{H}^s$ admissible, and the pairs $\left(\frac{dp}{2s}, \frac{2d^2p}{d^2p-8s}\right)$, $\left(\frac{d}{s}, \frac{2d^2}{d^2-4s}\right)$ are L^2 admissible, we obtain the last estimate. Similarly,

$$\begin{aligned}&\|D^sF_{\bar{z}}(v+\bar{w})\bar{w}\|_{L_t^{\frac{d}{2s}}L_x^{\frac{2d^2(p-1)}{d^2(p-1)+16}}}\\ &\qquad\lesssim \|D^sw\|_{S(L^2)}\big(\|u\|_{S(\dot{H}^s)}^{\frac{(p-1)(1+s-p)}{s}}\|D^su\|_{S(L^2)}^{\frac{(p-1)^2}{s}}+\|u\|_{S(\dot{H}^s)}^{p-1}\big).\end{aligned}$$

Therefore, Littlewood-Paley theory produces

$$\|F(u)-F(v)\|_{\dot{\beta}^s_{S'(L^2)}} \lesssim 2\|u-v\|_{\dot{\beta}^s_{S(L^2)}}\left(\|u\|_{\dot{\beta}^0_{S(\dot{H}^s)}}^{\frac{(p-1)(1+s-p)}{s}}\|u\|_{\dot{\beta}^s_{S(L^2)}}^{\frac{(p-1)^2}{s}}+\|u\|_{\dot{\beta}^0_{S(\dot{H}^s)}}^{p-1}\right),$$

and taking $\delta_4 \leq \sqrt[\frac{(p-1)(1+s-p)}{s}]{\frac{1}{2^{(p+1)}CA^{\frac{(p-1)^2}{s}}}}$ implies that Φ_{u_0} is a contraction.

From cases (a) and (b) choosing $\delta_{sd} \leq \min\left\{\delta_1, \delta_2, \delta_3, \delta_4\right\}$ implies that the map $u \mapsto \Phi_{u_0}(u)$ is a contraction which concludes the proof. □

We next illustrate the above cases when considering the estimate $\|D^s\big(F(u)-F(v)\big)\|_{S'(L^2)}$ in the above proof: we describe the $\dot{H}^{\frac{1}{2}}-$critical cases $\text{NLS}^+_{\frac{7}{3}}(\mathbb{R}^4)$, $\text{NLS}^+_{\frac{5}{3}}(\mathbb{R}^7)$ and $\text{NLS}^+_{\frac{13}{9}}(\mathbb{R}^{10})$ corresponding to the cases (a), (b)(i) and (b)(ii), respectively.

Example 2.14. Case (a): For $\text{NLS}^+_{\frac{7}{3}}(\mathbb{R}^4)$, the nonlinearity $F(u)=|u|^{\frac{4}{3}}u$ is $C^2(\mathbb{C})$. The pairs $(4, \frac{8}{7})$, $(\frac{28}{3}, \frac{56}{25})$, and $(\frac{28}{3}, \frac{28}{9})$ are L^2 dual admissible, L^2 admissible and $\dot{H}^{\frac{1}{2}}$ admissible, respectively.

$$\|D^{\frac{1}{2}}\big(F(u)-F(v)\big)\|_{S'(L^2)} \lesssim \|D^{\frac{1}{2}}(|u|^{\frac{4}{3}}u-|v|^{\frac{4}{3}}v)\|_{L_t^4L_x^{\frac{8}{7}}} \tag{2.26}$$

$$\lesssim \|D^{\frac{1}{2}}|u-v|\|_{L_t^{\frac{28}{3}}L_x^{\frac{56}{25}}}\left(\|u\|^{\frac{4}{3}}_{L_t^{\frac{28}{3}}L_x^{\frac{28}{9}}}+\|v\|^{\frac{4}{3}}_{L_t^{\frac{28}{3}}L_x^{\frac{28}{9}}}\right) \tag{2.27}$$

$$\lesssim \|D^{\frac{1}{2}}|u-v|\|_{S(L^2)}\left(\|u\|^{\frac{4}{3}}_{S(\dot{H}^{\frac{1}{2}})}+\|v\|^{\frac{4}{3}}_{S(\dot{H}^{\frac{1}{2}})}\right). \tag{2.28}$$

Since $L_t^4L_x^{\frac{8}{7}} \subseteq S'(L^2)$, we have (2.26). Applying (2.7), chain rule (Lemma 2.1) and Hölder's inequality, we obtain (2.26). Finally, (2.28) comes from the fact that $S(L^2) \subseteq L_t^{\frac{28}{3}}L_x^{\frac{56}{25}}$ and $S(\dot{H}^{\frac{1}{2}}) \subseteq L_t^{\frac{28}{3}}L_x^{\frac{28}{9}}$.

Example 2.15. Case (b) (i): The $\mathrm{NLS}^+_{\frac{5}{3}}(\mathbb{R}^7)$ is $\dot{H}^{\frac{1}{2}}-$critical so $s=\frac{1}{2}\leq\frac{2}{3}=p-1$. The nonlinearity $F(u)=|u|^{\frac{2}{3}}u$ is $C^1(\mathbb{C})$. The pairs $(\frac{35}{3},\frac{490}{233})$, $(14,\frac{98}{47})$ are L^2 admissible; the pairs $(\frac{35}{3},\frac{245}{99})$, $(\frac{28}{3},\frac{98}{39})$ are $\dot{H}^{\frac{1}{2}}$ admissible and the pair $(7,\frac{98}{73})$ is L^2 dual admissible. Bound $\|D^s\big(F(u)-F(v)\big)\|_{S'(L^2)}$ by looking at $\|D^{\frac{1}{2}}F_z(v+w)w\|_{L^7_tL^{\frac{98}{73}}_x}$ and its conjugate $\|D^{\frac{1}{2}}F_{\bar{z}}(v+\bar{w})\bar{w}\|_{L^7_tL^{\frac{98}{73}}_x}$, as follows

$$\|D^{\frac{1}{2}}F_z(u)w\|_{L^7_tL^{\frac{98}{73}}_x}\lesssim\|D^{\frac{1}{6}}F_z(u)w\|_{L^7_tL^{\frac{294}{205}}_x} \tag{2.29}$$

$$\lesssim\|D^{\frac{1}{6}}F_z(u)\|_{L^{\frac{35}{2}}_tL^{\frac{1470}{431}}_x}\|w\|_{L^{\frac{35}{3}}_tL^{\frac{245}{99}}_x}+\|F_z(u)\|_{L^{14}_tL^{\frac{49}{13}}_x}\|D^{\frac{1}{6}}w\|_{L^{14}_tL^{\frac{294}{127}}_x} \tag{2.30}$$

$$\lesssim\|u\|^{\frac{1}{3}}_{L^{\frac{35}{3}}_tL^{\frac{245}{99}}_x}\|D^{\frac{1}{2}}u\|^{\frac{1}{3}}_{L^{\frac{35}{3}}_tL^{\frac{490}{233}}_x}\|D^{\frac{1}{2}}w\|_{L^{\frac{35}{3}}_tL^{\frac{490}{233}}_x}+\|u\|^{\frac{1}{6}}_{L^{\frac{28}{3}}_tL^{\frac{98}{39}}_x}\|D^{\frac{1}{2}}w\|_{L^{14}_tL^{\frac{98}{47}}_x} \tag{2.31}$$

$$\lesssim\|D^{\frac{1}{2}}w\|_{S(L^2)}\Big(\|u\|^{\frac{1}{3}}_{S(\dot{H}^{\frac{1}{2}})}\|D^{\frac{1}{2}}u\|^{\frac{1}{3}}_{S(L^2)}+\|u\|^{\frac{1}{6}}_{S(\dot{H}^{\frac{1}{2}})}\Big),$$

where Remark 2.9 yields (2.29), Leibniz rule gives (2.30). Applying the chain rule for Hölder-continuous functions (Lemma 2.3) with $\rho:=\frac{2}{3}$, $\sigma:=\frac{1}{6}$ and $\nu:=\frac{1}{2}$ to the first term of (2.30) and Lemma 2.8 to the second term, we get (2.31). In a similar fashion, we obtain the estimate for the conjugate

$$\|D^sF_{\bar{z}}(v+\bar{w})\bar{w}\|_{L^7_tL^{\frac{98}{73}}_x}\lesssim\|D^{\frac{1}{2}}w\|_{S(L^2)}\Big(\|u\|^{\frac{1}{3}}_{S(\dot{H}^{\frac{1}{2}})}\|D^{\frac{1}{2}}u\|^{\frac{1}{3}}_{S(L^2)}+\|u\|^{\frac{1}{6}}_{S(\dot{H}^{\frac{1}{2}})}\Big).$$

Example 2.16. Case (b) (ii): Consider the $\dot{H}^{\frac{1}{2}}-$critical NLS in dimension 10, i.e., $\mathrm{NLS}^+_{\frac{13}{9}}(\mathbb{R}^{10})$, so $s=\frac{1}{2}>\frac{4}{9}=p-1$. Note that $F(u)=|u|^{\frac{4}{9}}u$ is $C^1(\mathbb{C})$. The pairs $(\frac{130}{9},\frac{1300}{567})$, $(\frac{80}{9},\frac{400}{171})$ are $\dot{H}^{\frac{1}{2}}$ admissible; the pairs $(\frac{130}{9},\frac{325}{158})$, $(20,\frac{100}{49})$ are L^2 admissible and the pair $(10,\frac{25}{17})$ is L^2 dual admissible. Estimate $\|D^s\big(F(u)-F(v)\big)\|_{S'(L^2)}$ by looking at $\|D^{\frac{1}{2}}F_z(v+w)w\|_{L^{10}_tL^{\frac{25}{17}}_x}$ and its conjugate $\|D^{\frac{1}{2}}F_{\bar{z}}(v+$

$\bar{w})\bar{w}\|_{L_t^{10}L_x^{\frac{25}{17}}}$, as follows

$$\|D^{\frac{1}{2}}F_z(u)w\|_{L_t^{10}L_x^{\frac{25}{17}}} \lesssim \|D^{\frac{16}{81}}F_z(u)w\|_{L_t^{10}L_x^{\frac{8100}{5263}}} \tag{2.32}$$

$$\lesssim \|D^{\frac{16}{81}}F_z(u)\|_{L_t^{\frac{65}{2}}L_x^{\frac{26325}{5623}}}\|w\|_{L_t^{\frac{130}{9}}L_x^{\frac{1300}{567}}} + \|u\|^{\frac{4}{9}}_{L_t^{\frac{80}{9}}L_x^{\frac{400}{171}}}\|D^{\frac{16}{81}}w\|_{L_t^{20}L_x^{\frac{2025}{931}}} \tag{2.33}$$

$$\lesssim \|u\|^{\frac{4}{81}}_{L_t^{\frac{130}{9}}L_x^{\frac{1300}{567}}}\|D^{\frac{1}{2}}u\|^{\frac{32}{81}}_{L_t^{\frac{130}{9}}L_x^{\frac{325}{158}}}\|D^{\frac{1}{2}}w\|_{L_t^{\frac{130}{9}}L_x^{\frac{325}{158}}} + \|u\|^{\frac{4}{9}}_{L_t^{\frac{80}{9}}L_x^{\frac{400}{171}}}\|D^{\frac{1}{2}}w\|_{L_t^{20}L_x^{\frac{100}{49}}} \tag{2.34}$$

$$\lesssim \|D^{\frac{1}{2}}w\|_{S(L^2)}\left(\|u\|^{\frac{4}{81}}_{S(\dot{H}^s)}\|D^{\frac{1}{2}}u\|^{\frac{32}{81}}_{S(L^2)} + \|u\|^{\frac{4}{9}}_{S(\dot{H}^s)}\right),$$

where as in case (b)(i) Remark 2.9 yields (2.32), Leibniz rule gives (2.33). Applying the chain rule for Hölder-continuous functions (Lemma 2.3) with $\rho := \frac{4}{9}$, $\sigma := \frac{16}{81}$ and $\nu := \frac{1}{2}$ to the first term of (2.33) and Lemma 2.8 to the second term, we obtain (2.34). In a similar fashion, we obtain the estimate for the conjugate

$$\|D^sF_{\bar{z}}(v+\bar{w})\bar{w}\|_{L_t^{10}L_x^{\frac{25}{17}}} \lesssim \|D^{\frac{1}{2}}w\|_{S(L^2)}\left(\|u\|^{\frac{4}{81}}_{S(\dot{H}^s)}\|D^{\frac{1}{2}}u\|^{\frac{32}{81}}_{S(L^2)} + \|u\|^{\frac{4}{9}}_{S(\dot{H}^s)}\right).$$

Note that the difference between the treatment of the case (b) (i) and (ii) is just the choice of the value ρ when applying the chain rule for Hölder-continuous functions (Lemma 2.3).

Proposition 2.17 (Long term perturbation)**.** *For each $A > 0$, there exist $\epsilon_0 = \epsilon_0(A) > 0$ and $c = c(A) > 0$ such that the following holds. Let $u = u(x,t) \in H^1(\mathbb{R}^d)$ solve $NLS_p^+(\mathbb{R}^d)$. Let $v = v(x,t) \in H^1(\mathbb{R}^d)$ for all t and satisfies $\tilde{e} = iv_t + \Delta v + |v|^{p-1}v$.*

If $\|v\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \leq A$, $\|\tilde{e}\|_{\dot{\beta}^0_{S'(\dot{H}^{-s})}} \leq \epsilon_0$ and $\|e^{i(t-t_0)\Delta}(u(t_0)-v(t_0))\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \leq \epsilon_0$, then $\|u\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \leq c$.

Proof. Let $F(u) = |u|^{p-1}u,\ w = u - v$, and $W(v,w) = F(u) - F(v) = F(v+w) - F(v)$. Therefore, w solves the equation

$$iw_t + \Delta w + W(v,w) + \tilde{e} = 0.$$

Since $\|v\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \le A$, split the interval $[t_0,\infty)$ into $K = K_A$ intervals $I_j = [t_j, t_{j+1}]$ such that for each j, $\|v\|_{\dot{\beta}^0_{S(\dot{H}^s, I_j)}} \le \delta$ with δ to be chosen later. Recall that the integral equation of w at time t_j is given by

$$w(t) = e^{i(t-t_j)\Delta} w(t_j) + i\int_{t_j}^{t} e^{i(t-\tau)\Delta}(W+\tilde{e})(\tau)d\tau. \tag{2.35}$$

Applying Kato Besov Strichartz estimate (2.12) on (2.35) for each I_j, we obtain

$$\begin{aligned}
\|w\|_{\dot{\beta}^0_{S(\dot{H}^s,I_j)}} &\lesssim \|e^{i(t-t_j)\Delta} w(t_j)\|_{\dot{\beta}^0_{S(\dot{H}^s,I_j)}} + \|\int_{t_j}^{t} e^{i(t-\tau)\Delta}(W+\tilde{e})(\tau)d\tau\|_{\dot{\beta}^0_{S(\dot{H}^s,I_j)}} \\
&\lesssim \|e^{i(t-t_j)\Delta} w(t_j)\|_{\dot{\beta}^0_{S(\dot{H}^s,I_j)}} + c\|W(v,w)\|_{\dot{\beta}^0_{S'(\dot{H}^{-s},I_j)}} + c\|\tilde{e}\|_{\dot{\beta}^0_{S'(\dot{H}^{-s},I_j)}} \\
&\lesssim \|e^{i(t-t_j)\Delta} w(t_j)\|_{S(\dot{H}^s,I_j)} + c\|W(v,w)\|_{\dot{\beta}^0_{S'(\dot{H}^{-s},I_j)}} + c\epsilon_0.
\end{aligned}$$

Thus, for each dyadic number $N \in 2^{\mathbb{Z}}$, the following estimate holds

$$\begin{aligned}
\|W(v,w)\|_{S'(\dot{H}^{-s},I_j)} &\lesssim \|F(v+w)-F(v)\|_{L_{I_j}^{\frac{12(d-2s)}{(8+3d-6s)(1-s)}} L_x^{\frac{6d(d-2s)}{3(d^2+2s^2)+9d(1-s)-2(5s+4)}}} \\
&\lesssim \|w\|_{L_{I_j}^{\frac{4}{1-s}} L_x^{\frac{2d}{d-s-1}}} \left(\|v\|^{p-1}_{L_{I_j}^{\frac{6}{1-s}} L_x^{\frac{6d}{3d-4s-2}}} + \|w\|^{p-1}_{L_{I_j}^{\frac{6}{1-s}} L_x^{\frac{6d}{3d-4s-2}}} \right) \qquad (2.36) \\
&\lesssim \|w\|_{S(\dot{H}^s,I_j)} \left(\|v\|^{p-1}_{S(\dot{H}^s,I_j)} + \|w\|^{p-1}_{S(\dot{H}^s,I_j)} \right) \\
&\le \|w\|_{S(\dot{H}^s,I_j)} \left(\delta_N^{p-1} + \|w\|^{p-1}_{S(\dot{H}^s,I_j)} \right), \qquad (2.37)
\end{aligned}$$

where we first observed that the pairs $(\frac{6}{1-s}, \frac{6d}{3d-4s-2})$, $(\frac{4}{1-s}, \frac{2d}{d-s-1})$ are $\dot{H}^s$ admissible; the pair $(\frac{12(d-2s)}{(8+3d-6s)(1-s)}, \frac{6d(d-2s)}{3(d^2+2s^2)+9d(1-s)-2(5s+4)})$ is $\dot{H}^{-s}$ admissible. Thus, we used (2.7) and Hölder's inequality to obtain (2.36). Since $\|v\|_{\dot{\beta}^0_{S(\dot{H}^s,I_j)}} \le \delta$ for each dyadic interval, there exists $\delta_N = \delta(N)$, so we have (2.37). Therefore,

$$\begin{aligned}
\|F(v+w)-F(v)\|_{\dot{\beta}^0_{S'(\dot{H}^{-s},I_j)}} &\lesssim \|w\|_{\dot{\beta}^0_{S(\dot{H}^s,I_j)}} \left(\|v\|^{p-1}_{\dot{\beta}^0_{S(\dot{H}^s,I_j)}} + \|w\|^{p-1}_{\dot{\beta}^0_{S(\dot{H}^s,I_j)}} \right) \\
&\le \|w\|_{\dot{\beta}^0_{S(\dot{H}^s,I_j)}} \left(\delta^{p-1} + \|w\|^{p-1}_{\dot{\beta}^0_{S(\dot{H}^s,I_j)}} \right).
\end{aligned}$$

Choosing $\delta = \sum_{N\in 2^{\mathbb{Z}}} \delta_N < \min\left\{1, \frac{1}{4c_1}\right\}$ and $\|e^{i(t-t_j)\Delta} w(t_j)\|_{\dot{\beta}^0_{S(\dot{H}^s,I_j)}} + c_1\epsilon_0 \le \min\left\{1, \frac{1}{2\sqrt[p]{4c_1}}\right\}$, it follows

$$\|w\|_{\dot{\beta}^0_{S(\dot{H}^s,I_j)}} \le 2\|e^{i(t-t_j)\Delta} w(t_j)\|_{\dot{\beta}^0_{S(\dot{H}^s,I_j)}} + 2c_1\epsilon_0.$$

Taking $t = t_{j+1}$, applying $e^{i(t-t_{j+1})\Delta}$ to both sides of (2.35) and repeating the Kato estimates (2.5) , we obtain

$$\|e^{i(t-t_{j+1})\Delta}w(t_{j+1})\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \leq 2\|e^{i(t-t_j)\Delta}w(t_j)\|_{\dot{\beta}^0_{S(\dot{H}^s,I_j)}} + 2c_1\epsilon_0.$$

Iterating this process until $j = 0$, we obtain

$$\begin{aligned}\|e^{i(t-t_{j+1})\Delta}w(t_{j+1})\|_{\dot{\beta}^0_{S(\dot{H}^s)}} &\leq 2^j\|e^{i(t-t_0)\Delta}w(t_0)\|_{\dot{\beta}^0_{S(\dot{H}^s,I_j)}} + (2^j-1)2c_1\epsilon_0 \\ &\leq 2^{j+2}c_1\epsilon_0.\end{aligned}$$

These estimates must hold for all intervals I_j for $0 \leq j \leq K-1$, therefore,

$$2^{K+2}c_1\epsilon_0 \leq \min\Big\{1, \frac{1}{2\sqrt[p]{4c_1}}\Big\},$$

which determines how small ϵ_0 has to be taken in terms of K (as well as, in terms of A). $\square$

As an illustration of how the estimate $\|W(v,w)\|_{S'(\dot{H}^{-s},I_j)}$ works for the cases considered in the proof of Proposition 2.13, we again consider the $\dot{H}^{\frac{1}{2}}$-critical cases: $\mathrm{NLS}_3^+(\mathbb{R}^3)$, $\mathrm{NLS}^+_{\frac{5}{3}}(\mathbb{R}^7)$ and $\mathrm{NLS}^+_{\frac{13}{9}}(\mathbb{R}^{10})$, corresponding to the cases (a), (b)(i) and (b)(ii), respectively.

Example 2.18. Case (a): For $\mathrm{NLS}_3^+(\mathbb{R}^3)$, the nonlinearity $F(u) = |u|^3u$ is $C^2(\mathbb{C})$. The pairs $(8,4)$, $(12,\frac{18}{5})$ are $\dot{H}^{\frac{1}{2}}$ admissible and the pair $(\frac{24}{7},\frac{36}{29})$ is $\dot{H}^{-\frac{1}{2}}$ admissible.

$$\begin{aligned}\|W(v,w)\|_{S'(\dot{H}^{-\frac{1}{2}},I_j)} &\lesssim \|W(v,w)\|_{L^{\frac{24}{7}}_{I_j}L^{\frac{36}{29}}_x} \lesssim \|w\|_{L^8_{I_j}L^4_x}\Big(\|v\|^2_{L^{12}_{I_j}L^{\frac{18}{5}}_x} + \|w\|^2_{L^{12}_{I_j}L^{\frac{18}{5}}_x}\Big) \qquad (2.38) \\ &\lesssim \|w\|_{S(\dot{H}^{\frac{1}{2}},I_j)}\Big(\|v\|^2_{S(\dot{H}^{\frac{1}{2}},I_j)} + \|w\|^2_{S(\dot{H}^{\frac{1}{2}},I_j)}\Big).\end{aligned}$$

We get (2.38) combining (2.7) and Hölder's inequality with the split $\frac{1}{8}+\frac{2}{12}=\frac{7}{24}$ and $\frac{1}{4}+\frac{10}{18}=\frac{29}{36}$.

Example 2.19. Case (b) (i): For $\mathrm{NLS}^+_{\frac{5}{3}}(\mathbb{R}^7)$, the nonlinearity $F(u) = |u|^{\frac{5}{3}}u$ is $C^1(\mathbb{C})$. The pairs $(8, \frac{28}{11})$, $(12, \frac{42}{17})$ are $\dot{H}^{\frac{1}{2}}$ admissible and the pair $(\frac{72}{13}, \frac{252}{167})$ is $\dot{H}^{-\frac{1}{2}}$ admissible.

$$\begin{aligned}
\|W(v,w)\|_{S'(\dot{H}^{-\frac{1}{2}},I_j)} &\lesssim \|W(v,w)\|_{L^{\frac{72}{13}}_{I_j}L^{\frac{252}{167}}_x} \\
&\lesssim \|w\|_{L^8_{I_j}L^{\frac{28}{11}}_x}\left(\|v\|^{\frac{2}{3}}_{L^{12}_{I_j}L^{\frac{42}{17}}_x} + \|w\|^{\frac{2}{3}}_{L^{12}_{I_j}L^{\frac{42}{17}}_x}\right) \qquad (2.39)\\
&\lesssim \|w\|_{S(\dot{H}^s,I_j)}\left(\|v\|^{\frac{2}{3}}_{S(\dot{H}^s,I_j)} + \|w\|^{\frac{2}{3}}_{S(\dot{H}^s,I_j)}\right).
\end{aligned}$$

As before in Case (a), we get (2.39) combining (2.7) and Hölder's inequality with indices $\frac{1}{8} + \frac{2}{36} = \frac{13}{72}$ and $\frac{11}{28} + \frac{34}{126} = \frac{167}{252}$.

Example 2.20. Case (b) (ii): For $\mathrm{NLS}^+_{\frac{13}{9}}(\mathbb{R}^{10})$, the nonlinearity $F(u) = |u|^{\frac{4}{9}}u$ is $C^1(\mathbb{C})$. The pairs $(8, \frac{40}{17})$, $(12, \frac{30}{13})$ are $\dot{H}^{\frac{1}{2}}$ admissible and the pair $(\frac{216}{35}, \frac{1080}{667})$ is $\dot{H}^{-\frac{1}{2}}$ admissible.

$$\begin{aligned}
\|W(v,w)\|_{S'(\dot{H}^{-\frac{1}{2}},I_j)} &\lesssim \|W(v,w)\|_{L^{\frac{216}{35}}_{I_j}L^{\frac{1080}{667}}_x} \\
&\lesssim \|w\|_{L^8_{I_j}L^{\frac{40}{17}}_x}\left(\|v\|^{\frac{4}{9}}_{L^{12}_{I_j}L^{\frac{30}{13}}_x} + \|w\|^{\frac{4}{9}}_{L^{\frac{160}{279}}_{I_j}L^{\frac{392}{45}}_x}\right) \qquad (2.40)\\
&\lesssim \|w\|_{S(\dot{H}^{\frac{1}{2}},I_j)}\left(\|v\|^{\frac{4}{9}}_{S(\dot{H}^s,I_j)} + \|w\|^{\frac{4}{9}}_{S(\dot{H}^s,I_j)}\right).
\end{aligned}$$

We get (2.40) combining (2.7) and Hölder's inequality with the split $\frac{1}{8} + \frac{1}{27} = \frac{35}{216}$ and $\frac{17}{40} + \frac{26}{135} = \frac{667}{1080}$.

Proposition 2.21 (H^1 scattering)**.** *Assume $u_0 \in H^1(\mathbb{R}^d)$. Let $u(t)$ be a global solution to $NLS^+_p(\mathbb{R}^d)$ with the initial condition u_0, globally finite $\dot{H}^s$ Besov Strichartz norm $\|u\|_{\dot{\beta}^0_{S(\dot{H}^s)}} < +\infty$ and uniformly bounded $H^1(\mathbb{R}^d)$ norm* $\sup_{t\in[0,+\infty)}\|u(t)\|_{H^1} \leq B$*. Then there exists $\phi_+ \in H^1(\mathbb{R}^d)$ such that (1.3) holds, i.e., $u(t)$ scatters in $H^1(\mathbb{R}^d)$ as $t \to +\infty$. Similar statement holds for negative time.*

Proof. Suppose $u(t)$ solves $\mathrm{NLS}^+_p(\mathbb{R}^d)$ with the initial datum u_0, and satisfies the integral equation (1.2).

The assumption $\|u\|_{\dot{\beta}^0_{S(\dot{H}^s)}} < +\infty$ implies that for each dyadic $N \in 2^{\mathbb{Z}}$ there exists $M = \|u\|_{L_t^{\frac{dp}{2s}} L_x^{\frac{d^2p(p-1)}{2(d+4)}}} < \infty$ and let $\tilde{M} \sim M^{\frac{np}{2s}}$. Decompose $[0, +\infty) = \cup_{j=1}^{\tilde{M}} I_j$, such that for each j, $\|u\|_{L_{I_j}^{\frac{dp}{2s}} L_x^{\frac{d^2p(p-1)}{2(d+4)}}} < \delta$. Hence, the triangle inequality and Strichartz estimates yield

$$\|u\|_{S(L^2)} \lesssim \|e^{it\triangle}u_0\|_{S(L^2)} + \|F(u)\|_{S'(L^2)},$$

$$\|\nabla u\|_{S(L^2)} \lesssim \|e^{it\triangle}\nabla u_0\|_{S(L^2)} + \|\nabla F(u)\|_{S'(L^2)}.$$

Therefore, the integral equation (1.2) on I_j, combined with the above inequalities, leads to

$$\|\nabla u\|_{S(L^2;I_j)} \lesssim B + \left\||u|^{p-1}\nabla u\right\|_{S'(L^2;I_j)} \lesssim B + \left\||u|^{p-1}\nabla u\right\|_{L_{I_j}^{\frac{d}{2s}} L_x^{\frac{2d^2(p-1)}{d^2(p-1)+16}}} \tag{2.41}$$

$$\lesssim B + \|u\|^{p-1}_{L_{I_j}^{\frac{dp}{2s}} L_x^{\frac{d^2p(p-1)}{2(d+4)}}} \|\nabla u\|_{L_{I_j}^{\frac{dp}{2s}} L_x^{\frac{2d^2p}{d^2p-8s}}} \tag{2.42}$$

$$\lesssim B + \delta^{p-1}\|\nabla u\|_{S(L^2;I_j)}. \tag{2.43}$$

The pairs $\left(\frac{d}{2s}, \frac{d^2p(p-1)}{2(d+4)}\right)$ and $\left(\frac{d}{2s}, \frac{2d^2p}{d^2p-8s}\right)$ are L^2 admissible and the pair $\left(\frac{d}{2s}, \frac{2d^2(p-1)}{d^2(p-1)+16}\right)$ is L^2 dual admissible; we obtain (2.42) applying Hölder's inequality to (2.41). Similarly, by dropping the gradient, it follows

$$\|u\|_{S(L^2;I_j)} \lesssim B + \delta^{p-1}\|u\|_{S(L^2;I_j)}. \tag{2.44}$$

Combining (2.43) and (2.44) and using the fact that δ can be chosen appropiately small, gives that $\|(1+|\nabla|)u\|_{S(L^2;I_j)} \lesssim 2B$. Summing over the $\tilde{M}$ intervals, leads to

$$\|(1+|\nabla|)u\|_{S(L^2)} \lesssim BM^{\frac{np}{2s}}.$$

Define the wave operator

$$\phi_+ = u_0 + i\int_0^{+\infty} e^{-i\tau\Delta}F(u(\tau))d\tau,$$

note that $\phi_+ \in H^1$, thus Strichartz estimates and hypothesis lead to

$$\|\phi_+\|_{H^1} \lesssim \|u_0\|_{H^1} + \big\||u|^{p-1}\nabla u\big\|_{S'(L^2)} \lesssim \|u_0\|_{H^1} + \big\||u|^{p-1}\nabla u\big\|_{L_t^{\frac{d}{2s}} L_x^{\frac{2d^2(p-1)}{d^2(p-1)+16}}}$$

$$\lesssim \|u_0\|_{H^1} + \|u\|^{p-1}_{L_t^{\frac{dp}{2s}} L_x^{\frac{d^2p(p-1)}{2(d+4)}}} \|\nabla u\|_{L_t^{\frac{dp}{2s}} L_x^{\frac{2d^2p}{d^2p-8s}}} \lesssim B + BM^{\frac{p(d+2s)-2s}{2s}}. \quad (2.45)$$

Additionally,

$$u(t) - e^{it\Delta}\phi^+ = -i\int_t^{+\infty} e^{i(t-\tau)\Delta}F(u(\tau))d\tau. \quad (2.46)$$

Therefore, estimating the L^2 norm of (2.46), Strichartz estimates and Hölder's inequality give

$$\|u(t) - e^{it\Delta}\phi_+\|_{L^2} \lesssim \Big\|\int_t^{+\infty} e^{i(t-\tau)\Delta}F(u(\tau))d\tau\Big\|_{S(L^2)}$$

$$\lesssim \big\|F(u(\tau))\big\|_{S'(L^2;[t,+\infty)} \lesssim \big\||u|^{p-1}\nabla u\big\|_{L_t^{\frac{d}{2s}} L_x^{\frac{2d^2(p-1)}{d^2(p-1)+16}}}, \quad (2.47)$$

and similary, estimating the $\dot{H}^1$ norm of (2.46), we obtain

$$\|\nabla(u(t) - e^{it\Delta}\phi_+)\|_{L^2} \lesssim \Big\|\int_t^{+\infty} e^{i(t-\tau)\Delta}F(u(\tau))d\tau\Big\|_{S(L^2)}$$

$$\lesssim \big\|F(u(\tau))\big\|_{S'(L^2;[t,+\infty))} \lesssim \big\||u|^{p-1}\nabla u\big\|_{L_{[t,\infty)}^{\frac{d}{2s}} L_x^{\frac{2d^2(p-1)}{d^2(p-1)+16}}}. \quad (2.48)$$

Using the Leibniz rule (Lemma 2.2) to estimate (2.47) and (2.48), yields

$$\big\||u|^{p-1}\nabla u\big\|_{L_{[t,\infty)}^{\frac{d}{2s}} L_x^{\frac{2d^2(p-1)}{d^2(p-1)+16}}} \lesssim \|u\|^{p-1}_{L_{[t,\infty)}^{\frac{dp}{2s}} L_x^{\frac{d^2p(p-1)}{2(d+4)}}} \|\nabla u\|_{L_{[t,\infty)}^{\frac{dp}{2s}} L_x^{\frac{2d^2p}{d^2p-8s}}}.$$

By (2.45) the term $\|u\|^{p-1}_{L_{[t,\infty)}^{\frac{dp}{2s}} L_x^{\frac{d^2p(p-1)}{2(d+4)}}} \|\nabla u\|_{L_{[t,\infty)}^{\frac{dp}{2s}} L_x^{\frac{2d^2p}{d^2p-8s}}}$ is bounded. Then as $t \to \infty$ the term $\|u\|_{L_{[t,\infty)}^{\frac{dp}{2s}} L_x^{\frac{d^2p(p-1)}{2(d+4)}}} \to 0$, thus, summing over all dyadic N, (1.3) is obtained. □

Combining Lemmas 2.7, 2.8 and Remark 2.9, we obtain the following version for the local theory propositions, we add $*$ to indicate to which proposition it corresponds to.

Proposition 2.13* (Small data). *Suppose* $\|u_0\|_{\dot{H}^s} \lesssim A$. *There exists* $\delta_{sd} = \delta_{sd}(A) > 0$ *such that if* $\|e^{it\triangle}u_0\|_{S(\dot{H}^s)} \lesssim \delta_{sd}$, *then* $u(t)$ *solving the* $NLS_p^+(\mathbb{R}^d)$ *is global in* $\dot{H}^s(\mathbb{R}^d)$ *and* $\|u\|_{S(\dot{H}^s)} \lesssim 2\|e^{it\triangle}u_0\|_{S(\dot{H}^s)}$, $\|D^s u\|_{S(L^2)} \lesssim 2c\|u_0\|_{\dot{H}^s}$.

Proposition 2.17* (Long term perturbation). *For each* $A > 0$, *there exist* $\epsilon_0 = \epsilon_0(A) > 0$ *and* $c = c(A) > 0$ *such that the following holds. Let* $u = u(x,t) \in H^1(\mathbb{R}^d)$ *solve* $NLS_p(\mathbb{R}^d)$. *Let* $v = v(x,t) \in H^1(\mathbb{R}^d)$ *for all* t *and satisfies* $\tilde{e} = iv_t + \Delta v + |v|^{p-1}v$.

If $\|v\|_{S(\dot{H}^s)} \leq A$, $\|\tilde{e}\|_{S'(\dot{H}^{-s})} \leq \epsilon_0$ *and* $\|e^{i(t-t_0)\Delta}(u(t_0) - v(t_0))\|_{S(\dot{H}^s)} \leq \epsilon_0$, *then* $\|u\|_{S(\dot{H}^s)} \leq c$.

Proposition 2.21* (H^1 scattering). *Assume* $u_0 \in H^1(\mathbb{R}^d)$, $u(t)$ *is a global solution to* $NLS_p^+(\mathbb{R}^d)$ *with the initial condition* u_0, *globally finite* $\dot{H}^s$ *norm* $\|u\|_{S(\dot{H}^s)} < +\infty$ *and uniformly bounded* $H^1(\mathbb{R}^d)$ *norm* $\sup_{t\in[0,+\infty)} \|u(t)\|_{H^1} \leq B$. *Then there exists* $\phi_+ \in H^1(\mathbb{R}^d)$ *such that* (1.3) *holds, i.e.,* $u(t)$ *scatters in* $H^1(\mathbb{R}^d)$ *as* $t \to +\infty$. *Similar statement holds for negative time.*

2.4 Properties of the Ground State

Recall that $Q = Q(x)$ is the ground state for the nonlinear elliptic equation

$$\alpha^2 \Delta Q - \beta Q + Q^p = 0, \tag{2.49}$$

where

$$\alpha = \frac{\sqrt{d(p-1)}}{2} \quad \text{and} \quad \beta = 1 - \frac{(d-2)(p-1)}{4}.$$

And $u_Q(x,t) = e^{i\beta t}Q(\alpha x)$ is a soliton solution of $\text{NLS}_p^{\pm}(\mathbb{R}^d)$ [10].

Weinstein [Weinstein, 1982] proved the Gagliardo-Nierberg inequality

$$\|u\|_{L^{p+1}}^{p+1} \leq C_{GN} \|\nabla u\|_{L^2}^{\frac{d(p-1)}{2}} \|u\|_{L^2}^{2-\frac{(d-2)(p-1)}{2}} \tag{2.50}$$

[10] Here, the elliptic equation (2.49) corresponds to (1.5) and $u_Q(x,t)$ as in (1.4).

with the sharp constant

$$C_{GN} = \frac{p+1}{2\|Q\|_{L^2}^{p-1}}. \tag{2.51}$$

This inequality is optimized by Q, i.e., $\|Q\|_{L^{p+1}}^{p+1} = \frac{p+1}{2}\|\nabla Q\|_{L^2}^{\frac{d(p-1)}{2}}\|Q\|_{L^2}^{2-\frac{d(p-1)}{2}}$.

Multiplying (1.5) by Q and integrating, gives

$$\|Q\|_{L^{p+1}}^{p+1} = \alpha^2\|\nabla Q\|_{L^2}^2 + \beta\|Q\|_{L^2}^2,$$

thus,

$$\frac{p+1}{2}\|\nabla Q\|_{L^2}^{\frac{d(p-1)}{2}}\|Q\|_{L^2}^2 - \alpha^2\|\nabla Q\|_{L^2}^2\|Q\|_{L^2}^{\frac{d(p-1)}{2}} - \beta\|Q\|_{L^2}^{2+\frac{d(p-1)}{2}} = 0.$$

The trivial solution of the above equation is $\|Q\|_{L^2}^2 = 0$, we exclude it and denote $z = \frac{\|\nabla Q\|_{L^2}}{\|Q\|_{L^2}}$. Thus obtaining

$$\frac{p+1}{2}z^{\frac{d(p-1)}{2}} - \frac{d(p-1)}{4}z^2 + \frac{(d-2)(p-1)}{4} - 1 = 0.$$

The only real root of the above equation is $z = 1$, hence,

$$\|\nabla Q\|_{L^2} = \|Q\|_{L^2},$$

and,

$$\| Q\|_{L^{p+1}}^{p+1} = \frac{p+1}{2}\|Q\|_{L^2}^2.$$

In addition,

$$\|u_Q\|_{L^2}^2 = \alpha^{-d}\|Q\|_{L^2}^2, \quad \|\nabla u_Q\|_{L^2}^2 = \alpha^{2-d}\|\nabla Q\|_{L^2}^2, \quad \text{and} \quad \|u_Q\|_{L^{p+1}}^{p+1} = \alpha^{-d}\|Q\|_{L^{p+1}}^{p+1}, \tag{2.52}$$

therefore, the scale invariant quantity becomes

$$\|u_Q\|_{L^2}^{1-s}\|\nabla u_Q\|_{L^2}^s = \alpha^{-\frac{2}{p-1}}\|Q\|_{L^2}, \tag{2.53}$$

and the mass-energy scale invariant quantity is

$$M[u_Q]^{1-s}E[u_Q]^s = \left(\alpha^{-d}\|Q\|_{L^2}^2\right)^{1-s}\left(\frac{\alpha^{2-d}}{2}\|\nabla Q\|_{L^2}^2 - \frac{\alpha^{-d}}{p+1}\|Q\|_{L^{p+1}}^{p+1}\right)^s \tag{2.54}$$

$$= \frac{\alpha^{-d}}{2^s}\left(\frac{(p-1)s}{2}\right)^s\|Q\|_{L^2}^2 \tag{2.55}$$

$$= \left(\frac{s}{d}\right)^s\left(\|u_Q\|_{L^2}^{1-s}\|\nabla u_Q\|_{L^2}^s\right)^2. \tag{2.56}$$

The energy definition yields (2.54), Pohozhaev identities (2.52) and (2.53) implies (2.55) and (2.56).

Notice that

$$\begin{aligned} M[u]^{1-s}E[u]^s &= (\|u\|_{L^2}^2)^{1-s}\left(\frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{p+1}\| \ u\|_{L^{p+1}}^{p+1}\right)^s \\ &\geq (\|u\|_{L^2}^{1-s}\|\nabla u\|_{L^2}^s)^2\left(\frac{1}{2} - \frac{C_{GN}}{p+1}(\|u\|_{L^2}^{1-s}\|\nabla u\|_{L^2}^s)^{p-1}\right)^s \\ &\geq \frac{1}{2^s}(\|u\|_{L^2}^{1-s}\|\nabla u\|_{L^2}^s)^2\left(1-\alpha^{-2}\left(\frac{\|u\|_{L^2}^{1-s}\|\nabla u\|_{L^2}^s}{\|u_Q\|_{L^2}^{1-s}\|\nabla u_Q\|_{L^2}^s}\right)^{p-1}\right)^s, \end{aligned}$$

therefore,

$$\frac{d}{2s}\left[\mathcal{G}_u(t)\right]^{\frac{2}{s}}\left(1-\frac{[\mathcal{G}_u(t)]^{p-1}}{\alpha^2}\right) \leq (\mathcal{ME}[u])^{\frac{1}{s}} \leq \frac{d}{2s}\left[\mathcal{G}_u(t)\right]^{\frac{2}{s}}. \tag{2.57}$$

Summarizing, the upper bound in (2.57) is obtained bounding the energy $E[u]$ above by the kinetic energy; and the lower bound is achieved using the definition of energy and the sharp Gagliardo-Nirenberg inequality (2.50) to bound the potential term.

2.5 Properties of the Momentum

Let u be a solution of $\mathrm{NLS}_p^+(\mathbb{R}^d)$ and assume that $P[u] \neq 0$. Let $\xi_0 \in \mathbb{R}^d$ to be chosen later and w be the Galilean transformation of u

$$w(x,t) = e^{ix\cdot\xi_0}e^{-it|\xi_0|^2}u(x-2\xi_0 t, t).$$

Then

$$\nabla w(x,t) = i\xi_0 \cdot e^{ix\cdot\xi_0}e^{-it|\xi_0|^2}u(x-2\xi_0 t,t) + e^{ix\cdot\xi_0}e^{-it|\xi_0|^2}\nabla u(x-2\xi_0 t,t),$$

therefore,

$$\|\nabla w\|_{L^2}^2 = |\xi_0|^2 M[u] + 2\xi_0 \cdot P[u] + \|\nabla u\|_{L^2}^2. \tag{2.58}$$

Observe that $M[w] = M[u]$, $P[w] = \xi_0 M[u] + P[u]$, and

$$E[w] = \frac{1}{2}|\xi_0|^2 M[u] + \xi_0 \cdot P[u] + E[u]. \tag{2.59}$$

Note that the value $\xi_0 = -\frac{P[u]}{M[u]}$ minimizes the expressions (2.58) and (2.59), with $P[w] = 0$, that is,

$$E[w] = E[u] - \frac{(P[u])^2}{2M[u]} \qquad \text{and} \qquad \|\nabla w\|_{L^2}^2 = \|\nabla u\|_{L^2}^2 - \frac{(P[u])^2}{M[u]}.$$

Thus, the conditions (1.9), (1.10) and (1.11) in Theorem 1.6 become

$$(\mathcal{ME}[w])^{\frac{1}{s}} = (\mathcal{ME}[u]) - \frac{d}{2s}\left(\mathcal{P}[u]\right)^{\frac{2}{s}} < 1, \qquad [\mathcal{G}_w(0)]^{\frac{2}{s}} = [\mathcal{G}_u(0)]^{\frac{2}{s}} - \mathcal{P}^{\frac{2}{s}}[u] < 1$$

and $[\mathcal{G}_w(0)]^{\frac{2}{s}} > 1$, hence we restate Theorem 1.6 as

Theorem 1.6* (Zero momentum)**.** *Let $u_0 \in H^1(\mathbb{R}^d)$ with $d \geq 1$ and $u(t)$ be the corresponding solution to* (1.1) *in $H^1(\mathbb{R}^d)$ with maximal time interval of existence (T_*, T^*) and $s := s_c \in (0,1)$. Assume $\mathcal{ME}[u] < 1$.*

I. If $\mathcal{G}_u(0) < 1$, then

(a) $\mathcal{G}_u(t) < 1$ for all $t \in \mathbb{R}$, thus, the solution is global in time (i.e., $T_* = -\infty$, $T^* = +\infty$) and

(b) u scatters in $H^1(\mathbb{R}^d)$, this means, there exists $\phi_\pm \in H^1(\mathbb{R}^d)$ such that

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta}\phi_\pm\|_{H^1(\mathbb{R}^d)} = 0.$$

II. If $\mathcal{G}_u(0) > 1$, then $\mathcal{G}_u(t) > 1$ for all $t \in (T_*, T^*)$ and if

(a) u_0 is radial (for $d \geq 3$ and in $d = 2$, $3 < p \leq 5$) or u_0 is of finite variance, i.e., $|x|u_0 \in L^2(\mathbb{R}^d)$, then the solution blows up in finite time (i.e., $T^* < +\infty$, $T_* > -\infty$).

(b) u_0 non-radial and of infinite variance, then either the solution blows up in finite time (i.e., $T^* < +\infty$, $T_* > -\infty$) or there exists a sequence of times $t_n \to +\infty$ (or $t_n \to -\infty$) such that $\|\nabla u(t_n)\|_{L^2(\mathbb{R}^d)} \to \infty$.

Thus, in the rest of the paper, we will assume that $P[u] = 0$ and prove only Theorem 1.6*. To illustrate the scenarios for global behavior of solutions given by Theorem 1.6*we provide Figure 2.1.

We plot $y = (\mathcal{ME}[u])^{\frac{1}{s_c}}$ vs. $[\mathcal{G}_u(t)]^{\frac{2}{s_c}}$ using the (2.57) restriction in Figure 1.

2.6 Global versus Blowup Dichotomy

In this section we establish the sharp threshold for the global existence and finite time blowup solutions of the $\mathrm{NLS}_p^+(\mathbb{R}^d)$. Theorem 2.1 and Corollary 2.5 of Holmer-Roudenko [Holmer and Roudenko, 2007] proved the general case for the mass-supercritical and energy-subcritical NLS equations with H^1 initial data, thus, establishing Theorem 1.6* I(a) and II(a) for finite variance data. We only included the proof of the blow up in finite time when $d = 2$ and $p = 5$ (i.e., Theorem 1.6* part II(a)) for the radial initial data, since it was not include in [Holmer and Roudenko, 2007] (they considered $p < 5$).

Lemma 2.22 (Gagliardo-Nirenberg estimate for radial functions [Ogawa and Tsutsumi, 1991])**.** *Let $d \geq 2$ and $u \in H^1(\mathbb{R}^d)$ be radially symmetric. Then for any $R > 0$, u satisfies*

$$\|u(x)\|^{p+1}_{L^{p+1}(R<|x|)} \leq \frac{c}{R^{\frac{(d-1)(p-1)}{2}}} \|u\|^{\frac{p+3}{2}}_{L^2(R<|x|)} \|\nabla u\|^{\frac{p-1}{2}}_{L^2(R<|x|)}, \qquad (2.60)$$

where c depends only on d.

Proof of Theorem 1.6 part II. (for radial data in the case $p = 5$ and $d = 2$). Recall that the variance is given by

$$V(t) = \int |x|^2 |u(x,t)|^2 dx.$$

The standard argument for finite variance data is to examine the derivative and show that

$$\partial_t^2 V(t) = 32E[u_0] - 8\|\nabla u(t)\|^2_{L^2} < 0,$$

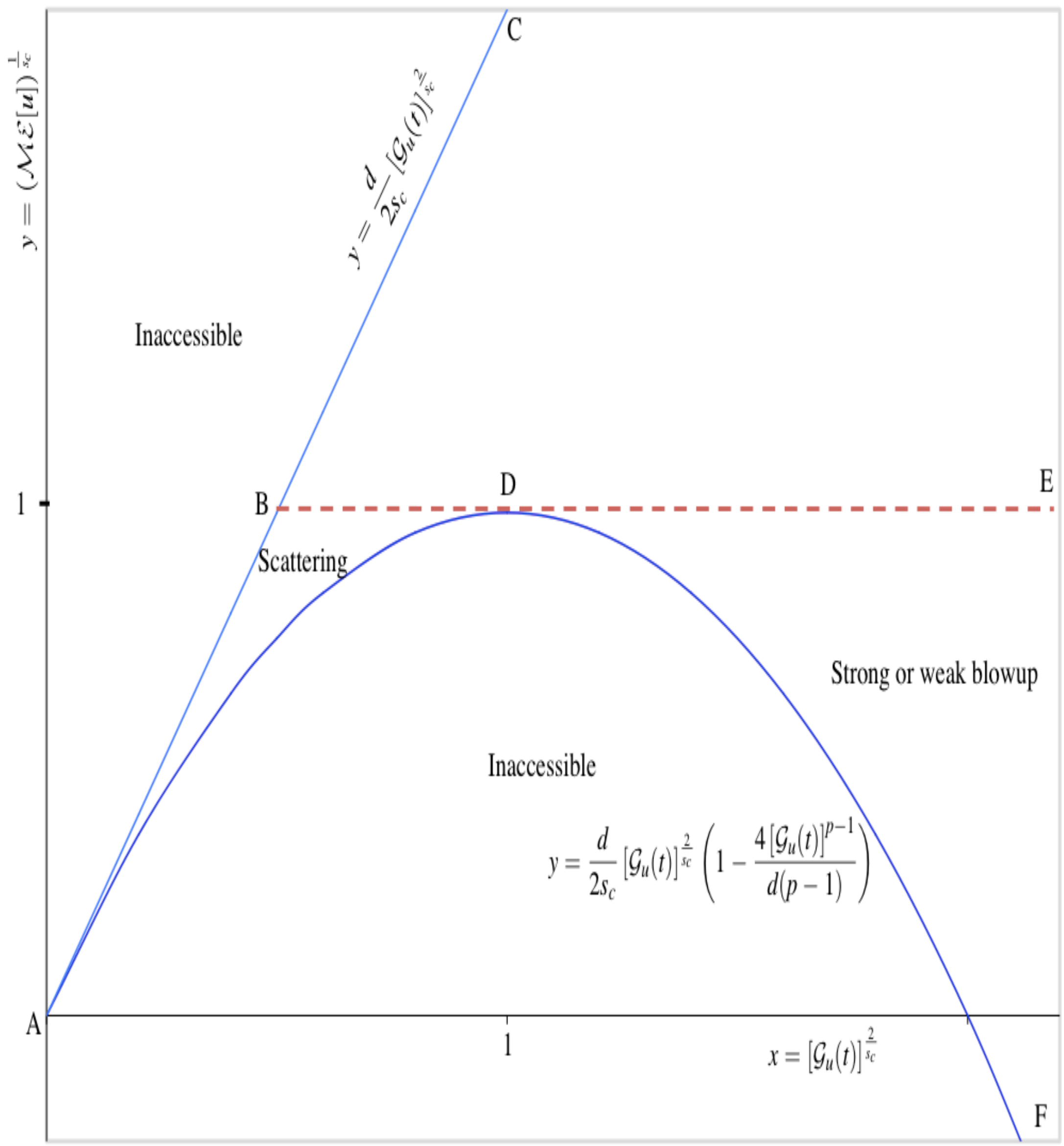

Figure 2.1: Plot of plot $y = (\mathcal{ME}[u])^{\frac{1}{s_c}}$ vs. $[\mathcal{G}_u(t)]^{\frac{2}{s_c}}$ where $\mathcal{G}_u(t)$ and $\mathcal{ME}[u]$ are defined by (1.6) and (1.8), respectively. The region above the line ABC and below the curve ADF are forbidden regions by (2.57). Global existence of solutions and scattering holds in the region ABD, which corresponds to Theorem 1.6* part I and the region EDF explains Theorem 1.6* part II (a), and the "weak" blowup Theorem 1.6 part II (b).

which by convexity implies the finite time existence of solutions. To obtain a wider range of blow up solutions, there are more delicate arguments (see [Lushnikov, 1995], [Holmer et al., 2010]).

Here, for infinite variance radial data, the argument of localized variance is used following Ogawa-Tsutsumi techniques [Ogawa and Tsutsumi, 1991].

Let $\chi \in C^\infty(\mathbb{R}^d)$ be radial,

$$\chi(r) = \begin{cases} r^2 & 0 \le r \le 1 \\ \text{smooth} & 1 < r < 4 \\ c & 4 \le r \end{cases}$$

such that $\partial_r^2 \chi(r) \le 2$ for all $r \ge 0$. Now, for $m > 0$ large, let $\chi_m(r) = m^2 \chi\left(\frac{r}{m}\right)$. Define the localized variance

$$V(t) = \int \chi(x)|u(x,t)|^2 dx$$

and consider the second derivative of the localized variance

$$\partial_t^2 V(t) = 4\int \chi'' |\nabla u|^2 - \int \triangle^2 \chi |u|^2 - \frac{4}{3}\int \triangle \chi |u|^{p+1}. \tag{2.61}$$

For $r \le m$ it follows that $\triangle \chi_m(r) = 4$ and $\triangle^2 \chi_m(r) = 0$. Each of the three terms in the inequality (2.61) are bounded as follows:

$$\begin{aligned} 4\int \chi_m'' |\nabla u|^2 &\le 8\int_{\mathbb{R}^d} |\nabla u|^2, \\ -\int \triangle^2 \chi_m |u|^2 &\le \frac{c_1}{m^2}\int_{m \le |x| \le 2m} |u|^2 \le \frac{c_1}{m^2}\int_{m \le |x|} |u|^2, \\ -\int \triangle \chi_m |u|^{p+1} &\le -4\int_{\mathbb{R}^d} |u|^{p+1} + c_2 \int_{m \le |x|} |u|^{p+1}. \end{aligned}$$

Thus, rewriting (2.61), we obtain

$$\begin{aligned} \partial_t^2 V(t) \le& 32E[u] - 8\|\nabla u\|_{L^2}^2 + \frac{c_1}{m^2}\|u\|_{L^2}^2 + c_3\|u\|_{L^6(|x| \ge m)}^6 \\ \le& 32E[u] - 8\|\nabla u\|_{L^2}^2 + \frac{c_1}{m^2}\|u\|_{L^2}^2 + \frac{c_4}{m^2}\|u\|_{L^2}^4 \|\nabla u\|_{L^2}^2, \end{aligned} \tag{2.62}$$

where $\|u\|_{L^6(|x|\geq m)}$ was estimated using (2.60).

Let $\epsilon > 0$, to be chosen later, pick $m_1 > \left(\frac{c_1}{\epsilon E[u_Q]}\right)^{\frac{1}{2}} \|u\|_{L^2}$, $m_2 > \left(\frac{c_4}{\epsilon}\right)^{\frac{1}{2}} \|u\|_{L^2}^2$ and $m = \max\{m_1, m_2\}$, we get

$$\partial_t^2 V(t) < 32E[u] - (8-\epsilon)\|\nabla u\|_{L^2}^2 + \epsilon E[u_Q]$$

Furthermore, the assumptions $\mathcal{ME}[u] < 1$ and $\mathcal{G}_u(0) > 1$ imply that there exists $\delta_1 > 0$ such that $\mathcal{ME}[u] < 1 - \delta_1$ and there exists $\delta_2 = \delta_2(\delta_1)$ such that $\mathcal{G}_u(t) > (1+\delta_2)$ for all $t \in I$. Multiplying both sides of (2.62) by $M[u_0]$, leads to

$$\begin{aligned} M[u_0]\partial_t^2 V(t) <& 32(1-\delta_1)M[u_Q]E[u_Q] - (8-\epsilon)(1+\delta_2)\|u_Q\|_{L^2}^2\|\nabla u_Q\|_{L^2}^2 \\ &+ \epsilon M[u_Q]E[u_Q] \\ <& [32(1-\delta_1) - 4(8-\epsilon)(1+\delta_2) + \epsilon]M[u_Q]E[u_Q], \end{aligned}$$

the last inequality follows since $4E[u_Q] = \|\nabla u_Q\|_{L^2}^2$. Choosing $\epsilon < \frac{32(\delta_1+\delta_2)}{5+4\delta_2}$ implies that the second derivative of the variance is bounded by a negative constant ($-A < 0$) for all $t \in \mathbb{R}$, i.e., $\partial_t^2 V(t) < -A$, and integrating twice over t, we have that $V(t) < -At^2 + Bt + C$. Thus, there exists T such that $V(T) < 0$ which is a contradiction. Therefore, radially symmetric solutions of the type described in Theorem 1.6* part II (a) must blow up in finite time.

□

2.7 Energy bounds and Existence of the Wave Operator

Lemma 2.23 (Comparison of Energy and Gradient)**.** *Let $u_0 \in H^1(\mathbb{R}^d)$ such that $\mathcal{G}_u(0) < 1$ and $\mathcal{ME}[u] < 1$. Then*

$$\frac{s}{d}\|\nabla u(t)\|_{L^2}^2 \leq E[u] \leq \frac{1}{2}\|\nabla u(t)\|_{L^2}^2. \tag{2.63}$$

Proof. The energy definition combined with $\mathcal{G}(0) < 1$ (and thus, by Theorem 1.6* part I (a) $\mathcal{G}_u(t) < 1$), the Gagliardo-Nirenberg inequality (2.50) and Pohozhaev identities (2.52) and (2.53) yield

$$\begin{aligned} E[u] &\geq \|\nabla u(t)\|_{L^2}^2 \left(\frac{1}{2} - \frac{C_{GN}}{p+1}\|\nabla u(t)\|_{L^2}^{\frac{d(p-1)}{2}-2}\|u\|_{L^2}^{2-\frac{(d-2)(p-1)}{2}}\right) \\ &\geq \|\nabla u(t)\|_{L^2}^2 \left(\frac{1}{2} - \frac{C_{GN}}{p+1}\left(\|\nabla u_{_Q}\|_{L^2}^{s}\|u_{_Q}\|_{L^2}^{(1-s)}\right)^{p-1}\right) \\ &= \|\nabla u(t)\|_{L^2}^2 \left(\frac{1}{2} - \frac{C_{GN}}{p+1}\alpha^{-2}\|Q\|_{L^2}^{p-1}\right) \\ &= \left(\frac{\alpha^2-1}{2\alpha^2}\right)\|\nabla u(t)\|_{L^2}^2 = \frac{s}{d}\|\nabla u(t)\|_{L^2}^2, \end{aligned} \tag{2.64}$$

where the equality (2.64) is obtained from combining (2.53), the sharp constant (2.51) and $\alpha = \frac{\sqrt{d(p-1)}}{2}$.

The second inequality of (2.63) follows directly from the definition of energy. □

Lemma 2.24 (Lower bound on the convexity of the variance)**.** *Let* $u_0 \in H^1(\mathbb{R}^d)$ *satisfy* $\mathcal{G}_u(0) < 1$ *and* $\mathcal{ME}[u] < 1$*. Then* $\mathcal{G}_u(t) \leq \omega$ *for all* t*, and*

$$16(1-\omega^{p-1})E[u] \leq 8(1-\omega^{p-1})\|\nabla u\|_{L^2}^2 \leq 8\|\nabla u\|_{L^2}^2 - \frac{4d(p-1)}{p+1}\|u\|_{L^{p+1}}^{p+1}, \tag{2.65}$$

where $\omega = \sqrt{\mathcal{ME}[u]}$.

Proof. The first inequality in (2.63) yields $\|\nabla u\|_{L^2}^2 \leq \frac{d}{s}E[u]$, multiplying it by $M^\theta[u]$, where $\theta = \frac{1-s}{s}$, normalizing by $\|\nabla u_{_Q}\|_{L^2}^2\|u_{_Q}\|_{L^2}^{2\theta}$ and using the fact that $\|\nabla u_{_Q}\|_{L^2}^2 \leq \frac{d}{s}E[u_{_Q}]$ leads to

$$[\mathcal{G}_u(t)]^2 \leq \mathcal{ME}[u], \qquad \text{i.e,} \qquad \mathcal{G}_u(t) \leq \omega.$$

Next, considering the right side of (2.65), applying Gagliardo-Nirenberg inequality (2.50), then the relation (2.53) and recalling that $\alpha = \frac{\sqrt{d(p-1)}}{2}$, we obtain

$$\begin{aligned} 8\|\nabla u\|_{L^2}^2 - \frac{4d(p-1)}{p+1}\|u\|_{L^{p+1}}^{p+1} &\geq \|\nabla u\|_{L^2}^2\left(8 - \frac{2d(p-1)}{\alpha^2}[\mathcal{G}_u(t)]^{p-1}\right) \\ &\geq 8\|\nabla u\|_{L^2}^2(1-\omega^{p-1}), \end{aligned} \tag{2.66}$$

which gives the middle inequality in (2.65).

Finally, combining (2.66) with the second inequality in (2.63), completes the proof. □

Proposition 2.25 (Existence of Wave Operators)**.** *Let* $\psi \in H^1(\mathbb{R}^d)$.

I. Then there exists $v_+ \in H^1$ *such that for some* $-\infty < T^* < +\infty$ *it produces a solution* $v(t)$ *to* $NLS_p^+(\mathbb{R}^d)$ *on time interval* $[T^*, \infty)$ *such that*

$$\|v(t) - e^{it\Delta}\psi\|_{H^1} \to 0 \qquad as \qquad t \to +\infty \tag{2.67}$$

Similarly, there exists $v_- \in H^1$ *such that for some* $-\infty < T_* < +\infty$ *it produces a solution* $v(t)$ *to* $NLS_p^+(\mathbb{R}^d)$ *on time interval* $(-\infty, T_*]$ *such that*

$$\|v(-t) - e^{-it\Delta}\psi\|_{H^1} \to 0 \qquad as \qquad t \to +\infty \tag{2.68}$$

II. Suppose that for some $0 < \sigma \leq \left(\frac{2s}{d}\right)^{\frac{s}{2}} < 1$

$$\|\psi\|_{L^2}^{2(1-s)} \|\nabla\psi\|_{L^2}^{2s} < \sigma^2 \left(\frac{d}{s}\right)^s M[u_Q]^{1-s} E[u_Q]^s\,. \tag{2.69}$$

Then there exists $v_0 \in H^1$ *such that* $v(t)$ *solving* $NLS_p^+(\mathbb{R}^d)$ *with initial data* v_0 *is global in* H^1 *with*

$$M[v] = \|\psi\|_{L^2}^2, \qquad E[v] = \frac{1}{2}\|\nabla\psi\|_{L^2}^2, \qquad \mathcal{G}_v(t) \leq \sigma < 1 \tag{2.70}$$

$$and \qquad \|v(t) - e^{it\triangle}\psi\|_{H^1} \to 0 \qquad as \qquad t \to \infty. \tag{2.71}$$

Moreover, if $\|e^{it\triangle}\psi\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \leq \delta_{sd}$*, then*

$$\|v_0\|_{\dot{H}^s} \leq 2\|\psi\|_{\dot{H}^s} \qquad and \qquad \|v\|_{\dot{H}^s} \leq 2\|e^{it\triangle}\psi\|_{\dot{\beta}^0_{S(\dot{H}^s)}}\,.$$

Proof. I. This is essentially Theorem 2 part (a) of [Strauss, 1981a] adapted to the case $0 < s < 1$ (see his Remark (36) and [Strauss, 1981b, Theorem 17]).

II. For this part, we consider the integral equation

$$v(t) = e^{it\triangle}\psi - i\int_t^{\infty} e^{i(t-t')\triangle}(|v|^{p-1}v)dt'. \tag{2.72}$$

We want to find a solution to (2.72) which exists for all t. Note that for $T > 0$ from the small data theory (Proposition 2.13) there exists $\delta_{sd} > 0$ such that $\|e^{it\triangle}\psi\|_{\dot{\beta}^0_{S([T,\infty),\dot{H}^s)}} \leq \delta_{sd}$. Thus, repeating the argument of Proposition 2.13, we first show that we can solve the equation (2.72) in $\dot{H}^s$ for $t \geq T$ with T large. So this solution will estimate $\|\nabla v\|_{S(L^2;[T,\infty))}$, which will also show that v is in H^1.

Observe that for any $v \in H^1$

$$\begin{aligned}\|\nabla|v|^{p-1}v\|_{S'(L^2)} &\lesssim \|\nabla|v|^{p-1}v\|_{L_t^{\frac{d}{2s}} L_x^{\frac{2d^2(p-1)}{d^2(p-1)+16}}} \\ &\lesssim \|v\|^{p-1}_{L_t^{\frac{dp}{2s}} L_x^{\frac{d^2p(p-1)}{2(d+4)}}} \|\nabla v\|_{L_t^{\frac{dp}{2s}} L_x^{\frac{2d^2p}{d^2p-8s}}} \lesssim \|v\|^{p-1}_{S(\dot{H}^s)}\|\nabla v\|_{S(L^2)}. \end{aligned} \tag{2.73}$$

Note that the pairs $\left(\frac{d}{2s}, \frac{d^2p(p-1)}{2(d+4)}\right)$ and $\left(\frac{d}{2s}, \frac{2d^2p}{d^2p-8s}\right)$ are L^2 admissibles and the pair $\left(\frac{d}{2s}, \frac{2d^2(p-1)}{d^2(p-1)+16}\right)$ is L^2 dual admissible. Thus, the Hölder's inequality yields (2.73). Now, the Strichartz (2.3) and Kato Strichartz (2.5) estimates imply

$$\begin{aligned}\|\nabla v\|_{S([T,\infty),L^2)} &\lesssim c_1\|\nabla\psi\|_{S([T,\infty),L^2)} + c\|\nabla(|v|^p v)\|_{S'([T,\infty),L^2)} \\ &\lesssim c_1\|\psi\|_{\dot{H}^1} + c_3\|v\|^{p-1}_{S([T,\infty),\dot{H}^s)}\|\nabla v\|_{S([T,\infty),L^2)}.\end{aligned}$$

Taking T large enough, so that $c_3\|v\|_{S([T,\infty),\dot{H}^s)} \leq \frac{1}{2}$, we obtain

$$\|\nabla v\|_{S([T,\infty),L^2)} \leq 2c_1\|\psi\|_{H^1}.$$

It now follows

$$\begin{aligned}\|\nabla(v - e^{it\Delta}\psi)\|_{S([T,\infty),L^2)} &\leq \|\nabla(|v|^{p-1}v)\|_{S'([T,\infty),L^2)} \\ &\leq \|\nabla v\|_{S([T,\infty),L^2)}\|v\|^{p-1}_{S([T,\infty),H^s)} \leq c\|\psi\|_{H^1},\end{aligned}$$

hence, $\|\nabla(v - e^{it\Delta}\psi)\|_{S(L^2([T,\infty)))} \to 0$ as $T \to \infty$.

On the other hand, Proposition 2.21 (H^1 Scattering) implies $v(t) \to e^{it\Delta}\psi$ in H^1 as $t \to \infty$, and the decay estimate (1.12) together with the embedding and $H^1(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ for $q \leq \frac{2d}{d-2}$ when $3 \leq d$, $q < \infty$ when $d = 2$ and $q \leq \infty$ when $d = 1$ imply

$$\|e^{it\Delta}\psi\|_{L_x^{p+1}} \leq |t|^{\frac{-(p-1)d}{2(p+1)}} \|\psi\|_{H^1},$$

thus, $\|e^{it\Delta}\psi\|_{L_x^{p+1}} \to 0$ as $t \to \infty$. Since $\|\nabla e^{it\Delta}\psi\|_{L^2} = \|\nabla\psi\|_{L^2}$, it follows

$$\begin{aligned} E[v] &= \frac{1}{2}\|\nabla v\|_{L^2}^2 - \frac{1}{p+1}\|v\|_{L_x^{p+1}}^{p+1} \\ &= \lim_{t\to\infty}\left(\frac{1}{2}\|\nabla e^{it\Delta}\psi\|_{L^2}^2 - \frac{1}{p+1}\|e^{it\Delta}\psi\|_{L_x^{p+1}}^{p+1}\right) = \frac{1}{2}\|\nabla\psi\|_{L^2}^2 \end{aligned}$$

and

$$M[v] = \lim_{t\to\infty}\|v(t)\|_{L^2}^2 = \lim_{t\to\infty}\|e^{it\Delta}\psi\|_{L^2}^2 = \|\psi\|_{L^2}^2.$$

From the hypothesis (2.69), we obtain

$$M[v]^{1-s}E[v]^s = \frac{1}{2^s}\|\psi\|_{L^2}^{2(1-s)}\|\nabla\psi\|_{L^2}^{2s} < \sigma^2\left(\frac{d}{2s}\right)^s M[u_Q]^{1-s}E[u_Q]^s$$

and thus,

$$\mathcal{ME}[v] < 1, \qquad \text{since} \qquad \sigma^2 < \left(\frac{2s}{d}\right)^s.$$

Furthermore,

$$\begin{aligned} \lim_{t\to\infty}\|v(t)\|_{L^2}^{2(1-s)}\|\nabla v(t)\|_{L^2}^{2s} &= \lim_{t\to\infty}\|e^{it\Delta}\psi\|_{L^2}^{2(1-s)}\|\nabla(e^{it\Delta}\psi)\|_{L^2}^{2s} = \|\psi\|_{L^2}^{2(1-s)}\|\nabla\psi\|_{L^2}^{2s} \\ &< \sigma^2\left(\frac{d}{s}\right)^s M[u_Q]^{1-s}E[u_Q]^s = \sigma^2\|u_Q\|_{L^2}^{2(1-s)}\|\nabla u_Q\|_{L^2}^{2s}, \end{aligned}$$

where, the inequality is due to (2.69) and the last equality is obtained using (2.56). Hence,

$$\lim_{t\to\infty}\mathcal{G}_v(t) \leq \sigma < 1.$$

We can take $T > 0$ large so that $\mathcal{G}_v(T) \leq 1$. Then applying Theorem 1.6* part I (a) (global existence of solutions with $\mathcal{ME}[v] < 1$ and $\mathcal{G}_v(t) < 1$), we evolve v from time T back to time 0 (we automatically get $\mathcal{G}_v(t) \leq 1$ for all $t \in [0, +\infty)$.) Thus, we obtain v with initial data $v_0 \in H^1$ and properties (2.70) and (2.71) as desired. □

Chapter 3

SCATTERING VIA CONCENTRATION COMPACTNESS

The goal of this chapter is to prove scattering in $H^1(\mathbb{R}^d)$ of global solutions of $\mathrm{NLS}_p^+(\mathbb{R}^d)$ from Theorem 1.6* part I (a).

Definition 3.1. Suppose $u_0 \in H^1(\mathbb{R}^d)$ and let u be the corresponding $H^1(\mathbb{R}^d)$ solution to (1.1) on $[0, T^*)$, the maximal (forward in time) interval of existence. We say that $SC(u_0)$ holds if $T^* = +\infty$ and $\|u\|_{\dot{\beta}^0_{S(\dot{H}^s)}} < \infty$.

3.1 Outline of Scattering via Concentration Compactness

Notice that $H^1(\mathbb{R}^d)$ scattering of $u(t) = \mathrm{NLS}(t)u_0$ is obtained when $SC(u_0)$ holds by Proposition 2.21. Therefore, to establish Theorem 1.6* part I (b), it will be enough to verify that the global-in-time $\dot{H}^s$ Besov-Strichartz norm is finite, i.e., $\|u\|_{\dot{\beta}^0_{S(\dot{H}^s)}} < \infty$, since the hypotheses provides an *a priori bound* for $\|\nabla u(t)\|_{L^2}$ (by Theorem 1.6* part I a), thus, the maximal forward time of existence is $T = +\infty$. In other words, it remains to show

Proposition 3.2. *If $\mathcal{G}_u(0) < 1$ and $\mathcal{ME}[u] < 1$, then $SC(u_0)$ holds.*

The technique to achieve the scattering property above (Proposition 3.2) is the induction argument on the mass-energy threshold as in [Holmer and Roudenko, 2008] and [Duyckaerts et al., 2008] (see also [Kenig and Merle, 2006]), and we describe it in steps 1, 2, 3.

Step 1: *Small Data.*

The equivalence of energy with the gradient (Lemma 2.23) yields

$$\|u_0\|_{\dot{H}^s}^{p+1} \leq (\|u_0\|_{L^2}^{1-s}\|\nabla u_0\|_{L^2}^{s})^{\frac{p+1}{2}} \leq \left(\left(\frac{d}{s}\right)^s M[u]^{1-s}E[u]^s\right)^{\frac{p+1}{4}}.$$

If $\mathcal{G}_u(0) < 1$ and $M[u]^{1-s}E[u]^s < \left(\frac{s}{d}\right)^s \delta_{sd}^4$, then using the above inequality one obtains $\|u_0\|_{\dot{H}^s} \leq \delta_{sd}$ and by Strichartz estimates $\|e^{it\Delta}u_0\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \leq c\delta_{sd}$. Hence, the small data (Proposition 2.13) yields $SC(u_0)$ property.

Observe that Step 1 gives the basis for induction:

Assume $\mathcal{G}_u(0) < 1$. Then for small $\delta > 0$ such that $M[u_0]^{1-s}E[u_0]^s < \delta$, we have that $SC(u_0)$ holds.

Let $(ME)_c$ be the supremum of all such δ for which $SC(u_0)$ holds, namely,

$$(ME)_c = \sup \big\{ \delta \mid u_0 \in H^1(\mathbb{R}^d) \text{ with the property:}$$
$$\mathcal{G}_u(0) < 1 \text{ and } M[u]^{1-s}E[u]^s < \delta \Rightarrow SC(u_0) \text{ holds} \big\}.$$

Thus, the goal is to show that $(ME)_c = M[u_Q]^{1-s}E[u_Q]^s$.

Remark 3.3. In the definition of $(ME)_c$, it should be considered $\mathcal{G}_u(0) \leq 1$ instead of the strict inequality $\mathcal{G}_u(0) < 1$. However, $\mathcal{G}_u(0)$=1 only when $\mathcal{ME}[u] = 1$ (see Figure 2.1 point D). In other words, $u_0 = u_Q(x)$ is a soliton solution to (1.1) and does not scatter, thus, it suffices to consider the strict inequality $\mathcal{G}_u(0) < 1$.

Step 2: *Induction on the scattering threshold and construction of the "critical" solution.*

Assume that $(ME)_c < M[u_Q]^{1-s}E[u_Q]^s$. This means that, there exists a sequence of initial data $\{u_{n,0}\}$ in $H^1(\mathbb{R}^d)$ which will approach the threshold $(ME)_c$ from above and produce solutions which do not scatter, i.e., there exists a sequence $u_{n,0} \in H^1(\mathbb{R}^d)$ with

$$\mathcal{G}_{u_n}(0) < 1 \quad \text{and} \quad M[u_{n,0}]^{1-s}E[u_{n,0}]^s \searrow (ME)_c \text{ as } n \to \infty \tag{3.1}$$
$$\text{and} \quad \|u\|_{\dot{\beta}^0_{S(\dot{H}^s)}} = +\infty,$$

i.e., $SC(u_{n,0})$ does not hold (this is possible by definition of supremum of $(ME)_c$).

Using a nonlinear profile decomposition on the sequence $\{u_{n,0}\}$ it will allow us to construct a "critical" solution of $\mathrm{NLS}_p^+(\mathbb{R}^d)$, denoted by $u_c(t)$, that will lie exactly at the threshold $(ME)_c$ and will not scatter, see Existence of the Critical solution (Proposition 3.11).

Step 3: *Localization properties of the critical solution.*

The critical solution $u_c(t)$ will have the property that $K = \{u_c(t) | t \in [0, +\infty)\}$ is precompact in $H^1(\mathbb{R}^d)$ (Lemma 3.12). Hence, its localization implies that for given $\epsilon > 0$, there exists an $R > 0$ such that $\|\nabla u(x,t)\|^2_{L^2(|x+x(t)|>R)} \leq \epsilon$ uniformly in t (Corollary 3.13); this combined with the zero momentum will give control on the growth of $x(t)$ (Lemma 3.14).

On the other hand, the rigidity theorem (Theorem 3.15) implies that such compact in $H^1(\mathbb{R}^d)$ solutions with the control on $x(t)$, can only be zero solutions, which contradicts the fact that u_c does not scatter. As a consequence, such u_c does not exist and the assumption that $(ME)_c < M[u_Q]E[u_Q]$ is not valid. This finishes the proof of scattering in Theorem 1.6*, Part 1(b).

In the rest of this chapter we proceed with the linear and nonlinear profile decomposition and the proof of the existence and properties of the critical solution described in Step 2 and Step 3.

3.2 Profile decomposition

This section contains the profile decomposition for linear and nonlinear flows for $\mathrm{NLS}_p^+(\mathbb{R}^d)$. The important point to make here is that these are general profile decompositions for bounded sequences on H^1.

Proposition 3.4 (Linear Profile decomposition)**.** *Let $\phi_n(x)$ be a uniformly bounded sequence in $H^1(\mathbb{R}^d)$. Then for each $M \in \mathbb{N}$ there exists a subsequence of ϕ_n (also denoted ϕ_n), such that, for each $1 \leq j \leq M$, there exist, fixed in n, a* *profile* ψ^j *in* $H^1(\mathbb{R}^d)$*, a sequence* t_n^j *of time shifts, a sequence* x_n^j *of space shifts and a sequence* $W_n^M(x)$ *of remainders*[11] *in* $H^1(\mathbb{R}^d)$*, such that*

$$\phi_n(x) = \sum_{j=1}^{M} e^{-it_n^j\Delta}\psi^j(x - x_n^j) + W_n^M(x)$$

with the properties:

- *Pairwise divergence for the time and space sequences. For* $1 \le k \ne j \le M$,

$$\lim_{n\to\infty} |t_n^j - t_n^k| + |x_n^j - x_n^k| = +\infty. \tag{3.2}$$

- *Asymptotic smallness for the remainder sequence*

$$\lim_{M\to\infty} \big(\lim_{n\to\infty} \|e^{it\Delta}W_n^M\|_{\dot\beta^0_{S(\dot H^s)}} \big) = 0. \tag{3.3}$$

- *Asymptotic Pythagorean expansion. For fixed* $M \in \mathbb{N}$ *and any* $0 \le s \le 1$*, we have*

$$\|\phi_n\|_{\dot H^s}^2 = \sum_{j=1}^{M} \|\psi^j\|_{\dot H^s}^2 + \|W_n^M\|_{\dot H^s}^2 + o_n(1). \tag{3.4}$$

Proof. Let ϕ_n be uniformly bounded in H^1, and $c_1 > 0$ such that $\|\phi_n\|_{H^1} \le c_1$.

For each dyadic $N \in 2^{\mathbb{N}}$, given (q, r) an $\dot H^s$ admissible pair, pick $\theta = \frac{4d(d+r)-2d^2r}{r^2(d-2s)(d-2)-2dr(d+2s-4)}$, so $0 < \theta < 1$.

Let $r_1 = \frac{r(d-2)+2d}{2(d-2)}$, and $q_1 = \frac{8(d-2)+4dr}{r(d-2s)(d-2)-2d(d+2s-4)}$, so (q_1, r_1) is $\dot H^s$ admissible pair, for $0 < s < 1$ and $d \ge 2$. Interpolation and Strichartz estimates (2.4) yield

$$\begin{aligned} \|e^{it\Delta}W_n^M\|_{L_t^qL_x^r} &\le \|e^{it\triangle}W_n^M\|_{L_t^{q_1}L_x^{r_1}}^{1-\theta} \|e^{it\triangle}W_n^M\|_{L_t^\infty L_x^{\frac{2d}{d-2s}}}^{\theta} \\ &\le c\|W_n^M\|_{\dot H^s}^{1-\theta} \|e^{it\Delta}W_n^M\|_{L_t^\infty L_x^{\frac{2d}{d-2s}}}^{\theta}. \end{aligned} \tag{3.5}$$

[11]Here, in Proposition 3.4 and Proposition 3.6, $W_n^M(x)$ and $\widetilde{W}_n^M(x)$ represent the remainders for the linear and nonlinear decompositions, respectively.

The goal is to write the profile ϕ_n as $\sum_{j=1}^{M} e^{-it_n^j\Delta}\psi^j(x-x_n^j)+W_n^M(x)$ with $\|W_n^M(x)\|_{\dot{H}^s} \leq c_1$, for some constant c_1. By (3.5), it suffices to show

$$\lim_{M\to+\infty}\left[\limsup_{n\to+\infty}\|e^{it\Delta}W_n^M\|_{L_t^\infty L_x^{\frac{2d}{d-2s}}}\right]=0\ .$$

We have $d \geq 2s$, since we are considering

$$\begin{cases} (i) & 0\leq s\leq 1 \quad \text{in} \quad d\geq 3 \\ (ii) & 0<s<1 \quad \text{in} \quad d=2 \\ (iii) & 0<s<\frac{1}{2} \quad \text{in} \quad d=1. \end{cases} \tag{3.6}$$

Construction of ψ_n^1 :

Let $A_1=\limsup_{n\to+\infty}\|e^{it\Delta}\phi_n\|_{L_t^\infty L_x^{\frac{2d}{d-2s}}}$. If $A_1=0$, taking $\psi^j=0$ for all j finishes the construction.

Suppose that $A_1>0$, and let $c_1=\limsup_{n\to+\infty}\|\phi_n\|_{H^1}<\infty$. Passing to a subsequence ϕ_n, we show that there exist sequences t_n^1 and x_n^1 and a function $\psi^1\in H^1$, such that

$$e^{it_n^1\Delta}\phi_n(\cdot+x_n^1)\rightharpoonup\psi^1 \quad \text{in} \quad H^1,$$

and a constant $K>0$, independent of all parameters, with

$$Kc_1^{\frac{d+2s-4s^2}{2s}}\|\psi^1\|_{\dot{H}^s}\geq A_1^{\frac{d+4s-4s^2}{2s}}. \tag{3.7}$$

Note that $d+2s-4s^2=d+2s(1-2s)>0$ by (3.6).

Let χ_r be a radial Schwartz function such that supp $\chi_r\subset\left[\frac{1}{2r},2r\right]$ and $\widehat{\chi}_r(\xi)=1$ for $\frac{1}{r}\leq|\xi|\leq r$. Note that $|1-\hat{\chi}_r|\leq 1$ and $\dot{H}^s\hookrightarrow L^{\frac{2d}{d-2s}}$ in $\mathbb{R}^d$ with $2s<d$, then

$$\begin{aligned}\|e^{it\Delta}\phi_n-\chi_r*e^{it\Delta}\phi_n\|^2_{L_t^\infty L_x^{\frac{2d}{d-2s}}} &\leq \int|\xi|(1-\hat{\chi}_r(\xi))^2|\hat{\phi}_n(\xi)|^2d\xi \\ &\leq \int_{|\xi|\leq\frac{1}{r}}|\xi||\hat{\phi}_n|^2d\xi+\int_{|\xi|\geq r}|\xi||\hat{\phi}_n(\xi)|^2d\xi \\ &\leq \frac{\|\phi_n\|^2_{L_x^2}+\|\phi_n\|^2_{\dot{H}_x^1}}{r}\leq\frac{c_1^2}{r}.\end{aligned}$$

Take $r = \frac{4c_1^2}{A_1^2}$, then $A_1 = \frac{2c_1}{\sqrt{r}}$. Using the definition of A_1, triangle inequality and the previous calculation, for large n we have

$$\frac{A_1}{2} \leq \|\chi_r * e^{it\Delta}\phi_n\|_{L_t^\infty L_x^{\frac{2d}{d-2s}}}. \tag{3.8}$$

Therefore, interpolation implies

$$\begin{aligned} \|\chi_r * e^{it\Delta}\phi_n\|^d_{L_t^\infty L_x^{\frac{2d}{d-2s}}} &\leq \|\chi_r * e^{it\Delta}\phi_n\|^{d-2s}_{L_t^\infty L_x^2}\|\chi_r * e^{it\Delta}\phi_n\|^{2s}_{L_t^\infty L_x^\infty} \\ &\leq \|\phi_n\|^{d-2s}_{L_x^2}\|\chi_r * e^{it\Delta}\phi_n\|^{2s}_{L_t^\infty L_x^\infty}, \end{aligned} \tag{3.9}$$

where the second inequality follows from the fact that $|\widehat{\chi_r}| \leq 1$ and L^2 isometry property of the linear Schrödinger operator.

Using the definition of c_1, combining (3.8) and (3.9), we get

$$\left(\frac{A_1}{2c_1^{\frac{d-2s}{d}}}\right)^{\frac{d}{2s}} \leq \|\chi_r * e^{it\Delta}\phi_n\|_{L_t^\infty L_x^\infty}.$$

Thus, there exists a sequence of $(x_n^1, t_n^1) \in \mathbb{R}^d \times \mathbb{R}_+^1$ satisfying

$$\left(\frac{A_1}{2c_1^{\frac{d-2s}{d}}}\right)^{\frac{d}{2s}} \leq |\chi_r * e^{it_n^1\Delta}\phi_n(x_n^1)|.$$

Since $e^{it\Delta}$ is an H^1 isometry and translation invariant[12], it follows that $\{e^{it_n^1\Delta}\phi_n(\cdot + x_n^1)\}$ is uniformly bounded in H^1 (with the same constant as ϕ_n's) and along a subsequence

$$\{e^{it_n^1\Delta}\phi_n(\cdot + x_n^1)\} \rightharpoonup \psi^1 \text{ with } \|\psi^1\|_{H^1} \leq c_1.$$

Observe that

$$\left(\frac{A_1}{2c_1^{\frac{d-2s}{d}}}\right)^{\frac{d}{2s}} \leq \Big|\int_{\mathbb{R}^2} \chi_r(x_n^1 - y)\psi^1(y)dy\Big| \leq \|\chi_r\|_{\dot{H}^{-s}}\|\psi^1\|_{\dot{H}^s} \leq r^{1-s}\|\psi^1\|_{\dot{H}^s},$$

since $\|\chi_r\|^2_{\dot{H}^{-s}} \lesssim r^{1-s}$ (by converting to radial coordinates) and the Hölder's inequality produces (3.7) with $K = 2^{\frac{d+4s-4s^2}{2s}}$.

[12] $\left|\left(e^{it\Delta}f(x+h)\right)^\wedge\right| = \left|e^{i\xi h}\left(e^{it\Delta}f(x)\right)^\wedge\right| = \left|\left(e^{it\Delta}f(x)\right)^\wedge\right|$

Define $W_n^1(x) = \phi_n(x) - e^{-it_n^1\Delta}\psi^1(x - x_n^1)$. Note that $e^{it_n^1\Delta}\phi_n(\cdot + x_n^1) \rightharpoonup \psi^1$ in H^1, therefore, for any $0 \le s \le 1$, we have

$$\langle \phi_n, e^{-it_n^1\Delta}\psi^1(\cdot - x_n^1)\rangle_{\dot{H}^s} = \langle e^{it_n^1\Delta}\phi_n, \psi^1(\cdot - x_n^1)\rangle_{\dot{H}^s} \to \|\psi^1\|^2_{\dot{H}^s},$$

and since $\|W_n^1\|^2_{\dot{H}^s} = \langle \phi_n - e^{-it_n^1\Delta}\psi(\cdot - x_n^1), \phi_n - e^{-it_n^1\Delta}\psi^1(\cdot - x_n^1)\rangle^2_{\dot{H}^s}$, we obtain

$$\lim_{n\to\infty} \|W_n^1\|^2_{\dot{H}^s} = \lim_{n\to\infty} \|e^{it_n^1\Delta}\phi_n\|^2_{\dot{H}^s} - \|\psi^1\|^2_{\dot{H}^s}.$$

Taking $s = 1$ and $s = 0$, yields $\|W_n^1\|_{H^1} \le c_1$.

Construction of ψ^j for $j \ge 2$ (Inductively we assume that ψ^{j-1} is known and construct ψ^j):

Let $M \ge 2$. Suppose that ψ^j, x_n^j, t_n^j and W_n^j are known for $j \in \{1, \cdots, M-1\}$. Consider

$$A_M = \limsup_n \|e^{it\Delta}W_n^{M-1}\|_{L_t^\infty L_x^{\frac{2d}{d-2s}}}.$$

If $A_M = 0$, then taking $\psi^j = 0$ for $j \ge M$ will end the construction.

Assume $A_M > 0$, we apply the previous step to W_n^{M-1}, and let $c_M = \limsup_n \|W_n^{M-1}\|_{H^1}$, thus, obtaining sequences (or subsequences) x_n^M, t_n^M and a function $\psi^M \in H^1$ such that

$$e^{it_n^M\Delta}W_n^{M-1}(\cdot + x_n^M) \rightharpoonup \psi^M \quad \text{in} \quad H^1 \quad \text{and} \quad Kc_M^{\frac{d+2s-4s^2}{2s}}\|\psi^M\|_{\dot{H}^s} \ge A_M^{\frac{d+4s-4s^2}{2s}}. \tag{3.10}$$

Define

$$W_n^M(x) = W_n^{M-1}(x) - e^{-it_n^M\Delta}\psi^M(x - x_n^M).$$

Then (3.2) and (3.4) follow from induction, i.e., we assume (3.4) holds at rank $M-1$, then expanding

$$\|W_n^M(x)\|^2_{\dot{H}^s} = \|e^{it_n^M\Delta}W^{M-1}(\cdot + x_n^M) - \psi^M\|^2_{\dot{H}^s},$$

the weak convergence yields (3.4) at rank M.

In the same fashion, we assume (3.2) is true for $j, k \in \{1, \ldots, M-1\}$ with $j \neq k$, that is $|t_n^j - t_n^k| + |x_n^j - x_n^k| \to +\infty$ as $n \to \infty$. Take $k \in \{1, \ldots, M-1\}$ and show that

$$|t_n^M - t_n^k| + |x_n^M - x_n^k| \to +\infty.$$

Passing to a subsequence, assume $t_n^M - t_n^k \to t^{M_1}$ and $x_n^M - x_n^k \to x^{M_1}$ finite, then as $n \to \infty$

$$\begin{aligned} e^{it_n^M \Delta} W_n^{M-1}(x + x_n^M) =& e^{i(t_n^M - t_n^j)\Delta}(e^{it_n^j \Delta} W_n^{j-1}(x + x_n^j) - \psi^j(x + x_n^j)) \\ & - \sum_{k=j+1}^{M-1} e^{i(t_n^j - t_n^k)\Delta} \psi_n^k(x + x_n^j - x_n^k). \end{aligned}$$

The orthogonality condition (3.2) implies that the right hand side goes to 0 weakly in H^1, while the left side converges weakly to a nonzero ψ^M, which is a contradiction. Note that the orthogonality condition (3.2) holds for $k = M$, and since (3.4) holds for all M, we have

$$\|\phi_n\|_{\dot{H}^s}^2 \geq \sum_{j=1}^{M} \|\psi^j\|_{\dot{H}^s}^2 + \| W_n^M \|_{\dot{H}^s}^2$$

and $c_M \leq c_1$. Fix s. If for all M, $A_M > 0$, then (3.10) yields

$$\sum_{M \geq 1} \left(\frac{A_M^{\frac{d+4s-4s^2}{2s}}}{K c_1^{\frac{d+2s-4s^2}{2s}}} \right)^2 \leq \sum_{n \geq 1} \|\psi^M\|_{\dot{H}^s}^2 \leq \limsup_n \|\phi_n\|_{\dot{H}^s}^2 < \infty,$$

therefore, $A_M \to 0$ as $M \to \infty$, and consequently, $\|e^{it\Delta} W_n^M\|_{S(\dot{H}^s)} \to 0$ as $n \to \infty$. Finally, summing over all dyadic N, yields (3.3).

□

Proposition 3.5 (Energy Pythagorean expansion)**.** *Under the hypothesis of Proposition 3.4, we have*

$$E[\phi_n] = \sum_{j=1}^{M} E[e^{-it_n^j \Delta} \psi^j] + E[W_n^M] + o_n(1). \tag{3.11}$$

Proof. By definition of $E[u]$ and (3.4) with $s = 1$, it suffices to prove that for all $M \leq 1$, we have

$$\|\phi_n\|_{L^{p+1}}^{p+1} = \sum_{j=1}^{M} \|e^{-it_n^j \Delta}\psi^j\|_{L^{p+1}}^{p+1} + o_n(1). \tag{3.12}$$

Step 1. Pythagorean expansion of a sum of orthogonal profiles. Fix $M \geq 1$. We want to show that the condition (3.2) yields

$$\left\| \sum_{j=1}^{M} e^{-it_n^j \Delta}\psi^j(\cdot - x_n^j) \right\|_{L_x^{p+1}}^{p+1} = \sum_{j=1}^{M} \|e^{-it_n^j \Delta}\psi^j\|_{L_x^{p+1}}^{p+1} + o_n(1). \tag{3.13}$$

By rearranging and reindexing, we can find $M_0 \leq M$ such that

(a) t_n^j is bounded in n whenever $1 \leq j \leq M_0$,

(b) $|t_n^j| \to \infty$ as $n \to \infty$ if $M_0 + 1 \leq j \leq M$.

For the case (a) take a subsequence and assume that for each $1 \leq j \leq M_0$, t_n^j converges (in n), then adjust the profiles ψ^j's such that $t_n^j = 0$. From (3.2) we have $|x_n^j - x_n^k| \to +\infty$ as $n \to \infty$, which implies

$$\left\| \sum_{j=1}^{M_0} \psi^j(\cdot - x_n^j) \right\|_{L_x^{p+1}}^{p+1} = \sum_{j=1}^{M_0} \|\psi^j\|_{L_x^{p+1}}^{p+1} + o_n(1). \tag{3.14}$$

For the case (b), i.e., for $M_0 \leq j \leq M$, $|t_n^j| \to \infty$ as $n \to \infty$ and for $\tilde{\psi} \in \dot{H}^{\frac{p}{p+1}} \cap L^{\frac{p}{p+1}}$, thus, the Sobolev embedding and the L^p space-time decay estimate yield

$$\|e^{-it_n^k \Delta}\psi^k\|_{L_x^{p+1}} \leq c\|\psi^k - \tilde{\psi}\|_{\dot{H}^{\frac{p}{p+1}}} + \frac{c}{|t_n^k|^{\frac{d(p-1)}{2(p+1)}}} \|\tilde{\psi}\|_{L_x^{\frac{p+1}{p}}},$$

and approximating ψ^k by $\tilde{\psi} \in C_{comp}^{\infty}$ in $\dot{H}^{\frac{p}{p+1}}$, we have

$$\|e^{-it_n^k \Delta}\psi^k\|_{L_x^{p+1}} \to 0 \text{ as } n \to \infty. \tag{3.15}$$

Thus, combining (3.14) and (3.15), we obtain (3.12).

Step 2. Finishing the proof. Note that

$$\begin{aligned}\|W_n^{M_1}\|_{L_x^{p+1}} &\leq \|W_n^{M_1}\|_{L_t^\infty L_x^{p+1}} \leq \|W_n^{M_1}\|^{1/2}_{L_t^\infty L_x^{\frac{2d}{d-2s}}} \|W_n^{M_1}\|^{1/2}_{L_t^\infty L_x^{\frac{2d(d+2-2s)}{d(d-2)+4s(1-s)}}} \\ &\leq \|W_n^{M_1}\|^{1/2}_{L_t^\infty L_x^{\frac{2d}{d-2s}}} \|W_n^{M_1}\|^{1/2}_{L_t^\infty \dot{H}_x^1} \leq \|W_n^{M_1}\|^{1/2}_{L_t^\infty L_x^{\frac{2d}{d-2s}}} \sup_n \|\phi_n\|^{1/2}_{H^1}.\end{aligned}$$

By (3.3) it follows that

$$\lim_{M_1\to+\infty}\left(\lim_{n\to+\infty}\|e^{it\Delta}W_n^{M_1}\|_{L^{p+1}}\right) = 0. \tag{3.16}$$

Let $M \geq 1$ and $\epsilon > 0$. The sequence of profiles $\{\psi^n\}$ is uniformly bounded in H^1 and in L^{p+1}. Hence, (3.16) implies that the sequence of remainders $\{W_n^M\}$ is also uniformly bounded in L_x^{p+1}. Pick $M_1 \geq M$ and n_1 such that for $n \geq n_1$, we have

$$\begin{aligned}&\left|\|\phi_n - W_n^{M_1}\|^{p+1}_{L_x^{p+1}} - \|\phi_n\|^{p+1}_{L_x^{p+1}}\right| + \left|\|W_n^M - W_n^{M_1}\|^{p+1}_{L_x^{p+1}} - \|W_n^M\|^{p+1}_{L_x^{p+1}}\right| \\ &\quad\leq C\Big(\big(\sup_n\|\phi_n\|^p_{L_x^{p+1}} + \sup_n\|W_n^M\|^p_{L_x^{p+1}}\big)\|W_n^{M_1}\|_{L_x^{p+1}} + \|W_n^{M_1}\|^{p+1}_{L_x^{p+1}}\Big) \leq \frac{\epsilon}{3}.\end{aligned} \tag{3.17}$$

Choose $n_2 \geq n_1$ such that $n \geq n_2$. Then (3.13) yields

$$\left|\|\phi_n - W_n^{M_1}\|^{p+1}_{L_x^{p+1}} - \sum_{j=1}^{M_1}\|e^{-it_n^j\Delta}\psi^j\|^{p+1}_{L_x^{p+1}}\right| \leq \frac{\epsilon}{3}. \tag{3.18}$$

Since $W_n^M - W_n^{M_1} = \sum_{j=M+1}^{M_1} e^{-it_n^j\Delta}\psi^j(\cdot - x_n^j)$, by (3.13), there exist $n_3 \geq n_2$ such that $n \geq n_3$,

$$\left|\|W_n^M - W_n^{M_1}\|^{p+1}_{L_x^{p+1}} - \sum_{j=M+1}^{M_1}\|e^{-it_n^j\Delta}\psi^j\|^{p+1}_{L_x^{p+1}}\right| \leq \frac{\epsilon}{3}. \tag{3.19}$$

Thus for $n \geq n_3$, (3.17), (3.18), and (3.19) yield

$$\left|\|\phi_n\|^{p+1}_{L_x^{p+1}} - \sum_{j=1}^{M}\|e^{-it_n^j\Delta}\psi^j\|^{p+1}_{L_x^{p+1}} - \|W_n^M\|^{p+1}_{L_x^{p+1}}\right| \leq \epsilon, \tag{3.20}$$

which concludes the proof. □

Proposition 3.6 (Nonlinear Profile decomposition)**.** *Let $\phi_n(x)$ be a uniformly bounded sequence in $H^1(\mathbb{R}^d)$. Then for each $M \in \mathbb{N}$ there exists a subsequence* *of* ϕ_n*, also denoted by* ϕ_n*, for each* $1 \leq j \leq M$*, there exist a (same for all* n*) nonlinear profile* $\widetilde{\psi}^j$ *in* $H^1(\mathbb{R}^d)$*, a sequence of time shifts* t_n^j*, and a sequence of space shifts* x_n^j *and in addition, a sequence (in* n*) of remainders* $\widetilde{W}_n^M(x)$ *in* $H^1(\mathbb{R}^d)$*, such that*

$$\phi_n(x) = \sum_{j=1}^{M} NLS(-t_n^j)\widetilde{\psi}^j(x - x_n^j) + \widetilde{W}_n^M(x), \tag{3.21}$$

where (as $n \to \infty$*)*

(a) for each j*, either* $t_n^j = 0, t_n^j \to +\infty$ *or* $t_n^j \to -\infty$*,*

(b) if $t_n^j \to +\infty$*, then* $\|NLS(-t)\widetilde{\psi}^j\|_{\dot{\beta}^0_{S([0,\infty);\dot{H}^s)}} < +\infty$ *and if* $t_n^j \to -\infty$*, then* $\|NLS(-t)\widetilde{\psi}^j\|_{\dot{\beta}^0_{S((-\infty,0];\dot{H}^s)}} < +\infty$*,*

(c) for $k \neq j$*, then* $|t_n^j - t_n^k| + |x_n^j - x_n^k| \to +\infty$*.*

The remainder sequence has the following asymptotic smallness property:

$$\lim_{M\to\infty}\big(\lim_{n\to\infty}\|NLS(t)\widetilde{W}_n^M\|_{\dot{\beta}^0_{S(\dot{H}^s)}}\big) = 0. \tag{3.22}$$

For fixed $M \in \mathbb{N}$ *and any* $0 \leq s \leq 1$*, we have the asymptotic Pythagorean expansion*

$$\|\phi_n\|^2_{\dot{H}^s} = \sum_{j=1}^{M}\|NLS(-t_n^j)\widetilde{\psi}^j\|^2_{\dot{H}^s} + \|\widetilde{W}_n^M\|^2_{\dot{H}^s} + o_n(1) \tag{3.23}$$

and the energy Pythagorean decomposition (note that $E[NLS(-t_n^j)\widetilde{\psi}^j] = E[\widetilde{\psi}^j]$*):*

$$E[\phi_n] = \sum_{j=1}^{M} E[\widetilde{\psi}^j] + E[\widetilde{W}_n^M] + o_n(1). \tag{3.24}$$

Proof. From Proposition 3.4, given that $\phi_n(x)$ is a uniformly bounded sequence in H^1, we have

$$\phi_n(x) = \sum_{j=1}^{M} e^{-it_n^j\Delta}\psi^j(x - x_n^j) + W_n^M(x) \tag{3.25}$$

satisfying (3.2), (3.3), (3.4) and (3.11). We will choose $M \in \mathbb{N}$ later. To prove this proposition, the idea is to replace a linear flow $e^{it\Delta}\psi^j$ by some nonlinear flow.

Now for each ψ^j we can apply the wave operator (Proposition 2.25) to obtain a function $\widetilde{\psi}^j \in H^1$, which we will refer to as the nonlinear profile (corresponding to the linear profile ψ^j) such that the following properties hold:

For a given j, there are two cases to consider: either t_n^j is bounded, or $|t_n^j| \to +\infty$.

Case $|t_n^j| \to +\infty$*:*

If $t_n^j \to +\infty$, Proposition 2.25 Part I equation (2.67) implies that

$$\|\mathrm{NLS}(-t_n^j)\widetilde{\psi}^j - e^{-it_n^j\Delta}\psi^j\|_{H^1} \to 0 \quad \text{as} \quad t_n^j \to +\infty$$

and so

$$\|\mathrm{NLS}(-t)\widetilde{\psi}^j\|_{\dot{\beta}^0_{S([0,+\infty),\dot{H}^s)}} < +\infty. \tag{3.26}$$

Similarly, if $t_n^j \to -\infty$, by (2.68) we obtain

$$\|\mathrm{NLS}(-t_n^j)\widetilde{\psi}^j - e^{-it_n^j\Delta}\psi^j\|_{H^1} \to 0 \quad \text{as} \quad t_n^j \to -\infty,$$

and hence,

$$\|\mathrm{NLS}(-t)\widetilde{\psi}^j\|_{\dot{\beta}^0_{S((-\infty,0],\dot{H}^s)}} < +\infty. \tag{3.27}$$

Case t_n^j *is bounded* (as $n \to \infty$): Adjusting the profiles ψ^j we reduce it to the case $t_n^j = 0$. Thus, (3.2) becomes $|x_n^j - x_n^k| \to +\infty$ as $n \to \infty$, and continuity of the *linear* flow in H^1, leads to $e^{-t_n^j\Delta}\psi^j \to \psi^j$ strongly in H^1 as $n \to \infty$. In this case, we simply let

$$\tilde{\psi}^j = \mathrm{NLS}(0)e^{-i(\lim_{n\to\infty} t_n^j)\Delta}\psi^j = e^{-i0\Delta}\psi^j = \psi^j.$$

Thus, in either case of sequence $\{t_n^j\}$, we have a new nonlinear profile $\widetilde{\psi}^j$ associated to each original linear profile ψ^j such that

$$\|\text{NLS}(-t_n^j)\widetilde{\psi}^j - e^{-it_n^j\Delta}\psi^j\|_{H^1} \to 0 \qquad \text{as} \qquad n \to +\infty. \tag{3.28}$$

Thus, we can substitute $e^{-it_n^j\Delta}\psi^j$ by $\text{NLS}(-t_n^j)\tilde{\psi}^j$ in (3.25) to obtain

$$\phi_n(x) = \sum_{j=1}^{M} \text{NLS}(-t_n^j)\widetilde{\psi}^j(x - x_n^j) + \widetilde{W}_n^M(x), \tag{3.29}$$

where

$$\begin{aligned}\widetilde{W}_n^M(x) &= W_n^M(x) + \sum_{j=1}^{M} \{e^{-it_n^j\Delta}\psi^j(x - x_n^j) - \text{NLS}(-t_n^j)\widetilde{\psi}^j(x - x_n^j)\} \\ &\equiv W_n^M(x) + \sum_{j=1}^{M} \mathcal{T}^j. \end{aligned} \tag{3.30}$$

The triangle inequality yields

$$\|e^{it\Delta}\widetilde{W}_n^M\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \leq \|e^{it\Delta}W_n^M\|_{\dot{\beta}^0_{S(\dot{H}^s)}} + c\sum_{j=1}^{M} \left\|e^{-it_n^j\Delta}\psi^j - \text{NLS}(-t_n^j)\widetilde{\psi}^j\right\|_{\dot{\beta}^0_{S(\dot{H}^s)}}.$$

By (3.28) we have that

$$\|e^{it\Delta}\widetilde{W}_n^M\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \leq \|e^{it\Delta}W_n^M\|_{\dot{\beta}^0_{S(\dot{H}^s)}} + c\sum_{j=1}^{M} o_n(1),$$

and thus,

$$\lim_{M\to\infty}\Big(\lim_{n\to\infty} \|e^{it\Delta}\widetilde{W}_n^M\|_{\dot{\beta}^0_{S(\dot{H}^s)}}\Big) = 0.$$

Now we are going to apply a nonlinear flow to $\phi_n(x)$ and approximate it by a combination of "nonlinear bumps" $\text{NLS}(t - t_n^j)\widetilde{\psi}^j(x - x_n^j)$, i.e.,

$$\text{NLS}(t)\phi_n(x) \approx \sum_{j=1}^{M} \text{NLS}(t - t_n^j)\widetilde{\psi}^j(x - x_n^j).$$

Obviously, this can not hold for any bounded in H^1 sequence $\{\phi_n\}$, since, for a example, a nonlinear flow can introduce finite time blowup solutions. However, under the proper conditions we can use the long term perturbation theory

(Proposition 2.17) to guarantee that a nonlinear flow behaves basically similar to the linear flow.

To simplify notation, introduce the nonlinear evolution of each separate initial condition $u_{n,0} = \phi_n$:

$$u_n(t,x) = \mathrm{NLS}(t)\phi_n(x),$$

the nonlinear evolution of each separate nonlinear profile ("bump"):

$$v^j(t,x) = \mathrm{NLS}(t)\widetilde{\psi}^j(x),$$

and a linear sum of nonlinear evolutions of "bumps":

$$\tilde{u}_n(t,x) = \sum_{j=1}^{M} v^j(t - t_n^j, x - x_n^j).$$

Intuitively, we think that $\phi_n = u_{n,0}$ is a sum of bumps $\widetilde{\psi}^j$ (appropriately transformed) and $u_n(t)$ is a nonlinear evolution of their entire sum. On the other hand, $\tilde{u}_n(t)$ is a sum of nonlinear evolutions of each bump so we now want to compare $u_n(t)$ with $\tilde{u}_n(t)$.

Note that if we had just the linear evolutions, then both $u_n(t)$ and $\tilde{u}_n(t)$ would be the same.

Thus, $u_n(t)$ satisfies

$$i\partial_t u_n + \Delta u_n + |u_n|^{p-1}u_n = 0,$$

and $\tilde{u}_n(t)$ satisfies

$$i\partial_t \tilde{u}_n + \Delta \tilde{u}_n + |\tilde{u}_n|^{p-1}\tilde{u}_n = \tilde{e}_n^M,$$

where

$$\tilde{e}_n^M = |\tilde{u}_n|^{p-1}\tilde{u}_n - \sum_{j=1}^{M} |v_n^j(t - t_n^j, \cdot - x_n^j)|^{p-1} v_n^j(t - t_n^j, \cdot - x_n^j).$$

Claim 3.7. *There exists a constant A independent of M, and for every M, there exists $n_0 = n_0(M)$ such that if $n > n_0$, then $\|\tilde{u}_n\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \leq A$.*

Claim 3.8. *For each M and $\epsilon > 0$, there exists $n_1 = n_1(M, \epsilon)$ such that if $n > n_1$, then $\|\tilde{e}_n^M\|_{\dot{\beta}^0_{S'(\dot{H}^{-s})}} \leq \epsilon$.*

Note $\tilde{u}_n(0,x) - u_n(0,x) = \widetilde{W}_n^M(x)$. Then for any $\tilde{\epsilon} > 0$ there exists $M_1 = M_1(\tilde{\epsilon})$ large enough such that for each $M > M_1$ there exists $n_2 = n_2(M)$ with $n > n_2$ implying

$$\|e^{it\Delta}(\tilde{u}_n(0) - u_n(0))\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \leq \tilde{\epsilon}.$$

Therefore, for M large enough and $n = \max(n_0, n_1, n_2)$, since

$$e^{it\Delta}(\tilde{u}_n(0)) = e^{it\Delta}\left(\sum_{j=1}^{M} v^j(-t_n^j, x - x_n^j)\right),$$

which are scattering by (3.28), Proposition 2.17 implies $\|u_n\|_{\dot{\beta}^0_{S(\dot{H}^s)}} < +\infty$, a contradiction.

Coming back to the nonlinear remainder $\widetilde{W}_n^M$, we estimate its nonlinear flow as follows (recall the notation of $\widetilde{W}_n^M$, W_n^M and $\mathcal{T}^j$ in (3.30)):

By Besov Strichartz estimates (2.12) and by the triangle inequality, we get

$$\begin{aligned}
\|\mathrm{NLS}(t)\widetilde{W}_n^M\|_{\dot{\beta}^0_{S(\dot{H}^s)}} &\leq \|e^{it\Delta}\widetilde{W}_n^M\|_{\dot{\beta}^0_{S(\dot{H}^s)}} + \left\|\left|\widetilde{W}_n^M\right|^{p-1}\widetilde{W}_n^M\right\|_{\dot{\beta}^0_{S'(\dot{H}^{-s})}} \\
&\leq \|e^{it\Delta}\widetilde{W}_n^M\|_{\dot{\beta}^0_{S(\dot{H}^s)}} + c\sum_{j=1}^{M}\|\mathcal{T}^j\|^{p-1}_{\dot{\beta}^0_{S(\dot{H}^s)}}\|\mathcal{T}^j\|_{\dot{\beta}^s_{S(L^2)}} \qquad (3.31) \\
&\leq \|e^{it\Delta}\widetilde{W}_n^M\|_{\dot{\beta}^0_{S(\dot{H}^s)}} + c\sum_{j=1}^{M}\|\mathcal{T}^j\|^{p-1}_{\dot{\beta}^0_{S(\dot{H}^s)}}\|\mathcal{T}^j\|_{\dot{\beta}^0_{S(\dot{H}^1)}}. \qquad (3.32)
\end{aligned}$$

We used (2.13) to obtain (3.31) and since $s < 1$ we have $\dot{H}^1 \hookrightarrow \dot{H}^s$, so it yields (3.32). Hence,

$$\|\mathrm{NLS}(t)\widetilde{W}_n^M\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \leq \|e^{it\Delta}\widetilde{W}_n^M\|_{\dot{\beta}^0_{S(\dot{H}^s)}} + c\sum_{j=1}^{M}\left\|e^{-it_n^j\Delta}\psi^j - \mathrm{NLS}(-t_n^j)\widetilde{\psi}^j\right\|^p_{H^1}$$

and by (3.28) and then applying (3.3), we obtain

$$\lim_{n\to\infty} \|e^{it\Delta} W_n^M\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \to 0 \qquad \text{as} \qquad M \to \infty.$$

Thus we proved (3.29), (3.22). This also gives (3.23).

Next, we substitute the linear flow in Lemma 3.5 by the nonlinear and repeat the above long term perturbation argument to obtain

$$\|\phi_n\|_{L^{p+1}}^{p+1} = \sum_{j=1}^{M} \|\mathrm{NLS}(-t_n^j)\psi^j\|_{L^{p+1}}^{p+1} + \|\widetilde{W}_n^M\|_{L^{p+1}}^{p+1} + o_n(1), \tag{3.33}$$

which yields the energy Pythagorean decomposition (3.24). The proof will be concluded after we prove the Claims 3.7 and 3.8.

Proof of Claim 3.7. We show that for a large constant A independent of M and if $n > n_0 = n_0(M)$, then

$$\|\tilde{u}_n\|_{S(\dot{H}^s)} \leq A. \tag{3.34}$$

Let M_0 be a large enough such that $\|e^{it\Delta}\widetilde{W}_n^{M_0}\|_{S(\dot{H}^s)} \leq \delta_{sd}$. Then, by (3.30), for each $j > M_0$, we have $\|e^{it\Delta}\psi^j\|_{S(\dot{H}^s)} \leq \delta_{sd}$, thus, Proposition 2.25 yields

$$\|v^j\|_{S(\dot{H}^s)} \leq 2\|e^{it\Delta}\psi^j\|_{S(\dot{H}^s)} \;\text{ for }\; j > M_0.$$

Assume both $s \neq \frac{1}{2}$ and $d \neq 2$, the pairs $\left(\frac{2(d+2)}{d-2s}, \frac{2(d+2)}{d-2s}\right)$, $(\infty, \frac{2d}{d-2s})$, $\left(\frac{6}{1-s}, \frac{6d}{3d-4s-2}\right)$ and $\left(\frac{4}{1-s}, \frac{2d}{d-s-1}\right)$, are $\dot{H}^s$ admissible. Hence, we have

$$\begin{aligned}
&\|\tilde{u}_n\|^{\frac{2(d+2)}{d-2s}}_{L_t^{\frac{2(d+2)}{d-2s}} L_x^{\frac{2(d+2)}{d-2s}}} = \\
&= \sum_{j=1}^{M_0} \|v^j\|^{\frac{2(d+2)}{d-2s}}_{L_t^{\frac{2(d+2)}{d-2s}} L_x^{\frac{2(d+2)}{d-2s}}} + \sum_{j=M_0+1}^{M} \|v^j\|^{\frac{2(d+2)}{d-2s}}_{L_t^{\frac{2(d+2)}{d-2s}} L_x^{\frac{2(d+2)}{d-2s}}} + \text{cross terms} \\
&\leq \sum_{j=1}^{M_0} \|v^j\|^{\frac{2(d+2)}{d-2s}}_{L_t^{\frac{2(d+2)}{d-2s}} L_x^{\frac{2(d+2)}{d-2s}}} + 2^{\frac{2(d+2)}{d-2s}} \sum_{j=M_0+1}^{M} \|e^{it\Delta}\psi^j\|^{\frac{2(d+2)}{d-2s}}_{L_t^{\frac{2(d+2)}{d-2s}} L_x^{\frac{2(d+2)}{d-2s}}} + \text{cross terms,}
\end{aligned} \tag{3.35}$$

note that by (3.25) we have

$$\|e^{it\Delta}\phi_n\|^{\frac{2(d+2)}{d-2s}}_{L_t^{\frac{2(d+2)}{d-2s}}L_x^{\frac{2(d+2)}{d-2s}}} =$$

$$= \sum_{j=1}^{M_0} \|e^{it\Delta}\psi^j\|^{\frac{2(d+2)}{d-2s}}_{L_t^{\frac{2(d+2)}{d-2s}}L_x^{\frac{2(d+2)}{d-2s}}} + 2^{\frac{2(d+2)}{d-2s}} \sum_{j=M_0+1}^{M} \|e^{it\Delta}\psi^j\|^{\frac{2(d+2)}{d-2s}}_{L_t^{\frac{2(d+2)}{d-2s}}L_x^{\frac{2(d+2)}{d-2s}}} + \text{cross terms.} \tag{3.36}$$

Observe that by (3.2) and taking $n_0 = n_0(M)$ large enough, we can consider $\{u_n\}_{n>n_0}$ and thus, make "the cross terms" ≤ 1. Then (3.36) and

$$\|e^{it\Delta}\phi_n\|_{L_t^{\frac{2(d+2)}{d-2s}}L_x^{\frac{2(d+2)}{d-2s}}} \leq c\|\phi_n\|_{\dot{H}^s} \leq c_1$$

imply $\sum_{j=M_0+1}^{M} \|e^{it\Delta}\psi^j\|^{\frac{2(d+2)}{d-2s}}_{L_t^{\frac{2(d+2)}{d-2s}}L_x^{\frac{2(d+2)}{d-2s}}}$ is bounded independent of M provided $n > n_0$. If $n > n_0$ then $\|\tilde{u}_n\|_{L_t^{\frac{2(d+2)}{d-2s}}L_x^{\frac{2(d+2)}{d-2s}}}$ is also bounded independent of M by (3.35).

In a similar fashion, one can prove that $\|\tilde{u}_n\|_{L_t^\infty L_x^{\frac{2d}{d-2s}}}$ is bounded independent of M provided $n > n_0$. Interpolation between these exponents gives $\|\tilde{u}_n\|_{L_t^{\frac{6}{1-s}}L_x^{\frac{6d}{3d-4s-2}}}$ and $\|\tilde{u}_n\|_{L_t^{\frac{4}{1-s}}L_x^{\frac{2d}{d-s-1}}}$ are bounded independent of M for $n > n_0$.

When $s = \frac{1}{2}$ and $d = 2$, the previous argument takes the pair $(2, \infty)$ which is not an admissible pair in dimension 2, instead we estimate $\|\tilde{u}\|_{L_x^8, L_x^8}$ and $\|\tilde{u}\|_{L_x^\infty, L_x^4}$, as previously done, and interpolate between them to get that $\|\tilde{u}\|_{L_x^{12}, L_x^6}$ is bounded independent of M provided $n > n_0$.

To close the argument, we apply Kato estimate (2.5) to the integral equation of

$$i\partial_t \tilde{u}_n + \Delta\tilde{u}_n + |\tilde{u}_n|^{p-1}\tilde{u}_n = \tilde{e}_n^M.$$

Claiming $\|\tilde{e}_n^M\|_{\dot{\beta}^0_{S'(\dot{H}^{-s})}} \leq 1$ (see Claim 3.8) , as in Proposition 2.17, we obtain that $\|\tilde{u}_n\|_{\dot{\beta}^0_{S(\dot{H}^s)}}$ is as well bounded independent of M provided $n > n_0$. Thus, Claim 3.7 is proved.

Proof of Claim 3.8. Note that the pairs $(\frac{6}{1-s}, \frac{6d}{3d-4s-2})$, $(\frac{4}{1-s}, \frac{2d}{d-s-1})$ are $\dot{H}^s$ admissible and the pair $(\frac{12(d-2s)}{(8+3d-6s)(1-s)}, \frac{6d(d-2s)}{3(d^2+2s^2)+9d(1-s)-2(5s+4)})$ is $\dot{H}^{-s}$ admissible. Recall the elemental inequality: for $a_j, a_k \in \mathbb{C}$,

$$\left| \left| \sum_{j=1}^{M} a_j \right|^{p-1} \sum_{k=1}^{M} a_k - \sum_{j=1}^{M} |a_j|^{p-1} a_j \right| \leq c_{p,M} \sum_{j=1}^{M} \sum_{\substack{k=1 \\ k\neq j}}^{M} |a_k|^{p-1} |a_j|,$$

which combined with the Hölder's inequality, for each dyadic number $N \in 2^{\mathbb{Z}}$, leads to

$$\begin{aligned} \|\tilde{e}_n^M\|_{S'(\dot{H}^{-s})} &\leq \|\tilde{e}_n^M\|_{L_t^{\frac{12(d-2s)}{(8+3d-6s)(1-s)}} L_x^{\frac{6d(d-2s)}{3(d^2+2s^2)+9d(1-s)-2(5s+4)}}} \\ &\leq \sum_{j=1}^{M} \sum_{\substack{k=1 \\ k\neq j}}^{M} \|v^k(t-t_n^k, x-x^k)\|^{p-1}_{L_t^{\frac{6}{1-s}} L_x^{\frac{6d}{3d-4s-2}}} \|v^j(t-t_n^j, x-x^j)\|_{L_t^{\frac{4}{1-s}} L_x^{\frac{2d}{d-s-1}}}. \end{aligned}$$

Here, we used the following Hölder split

$$\frac{(p-1)(1-s)}{6} + \frac{1-s}{4} = \frac{(8+3d-6s)(1-s)}{12(d-2s)},$$

$$\frac{(p-1)(3d-4s-2)}{6d} + \frac{d-s-1}{2d} = \frac{3(d^2+2s^2)+9d(1-s)-2(5s+4)}{6d(d-2s)}.$$

Note that either $\{t_n^k\} \to \pm\infty$ or $\{t_n^k\}$ is bounded.

If $\{t_n^j\} \to \pm\infty$, without loss of generality assume $|t_n^k - t_n^j| \to \infty$ as $n \to \infty$ and by adjusting the profiles that $|x_n^k - x_n^j| \to 0$ as $n \to \infty$. Since $v^k \in L_t^{\frac{6}{1-s}} L_x^{\frac{6d}{3d-4s-2}}$ and $v^{j_2} \in L_{I_j}^{\frac{4}{1-s}} L_x^{\frac{2d}{d-s-1}}$, then

$$\|v^k(t-t_n^k, x-x^k)\|^{p-1}_{L_t^{\frac{6}{1-s}} L_x^{\frac{6d}{3d-4s-2}}} \|v^j(t-t_n^j, x-x^j)\|_{L_t^{\frac{4}{1-s}} L_x^{\frac{2d}{d-s-1}}} \to 0.$$

If $\{t_n^j\}$ is bounded, without loss of generality, assume $|x_n^j - x_n^k| \to \infty$ as $n \to \infty$, then

$$\|v^k(t-t_n^k, x-x^k)\|^{p-1}_{L_t^{\frac{6}{1-s}} L_x^{\frac{6d}{3d-4s-2}}} \|v^j(t-t_n^j, x-x^j)\|_{L_t^{\frac{4}{1-s}} L_x^{\frac{2d}{d-s-1}}} \to 0.$$

Thus, in either case we obtain Claim 3.8.

This finishes the proof of Proposition 3.6 □

Observe that (3.23) gives $\dot{H}^1$ asymptotic orthogonality at $t = 0$ and the following lemma extends it to the bounded NLS flow for $0 \leq t \leq T$.

Lemma 3.9 ($\dot{H}^1$ Pythagorean decomposition along the bounded NLS flow)**.** *Suppose ϕ_n is a bounded sequence in $H^1(\mathbb{R}^d)$. Let $T \in (0,\infty)$ be a fixed time. Assume that $u_n(t) \equiv NLS(t)\phi_n$ exists up to time T for all n, and $\lim_{n\to\infty} \|\nabla u_n(t)\|_{L^\infty_{[0,T]}L^2_x} < \infty$. Consider the nonlinear profile decomposition from Proposition 3.6. Denote $W_n^M(t) \equiv NLS(t)W_n^M$. Then for all j, the nonlinear profiles $v^j(t) \equiv NLS(t)\widetilde{\psi}^j$ exist up to time T and for all $t \in [0,T]$,*

$$\|\nabla u_n(t)\|_{L^2}^2 = \sum_{j=1}^{M} \|\nabla v^j(t - t_n^j)\|_{L^2}^2 + \|\nabla W_n^M(t)\|_{L^2_x}^2 + o_n(1), \tag{3.37}$$

where $o_n(1) \to 0$ uniformly on $0 \le t \le T$.

Proof. We use Propositon 3.6 to obtain profiles $\{\widetilde{\psi}^j\}$ and the nonlinear profile decomposition (3.21). Note that $\lim_{n\to\infty} \|\text{NLS}(t)W_n^M\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \to 0$ as $M \to \infty$, so by choosing a large M we can make $\|\text{NLS}(t)W_n^M\|_{\dot{\beta}^0_{S(\dot{H}^s)}}$ small.

Let M_0 be such that for $M \ge M_0$ (and for n large), we have $\|\text{NLS}(t)W_n^M\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \le \delta_{sd}$ (recall δ_{sd} from Proposition 2.13). Reorder the first M_0 profiles and let M_2, $0 \le M_2 \le M$, be such that

1. For each $1 \le j \le M_2$, we have $t_n^j = 0$. Observe that if $M_2 = 0$, there are no j in this case.

2. For each $M_2 + 1 \le j \le M_0$, we have $|t_n^j| \to \infty$. If $M_2 = M_0$, then it means that there are no j in this case.

From Proposition 3.6 and the profile decomposition (3.21) we have that $v^j(t)$ for $j > M_0$ are scattering, and for $M_2+1 \le j \le M_0$ we have $\|v^j(t-t_n^j)\|_{S(\dot{H}^s;[0,T])} \to 0$ as $n \to +\infty$.

In fact, taking $t_n^j \to +\infty$ and $\|v^j(-t)\|_{S(\dot{H}^s;[0,+\infty))} < \infty$, dominated convergence leads to $\|v^j(-t)\|_{L^q_{[0,+\infty)}L^r_x} < \infty$, for $q < \infty$, where (r,q) is an $\dot{H}^s$ admissible pair, and consequently, $\|v^j(t - t_n^j)\|_{L^q_{[0,T]}L^r_x} \to 0$ as $n \to \infty$. As $v^j(t)$ has been constructed via the existence of wave operators to converge in H^1 to a linear flow, the L^r_x decay of the linear flow

$$\|v^j(t-t^j)\|_{L^\infty_{[0,T]}L^r_x} \to 0,$$

with

$$r = \begin{cases} \frac{2d}{d-2s} & d \geq 3 \\ \frac{2}{1-s} & d = 2 \\ \frac{2}{1-2s} & d = 1 \end{cases} \qquad \text{and} \quad s \quad \text{as in} \quad (3.6).$$

Let $B = \max\{1, \lim_n \|\nabla u_n(t)\|_{L^\infty_{[0,T]}L^2_x}\} < \infty$. For each $1 \leq j \leq M_2$, let $T^j \leq T$ be the maximal forward time such that $\|\nabla v^j\|_{L^\infty_{[0,T^j]}L^2_x} \leq 2B$, and $\tilde{T} = \min_{1\leq j \leq M_2} T^j$ or $\tilde{T} = T$ if $M_2 = 0$. It is sufficient to prove that (3.37) holds for $\tilde{T} = T$, since for each $1 \leq j \leq M_2$, we have $T^j = T$, and therefore, $\tilde{T} = T$. Thus, let's consider $[0, \tilde{T}]$. For each $1 \leq j \leq M_2$, we have for $d \geq 3$:

$$\|v^j(t)\|_{S(\dot{H}^s;[0,\tilde{T}])} \lesssim \|v^j\|_{L^{\frac{2}{1-s}}_{[0,\tilde{T}]}L^{\frac{2d}{d-2}}_x} + \|v^j\|_{L^\infty_{[0,\tilde{T}]}L^{\frac{2d}{d-2s}}_x} \tag{3.38}$$

$$\lesssim \|v^j\|_{L^{\frac{2}{1-s}}_{[0,\tilde{T}]}L^\infty_x} \|v^j\|_{L^\infty_{[0,\tilde{T}]}L^{\frac{2d}{d-2}}_x} + \|v^j\|^{1-s}_{L^\infty_{[0,\tilde{T}]}L^2_x} \|v^j\|^s_{L^\infty_{[0,\tilde{T}]}L^{\frac{2d}{d-2}}_x} \tag{3.39}$$

$$\lesssim \left(\tilde{T}^{\frac{1-s}{2}} + c^{1-s}\right)\|\nabla v^j\|_{L^\infty_{[0,\tilde{T}]}L^2_x} \lesssim \langle \tilde{T}^{\frac{1-s}{2}} \rangle B, \tag{3.40}$$

note that (3.38) comes from the "end point" admissible $S(\dot{H}^s)$ Strichartz norms ($L^{\frac{2}{1-s}}_t L^{\frac{2d}{d-2s}}_x$ and $L^\infty_t L^{\frac{2d}{d-2s}}_x$) since all the other $S(\dot{H}^s)$ norms will be bounded by interpolation; the Hölder's inequality yields (3.39) and the Sobolev's embedding $\dot{H}^1(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2s}}(\mathbb{R}^d)$ together with $\|v^j\|_{L^\infty_{[0,\tilde{T}]}L^2_x} = \|\psi^j\|_{L^2_x} \leq \|\phi_n\|_{L^2}$, from (3.23) with $s = 0$, gives (3.40).

For $d = 2$:

$$\|v^j(t)\|_{S(\dot{H}^s;[0,\tilde{T}])} \lesssim \|v^j\|_{L^\infty_{[0,\tilde{T}]} L_x^{\frac{2}{1-s}}} + \|v^j\|_{L^{\frac{2}{1-s}}_{[0,\tilde{T}]} L^r_x} \tag{3.41}$$

$$\lesssim \|v^j\|^{1-s}_{L^\infty_{[0,\tilde{T}]} L^2_x} \|v^j\|_{L^\infty_{[0,\tilde{T}]} L^\infty_x} + \|v^j\|^{1-s}_{L^2_{[0,\tilde{T}]} L^\infty_x} \|v^j\|_{L^\infty_{[0,\tilde{T}]} L^r_x} \tag{3.42}$$

$$\lesssim \|v^j\|^{1-s}_{L^\infty_{[0,\tilde{T}]} L^2_x} \|\nabla v^j\|_{L^\infty_{[0,\tilde{T}]} L^2_x} + \|v^j\|^{1-s}_{L^2_{[0,\tilde{T}]} L^\infty_x} \|v^j\|_{L^\infty_{[0,\tilde{T}]} \dot{H}_x^{1-\frac{2}{r}}} \tag{3.43}$$

$$\lesssim \left(\|v^j\|^{1-s}_{L^\infty_{[0,\tilde{T}]} L^2_x} + \|v^j\|^{1-s}_{L^2_{[0,\tilde{T}]} L^\infty_x} \right) \|\nabla v^j\|_{L^\infty_{[0,\tilde{T}]} L^2_x} \tag{3.44}$$

$$\lesssim \big(\tilde{T}^{\frac{1-s}{2}} + c^{1-s}\big) \|\nabla v^j\|_{L^\infty_{[0,\tilde{T}]} L^2_x} \lesssim \langle \tilde{T}^{\frac{1-s}{2}} \rangle B, \tag{3.45}$$

where $r = \left(\left(\frac{2}{1-s}\right)^+\right)'$. Note that (3.41) comes from the "end point" admissible Strichartz norms ($L^\infty_t L_x^{\frac{2}{1-s}}$ and $L_t^{\frac{2}{1-s}} L^r_x$); Hölder's inequality yields (3.42); the Sobolev's embeddings $\dot{H}^1(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ and $\dot{H}^{1-\frac{2}{r}}(\mathbb{R}^2) \hookrightarrow L^r(\mathbb{R}^2)$ leads to (3.43); since r is large we have the Sobolev's embedding $\dot{H}^1(\mathbb{R}^2) \hookrightarrow \dot{H}^{1-\frac{2}{r}}(\mathbb{R}^2)$, which implies (3.44), and finally, since $\|v^j\|_{L^\infty_{[0,\tilde{T}]} L^2_x} = \|\psi^j\|_{L^2_x} \leq \|\phi_n\|_{L^2}$ by (3.23) with $s = 0$ we get (3.45).

For $d = 1$:

$$\|v^j(t)\|_{S(\dot{H}^s;[0,\tilde{T}])} \lesssim \|v^j\|_{L^\infty_{[0,\tilde{T}]} L_x^{\frac{2}{1-2s}}} + \|v^j\|_{L^{\frac{4}{1-2s}}_{[0,\tilde{T}]} L^\infty_x} \tag{3.46}$$

$$\lesssim \|v^j\|^{1-2s}_{L^\infty_{[0,\tilde{T}]} L^2_x} \|v^j\|_{L^\infty_{[0,\tilde{T}]} L^\infty_x} + \|v^j\|^{\frac{1-2s}{2}}_{L^2_{[0,\tilde{T}]} L^\infty_x} \|v^j\|_{L^\infty_{[0,\tilde{T}]} L^\infty_x} \tag{3.47}$$

$$\lesssim \left(\|v^j\|^{1-2s}_{L^\infty_{[0,\tilde{T}]} L^2_x} + \|v^j\|^{\frac{1-2s}{2}}_{L^2_{[0,\tilde{T}]} L^\infty_x} \right) \|\nabla v^j\|_{L^\infty_{[0,\tilde{T}]} L^2_x}$$

$$\lesssim \big(\tilde{T}^{\frac{1-2s}{2}} + c^{\frac{1-2s}{2}}\big) \|\nabla v^j\|_{L^\infty_{[0,\tilde{T}]} L^2_x} \lesssim \langle \tilde{T}^{\frac{1-2s}{2}} \rangle B, \tag{3.48}$$

note that (3.46) comes from the "end point" admissible Strichartz norms ($L^\infty_t L_x^{\frac{2}{1-2s}}$ and $L_t^{\frac{4}{1-2s}} L^\infty_x$); Hölder's inequality yields (3.47); the Sobolev's embeddings $\dot{H}^1(\mathbb{R}^1) \hookrightarrow L^\infty(\mathbb{R}^1)$ implies (3.47), and finally, $\|v^j\|_{L^\infty_{[0,\tilde{T}]} L^2_x} = \|\psi^j\|_{L^2_x} \leq \|\phi_n\|_{L^2}$ leads to (3.48).

As in the proof of Proposition 3.6, set $\tilde{u}_n(t,x) = \sum_{j=1}^M v^j(t - t^j_n, x - x^j_n)$ and, a linear sum of nonlinear flows of nonlinear profiles $\widetilde{\psi}^j$, $\tilde{e}^M_n = i\partial_t \tilde{u}_n + \Delta \tilde{u}_n +$

$|\tilde{u}_n|^{p-1}\tilde{u}_n$. Thus, for $M > M_0$ we have

Claim 3.7: There exist a constant $A = A(\tilde{T})$ independent of M, and for every M, there exists $n_0 = n_0(M)$ such that if $n > n_0$, then $\|\tilde{u}_n\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \leq A$.

Claim 3.8: For each M and $\epsilon > 0$, there exists $n_1 = n_1(M, \epsilon)$ such that if $n > n_1$, then $\|\tilde{e}_n^M\|_{\dot{\beta}^0_{S'(\dot{H}^{-s})}}$.

Remark 3.10. Note since $u(0) - \tilde{u}_n(0) = \widetilde{W}_n^M$, there exists $M' = M'(\epsilon)$ large enough so that for each $M > M'$ there exists $n_2 = n_2(M)$ such that $n > n_2$ implies

$$\|e^{it\Delta}(u(0) - \tilde{u}_n(0))\|_{\dot{\beta}^0_{S(\dot{H}^s;[0,\tilde{T}])}} \leq \epsilon.$$

We will next apply the long term perturbation argument (Proposition 2.17); note that in Proposition 2.17, $T = +\infty$, while here, it is not necessary. However, T does not form part of the parameter dependence, since ϵ_0 depends only on $A = A(T)$, not on T, that is, there will be dependence on T, but it is only through A.

Thus, the long term perturbation argument (Proposition 2.17) gives us $\epsilon_0 = \epsilon_0(A)$. Selecting an arbitrary $\epsilon \leq \epsilon_0$, and from Remark 3.10 take $M' = M'(\epsilon)$. Now select an arbitrary $M > M'$ and take $n' = \max(n_0, n_1, n_2)$. Then combining claims 3.7 - 3.8, Remark 3.10 and Proposition 3.6, we obtain that for $n > n'(M, \epsilon)$ with $c = c(A) = c(\tilde{T})$ we have

$$\|u_n - \tilde{u}_n\|_{S(\dot{H}^s;[0,\tilde{T}])} \leq c(\tilde{T})\epsilon. \tag{3.49}$$

We will next prove (3.37) for $0 \leq t \leq \tilde{T}$. Recall that for each dyadic number $N \in 2^{\mathbb{Z}}$, $\|v^j(t - t_n^j)\|_{S(\dot{H}^s;[0,\tilde{T}])} \to 0$ as $n \to \infty$ and for each $1 \leq j \leq M_2$, we have $\|\nabla v^j\|_{L^\infty_{[0,T^j]}L^2_x} \leq 2B$. Strichartz estimates imply

$$\|\nabla v^j(t - t_n^j)\|_{L^\infty_{[0,\tilde{T}])}L^2_x} \lesssim \|\nabla v^j(-t_n^j)\|_{L^\infty_{[0,\tilde{T}])}L^2_x},$$

then

$$\begin{aligned}\|\nabla \tilde{u}(t)\|^2_{L^\infty_{[0,\tilde{T}]}L^2_x} &= \sum_{j=1}^{M_2} \|\nabla v^j(t)\|^2_{L^\infty_{[0,\tilde{T}]}L^2_x} + \sum_{j=M_2+1}^{M} \|\nabla v^j(t-t_n^j)\|^2_{L^\infty_{[0,\tilde{T}]}L^2_x} + o_n(1)\\ &\lesssim M_2B^2 + \sum_{j=M_2+1}^{M} \|\nabla \mathrm{NLS}(-t_n^j)\psi^j\|^2_{L^2_x} + o_n(1)\\ &\lesssim M_2B^2 + \|\nabla\phi_n\|^2_{L^2_x} + o_n(1) \lesssim M_2B^2 + B^2 + o_n(1).\end{aligned}$$

Using (3.49), we obtain for $d \geq 3$:

$$\|u_n - \tilde{u}_n\|_{L^\infty_{[0,\tilde{T}]}L^{p+1}_x} \lesssim \|u_n - \tilde{u}_n\|^{\frac{2}{d-2s+2}}_{L^\infty_{[0,\tilde{T}]}L_x^{\frac{2d}{d-2s}}} \|u_n - \tilde{u}_n\|^{\frac{d-2s}{d-2s+2}}_{L^\infty_{[0,\tilde{T}]}L_x^{\frac{2d}{d-2}}} \tag{3.50}$$

$$\lesssim \|u_n - \tilde{u}_n\|^{\frac{2}{d-2s+2}}_{S(\dot{H}^s;[0,\tilde{T}])} \|\nabla(u_n - \tilde{u}_n)\|^{\frac{d-2s}{d-2s+2}}_{L^\infty_{[0,\tilde{T}]}L^2_x} \tag{3.51}$$

$$\lesssim c(\tilde{T})^{\frac{2}{d-2s+2}} (M_2B^2 + B^2 + o(1))^{\frac{d-2s}{d-2s+2}} \epsilon^{\frac{2}{d-2s+2}},$$

in this case, we used Hölder's inequality to get (3.50) and the Sobolev embedding $\dot{H}^1(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2}}(\mathbb{R}^d)$ to obtain (3.51).

For $d = 2$:

$$\|u_n - \tilde{u}_n\|_{L^\infty_{[0,\tilde{T}]}L^{p+1}_x} \lesssim \|u_n - \tilde{u}_n\|^{\frac{1}{2-s}}_{L^\infty_{[0,\tilde{T}]}L_x^{\frac{2}{1-s}}} \|u_n - \tilde{u}_n\|^{\frac{1-s}{2-s}}_{L^\infty_{[0,\tilde{T}]}L^\infty_x} \tag{3.52}$$

$$\lesssim \|u_n - \tilde{u}_n\|^{\frac{1}{2-s}}_{S(\dot{H}^s;[0,\tilde{T}])} \|\nabla(u_n - \tilde{u}_n)\|^{\frac{1-s}{2-s}}_{L^\infty_{[0,\tilde{T}]}L^2_x} \tag{3.53}$$

$$\lesssim c(\tilde{T})^{\frac{1}{2-s}} (M_2B^2 + B^2 + o(1))^{\frac{1-s}{2-s}} \epsilon^{\frac{1}{2-s}},$$

here, we used Hölder's inequality to get (3.52) and the Sobolev embedding $\dot{H}^1(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ to obtain (3.53).

For $d = 1$:

$$\|u_n - \tilde{u}_n\|_{L^\infty_{[0,\tilde{T}]}L^{p+1}_x} \lesssim \|u_n - \tilde{u}_n\|^{\frac{2}{3-2s}}_{L^\infty_{[0,\tilde{T}]}L_x^{\frac{4}{1-2s}}} \|u_n - \tilde{u}_n\|^{\frac{1-2s}{3-2s}}_{L^\infty_{[0,\tilde{T}]}L^\infty_x} \tag{3.54}$$

$$\lesssim \|u_n - \tilde{u}_n\|^{\frac{2}{3-2s}}_{S(\dot{H}^s;[0,\tilde{T}])} \|\nabla(u_n - \tilde{u}_n)\|^{\frac{1-2s}{3-2s}}_{L^\infty_{[0,\tilde{T}]}L^2_x} \tag{3.55}$$

$$\lesssim c(\tilde{T})^{\frac{2}{3-2s}} (M_2B^2 + B^2 + o(1))^{\frac{1-2s}{3-2s}} \epsilon^{\frac{2}{3-2s}},$$

here, we used Hölder's inequality to get (3.54) and the Sobolev embedding $\dot{H}^1(\mathbb{R}^1) \hookrightarrow L^\infty(\mathbb{R}^1)$ to obtain (3.55).

Similar to the argument in the proof of (3.33), we establish that for $0 \leq t \leq \tilde{T}$

$$\|u_n(t)\|_{L^{p+1}}^{p+1} = \sum_{j=1}^{M} \|v^j(t - t_n^j)\|_{L^{p+1}}^{p+1} + \|W_n^M(t)\|_{L^{p+1}}^{p+1} + o_n(1). \tag{3.56}$$

Energy conservation and (3.24) give us

$$\begin{aligned} E[u_n(t)] &= \sum_{j=1}^{M} E[v^j(t - t^j)] + E[W_n^M] + o_n(1) \\ &= \sum_{j=1}^{M} E[\psi^j] + E[W_n^M] + o_n(1). \end{aligned} \tag{3.57}$$

Combining (3.56) and (3.57), completes the proof of (3.37). □

We now have all the profile decomposition tools to apply to our particular situation in part I (a) of Theorem 1.6*.

Proposition 3.11 (Existence of a critical solution.). *Let* $\theta = \frac{1-s}{s}$, *with* $0 < s < 1$. *There exists a global* ($T = +\infty$) H^1 *solution* $u_{\rm c}(t) \in H^1(\mathbb{R}^d)$ *with initial datum* $u_{c,0} \in H^1(\mathbb{R}^d)$ *such that*

$$\|u_{\rm c,0}\|_{L^2} = 1, \qquad E[u_{\rm c}] = (ME)_{\rm c} < M[u_Q]^\theta E[u_Q],$$

$$\mathcal{G}_{u_{\rm c}}(t) < 1 \qquad \text{for all} \qquad 0 \leq t < +\infty,$$

$$\|u_{\rm c}\|_{\dot{\beta}^0_{S(\dot{H}^s)}} = +\infty. \tag{3.58}$$

Proof. Consider a sequence of solutions $u_n(t)$ to $\mathrm{NLS}_p^+(\mathbb{R}^d)$ with corresponding initial data $u_{n,0}$ such that $\mathcal{G}_{u_n}(0) < 1$ and $M[u_n]^\theta E[u_n] \searrow (ME)_{\rm c}$ as $n \to +\infty$, for which $SC(u_{n,0})$ does not hold for any n.

Without lost of generality, rescale the solutions so that $\|u_{n,0}\|_{L^2} = 1$, thus,

$$\|\nabla u_{n,0}\|_{L^2} < \|u_Q\|_{L^2}^\theta \|\nabla u_Q\|_{L^2} \quad \text{and} \quad E[u_n] \searrow (ME)_{\rm c}.$$

By construction, $\|u_n\|_{\dot{\beta}^0_{S(\dot{H}^s)}} = +\infty$. Note that the sequence $\{u_{n,0}\}$ is uniformly bounded on H^1. Thus, applying the nonlinear profile decomposition (Proposition 3.6), we have

$$u_{n,0}(x) = \sum_{j=1}^{M} \mathrm{NLS}(-t_n^j)\widetilde{\psi}^j(x - x_n^j) + \widetilde{W}_n^M(x). \tag{3.59}$$

Now we will refine the profile decomposition property (b) in Proposition 3.6 by using part II of Proposition 2.25 (wave operator), since it is specific to our particular setting here.

Recall that in nonlinear profile decomposition we consider 2 cases when $|t_n^j| \to \infty$ and $|t_n^j|$ is bounded. In the first case, we can refine it to the following.

First note that we can obtain $\widetilde{\psi}^j$ (from linear ψ^j) such that

$$\|\mathrm{NLS}(-t_n^j)\widetilde{\psi}^j - e^{-it_n^j\Delta}\psi^j\|_{H^1} \to 0 \qquad \text{as} \qquad n \to +\infty$$

with properties (2.70), since the linear profiles ψ^j's satisfy

$$\|\psi\|_{L^2}^{2(1-s)}\|\nabla\psi\|_{L^2}^{2s} < \sigma^2 \left(\frac{d}{s}\right)^s M[u_Q]^{1-s}E[u_Q]^s,$$

and since

$$\sum_{j=1}^{M} M[e^{-it^j\Delta}\psi^j] + \lim_{n\to+\infty} M[W_n^M] = \lim_{n\to+\infty} M[u_{n,0}] = 1.$$

$$\sum_{j=1}^{M} \lim_{n\to+\infty} E[e^{-it_n^j\Delta}\psi^j] + \lim_{n\to+\infty} E[W_n^M] = \lim_{n\to+\infty} E[u_{n,0}] = (ME)_{\mathrm{c}},$$

we also have,

$$\frac{1}{2}\|\psi^j\|_{L^2}^{\theta}\|\nabla\psi^j\|_{L^2} \leq (ME)_{\mathrm{c}}.$$

The properties (2.70) for $\widetilde{\psi}^j$ imply that $(\mathcal{ME}[\widetilde{\psi}^j])^{\frac{1}{s}} < (ME)_{\mathrm{c}}$, and thus, we get that

$$\|\mathrm{NLS}(t)\widetilde{\psi}^j(\cdot - x_n^j)\|_{\dot{\beta}^0_{S(\dot{H}^s)}} < +\infty. \tag{3.60}$$

This fact will be essential for case 1 below. Otherwise, in nonlinear decomposition 3.59 we also have the Pythagorean decomposition for mass and energy:

$$\sum_{j=1}^{M} \lim_{n\to+\infty} E[\widetilde{\psi}^j] + \lim_{n\to+\infty} E[\widetilde{W}_n^M] = \lim_{n\to+\infty} E[u_{n,0}] = (ME)_{\text{c}}.$$

Since each energy is greater than 0 (Lemma 2.23), for all j we obtain

$$E[\widetilde{\psi}^j] \leq (ME)_{\text{c}}. \tag{3.61}$$

Furthermore, $s = 0$ in (3.23) imply

$$\sum_{j=1}^{M} M[\widetilde{\psi}^j] + \lim_{n\to+\infty} M[\widetilde{W}_n^M] = \lim_{n\to+\infty} M[u_{n,0}] = 1. \tag{3.62}$$

We show that in the profile decomposition (3.59) either more than one profiles $\widetilde{\psi}^j$ are non-zero, or only one profile $\widetilde{\psi}^j$ is non-zero and the rest $(M-1)$ profiles are zero. The first case will give a contradiction to the fact that each $u_n(t)$ does not scatter, consequently, only the second possibility holds. That non-zero profile $\widetilde{\psi}^j$ will be the initial data $u_{c,0}$ and will produce the critical solutiton $u_{\text{c}}(t) = \text{NLS}(t)u_{c,0}$, such that $\|u_{\text{c}}\|_{\dot{\beta}^0_{S(\dot{H}^s)}} = +\infty$.

Case 1: More than one $\widetilde{\psi}^j \neq 0$. For each j, (3.62) gives $M[\widetilde{\psi}^j] < 1$ and for a large enough n, (3.61) and (3.62) yield

$$M[\text{NLS}(t)\widetilde{\psi}^j]^\theta E[\text{NLS}(t)\widetilde{\psi}^j] = M[\widetilde{\psi}^j]^\theta E[\widetilde{\psi}^j] < (ME)_{\text{c}}.$$

Recall (3.60), we have

$$\|\text{NLS}(t-t^j)\widetilde{\psi}^j(\cdot - x_n^j)\|_{\dot{\beta}^0_{S(\dot{H}^s)}} < +\infty, \qquad \text{for large enough } n,$$

and thus, the right hand side in (3.59) is finite in $S(\dot{H}^s)$, since (3.22) holds for the remainder $\widetilde{W}_n^M(x)$. This contradicts the fact that $\|\text{NLS}(t)u_{n,0}\|_{\dot{\beta}^0_{S(\dot{H}^s)}} = +\infty$.

Case 2: Thus, we have that only one profile $\widetilde{\psi}^j$ is non-zero, renamed to be $\widetilde{\psi}^1$,

$$u_{n,0} = \text{NLS}(-t_n^1)\widetilde{\psi}^1(\cdot - x_n^1) + \widetilde{W}_n^1, \tag{3.63}$$

with

$$M[\widetilde{\psi}^1] \le 1, \quad E[\widetilde{\psi}^1] \le (ME)_{\rm c} \quad \text{and} \quad \lim_{n\to+\infty} \|\mathrm{NLS}(t)\widetilde{W}_n^1\|_{\dot{\beta}^0_{S(\dot{H}^s)}} = 0.$$

Let $u_{\rm c}$ be the solution to $\mathrm{NLS}_p^+(\mathbb{R}^d)$ with the initial condition $u_{c,0} = \widetilde{\psi}^1$. Applying NLS$(t)$ to both sides of (3.63) and estimating it in $\dot{\beta}^0_{S(\dot{H}^s)}$, we obtain (by the nonlinear profile decomposition Proposition 3.6) that

$$\begin{aligned} \|u_{\rm c}\|_{\dot{\beta}^0_{S(\dot{H}^s)}} &= \|\mathrm{NLS}(t-t_n^1)\widetilde{\psi}^1\|_{\dot{\beta}^0_{S(\dot{H}^s)}} = \lim_{n\to\infty} \|\mathrm{NLS}(t)u_{n,0}\|_{\dot{\beta}^0_{S(\dot{H}^s)}} \\ &= \lim_{n\to\infty} \|u_n(t)\|_{\dot{\beta}^0_{S(\dot{H}^s)}} = +\infty, \end{aligned}$$

since by construction $\|u_n\|_{\dot{\beta}^0_{S(\dot{H}^s)}} = +\infty$, completing the proof. □

Lemma 3.12 (Precompactness of the flow of the critical solution)**.** *Assume $u_{\rm c}$ as in Proposition 3.11. Then there exists a continuous path $x(t)$ in $\mathbb{R}^d$ such that*

$$K = \{u_{\rm c}(\cdot - x(t), t) | t \in [0, +\infty)\}$$

is precompact in $H^1(\mathbb{R}^d)$.

Proof. Let a sequence $\tau_n \to +\infty$ and $\phi_n = u_{\rm c}(\tau_n)$ be a uniformly bounded sequence in H^1; we want to show that $u_{\rm c}(\tau_n)$ has a convergent subsequence in H^1.

The nonlinear profile decomposition (Proposition 3.6) implies the existence of profiles $\widetilde{\psi}^j$, the time and space sequences$\{t_n^j\}$, $\{x_n^j\}$ and an error $\widetilde{W}_n^M$ such that

$$u_{\rm c}(\tau_n) = \sum_{j=1}^{M} \mathrm{NLS}(-t_n^j)\widetilde{\psi}^j(x - x_n^j) + \widetilde{W}_n^M(x), \tag{3.64}$$

with $|t_n^j - t_n^k| + |x_n^j - x_n^k| \to +\infty$ as $n \to +\infty$ for fixed $j \neq k$. In addition,

$$\sum_{j=1}^{M} E[\widetilde{\psi}^j] + E[\widetilde{W}_n^M] = E[u_{\rm c}] = (ME)_{\rm c},$$

since each energy is nonnegative, we have

$$\lim_{n\to\infty} E[\mathrm{NLS}(-t_n^j)\widetilde{\psi}^j(x - x_n^j)] \le (ME)_{\rm c}.$$

Taking $s=0$ in (3.23)

$$\sum_{j=1}^{M} M[\widetilde{\psi}^j(x-x_n^j)] + \lim_{n\to\infty} \|\widetilde{W}_n^M\|_{L^2}^2 = M[u_c] = 1.$$

Note that, in the decomposition (3.64) either have more than one $\widetilde{\psi}^j \neq 0$ or only one $\widetilde{\psi}^1 \neq 0$ and $\widetilde{\psi}^j = 0$ for all $2 \leq j < M$. Following the argument of Proposition 3.11, we show that only the second case occurs:

$$u_{\rm c}(\tau_n) = \mathrm{NLS}(-t_n^1)\widetilde{\psi}^1(x-x_n^1) + \widetilde{W}_n^1(x) \tag{3.65}$$

such that

$$M[\widetilde{\psi}^1] = 1, \qquad \lim_{n\to\infty} E[\mathrm{NLS}(-t_n^1)\widetilde{\psi}^j(x-x_n^1)] = (ME)_{\rm c},$$

$$\lim_{n\to\infty} M[\widetilde{W}_n^M] = 0 \qquad \text{and} \qquad \lim_{n\to\infty} E[\widetilde{W}_n^M] = 0.$$

Lemma 2.23 implies

$$\lim_{n\to\infty} \|\widetilde{W}_n^M\|_{H^1} = 0. \tag{3.66}$$

The sequence x_1^n will create a path $x(t)$ by continuity. We now show that t_1^n has a convergence subsequence $\widetilde{t_1^n}$.

Assume that $\widetilde{t_n^1} \to -\infty$, apply NLS$(t)$ to (3.65) implies then triangle inequality yields

$$\begin{aligned}\|\mathrm{NLS}(t)u_{\rm c}(\tau_n)\|_{\dot\beta^0_{S(\dot H^s;[0,+\infty))}} \leq & \|\mathrm{NLS}(t-\widetilde{t_n^1})\widetilde{\psi}^1(x-x_n^1)\|_{\dot\beta^0_{S(\dot H^s;[0,+\infty))}} \\ & + \|\mathrm{NLS}(t)\widetilde{W}_n^M(x)\|_{\dot\beta^0_{S(\dot H^s;[0,+\infty))}}.\end{aligned}$$

Note

$$\begin{aligned}&\lim_{n\to+\infty} \|\mathrm{NLS}(t-\widetilde{t_n^1})\widetilde{\psi}^1(x-x_n^1)\|_{\dot\beta^0_{S(\dot H^s;[0,+\infty))}} \\ &\qquad = \lim_{n\to+\infty} \|\mathrm{NLS}(t)\widetilde{\psi}^1(x-x_n^1)\|_{\dot\beta^0_{S(\dot H^s;[t_n^1,+\infty))}} = 0,\end{aligned}$$

and

$$\|\mathrm{NLS}(t)\widetilde{W}_n^M\|_{\dot\beta^0_{S(\dot H^s)}} \leq \frac{1}{2}\,\delta_{\rm sd},$$

thus, taking n sufficiently large, the small data scattering theory (Proposition 2.13) implies $\|u_{\rm c}\|_{\dot\beta^0_{S(\dot H^s;(-\infty,\tau_n))}} \leq \delta_{\rm sd}$ a contradiction.

In a similar fashion, assuming that $\widetilde{t^1_n} \to +\infty$, we obtain that for n large, $\|\mathrm{NLS}(t)u_{\rm c}(\tau_n)\|_{\dot\beta^0_{S(\dot H^s;(-\infty,0])}} \leq \frac{1}{2}\delta_{\rm sd}$, and thus, the small data scattering theory (Proposition 2.13) shows that

$$\|u_{\rm c}\|_{\dot\beta^0_{S(\dot H^s;(-\infty,\tau_n])}} \leq \delta_{\rm sd}. \tag{3.67}$$

Taking $n \to +\infty$ implies $\tau_n \to +\infty$, thus (3.67) becomes $\|u_{\rm c}\|_{\dot\beta^0_{S(\dot H^s;(-\infty,+\infty))}} \leq \delta_{\rm sd}$, a contradiction. Thus, $\widetilde{t^1_n}$ must converge to some finite t^1.

Since (3.66) holds and $\mathrm{NLS}(\tilde{t^1_n})\widetilde{\psi}^1 \to \mathrm{NLS}(t^1)\widetilde{\psi}^1$ in H^1, (3.65) implies $u_{\rm c}(\tau_n)$ converges in H^1. □

Corollary 3.13. *(Precompactness of the flow implies uniform localization.) Assume u is a solution to* (1.1) *such that*

$$K = \{u(\cdot - x(t), t) | t \in [0, +\infty)\}$$

is precompact in $H^1(\mathbb{R}^d)$. Then for each $\epsilon > 0$, there exists $R > 0$, so that for all $0 \leq t < \infty$

$$\int_{|x+x(t)|>R} |\nabla u(x,t)|^2 + |u(x,t)|^2 + |u(x,t)|^{p+1} dx < \epsilon. \tag{3.68}$$

Furthermore, $\|u(t, \cdot - x(t))\|_{H^1(|x|>R)} < \epsilon$.

Proof. Assume (3.68) does not hold, i.e., there exists $\epsilon > 0$ and a sequence of times t_n such that for any $R > 0$, we have

$$\int_{|x+x(t_n)|>R} |\nabla u(x,t_n)|^2 + |u(x,t_n)|^2 + |u(x,t_n)|^{p+1}\, dx \geq \epsilon.$$

Changing variables, we get

$$\int_{|x|>R} |\nabla u(x-x(t_n),t_n)|^2 + |u(x-x(t_n),t_n)|^2 + |u(x-x(t_n),t_n)|^{p+1}\, dx \geq \epsilon. \tag{3.69}$$

Note that since K is precompact, there exists $\phi \in H^1$ such that, passing to a subsequence of t_n, we have $u(\cdot - x(t_n), t_n) \to \phi$ in H^1. For all $R > 0$, (3.69) implies

$$\forall R > 0, \quad \int_{|x|>R} |\nabla\phi(x)|^2 + |\phi(x)|^2 + |\phi(x)|^4 \geq \epsilon,$$

which is a contradiction with the fact that $\phi \in H^1$. Thus, (3.68) and $\|u(t, \cdot - x(t))\|_{H^1(|x|>R)} < \epsilon$ hold. □

Lemma 3.14. *Let $u(t)$ be a solution of $NLS_p^+(\mathbb{R}^d)$ defined on $[0, +\infty)$ such that $P[u] = 0$ and either*

(a) $K = \{u(\cdot - x(t), t) | t \in [0, +\infty)\}$ is precompact in $H^1(\mathbb{R}^d)$, or

(b) for all $0 < t$,

$$\|u(t) - e^{i\theta(t)} u_Q(\cdot - x(t))\|_{H^1} \leq \epsilon_1 \tag{3.70}$$

for some continuous function $\theta(t)$ and $x(t)$. Then

$$\lim_{t\to+\infty} \frac{x(t)}{t} = 0. \tag{3.71}$$

Proof. Without loss of generality, suppose $x(0) = 0$.

(a) Assume $K = \{u(\cdot - x(t), t) | t \in [0, +\infty)\}$ is precompact in $H^1(\mathbb{R}^d)$.

We proceed by contradiction assumming that (3.71) does not hold, i.e., there exists a sequence $t_n \to +\infty$ such that $|x(t_n)|/t_n \geq \epsilon_0$ for some $\epsilon_0 > 0$. Given $R > 0$, let

$$t_0(R) = \inf\{t \geq 0 : |x(t)| \geq R\}.$$

Since $x(t)$ is continuous, the value of $t_0(R)$ is well-defined and satisfies

(i) $t_0(R) > 0$,

(ii) $|x(t)| < R$ for $0 \leq t < t_0(R)$, and

(iii) $|x(t_0(R))| = R$.

Let $R_n = |x(t_n)|$ and $\tilde{t}_n = t_0(R_n)$, hence, $t_n \geq \tilde{t}_n$. The assumption $t_n \to +\infty$ and $|x(t_n)|/t_n \geq \epsilon_0$ imply $R_n/\tilde{t}_n \geq \epsilon_0$, $R_n = |x(t_n)| \to +\infty$ and $\tilde{t}_n = t_0(R_n) \to +\infty$; allowing to forget about t_n and work on the time interval $[0, \tilde{t}_n]$ with the properties:

1. for $0 \leq t < \tilde{t}_n$, we have $|x(t)| < R_n$;
2. $|x(\tilde{t}_n)| = R_n$;
3. $\dfrac{R_n}{\tilde{t}_n} \geq \epsilon_0$ and $\tilde{t}_n \to +\infty$.

The precompactness of K and Corollary 3.13 imply that for any $\epsilon > 0$ there exists $R_0(\epsilon) \geq 0$ such that for any $t \geq 0$,

$$\int_{|x+x(t)|\geq R_0(\epsilon)} \left(|u|^2 + |\nabla u|^2\right)\, dx \leq \epsilon. \tag{3.72}$$

Let $\theta(x) \in C^\infty_{comp}(\mathbb{R})$ such that

$$\theta(x) = \begin{cases} x & -1 \leq x \leq 1 \\ 0 & |x| \geq 2^{1/d} \end{cases},$$

$|\theta(x)| \leq |x|$, $\|\theta'\|_\infty \leq 4$, and $\|\theta\|_\infty \leq 2$.

Set $\phi(x) = (\theta(x_1), \theta(x_2), \cdots, \theta(x_d))$, where $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$. Then $\phi(x) = x$ for $|x| \leq 1$ and $\|\phi\|_\infty \leq 2$. For $R > 0$, let $\phi_R(x) = R\phi(x/R)$. Let $z_R : \mathbb{R} \to \mathbb{R}^d$ be the truncated center of mass given by

$$z_R(t) = \int \phi_R(x)\, |u(x,t)|^2\, dx\, .$$

Then $z_R'(t) = ([z_R'(t)]_1, [z_R'(t)]_2, \cdots, [z_R'(t)]_d)$, where

$$[z_R'(t)]_j = 2\,\mathrm{Im} \int \theta'(x_j/R)\, \partial_j u\, \bar{u}\, dx.$$

Since $\theta'(x_j/R) = 1$ for $|x_j| \le 1$, the zero momentum property implies

$$\operatorname{Im} \int_{|x_j| \le R} \partial_j u \, \bar{u} = -\operatorname{Im} \int_{|x_j| > R} \partial_j u \, \bar{u},$$

and thus,

$$[z_R'(t)]_j = -2 \operatorname{Im} \int_{|x_j| \ge R} \partial_j u \; \bar{u} \, dx + 2 \operatorname{Im} \int_{|x_j| \ge R} \theta'(x_j/R) \partial_j u \; \bar{u} \, dx \,,$$

and Cauchy-Schwarz yields

$$|z_R'(t)| \le 5 \int_{|x| \ge R} (|\nabla u|^2 + |u|^2) dx \,. \tag{3.73}$$

Let $\tilde{R}_n = R_n + R_0(\epsilon)$. For $|x| > \tilde{R}_n$ and $0 \le t \le \tilde{t}_n$ we have $|x + x(t)| \ge \tilde{R}_n - R_n = R_0(\epsilon)$, therefore (3.73) and (3.72) yield

$$|z_{\tilde{R}_n}'(t)| \le 5 \, \epsilon. \tag{3.74}$$

Note that

$$z_{\tilde{R}_n}(0) = \int_{|x| < R_0(\epsilon)} \phi_{\tilde{R}_n}(x) \, |u_0(x)|^2 \, dx + \int_{|x + x(0)| \ge R_0(\epsilon)} \phi_{\tilde{R}_n}(x) |u_0(x)|^2 \, dx \,,$$

thus, (3.72) implies

$$|z_{\tilde{R}_n}(0)| \le R_0(\epsilon) M[u] + 2 \tilde{R}_n \, \epsilon. \tag{3.75}$$

For $0 \le t \le \tilde{t}_n$, we split $z_{\tilde{R}_n}(t)$ as

$$\begin{aligned} z_{\tilde{R}_n}(t) &= \int_{|x + x(t)| \ge R_0(\epsilon)} \phi_{\tilde{R}_n}(x) \, |u(x,t)|^2 \, dx + \int_{|x + x(t)| \le R_0(\epsilon)} \phi_{\tilde{R}_n}(x) \, |u(x,t)|^2 \, dx. \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

To estimate I, observe that $|\phi_{\tilde{R}_n}(x)| \le 2 \tilde{R}_n$, combining it with (3.72), yields

$$|\mathrm{I}| = \Big| \int_{|x + x(t)| \ge R_0(\epsilon)} \phi_{\tilde{R}_n}(x) \, |u(x,t)|^2 \, dx \Big| \le 2 \tilde{R}_n \epsilon.$$

To estimate II, note that $|x| \le |x + x(t)| + |x(t)| \le R_0(\epsilon) + R_n = \tilde{R}_n$, therefore, $\phi_{\tilde{R}_n}(x) = x$, and we rewrite II as

$$\begin{aligned} \mathrm{II} &= \int_{|x + x(t)| \le R_0(\epsilon)} (x + x(t)) \, |u(x,t)|^2 \, dx - x(t) \int_{|x + x(t)| \le R_0(\epsilon)} |u(x,t)|^2 \, dx \\ &= \int_{|x + x(t)| \le R_0(\epsilon)} (x + x(t)) \, |u(x,t)|^2 \, dx - x(t) M[u] + x(t) \int_{|x + x(t)| \ge R_0(\epsilon)} |u(x,t)|^2 \, dx \\ &= \mathrm{IIA} + \mathrm{IIB} + \mathrm{IIC}. \end{aligned}$$

Note $|\mathrm{IIA}| \le R_0(\epsilon)M[u]$, and (3.72) yields $|\mathrm{IIC}| \le |x(t)|\epsilon \le \tilde{R}_n\epsilon$. Thus,

$$\begin{aligned} |z_{\tilde{R}_n}(t)| &\ge |\mathrm{IIB}| - |\mathrm{I}| - |\mathrm{IIA}| - |\mathrm{IIC}| \\ &\ge |x(t)|M[u] - R_0(\epsilon)M[u] - 3\tilde{R}_n\epsilon\,. \end{aligned}$$

Taking $t = \tilde{t}_n$, we get

$$|z_{\tilde{R}_n}(\tilde{t}_n)| \ge \tilde{R}_n(M[u] - 3\epsilon) - R_0(\epsilon)M[u]\,. \tag{3.76}$$

Combining (3.74), (3.75), and (3.76), we have

$$\begin{aligned} 5\,\epsilon\,\tilde{t}_n \ge \int_0^{\tilde{t}_n} |z'_{\tilde{R}_n}(t)|\,dt \ge \left|\int_0^{\tilde{t}_n} z'_{\tilde{R}_n}(t)\,dt\right| \ge |z_{\tilde{R}_n}(\tilde{t}_n) - z_{\tilde{R}_n}(0)| \\ \ge \tilde{R}_n(M[u] - 5\epsilon) - 2R_0(\epsilon)M[u]\,. \end{aligned}$$

Dividing by $\tilde{t}_n$ and using that $\tilde{R}_n \ge R_n$ (assume $\epsilon \le \frac{1}{5}M[u]$), we obtain

$$5\,\epsilon \ge \frac{R_n}{\tilde{t}_n}(M[u] - 5\epsilon) - \frac{2R_0(\epsilon)M[u]}{\tilde{t}_n}\,.$$

Since $R_n/\tilde{t}_n \ge \epsilon_0$, we have

$$5\,\epsilon \ge \epsilon_0(M[u] - 5\epsilon) - \frac{2R_0(\epsilon)M[u]}{\tilde{t}_n}\,.$$

Take $\epsilon = M[u]\epsilon_0/11$ (assume $\epsilon_0 \le 1$), and then send $n \to +\infty$. Since $\tilde{t}_n \to +\infty$, we get a contradiction.

(b) For all $t > 0$,

$$\|u(t) - e^{i\theta(t)}u_Q(\cdot - x(t))\|_{H^1} \le \epsilon_1$$

for some continuous function $\theta(t)$ and $x(t)$.

Let $R(T) = \max\limits_{0\le t\le T} |x(t)|$. It suffices to prove that there exists an absolute constant $c > 0$ such that for each T with $R(T) = |x(T)| \gg 1$, we have

$$|x(T)| \le c\,T\left(e^{-|x(T)|} + \epsilon\right)^2\,. \tag{3.77}$$

Fix $T > 0$, thus $|x(t)| \leq R(T)$ for $0 \leq t \leq T$, and by (3.73) there is an absolute constant c_1 such that

$$|z'_{2R(T)}(t)| \leq c_1 \int_{|x|\geq 2R(T)} \left(|\nabla u(t)|^2 + |u(t)|^2\right) dx.$$

In addition, for all $0 \leq t \leq T$, (3.70) implies

$$|z'_{2R(T)}(t)| \leq c_2 \left(\epsilon + \|Q\|_{H^1(|x|\geq R(T))}\right)^2 .$$

Due to the exponential decay at ∞ of $Q(x)$, we have (upon integrating the above inequality over $[0, T]$) the bound

$$|z_{2R(T)}(t) - z_{2R(T)}(0)| \leq c_3 T \left(\epsilon + e^{-R(T)}\right)^2 . \tag{3.78}$$

Since $|x(T)| = R(T)$, there exists an absolute constant c_4 such that

$$|z_{2R(T)}(T)| \geq c_4 R(T) . \tag{3.79}$$

In addition, $x(0) = 0$ implies

$$|z_{2R(T)}(0)| \leq c_5 \left(1 + R(T)\epsilon^2\right) . \tag{3.80}$$

By combining (3.78), (3.79), and (3.80), we obtain (3.77).

□

Theorem 3.15. *(Rigidity Theorem.) Let $u_0 \in H^1$ satisfy $P[u_0] = 0$, $\mathcal{ME}[u_0] < 1$ and $\mathcal{G}_u(0) < 1$. Let u be the global $H^1(\mathbb{R}^d)$ solution of $NLS_p^+(\mathbb{R}^d)$ with initial data u_0 and suppose that $K = \{u_c(\cdot - x(t), t) | t \in [0, +\infty)\}$ is precompact in H^1, then $u_0 \equiv 0$.*

Proof. Let $\phi \in C_0^\infty$ be radial, with

$$\phi(x) = \begin{cases} |x|^2 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| \geq 2. \end{cases}$$

For $R > 0$ define

$$z_R(t) = \int R^2 \phi(\frac{x}{R}) |u(x,t)|^2 dx. \tag{3.81}$$

Then

$$z_R'(t) = 2\,\mathrm{Im} \int R \nabla \phi(\frac{x}{R}) \cdot \nabla u(t) \bar{u}(t) dx, \tag{3.82}$$

and Hölder's inequality yields

$$|z_R'(t)| \le cR \int_{\{|x| \le 2R\}} |\nabla u(t)||u(t)| dx \le cR \|u(t)\|_{L^2}^{2(1-s)} \|\nabla u(t)\|_{L^2}^{2s}. \tag{3.83}$$

Note that,

$$\begin{aligned} z_R''(t) = 4 \sum_{j,k} \int \frac{\partial^2 \phi}{\partial x_j \partial x_k} \left(\frac{|x|}{R} \right) \frac{\partial u}{\partial x_j} \frac{\partial \bar{u}}{\partial x_k} - \frac{1}{R^2} \int \Delta^2 \phi \left(\frac{|x|}{R} \right) |u|^2 \\ - 4 \left(\frac{1}{2} - \frac{1}{p+1} \right) \int \Delta \phi \left(\frac{|x|}{R} \right) |u|^{p+1}. \end{aligned} \tag{3.84}$$

Since ϕ is radial, we have

$$z_R''(t) = 8 \int |\nabla u|^2 - \frac{4d(p-1)}{p+1} \int |u|^{p+1} + A_R(u(t)), \tag{3.85}$$

where

$$\begin{aligned} A_R(u(t)) = 4 \sum_j \int \left(\partial_{x_j}^2 \phi \left(\frac{|x|}{R} \right) - 2 \right) \left| \frac{\partial u}{\partial x_j} \right|^2 + 4 \sum_{j \ne k} \int_{R \le |x| \le 2R} \frac{\partial^2 \phi}{\partial x_j \partial x_k} \left(\frac{|x|}{R} \right) \\ - \frac{1}{R^2} \int \Delta^2 \phi \left(\frac{|x|}{R} \right) |u|^2 - 4 \left(\frac{1}{2} - \frac{1}{p+1} \right) \int \left(\Delta \phi \left(\frac{|x|}{R} \right) - 2d \right) |u|^{p+1}. \end{aligned} \tag{3.86}$$

Thus,

$$\big| A_R(u(t)) \big| = c \int_{|x| \ge R} \left(|\nabla u(t)|^2 + \frac{1}{R^2} |u(t)|^2 + |u(t)|^{p+1} \right) dx. \tag{3.87}$$

Choosing R large enough, over a suitably chosen time interval $[t_0, t_1]$, with $0 \ll t_0 \ll t_1 < \infty$, combining (3.85) and (2.65), we obtain

$$|z_R''(t)| \ge 16(1 - \omega^{p-1}) E[u] - |A_R(u(t))|. \tag{3.88}$$

From Corollary 3.13 , letting $\epsilon = \frac{1-\omega^{p-1}}{c}$, with c as in (3.87), we can obtain $R_0 \geq 0$ such that for all t,

$$\int_{|x+x(t)|>R_0} \left(|\nabla u(t)|^2 + |u(t)|^2 + |u(t)|^{p+1}\right) \leq \frac{1-\omega^{p-1}}{c} E[u]. \tag{3.89}$$

Thus combining (3.87), (3.88) and (3.89), and taking $R \geq R_0 + \sup_{t_0 \leq t \leq t_1} |x(t)|$, gives that for all $t_0 \leq t \leq t_1$,

$$|z''(t)| \geq 8(1-\omega^{p-1})E[u]. \tag{3.90}$$

By Lemma 3.14, there exists $t_0 \geq 0$ such that for all $t \geq t_0$, we have $|x(t)| \leq \gamma t$. Taking $R = R_0 + \gamma t_1$, we have that (3.90) holds for all $t \in [t_0, t_1]$. Thus, integrating it over this interval, we obtain

$$|z_R'(t_1) - z_R'(t_0)| \geq 8(1-\omega^{p-1})E[u](t_1 - t_0). \tag{3.91}$$

In addition, for all $t \in [t_0, t_1]$, combining (3.83), $\mathcal{G}_u(0) < 1$, and Lemma 2.24 we have

$$\begin{aligned} |z_R'(t)| &\leq cR\|u(t)\|_{L^2}^{2(1-s)}\|\nabla u(t)\|_{L^2}^{2s} \leq 2cR\|u_Q\|_{L^2}^{2(1-s)}\|\nabla u_Q\|_{L^2}^{2s} \\ &\leq c\|u_Q\|_{L^2}^{2(1-s)}\|\nabla u_Q\|_{L^2}^{2s}(R_0 + \gamma t_1). \end{aligned} \tag{3.92}$$

Combining (3.91) and (3.92) yields

$$8(1-\omega^{p-1})E[u](t_1 - t_0) \leq 2c\|u_Q\|_{L^2}^{2(1-s)}\|\nabla u_Q\|_{L^2}^{2s}(R_0 + \gamma t_1). \tag{3.93}$$

Observe that, ω, and R_0 are constants depending on $\mathcal{ME}[u]$, S, and $t_0 = t(\gamma)$. Let $\gamma = \frac{(1-\omega^{p-1})E[u]}{c\|u_Q\|_{L^2}^{2(1-s)}\|\nabla u_Q\|_{L^2}^{2s}} > 0$, thus, (3.93) yields

$$6(1-\omega^{p-1})E[u]t_1 \leq 2c\|u_Q\|_{L^2}^{2(1-s)}\|\nabla u_Q\|_{L^2}^{2s}R_0 + 8(1-\omega^{p-1})E[u]t_0, \tag{3.94}$$

sending $t_1 \to +\infty$ implies that the left hand side of (3.94) goes to ∞ and the right hand side is bounded which is a contradiction unless $E[u] = 0$ which implies $u \equiv 0$.

□

Chapter 4

WEAK BLOWUP VIA CONCENTRATION COMPACTNESS

In this chapter, we complete the proof of Theorem 1.6* part II (b), i.e., if under the mass-energy threshold $\mathcal{ME}[u] < 1$, a solution $u(t)$ to $\mathrm{NLS}_p^+(\mathbb{R}^d)$ with the initial condition $u_0 \in H^1$ such that $\mathcal{G}_u(0) > 1$ exists globally for all positive time, then there exists a sequence of times $t_n \to +\infty$ such that $\mathcal{G}_u(t_n) \to +\infty$. We call this solution a "weak blowup" solution.

Recall that $u_Q(x,t) = e^{i\beta t}Q(\alpha x)$ is a soliton solution of $\mathrm{NLS}_p^\pm(\mathbb{R}^d)$, where $\alpha = \frac{\sqrt{d(p-1)}}{2}$ and $\beta = 1 - \frac{(d-2)(p-1)}{4}$.

Definition 4.1. Let $\lambda > 0$. The horizontal line for which

$$M[u] = M[u_Q] \qquad \text{and} \qquad \frac{E[u]}{E[u_Q]} = \frac{d}{2s}\lambda^{\frac{2}{s}}\left(1 - \frac{\lambda^{p-1}}{\alpha^2}\right)$$

is called the " *mass-energy*" line for λ.

Notice that in definition 4.1, the renormalized energy definition comes naturally by expressing the energy in term of the gradient which is assumed to be λ. We illustrate the mass-energy line notion in Figure 4.1.

4.1 Outline for Weak blowup via Concentration Compactness

Suppose that there is no finite time blowup for a nonradial and infinite variance solution (from Theorem 1.6* part II), thus, the existence on time (say, in forward direction) is infinite ($T^* = +\infty$). Now, under the assumption of global existence, we study the behavior of $\mathcal{G}_u(t)$ as $t \to +\infty$, and use a concentration compactness type argument for establishing the divergence of $\mathcal{G}_u(t)$ in H^1-norm as it was developed in [Holmer and Roudenko, 2010c], note that the concentration compactness and rigidity argument is not used here for scattering but for a blowup property. The description of this argument is in steps 1, 2 and 3.

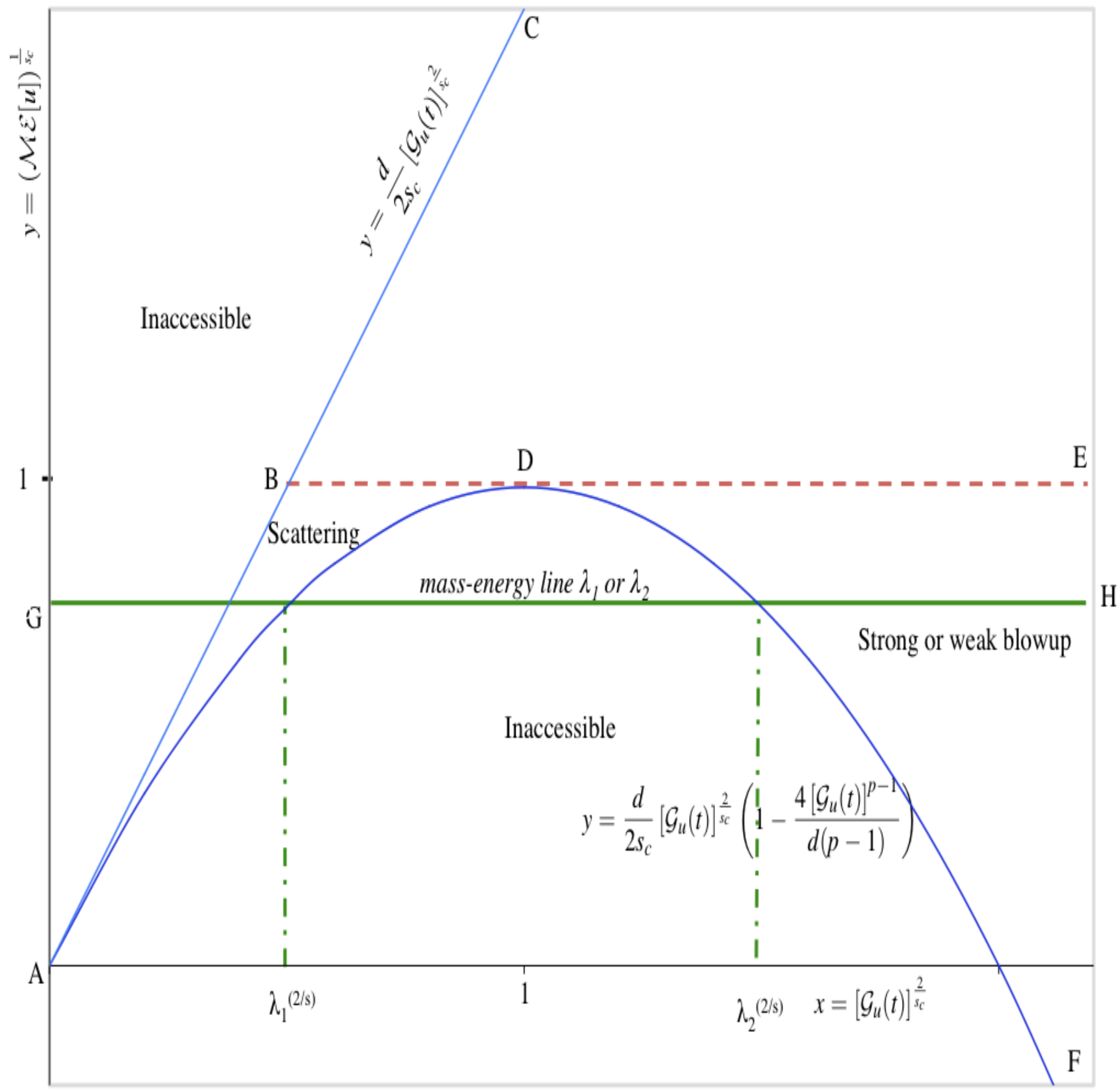

Figure 4.1: This is a graphical representation of restrictions on energy and gradient.For a given $\lambda > 0$ the horizontal line GH is referred to as the "mass-energy" line for this λ. Observe that this horizontal line can intersect the parabola $y = \frac{d}{2s}[\mathcal{G}_u(t)]^{\frac{2}{s}}\left(1 - \frac{[\mathcal{G}_u(t)]^{p-1}}{\alpha^2}\right)$ twice, i.e., it can be a "mass-energy" line for $0 < \lambda_1 < 1$ and $1 < \lambda_2 < \infty$, the first case produces solutions which are global and are scattering (by Theorem 1.6* part I) and the second case produces solutions which either blow up in finite time or diverge in infinite time ("weak blowup") as shown in Chapter 4.

Step 1: Near boundary behavior.

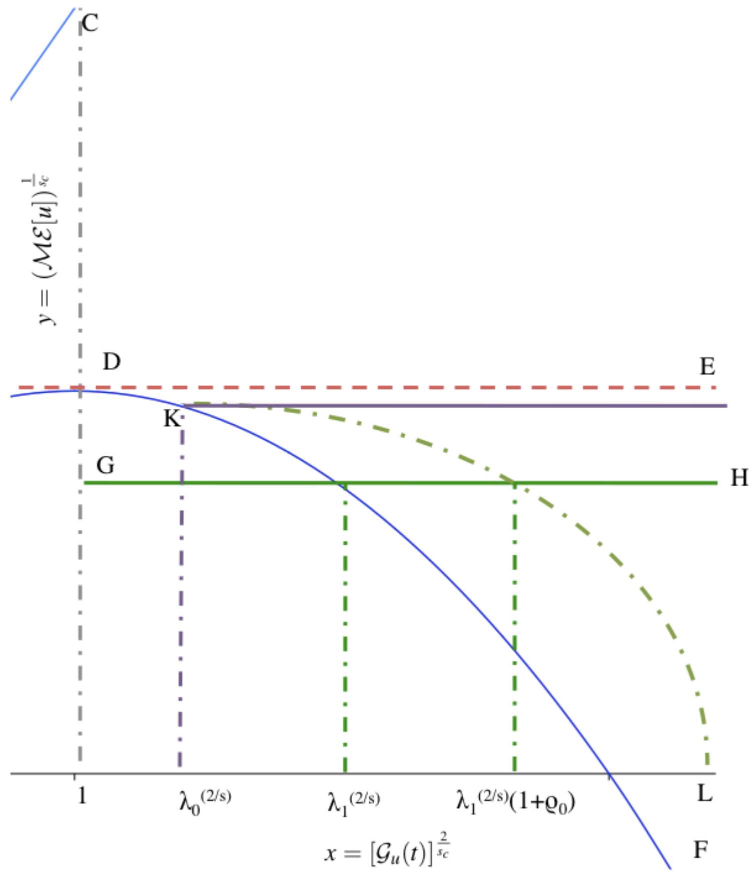

Figure 4.2: Near boundary behavior of $\mathcal{G}(t)$. We investigate whether the solution can remain close to the boundary (see the dash line KL) for all time

Theorem 1.6* II part (a) yields $\mathcal{G}_u(t) > 1$ for all $t \in (T_*, T^*)$ whenever $\mathcal{G}_u(0) > 1$ on the "mass-energy" line for some $\lambda > 1$. We illustrate this in Figure 4.1: given $u_0 \in H^1$, we first determine $M[u_0]$ and $E[u_0]$ which specifies the "mass-energy" line GH. Then the gradient $\mathcal{G}_u(t)$ of a solution $u(t)$ lives on the line GH. Note that $\mathcal{G}_u(t) > \lambda_2 > 1$ if $\mathcal{G}_u(0) > 1$. A natural question is whether $\mathcal{G}_u(t)$ can be, with time, much larger than 1 or λ_2. Proposition 4.6 shows that it can not. Thus, we prove that the renormalized gradient $\mathcal{G}_u(t)$ can not forever remain near the boundary if originally $\mathcal{G}_u(0)$ is very close to it, that is, if $\lambda_0 > 1$, there exists $\rho_0(\lambda_0) > 0$ such that for all $\lambda > \lambda_0$ there is NO solution at the "mass-energy" line for λ satisfying

$$\lambda \le \mathcal{G}_u(t) \le \lambda(1+\rho_0).$$

Using the Figure 4.2, this means that the solution $u(t)$ would have a gradient $\mathcal{G}_u(t)$ very close to the boundary DF (for all times), i.e., between the boundary DF and the dashed line KL. We will show that $\mathcal{G}_u(t)$ on any "mass-energy" line with $\mathcal{ME}[u] < 1$ and $\mathcal{G}_u(0) > 1$ will escape to infinity (along this line). By contradiction, assume that all solutions (starting from some mass-energy line corresponding to the initial renormalized gradient $\mathcal{G}_u(0) = \lambda_0 > 1$) are bounded in renormalized gradient for all $t > 0$.

Step 1 gives the basis for induction, giving that when $\lambda > 1$, any solution $u(t)$ of $\mathrm{NLS}_p^+(\mathbb{R}^d)$ at the "mass-energy" line for this λ can not have a renormalized gradient $\mathcal{G}_u(t)$ bounded near the boundary DF for all time (see Figure 4.2). We will show that $\mathcal{G}_u(t)$, in fact, will tend to $+\infty$ (at least along an infinite time sequence).

Definition 4.2. Let $\lambda > 1$. We say *the property* GBG(λ, σ) *holds*[13] if there exists a solution $u(t)$ of $\mathrm{NLS}_p^+(\mathbb{R}^d)$ at the mass-energy line λ (i.e., $M[u] = M[u_Q]$ and $\frac{E[u]}{E[u_Q]} = \frac{d}{2s}\lambda^{\frac{2}{s}}\left(1 - \frac{\lambda^{p-1}}{\alpha^2}\right)$) such that $\lambda \le \mathcal{G}_u(t) \le \sigma$ for all $t \ge 0$. Figure 4.3 illustrates this definition.

In other words, GBG(λ, σ) is not true if for every solution $u(t)$ of $\mathrm{NLS}_p^+(\mathbb{R}^d)$ at the "mass-energy" line for λ, such that $\lambda \le \mathcal{G}_u(t)$ for all $t > 0$, there exists t^* such that $\sigma < \mathcal{G}_u(t^*)$. Iterating, we conclude that, there exists a sequence $\{t_n\} \to \infty$ with $\sigma < \mathcal{G}_u(t_n)$ for all n.

Note that, if GBG(λ, σ) does not hold, then for any $\sigma' < \sigma$, GBG(λ, σ') does not hold either. This will allow us induct on the GBG notion.

[13]GBG stands for *globally bounded gradient.*

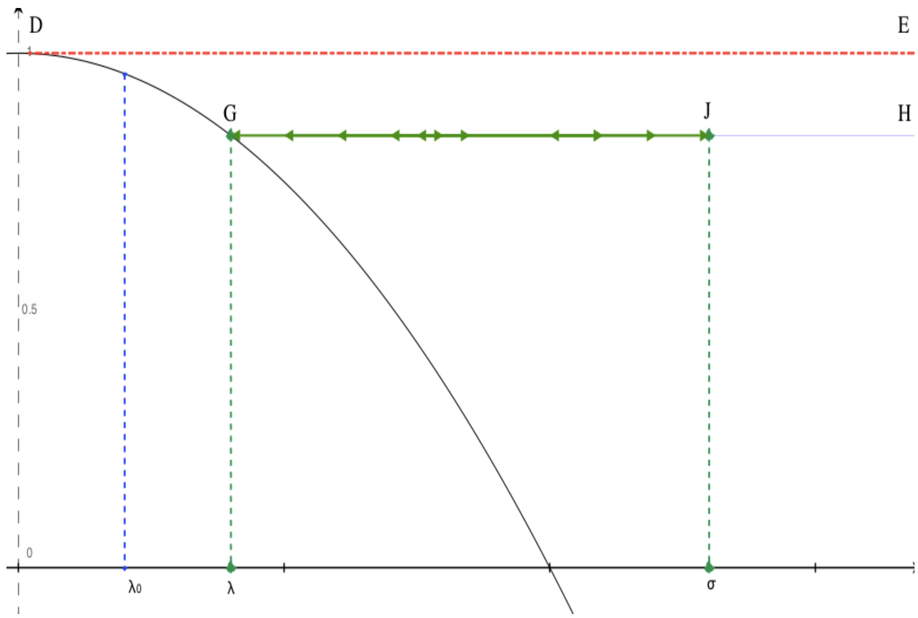

Figure 4.3: In the graph the statement "GBG(λ, σ) *holds*" implies $\mathcal{G}(t)$ only on the segment GJ.

Definition 4.3. Let $\lambda_0 > 1$. We define the critical threshold σ_c by

$$\sigma_c = \sup\Big\{\sigma | \sigma > \lambda_0 \text{ and GBG}(\lambda, \sigma) \text{ does NOT hold for all } \lambda \text{ with } \lambda_0 \leq \lambda \leq \sigma\Big\}.$$

Note that $\sigma_c = \sigma_c(\lambda_0)$ stands for "σ-critical".

From the step 1 (Proposition 4.6) we have that GBG$(\lambda, \lambda(1 + \rho_0(\lambda_0))$ does not hold for all $\lambda \geq \lambda_0$.

Step 2: Induction argument.

Let $\lambda_0 > 1$. We would like to show that $\sigma_c(\lambda_0) = +\infty$. Arguing by contradiction, we assume $\sigma_c(\lambda_0)$ is finite.

Let $u(t)$ be a solution to $\mathrm{NLS}_p^+(\mathbb{R}^d)$ with initial data $u_{n,0}$ at the "mass-energy" line for $\lambda > \lambda_0$, i.e.,$\frac{E[u]}{E[u_Q]} = \frac{d}{2s}\lambda^{\frac{2}{s}}\left(1 - \frac{\lambda^{p-1}}{\alpha^2}\right)$, $M[u] = M[u_Q]$ and $\mathcal{G}_u(0) > 1$. We want to show that there exists a sequence of times $\{t_n\} \to +\infty$ such that $\mathcal{G}_u(t_n) \to \infty$. Suppose the opposite, that is, such sequence of times does not exist.

Then there exists $\sigma < \infty$ satisfying $\lambda \leq \mathcal{G}_u(t) \leq \sigma$ for all $t \geq 0$, i.e., GBG(λ, σ) holds with $\sigma_c(\lambda_0) \leq \sigma < \infty$. At this point we can apply Proposition 3.6 (the nonlinear profile decomposition).

The nonlinear profile decomposition of the sequence $\{u_{n,0}\}$ and profile reordering will allow us to construct a "*critical threshold solution*" $u(t) = u_c(t)$ to $\text{NLS}_p^+(\mathbb{R}^d)$ at the "mass-energy" line λ_c, where $\lambda_0 < \lambda_c < \sigma_c(\lambda_0)$ and $\lambda_c < \mathcal{G}_{u_c}(t) < \sigma_c(\lambda_0)$ for all $t > 0$ (see Existence of threshold solution Lemma 4.8).

Step 3: Localization properties of critical threshold solution.

By construction, the critical threshold solution $u_c(t)$ will have the property that the set $K = \{u(\cdot - x(t), t) | t \in [0, +\infty)\}$ has a compact closure in H^1 (Lemma 4.9). Thus, we will have uniform concentration of $u_c(t)$ in time, which together with the localization property (Corollary 3.13) implies that for a given $\epsilon > 0$, there exists an $R > 0$ such that $\|\nabla u(x,t)\|^2_{L^2(|x+x(t)|>R)} \leq \epsilon$ uniformly in t ; as a consequence, $u_c(t)$ blows up in finite time (Lemma (4.10)), that is, $\sigma_c = +\infty$, which contradicts the fact that $u_c(t)$ is bounded in H^1. Thus, $u_c(t)$ can not exist since our assumption that $\sigma_c(\lambda_0) < \infty$ is false, and this ends the proof of the "weak blowup".

In the rest of this chapter we proceed with the proof of claims described in Step 1, 2 and 3.

First, recall variational characterization of the ground state.

4.2 Variational Characterization of the Ground State

Propositon 4.4 is a restatement of Proposition 4.4 [Holmer and Roudenko, 2010c] adjusted for our general case, and shows that if a solution $u(t,x)$ is close to $u_Q(t,x)$ in mass and energy, then it is close to u_Q in $H^1(\mathbb{R}^d)$, up to a phase and shift in space. The proof is identical so we omit it.

Proposition 4.4. *There exists a function $\epsilon(\rho)$ defined for small $\rho > 0$ with $\lim_{\rho\to 0}\epsilon(\rho) = 0$, such that for all $u \in H^1(\mathbb{R}^d)$ with*

$$\big|\|u\|_{L^{p+1}} - \|u_Q\|_{L^{p+1}}\big| + \big|\|u\|_{L^2} - \|u_Q\|_{L^2}\big| + \big|\|\nabla u\|_{L^2} - \|\nabla u_Q\|_{L^2}\big| \le \rho,$$

there is $\theta_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d$ such that

$$\|u - e^{i\theta_0}u_Q(\cdot - x_0)\|_{H^1} \le \epsilon(\rho). \tag{4.1}$$

The Proposition 4.5 is a variant of Proposition 4.1 [Holmer and Roudenko, 2010c], rephrased for our case.

Proposition 4.5. *There exists a function $\epsilon(\rho)$ such that $\epsilon(\rho) \to 0$ as $\rho \to 0$ satisfying the following: Suppose there exists $\lambda > 0$ such that*

$$\left|(\mathcal{M}\mathcal{E}[u])^{\frac{1}{s}} - \frac{d}{2s}\lambda^{\frac{2}{s}}\left(1 - \frac{\lambda^{p-1}}{\alpha^2}\right)\right| \le \rho\lambda^{\frac{2(p-1)}{s}} \tag{4.2}$$

and

$$\left|[\mathcal{G}_u(t)]^{\frac{1}{s}} - \lambda\right| \le \rho\begin{cases}\lambda^{\frac{2}{s}} & \text{if} \quad \lambda \le 1\\ \lambda & \text{if} \quad \lambda \ge 1\end{cases}. \tag{4.3}$$

Then there exist $\theta_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d$ such that

$$\left\|u(x) - e^{i\theta_0}\lambda\kappa^{-\frac{s}{1-s}}u_Q\big(\lambda(\kappa^{-\frac{3s}{d(1-s)}}x - x_0)\big)\right\|_{L^2} \le \kappa^{\frac{s}{2(1-s)}}\epsilon(\rho),$$

and

$$\left\|\nabla\left[u(x) - e^{i\theta_0}\lambda\kappa^{-\frac{s}{1-s}}u_Q\big(\lambda(\kappa^{-\frac{3s}{d(1-s)}}x - x_0)\right]\right\|_{L^2} \le \lambda\kappa^{-\frac{s}{2(1-s)}}\epsilon(\rho),$$

where $\kappa = \left(\frac{M[u]}{M[u_Q]}\right)^{\frac{1-s}{s}}$.

Proof. Set $v(x) = \kappa^{\frac{s}{1-s}}u(\kappa^{\frac{3s}{(1-s)d}}x)$, hence $M[v] = \kappa^{-\frac{s}{1-s}}M[u]$. Assume $M[v] = M[u_Q]$. Then there exists $\lambda > 0$ such that (4.2) and (4.3) become

$$\left|\frac{E[v]}{E[u_Q]} - \frac{d}{2s}\lambda^{\frac{2}{s}}\left(1 - \frac{\lambda^{p-1}}{\alpha^2}\right)\right| \le \rho_0\lambda^{\frac{2(p-1)}{s}}, \tag{4.4}$$

and

$$\left|\frac{\|\nabla v\|_{L^2}}{\|\nabla u_Q\|_{L^2}}-\lambda\right|\le \rho_0\begin{cases}\lambda^{\frac{2}{s}} \text{ if } \lambda\le 1\\ \lambda \text{ if } \lambda\ge 1\end{cases}. \tag{4.5}$$

Letting $\tilde{u}(x)=\lambda^{-\frac{2((p-1)(d-2)-ds)}{((p-1)(d-2)-4)s}}v(\lambda^{-\frac{2((p-1)(2-s)+2s)}{((p-1)(d-2)-4)s}}x)$, we have

$$\left|\frac{\|\nabla \tilde{u}\|_{L^2}}{\|\nabla u_Q\|_{L^2}}-1\right|\le \rho_0\begin{cases}\lambda^{\frac{2}{s}-1} \text{ if } \lambda\le 1\\ 1 \quad \text{ if } \lambda\ge 1\end{cases}\le\rho_0. \tag{4.6}$$

Combining Pohozhaev identities, (4.4) and (4.5), gives

$$\begin{aligned}\frac{d}{2s\alpha^2}\left|\frac{\|\ v\|_{L^{p+1}}^{p+1}}{\|u_Q\|_{L^{p+1}}^{p+1}}-\lambda^{\frac{2(p-1)}{s}}\right| &\le \left|\frac{E[v]}{E[u_Q]}-\left(\frac{d}{2s}\lambda^{\frac{2}{s}}\left(1-\frac{\lambda^{p-1}}{\alpha^2}\right)\right)\right|\\ &\quad+\frac{d}{2s}\left|\frac{\|\nabla v\|_{L^2}^2}{\|\nabla u_Q\|_{L^2}^2}-\lambda^2\right|\\ &\le \rho_0\left(\lambda^{\frac{2(p-1)}{s}}+\frac{d}{2s}\begin{cases}\lambda^{\frac{2(p-1)}{s}} \text{ if } \lambda\le 1\\ \lambda^{\frac{2}{s}} \text{ if } \lambda\ge 1\end{cases}\right)\\ &\le \frac{d+2s}{2s}\rho_0\lambda^{\frac{2(p-1)}{s}}.\end{aligned}$$

This yields

$$\left|\frac{\|\tilde{u}\|_{L^{p+1}}^{p+1}}{\|u_Q\|_{L^{p+1}}^{p+1}}-1\right|\le\frac{\alpha^2(d+2s)}{d}\rho_0. \tag{4.7}$$

From (4.6) and (4.7) we have

$$\left|\|\tilde{u}\|_{L^{p+1}}-\|u_Q\|_{L^{p+1}}\right|+\left|\|\tilde{u}\|_{L^2}-\|u_Q\|_{L^2}\right|+\left|\|\nabla\tilde{u}\|_{L^2}-\|\nabla u_Q\|_{L^2}\right|\le C(\|u_Q\|_{L^2})\rho_0.$$

Let $\rho=\frac{\rho_0}{C(\|u_Q\|_{L^2})}$, then by Proposition 4.4 there exist $\theta\in\mathbb{R}$ and $x_0\in\mathbb{R}^d$ such that (4.1) holds for $\tilde{u}$. Rescaling to v and then to u, completes the proof. □

Next proposition is "close to the boundary" behavior.

Proposition 4.6. *Fix $\lambda_0>1$. There exists $\rho_0=\rho_0(\lambda_0)>0$ (with the property that $\rho_0\to 0$ as $\lambda_0\searrow 1$) such that for any $\lambda\ge\lambda_0$, there is NO solution $u(t)$ of $NLS_p^+(\mathbb{R}^d)$ with P[u]=0 satisfying $\|u\|_{L^2}=\|u_Q\|_{L^2}$, and $\frac{E[u]}{E[u_Q]}=\frac{d}{2s}\lambda^{\frac{2}{s}}\left(1-\frac{\lambda^{p-1}}{\alpha^2}\right)$ (i.e., on any "mass-energy" line corresponding to $\lambda\ge\lambda_0$ and $\mathcal{ME}<1$) with $\lambda\le\mathcal{G}_u(t)\le\lambda(1+\rho_0)$ for all $t\ge 0$. A similar statement holds for $t\le 0$.*

Proof. To the contrary, assume that there exists a solution $u(t)$ of (1.1) with $\|u\|_{L^2} = \|u_Q\|_{L^2}$, $\frac{E[u]}{E[u_Q]} = \frac{d}{2s}\lambda^{\frac{2}{s}}\left(1 - \frac{\lambda^{p-1}}{\alpha^2}\right)$ and $\mathcal{G}_u(t) \in [\lambda, \lambda(1+\rho_0)]$.

By continuity of the flow $u(t)$ and Proposition 4.5, there are continuous $x(t)$ and $\theta(t)$ such that

$$\left\| u(x) - e^{i\theta_0}\lambda u_Q\big(\lambda(x - x_0)\big) \right\|_{L^2} \leq \epsilon(\rho), \tag{4.8}$$

and

$$\left\| \nabla \Big[u(x) - e^{i\theta_0}\lambda u_Q\big(\lambda(x - x_0)\Big] \right\|_{L^2} \leq \lambda\epsilon(\rho). \tag{4.9}$$

Define $R(T) = \max\Big\{ \max_{0\leq t\leq T} |x(t)|, \log \epsilon(\rho)^{-1} \Big\}$. Consider the localized variance (3.81). Note

$$\frac{d}{s}\lambda^{\frac{2}{s}} E[u_Q] = \lambda^{\frac{2}{s}} \|\nabla u_Q\|_{L^2}^2 \leq \|\nabla u(t)\|_{L^2}^2,$$

then,

$$\begin{aligned} z_R'' &= 4d(p-1)E[u] - \big(2d(p-1) - 8\big)\|\nabla u\|_{L^2}^2 + A_R(u(t)) \\ &= 16\alpha^2 E[u] - 8\big(\alpha^2 - 1\big)\|\nabla u\|_{L^2}^2 + A_R(u(t)) \leq -8\frac{d}{s}\lambda^{\frac{2}{s}}(\lambda^{p-1} - 1)E[u_Q] + A_R(u(t)), \end{aligned}$$

where $A_R(u(t))$ is given by (3.86).

Let $T > 0$ and for the local virial identity (3.84) assume $R = 2R(T)$. Therefore, (4.8) and (4.9) assure that there exists $c_1 > 0$ such that

$$|A_R(u(t))| \leq c_1\lambda^2\big(\epsilon(\rho) + e^{-R(T)}\big)^2 \leq \tilde{c}_1\lambda^2\epsilon(\rho)^2.$$

Taking a suitable ρ_0 small (i.e. $\lambda > 1$ is taken closer to 1), such that for $0 \leq t \leq T$, $\epsilon(\rho)$ is small enough, we get

$$z_R''(t) \leq -8\frac{d}{s}\lambda^{\frac{2}{s}}(\lambda^{p-1} - 1)E[u_Q].$$

Integrating $z_R''(t)$ in time over $[0, T]$ twice, we obtain

$$\frac{z_R(T)}{T^2} \leq \frac{z_R(0)}{T^2} + \frac{z_R'(0)}{T} - 8\frac{d}{s}\lambda^{\frac{2}{s}}(\lambda^{p-1} - 1)E[u_Q].$$

Note $\sup_{x\in\mathbb{R}^d}\phi(x)$ from (3.81), is bounded, say by $c_2 > 0$. Then from (3.81) we have

$$|z_R(0)| \le c_2 R^2 \|u_0\|_{L^2}^2 = c_2 R^2 \|u_Q\|_{L^2}^2,$$

and by (3.83)

$$|z_R'(0)| \le c_3 R \|u_0\|_{L^2}^{2(1-s)} \|\nabla u_0\|_{L^2}^{2s} \le c_3 R \|u_Q\|_{L^2}^{2(1-s)} \|\nabla u_Q\|_{L^2}^{2s} \lambda^{\frac{1}{s}}(1+\rho_0).$$

Taking T large enough so that by Lemma 3.14 we have $\frac{R(T)}{T} < \epsilon(\rho)$, we estimate

$$\begin{aligned}\frac{z_{2R(T)}(T)}{T^2} \le & c_4\Big(\frac{R(T)^2}{T^2} + \frac{R(T)}{T}\Big) - 4\frac{d}{s}\lambda^{\frac{2}{s}}(\lambda^{p-1}-1)E[u_Q] \\ \le & C(\epsilon(\rho)^2 + \epsilon(\rho)) - 4\frac{d}{s}\lambda^{\frac{2}{s}}(\lambda^{p-1}-1)E[u_Q].\end{aligned}$$

We can initially choose ρ_0 small enough (and thus, $\epsilon(\rho_0)$) such that $C(\epsilon(\rho)^2 + \epsilon(\rho)) < 4\frac{d}{s}\lambda^{\frac{2}{s}}(\lambda^{p-1}-1)E[u_Q]$. We obtain $0 \le z_{2R(T)}(T) < 0$, which is a contradiction, showing that our initial assumption about the existence of a solution to (1.1) with bounded $\mathcal{G}_u(t)$ does not hold. □

Before we exhibit the existence of a critical element/solution, we return to the nonlinear profile decomposition (Proposition 3.6) and introduce reordering.

Lemma 4.7 (Profile reordering)**.** *Suppose $\phi_n = \phi_n(x)$ is a bounded sequence in $H^1(\mathbb{R}^d)$. Let $\lambda_0 > 1$. Assume that $M[\phi_n] = M[u_Q]$ and $\frac{E[\phi_n]}{E[u_Q]} = \frac{d}{2s}\lambda_n^{\frac{2}{s}}\big(1 - \frac{\lambda_n^{p-1}}{\alpha^2}\big)$ such that $1 < \lambda_0 \le \lambda_n$ and $\lambda_n \le \mathcal{G}_{\phi_n}(t)$ for each n. Apply Proposition 3.6 to the sequence $\{\psi_n\}$ and obtain nonlinear profiles $\{\widetilde{\psi}^j\}$. Then, these profiles $\widetilde{\psi}^j$ can be reordered so that there exist $1 \le M_1 \le M_2 \le M$ and*

1. *For each $1 \le j \le M_1$, we have $t_n^j = 0$ and $v^j(t) \equiv NLS(t)\widetilde{\psi}^j$ does not scatter as $t \to +\infty$. (In particular, there is at least one such j)*
2. *For each $M_1 + 1 \le j \le M_2$, we have $t_n^j = 0$ and $v^j(t)$ scatters as $t \to +\infty$. (If $M_1 = M_2$, there are no j with this property.)*

3. *For each $M_2+1 \le j \le M$, we have $|t_n^j| \to \infty$ and $v^j(t)$ scatters as $t \to +\infty$. (If $M_2 = M$, there are no j with this property.)*

Proof. Pohozhaev identities (2.52) and energy definition yield

$$\left(\frac{\|\phi_n\|_{L^{p+1}}}{\|u_Q\|_{L^{p+1}}}\right)^{p+1} = \frac{d}{d-2s}[\mathcal{G}_{\phi_n}(t)]^{\frac{2}{s}} - \frac{2s}{d-2s}\frac{E[\phi_n]}{E[u_Q]} \ge \lambda_n^{\frac{2(p-1)}{s}} \ge \lambda_0^{\frac{2(p-1)}{s}} > 1.$$

Notice that if j is such that $|t_n^j| \to \infty$, then $\|\mathrm{NLS}(-t_n^j)\widetilde{\psi}^j\|_{L^{p+1}} \to 0$ and by (3.33) we have that $\frac{\|\phi_n\|_{L^{p+1}}}{\|u_Q\|_{L^{p+1}}} \to 0$. Therefore, there exists at least one j such that t_n^j converges. Without loss of generality, assume $t_n^j = 0$, and reorder the profiles such that for $1 \le j \le M_2$, we have $t_n^j = 0$ and for $M_2+1 \le j \le M$, we have $|t_n^j| \to 0$.

It is left to prove that there exists at least one j, $1 \le j \le M_2$ such that $v^j(t)$ is not scattering. Assume that for all $1 \le j \le M_2$ we have that all v^j are scattering, and thus, $\|v^j(t)\|_{L^{p+1}} \to 0$ as $t \to +\infty$. Let $\epsilon > 0$ and t_0 large enough such that for all $1 \le j \le M_2$ we have $\|v^j(t)\|_{L^{p+1}}^{p+1} \le \epsilon/M_2$. Using L^{p+1} orthogonality (3.56) along the NLS flow, and letting $n \to +\infty$, we obtain

$$\begin{aligned}\lambda_0^{\frac{2(p-1)}{s}}\|u_Q\|_{L^{p+1}}^{p+1} &\le \|u_n(t)\|_{L^{p+1}}^{p+1} \\ &= \sum_{j=1}^{M_2}\|v^j(t_0)\|_{L^{p+1}}^{p+1} + \sum_{j=M_2+1}^{M}\|v^j(t_0-t_n^j)\|_{L^{p+1}}^{p+1} + \|W_n^M(t)\|_{L^{p+1}}^{p+1} + o_n(1) \\ &\le \epsilon + \|W_n^M(t)\|_{L^{p+1}}^{p+1} + o_n(1).\end{aligned}$$

The last line is obtained since $\sum_{j=M_2+1}^{M}\|v^j(t_0-t_n^j)\|_{L^{p+1}}^{p+1} \to 0$ as $n \to \infty$, and gives a contradiction. □

Recall that we have a fixed $\lambda_0 > 1$.

Lemma 4.8 (Existence of threshold solution)**.** *There exists initial data $u_{c,0} \in H^1(\mathbb{R}^d)$ and $1 < \lambda_0 \le \lambda_c \le \sigma_c(\lambda_0)$ such that $u_c(t) \equiv NLS(t)u_{c,0}$ is a global solution with $M[u_c] = M[u_Q]$, $\frac{E[u_c]}{E[u_Q]} = \frac{d}{2s}\lambda_c^{\frac{2}{s}}\left(1 - \frac{\lambda_c^{p-1}}{\alpha^2}\right)$ and, moreover, $\lambda_c \le \mathcal{G}_{u_c}(t) \le \sigma_c$ for all $t \ge 0$.*

Proof. Definition of σ_c implies the existence of sequences $\{\lambda_n\}$ and $\{\sigma_n\}$ with $\lambda_0 \le \lambda_n \le \sigma_n$ and $\sigma_n \searrow \sigma_c$ such that $\mathrm{GBG}(\lambda_n, \sigma_n)$ is false. This means that there exists $u_{n,0}$ with $M[u] = M[u_Q]$, $\frac{E[u_{n,0}]}{E[u_Q]} = \frac{d}{2s}\lambda^{\frac{2}{s}}\left(1 - \frac{\lambda^{p-1}}{\alpha^2}\right)$ and $\lambda_c \le \frac{\|\nabla u\|_{L^2}}{\|\nabla u_Q\|_{L^2}} = [\mathcal{G}_u(t)]^{\frac{1}{s}} \le \sigma_c$, such that $u_n(t) = \mathrm{NLS}(t)u_{n,0}$ is global.

Note that the sequence $\{\lambda_n\}$ is bounded, thus we pass to a convergent subsequence $\{\lambda_{n_k}\}$. Assume $\lambda_{n_k} \to \lambda'$ as $n_k \to \infty$, thus $\lambda_0 \le \lambda' \le \sigma_c$.

We apply the nonlinear profile decomposition (Proposition 3.6) and reordering (Lemma 4.7).

In Lemma 4.7, let $\phi_n = u_{n,0}$. Recall that $v^j(t)$ scatters as $t \to \infty$ for $M_1 + 1 \le j \le M_2$, and by Proposition 3.6, $v^j(t)$ also scatter in one or the other time direction for $M_2 + 1 \le j \le M$ and $E[\widetilde{\psi}^j] = E[v^j] \ge 0$. Thus, by the Pythagorean decomposition for the nonlinear flow (3.24) we have

$$\sum_{j=1}^{M_1} E[\widetilde{\psi}^j] \le E[\phi_n] + o_n(1)$$

For at least one $1 \le j \le M_1$, we have $E[\widetilde{\psi}^j] \le \max\{\lim_n E[\phi_n], 0\}$. Without loss of generality, we may assume $j = 1$. Since $1 = M[\widetilde{\psi}^1] \le \lim_n M[\phi_n] = M[u_Q] = 1$, it follows

$$\left(\mathcal{ME}[\widetilde{\psi}^1]\right)^{\frac{1}{s}} \le \max\left(\lim_n \frac{E[\phi_n]}{E[u_Q]}\right),$$

thus, for some $\lambda_1 \ge \lambda_0$, we have

$$\left(\mathcal{ME}[\widetilde{\psi}^1]\right)^{\frac{1}{s}} = \frac{d}{2s}\lambda_1^{\frac{2}{s}}\left(1 - \frac{\lambda^{p-1}}{\alpha^2}\right).$$

Recall $\widetilde{\psi}^1$ is a nonscattering solution, thus $[\mathcal{G}_{\psi^1}(t)]^{\frac{1}{s}} > \lambda$, otherwise it will contradict Theorem 1.6* Part I (b). We have two cases: either $\lambda_1 \le \sigma_c$ or $\lambda_1 > \sigma_c$.

Case 1. $\lambda_1 \le \sigma_c$. Since the statement "$\mathrm{GBG}(\lambda_1, \sigma_c - \delta)$ is false" implies for each $\delta > 0$, there is a nondecreasing sequence t_k of times such that

$$\lim [\mathcal{G}_{v^1}(t_k)]^{\frac{1}{s}} \ge \sigma_c,$$

thus,

$$
\begin{aligned}
\sigma_c^2 - o_k(1) &\leq \lim[\mathcal{G}_{v^1}(t_k)]^{\frac{2}{s}} \\
&\leq \frac{\|\nabla v^1(t_k)\|_{L^2}^2}{\|\nabla u_Q\|_{L^2}^2} \\
&\leq \frac{\sum_{j=1}^M \|\nabla v^1(t_k - t_n)\|_{L^2}^2 + \|W_n^M(t_k)\|_{L^2}^2}{\|\nabla u_Q\|_{L^2}^2} \qquad (4.10)\\
&\leq \frac{\|\nabla u_n(t)\|_{L^2}^2}{\|\nabla u_Q\|_{L^2}^2} + o_n(1) \\
&\leq \sigma_c^2 + o_n(1).
\end{aligned}
$$

Taking $k \to \infty$, we obtain $\sigma_c^2 - o_n(1) = \sigma_c^2 + o_k(1)$. Thus, $\|W_n^M(t_k)\|_{H^1} \to 0$ and $M[v^1] = M[u_Q]$. Then, Lemma 3.9 yields that for all t,

$$\frac{\|\nabla v^1(t)\|_{L^2}^2}{\|\nabla u_Q\|_{L^2}^2} \leq \lim_n \frac{\|u_n(t)\|_{L^2}^2}{\|\nabla u_Q\|_{L^2}^2} \leq \sigma_c.$$

Take $u_{c,0} = v^1(0)(= \psi^1)$, and $\lambda_c = \lambda_1$.

Case 2. $\lambda_1 \geq \sigma_c$. Note that

$$\lambda_1^2 \leq \lim[\mathcal{G}_{v^1}(t_k)]^{\frac{2}{s}}. \qquad (4.11)$$

Replacing the first line of (4.10) by (4.11), taking $t_k = 0$ and sending $n \to +\infty$, we obtain

$$
\begin{aligned}
\lambda_1^2 &\leq \frac{\|v^1(t_k)\|_{L^2}^2 \|\nabla v^1(t_k)\|_{L^2}^2}{\|u_Q\|_{L^2}^2 \|\nabla u_Q\|_{L^2}^2} \\
&\leq \frac{\|\nabla v^1(t_k)\|_{L^2}^2}{\|\nabla u_Q\|_{L^2}^2} \\
&\leq \frac{\sum_{j=1}^M \|\nabla v^1(t_k - t_n^j)\|_{L^2}^2 + \|W_n^M(t_k)\|_{L^2}^2}{\|\nabla u_Q\|_{L^2}^2} \\
&\leq \frac{\|\nabla u_n(t)\|_{L^2}^2}{\|\nabla u_Q\|_{L^2}^2} + o_n(1) \\
&\leq \sigma_c^2 + o_n(1).
\end{aligned}
$$

Thus, we have $\lambda_1 \leq \sigma_c$, which is a contradiction. Thus, this case cannot happen.

□

Lemma 4.9. *Assume $u(t) = u_c(t)$ to be the critical solution provided by Lemma 4.8. Then there exists a path $x(t)$ in $\mathbb{R}^d$ such that*

$$K = \{u(\cdot - x(t), t) | t \geq 0\}$$

has a compact closure in $H^1(\mathbb{R}^d)$.

Proof. As we proved in Lemma 3.13, it suffices to show that for each sequence of times $t_n \to \infty$, passing to a subsequence, there exists a sequence x_n such that $u(\cdot - x_n, t_n)$ converges in H^1. Let $\phi_n = u(t_n)$ as in Proposition 4.7, and apply the proof of Lemma 4.8. It follows for $j \geq 2$ we have $\psi_j = 0$ and $\widetilde{W}_n^M \to 0$ in H^1 as $n \to \infty$. And thus, $u(\cdot - x_n, t_n) \to \psi^1$ in H^1. □

Lemma 4.10 (Blow up for *a priori* localized solutions)**.** *Suppose u is a solution of the $NLS_p^+(\mathbb{R}^d)$ at the mass-energy line $\lambda > 1$, with $\mathcal{G}_u(0) > 1$. Select κ such that $0 < \kappa < \min(\lambda - 1, \kappa_0)$, where κ_0 is an absolute constant. Assume that there is a radius $R \gtrsim \kappa^{-1/2}$ such that for all t, we have*

$$\mathcal{G}_{u_R}(t) := \frac{\|u\|_{L^2(|x| \geq R)}^{1-s} \|\nabla u(t)\|_{L^2(|x| \geq R)}^{s}}{\|u_Q\|_{L^2(|x| \geq R)}^{1-s} \|\nabla u_Q\|_{L^2(|x| \geq R)}^{s}} \lesssim \kappa.$$

Define $\tilde{r}(t)$ to be the scaled local variance:

$$r(t) = \frac{z_R(t)}{32\alpha^2 E[u_Q] \left(\frac{d}{2s}\lambda^{\frac{2}{s}}\left(1 - \frac{\lambda^{p-1}}{\alpha^2} - \kappa\right)\right)}.$$

Then blowup occurs in forward time before t_b (i.e., $T^* \leq t_b$), where

$$t_b = r'(0) + \sqrt{r'(0)^2 + 2r(0)}\,.$$

Proof. By the local virial identity (3.85),

$$r''(t) = \frac{16\alpha^2 E[u] - 8(\alpha^2 - 1)\|\nabla u\|_{L^2}^2 + A_R(u(t))}{16\alpha^2 E[u_Q] \left(\frac{d}{2s}\lambda^{\frac{2}{s}}\left(1 - \frac{\lambda^{p-1}}{\alpha^2}\right) - \kappa\right)},$$

where

$$\left|A_R(u(t))\right| = \|\nabla u(t)\|_{L^2(|x| \geq R)}^2 + \frac{1}{R^2}\|u(t)\|_{L^2(|x| \geq R)}^2 + \|u(t)\|_{L^{p+1}(|x| \geq R)}^{p+1}.$$

Note that, $E[u_Q] = \frac{d}{s}\|\nabla u_Q\|_{L^2}^2$ and definition of the mass-energy line yield

$$\frac{16\alpha^2 E[u] - 8(\alpha^2-1)\|\nabla u\|_{L^2}^2}{16\alpha^2 E[u_Q]} = \frac{E[u]}{E[u_Q]} - \frac{d\|\nabla u\|_{L^2}^2}{sE[u_Q]}$$

$$= \frac{E[u]}{E[u_Q]} - \frac{\|\nabla u\|_{L^2}^2}{\|\nabla u_Q\|_{L^2}^2} \tag{4.12}$$

$$\leq \frac{d}{2s}\lambda^{\frac{2}{s}}\left(1 - \frac{\lambda^{p-1}}{\alpha^2}\right) - [\mathcal{G}_u(t)]^2. \tag{4.13}$$

In addition, we have the following estimates

$$\|\nabla u(t)\|_{L^2(|x|\geq R)}^2 \lesssim \kappa, \qquad \frac{\|u(t)\|_{L^2(|x|\geq R)}^2}{R^2} = \frac{\|u_Q\|_{L^2}^2}{R^2} \lesssim \kappa,$$

$$\|u(t)\|_{L^{p+1}(|x|\geq R)}^{p+1} \lesssim \|\nabla u\|_{L^2(|x|\geq R)}^{\frac{d(p-1)}{2}} \|u\|_{L^2(|x|\geq R)}^{2-\frac{(d-2)(p-1)}{2}} \tag{4.14}$$

$$\lesssim [\mathcal{G}_{u_R}(t)]^2 \left(\|\nabla u_Q\|_{L^2}^s \|u_Q\|_{L^2}^{1-s}\right)^{p-1} \lesssim \kappa.$$

We used the Gagliardo-Nirenberg to obtain (4.14) and noticing that $\|\nabla u_Q\|_{L^2}^s$ and $\|u_Q\|_{L^2}^{1-s}$ are constants, the last expression is estimated by κ (up to a constant). In addition, $\mathcal{G}_u(t) > 1$, then $\kappa \lesssim \kappa[\mathcal{G}_u(t)]^2$. Applying the above estimates, it follows

$$r''(t) \lesssim \frac{\frac{d}{2s}\lambda^{\frac{2}{s}}\left(1 - \frac{\lambda^{p-1}}{\alpha^2}\right) - [\mathcal{G}_u(t)]^2(1-\kappa)}{\frac{d}{2s}\lambda^{\frac{2}{s}}\left(1 - \frac{\lambda^{p-1}}{\alpha^2}\right) - \kappa}.$$

Since $\mathcal{G}_u(t) \geq \lambda$, we obtain $\tilde{r}''(t) \leq -1$. which is a contradiction. Now integrating in time twice gives

$$r(t) \leq -\frac{1}{2}t^2 + r'(0)t + r(0)\,.$$

The positive root of the polynomial on the right-hand side is

$$t_b = r'(0) + \sqrt{r'(0)^2 + 2r(0)}.$$

□

This concludes all the claims in steps 1, 2 and 3 in section 4.1 and finishes the proof of Theorem 1.6* part II (b).

Chapter 5

FUTURE PROJECTS ON NONLINEAR SCHRÖDINGER EQUATION

Let's return to Figure 2.1 to summarize our results and postulate future directions of research. In this work we completely described the behavior of solutions under the line BE. The behavior of solutions on the line BE (i.e. $\mathcal{ME}[u] = 1$) is only known in the case $p = 3,\ N = 3$ ($s_c = \frac{1}{2}$), see [Duyckaerts and Roudenko, 2010]. The behavior of solutions above the line BE is largely unknown, there are blow up criteria for the case $p = 3, N = 3$ in [Holmer and Roudenko, 2007].

Thus, one could ask if further characterizations of solutions to NLS can be investigated:

Question1. If the mass-energy threshold is dropped and one analyzes the gradient of the solution u of the Cauchy problem (1.1) with $u_0 \in H^1$, is it possible to find a bound on the gradient so that there is global existence and scattering? Is there $B > 0$ if $\sup_{t\in(T_*,T^*)} \mathcal{G}_u(t) < B$, then $T* = +\infty$ and scattering holds? It seems that $B = 1$ will give a criterion, but could we go beyond 1? Is there any relationship between B and the conserved quantities?

Question 2. Can we find $C > 0$ such that if $\inf_{t\in(T_*,T^*)} \mathcal{G}_u(t) < C$, then $T* < +\infty$, i.e., a finite blowup occurs? Is it possible that $C = B$ or on the interval $[C, B]$ any behavior of solutions happen?

Question 3. In Figure 2.1, in the region DEF, it has been proven that there are "weak" blowup solutions. Could a "weak" blowup solution turn into a "strong" blowup, i.e., is it true that for any sequence of times t_n the gradient $\|\nabla u(t_n)\|_L^2 \to 0$ as $t_n \to \infty$. In other words, the existence of solution is global in time but with an exploding gradient along any time sequence?

Question 4. From the local $\dot{H}^{s_c}$ theory, it is known that for small (in $\dot{H}^{s_c}$) initial data there is scattering (i.e., for δ small, $\|u_0\|_{\dot{H}^{s_c}} < \delta$, then scattering holds, see [Cazenave and Weissler, 1990]). One could ask, if it is possible to find a threshold (i.e., the supremum of all such δ) that guarantees global existence and scattering? Could $\sup_{0<t<T*} \|u(t)\|_{\dot{H}^{s_c}} < \|u_Q\|_{\dot{H}^{s_c}}$ be such a threshold? This question is in the spirit of Kenig-Merle [Kenig and Merle, 2010] work for for the defocusing cubic NLS in 3 dimensions ($\mathrm{NLS}_3^-(\mathbb{R}^3)$).

Question 5. Extend the characterization of solution behavior on line GH (as in [Duyckaerts and Roudenko, 2010]) to other NLS equations with $0 < s < 1$.

Question 6. If nonlinearity is combined-type (such as 2 different powers or some potential is introduced), how does this influence the scattering or "weak blowup" behavior.

REFERENCES

[Berestycki and Lions, 1983a] Berestycki, H. and Lions, P.-L. (1983a). Nonlinear scalar field equations. I. Existence of a ground state. *Arch. Rational Mech. Anal.*, 82(4):313–345.

[Berestycki and Lions, 1983b] Berestycki, H. and Lions, P.-L. (1983b). Nonlinear scalar field equations. II. Existence of infinitely many solutions. *Arch. Rational Mech. Anal.*, 82(4):347–375.

[Bourgain, 1999] Bourgain, J. (1999). Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case. *J. Amer. Math. Soc.*, 12(1):145–171.

[Brezis and Coron, 1985] Brezis, H. and Coron, J.-M. (1985). Convergence of solutions of H-systems or how to blow bubbles. *Arch. Rational Mech. Anal.*, 89(1):21–56.

[Carreon and Guevara, 2011] Carreon, F. and Guevara, C. (2011). Scattering and blow-up for the 2d quintic nonlinear schrödinger equation. Available at *http://mathpost.asu.edu/∼guevara/*.

[Cazenave, 2003] Cazenave, T. (2003). *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York.

[Cazenave and Weissler, 1990] Cazenave, T. and Weissler, F. B. (1990). The Cauchy problem for the critical nonlinear Schrödinger equation in H^s. *Nonlinear Anal.*, 14(10):807–836.

[Colliander et al., 2008] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., and Tao, T. (2008). Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$. *Ann. of Math. (2)*, 167(3):767–865.

[Dalfovo et al., 1999] Dalfovo, F., Giorgini, S., Pitaevskii, L. P., and Stringari, S. (1999). Theory of Bose-Einstein Condensate in trapped gases. *Rev. Mod. Phys.*, 71:463–512.

[Dodson, 2009] Dodson, B. (2009). Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear schrödinger equation when $d \geq 3$. *arxiv.org/abs/0912.2467*.

[Dodson, 2010a] Dodson, B. (2010a). Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear schrödinger equation when $d = 1$. *arxiv.org/abs/1010.0040*.

[Dodson, 2010b] Dodson, B. (2010b). Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear schrödinger equation when $d = 2$. *arxiv.org/abs/1006.1375*.

[Duyckaerts et al., 2008] Duyckaerts, T., Holmer, J., and Roudenko, S. (2008). Scattering for the non-radial 3D cubic nonlinear Schrödinger equation. *Math. Res. Lett.*, 15(6):1233–1250.

[Duyckaerts and Roudenko, 2010] Duyckaerts, T. and Roudenko, S. (2010). Threshold solutions for the focusing 3D cubic Schrödinger equation. *Rev. Mat. Iberoam.*, 26(1):1–56.

[Escauriaza et al., 2003] Escauriaza, L., Serëgin, G. A., and Shverak, V. (2003). $L_{3,\infty}$-solutions of Navier-Stokes equations and backward uniqueness. *Uspekhi Mat. Nauk*, 58(2(350)):3–44.

[Foschi, 2005] Foschi, D. (2005). Inhomogeneous Strichartz estimates. *J. Hyper. Diff. Eq.*, 2(1):1–24.

[Gérard, 1996] Gérard, P. (1996). Oscillations and concentration effects in semilinear dispersive wave equations. *J. Funct. Anal.*, 141(1):60–98.

[Ginibre and Velo, 1979a] Ginibre, J. and Velo, G. (1979a). On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case. *J. Funct. Anal.*, 32(1):1–32.

[Ginibre and Velo, 1979b] Ginibre, J. and Velo, G. (1979b). On a class of nonlinear Schrödinger equations. II. Scattering theory, general case. *J. Funct. Anal.*, 32(1):33–71.

[Ginibre and Velo, 1985] Ginibre, J. and Velo, G. (1985). Scattering theory in the energy space for a class of nonlinear Schrödinger equations. *J. Math. Pures Appl. (9)*, 64(4):363–401.

[Glangetas and Merle, 1995] Glangetas, L. and Merle, F. (1995). A geometrical approach of existence of blow up solutions in h^1 for nonlinear schrödinger equation. *Rep. No. R95031, Laboratoire d'Analyse Num erique, Univ. Pierre and Marie Curie.*

[Grillakis, 2000] Grillakis, M. G. (2000). On nonlinear Schrödinger equations. *Comm. PDE*, 25(9-10):1827–1844.

[Holmer et al., 2010] Holmer, J., Platte, R., and Roudenko, S. (2010). Blow-up criteria for the 3D cubic nonlinear Schrödinger equation. *Nonlinearity*, 23(4):977–1030.

[Holmer and Roudenko, 2007] Holmer, J. and Roudenko, S. (2007). On blow-up solutions to the 3D cubic nonlinear Schrödinger equation. *Appl. Math. Res. Express. AMRX*, 15(1):Art. ID abm004, 31.

[Holmer and Roudenko, 2008] Holmer, J. and Roudenko, S. (2008). A sharp condition for scattering of the radial 3D cubic nonlinear Schrödinger equation. *Comm. Math. Phys.*, 282(2):435–467.

[Holmer and Roudenko, 2010a] Holmer, J. and Roudenko, S. (2010a). Blow-up solutions on a sphere for the 3d quintic NLS in the energy space. *to appear in Analysis & PDE.*

[Holmer and Roudenko, 2010b] Holmer, J. and Roudenko, S. (2010b). A class of solutions to the 3d cubic nonlinear Schrödinger equation that blow-up on a circle. *AMRX Appl. Math. Res. eXpress (to appear).*

[Holmer and Roudenko, 2010c] Holmer, J. and Roudenko, S. (2010c). Divergence of infinite-variance nonradial solutions to the 3D NLS equation. *Comm. PDE*, 35(5):878–905.

[Kato, 1987] Kato, T. (1987). On nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Phys. Théor.*, 46(1):113–129.

[Kato, 1994] Kato, T. (1994). An $L^{q,r}$-theory for nonlinear Schrödinger equations. In *Spectral and scattering theory and applications*, volume 23 of *Adv. Stud. Pure*

Math., pages 223–238. Math. Soc. Japan, Tokyo.

[Keel and Tao, 1998] Keel, M. and Tao, T. (1998). Endpoint Strichartz estimates. *Amer. J. Math.*, 120(5):955–980.

[Kenig and Koch, 2009] Kenig, C. E. and Koch, G. S. (2009). An alternative approach to regularity for the navier-stokes equations in critical spaces. *arxiv.org/abs/0908.3349*.

[Kenig and Merle, 2006] Kenig, C. E. and Merle, F. (2006). Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case. *Invent. Math.*, 166(3):645–675.

[Kenig and Merle, 2010] Kenig, C. E. and Merle, F. (2010). Scattering for $\dot{H}^{1/2}$ bounded solutions to the cubic, defocusing NLS in 3 dimensions. *Trans. Amer. Math. Soc.*, 362(4):1937–1962.

[Kenig et al., 1993] Kenig, C. E., Ponce, G., and Vega, L. (1993). Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. *Comm. Pure Appl. Math.*, 46(4):527–620.

[Keraani, 2001] Keraani, S. (2001). On the defect of compactness for the Strichartz estimates of the Schrödinger equations. *J. Differential Equations*, 175(2):353–392.

[Killip et al., 2009] Killip, R., Tao, T., and Visan, M. (2009). The cubic nonlinear Schrödinger equation in two dimensions with radial data. *J. Eur. Math. Soc. (JEMS)*, 11(6):1203–1258.

[Killip and Visan, 2010] Killip, R. and Visan, M. (2010). The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher. *Amer. J. Math.*, 132(2):361–424.

[Killip and Visan, 2011] Killip, R. and Visan, M. (2011). Global well-posedness and scattering for the defocusing quintic NLS in three dimensions. *arxiv.org/abs/1102.1192*.

[Killip et al., 2008] Killip, R., Visan, M., and Zhang, X. (2008). The mass-critical nonlinear Schrödinger equation with radial data in dimensions three and higher. *Anal. PDE*, 1(2):229–266.

[Lions, 1984] Lions, P.-L. (1984). The concentration-compactness principle in the calculus of variations. The locally compact case. II. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(4):223–283.

[Lushnikov, 1995] Lushnikov, P. M. (1995). Dynamic criterion for collapse. *Pis'ma Zh. E ksp. Teor. Fiz.*, 62:447–452.

[Merle, 1993] Merle, F. (1993). Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power. *Duke Math. J.*, 69(2):427–454.

[Ogawa and Tsutsumi, 1991] Ogawa, T. and Tsutsumi, Y. (1991). Blow-up of H^1 solution for the nonlinear Schrödinger equation. *J. Differential Equations*, 92(2):317–330.

[Raphaël, 2006] Raphaël, P. (2006). Existence and stability of a solution blowing up on a sphere for an L^2-supercritical nonlinear Schrödinger equation. *Duke*

Math. J., 134(2):199–258.

[Raphaël and Szeftel, 2009] Raphaël, P. and Szeftel, J. (2009). Standing ring blow up solutions to the N-dimensional quintic nonlinear Schrödinger equation. *Comm. Math. Phys.*, 290(3):973–996.

[Ryckman and Visan, 2007] Ryckman, E. and Visan, M. (2007). Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$. *Amer. J. Math.*, 129(1):1–60.

[Strauss, 1981a] Strauss, W. A. (1981a). Nonlinear scattering theory at low energy. *J. Funct. Anal.*, 41(1):110–133.

[Strauss, 1981b] Strauss, W. A. (1981b). Nonlinear scattering theory at low energy: sequel. *J. Funct. Anal.*, 43(3):281–293.

[Sulem and Sulem, 1999] Sulem, C. and Sulem, P.-L. (1999). *The nonlinear Schrödinger equation*, volume 139 of *Applied Mathematical Sciences*. Springer-Verlag, New York. Self-focusing and wave collapse.

[Tao, 2005] Tao, T. (2005). Global well-posedness and scattering for the higher-dimensional energy-critical nonlinear Schrödinger equation for radial data. *New York J. Math.*, 11:57–80.

[Tao, 2006] Tao, T. (2006). *Nonlinear dispersive equations*, volume 106 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC. Local and global analysis.

[Tao et al., 2008] Tao, T., Visan, M., and Zhang, X. (2008). Minimal-mass blowup solutions of the mass-critical NLS. *Forum Math.*, 20(5):881–919.

[Triebel, 1978] Triebel, H. (1978). *Interpolation theory, function spaces, differential operators*, volume 18 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam.

[Visan, 2007] Visan, M. (2007). The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions. *Duke Math. J.*, 138(2):281–374.

[Weinstein, 1982] Weinstein, M. I. (1982). Nonlinear Schrödinger equations and sharp interpolation estimates. *Comm. Math. Phys.*, 87(4):567–576.

[Zakharov, 1972] Zakharov, V. E. (1972). Collapse of langmuir waves. *Soviet Physics JETP (translation of the Journal of Experimental and Theoretical Physics of the Academy of Sciences of the USSR)*, 35:908–914.

[Zwiers, 2010] Zwiers, I. (2010). Standing ring blowup solutions for cubic NLS. *to appear in Analysis & PDE*.