Solvable Time-Dependent Models in Quantum Mechanics

by

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ABSTRACT

In the traditional setting of quantum mechanics, the Hamiltonian operator does not depend on time. While some Schrödinger equations with time-dependent Hamiltonians have been solved, explicitly solvable cases are typically scarce. This thesis is a collection of papers in which this first author along with Suslov, Suazo, and Lopez, has worked on solving a series of Schrödinger equations with a time-dependent quadratic Hamiltonian that has applications in problems of quantum electrodynamics, lasers, quantum devices such as quantum dots, and external varying fields.

In particular the author discusses a new completely integrable case of the time-dependent Schrödinger equation in R^n with variable coefficients for a modified oscillator, which is dual with respect to the time inversion to a model of the quantum oscillator considered by Meiler, Cordero-Soto, and Suslov. A second pair of dual Hamiltonians is found in the momentum representation. Our examples show that in mathematical physics and quantum mechanics a change in the direction of time may require a total change of the system dynamics in order to return the system back to its original quantum state.

The author also considers several models of the damped oscillators in nonrelativistic quantum mechanics in a framework of a general approach to the dynamics of the time-dependent Schrödinger equation with variable quadratic Hamiltonians. The Green functions are explicitly found in terms of elementary functions and the corresponding gauge transformations are discussed. The factorization technique is applied to the case of a shifted harmonic oscillator. The time-evolution of the expectation values of the energy related operators is determined for two models of the quantum damped oscillators under consideration. The classical equations of motion for the damped oscillations are derived for the corresponding expectation values of the position operator.

Finally, the author constructs integrals of motion for several models of the quantum damped oscillators in a framework of a general approach to the time-dependent Schrödinger equation with variable quadratic Hamiltonians. An extension of the Lewis–Riesenfeld dynamical invariant is given. The time-evolution of the expectation values of the energy related positive operators is determined for the oscillators under consideration. A proof of uniqueness of the corresponding Cauchy initial value problem is discussed as an application.

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INTRODUCTION

1.1 The Schrödinger Equation

For nearly two hundred years, the classical mechanics proposed by Isaac Newton proved to be sufficient in understanding the physical wold that surrounds us. In fact, it was generally believed among the scientific community that most of the answers to the deepest physical phenomenon were to be understood through classical mechanics. However, towards the end of the 19th century, experiments at the atomic level started to show discrepancies with the clasically-expected results. It was then that scientists such as Planck, Bohr, and Einstein ushered in a new era that would help understand these results. Eventually this movement was formalized into a field of modern physics in its own right known as quantum mechanics. This field that seemingly defied logic was able to accurately explain the rare occurrences related to very small particles such as electrons and photons. The mathematical equation used to predict these quantum effects is known as the Schrödinger Equation; a fundamental law of nature in its purest mathematical form:

$$\begin{aligned} i\hbar\frac{\partial\Psi}{\partial t} &= \frac{-\hbar^2}{2m}\nabla^2\Psi + V(x)\Psi = H\Psi \\ \Psi_0 &= \Psi(0,t), \end{aligned} \tag{1.1}$$

where H is the so-called Hamiltonian Operator (see [56]) and V is an arbitrary potential (energy) function. It was derived by Erwin Schrödinger in 1925-1926, and its fidelity to reality earned him the Nobel Prize. The so-called wave function (solution) represents the state of a particle, where the particle has a wave-particle duality in its behavior. However, to use the wave function Ψ for physical meaning, one must use $|\Psi|^2$ as a probability density function as suggested by Max Born. Thus, the probability of finding a particle between point a and b on an interval (or space in 3 dimensions) is given by

$$P(a,b) = \int_b^a |\Psi(x,t)|^2 \, dx. \tag{1.2}$$

Given this definition, the probability of finding the particle anywhere should be 1:

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = 1. \tag{1.3}$$

Ergo, (1.2) and (1.3) constitute Max Born's so called statistical interpretation.

Time-Independent Hamiltonian

The Hamiltonian operator corresponds to the total energy of the system and its corresponding spectrum yields the set of possible outcomes when one measures the total energy of a system (see [89] and [56]). It is often expressed as the sum of operators that correspond to kinetic and potential energy:

$$H = T + V \tag{1.4}$$

where T is expressed as the formula of kinetic energy,

$$T = \frac{p^2}{2m} \tag{1.5}$$

by the following so-called canonical substitution

$$p \longrightarrow -i\hbar\nabla. \tag{1.6}$$

Thus (1.4) is equivalent to our previous definition,

$$\frac{-\hbar^2}{2m}\nabla^2 + V(x) = H \tag{1.7}$$

via (1.6), where the potential function $V(x)$ is a multiplicative operator.

Different potential functions result in different models that pertain to certain physical assumptions. For example, if $V = 0$ the assumption is that the modeled particle is unconstrained or free. Nonetheless, we will focus on a particular case known as the quantum harmonic oscillator. In classical mechanics, a harmonic oscillator is a system which when displaced from its resting or equilibrium position, experiences a restoring force

$$F = -kx, \; k > 0. \tag{1.8}$$

A good example of a harmonic oscillator is a mass attached to a spring. To derive the potential energy for a harmonic oscillator, we calculate the work done:

$$V(x) = -\int F \cdot dx = \frac{1}{2}kx^2. \tag{1.9}$$

While a harmonic oscillator in classical mechanics can be unrealistic without considering a dampening effect, it turns out one can approximate any potential function with (1.9) by means of a Taylor expansion around an equilibrium, hence its importance in quantum mechanics.

Quadratic Time-Dependent Hamiltonian

The Hamiltonian (1.7) does not depend on time. Despite the fact that time-dependent Hamiltonians have been studied in both a mathematical and physical context, the explicit solvability of the corresponding Schrödinger equation is still a difficult task. Recently, this author along with Suslov, Suazo, and Lopez, has worked on solving the time-dependent Schrödinger Equation with a quadratic Hamiltonian of the form

$$H(t) = -a(t)\frac{\partial^2}{\partial x^2} + b(t)x^2 - i\left(c(t)x\frac{\partial}{\partial x} + d(t)\right) - f(t)x + ig(t)\frac{\partial}{\partial x} \tag{1.10}$$

(see [49]). Here, the Hamiltonian operator (1.10) is applied as follows:

$$H(t)\psi = -a(t)\frac{\partial^2\psi}{\partial x^2} + b(t)x^2\psi - i\left(c(t)x\frac{\partial\psi}{\partial x} + d(t)\psi\right) - f(t)x\psi + ig(t)\frac{\partial\psi}{\partial x} \tag{1.11}$$

Here $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f(t)$, and $g(t)$ are given real-valued functions of time t only. We find the Green's function by using the following substitution

$$\psi = Ae^{iS} = A(t)e^{iS(x,y,t)}, \qquad A = A(t) = \frac{1}{\sqrt{2\pi i\mu(t)}} \tag{1.12}$$

Adding a time dependence to the Hamiltonian is a natural way of incorporating an acting external force that depends on time. Quadratic Hamiltonians for one, have attracted substantial attention over the years in view of their great importance to many advanced quantum problems. Examples can be found in quantum and physical optics [58], [113], [161], [176], physics of lasers and masers [182], [205], [186], [214], molecular spectroscopy [67], quantum chemistry, and Hamiltonian cosmology [102], [161], [177], [178], [180]. They include coherent states [140], [137], [138], [113] and Berry's phase [15], [16], [38], [94], [125], [150], asymptotic and numerical methods [88], [106], [118], [147], [152], charged particle traps [136] and motion in uniform magnetic fields [49], [65], [121], [129], [130], [132], [138], polyatomic molecules in varying external fields, crystals through which an electron is passing and exciting the oscillator modes, and other interactions of the modes with external fields [81]. Quadratic

Hamiltonians have particular applications in quantum electrodynamics because the electromagnetic field can be represented as a set of forced harmonic oscillators [19], [81], [65], [87], [103], and [145]. Nonlinear oscillators play a central role in the novel theory of Bose–Einstein condensation [54] based on the nonlinear Schrödinger (or Gross–Pataevskii) equation [104], [105], [112], [166].

1.2 Solution of a Cauchy Initial Value Problem

In theory, the time-dependent Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}=H(t)\,\psi \tag{1.13}$$

can be solved using the time evolution operator given formally by

$$U(t,t_0)=\mathrm{T}\left(\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}H\left(t'\right)\,dt'\right)\right), \tag{1.14}$$

where T is the time ordering operator which orders operators with larger times to the left. This unitary operator takes a state at time t_0 to a state at time t, so that

$$\psi(x,t)=U(t,t_0)\,\psi(x,t_0). \tag{1.15}$$

However, an explicit construction of (1.14) is rarely possible. In [49], the authors are able to construct the time evolution operator for a specific form of a quadratic Hamiltonian. The fundamental solution of the time-dependent Schrödinger equation with the quadratic Hamiltonian of the form

$$i\frac{\partial\psi}{\partial t}=-a(t)\frac{\partial^2\psi}{\partial x^2}+b(t)x^2\psi-i\left(c(t)x\frac{\partial\psi}{\partial x}+d(t)\,\psi\right)-f(t)x\psi+ig(t)\frac{\partial\psi}{\partial x}, \tag{1.16}$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f(t)$, and $g(t)$ are given real-valued functions of time t only, can be found with the help of a familiar substitution

$$\psi=Ae^{iS}=A(t)\,e^{iS(x,y,t)},\qquad A=A(t)=\frac{1}{\sqrt{2\pi i\mu(t)}} \tag{1.17}$$

with

$$S=S(x,y,t)=\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2+\delta(t)x+\varepsilon(t)y+\kappa(t), \tag{1.18}$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$, $\varepsilon(t)$, and $\kappa(t)$ are differentiable real-valued functions of time t only. Indeed,

$$\frac{\partial S}{\partial t}=-a\left(\frac{\partial S}{\partial x}\right)^2-bx^2+fx+(g-cx)\frac{\partial S}{\partial x} \tag{1.19}$$

by choosing

$$\frac{\mu'}{2\mu}=a\frac{\partial^2 S}{\partial x^2}+d=2\alpha\left(t\right)a\left(t\right)+d\left(t\right). \tag{1.20}$$

Equating the coefficients of all admissible powers of $x^m y^n$ with $0\leq m+n\leq 2$, gives the following system of ordinary differential equations

$$\frac{d\alpha}{dt}+b\left(t\right)+2c\left(t\right)\alpha+4a\left(t\right)\alpha^2=0, \tag{1.21}$$

$$\frac{d\beta}{dt}+\left(c\left(t\right)+4a\left(t\right)\alpha\left(t\right)\right)\beta=0, \tag{1.22}$$

$$\frac{d\gamma}{dt}+a\left(t\right)\beta^2\left(t\right)=0, \tag{1.23}$$

$$\frac{d\delta}{dt}+\left(c\left(t\right)+4a\left(t\right)\alpha\left(t\right)\right)\delta=f\left(t\right)+2\alpha\left(t\right)g\left(t\right), \tag{1.24}$$

$$\frac{d\varepsilon}{dt}=\left(g\left(t\right)-2a\left(t\right)\delta\left(t\right)\right)\beta\left(t\right), \tag{1.25}$$

$$\frac{d\kappa}{dt}=g\left(t\right)\delta\left(t\right)-a\left(t\right)\delta^2\left(t\right), \tag{1.26}$$

where the first equation is the familiar Riccati nonlinear differential equation; see, for example, [91], [171], [213] and references therein. Substitution of (1.20) into (1.21) results in the second order linear equation

$$\mu''-\tau\left(t\right)\mu'+4\sigma\left(t\right)\mu=0 \tag{1.27}$$

with

$$\tau\left(t\right)=\frac{a'}{a}-2c+4d, \qquad \sigma\left(t\right)=ab-cd+d^2+\frac{d}{2}\left(\frac{a'}{a}-\frac{d'}{d}\right), \tag{1.28}$$

which must be solved subject to the initial data

$$\mu\left(0\right)=0, \qquad \mu'\left(0\right)=2a\left(0\right)\neq 0 \tag{1.29}$$

in order to satisfy the initial condition for the corresponding Green function; see the asymptotic formula (1.37) below. The authors in ([49]) refer to equation (1.27) as the *characteristic equation* and its solution $\mu\left(t\right)$, subject to (1.29), as the *characteristic function.* As the special case (1.27) contains the generalized equation of hypergeometric type, whose solutions are studied in detail in [158]; see also [2], [157], [197], and [213].

Thus, the Green function (fundamental solution or propagator) is explicitly given in terms of the characteristic function

$$\psi=G\left(x,y,t\right)=\frac{1}{\sqrt{2\pi i\mu\left(t\right)}}\ e^{i\left(\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2+\delta(t)x+\varepsilon(t)y+\kappa(t)\right)}. \tag{1.30}$$

Here

$$\alpha(t)=\frac{1}{4a(t)}\frac{\mu'(t)}{\mu(t)}-\frac{d(t)}{2a(t)}, \tag{1.31}$$

$$\beta(t)=-\frac{1}{\mu(t)}\ \exp\left(-\int_0^t (c(\tau)-2d(\tau))\ d\tau\right), \tag{1.32}$$

$$\begin{aligned}\gamma(t) &= \frac{a(t)}{\mu(t)\mu'(t)}\ \exp\left(-2\int_0^t (c(\tau)-2d(\tau))\ d\tau\right)+\frac{d(0)}{2a(0)} \\ &\quad -4\int_0^t \frac{a(\tau)\sigma(\tau)}{(\mu'(\tau))^2}\left(\exp\left(-2\int_0^\tau (c(\lambda)-2d(\lambda))\ d\lambda\right)\right)\ d\tau,\end{aligned} \tag{1.33}$$

$$\begin{aligned}\delta(t) &= \frac{1}{\mu(t)}\ \exp\left(-\int_0^t (c(\tau)-2d(\tau))\ d\tau\right) \\ &\quad\times\int_0^t \exp\left(\int_0^\tau (c(\lambda)-2d(\lambda))\ d\lambda\right) \\ &\quad\times\left(\left(f(\tau)-\frac{d(\tau)}{a(\tau)}g(\tau)\right)\mu(\tau)+\frac{g(\tau)}{2a(\tau)}\mu'(\tau)\right)\ d\tau,\end{aligned} \tag{1.34}$$

$$\begin{aligned}\varepsilon(t) &= -\frac{2a(t)}{\mu'(t)}\delta(t)\ \exp\left(-\int_0^t (c(\tau)-2d(\tau))\ d\tau\right) \\ &\quad+8\int_0^t \frac{a(\tau)\sigma(\tau)}{(\mu'(\tau))^2}\exp\left(-\int_0^\tau (c(\lambda)-2d(\lambda))\ d\lambda\right)(\mu(\tau)\delta(\tau))\ d\tau \\ &\quad+2\int_0^t \frac{a(\tau)}{\mu'(\tau)}\exp\left(-\int_0^\tau (c(\lambda)-2d(\lambda))\ d\lambda\right)\left(f(\tau)-\frac{d(\tau)}{a(\tau)}g(\tau)\right)\ d\tau,\end{aligned} \tag{1.35}$$

$$\begin{aligned}\kappa(t) &= \frac{a(t)\mu(t)}{\mu'(t)}\delta^2(t)-4\int_0^t \frac{a(\tau)\sigma(\tau)}{(\mu'(\tau))^2}(\mu(\tau)\delta(\tau))^2\ d\tau \\ &\quad-2\int_0^t \frac{a(\tau)}{\mu'(\tau)}(\mu(\tau)\delta(\tau))\left(f(\tau)-\frac{d(\tau)}{a(\tau)}g(\tau)\right)\ d\tau\end{aligned} \tag{1.36}$$

with $\delta(0)=g(0)/(2a(0))$, $\varepsilon(0)=-\delta(0)$, and $\kappa(0)=0$. Integration by parts are used to resolve the singularities of the initial data. Then the corresponding asymptotic formula is

$$\begin{aligned}G(x,y,t) &= \frac{e^{iS(x,y,t)}}{\sqrt{2\pi i\mu(t)}}\sim\frac{1}{\sqrt{2\pi ia(0)t}}\exp\left(i\frac{(x-y)^2}{4a(0)t}\right)\exp\left(i\frac{g(0)}{2a(0)}(x-y)\right) \\ &\quad\times\exp\left(-i\frac{c(0)}{4a(0)}\left(x^2-y^2\right)\right)\exp\left(-i\frac{a'(0)}{8a^2(0)}(x-y)^2\right)\end{aligned} \tag{1.37}$$

as $t\to 0^+$. Notice that the first term on the right hand side is a familiar free particle propagator.

By the superposition principle, one obtain an explicit solution of the Cauchy initial value problem

$$i\frac{\partial\psi}{\partial t}=H(t)\,\psi,\qquad \psi(x,t)|_{t=0}=\psi_0(x) \tag{1.38}$$

on the infinite interval $-\infty < x < \infty$ with the general quadratic Hamiltonian as in (1.16) in the form

$$\psi(x,t) = \int_{-\infty}^{\infty} G(x,y,t)\ \psi_0(y)\ dy. \tag{1.39}$$

This yields the time evolution operator (1.14) explicitly as an integral operator. Properties of similar oscillatory integrals are discussed in [191].

1.3 Organization of the Thesis

This thesis is a collection of papers first-authored by the author of the thesis. These papers are a continued study of the mathematical properties and physical applications of the time-dependent Schrödinger equation proposed in [49]. Mathematically we study conditions for the time-invertibility of the proposed equation and Uniqueness results via Quadratic Invariants. Physically, we propose several damped oscillator models and other models that may be used in quantum optics, specifically applied to quantum devices such as quantum dots [71], [96].

In **Chapter 2** we discuss a new completely integrable case of the time-dependent Schrödinger equation in R^n with variable coefficients for a modified oscillator, which is dual with respect to the time inversion to a model of the quantum oscillator considered by Meiler, Cordero-Soto, and Suslov in [143]. A second pair of dual Hamiltonians is found in the momentum representation. Our examples show that in mathematical physics and quantum mechanics a change in the direction of time may require a total change of the system dynamics in order to return the system back to its original quantum state. Particular solutions of the corresponding nonlinear Schrödinger equations are obtained. A Hamiltonian structure of the classical integrable problem and its quantization are also discussed.

In **Chapter 3** we consider several models of the damped oscillators in nonrelativistic quantum mechanics in a framework of a general approach to the dynamics of the time-dependent Schrödinger equation with variable quadratic Hamiltonians. The Green functions are explicitly found in terms of elementary functions and the corresponding gauge transformations are discussed. The factorization technique is applied to the case of a shifted harmonic oscillator. The time-evolution of the expectation values of the energy related operators is determined for two models of the quantum damped oscillators under consideration. The classical equations of motion for the damped oscillations are derived for the corresponding expectation values of the position operator.

In **Chapter 4** we construct integrals of motion for several models of the quantum damped oscillators in a framework of a general approach to the time-dependent Schrödinger equation with variable quadratic Hamiltonians. An extension of the Lewis–Riesenfeld dynamical invariant is given. The time-evolution of the expectation values of the energy related positive operators is determined for the oscillators under consideration. A proof of uniqueness of the corresponding Cauchy initial value problem is discussed as an application.

In **Chapter 5** We present some of the connections between a so-called reduced characteristic equation and a quadratic invariant of Time-Dependent Schrödinger equation. The solution of this equation previously obtained by the authors, presents practical difficulties since the Green's function of the equation requires solving a second-order differential equation with time-dependent coefficients called the characteristic equation. In the present paper, the authors use a gauge transformation lemma to generate a reduced characteristic equation. The transformation itself provides a simpler form of the general solution. The dynamical invariant of the corresponding Hamiltonian is explicit up to the solution of the very same characteristic equation. We conclude by using the gauge transformation to obtain a uniqueness result for a general quantum dot model by using dynamical invariants, and we state some of the possible future projects.

Chapter 2

THE TIME REVERSAL FOR MODIFIED OSCILLATORS

citation: R. Cordero-Soto and S. K. Suslov, Teoret. Mat. Fiz., 2010, **Volume 162** (2010) # 3, 345-380.

2.1 Introduction

The Cauchy initial value problem for the Schrödinger equation

$$i\frac{\partial\psi}{\partial t}=H(t)\,\psi \tag{2.1}$$

for a certain modified oscillator is explicitly solved in Ref. [143] for the case of n dimensions in R^n. When $n=1$ the Hamiltonian considered by Meiler, Cordero-Soto, and Suslov has the form

$$H(t)=\frac{1}{2}\left(aa^{\dagger}+a^{\dagger}a\right)+\frac{1}{2}e^{2it}a^2+\frac{1}{2}e^{-2it}\left(a^{\dagger}\right)^2, \tag{2.2}$$

where the creation and annihilation operators are defined as in [85]:

$$a^{\dagger}=\frac{1}{i\sqrt{2}}\left(\frac{\partial}{\partial x}-x\right),\qquad a=\frac{1}{i\sqrt{2}}\left(\frac{\partial}{\partial x}+x\right). \tag{2.3}$$

The corresponding time evolution operator is found in [143] as an integral operator

$$\psi(x,t)=U(t)\,\psi(x,0)=\int_{-\infty}^{\infty}G(x,y,t)\ \psi(y,0)\ dy \tag{2.4}$$

with the kernel (Green's function or propagator) given in terms of trigonometric and hyperbolic functions as follows

$$\begin{aligned}G(x,y,t) &= \frac{1}{\sqrt{2\pi i\left(\cos t\sinh t+\sin t\cosh t\right)}}\\ &\quad\times\exp\left(\frac{\left(x^2-y^2\right)\sin t\sinh t+2xy-\left(x^2+y^2\right)\cos t\cosh t}{2i\left(\cos t\sinh t+\sin t\cosh t\right)}\right).\end{aligned} \tag{2.5}$$

It is worth noting that the time evolution operator is known explicitly only in a few special cases. An important example of this source is the forced harmonic oscillator originally considered by Richard Feynman in his path integrals approach to the nonrelativistic quantum mechanics [77], [78], [79], [80], and [81]; see also [134]. Since then this problem and its special and limiting cases were discussed by many authors; see Refs. [14], [87], [99], [142], [145],

[207] for the simple harmonic oscillator and Refs. [4], [21], [98], [154], [173] for the particle in a constant external field and references therein. Furthermore, in Ref. [49] the time evolution operator for the one-dimensional Schrödinger equation (2.1) has been constructed in a general case when the Hamiltonian is an arbitrary quadratic form of the operator of coordinate and the operator of linear momentum. In this approach, the above mentioned exactly solvable models, including the modified oscillator of [143], are classified in terms of elementary solutions of a certain characteristic equation related to the Riccati differential equation.

In the present paper we find the time evolution operator for a "dual" time-dependent Schrödinger equation of the form

$$i\frac{\partial \psi}{\partial \tau} = H(\tau)\,\psi, \qquad \tau = \frac{1}{2}\sinh(2t) \tag{2.6}$$

with another "exotic" Hamiltonian of a modified oscillator given by

$$H(\tau) = \frac{1}{2}\left(aa^{\dagger} + a^{\dagger}a\right) + \frac{1}{2}e^{-i\arctan(2\tau)}a^{2} + \frac{1}{2}e^{i\arctan(2\tau)}\left(a^{\dagger}\right)^{2}. \tag{2.7}$$

We show that the corresponding propagator can be obtain from expression (2.5) by interchanging the coordinates $x \leftrightarrow y$. This implies that these two models are related to each other with respect to the inversion of time, which is the main result of this article.

The paper is organized as follows. In section 2 we derive the propagators for the Hamiltonians (2.2) and (2.7) following the method of [49] — expression (2.5) was obtain in [143] by a totally different approach using $SU(1,1)$-symmetry of the n-dimensional oscillator wave functions and the Meixner–Pollaczek polynomials. Another pair of completely integrable dual Hamiltonians is also discussed here. The "hidden" symmetry of quadratic propagators is revealed in section 3. The next section is concerned with the complex form of the propagators, which unifies Green's functions for several classical models by geometric means. In section 5 we consider the inverses of the corresponding time evolution operators and its relation with the inversion of time. A transition to the momentum representation in section 6 gives the reader a new insight on the symmetries of the quadratic Hamiltonians under consideration together with a set of identities for the corresponding time evolution operators. The n-dimensional case is discussed in sections 7 and 8. Particular solutions of the corresponding nonlinear Schrödinger equations are constructed in section 9. The last section is concerned with the ill-posedness of the Schrödinger equations. Three Appendixes at the end of the paper deal with required solutions of a certain type of characteristic equations, a quantum Hamiltonian transformation, and a Hamiltonian structure of the characteristic equations under consideration, respectively.

As in [49], [122], [143] and [194], we are dealing here with solutions of the time-dependent Schrödinger equation with variable coefficients. The case of a corresponding diffusion-type equation is investigated in [196]. These exactly solvable models are of interest in a general treatment of the nonlinear evolution equations; see [24], [31], [36], [37], [76], [119], [144], [203] and [10], [27], [28], [29], [30], [32], [33], [109], [153], [166], [174], [175], [184], [190] and references therein. They facilitate, for instance, a detailed study of problems related to global existence and uniqueness of solutions for the nonlinear Schrödinger equations with general quadratic Hamiltonians. Moreover, these explicit solutions can also be useful when testing numerical methods of solving the Schrödinger and diffusion-type equations with variable coefficients.

2.2 Derivation of The Propagators

The fundamental solution of the time-dependent Schrödinger equation with the quadratic Hamiltonian of the form

$$i\frac{\partial \psi}{\partial t} = -a(t)\frac{\partial^2 \psi}{\partial x^2} + b(t)x^2\psi - i\left(c(t)x\frac{\partial \psi}{\partial x} + d(t)\psi\right) \tag{2.8}$$

in two interesting special cases, namely,

$$a = \cos^2 t, \quad b = \sin^2 t, \quad c = 2d = \sin(2t) \tag{2.9}$$

and

$$a = \cosh^2 t, \quad b = \sinh^2 t, \quad c = 2d = -\sinh(2t), \tag{2.10}$$

corresponding to the Hamiltonians (2.2) and (2.7), respectively, (we give details of the proof in the Appendix B), can be found by the method proposed in [49] in the form

$$\psi = G(x,y,t) = \frac{1}{\sqrt{2\pi i\mu(t)}}\, e^{i\left(\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2\right)}, \tag{2.11}$$

where

$$\alpha(t) = \frac{1}{4a(t)} \frac{\mu'(t)}{\mu(t)} - \frac{d(t)}{2a(t)}, \tag{2.12}$$

$$\beta(t) = -\frac{1}{\mu(t)}, \qquad \frac{d\gamma}{dt} + \frac{a(t)}{\mu(t)^2} = 0, \tag{2.13}$$

$$\gamma(t) = \frac{a(t)}{\mu(t)\mu'(t)} + \frac{d(0)}{2a(0)} - 4\int_0^t \frac{a(\tau)\sigma(\tau)}{(\mu'(\tau))^2}\, d\tau, \tag{2.14}$$

and the function $\mu(t)$ satisfies the characteristic equation

$$\mu'' - \tau(t)\mu' + 4\sigma(t)\mu = 0 \tag{2.15}$$

with

$$\tau(t) = \frac{a'}{a} - 2c + 4d, \qquad \sigma(t) = ab - cd + d^2 + \frac{d}{2}\left(\frac{a'}{a} - \frac{d'}{d}\right) \tag{2.16}$$

subject to the initial data

$$\mu(0) = 0, \qquad \mu'(0) = 2a(0) \neq 0. \tag{2.17}$$

Equation (2.12) (more details can be found in [49]) allows us to integrate the familiar Riccati nonlinear differential equation emerging when one substitutes (2.11) into (2.8). See, for example, [91], [148], [171], [172], [213] and references therein. A Hamiltonian structure of these characteristic equations is discussed in Appendix C.

In the case (2.9), the characteristic equation has a special form of Ince's equation [135]

$$\mu'' + 2\tan t\ \mu' - 2\mu = 0. \tag{2.18}$$

Two linearly independent solutions are found in [49]:

$$\mu_1 = \cos t \cosh t + \sin t \sinh t = W(\cos t, \sinh t), \tag{2.19}$$

$$\mu_2 = \cos t \sinh t + \sin t \cosh t = W(\cos t, \cosh t) \tag{2.20}$$

with the Wronskian $W(\mu_1, \mu_2) = 2\cos^2 t = 2a$. Another method of integration of all characteristic equations from this section is discussed in the Appendix A; see Table 1 at the end of the paper for the sets of fundamental solutions. The second case (2.10) gives

$$\mu'' - 2\tanh t\ \mu' + 2\mu = 0 \tag{2.21}$$

and the two linearly independent solutions are [196]

$$\mu_2 = \cos t \sinh t + \sin t \cosh t = W(\cos t, \cosh t), \tag{2.22}$$

$$\mu_3 = \sin t \sinh t - \cos t \cosh t = W(\sin t, \cosh t) \tag{2.23}$$

with $W(\mu_2, \mu_3) = 2\cosh^2 t = 2a$. Equation (2.21) can be obtain from (2.18) as a result of the substitution $t \to it$. Also, $W(\mu_1, \mu_3) = \sin(2t) + \sinh(2t)$. The common solution of the both characteristic equations, namely,

$$\mu(t) = \mu_2 = \cos t \sinh t + \sin t \cosh t, \tag{2.24}$$

satisfies the initial conditions (2.17).

From (2.11)–(2.14), as a result of elementary calculations, one arrives at the Green function (2.5) in the case (2.9) and has to interchange there $x \leftrightarrow y$ in the second case (2.10). The reader can see some calculation details in section 9, where more general solutions are found in a similar way. The next section explains this unusual symmetry between two propagators from a more general point of view.

Two more completely integrable cases of the dual quadratic Hamiltonians occur when

$$a = \sin^2 t, \quad b = \cos^2 t, \quad c = 2d = -\sin(2t) \tag{2.25}$$

and

$$a = \sinh^2 t, \quad b = \cosh^2 t, \quad c = 2d = \sinh(2t). \tag{2.26}$$

The corresponding characteristic equations are

$$\mu'' - 2\cot t \ \mu' - 2\mu = 0 \tag{2.27}$$

and

$$\mu'' - 2\coth t \ \mu' + 2\mu = 0, \tag{2.28}$$

respectively, with a common solution

$$\mu(t) = \mu_4 = \sin t \cosh t - \cos t \sinh t = W(\sin t, \sinh t) \tag{2.29}$$

such that $\mu(0)=\mu'(0)=\mu''(0)=0$ and $\mu'''(0)=4$. Once again, from (2.11)–(2.14) one arrives at the Green function

$$G(x,y,t) = \frac{1}{\sqrt{2\pi i(\sin t\cosh t-\cos t\sinh t)}} \times \exp\left(\frac{\left(x^2+y^2\right)\cos t\cosh t-2xy+\left(x^2-y^2\right)\sin t\sinh t}{2i(\cos t\sinh t-\sin t\cosh t)}\right) \tag{2.30}$$

in the case (2.25) and has to interchange there $x\leftrightarrow y$ in the second case (2.26). The corresponding asymptotic formula takes the form

$$G(x,y,t)=\frac{e^{i\left(\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2\right)}}{\sqrt{2\pi i\mu(t)}}\sim\frac{1}{\sqrt{4\pi i\varepsilon}}\exp\left(i\frac{(x-y)^2}{4\varepsilon}\right) \tag{2.31}$$

as $\varepsilon=t^3/3\to 0^+$. We will show in section 6, see Eqs. (2.84) and (2.91), that our cases (2.9)–(2.10) and (2.25)–(2.26) are related to each other by means of the Fourier transform.

We have considered some elementary solutions of the characteristic equation (2.15), which are of interest in this paper. Generalizations to the forced modified oscillators are obvious; see Ref. [143]. More complicated cases may include special functions, like Bessel, hypergeometric or elliptic functions [2], [49], [122], [158], [170], [197], and [213].

2.3 On A "Hidden" Symmetry of Quadratic Propagators

Here we shall elaborate on the symmetry of propagators with respect to the substitution $x\leftrightarrow y$.

Lemma 1 *Consider two time-dependent Schrödinger equations with quadratic Hamiltonians*

$$i\frac{\partial\psi}{\partial t}=-a_k(t)\frac{\partial^2\psi}{\partial x^2}+b_k(t)x^2\psi-i\left(c_k(t)x\frac{\partial\psi}{\partial x}+d_k(t)\psi\right)\quad(k=1,2), \tag{2.32}$$

where $c_1-2d_1=c_2-2d_2=\varepsilon(t)$ and $d_k(0)=0$. Suppose that the initial value problems for corresponding characteristic equations

$$\mu''-\tau_k(t)\mu'+4\sigma_k(t)\mu=0,\qquad \mu(0)=0,\quad \mu'(0)=2a_k(0)\neq 0 \tag{2.33}$$

with

$$\tau_k(t)=\frac{a_k'}{a_k}-2c_k+4d_k,\quad \sigma_k(t)=a_kb_k-c_kd_k+d_k^2+\frac{d_k}{2}\left(\frac{a_k'}{a_k}-\frac{d_k'}{d_k}\right) \tag{2.34}$$

have a joint solution $\mu(t)$ and, in addition, the following relations hold

$$4\left(a_1b_1-c_1d_1+d_1^2\right)=\frac{4a_1a_2h^2-(\mu')^2}{\mu^2}-2\varepsilon\frac{\mu'}{\mu}=4\left(a_2b_2-c_2d_2+d_2^2\right), \tag{2.35}$$

where $h(t) = \exp\left(-\int_0^t \varepsilon(\tau)\ d\tau\right)$. *Then the corresponding fundamental solutions*

$$\psi_k = G_k(x,y,t) = \frac{1}{\sqrt{2\pi i\mu(t)}}\ e^{i\left(\alpha_k(t)x^2+\beta_k(t)xy+\gamma_k(t)y^2\right)} \tag{2.36}$$

possess the following symmetry

$$\alpha(t) = \alpha_1(t) = \gamma_2(t), \quad \gamma(t) = \gamma_1(t) = \alpha_2(t), \quad \beta(t) = \beta_1(t) = \beta_2(t) \tag{2.37}$$

and

$$G_1(x,y,t) = G_2(y,x,t). \tag{2.38}$$

This property holds for a single Schrödinger equation under the single hypothesis (2.35).

Indeed, according to Ref. [49],

$$\beta(t) = \beta_1(t) = \beta_2(t) = -\frac{h(t)}{\mu(t)} \tag{2.39}$$

in the case of a joint solution $\mu(t)$ of two characteristic equations. In view of the structure of propagators for general quadratic Hamiltonians found in [49], the symmetry under consideration holds if we have

$$\alpha = \frac{1}{4a_1}\frac{\mu'}{\mu} - \frac{d_1}{2a_1}, \qquad \frac{d\alpha}{dt} + a_2\frac{h^2}{\mu^2} = 0 \tag{2.40}$$

and

$$\gamma = \frac{1}{4a_2}\frac{\mu'}{\mu} - \frac{d_2}{2a_2}, \qquad \frac{d\gamma}{dt} + a_1\frac{h^2}{\mu^2} = 0, \tag{2.41}$$

simultaneously. Excluding α from (2.40), one gets

$$\mu'' - \frac{a_1'}{a_1}\mu' + 2d_1\left(\frac{a_1'}{a_1} - \frac{d_1'}{d_1}\right)\mu = \frac{(\mu')^2 - 4a_1a_2h^2}{\mu}. \tag{2.42}$$

Comparison with the characteristic equation results in the first condition in (2.35). The case of γ, which gives the second condition, is similar. This completes the proof.

A few examples are in order. When $a = 1/2$, $b = c = d = 0$, and $\mu'' = 0$, $\mu = t$, one gets

$$G(x,y,t) = \frac{1}{\sqrt{2\pi it}}\ \exp\left(\frac{i(x-y)^2}{2t}\right) \tag{2.43}$$

as the free particle propagator [81] with an obvious symmetry under consideration. Our criteria (2.35), namely, $4a^2 = (\mu')^2$, stands.

The simple harmonic oscillator with $a=b=1/2$, $c=d=0$ and $\mu''+\mu=0$, $\mu=\sin t$ has the familiar propagator of the form

$$G(x,y,t)=\frac{1}{\sqrt{2\pi i\sin t}}\ \exp\left(\frac{i}{2\sin t}\left(\left(x^2+y^2\right)\cos t-2xy\right)\right), \tag{2.44}$$

which is studied in detail at [14], [87], [99], [142], [145], [207]. (For an extension to the case of the forced harmonic oscillator including an extra velocity-dependent term and a time-dependent frequency, see [77], [78], [81] and [134].) Our condition (2.35) takes the form of the trigonometric identity

$$4ab=\frac{4a^2-(\mu')^2}{\mu^2}, \tag{2.45}$$

which confirms the symmetry of the propagator.

For the quantum damped oscillator [50], $a=b=\omega_0/2$, $c=d=-\lambda$ and

$$\begin{aligned} G(x,y,t) &= \sqrt{\frac{\omega e^{\lambda t}}{2\pi i\omega_0\sin\omega t}}\ \exp\left(\frac{i\omega}{2\omega_0\sin\omega t}\left(\left(x^2+y^2\right)\cos\omega t-2xy\right)\right) \\ &\quad\times\exp\left(\frac{i\lambda}{2\omega_0}\left(x^2-y^2\right)\right) \end{aligned} \tag{2.46}$$

with $\omega=\sqrt{\omega_0^2-\lambda^2}>0$ and $\mu=(\omega_0/\omega)\,e^{-\lambda t}\sin\omega t$. The criterion

$$4ab=\frac{4\,(ah)^2-(\mu')^2}{\mu^2}-2\varepsilon\frac{\mu'}{\mu}, \tag{2.47}$$

where $\varepsilon=c-2d=\lambda$ and $h=e^{-\lambda t}$, holds. But here $d(0)=-\lambda\neq 0$ and a more detailed analysis of asymptotics gives an extra antisymmetric term in the propagator above; see [50] for more details.

The case of Hamiltonians (2.2) and (2.7) corresponds to

$$a_1=\cos^2 t,\quad b_1=\sin^2 t,\quad c_1=2d_1=\sin(2t) \tag{2.48}$$

and

$$a_2=\cosh^2 t,\quad b_2=\sinh^2 t,\quad c_2=2d_2=-\sinh(2t), \tag{2.49}$$

respectively. Our criteria (2.35) are satisfied in view of an obvious identity

$$4a_1a_2=\left(\mu'\right)^2. \tag{2.50}$$

The characteristic function is given by (2.24). This explains the propagator symmetry found in the previous section.

Our last dual pair of quadratic Hamiltonians has the following coefficients

$$a_1 = \sin^2 t, \quad b_1 = \cos^2 t, \quad c_1 = 2d_1 = -\sin(2t) \tag{2.51}$$

and

$$a_2 = \sinh^2 t, \quad b_2 = \cosh^2 t, \quad c_2 = 2d_2 = \sinh(2t). \tag{2.52}$$

The criteria (2.35) are satisfied in view of the identity (2.50) with the characteristic function (2.29) and, therefore, the propagator (2.30) obeys the symmetry under the substitution $x \leftrightarrow y$.

Remark 1 *A simple relation*

$$\frac{\mu'}{\mu} = 4\frac{\sigma_1 - \sigma_2}{\tau_1 - \tau_2}, \tag{2.53}$$

which is valid for a joint solution of two characteristic equations, can be used in our criteria (2.35).

Although we have formulated the hypotheses of our lemma for the Green functions only, it can be applied to solutions with regular initial data. For instance, a pair of characteristic equations (2.18) and (2.28) has a joint solution given by (2.19), which does not satisfy initial conditions required for the Green functions. The coefficients of the corresponding quadratic Hamiltonians are

$$a_1 = \cos^2 t, \quad b_1 = \sin^2 t, \quad c_1 = 2d_1 = \sin(2t) \tag{2.54}$$

and

$$a_2 = \sinh^2 t, \quad b_2 = \cosh^2 t, \quad c_2 = 2d_2 = \sinh(2t). \tag{2.55}$$

The criteria (2.35) are satisfied in view of the identity (2.50) and the particular solution

$$\begin{aligned} \psi = K(x,y,t) &= \frac{1}{\sqrt{2\pi(\cos t\cosh t + \sin t\sinh t)}} \\ &\times \exp\left(\frac{\left(x^2+y^2\right)\sin t\cosh t - 2xy - \left(x^2-y^2\right)\cos t\sinh t}{2i(\cos t\cosh t + \sin t\sinh t)}\right) \end{aligned} \tag{2.56}$$

obeys the symmetry under the substitution $x \leftrightarrow y$. The initial condition is the standing wave $K(x,y,0) = e^{ixy}/\sqrt{2\pi}$.

In a similar fashion, the characteristic equations (2.27) and (2.21) have a common solution

$$\mu = -\mu_3 = \cos t \cosh t - \sin t \sinh t. \tag{2.57}$$

The coefficients of the corresponding Hamiltonians are

$$a_1 = \sin^2 t, \quad b_1 = \cos^2 t, \quad c_1 = 2d_1 = -\sin(2t) \tag{2.58}$$

and

$$a_2 = \cosh^2 t, \quad b_2 = \sinh^2 t, \quad c_2 = 2d_2 = -\sinh(2t). \tag{2.59}$$

The criteria (2.35) are satisfied once again and the particular solution is given by

$$\begin{aligned} \psi = K(x,y,t) = & \frac{1}{\sqrt{2\pi(\cos t \cosh t - \sin t \sinh t)}} \\ & \times \exp\left(\frac{(x^2+y^2)\sin t \cosh t + 2xy + (x^2-y^2)\cos t \sinh t}{2i(\cos t \cosh t - \sin t \sinh t)} \right) \end{aligned} \tag{2.60}$$

with $K(x,y,0) = e^{-ixy}/\sqrt{2\pi}$. We shall discuss in section 6 how these solutions are related to the corresponding time evolution operators.

2.4 Complex Form of The Propagators

It is worth noting that the propagator in (2.5) can be rewritten in terms of the Wronskians of trigonometric and hypergeometric functions as

$$\begin{aligned} G(x,y,t) &= \frac{1}{\sqrt{2\pi i W(\cos t, \cosh t)}} \\ &\quad \times \exp\left(\frac{W(\sin t, \cosh t)\, x^2 + 2xy - W(\cos t, \sinh t)\, y^2}{2iW(\cos t, \cosh t)} \right). \end{aligned} \tag{2.61}$$

This simply means that our propagator has the following structure

$$G = \sqrt{\frac{c_2 - c_3}{4\pi i}} \exp\left(\frac{c_1 x^2 + (c_2 - c_3)\, xy - c_4 y^2}{2i} \right), \tag{2.62}$$

where the coefficients are solutions of the system of linear equations

$$\begin{aligned} c_1 \cos t + c_2 \cosh t &= \sin t, \\ -c_1 \sin t + c_2 \sinh t &= \cos t, \\ c_3 \cos t + c_4 \cosh t &= \sinh t, \\ -c_3 \sin t + c_4 \sinh t &= \cosh t \end{aligned} \tag{2.63}$$

obtained by Cramer's rule. A complex form of this system is

$$c_1 z^* + c_2 \zeta = i z^*, \quad c_3 z^* + c_4 \zeta = i \zeta^*, \tag{2.64}$$

where we introduce two complex variables

$$z = \cos t + i \sin t, \qquad \zeta = \cosh t + i \sinh t \tag{2.65}$$

and use the star for complex conjugate. Taking the complex conjugate of the system (2.64), which has the real-valued solutions, and using Cramer's rule once again, one gets

$$\begin{aligned} c_1 &= \frac{z\zeta + z^*\zeta^*}{i\left(z\zeta - z^*\zeta^*\right)}, \qquad c_2 = \frac{2i}{z\zeta - z^*\zeta^*}, \\ c_3 &= \frac{2}{i\left(z\zeta - z^*\zeta^*\right)}, \qquad c_4 = -\frac{z\zeta^* + z^*\zeta}{i\left(z\zeta - z^*\zeta^*\right)}. \end{aligned} \tag{2.66}$$

As a result, we obtain a compact symmetric expression of the propagator (2.5) as a function of two complex variables

$$\begin{aligned} G(x,y,t) &= G(x,y,z,\zeta) = \frac{1}{\sqrt{\pi\left(z\zeta - z^*\zeta^*\right)}} \\ &\times \exp\left(\frac{\left(z\zeta + z^*\zeta^*\right)x^2 - 4xy + \left(z\zeta^* + z^*\zeta\right)y^2}{2\left(z^*\zeta^* - z\zeta\right)}\right). \end{aligned} \tag{2.67}$$

This function takes a familiar form

$$G = \frac{1}{\sqrt{2\pi i\left(x_1x_4 + x_2x_3\right)}} \exp\left(\frac{\left(x^2 - y^2\right)x_2x_4 + 2xy - \left(x^2 + y^2\right)x_1x_3}{2i\left(x_1x_4 + x_2x_3\right)}\right), \tag{2.68}$$

in a real-valued four-dimensional vector space, if we set $z = x_1 + ix_2$ and $\zeta = x_3 + ix_4$ with $x_1' = -x_2$, $x_2' = x_1$, $x_3' = x_4$, $x_4' = x_3$, and solve the following initial value problem

$$\begin{aligned} &x_1'' + x_1 = 0, \qquad x_1(0) = 1, \quad x_1'(0) = 0, \\ &x_2'' + x_2 = 0, \qquad x_2(0) = 0, \quad x_2'(0) = 1, \\ &x_3'' - x_3 = 0, \qquad x_3(0) = 1, \quad x_3'(0) = 0, \\ &x_4'' - x_4 = 0, \qquad x_4(0) = 0, \quad x_4'(0) = 1, \end{aligned} \tag{2.69}$$

the solution of which can be interpreted as a trajectory of a classical "particle" moving in this space; cf. (2.5).

It is worth noting that our propagators expression (2.67), extended to a function of two independent complex variables z and ζ, allows us to unify several exactly solvable quantum mechanical models in geometrical terms, namely, by choosing different contours, with certain "synchronized" parametrization, in the pair of complex "time" planes under consideration. Indeed, the free particle propagator (2.43) appears in this way when one chooses $z = 1$ and $\zeta = 1 + it$. The simple harmonic oscillator propagator (2.44) corresponds to the case $z = 1$ and $\zeta = e^{it}$. As we have seen in this section, the propagator (2.5) is also a special case of (2.67). This is why we may refer to the Hamiltonians under consideration as the ones of modified oscillators. By a vague analogy with the special theory of relativity, one may also say that in this case there are two synchronized "clocks", namely, the two contour parameterized by (2.65), one in Euclidean and another one in the pseudo-Euclidean two dimensional spaces, respectively, which geometrically describes a time evolution for the Hamiltonians of modified oscillators. This idea of introducing a geometric structure of time in the problem under consideration may be useful for other types of evolutionary equations.

In a similar fashion, our new propagator (2.30) can be rewritten in terms of the Wronskians as

$$\begin{aligned} G(x,y,t) &= \frac{1}{\sqrt{2\pi i W(\sin t, \sinh t)}} \\ &\quad \times \exp\left(\frac{W(\cos t, \sinh t)\, x^2 - 2xy - W(\sin t, \cosh t)\, y^2}{2iW(\sinh t, \sin t)}\right). \end{aligned} \tag{2.70}$$

The corresponding complex form is

$$\begin{aligned} G(x,y,t) &= G(x,y,z,\zeta) = \frac{1}{\sqrt{\pi\left(z\zeta^* - z^*\zeta\right)}} \\ &\quad \times \exp\left(\frac{\left(z\zeta^* + z^*\zeta\right)x^2 - 4xy + \left(z\zeta + z^*\zeta^*\right)y^2}{2\left(z^*\zeta - z\zeta^*\right)}\right). \end{aligned} \tag{2.71}$$

We leave the details to the reader.

2.5 The Inverse Operator and Time Reversal

We follow the method suggested in [194] for general quadratic Hamiltonians with somewhat different details. The left inverse of the integral operator defined by (2.4)–(2.5), namely,

$$U(t)\,\psi(x) = \int_{-\infty}^{\infty} G(x,y,t)\ \psi(y)\ dy, \tag{2.72}$$

is

$$U^{-1}(t)\,\chi(x)=\int_{-\infty}^{\infty}G(y,x,-t)\ \chi(y)\ dy \tag{2.73}$$

in view of $U^{-1}=U^{\dagger}$. Indeed, when $s<t$, by the Fubini theorem

$$\begin{aligned} U^{-1}(s)\,(U(t)\,\psi) &= U^{-1}(s)\,\chi=\int_{-\infty}^{\infty}G(z,x,-s)\ \chi(z)\ dz \\ &= \int_{-\infty}^{\infty}G(z,x,-s)\ \left(\int_{-\infty}^{\infty}G(z,y,t)\ \psi(y)\ dy\right)\ dz \\ &= \int_{-\infty}^{\infty}G(x,y,s,t)\ \psi(y)\ dy. \end{aligned} \tag{2.74}$$

Here

$$\begin{aligned} G(x,y,s,t) &= \int_{-\infty}^{\infty}G(z,x,-s)\,G(z,y,t)\ dz \\ &= \frac{e^{i\left(\gamma(t)y^2-\gamma(s)x^2\right)}}{2\pi\sqrt{\mu(s)\,\mu(t)}}\int_{-\infty}^{\infty}e^{i\left((\alpha(t)-\alpha(s))z^2+(\beta(t)y-\beta(s)x)z\right)}\ dz \\ &= \frac{1}{\sqrt{4\pi i\mu(s)\,\mu(t)\,(\alpha(s)-\alpha(t))}} \\ &\quad\times\exp\left(\frac{(\beta(t)\,y-\beta(s)\,x)^2-4\,(\alpha(t)-\alpha(s))\left(\gamma(t)\,y^2-\gamma(s)\,x^2\right)}{4i\,(\alpha(t)-\alpha(s))}\right) \end{aligned} \tag{2.75}$$

by the familiar Gaussian integral [19], [162] and [179]:

$$\int_{-\infty}^{\infty}e^{i\left(az^2+2bz\right)}\,dz=\sqrt{\frac{\pi i}{a}}\,e^{-ib^2/a}. \tag{2.76}$$

In view of (2.13),

$$(\beta(t)\,y-\beta(s)\,x)^2=\left(\frac{x}{\mu(s)}-\frac{y}{\mu(t)}\right)^2=\frac{\left(\sqrt{\frac{\mu(t)}{\mu(s)}}x-\sqrt{\frac{\mu(s)}{\mu(t)}}y\right)^2}{\mu(s)\,\mu(t)}, \tag{2.77}$$

and a singular part of (2.75) becomes

$$\begin{aligned} &\frac{1}{\sqrt{4\pi i\mu(s)\,\mu(t)\,(\alpha(s)-\alpha(t))}}\exp\left(\frac{(\beta(t)\,y-\beta(s)\,x)^2}{4i\,(\alpha(t)-\alpha(s))}\right) \\ &=\frac{1}{\sqrt{4\pi i\mu(s)\,\mu(t)\,(\alpha(s)-\alpha(t))}}\exp\left(\frac{\left(\sqrt{\frac{\mu(t)}{\mu(s)}}x-\sqrt{\frac{\mu(s)}{\mu(t)}}y\right)^2}{4i\mu(s)\,\mu(t)\,(\alpha(t)-\alpha(s))}\right). \end{aligned}$$

Thus, in the limit $s\to t^-$, one can obtain formally the identity operator in the right hand side of (2.74). The leave the details to the reader.

On the other hand, the integral operator in (2.4)–(2.5), namely,

$$\chi(x) = \frac{1}{\sqrt{2\pi i\mu(t)}} \int_{-\infty}^{\infty} e^{i\left(\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2\right)}\ \psi(y)\ dy \tag{2.78}$$

is essentially the Fourier transform and its inverse is given by

$$\psi(y) = \frac{1}{\sqrt{-2\pi i\mu(t)}} \int_{-\infty}^{\infty} e^{-i\left(\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2\right)}\ \chi(x)\ dx \tag{2.79}$$

in correspondence with our definition (2.73) in view of (2.13).

The Schrödinger equation (2.1) retains the same form if we replace t in it by $-t$ and, at the same time, take complex conjugate provided that $(H(-t)\varphi)^* = H(t)\varphi^*$. The last relation holds for both Hamiltonians (2.2) and (2.7). Hence the function $\chi(x,t) = \psi^*(x,-t)$ does satisfy the same equation as the original wave function $\psi(x,t)$. This property is usually known as the symmetry with respect to time inversion (time reversal) in quantum mechanics [87], [121], [145], [216]. This fact is obvious from a general solution given by (2.4)–(2.5) for our Hamiltonians.

On the other hand, by the definition, the (left) inverse $U^{-1}(t)$ of the time evolution operator $U(t)$ returns the system to its initial quantum state:

$$\psi(x,t) = U(t)\,\psi(x,0), \tag{2.80}$$

$$U^{-1}(t)\,\psi(x,t) = U^{-1}(t)\left(U(t)\,\psi(x,0)\right) = \psi(x,0). \tag{2.81}$$

Our analysis of two oscillator models under consideration shows that this may be related to the reversal of time in the following manner. The left inverse of the time evolution operator (2.4) for the Schrödinger equation (2.1) with the original Hamiltonian of a modified oscillator (2.2) can be obtained by the time inversion $t \to -t$ in the evolution operator corresponding to the new “dual” Hamiltonian (2.7) (and vise versa). The same is true for the second pair of dual Hamiltonians. More details will be given in section 6. This is an example of a situation in mathematical physics and quantum mechanics when a change in the direction of time may require a total change of the system dynamics in order to return the system back to its original quantum state. Moreover, moving backward in time the system will repeat the same quantum states only when

$$\psi(x,t-s) = U(t-s)\,\psi(x,0) = U^{-1}(s)\,U(t)\,\psi(x,0), \qquad 0 \le s \le t, \tag{2.82}$$

which is equivalent to the semi-group property

$$U(s)U(t-s)=U(t) \tag{2.83}$$

for the time evolution operator. This seems not true for propagators (2.5) and (2.30).

2.6 The Momentum Representation

The time-dependent Schrödinger equation (2.8) can be rewritten in terms of the operator of coordinate x and the operator of linear momentum $p_x = i^{-1}\partial/\partial x$ as follows

$$i\frac{\partial\psi}{\partial t}=\left(a(t)\,p_x^2+b(t)\,x^2+d(t)\,(xp_x+p_xx)\right)\psi \tag{2.84}$$

with $c=2d$. The corresponding quadratic Hamiltonian

$$H=a(t)\,p_x^2+b(t)\,x^2+d(t)\,(xp_x+p_xx) \tag{2.85}$$

obeys a special symmetry, namely, it formally preserves this structure under the permutation $x\leftrightarrow p_x$. This fact is well-known for the simple harmonic oscillator [87], [121], [145].

In order to interchange the coordinate and momentum operators in quantum mechanics one switches between the coordinate and momentum representations by means of the Fourier transform

$$\psi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ixy}\chi(y)\ dy=F[\chi] \tag{2.86}$$

and its inverse

$$\chi(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ixy}\psi(x)\ dx=F^{-1}[\psi]. \tag{2.87}$$

Indeed, the familiar properties

$$p_x\psi=p_xF[\chi]=F[y\chi],\qquad x\psi=xF[\chi]=-F[p_y\chi] \tag{2.88}$$

imply

$$p_x^2\psi=F\left[y^2\chi\right],\qquad x^2\psi=F\left[p_y^2\chi\right] \tag{2.89}$$

and

$$(xp_x+p_xx)\,\psi=-F\left[(p_yy+yp_y)\,\chi\right]. \tag{2.90}$$

Therefore,

$$\begin{aligned} H\psi &= \left(ap_x^2 + bx^2 + d\left(xp_x + p_x x\right)\right) F\left[\chi\right] \\ &= F\left[\left(bp_y^2 + ay^2 - d\left(yp_y + p_y y\right)\right)\chi\right] \end{aligned}$$

by the linearity of the Fourier transform. In view of

$$\frac{\partial\psi}{\partial t} = F\left[\frac{\partial\chi}{\partial t}\right]$$

the Schrödinger equation (2.84) takes the form

$$i\frac{\partial\chi}{\partial t} = \left(b\left(t\right)p_y^2 + a\left(t\right)y^2 - d\left(t\right)\left(yp_y + p_y y\right)\right)\chi \tag{2.91}$$

with $a \leftrightarrow b$ and $d \to -d$ in the momentum representation.

This property finally reveals that our quadratic Hamiltonians (2.9) and (2.25), similarly (2.10) and (2.26), corresponds to the same Schrödinger equation written in the coordinate and momentum representations, respectively. Thus, in section 2, we have solved the Cauchy initial value problem for modified oscillators both in the coordinate and momentum representations.

In this paper the creation and annihilation operators are defined by

$$a^\dagger = \frac{p_x + ix}{\sqrt{2}} = \frac{1}{i\sqrt{2}}\left(\frac{\partial}{\partial x} - x\right), \tag{2.92}$$

$$a = \frac{p_x - ix}{\sqrt{2}} = \frac{1}{i\sqrt{2}}\left(\frac{\partial}{\partial x} + x\right) \tag{2.93}$$

with the familiar commutator $\left[a, a^\dagger\right] = aa^\dagger - a^\dagger a = 1$ [85]. One can see that

$$a_x\psi = a_x F\left[\chi\right] = F\left[ia_y\chi\right], \tag{2.94}$$

$$a_x^\dagger\psi = a_x^\dagger F\left[\chi\right] = F\left[-ia_y^\dagger\chi\right],$$

or

$$a_x \to ia_y, \qquad a_x^\dagger \to -ia_y^\dagger \tag{2.95}$$

under the Fourier transform. This observation will be important in the next section.

Finally we summarize all results on solution of the Cauchy initial value problems for the modified oscillator under consideration. We denote

$$
\begin{aligned}
U(t)\,\psi(x) &= \int_{-\infty}^{\infty} G_U(x,y,t)\ \psi(y)\ dy, && (2.96)\\
K(t)\,\psi(x) &= \int_{-\infty}^{\infty} K_U(x,y,t)\ \psi(y)\ dy, && (2.97)\\
V(t)\,\psi(x) &= \int_{-\infty}^{\infty} G_V(x,y,t)\ \psi(y)\ dy, && (2.98)\\
L(t)\,\psi(x) &= \int_{-\infty}^{\infty} K_V(x,y,t)\ \psi(y)\ dy. && (2.99)
\end{aligned}
$$

The kernels of these integral operators are defined as follows. Here $G_U(x,y,t)$ and $G_V(x,y,t)$ are the Green functions in (2.5) and (2.30), respectively. The kernels $K_U(x,y,t)$ and $K_V(x,y,t)$ are given by (2.56) and (2.60), respectively. The following operator identities hold

$$U(t) = K(t)F^{-1} = FL(t) = FV(t)F^{-1}, \tag{2.100}$$

$$V(t) = L(t)F = F^{-1}K(t) = F^{-1}U(t)F, \tag{2.101}$$

$$U^{-1}(t) = FK^{-1}(t) = L^{-1}(t)F^{-1} = FV^{-1}(t)F^{-1}, \tag{2.102}$$

$$V^{-1}(t) = F^{-1}L^{-1}(t) = K^{-1}(t)F = F^{-1}U^{-1}(t)F, \tag{2.103}$$

$$K(t) = FL(t)F, \qquad L(t) = F^{-1}K(t)F^{-1}, \tag{2.104}$$

$$K^{-1}(t) = F^{-1}L^{-1}(t)F^{-1}, \quad L^{-1}(t) = FK^{-1}(t)F. \tag{2.105}$$

Here F and F^{-1} are the operators of Fourier transform and its inverse, respectively, which relate the wave functions in the coordinate and momentum representations

$$\psi = F[\chi], \qquad \chi = F^{-1}[\psi]$$

at any given moment of time. The time evolution operators $U(t)$, $V(t)$ and their inverses $U^{-1}(t)$, $V^{-1}(t)$ obey the symmetry with respect to the time reversal, which has been discussed in section 5.

Consider the particular solution (2.56). A more general solution of the Schrödinger equation (2.8)–(2.9) can be obtained by the superposition principle in the form

$$\psi(x,t) = \int_{-\infty}^{\infty} K_U(x,y,t)\ \chi(y,0)\ dy, \tag{2.106}$$

where χ is a suitable arbitrary function, independent of time, such that the integral converges and one can interchange the differentiation and integration. In view of the continuity of the kernel at $t=0$, we get

$$\psi(x,0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ixy}\,\chi(y,0)\ dy, \tag{2.107}$$

which simply relates the initial data in the coordinate and momentum representations. Then solution of the initial value problem is given by the inverse of the Fourier transform

$$\chi(y,0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ixy}\,\psi(x,0)\ dx \tag{2.108}$$

followed by the back substitution of this expression into (2.106). This implies the above factorization $U(t)=K(t)F^{-1}$ of the corresponding time evolution operator; see (2.100). The Green function (2.5) can be derived as

$$G_U(x,y,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}K_U(x,z,t)\ e^{-iyz}\,dz \tag{2.109}$$

with the help of the integral (2.76). The second equation, $U(t)=FL(t)$, is related to following integral

$$G_U(x,y,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ixz}\,K_V(z,y,t)\ dz. \tag{2.110}$$

The meaning of the operator $L(t)$, is established in a similar fashion. One can see that the relation $V(t)=L(t)F$ in (2.101) follows from the elementary integral

$$G_V(x,y,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}K_V(x,z,t)\ e^{iyz}\,dz \tag{2.111}$$

and $K(t)=FV(t)$ corresponds to

$$K_U(x,y,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ixz}\,G_V(z,y,t)\ dz. \tag{2.112}$$

This proves (2.100)–(2.101). The inverses $K^{-1}(t)$ and $L^{-1}(t)$ are found, for instance, by the inverse of Fourier transform similar to (2.78)–(2.79). They are not directly related to the reversal of time.

2.7 The Case of n-Dimensions

In the case of R^n with an arbitrary number of dimensions, the Schrödinger equation for a modified oscillator (2.1) with the original Hamiltonian

$$H(t)=\frac{1}{2}\sum_{s=1}^{n}\left(a_s a_s^{\dagger}+a_s^{\dagger}a_s\right)+\frac{1}{2}e^{2it}\sum_{s=1}^{n}\left(a_s\right)^2+\frac{1}{2}e^{-2it}\sum_{s=1}^{n}\left(a_s^{\dagger}\right)^2, \tag{2.113}$$

considered by Meiler, Cordero-Soto, and Suslov [143], has the Green function of the form

$$\begin{aligned} G_t(x,x') &= \prod_{s=1}^{n} G_t(x_s,x'_s) \qquad (2.114) \\ &= \left(\frac{1}{2\pi i(\cos t\sinh t+\sin t\cosh t)}\right)^{n/2} \\ &\quad \times \exp\left(\frac{(x^2-x'^2)\sin t\sinh t+2x\cdot x'-(x^2+x'^2)\cos t\cosh t}{2i(\cos t\sinh t+\sin t\cosh t)}\right). \end{aligned}$$

Solution of the Cauchy initial value problem can be written as

$$\psi(x,t)=\int_{R^n} G_t(x,x')\ \psi(x',0)\ dx', \qquad (2.115)$$

where $dv'=dx'=dx'_1\cdot\ldots\ \cdot dx'_n$. The propagator expansion in the hyperspherical harmonics is given by

$$G_t(x,x')=\sum_{K\nu} Y_{K\nu}(\Omega)\ Y^*_{K\nu}(\Omega')\ \mathcal{G}^K_t(r,r') \qquad (2.116)$$

with

$$\begin{aligned} \mathcal{G}^K_t(r,r') &= \frac{e^{-i\pi(2K+n)/4}}{2^{K+n/2-1}\Gamma(K+n/2)}\frac{(rr')^K}{(\cos t\sinh t+\sin t\cosh t)^{K+n/2}} \qquad (2.117) \\ &\quad \times \exp\left(i\frac{\left(r^2+(r')^2\right)\cos t\cosh t-\left(r^2-(r')^2\right)\sin t\sinh t}{2(\cos t\sinh t+\sin t\cosh t)}\right) \\ &\quad \times {}_0F_1\left(\begin{matrix} - \\ K+n/2 \end{matrix};-\frac{(rr')^2}{4(\cos t\sinh t+\sin t\cosh t)^2}\right). \end{aligned}$$

Here $Y_{K\nu}(\Omega)$ are the hyperspherical harmonics constructed by the given tree T in the graphical approach of Vilenkin, Kuznetsov and Smorodinskiĭ [157], the integer K corresponds to the constant of separation of the variables at the root of T (denoted by K due to the tradition of the method of K-harmonics in nuclear physics [188]) and $\nu=\{l_1,l_2,\ldots,l_p\}$ is the set of all other subscripts corresponding to the remaining vertexes of the binary tree T. These formulas imply the familiar expansion of a plane wave in R^n in terms of the hyperspherical harmonics

$$e^{ix\cdot x'}=rr'\left(\frac{2\pi}{rr'}\right)^{n/2}\sum_{K\nu} i^K\ Y^*_{K\nu}(\Omega)\ Y_{K\nu}(\Omega')\ J_{K+n/2-1}(rr'), \qquad (2.118)$$

where

$$J_\mu(z)=\frac{(z/2)^\mu}{\Gamma(\mu+1)}\ {}_0F_1\left(\begin{matrix} - \\ \mu+1 \end{matrix};-\frac{z^2}{4}\right) \qquad (2.119)$$

is the Bessel function. See [143] and references therein for more details. It is worth noting that the Green function (2.5) was originally found by Meiler, Cordero-Soto, and Suslov as the special case $n = 1$ of the expansion (2.116)–(2.117). The dynamical $SU(1,1)$ symmetry of the harmonic oscillator wave functions, Bargmann's functions for the discrete positive series of the irreducible representations of this group, the Fourier integral of a weighted product of the Meixner–Pollaczek polynomials, a Hankel-type integral transform and the hyperspherical harmonics were utilized in order to derive the n-dimensional Green function.

Our results show that the "dual" Schrödinger equation (2.6) with a new Hamiltonian of the form

$$H(\tau) = \frac{1}{2}\sum_{s=1}^{n}\left(a_s a_s^{\dagger} + a_s^{\dagger} a_s\right) + \frac{1}{2}e^{-i\arctan(2\tau)}\sum_{s=1}^{n}\left(a_s\right)^2 + \frac{1}{2}e^{i\arctan(2\tau)}\sum_{s=1}^{n}\left(a_s^{\dagger}\right)^2 \tag{2.120}$$

has the propagator that is almost identical to (2.114) but with $x \leftrightarrow x'$. Indeed, in the case of n-dimensions one has

$$H(\tau) = \sum_{s=1}^{n} H_s(\tau), \tag{2.121}$$

where we denote

$$H_s(\tau) = \frac{1}{2}\left(a_s a_s^{\dagger} + a_s^{\dagger} a_s\right) + \frac{1}{2}e^{-i\arctan(2\tau)}\left(a_s\right)^2 + \frac{1}{2}e^{i\arctan(2\tau)}\left(a_s^{\dagger}\right)^2. \tag{2.122}$$

If one chooses

$$\begin{aligned}
\psi_s &= \psi_s(x_s, t) = G_t\left(x_s', x_s\right) \qquad (2.123)\\
&= \left(\frac{1}{2\pi i\left(\cos t \sinh t + \sin t \cosh t\right)}\right)^{1/2}\\
&\quad \times \exp\left(\frac{\left(x_s'^2 - x_s^2\right)\sin t \sinh t + 2x_s' x_s - \left(x_s'^2 + x_s^2\right)\cos t \cosh t}{2i\left(\cos t \sinh t + \sin t \cosh t\right)}\right)
\end{aligned}$$

with

$$i\frac{\partial \psi_s}{\partial \tau} = H_s(\tau)\,\psi_s, \qquad t = \frac{1}{2}\sinh(2\tau) \tag{2.124}$$

and denotes

$$\psi = \prod_{k=1}^{n}\psi_k = \prod_{k=1}^{n} G_t\left(x_k', x_k\right) = G_t\left(x', x\right), \tag{2.125}$$

then

$$i\frac{\partial \psi}{\partial \tau} = \sum_{s=1}^{n}\left(i\frac{\partial \psi_s}{\partial \tau}\right)\prod_{k\neq s}\psi_k \tag{2.126}$$

and

$$H(\tau)\psi=\sum_{s=1}^{n}(H_s(\tau)\psi_s)\prod_{k\neq s}\psi_k. \tag{2.127}$$

As a result,

$$\left(i\frac{\partial}{\partial\tau}-H(\tau)\right)\psi=\sum_{s=1}^{n}\left(i\frac{\partial\psi_s}{\partial\tau}-H_s(\tau)\psi_s\right)\prod_{k\neq s}\psi_k\equiv 0, \tag{2.128}$$

and Eq. (2.6) for the n-dimensional propagator is satisfied. For the initial data, formally,

$$\lim_{t\to 0^+}G_t\left(x',x\right)=\prod_{k=1}^{n}\lim_{t\to 0^+}G_t\left(x_k',x_k\right)=\prod_{k=1}^{n}\delta\left(x_k'-x_k\right)=\delta\left(x-x'\right), \tag{2.129}$$

where $\delta(x)$ is the Dirac delta function in R^n. Further details are left to the reader.

The n-dimensional version of the Hamiltonian corresponding to the coefficients (2.25) is given by

$$H(t)=\frac{1}{2}\sum_{s=1}^{n}\left(a_sa_s^{\dagger}+a_s^{\dagger}a_s\right)-\frac{1}{2}e^{2it}\sum_{s=1}^{n}(a_s)^2-\frac{1}{2}e^{-2it}\sum_{s=1}^{n}\left(a_s^{\dagger}\right)^2 \tag{2.130}$$

with the propagator

$$\begin{aligned}G_t\left(x,x'\right) &= \left(\frac{1}{2\pi i(\sin t\cosh t-\cos t\sinh t)}\right)^{n/2} \\ &\quad\times\exp\left(\frac{\left(x^2+x'^2\right)\cos t\cosh t-2x\cdot x'+\left(x^2-x'^2\right)\sin t\sinh t}{2i(\cos t\sinh t-\sin t\cosh t)}\right).\end{aligned} \tag{2.131}$$

The dual counterpart of this Hamiltonian with respect to time reversal has the form

$$H(\tau)=\frac{1}{2}\sum_{s=1}^{n}\left(a_sa_s^{\dagger}+a_s^{\dagger}a_s\right)-\frac{1}{2}e^{-i\arctan(2\tau)}\sum_{s=1}^{n}(a_s)^2-\frac{1}{2}e^{i\arctan(2\tau)}\sum_{s=1}^{n}\left(a_s^{\dagger}\right)^2 \tag{2.132}$$

and one has to interchange $x\leftrightarrow x'$ in (2.131) in order to obtain the corresponding Green function. It is worth noting that the Hamiltonians (2.113) and (2.130) (respectively, (2.120) and (2.132)) are transforming into each other under the substitution $a_s\to ia_s$, $a_s^{\dagger}\to -ia_s^{\dagger}$, which preserves the commutation relations of the creation and annihilation operators. As we have seen in the previous section this property is related to solving the problem in the coordinate and momentum representations.

Combining all four cases together, one may summarize that two Hamiltonians,

$$H_{\pm}(t)=\frac{1}{2}\sum_{s=1}^{n}\left(a_sa_s^{\dagger}+a_s^{\dagger}a_s\right)\pm\frac{1}{2}e^{2it}\sum_{s=1}^{n}(a_s)^2\pm\frac{1}{2}e^{-2it}\sum_{s=1}^{n}\left(a_s^{\dagger}\right)^2, \tag{2.133}$$

and their duals with respect to the time reversal,

$$H_{\pm}(\tau)=\frac{1}{2}\sum_{s=1}^{n}\left(a_sa_s^{\dagger}+a_s^{\dagger}a_s\right)\pm\frac{1}{2}e^{-i\arctan(2\tau)}\sum_{s=1}^{n}(a_s)^2\pm\frac{1}{2}e^{i\arctan(2\tau)}\sum_{s=1}^{n}\left(a_s^{\dagger}\right)^2 \tag{2.134}$$

with $\tau = \frac{1}{2}\sinh(2t)$, have the following Green functions:

$$G_t^{\pm}(x,x') = \left(\frac{1}{2\pi i(\sin t\cosh t \pm \cos t\sinh t)}\right)^{n/2} \times \exp\left(\frac{\pm(x^2-x'^2)\sin t\sinh t + 2x\cdot x' - (x^2+x'^2)\cos t\cosh t}{2i(\sin t\cosh t\pm\cos t\sinh t)}\right). \tag{2.135}$$

This expression is valid for two Hamiltonians (2.133), respectively. One has to interchange $x \leftrightarrow x'$ for the case of the dual Hamiltonians (2.134).

In a similar fashion, the n-dimensional form of the kernels (2.56) and (2.60) is

$$K_t^{\pm}(x,x') = \left(\frac{1}{2\pi(\cos t\cosh t\pm\sin t\sinh t)}\right)^{n/2} \times \exp\left(\frac{(x^2+x'^2)\sin t\cosh t \mp 2x\cdot x' \mp (x^2-x'^2)\cos t\sinh t}{2i(\cos t\cosh t\pm\sin t\sinh t)}\right) \tag{2.136}$$

and

$$\begin{aligned} G_t^{\pm}(x,x') &= \frac{1}{(2\pi)^{n/2}}\int_{R^n} K_t^{\pm}(x,x'')\ e^{\mp ix'\cdot x''}\ dx'' \\ &= \frac{1}{(2\pi)^{n/2}}\int_{R^n} e^{\pm ix\cdot x''}\ K_t^{\mp}(x'',x')\ dx''. \end{aligned} \tag{2.137}$$

We denote

$$\begin{aligned} U_{\pm}(t)\psi(x) &= \int_{R^n} G_t^{\pm}(x,x')\ \psi(x')\ dx', \\ K_{\pm}(t)\psi(x) &= \int_{R^n} K_t^{\pm}(x,x')\ \psi(x')\ dx', \end{aligned} \tag{2.138}$$

and

$$F_{\pm}\psi(x) = \frac{1}{(2\pi)^{n/2}}\int_{R^n} e^{\pm ix\cdot x'}\ \psi(x')\ dx'. \tag{2.139}$$

The corresponding relations are

$$\begin{aligned} &U_{\pm}(t) = K_{\pm}(t)F_{\mp} = F_{\pm}K_{\mp}(t) = F_{\pm}U_{\mp}(t)F_{\mp}, \\ &U_{\pm}^{-1}(t) = F_{\pm}K_{\pm}^{-1}(t) = K_{\mp}^{-1}(t)F_{\mp} = F_{\pm}U_{\mp}^{-1}(t)F_{\mp}. \end{aligned} \tag{2.140}$$

We leave further details to the reader.

A certain time-dependent Schrödinger equation with variable coefficients was considered in [143] in a pure algebraic manner in connection with representations of the group $SU(1,1)$ in an abstract Hilbert space. Our Hamiltonians (2.133) and (2.134) belong to the same class thus providing new explicit realizations of this model in addition to several cases already discussed by Meiler, Cordero-Soto, and Suslov.

2.8 Eigenfunction Expansions

The normalized wave functions of the n-dimensional harmonic oscillator

$$H_0\Psi = E\Psi, \qquad H_0 = \frac{1}{2}\sum_{s=1}^{n}\left(-\frac{\partial^2}{\partial x_s^2} + x_s^2\right) \tag{2.141}$$

have the form

$$\Psi(x) = \Psi_{NK\nu}(r,\Omega) = Y_{K\nu}(\Omega)\ R_{NK}(r), \tag{2.142}$$

where $Y_{K\nu}(\Omega)$ are the hyperspherical harmonics associated with a binary tree T, the integer number K corresponds to the constant of separation of the variables at the root of T and $\nu = \{l_1, l_2, \dots, l_p\}$ is the set of all other subscripts corresponding to the remaining vertexes of the binary tree T; see [157], [188], [212] for a graphical approach of Vilenkin, Kuznetsov and Smorodinskiĭ to the theory of spherical harmonics. The radial functions are given by

$$R_{NK}(r) = \sqrt{\frac{2\left[(N-K)/2\right]!}{\Gamma\left[(N+K+n)/2\right]}}\ \exp\left(-r^2/2\right)\ r^K\ L_{(N-K)/2}^{K+n/2-1}\left(r^2\right), \tag{2.143}$$

where $L_k^{\alpha}(\xi)$ are the Laguerre polynomials. The corresponding energy levels are

$$E = E_N = N + n/2, \qquad (N-K)/2 = k = 0, 1, 2, \dots \tag{2.144}$$

and we can use the $SU(1,1)$-notation for the wave function as follows

$$\psi_{jm\{\nu\}}(x) = \Psi_{NK\nu}(r,\Omega) = Y_{K\nu}(\Omega)\ R_{NK}(r), \tag{2.145}$$

where the new quantum numbers are given by $j = K/2 + n/4 - 1$ and $m = N/2 + n/4$ with $m = j+1,\ j+2, \dots$. The inequality $m \geq j+1$ holds because of the quantization rule (2.144), which gives $N = K, K+2, K+4, \dots$. See [143], [157] and [188] for more details on the group theoretical properties of the n-dimensional harmonic oscillator wave functions.

The Cauchy initial value problem for the Schrödinger equation (2.1) with the Hamiltonian of a modified oscillator (2.113) has also the eigenfunction expansion form of the solution [143]:

$$\psi(x,t) = \sum_{j\{\nu\}}\sum_{m=j+1}^{\infty} c_m(t)\ \psi_{jm\{\nu\}}(x) \tag{2.146}$$

with the time dependent coefficients

$$c_m(t) = e^{-2imt} \sum_{m'=j+1}^{\infty} i^{m'-m} v_{m'm}^{j}(2t) \int_{R^n} \psi_{jm'\{v\}}^{*}(x')\, \psi(x',0)\ dv' \tag{2.147}$$

given in terms of the Bargmann functions [11], [157] and [212]

$$v_{mm'}^{j}(\mu) = \frac{(-1)^{m-j-1}}{\Gamma(2j+2)} \sqrt{\frac{(m+j)!\,(m'+j)!}{(m-j-1)!\,(m'-j-1)!}} \left(\sinh\frac{\mu}{2}\right)^{-2j-2} \left(\tanh\frac{\mu}{2}\right)^{m+m'}$$

$$\times\ {}_2F_1\left(\begin{array}{c} -m+j+1,\ -m'+j+1 \\ 2j+2 \end{array} ;\ -\frac{1}{\sinh^2(\mu/2)}\right). \tag{2.148}$$

Choosing the initial data in (2.115) and (2.146)–(2.147) as $\psi(x,0) = \delta(x-x')$, we arrive at the eigenfunction expansion for the Green function

$$G_t(x,x') = \sum_{j\{v\}} \sum_{m,m'=j+1}^{\infty} e^{-2imt}\ i^{m'-m}\ v_{m'm}^{j}(2t)\ \psi_{jm\{v\}}(x)\, \psi_{jm'\{v\}}^{*}(x'), \tag{2.149}$$

where by (2.131) the following symmetry property holds

$$G_t(x,x') = G_{-t}^{*}(x,x'). \tag{2.150}$$

In this paper we have found solution of the Cauchy initial value problem for the new Hamiltonian (2.120) in an integral form

$$\psi(x,t) = \int_{R^n} G_t(x',x)\ \psi(x',0)\ dx'. \tag{2.151}$$

In view of (2.149)–(2.150), the eigenfunction expansion of this solution is given by

$$\psi(x,t) = \sum_{j\{v\}} \sum_{m=j+1}^{\infty} c_m(t)\ \psi_{jm\{v\}}(x), \tag{2.152}$$

where

$$c_m(t) = \sum_{m'=j+1}^{\infty} (-i)^{m-m'} e^{-2im't} \left(v_{m'm}^{j}(-2t)\right)^{*} \int_{R^n} \psi_{jm'\{v\}}^{*}(x')\, \psi(x',0)\ dv'. \tag{2.153}$$

This expansion is in agreement with the unitary infinite matrix of the inverse operator in the basis of the harmonic oscillator wave functions; see section 5.

The cases of the Hamiltonian (2.130) and its dual (2.132) can be investigated by taking the Fourier transform of the expansions (2.146)–(2.147) and (2.152)–(2.153), respectively. The corresponding transformations of the oscillator wave functions are

$$i^{\pm N} \Psi_{NKv}(x) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{\pm ix\cdot x'}\ \Psi_{NKv}(x')\ dx'. \tag{2.154}$$

This can be evaluated in hyperspherical coordinates with the help of expansion (2.118)–(2.119), or by adding the $SU(1,1)$-momenta according to the tree T [157], [188] and using linearity of the Fourier transform. One can use

$$e^{ix\cdot x'} = (2\pi)^{n/2} \sum_{K\nu} i^K \; Y^*_{K\nu}(\Omega) \; Y_{K\nu}(\Omega') \; S_{-1}(r,r') \tag{2.155}$$

with

$$S_{-1}(r,r') = \frac{(rr')^K}{2^{K+n/2-1}\Gamma(K+n/2)} \; {}_0F_1\left(\begin{array}{c} - \\ K+n/2 \end{array} ; -\frac{(rr')^2}{4} \right) \tag{2.156}$$

and

$$(-1)^{(N-K)/2} R_{NK}(r) = \int_0^\infty S_{-1}(r,r') \; R_{NK}(r') \, (r')^{n-1} \, dr' \tag{2.157}$$

as a special case of Eqs. (7.3) and (7.6) of Ref. [143] together with the orthogonality property of hyperspherical harmonics. We leave further details to the reader.

2.9 Particular Solutions of The Nonlinear Schrödinger Equations

The method of solving the equation (2.8) is extended in [49] to the nonlinear Schrödinger equation of the form

$$i\frac{\partial \psi}{\partial t} = -a(t)\frac{\partial^2 \psi}{\partial x^2} + b(t)x^2\psi - i\left(c(t)x\frac{\partial \psi}{\partial x} + d(t)\psi\right) + h(t)|\psi|^{2s}\psi, \qquad s \geq 0. \tag{2.158}$$

We elaborate first on two cases (2.9) and (2.10). A particular solution takes the form

$$\psi = \psi(x,t) = K_h(x,y,t) = \frac{e^{i\phi}}{\sqrt{\mu(t)}} \, e^{i\left(\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2+\kappa(t)\right)}, \qquad \phi = \text{constant}, \tag{2.159}$$

where equations (2.12)–(2.14) hold and, in addition,

$$\frac{d\kappa}{dt} = -\frac{h(t)}{\mu^s(t)}, \qquad \kappa(t) = \kappa(0) - \int_0^t \frac{h(\tau)}{\mu^s(\tau)}\, d\tau, \tag{2.160}$$

provided that the integral converges.

In the first case (2.9), by the superposition principle, the general solution of the characteristic equation (2.18) has the form

$$\begin{aligned} \mu &= c_1\mu_1(t) + c_2\mu_2(t) \qquad (2.161) \\ &= \cos t\,(c_1 \cosh t + c_2 \sinh t) + \sin t\,(c_1 \sinh t + c_2 \cosh t) \\ &= \frac{c_1+c_2}{\sqrt{2}}\; e^t \sin\left(t+\frac{\pi}{4}\right) + \frac{c_1-c_2}{\sqrt{2}}\; e^{-t}\cos\left(t+\frac{\pi}{4}\right) \end{aligned}$$

with $\mu' = 2\cos t\,(c_1 \sinh t + c_2 \cosh t)$ and

$$\mu(0) = c_1, \qquad \mu'(0) = 2c_2. \tag{2.162}$$

Then

$$\alpha(t) = \frac{\cos t\,(c_1 \sinh t + c_2 \cosh t) - \sin t\,(c_1 \cosh t + c_2 \sinh t)}{2\,(\cos t\,(c_1 \cosh t + c_2 \sinh t) + \sin t\,(c_1 \sinh t + c_2 \cosh t))}, \tag{2.163}$$

$$\beta(t) = \frac{c_1 \beta(0)}{\cos t\,(c_1 \cosh t + c_2 \sinh t) + \sin t\,(c_1 \sinh t + c_2 \cosh t)}, \tag{2.164}$$

and

$$\gamma(t) = \gamma(0) - \frac{c_1 \beta^2(0)\,(\cos t \sinh t + \sin t \cosh t)}{2\,(\cos t\,(c_1 \cosh t + c_2 \sinh t) + \sin t\,(c_1 \sinh t + c_2 \cosh t))} \tag{2.165}$$

as a result of elementary but somewhat tedious calculations. The first two equations follow directly from (2.12) and a constant multiple of the first equation (2.13), respectively. One should use

$$\frac{d\gamma}{dt} + a(t)\,\beta^2 = 0, \tag{2.166}$$

see [49], integration by parts as in (2.14), and an elementary integral

$$\int \frac{dt}{(c_1 \sinh t + c_2 \cosh t)^2} = \frac{\sinh t}{c_2\,(c_1 \sinh t + c_2 \cosh t)} + C \tag{2.167}$$

in order to derive (2.165).

Two special cases are as follows. The original propagator (2.5) appears in the limit $c_1 \to 0$ when $\beta(0) = -(c_1)^{-1}$ and $\gamma(0) = (2c_1c_2)^{-1}$. The solution with the standing wave initial data $\psi(x,0) = e^{ixy}$ found in [49] corresponds to $c_1 = 1$ and $c_2 = 0$.

Equation (2.160) can be explicitly integrated in some special cases, say, when $h(t) = \lambda \mu'(t)$:

$$\kappa(t) = \begin{cases} \kappa(0) - \dfrac{\lambda}{1-s}\left(\mu^{1-s}(t) - \mu^{1-s}(0)\right), & \text{when } s \neq 1, \\ \\ \kappa(0) - \lambda \ln\left(\dfrac{\mu(t)}{\mu(0)}\right), & \text{when } s = 1. \end{cases} \tag{2.168}$$

Here $\mu(0) \neq 0$; cf. [49]. One may treat the general particular solution of the form (2.159) with the coefficients (2.163)–(2.165) and (2.168) as an example of application of yet unknown "nonlinear" superposition principle for the Schrödinger equation under consideration for two particular solutions of a similar form with $c_1 \neq 0$, $c_2 = 0$ and $c_1 = 0$, $c_2 \neq 0$.

It is worth noting that function (2.159) with the coefficients given by (2.161)–(2.168) does also satisfy the following linear Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = -a(t)\,\frac{\partial^2 \psi}{\partial x^2} + b(t)\,x^2\psi - i\left(c(t)\,x\frac{\partial \psi}{\partial x} + d(t)\,\psi\right) + \frac{h(t)}{\mu^s(t)}\,\psi, \qquad s \geq 0. \tag{2.169}$$

Then a more general solution of this equation can be obtained by the superposition principle as follows

$$\psi(x,t) = \int_{-\infty}^{\infty} K_h(x,y,t)\ \chi(y)\ dy, \tag{2.170}$$

where χ is an arbitrary function such that the integral converges and one can interchange differentiation and integration. Solution of the Cauchy initial value problem simply requires an inversion of the integral

$$\psi(x,0) = \int_{-\infty}^{\infty} K_h(x,y,0)\ \chi(y)\ dy \tag{2.171}$$

that is

$$\chi(y) = \frac{c_1\beta(0)}{2\pi}\int_{-\infty}^{\infty} K_h^*(x,y,0)\ \psi(x,0)\ dx, \tag{2.172}$$

say, by the inverse of the Fourier transform. Thus our equations (2.170) and (2.172) solve the initial value problem for the above linear Schrödinger equation (2.169) as a double integral with the help of the kernel $K_h(x,y,t)$ that is regular at $t = 0$, when $\mu(0) = c_1 \neq 0$.

On the other hand,

$$K_h(x,y,t) = \int_{-\infty}^{\infty} G_h(x,z,t)\ K_h(z,y,0)\ dz, \tag{2.173}$$

where $G_h(x,y,t)$ is the Green function, which can be obtain from our solution (2.159) in the limit $c_1 \to 0$ with a proper normalization as in the propagator (2.5). Therefore, substitution of (2.172) into (2.171) gives the traditional single integral form of the solution in terms of the Green function

$$\psi(x,t) = \int_{-\infty}^{\infty} G_h(x,y,t)\ \psi(y,0)\ dy \tag{2.174}$$

by (2.173).

The case of the new Hamiltonian, corresponding to (2.10), is similar. The general solution of characteristic equation (2.21) is given by

$$\begin{aligned}
\mu &= c_2\mu_2(t)+c_3\mu_3(t) && (2.175)\\
&= \cos t\,(c_2\sinh t-c_3\cosh t)+\sin t\,(c_2\cosh t+c_3\sinh t)\\
&= \frac{1}{\sqrt{2}}\,e^{t}\left(c_2\sin\left(t+\frac{\pi}{4}\right)-c_3\cos\left(t+\frac{\pi}{4}\right)\right)\\
&\quad -\frac{1}{\sqrt{2}}\,e^{-t}\left(c_2\cos\left(t+\frac{\pi}{4}\right)+c_3\sin\left(t+\frac{\pi}{4}\right)\right)
\end{aligned}$$

and $\mu'=2\cosh t\,(c_2\cos t+c_3\sin t)$ with $\mu(0)=-c_3$, $\mu'(0)=2c_2$. The first three coefficients of the quadratic form in the solution (2.159) are

$$\alpha(t) = \frac{\cos t\,(c_2\cosh t-c_3\sinh t)+\sin t\,(c_2\sinh t+c_3\cosh t)}{2\left(\cos t\,(c_2\sinh t-c_3\cosh t)+\sin t\,(c_2\cosh t+c_3\sinh t)\right)}, \tag{2.176}$$

$$\beta(t) = \frac{-c_3\beta(0)}{\cos t\,(c_2\sinh t-c_3\cosh t)+\sin t\,(c_2\cosh t+c_3\sinh t)}, \tag{2.177}$$

$$\gamma(t) = \gamma(0)+\frac{c_3\beta^2(0)\,(\cos t\sinh t+\sin t\cosh t)}{2\left(\cos t\,(c_2\sinh t-c_3\cosh t)+\sin t\,(c_2\cosh t+c_3\sinh t)\right)} \tag{2.178}$$

and one can use formula (2.168) for the last coefficient. The corresponding elementary integral is

$$\int\frac{dt}{(A\cos t+B\sin t)^2}=\frac{\sin t}{A\,(A\cos t+B\sin t)}+C. \tag{2.179}$$

The cases (2.25) and (2.26) can be considered in a similar fashion. The results are

$$\begin{aligned}
\mu &= c_3\mu_3(t)+c_4\mu_4(t) && (2.180)\\
&= \sin t\,(c_3\sinh t+c_4\cosh t)-\cos t\,(c_3\cosh t+c_4\sinh t),
\end{aligned}$$

$$\alpha(t) = \frac{\sin t\,(c_3\cosh t+c_4\sinh t)+\cos t\,(c_3\sinh t+c_4\cosh t)}{2\left(\sin t\,(c_3\sinh t+c_4\cosh t)-\cos t\,(c_3\cosh t+c_4\sinh t)\right)}, \tag{2.181}$$

$$\beta(t) = \frac{-c_3\beta(0)}{\sin t\,(c_3\sinh t+c_4\cosh t)-\cos t\,(c_3\cosh t+c_4\sinh t)}, \tag{2.182}$$

$$\gamma(t) = \gamma(0)+\frac{c_3\beta^2(0)\,(\sin t\cosh t-\cos t\sinh t)}{2\left(\sin t\,(c_3\sinh t+c_4\cosh t)-\cos t\,(c_3\cosh t+c_4\sinh t)\right)} \tag{2.183}$$

and

$$\begin{aligned}
\mu &= c_1\mu_1(t)+c_4\mu_4(t) && (2.184)\\
&= \cos t\,(c_1\cosh t-c_4\sinh t)+\sin t\,(c_1\sinh t+c_4\cosh t),
\end{aligned}$$

$$\alpha(t) = -\frac{\sinh t\,(c_1\cos t + c_4\sin t) + \cosh t\,(c_1\sin t - c_4\cos t)}{2\,(\sinh t\,(c_1\sin t - c_4\cos t) + \cosh t\,(c_1\cos t + c_4\sin t))}, \tag{2.185}$$

$$\beta(t) = \frac{c_1\beta(0)}{\sinh t\,(c_1\sin t - c_4\cos t) + \cosh t\,(c_1\cos t + c_4\sin t)}, \tag{2.186}$$

$$\gamma(t) = \gamma(0) + \frac{c_1\beta^2(0)\,(\cos t\sinh t - \sin t\cosh t)}{2\,(\sinh t\,(c_1\sin t - c_4\cos t) + \cosh t\,(c_1\cos t + c_4\sin t))}, \tag{2.187}$$

respectively. One can use once again formula (2.168) for the last coefficient. We leave further details to the reader.

2.10 A Note on The Ill-Posedness of The Schrödinger Equations

The same method shows that the joint solution of the both linear and nonlinear Schrödinger equations (2.169) and (2.158), respectively, corresponding to the initial data

$$\psi|_{t=0} = \delta_\varepsilon(x-y) = \frac{1}{\sqrt{2\pi i\varepsilon}}\ \exp\left(\frac{i(x-y)^2}{2\varepsilon}\right), \quad \varepsilon > 0, \tag{2.188}$$

has the form

$$\psi = G_\varepsilon(x,y,t) = \frac{1}{\sqrt{i\mu_\varepsilon(t)}}\ e^{i\left(\alpha_\varepsilon(t)x^2 + \beta_\varepsilon(t)xy + \gamma_\varepsilon(t)y^2 + \kappa_\varepsilon(t)\right)} \tag{2.189}$$

with the characteristic function $\mu_\varepsilon(t) = 2\pi(\varepsilon\mu_1(t) + \mu_2(t))$. The coefficients of the quadratic form are given by

$$\alpha_\varepsilon(t) = \frac{\cos t\,(\varepsilon\sinh t + \cosh t) - \sin t\,(\varepsilon\cosh t + \sinh t)}{2\,(\cos t\,(\varepsilon\cosh t + \sinh t) + \sin t\,(\varepsilon\sinh t + \cosh t))}, \tag{2.190}$$

$$\beta_\varepsilon(t) = -\frac{1}{\cos t\,(\varepsilon\cosh t + \sinh t) + \sin t\,(\varepsilon\sinh t + \cosh t)}, \tag{2.191}$$

$$\gamma_\varepsilon(t) = \frac{\cos t\cosh t + \sin t\sinh t}{2\,(\cos t\,(\varepsilon\cosh t + \sinh t) + \sin t\,(\varepsilon\sinh t + \cosh t))}. \tag{2.192}$$

We simply choose $c_1 = 2\pi\varepsilon > 0$, $c_2 = 2\pi$ and $e^{i\varphi} = 1/\sqrt{i}$ and the initial data $\alpha(0) = \gamma(0) = -\beta(0)/2 = 1/(2\varepsilon)$ in a general solution (2.163)–(2.165). The case $\varepsilon = 0$, $t > 0$ corresponds to the original propagator (2.5), while $\varepsilon > 0$, $t = 0$ gives the delta sequence (2.188).

If $h = h_\varepsilon(t) = (\lambda/2\pi)\,\mu'_\varepsilon = 2\lambda\cos t\,(\varepsilon\sinh t + \cosh t)$, then

$$\kappa_\varepsilon(t) = \begin{cases} -\dfrac{\lambda}{(2\pi)^s}\dfrac{(\varepsilon\mu_1(t) + \mu_2(t))^{1-s} - \varepsilon^{1-s}}{1-s}, & \text{when } 0 \le s < 1, \\[2ex] -\dfrac{\lambda}{2\pi}\ln\left(\mu_1(t) + \dfrac{\mu_2(t)}{\varepsilon}\right), & \text{when } s = 1 \end{cases} \tag{2.193}$$

with $\kappa_\varepsilon(0) = 0$ provided $\varepsilon > 0$.

In this example, the initial data $\psi|_{t=0} = G_\varepsilon(x, y, 0) = \delta_\varepsilon(x - y)$ converge to the Dirac delta function $\delta(x-y)$ as $\varepsilon \to 0^+$ in the distributional sense [34], [187], [203], [210]

$$\lim_{\varepsilon\to 0^+} \int_{-\infty}^{\infty} G_\varepsilon(x, y, 0)\ \varphi(y)\ dy = \varphi(x). \tag{2.194}$$

On the other hand,

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} \int_{-\infty}^{\infty} G_\varepsilon(x, y, t)\ \varphi(y)\ dy \\ &\quad = e^{i\lim_{\varepsilon\to 0^+} \kappa_\varepsilon(t)} \int_{-\infty}^{\infty} G_0(x, y, t)\ \varphi(y)\ dy \end{aligned} \tag{2.195}$$

with $t > 0$. When $s = 1$ the solution $\psi = G_\varepsilon(x, y, t)$, $t > 0$ does not have a limit because of divergence of the logarithmic phase factor $\kappa_\varepsilon(t)$ as $\varepsilon \to 0^+$. See also Refs. [10] and [109] on the ill-posedness of some canonical dispersive equations.

The second case, corresponding to (2.10), is similar. One can choose $\mu_\varepsilon(t) = 2\pi(\mu_2(t) - \varepsilon\mu_3(t))$ and obtain

$$\alpha_\varepsilon(t) = \frac{\cos t(\cosh t + \varepsilon \sinh t) + \sin t(\sinh t - \varepsilon \cosh t)}{2(\cos t(\sinh t + \varepsilon \cosh t) + \sin t(\cosh t - \varepsilon \sinh t))}, \tag{2.196}$$

$$\beta_\varepsilon(t) = -\frac{1}{\cos t(\sinh t + \varepsilon \cosh t) + \sin t(\cosh t - \varepsilon \sinh t)}, \tag{2.197}$$

$$\gamma_\varepsilon(t) = \frac{\cos t \cosh t - \sin t \sinh t}{2(\cos t(\sinh t + \varepsilon \cosh t) + \sin t(\cosh t - \varepsilon \sinh t))}. \tag{2.198}$$

If $h_\varepsilon(t) = (\lambda/2\pi)\mu'_\varepsilon = 2\lambda \cosh t(\cos t - \varepsilon \sin t)$, then

$$\kappa_\varepsilon(t) = \begin{cases} -\dfrac{\lambda}{(2\pi)^s} \dfrac{(\mu_2(t) - \varepsilon\mu_3(t))^{1-s} - \varepsilon^{1-s}}{1-s}, & \text{when } 0 \le s < 1, \\[2ex] -\dfrac{\lambda}{2\pi} \ln\left(\dfrac{\mu_2(t)}{\varepsilon} - \mu_3(t)\right), & \text{when } s = 1 \end{cases} \tag{2.199}$$

and $\kappa_\varepsilon(0) = 0$ when $\varepsilon > 0$. Formulas (2.189) and (2.196)–(2.199) describe actual (nonlinear) evolution for initial data as in (2.188). One can observe once again a discontinuity with respect to these initial data as $\varepsilon \to 0^+$.

The cases (2.25) and (2.26) are as follows. One gets $\mu_\varepsilon(t) = 2\pi(\mu_4(t) - \varepsilon\mu_3(t))$,

$$\alpha_\varepsilon(t) = \frac{\sin t(\sinh t - \varepsilon \cosh t) + \cos t(\cosh t - \varepsilon \sinh t)}{2(\sin t(\cosh t - \varepsilon \sinh t) - \cos t(\sinh t - \varepsilon \cosh t))}, \tag{2.200}$$

$$\beta_\varepsilon(t) = -\frac{1}{\sin t(\cosh t - \varepsilon \sinh t) - \cos t(\sinh t - \varepsilon \cosh t)}, \tag{2.201}$$

$$\gamma_\varepsilon(t) = \frac{\cos t \cosh t - \sin t \sinh t}{2(\sin t(\cosh t - \varepsilon \sinh t) - \cos t(\sinh t - \varepsilon \cosh t))} \tag{2.202}$$

and $h_\varepsilon(t) = (\lambda/2\pi)\,\mu'_\varepsilon = 2\lambda\sin t\,(\sinh t - \varepsilon\cosh t)$,

$$\kappa_\varepsilon(t) = \begin{cases} -\dfrac{\lambda}{(2\pi)^s}\dfrac{(\mu_4(t) - \varepsilon\mu_3(t))^{1-s} - \varepsilon^{1-s}}{1-s}, & \text{when } 0 \le s < 1, \\ -\dfrac{\lambda}{2\pi}\ln\left(\dfrac{\mu_4(t)}{\varepsilon} - \mu_3(t)\right), & \text{when } s = 1 \end{cases} \tag{2.203}$$

with $\kappa_\varepsilon(0) = 0$, $\varepsilon > 0$ in the case (2.25). Also $\mu_\varepsilon(t) = 2\pi(\mu_4(t) + \varepsilon\mu_1(t))$,

$$\alpha_\varepsilon(t) = \frac{\sinh t(\sin t + \varepsilon\cos t) - \cosh t(\cos t - \varepsilon\sin t)}{2(\sinh t(\cos t - \varepsilon\sin t) - \cosh t(\sin t + \varepsilon\cos t))}, \tag{2.204}$$

$$\beta_\varepsilon(t) = \frac{1}{\sinh t(\cos t - \varepsilon\sin t) - \cosh t(\sin t + \varepsilon\cos t)}, \tag{2.205}$$

$$\gamma_\varepsilon(t) = -\frac{\cos t\cosh t + \sin t\sinh t}{2(\sinh t(\cos t - \varepsilon\sin t) - \cosh t(\sin t + \varepsilon\cos t))} \tag{2.206}$$

and $h_\varepsilon(t) = (\lambda/2\pi)\,\mu'_\varepsilon = 2\lambda\sinh t\,(\sin t + \varepsilon\cos t)$,

$$\kappa_\varepsilon(t) = \begin{cases} -\dfrac{\lambda}{(2\pi)^s}\dfrac{(\mu_4(t) + \varepsilon\mu_1(t))^{1-s} - \varepsilon^{1-s}}{1-s}, & \text{when } 0 \le s < 1, \\ -\dfrac{\lambda}{2\pi}\ln\left(\dfrac{\mu_4(t)}{\varepsilon} + \mu_1(t)\right), & \text{when } s = 1 \end{cases} \tag{2.207}$$

with $\kappa_\varepsilon(0) = 0$, $\varepsilon > 0$ in the case (2.26). We leave the details to the reader.

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2.11 Appendix A. Fundamental Solutions of The Characteristic Equations

We denote

$$u_1 = \cos t, \qquad u_2 = \sin t, \qquad v_1 = \cosh t, \qquad v_2 = \sinh t \tag{2.208}$$

such that $u_1' = -u_2$, $u_2' = u_1$, $v_1' = v_2$, $v_2' = v_1$ and study differential equations satisfied by the following set of the Wronskians of trigonometric and hyperbolic functions

$$\left\{W\left(u_\alpha, v_\beta\right)\right\}_{\alpha,\beta=1,2} = \left\{W\left(u_1,v_1\right),\ W\left(u_1,v_2\right),\ W\left(u_2,v_1\right),\ W\left(u_2,v_2\right)\right\}. \tag{2.209}$$

Let us take, for example,

$$y = W\left(u_1, v_1\right) = u_1 v_2 + u_2 v_1. \tag{2.210}$$

Then

$$y' = 2u_1 v_1, \qquad y'' = 2u_1 v_2 - 2u_2 v_1 \tag{2.211}$$

and

$$y'' - \tau y' + 4\sigma y = (4\sigma + 2)\, u_1 v_2 + (4\sigma - 2)\, u_2 v_1 - 2\tau u_1 v_1 = 0. \tag{2.212}$$

The last equation is satisfied when $\sigma = 1/2$, $\tau = 2v_2/v_1$ and $\sigma = -1/2$, $\tau = -2u_2/u_1$. All other cases are similar and the results are presented in Table 1.

Our calculations reveal the following identities

$$\begin{aligned} W''\left(u_1,v_1\right) &= -2W\left(u_2,v_2\right), \qquad & W''\left(u_1,v_2\right) &= -2W\left(u_2,v_1\right), \\ W''\left(u_2,v_1\right) &= 2W\left(u_1,v_2\right), \qquad & W''\left(u_2,v_2\right) &= 2W\left(u_1,v_1\right), \end{aligned} \tag{2.213}$$

for the Wronskians under consideration. This implies that the set of Wronskians (2.209) provides the fundamental solutions of the fourth order differential equation

$$W^{(4)} + 4W = 0 \tag{2.214}$$

with constant coefficients. The corresponding characteristic equation, $\lambda^4 + 4 = 0$, has four roots, $\lambda_1 = 1+i$, $\lambda_2 = 1-i$, $\lambda_3 = -1+i$, $\lambda_4 = -1-i$, and the fundamental solution set is given by

$$\left\{u_\alpha v_\beta\right\}_{\alpha,\beta=1,2} = \left\{u_1 v_1,\ u_1 v_2,\ u_2 v_1,\ u_2 v_2\right\}. \tag{2.215}$$

These solutions of the bi-harmonic equation (2.214) are even or odd functions of time. They do not satisfy our second order characteristic equations. For example, let $w_1 = u_1 v_2 = \cos t \sinh t$ and $w_1 = u_2 v_1 = \sin t \cosh t$. Then, by a direct calculation,

$$
\begin{aligned}
L(w_1) &= w_1'' + 2\tan t\ w_1' - 2w_1 = -2\frac{\sinh t}{\cos t}, \\
L(w_2) &= w_2'' + 2\tan t\ w_2' - 2w_2 = 2\frac{\sinh t}{\cos t}.
\end{aligned}
\tag{2.216}
$$

Thus, separately, these solutions of (2.214) satisfy nonhomogeneous characteristic equations. But together,

$$
L(y_1) = L(w_1 + w_2) = L(w_1) + L(w_2) = -2\frac{\sinh t}{\cos t} + 2\frac{\sinh t}{\cos t} = 0. \tag{2.217}
$$

A similar property holds for all other solutions of the characteristic equations from Table 1.

Characteristic equation $y'' - \tau y' + 4\sigma y = 0$	Fundamental solution set $\{y_i, y_k\}_{i<k}$
$u'' + u = 0$ $(\sigma = 1/4, \quad \tau = 0)$	$u_1 = \cos t, \qquad u_2 = \sin t$ $(u_1' = -u_2, \quad u_2' = u_1)$
$v'' - v = 0$ $(\sigma = -1/4, \quad \tau = 0)$	$v_1 = \cosh t, \qquad v_2 = \sinh t$ $(v_1' = v_2, \quad v_2' = v_1)$
$y'' + 2\tan t\, y' - 2y = 0$ $(\sigma = -1/2, \quad \tau = -2u_2/u_1)$	$y_1 = W(u_1, v_1) = u_1v_2 + u_2v_1$ $y_2 = W(u_1, v_2) = u_1v_1 + u_2v_2$
$y'' - 2\cot t\, y' - 2y = 0$ $(\sigma = -1/2, \quad \tau = 2u_2/u_1)$	$y_3 = W(u_2, v_2) = u_2v_1 - u_1v_2$ $y_4 = W(u_2, v_1) = u_2v_2 - u_1v_1$
$y'' - 2\tanh t\, y' + 2y = 0$ $(\sigma = 1/2, \quad \tau = 2v_2/v_1)$	$y_1 = W(u_1, v_1) = u_1v_2 + u_2v_1$ $y_4 = W(u_2, v_1) = u_2v_2 - u_1v_1$
$y'' - 2\coth t\, y' + 2y = 0$ $(\sigma = 1/2, \quad \tau = 2v_1/v_2)$	$y_2 = W(u_1, v_2) = u_1v_1 + u_2v_2$ $y_3 = W(u_2, v_2) = u_2v_1 - u_1v_2$

Table 2.1: Fundamental solutions of the characteristic equations.

Linear operators	u_1v_1	u_1v_2	u_2v_1	u_2v_2
$\dfrac{d}{dt}$	$u_1v_2 - u_2v_1$	$u_1v_1 - u_2v_2$	$u_1v_1 + u_2v_2$	$u_1v_2 + u_2v_1$
$\dfrac{d^2}{dt^2}$	$-2u_2v_2$	$-2u_2v_1$	$2u_1v_2$	$2u_1v_1$
L_1	$-2\dfrac{v_1}{u_1}$	$-2\dfrac{v_2}{u_1}$	$2\dfrac{v_2}{u_1}$	$2\dfrac{v_1}{u_1}$
L_2	$-2\dfrac{v_2}{u_2}$	$-2\dfrac{v_1}{u_2}$	$-2\dfrac{v_1}{u_2}$	$-2\dfrac{v_2}{u_2}$
L_3	$2\dfrac{u_1}{v_1}$	$-2\dfrac{u_2}{v_1}$	$2\dfrac{u_2}{v_1}$	$2\dfrac{u_1}{v_1}$
L_4	$2\dfrac{u_2}{v_2}$	$-2\dfrac{u_1}{v_2}$	$-2\dfrac{u_1}{v_2}$	$-2\dfrac{u_2}{v_2}$

Table 2.2: Construction of the fundamental solutions.

In order to obtain the fundamental solutions in an algebraic manner, we denote

$$\begin{aligned} L_1 &= \frac{d^2}{dt^2} + 2\frac{u_2}{u_1}\frac{d}{dt} - 2, \quad L_2 = \frac{d^2}{dt^2} - 2\frac{u_1}{u_2}\frac{d}{dt} - 2, \\ L_3 &= \frac{d^2}{dt^2} - 2\frac{v_2}{v_1}\frac{d}{dt} + 2, \quad L_4 = \frac{d^2}{dt^2} - 2\frac{v_1}{v_2}\frac{d}{dt} + 2 \end{aligned} \tag{2.218}$$

and compute the actions of these second order linear differential operators L_k on the four basis vectors $\{u_\alpha v_\beta\}_{\alpha,\beta=1,2}$, namely, $L_k(u_\alpha v_\beta)$. The results are presented in Table 2.

Therefore

$$
\begin{aligned}
L_1\left(u_1v_1+u_2v_2\right) &= L_1\left(u_1v_2+u_2v_1\right) \\
&= L_2\left(u_2v_2-u_1v_1\right) = L_2\left(u_2v_1-u_1v_2\right) \\
&= L_3\left(u_1v_2+u_2v_1\right) = L_3\left(u_2v_2-u_1v_1\right) \\
&= L_4\left(u_1v_1+u_2v_2\right) = L_4\left(u_2v_1-u_1v_2\right) = 0
\end{aligned} \tag{2.219}
$$

as has been stated in Table 1.

All our characteristic equations in this paper obey certain periodicity properties. For instance, equations

$$y'' + 2\tan t\; y' - 2y = 0 \tag{2.220}$$

and

$$y'' - 2\cot t\; y' - 2y = 0 \tag{2.221}$$

are invariant under the shifts $t \to t \pm \pi$ and interchange one into another when $t \to t \pm \pi/2$. Since only two solutions of a linear second order differential equation may be linearly independent, the corresponding fundamental solutions satisfy the following relations

$$\begin{pmatrix} y_1\left(t\pm\pi\right) \\ y_2\left(t\pm\pi\right) \end{pmatrix} = -\begin{pmatrix} \cosh\pi & \pm\sinh\pi \\ \pm\sinh\pi & \cosh\pi \end{pmatrix} \begin{pmatrix} y_1\left(t\right) \\ y_2\left(t\right) \end{pmatrix} \tag{2.222}$$

and

$$\begin{pmatrix} y_1\left(t\pm\pi/2\right) \\ y_2\left(t\pm\pi/2\right) \end{pmatrix} = -\begin{pmatrix} \sinh\left(\pi/2\right) & \pm\cosh\left(\pi/2\right) \\ \pm\cosh\left(\pi/2\right) & \sinh\left(\pi/2\right) \end{pmatrix} \begin{pmatrix} y_3\left(t\right) \\ y_4\left(t\right) \end{pmatrix}, \tag{2.223}$$

respectively. Two other characteristic equations have pure imaginary periods. We leave the details to the reader.

2.12 Appendix B. On A Transformation of The Quantum Hamiltonians

Our definition of the creation and annihilation operators given by (2.3) implies the following operator identities

$$x^2 = \frac{1}{2}\left(aa^\dagger + a^\dagger a\right) - \frac{1}{2}\left(a^2 + \left(a^\dagger\right)^2\right), \tag{2.224}$$

$$\frac{\partial^2}{\partial x^2} = -\frac{1}{2}\left(aa^\dagger + a^\dagger a\right) - \frac{1}{2}\left(a^2 + \left(a^\dagger\right)^2\right), \tag{2.225}$$

$$2x\frac{\partial}{\partial x}+1=-a^{2}+\left(a^{\dagger}\right)^{2} \tag{2.226}$$

(and vise versa), which allows us to transform the time-dependent Schrödinger equation (2.8) into a Hamiltonian form (2.1), where the Hamiltonian is written in terms of the creation and annihilation operators as follows

$$\begin{aligned} H &= \frac{1}{2}\left(a\left(t\right)+b\left(t\right)\right)\left(aa^{\dagger}+a^{\dagger}a\right) \\ &\quad +\frac{1}{2}\left(a\left(t\right)-b\left(t\right)+2id\left(t\right)\right)a^{2}+\frac{1}{2}\left(a\left(t\right)-b\left(t\right)-2id\left(t\right)\right)\left(a^{\dagger}\right)^{2}, \end{aligned} \tag{2.227}$$

when $c=2d$. This helps to transform the Hamiltonians of modified oscillators under consideration into different equivalent forms, which are used in the paper.

The trigonometric cases (2.9) and (2.25) results in the Hamiltonians (2.2) and (2.130) with $n=1$, respectively. In the first hyperbolic case (2.10) one gets

$$H=\frac{1}{2}\cosh\left(2t\right)\left(aa^{\dagger}+a^{\dagger}a\right)+\frac{1}{2}\left(1-i\sinh\left(2t\right)\right)a^{2}+\frac{1}{2}\left(1+i\sinh\left(2t\right)\right)\left(a^{\dagger}\right)^{2}, \tag{2.228}$$

where

$$1\pm i\sinh\left(2t\right)=\cosh\left(2t\right)e^{\pm i\arctan(2\tau)},\qquad \tau=\frac{1}{2}\sinh\left(2t\right), \tag{2.229}$$

which implies the Schrödinger equation (2.6)–(2.7). The second hyperbolic case (2.26) is similar. We leave the details to the reader.

2.13 Appendix C. On A Hamiltonian Structure of The Characteristic Equations

The Hamilton equations of classical mechanics [120],

$$\dot{q}=\frac{\partial H}{\partial p},\qquad \dot{p}=-\frac{\partial H}{\partial q}, \tag{2.230}$$

with a general quadratic Hamiltonian

$$H=a\left(t\right)p^{2}+b\left(t\right)q^{2}+2d\left(t\right)pq \tag{2.231}$$

are

$$\dot{q}=2ap+2dq,\qquad \dot{p}=-2bq-2dp. \tag{2.232}$$

We denote, as is customary, differentiation with respect to time by placing a dot above the canonical variables p and q. Elimination of the generalized momentum p from this system results in the second order equation with respect to the generalized coordinate

$$\ddot{q} - \frac{\dot{a}}{a}\,\dot{q} + 4\left(ab - d^2 + \frac{d}{2}\left(\frac{\dot{a}}{a} - \frac{\dot{d}}{d}\right)\right)q = 0. \tag{2.233}$$

It coincides with the characteristic equation (2.15)–(2.16) with $c = 2d$. Our choice of the coefficients (2.9)–(2.10) and (2.25)–(2.26) in the classical Hamiltonian (2.231) corresponds to the following models of modified classical oscillators

$$\ddot{q} + 2\tan t\ \dot{q} - 2q = 0, \tag{2.234}$$

$$\ddot{q} - 2\tanh t\ \dot{q} + 2q = 0, \tag{2.235}$$

$$\ddot{q} - 2\cot t\ \dot{q} - 2q = 0, \tag{2.236}$$

$$\ddot{q} - 2\coth t\ \dot{q} + 2q = 0, \tag{2.237}$$

respectively; see Appendix A for their fundamental solutions.

The standard quantization of the classical integrable systems under consideration, namely,

$$q \to x, \qquad p \to i^{-1}\frac{\partial}{\partial x}, \qquad [x,p] = xp - px = i \tag{2.238}$$

and

$$H \to ap^2 + bx^2 + d\left(px + xp\right), \qquad i\frac{\partial \psi}{\partial t} = H\psi, \tag{2.239}$$

leads to the quantum exactly solvable models of modified oscillators discussed in this paper.

Another example is a damped oscillator with the variable coefficients $a = (\omega_0/2)\,e^{-2\lambda t}$, $b = (\omega_0/2)\,e^{2\lambda t}$ and $c = d = 0$ [50]. The classical equation

$$\ddot{q} + 2\lambda\ \dot{q} + \omega_0^2\ q = 0 \tag{2.240}$$

describes damped oscillations [120]. The corresponding quantum propagator has the form (2.11) with

$$\mu = \frac{\omega_0}{\omega}e^{-\lambda t}\sin\omega t, \qquad \omega^2 = \omega_0^2 - \lambda^2 > 0 \tag{2.241}$$

and

$$\alpha(t) = \frac{\omega\cos\omega t - \lambda\sin\omega t}{2\omega_0\sin\omega t}e^{2\lambda t}, \tag{2.242}$$

$$\beta(t) = -\frac{\omega}{\omega_0\sin\omega t}e^{\lambda t}, \tag{2.243}$$

$$\gamma(t) = \frac{\omega\cos\omega t + \lambda\sin\omega t}{2\omega_0\sin\omega t}. \tag{2.244}$$

The Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = \frac{\omega_0}{2}\left(-e^{-2\lambda t}\frac{\partial^2 \psi}{\partial x^2} + e^{2\lambda t}x^2\psi\right) \tag{2.245}$$

describes the linear oscillator with a variable unit of length $x \to xe^{\lambda t}$. See [50] for more details.

Chapter 3

MODELS OF DAMPED OSCILLATORS IN QUANTUM MECHANICS

citation: R. Cordero-Soto, Erwin Suazo and S. K. Suslov, Journal of Physical Mathematics, **1** (2009), S090603.

3.1 An Introduction

We continue an investigation of the one-dimensional Schrödinger equations with variable quadratic Hamiltonians of the form

$$i\frac{\partial \psi}{\partial t} = -a(t)\,\frac{\partial^2 \psi}{\partial x^2} + b(t)\,x^2\psi - i\left(c(t)\,x\frac{\partial \psi}{\partial x} + d(t)\,\psi\right), \tag{3.1}$$

where $a(t)$, $b(t)$, $c(t)$, and $d(t)$ are real-valued functions of time t only; see Refs. [49], [52], [122], [134], [143], [194], [195], and [196] for a general approach and currently known explicit solutions. Here we discuss elementary cases related to the models of damped oscillators. The corresponding Green functions, or Feynman's propagators, can be found as follows [49], [195]:

$$\psi = G(x,y,t) = \frac{1}{\sqrt{2\pi i\mu(t)}}\; e^{i\left(\alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2\right)}, \tag{3.2}$$

where

$$\alpha(t) = \frac{1}{4a(t)}\frac{\mu'(t)}{\mu(t)} - \frac{d(t)}{2a(t)}, \tag{3.3}$$

$$\beta(t) = -\frac{h(t)}{\mu(t)}, \qquad h(t) = \exp\left(-\int_0^t \left(c(\tau) - 2d(\tau)\right)\,d\tau\right), \tag{3.4}$$

$$\gamma(t) = \frac{a(t)\,h^2(t)}{\mu(t)\,\mu'(t)} + \frac{d(0)}{2a(0)} - 4\int_0^t \frac{a(\tau)\,\sigma(\tau)\,h^2(\tau)}{\left(\mu'(\tau)\right)^2}\,d\tau, \tag{3.5}$$

and the function $\mu(t)$ satisfies the characteristic equation

$$\mu'' - \tau(t)\,\mu' + 4\sigma(t)\,\mu = 0 \tag{3.6}$$

with

$$\tau(t) = \frac{a'}{a} - 2c + 4d, \qquad \sigma(t) = ab - cd + d^2 + \frac{d}{2}\left(\frac{a'}{a} - \frac{d'}{d}\right) \tag{3.7}$$

subject to the initial data

$$\mu(0) = 0, \qquad \mu'(0) = 2a(0) \neq 0. \tag{3.8}$$

More details can be found in Refs. [49] and [195]. The corresponding Hamiltonian structure is discussed in Ref. [52].

The simple harmonic oscillator is of interest in many advanced quantum problems [81], [121], [145], and [183]. The forced harmonic oscillator was originally considered by Richard Feynman in his path integrals approach to the nonrelativistic quantum mechanics [77], [78], [79], [80], and [81]; see also [134]. Its special and limiting cases were discussed by many authors; see Refs. [14], [87], [99], [142], [145], [207] for the simple harmonic oscillator and Refs. [4], [21], [98], [154], [173] for the particle in a constant external field and references therein.

The damped oscillations have been analyzed to a great extent in classical mechanics; see, for example, Refs. [13] and [120]. In the present paper we consider the time-dependent Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = H\psi \tag{3.9}$$

with the following nonself-adjoint Hamiltonians

$$H = H_1 = \frac{\omega_0}{2}\left(p^2 + x^2\right) - \lambda\, px \tag{3.10}$$

and

$$H = H_2 = \frac{\omega_0}{2}\left(p^2 + x^2\right) - \lambda xp, \tag{3.11}$$

where $p = -i\partial/\partial x,$ as quantum analogs of the damped oscillator. A related self-adjoint Hamiltonian

$$H = H_0 = \frac{\omega_0}{2}\left(p^2 + x^2\right) - \frac{\lambda}{2}\left(px + xp\right) \tag{3.12}$$

is also analyzed. Although discussion of a quantum damped oscillator is usually missing in the standard classical textbooks [121], [145], and [183] among others, we believe that the models presented here have a significant value from the pedagogical and mathematical points of view. For instance, one of these models was crucial for our understanding of a "hidden" symmetry of the quadratic propagators in Ref. [52]. Moreover, our models show that fundamentals of quantum mechanics, such as evolution of the expectation values of operators and Ehrenfest's theorem, can be extended to the case of nonself-adjoint Hamiltonians. This provides, in our opinion, a somewhat better understanding of the mathematical foundations of quantum mechanics and can be used in the classroom.

The paper is organized as follows. In section 2 we derive the propagators for the models of the damped oscillator (3.10) and (3.11) following the method of Ref. [49]. The corresponding gauge transformations are discussed in section 3. The next section is concerned with the separation of the variables for related model of a "shifted" linear harmonic oscillator (3.12). The factorization technique is applied to this oscillator in section 5. The time evolution of the expectation values of the energy related operators is determined for these quantum damped oscillators in section 6. The classical equations for the damped oscillations are derived for the expectation values of the position operator in the next section. One more model of the damped oscillator with a variable quadratic Hamiltonian is introduced in section 8. The last section contains some remarks on the momentum representation.

3.2 The First Two Models

For the time-dependent Schrödinger equation:

$$i\frac{\partial\psi}{\partial t}=\frac{\omega_0}{2}\left(-\frac{\partial^2\psi}{\partial x^2}+x^2\psi\right)+i\lambda\left(x\frac{\partial\psi}{\partial x}+\psi\right) \tag{3.13}$$

with $a=b=\omega_0/2$ and $c=d=-\lambda$, the characteristic equation (3.6) takes the form of the classical equation of motion for the damped oscillator [13], [120]:

$$\mu''+2\lambda\mu'+\omega_0^2\mu=0, \tag{3.14}$$

whose suitable solution is as follows

$$\mu=\frac{\omega_0}{\omega}e^{-\lambda t}\sin\omega t,\qquad \omega=\sqrt{\omega_0^2-\lambda^2}>0. \tag{3.15}$$

The corresponding propagator is given by

$$\begin{aligned} G(x,y,t) &= \sqrt{\frac{\omega e^{\lambda t}}{2\pi i\omega_0\sin\omega t}}\exp\left(\frac{i\omega}{2\omega_0\sin\omega t}\left(\left(x^2+y^2\right)\cos\omega t-2xy\right)\right) \\ &\quad\times\exp\left(\frac{i\lambda}{2\omega_0}\left(x^2-y^2\right)\right). \end{aligned} \tag{3.16}$$

Indeed, directly from (3.3)–(3.4):

$$\alpha(t)=\frac{\omega\cos\omega t+\lambda\sin\omega t}{2\omega_0\sin\omega t},\qquad \beta(t)=-\frac{\omega}{\omega_0\sin\omega t}. \tag{3.17}$$

The integral in (3.5) can be evaluated with the help of a familiar antiderivative

$$\int \frac{dt}{(A\cos t + B\sin t)^2} = \frac{\sin t}{A(A\cos t + B\sin t)} + C. \tag{3.18}$$

It gives

$$\gamma(t) = \frac{\omega \cos \omega t - \lambda \sin \omega t}{2\omega_0 \sin \omega t} \tag{3.19}$$

with the help of the following identity

$$\omega^2 - \omega_0^2 \sin^2 \omega t = \omega^2 \cos^2 \omega t - \lambda^2 \sin^2 \omega t \tag{3.20}$$

and the propagator (3.16) is verified. A "hidden" symmetry of this propagator is discussed in Ref. [52].

The time-evolution of the squared norm of the wave function is given by

$$\|\psi(x,t)\|^2 = \int_{-\infty}^{\infty} |\psi(x,t)|^2 \ dx = e^{\lambda t} \|\psi(x,0)\|^2 . \tag{3.21}$$

It is derived in section 6 among other things. We have discussed here the case $\omega_0^2 > \lambda^2$. Two more cases, when $\omega_0^2 = \lambda^2$ and $\omega_0^2 < \lambda^2$, are similar and the details are left to the reader.

In a similar fashion, the time-dependent Schrödinger equation of the form

$$i\frac{\partial \psi}{\partial t} = \frac{\omega_0}{2}\left(-\frac{\partial^2 \psi}{\partial x^2} + x^2\psi\right) + i\lambda x\frac{\partial \psi}{\partial x} \tag{3.22}$$

with $a = b = \omega_0/2$ and $c = -\lambda$, $d = 0$, has the characteristic equation

$$\mu'' - 2\lambda\mu' + \omega_0^2\mu = 0 \tag{3.23}$$

with the solution

$$\mu = \frac{\omega_0}{\omega} e^{\lambda t} \sin \omega t, \qquad \omega = \sqrt{\omega_0^2 - \lambda^2} > 0. \tag{3.24}$$

The corresponding propagator is given by

$$\begin{aligned} G(x,y,t) \quad &= \quad \sqrt{\frac{\omega e^{-\lambda t}}{2\pi i \omega_0 \sin \omega t}} \exp\left(\frac{i\omega}{2\omega_0 \sin \omega t}\left(\left(x^2+y^2\right)\cos \omega t - 2xy\right)\right) \\ &\quad \times \exp\left(\frac{i\lambda}{2\omega_0}\left(x^2-y^2\right)\right) \end{aligned} \tag{3.25}$$

and the evolution of the squared norm is as follows

$$\|\psi(x,t)\|^2 = e^{-\lambda t} \|\psi(x,0)\|^2 . \tag{3.26}$$

The solution of the Cauchy initial value problem

$$i\frac{\partial \psi}{\partial t}=H\psi, \qquad \psi(x,0)=\chi(x) \tag{3.27}$$

for our models (3.13) and (3.22) is given by the superposition principle in an integral form

$$\psi(x,t)=\int_{-\infty}^{\infty}G(x,y,t)\ \chi(y)\ dy \tag{3.28}$$

for a suitable initial function χ on R; a rigorous proof is given in Ref. [195].

3.3 The Gauge Transformations

The time-dependent Schrödinger equation

$$i\frac{\partial \psi}{\partial t}=\left(\frac{\omega_0}{2}(p-A)^2+U+(p-A)V+W(p-A)\right)\psi, \tag{3.29}$$

where $p=i^{-1}\partial/\partial x$ is the linear momentum operator and $A=A(x,t)$, $U=U(x,t)$, $V=V(x,t)$, $W=W(x,t)$ are real-valued functions, with the help of the gauge transformation

$$\psi=e^{-if(x,t)}\widetilde{\psi} \tag{3.30}$$

can be transformed into a similar form

$$i\frac{\partial \widetilde{\psi}}{\partial t}=\left(\frac{\omega_0}{2}\left(p-\widetilde{A}\right)^2+\widetilde{U}+\left(p-\widetilde{A}\right)\widetilde{V}+\widetilde{W}\left(p-\widetilde{A}\right)\right)\widetilde{\psi} \tag{3.31}$$

with the new vector and scalar potentials given by

$$\widetilde{A}=A+\frac{\partial f}{\partial x}, \qquad \widetilde{U}=U-\frac{\partial f}{\partial t}, \qquad \widetilde{V}=V, \qquad \widetilde{W}=W. \tag{3.32}$$

Here we consider the one-dimensional case only and may think of f as being an arbitrary complex-valued differentiable function. Also, the Hamiltonian in the right hand side of equation (3.29) is not assumed to be self-adjoint. See Refs. [121] and [145] for discussion of the traditional case, when $V=W\equiv 0$.

An interesting special case of the gauge transformation related to this paper is given by

$$A=0, \qquad U=\frac{\omega_0}{2}x^2, \qquad V=-\lambda x, \qquad W=0, \qquad f=\frac{i\lambda t}{2}, \tag{3.33}$$

$$\widetilde{A}=0, \qquad \widetilde{U}=\frac{\omega_0}{2}x^2-\frac{i\lambda}{2}, \qquad \widetilde{V}=-\lambda x, \qquad \widetilde{W}=0, \tag{3.34}$$

when the new Hamiltonian is

$$\begin{aligned} \widetilde{H} &= \frac{\omega_0}{2}\left(p-\widetilde{A}\right)^2+\widetilde{U}+p\widetilde{V} \\ &= \frac{\omega_0}{2}\left(-\frac{\partial^2}{\partial x^2}+x^2\right)+i\frac{\lambda}{2}\left(2x\frac{\partial}{\partial x}+1\right), \end{aligned} \tag{3.35}$$

and equation (3.13) takes the form

$$i\frac{\partial\psi}{\partial t}=\frac{\omega_0}{2}\left(-\frac{\partial^2\psi}{\partial x^2}+x^2\psi\right)+i\frac{\lambda}{2}\left(2x\frac{\partial\psi}{\partial x}+\psi\right). \tag{3.36}$$

The corresponding Green function is given by

$$\begin{aligned} G(x,y,t) &= \sqrt{\frac{\omega}{2\pi i\omega_0\sin\omega t}}\exp\left(\frac{i\omega}{2\omega_0\sin\omega t}\left(\left(x^2+y^2\right)\cos\omega t-2xy\right)\right) \\ &\quad\times\exp\left(\frac{i\lambda}{2\omega_0}\left(x^2-y^2\right)\right), \qquad \omega=\sqrt{\omega_0^2-\lambda^2}>0 \end{aligned} \tag{3.37}$$

and the norm of the wave function is conserved with time. This can be established once again directly from our equations (3.2)–(3.8). We leave the details to the reader. A traditional method of separation of the variables and using the Mehler formula for Hermite polynomials is discussed in the next section. The factorization technique is applied to this Hamiltonian in section 5.

Equation (3.36), in turn, admits another local gauge transformation:

$$A=0, \qquad U=\frac{\omega_0}{2}x^2, \qquad V=W=-\frac{\lambda x}{2}, \qquad f=-\frac{\lambda x^2}{2\omega_0}, \tag{3.38}$$

$$\widetilde{A}=-\frac{\lambda x}{\omega_0}, \qquad \widetilde{U}=\frac{\omega_0}{2}x^2, \qquad \widetilde{V}=\widetilde{W}=-\frac{\lambda x}{2} \tag{3.39}$$

and the Hamiltonian becomes

$$\begin{aligned} \widetilde{H} &= \frac{\omega_0}{2}\left(p-\widetilde{A}\right)^2+\widetilde{U}+\left(p-\widetilde{A}\right)\widetilde{V}+\widetilde{W}\left(p-\widetilde{A}\right) \\ &= \frac{\omega_0}{2}\left(p+\frac{\lambda x}{\omega_0}\right)^2+\frac{\omega_0}{2}x^2 \\ &\quad+\left(p+\frac{\lambda x}{\omega_0}\right)\left(-\frac{\lambda x}{\omega_0}\right)+\left(-\frac{\lambda x}{\omega_0}\right)\left(p+\frac{\lambda x}{\omega_0}\right) \\ &= \frac{\omega_0}{2}p^2+\frac{\omega_0^2-\lambda^2}{2\omega_0}x^2. \end{aligned} \tag{3.40}$$

As a result, equation (3.36) takes the form of equation for the harmonic oscillator:

$$i\frac{\partial\psi}{\partial t}=\frac{\omega_0}{2}\left(-\frac{\partial^2\psi}{\partial x^2}+\frac{\omega^2}{\omega_0^2}x^2\psi\right), \qquad \omega^2=\omega_0^2-\lambda^2>0 \tag{3.41}$$

and can be solved, once again, by the traditional method of separation of the variables or by the factorization technique.

We shall refer to the case (3.36) as one of a shifted linear harmonic oscillator. The Ansatz

$$\psi(x,t) = e^{-iEt}\varphi(x) \tag{3.42}$$

in the time-dependent Schrödinger equation results in the stationary Schrödinger equation

$$H\varphi = E\varphi \tag{3.43}$$

with the Hamiltonian (3.35). The last equation, namely,

$$-\varphi'' + x^2\varphi + \frac{i\lambda}{\omega_0}\left(2x\varphi' + \varphi\right) = \frac{2E}{\omega_0}\varphi, \tag{3.44}$$

with the help of the substitution

$$\varphi = \exp\left(\frac{i\lambda x^2}{2\omega_0}\right)u(x) \tag{3.45}$$

is reduced to the following equation

$$-u'' + \frac{\omega^2}{\omega_0^2}x^2 u = \frac{2E}{\omega_0}u. \tag{3.46}$$

The change of the variable

$$u(x) = v(\xi), \qquad x = \xi\sqrt{\frac{\omega_0}{\omega}} \tag{3.47}$$

gives us the stationary Schrödinger equation for the simple harmonic oscillator [121], [145], [158], [183]:

$$v'' + \left(2\varepsilon - \xi^2\right)v = 0 \tag{3.48}$$

with $\varepsilon = E/\omega$, whose eigenfunctions are given in terms of the Hermite polynomials as follows

$$v_n = C_n e^{-\xi^2/2}H_n(\xi), \tag{3.49}$$

and the corresponding eigenvalues are

$$\varepsilon_n = n + \frac{1}{2}, \qquad E_n = \omega\left(n + \frac{1}{2}\right) \qquad (n = 0, 1, 2, \ldots). \tag{3.50}$$

Thus the normalized wave functions of our shifted oscillator (3.36) are given by

$$\psi_n(x,t) = e^{-i\omega(n+1/2)t}\varphi_n(x), \tag{3.51}$$

where

$$\varphi_n(x) = C_n \exp\left(\frac{i\lambda x^2}{2\omega_0}\right) e^{-\xi^2/2} H_n(\xi), \qquad \xi = x\sqrt{\frac{\omega}{\omega_0}} \tag{3.52}$$

and

$$|C_n|^2 = \sqrt{\frac{\omega}{\omega_0}} \frac{1}{\sqrt{\pi}2^n n!} \tag{3.53}$$

in view of the orthogonality relation

$$\int_{-\infty}^{\infty} \varphi_n^*(x)\, \varphi_m(x)\ dx = \delta_{nm}. \tag{3.54}$$

We use the star for complex conjugate.

Solution of the initial value problem (3.27) can be found by the superposition principle in the form

$$\psi(x,t) = \sum_{n=0}^{\infty} c_n\ \psi_n(x,t), \tag{3.55}$$

where

$$\psi(x,0) = \chi(x) = \sum_{n=0}^{\infty} c_n\ \varphi_n(x) \tag{3.56}$$

and

$$c_n = \int_{-\infty}^{\infty} \varphi_n^*(y)\, \chi(y)\ dy \tag{3.57}$$

in view of the orthogonality property (3.54). Substituting (3.57) into (3.55) and changing the order of the summation and integration, one gets

$$\psi(x,t) = \int_{-\infty}^{\infty} G(x,y,t)\, \chi(y)\ dy, \tag{3.58}$$

where the Green function is given as the eigenfunction expansion:

$$G(x,y,t) = \sum_{n=0}^{\infty} e^{-i\omega(n+1/2)t} \varphi_n(x)\, \varphi_n^*(y). \tag{3.59}$$

This infinite series is summable with the help of the Poisson kernel for the Hermite polynomials (Mehler's formula) [170]:

$$\sum_{n=0}^{\infty} \frac{H_n(x)\, H_n(y)}{2^n n!} r^n = \frac{1}{\sqrt{1-r^2}} \exp\left(\frac{2xyr - \left(x^2+y^2\right) r^2}{1-r^2}\right), \quad |r| < 1. \tag{3.60}$$

The result is given, of course, by equation (3.37).

3.5 The Factorization Method for Shifted Harmonic Oscillator

It is worth applying the well-known factorization technique (see, for example, [6], [7], [9], [68] and [145]) to the Hamiltonian (3.35). The corresponding ladder operators can be found in the forms

$$a = (\alpha + i\beta)x + \gamma\frac{\partial}{\partial x}, \tag{3.61}$$

$$a^{\dagger} = (\alpha - i\beta)x - \gamma\frac{\partial}{\partial x}, \tag{3.62}$$

where α, β and γ are real numbers to be determined as follows. One gets

$$aa^{\dagger}\psi = \left(\alpha^2+\beta^2\right)x^2\psi + (\alpha - i\beta)\gamma\psi - 2i\beta\gamma x\frac{\partial\psi}{\partial x} - \gamma^2\frac{\partial^2\psi}{\partial x^2}, \tag{3.63}$$

$$a^{\dagger}a\psi = \left(\alpha^2+\beta^2\right)x^2\psi - (\alpha + i\beta)\gamma\psi - 2i\beta\gamma x\frac{\partial\psi}{\partial x} - \gamma^2\frac{\partial^2\psi}{\partial x^2}, \tag{3.64}$$

whence

$$\left(aa^{\dagger} - a^{\dagger}a\right)\psi = 2\alpha\gamma\psi \tag{3.65}$$

and

$$\frac{1}{2}\left(aa^{\dagger} + a^{\dagger}a\right)\psi = -\gamma^2\frac{\partial^2\psi}{\partial x^2} + \left(\alpha^2+\beta^2\right)x^2\psi - i\beta\gamma\left(2x\frac{\partial\psi}{\partial x} + \psi\right). \tag{3.66}$$

The canonical commutation relation occurs and the Hamiltonian (3.35) takes the standard form:

$$H = \frac{\omega}{2}\left(aa^{\dagger} + a^{\dagger}a\right), \tag{3.67}$$

if

$$2\alpha\gamma = 1, \qquad \omega\left(\alpha^2+\beta^2\right) = \omega\gamma^2 = \frac{1}{2}\omega_0, \qquad \omega\beta\gamma = -\frac{1}{2}\lambda. \tag{3.68}$$

The relation $\omega_0^2 = \omega^2 + \lambda^2$, which defines the new oscillator frequency, holds. As a result, the explicit form of the annihilation and creation operators is given by

$$\sqrt{2}a = \left(\sqrt{\frac{\omega}{\omega_0}} - \frac{i\lambda}{\sqrt{\omega_0\omega}}\right)x + \sqrt{\frac{\omega_0}{\omega}}\frac{\partial}{\partial x}, \tag{3.69}$$

$$\sqrt{2}a^{\dagger} = \left(\sqrt{\frac{\omega}{\omega_0}} + \frac{i\lambda}{\sqrt{\omega_0\omega}}\right)x - \sqrt{\frac{\omega_0}{\omega}}\frac{\partial}{\partial x}. \tag{3.70}$$

The special case $\lambda = 0$ and $\omega = \omega_0$ gives a traditional form of these operators.

The oscillator spectrum (3.50) and the corresponding stationary wave functions (3.52) can be obtain now in a standard way by using the Heisenberg–Weyl algebra of the rasing and lowering operators. In addition, the n-dimensional oscillator wave functions form a basis of the irreducible unitary representation of the Lie algebra of the noncompact group $SU(1,1)$ corresponding to the discrete positive series $\mathscr{D}_{+}^{j}$; see [143], [157] and [188]. Our operators (3.69)–(3.70) allow us to extend these group-theoretical properties for the case of the shifted oscillators. We leave the details to the reader.

3.6 Dynamics of Energy Related Expectation Values

The expectation value of an operator A in quantum mechanics is given by the formula

$$\langle A\rangle = \int_{-\infty}^{\infty} \psi^{*}(x,t)\ A(t)\,\psi(x,t)\ dx, \tag{3.71}$$

where the wave function satisfies the time-dependent Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = H\psi. \tag{3.72}$$

The time derivative of this expectation value can be written as

$$i\frac{d}{dt}\langle A\rangle = i\left\langle \frac{\partial A}{\partial t}\right\rangle + \left\langle AH - H^{\dagger}A\right\rangle, \tag{3.73}$$

where $H^{\dagger}$ is the Hermitian adjoint of the Hamiltonian operator H. Our formula is a simple extension of the well-known expression [121], [145], [183] to the case of a nonself-adjoint Hamiltonian.

We apply formula (3.73) to the Hamiltonian

$$H = \frac{\omega_0}{2}\left(p^2 + x^2\right) - \lambda px, \qquad p = -i\frac{\partial}{\partial x} \tag{3.74}$$

in equation (3.13). A few examples will follow. In the case of the identity operator $A = 1$, one gets

$$AH - H^{\dagger}A = \lambda\,(xp - px) = i\lambda \tag{3.75}$$

by the Heisenberg commutation relation

$$[x,p] = xp - px = i. \tag{3.76}$$

As a result,

$$\frac{d}{dt}\|\psi\|^2 = \lambda\,\|\psi\|^2, \tag{3.77}$$

and time-evolution of the squared norm of the wave function for our model of the damped quantum oscillator is given by equation (3.21).

In a similar fashion, if $A = H$, then

$$H^2 - H^\dagger H = \left(H - H^\dagger\right) H = i\lambda H, \tag{3.78}$$

and

$$\frac{d}{dt} \langle H \rangle = \lambda \langle H \rangle, \qquad \langle H \rangle = \langle H \rangle_0 e^{\lambda t}. \tag{3.79}$$

Moreover,

$$\frac{d}{dt} \langle H^n \rangle = \lambda \langle H^n \rangle, \qquad \langle H^n \rangle = \langle H^n \rangle_0 e^{\lambda t} \qquad (n = 0, 1, 2, \ldots), \tag{3.80}$$

which unifies the both of the previous cases.

Now we choose $A = p^2$, $A = x^2$ and $A = px + xp$, respectively, in order to obtain the following system:

$$\begin{aligned}
\frac{d}{dt} \left\langle p^2 \right\rangle &= 3\lambda \left\langle p^2 \right\rangle - \omega_0 \langle px + xp \rangle, \\
\frac{d}{dt} \left\langle x^2 \right\rangle &= -\lambda \left\langle x^2 \right\rangle + \omega_0 \langle px + xp \rangle, \\
\frac{d}{dt} \langle px + xp \rangle &= 2\omega_0 \left(\left\langle p^2 \right\rangle - \left\langle x^2 \right\rangle \right) + \lambda \langle px + xp \rangle.
\end{aligned} \tag{3.81}$$

Indeed,

$$\begin{aligned}
p^2 H - H^\dagger p^2 &= \frac{\omega_0}{2} \left[p^2, x^2\right] + \lambda \left[x, p^3\right] \\
&= 3i\lambda p^2 - i\omega_0 (px + xp),
\end{aligned} \tag{3.82}$$

$$\begin{aligned}
x^2 H - H^\dagger x^2 &= \frac{\omega_0}{2} \left[x^2, p^2\right] - \lambda x [x, p] x \\
&= i\omega_0 (px + xp) - i\lambda x^2,
\end{aligned} \tag{3.83}$$

and

$$\begin{aligned}
&(px + xp) H - H^\dagger (px + xp) \\
&= \frac{\omega_0}{2} \left(\left[p, x^3\right] + \left[x, p^3\right] \right) \\
&\quad + \frac{\omega_0}{2} \left(p [x, p] p - x [x, p] x \right) \\
&\quad + \lambda \left((xp)^2 - (px)^2 \right) \\
&= 2i\omega_0 \left(p^2 - x^2\right) + i\lambda (px + xp),
\end{aligned} \tag{3.84}$$

which results in (3.81).

The system can be solved explicitly, thus providing the complete dynamics of these expectation values. The eigenvalues are given by $r_0 = \lambda$, $r_\pm = \lambda \pm 2i\omega$ and the corresponding linearly independent eigenvectors are

$$x_0 = \begin{pmatrix} \omega_0 \\ \omega_0 \\ 2\lambda \end{pmatrix}, \qquad x_\pm = \begin{pmatrix} (\lambda \pm i\omega)^2 \\ \omega_0^2 \\ 2\omega_0\,(\lambda \pm i\omega) \end{pmatrix} \tag{3.85}$$

with the determinant

$$\begin{vmatrix} \omega_0 & (\lambda + i\omega)^2 & (\lambda - i\omega)^2 \\ \omega_0 & \omega_0^2 & \omega_0^2 \\ 2\lambda & 2\omega_0\,(\lambda + i\omega) & 2\omega_0\,(\lambda - i\omega) \end{vmatrix} = -8i\omega_0^2\omega^3 \neq 0. \tag{3.86}$$

The general solution of the system (3.81) can be obtain in a complex form as follows

$$\begin{pmatrix} \langle p^2 \rangle \\ \langle x^2 \rangle \\ \langle px + xp \rangle \end{pmatrix} = C_0 e^{\lambda t} \begin{pmatrix} \omega_0 \\ \omega_0 \\ 2\lambda \end{pmatrix} \tag{3.87}$$
$$+ C_+ e^{(\lambda + 2i\omega)t} \begin{pmatrix} (\lambda + i\omega)^2 \\ \omega_0^2 \\ 2\omega_0\,(\lambda + i\omega) \end{pmatrix} + C_- e^{(\lambda - 2i\omega)t} \begin{pmatrix} (\lambda - i\omega)^2 \\ \omega_0^2 \\ 2\omega_0\,(\lambda - i\omega) \end{pmatrix},$$

where C_0 and $C_\pm$ are constants. The corresponding solution of the initial value problem is given by

$$\begin{pmatrix} \langle p^2\rangle \\ \langle x^2\rangle \\ \langle px+xp\rangle \end{pmatrix} = \frac{1}{2\omega^2}\left(\omega_0\left(\left\langle p^2\right\rangle_0+\left\langle x^2\right\rangle_0\right)-\lambda\left\langle px+xp\right\rangle_0\right)e^{\lambda t}\begin{pmatrix} \omega_0 \\ \omega_0 \\ 2\lambda \end{pmatrix} \tag{3.88}$$
$$+\frac{1}{2\omega^2}\left(\frac{\lambda}{\omega_0}\langle px+xp\rangle_0+\frac{\omega^2-\lambda^2}{\omega_0^2}\left\langle x^2\right\rangle_0-\left\langle p^2\right\rangle_0\right)$$
$$\times e^{\lambda t}\begin{pmatrix} \left(\lambda^2-\omega^2\right)\cos 2\omega t-2\lambda\omega\sin 2\omega t \\ \omega_0^2\cos 2\omega t \\ 2\lambda\omega_0\cos 2\omega t-2\omega_0\omega\sin 2\omega t \end{pmatrix}$$
$$+\frac{1}{2\omega_0\omega}\left(\langle px+xp\rangle_0-\frac{2\lambda}{\omega_0}\left\langle x^2\right\rangle_0\right)$$
$$\times e^{\lambda t}\begin{pmatrix} 2\lambda\omega\cos 2\omega t+\left(\lambda^2-\omega^2\right)\sin 2\omega t \\ \omega_0^2\sin 2\omega t \\ 2\omega_0\omega\cos 2\omega t+2\lambda\omega_0\sin 2\omega t \end{pmatrix}.$$

The mechanical energy operator E can be conveniently introduced as the Hamiltonian of our shifted linear harmonic oscillator (3.35):

$$E=H_0=\frac{\omega_0}{2}\left(p^2+x^2\right)-\frac{\lambda}{2}\left(px+xp\right), \tag{3.89}$$

so that

$$H=H_0+i\frac{\lambda}{2}. \tag{3.90}$$

Then

$$\frac{d}{dt}\langle E\rangle = \frac{\omega_0}{2}\left(\frac{d}{dt}\left\langle p^2\right\rangle+\frac{d}{dt}\left\langle x^2\right\rangle\right) \tag{3.91}$$
$$-\frac{\lambda}{2}\frac{d}{dt}\langle px+xp\rangle$$
$$= \lambda\left\langle\frac{\omega_0}{2}\left(p^2+x^2\right)-\frac{\lambda}{2}\left(px+xp\right)\right\rangle$$

with the help of our system (3.81). Therefore,

$$\frac{d}{dt}\langle E\rangle=\lambda\langle E\rangle, \qquad \langle E\rangle=\langle E\rangle_0 e^{\lambda t} \tag{3.92}$$

for the expectation value of the mechanical energy of the damped oscillator under consideration.

The case of the second Hamiltonian:

$$H=\frac{\omega_0}{2}\left(p^2+x^2\right)-\lambda xp=H_0-i\frac{\lambda}{2}, \tag{3.93}$$

which is the Hermitian adjoint of the Hamiltonian (3.74), is similar. Here

$$H^{n+1}-H^{\dagger}H^n=\left(H-H^{\dagger}\right)H^n=\lambda\,[p,x]\,H^n=-i\lambda H^n$$

and

$$\frac{d}{dt}\langle H^n\rangle=-\lambda\,\langle H^n\rangle\,, \qquad \langle H^n\rangle=\langle H^n\rangle_0\,e^{-\lambda t} \qquad (n=0,1,2,\ldots)\,. \tag{3.94}$$

Moreover,

$$\begin{aligned} p^2H-H^{\dagger}p^2 &= \frac{\omega_0}{2}\left[p^2,x^2\right]+\lambda\,p\,[x,p]\,p \\ &= i\lambda p^2-i\omega_0\,(px+xp)\,, \end{aligned} \tag{3.95}$$

$$\begin{aligned} x^2H-H^{\dagger}x^2 &= \frac{\omega_0}{2}\left[x^2,p^2\right]+\lambda\,\left[p,x^3\right] \\ &= -3i\lambda x^2+i\omega_0\,(px+xp)\,, \end{aligned} \tag{3.96}$$

$$\begin{aligned} &(px+xp)\,H-H^{\dagger}\,(px+xp) \\ &=\frac{\omega_0}{2}\left(\left[p,x^3\right]+\left[x,p^3\right]\right) \\ &\quad+\frac{\omega_0}{2}\,(p\,[x,p]\,p-x\,[x,p]\,x) \\ &\quad-\lambda\left((xp)^2-(px)^2\right) \\ &=2i\omega_0\left(p^2-x^2\right)-i\lambda\,(px+xp)\,, \end{aligned} \tag{3.97}$$

and the corresponding system has the form

$$\begin{aligned} &\frac{d}{dt}\left\langle p^2\right\rangle=\lambda\left\langle p^2\right\rangle-\omega_0\,\langle px+xp\rangle\,, \\ &\frac{d}{dt}\left\langle x^2\right\rangle=-3\lambda\left\langle x^2\right\rangle+\omega_0\,\langle px+xp\rangle\,, \\ &\frac{d}{dt}\langle px+xp\rangle=2\omega_0\left(\left\langle p^2\right\rangle-\left\langle x^2\right\rangle\right)-\lambda\,\langle px+xp\rangle\,. \end{aligned} \tag{3.98}$$

The change $p\leftrightarrow x$, $\lambda\rightarrow-\lambda$, $\omega_0\rightarrow-\omega_0$ transforms formally this system back into (3.81). This observation allows us to obtain solution of the initial value problem from the previous solution given by (3.88). For the mechanical energy operator E introduced by equation (3.89) one gets

$$\frac{d}{dt}\langle E\rangle=-\lambda\,\langle E\rangle\,, \qquad \langle E\rangle=\langle E\rangle_0\,e^{-\lambda t} \tag{3.99}$$

with the help of (3.98).

The case of a general variable quadratic Hamiltonian of the form

$$H = a(t)\,p^2 + b(t)\,x^2 + c(t)\,px + d(t)\,xp, \tag{3.100}$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$ are real-valued functions of time only, is considered in a similar fashion. One gets

$$H^{n+1} - H^{\dagger}H^n = \left(H - H^{\dagger}\right)H^n = (c-d)\,[p,x]\,H^n = i\,(d-c)\,H^n \tag{3.101}$$

and

$$\frac{d}{dt}\left\langle H^n\right\rangle = \left\langle \frac{\partial H^n}{\partial t}\right\rangle + (d(t) - c(t))\left\langle H^n\right\rangle. \tag{3.102}$$

The cases $n = 0$ and $n = 1$ result in

$$\langle 1\rangle = \langle 1\rangle_0 \exp\left(\int_0^t (d(\tau) - c(\tau))\;d\tau\right) \tag{3.103}$$

and

$$\frac{d}{dt}\langle H\rangle = \left\langle \frac{\partial H}{\partial t}\right\rangle + (d(t) - c(t))\langle H\rangle, \tag{3.104}$$

respectively.

Moreover,

$$\begin{aligned} p^2H - H^{\dagger}p^2 &= b\left[p^2,x^2\right] + c\left[p^3,x\right] + d\,p\,[p,x]\,p \\ &= -i\,(3c+d)\,p^2 - 2ib\,(px+xp), \end{aligned} \tag{3.105}$$

$$\begin{aligned} x^2H - H^{\dagger}x^2 &= a\left[x^2,p^2\right] + cx\,[x,p]\,x + d\left[x^3,p\right] \\ &= i\,(3d+c)\,x^2 + 2ia\,(px+xp), \end{aligned} \tag{3.106}$$

$$\begin{aligned} &(px+xp)\,H - H^{\dagger}(px+xp) \\ &= a\left(\left[x,p^3\right] + p\,[x,p]\,p\right) \\ &\quad + b\left(\left[p,x^3\right] + x\,[p,x]\,x\right) \\ &\quad + (c-d)\left((px)^2 - (xp)^2\right) \\ &= 4iap^2 - 4ibx^2 - i\,(c-d)\,(px+xp), \end{aligned} \tag{3.107}$$

and the corresponding system has the form

$$\begin{aligned}
\frac{d}{dt}\left\langle p^2\right\rangle &= -\left(3c+d\right)\left\langle p^2\right\rangle - 2b\left\langle px+xp\right\rangle, \\
\frac{d}{dt}\left\langle x^2\right\rangle &= \left(c+3d\right)\left\langle x^2\right\rangle + 2a\left\langle px+xp\right\rangle, \\
\frac{d}{dt}\left\langle px+xp\right\rangle &= 4a\left\langle p^2\right\rangle - 4b\left\langle x^2\right\rangle + \left(d-c\right)\left\langle px+xp\right\rangle.
\end{aligned} \tag{3.108}$$

We have used the familiar identities

$$[x,p]=i, \qquad (xp)^2-(px)^2=i\left(px+xp\right), \tag{3.109}$$

$$\left[x^2,p^2\right]=2i\left(px+xp\right), \qquad \left[x,p^3\right]=3ip^2, \qquad \left[x^3,p\right]=3ix^2 \tag{3.110}$$

once again.

3.7 A Relation with the Classical Damped Oscillations

Application of formula (3.71) to the position x and momentum p operators allows to modify the Ehrenfest theorem [69], [145], [183] for the models of damped oscillators under consideration. For the Hamiltonian (3.74) one gets

$$xH-H^{\dagger}x = \frac{\omega_0}{2}\left[x,p^2\right]=i\omega_0 p, \tag{3.111}$$

$$pH-H^{\dagger}p = \frac{\omega_0}{2}\left[p,x^2\right]+\lambda\left[x,p^2\right]=-i\omega_0 x+2i\lambda p \tag{3.112}$$

and

$$\frac{d}{dt}\langle x\rangle=\omega_0\langle p\rangle, \qquad \frac{d}{dt}\langle p\rangle=-\omega_0\langle x\rangle+2\lambda\langle p\rangle. \tag{3.113}$$

Elimination of the expectation value $\langle p\rangle$ from this system results in

$$\frac{d^2}{dt^2}\langle x\rangle-2\lambda\frac{d}{dt}\langle x\rangle+\omega_0^2\langle x\rangle=0, \tag{3.114}$$

which is a classical equation of motion for a damped oscillator [13], [120].

For the second Hamiltonian (3.93) we obtain

$$\frac{d}{dt}\langle x\rangle=\omega_0\langle p\rangle-2\lambda\langle x\rangle, \qquad \frac{d}{dt}\langle p\rangle=-\omega_0\langle x\rangle, \tag{3.115}$$

which gives

$$\frac{d^2}{dt^2}\langle x\rangle+2\lambda\frac{d}{dt}\langle x\rangle+\omega_0^2\langle x\rangle=0 \tag{3.116}$$

in a similar fashion.

Finally, our model of the shifted harmonic oscillator (3.36), when the Hamiltonian is given by (3.89), results in

$$\frac{d^2}{dt^2}\langle x\rangle + \left(\omega_0^2 - \lambda^2\right)\langle x\rangle = 0. \tag{3.117}$$

We leave the details to the reader.

3.8 The Third Model

For the time-dependent Schrödinger equation with variable quadratic Hamiltonian:

$$i\frac{\partial\psi}{\partial t} = \frac{\omega_0}{2}\left(-e^{-2\lambda t}\frac{\partial^2\psi}{\partial x^2} + e^{2\lambda t}x^2\psi\right), \tag{3.118}$$

where $a = (\omega_0/2)\, e^{-2\lambda t}$, $b = (\omega_0/2)\, e^{2\lambda t}$ and $c = d = 0$, the characteristic equation takes the form (3.14) with the same solution (3.15). The corresponding propagator has the form (3.2) with

$$\alpha(t) = \frac{\omega\cos\omega t - \lambda\sin\omega t}{2\omega_0\sin\omega t}e^{2\lambda t}, \tag{3.119}$$

$$\beta(t) = -\frac{\omega}{\omega_0\sin\omega t}e^{\lambda t}, \tag{3.120}$$

$$\gamma(t) = \frac{\omega\cos\omega t + \lambda\sin\omega t}{2\omega_0\sin\omega t}. \tag{3.121}$$

This can be derived directly from equations (3.2)–(3.8) with the help of identity (3.20). We leave the details to the reader. It is worth noting that equation (3.118) can be obtain by introducing a variable unit of length $x \to xe^{\lambda t}$ in the Hamiltonian of the linear oscillator.

3.9 Momentum Representation

The time-dependent Schrödinger equations for the damped oscillators are also solved in the momentum representation. One can easily verify that under the Fourier transform our first Hamiltonian (3.74) takes the form of the second Hamiltonian (3.93) with $\lambda \to -\lambda$ and visa versa (see, for example, Ref. [52] for more details). Moreover, the inverses of the corresponding time evolution operators are obtained by the time reversal. Therefore, all identities of the commutative evolution diagram introduced in Ref. [52] for the modified oscillators are also valid for the quantum damped oscillators under consideration. We leave further details to the reader.

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Chapter 4

QUANTUM INTEGRALS OF MOTION FOR VARIABLE QUADRATIC HAMILTONIANS

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4.1 An Introduction

Evolution of a nonrelativistic quantum system from a given initial state to the final state is governed by the (time-dependent) Schrödinger equation. Unfortunately, its explicit solutions are available only for the simplest Hamiltonians and, in general, one has to rely on a variety of approximation, asymptotic and numerical methods. Luckily among the integrable cases are the so-called quadratic Hamiltonians that attracted substantial attention over the years in view of their great importance to many advanced quantum problems. Examples can be found in quantum and physical optics [58], [113], [161], [176], physics of lasers and masers [182], [205], [186], [214], molecular spectroscopy [67], quantum chemistry, quantization of mechanical systems [57], [75], [77], [78], [81], [114], [116] and Hamiltonian cosmology [17], [83], [84], [95], [102], [161], [177], [178], [180]. They include coherent states [140], [137], [138], [113] and Berry's phase [15], [16], [38], [94], [125], [150], asymptotic and numerical methods [88], [106], [118], [147], [152], charged particle traps [136] and motion in uniform magnetic fields [49], [48], [65], [121], [129], [130], [132], [138], polyatomic molecules in varying external fields, crystals through which an electron is passing and exciting the oscillator modes and other interactions of the modes with external fields [81]. Quadratic Hamiltonians have particular applications in quantum electrodynamics because the electromagnetic field can be represented as a set of forced harmonic oscillators [19], [81], [65], [87], [103] and [145]. Nonlinear oscillators play a central role in the novel theory of Bose–Einstein condensation [54] based on the nonlinear Schrödinger (or Gross–Pitaevskii) equation [104], [105], [112], [166].

The one-dimensional Schrödinger equation with variable quadratic Hamiltonians of the form

$$i\frac{\partial \psi}{\partial t} = -a(t)\frac{\partial^2 \psi}{\partial x^2} + b(t)x^2\psi - i\left(c(t)x\frac{\partial \psi}{\partial x} + d(t)\psi\right), \tag{4.1}$$

where $a(t)$, $b(t)$, $c(t)$, and $d(t)$ are real-valued functions of time t only, can be integrated in the following manner (see, for example, [49], [50], [52], [62], [122], [134], [143], [193], [194], [195], [196], [218], and [219] for a general approach and some elementary solutions). The Green functions, or Feynman's propagators, are given by [49], [195]:

$$\psi = G(x,y,t) = \frac{1}{\sqrt{2\pi i \mu(t)}}\ e^{i\left(\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2\right)}, \tag{4.2}$$

where

$$\alpha(t) = \frac{1}{4a(t)}\frac{\mu'(t)}{\mu(t)} - \frac{d(t)}{2a(t)}, \tag{4.3}$$

$$\beta(t) = -\frac{h(t)}{\mu(t)}, \qquad h(t) = \exp\left(-\int_0^t (c(\tau)-2d(\tau))\ d\tau\right), \tag{4.4}$$

$$\gamma(t) = \frac{a(t)h^2(t)}{\mu(t)\mu'(t)} + \frac{d(0)}{2a(0)} - 4\int_0^t \frac{a(\tau)\sigma(\tau)h^2(\tau)}{(\mu'(\tau))^2}\ d\tau \tag{4.5}$$

and the function $\mu(t)$ satisfies the so-called characteristic equation

$$\mu'' - \tau(t)\mu' + 4\sigma(t)\mu = 0 \tag{4.6}$$

with

$$\tau(t) = \frac{a'}{a} - 2c + 4d, \qquad \sigma(t) = ab - cd + d^2 + \frac{d}{2}\left(\frac{a'}{a} - \frac{d'}{d}\right) \tag{4.7}$$

subject to the initial data

$$\mu(0) = 0, \qquad \mu'(0) = 2a(0) \neq 0. \tag{4.8}$$

(More details can be found in Refs. [49], [195] and a Hamiltonian structure is considered in Refs. [15], [52].) Then, by the superposition principle, solution of the Cauchy initial value problem can be presented in an integral form

$$\psi(x,t) = \int_{-\infty}^{\infty} G(x,y,t)\ \varphi(y)\ dy, \quad \lim_{t\to 0^+}\psi(x,t) = \varphi(x) \tag{4.9}$$

for a suitable initial function φ on $\mathbb{R}$ (a rigorous proof is given in Ref. [195] and uniqueness is analyzed in this paper).

We discuss integrals of motion for several particular models of the damped and generalized quantum oscillators. The simple harmonic oscillator is of interest in many quantum problems [81], [121], [145], and [183]. The forced harmonic oscillator was originally considered by Richard Feynman in his path integrals approach to the nonrelativistic quantum mechanics

[77], [78], [79], [80], and [81]; see also [134]. Its special and limiting cases were discussed in Refs. [14], [87], [99], [142], [145], [207] for the simple harmonic oscillator and in Refs. [4], [21], [98], [154], [173] for the particle in a constant external field; see also references therein. The damped oscillations have been studied to a great extent in classical mechanics [12], [13] and [120]. Their quantum analogs are introduced and analyzed from different viewpoints by many authors; see, for example, [22], [39], [44], [45], [46], [50], [59], [60], [61], [63], [66], [123], [124], [107], [149], [156], [199], [200], [204], [209], and references therein. The quantum parametric oscillator with variable frequency is also largely studied in view of its physical importance; see, for example, [42], [65], [100], [101], [122], [140], [138], [164], [165], [168], [169], [185], and [189]; a detailed bibliography is given in [23].

In the present paper we revisit a familiar topic of the quantum integrals of motion for the time-dependent Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = H(t)\,\psi \tag{4.10}$$

with variable quadratic Hamiltonians of the form

$$H = a(t)\,p^2 + b(t)\,x^2 + d(t)\,(px + xp)\,, \tag{4.11}$$

where $p = -i\partial/\partial x$, $\hbar = 1$ and $a(t)\,,\ b(t)\,,\ c(t) = 2d(t)$ are some real-valued functions of time only (see, for example, [62], [125], [132], [138], [139], [218], [219] and references therein). A related energy operator E is defined in a traditional way as a quadratic in p and x operator that has constant expectation values [65]:

$$\frac{d}{dt}\langle E\rangle = \frac{d}{dt}\int_{-\infty}^{\infty} \psi^* E\psi\ dx = 0. \tag{4.12}$$

It is well-known that such quadratic invariants are not unique. Although an elegant general solution is known, say, for the parametric oscillator, it involves an integration of nonlinear Ermakov's equation [132]. Here the simplest energy operators are constructed for several integrable models of the damped and modified quantum oscillators. Then an extension of the familiar Lewis–Riesenfeld quadratic invariant is given to the most general case of the variable non-self-adjoint quadratic Hamiltonian (see also [125], [218], [219], we do not use canonical transformations and deal only with real-valued solutions of the corresponding generalized Ermakov system), which seems to be missing in the available literature and may be considered as the main result of this paper. (An attempt to collect relevant references is made[1].) Group-theoretical aspects will be discussed elsewhere, we only provide the factorization of the general quadratic invariant (see also [198]).

In general the average $\langle E \rangle$ is not positive. A complete dynamics of the expectation values of some energy-related positive operators is found instead for each model, which is a somewhat interesting result on its own. In addition to other works [15], [65], [62], [94], [132], [140], [139], [218], [219] these advances allow us to discuss uniqueness of the corresponding Cauchy initial value problem for the special models and for the general quadratic Hamiltonian under consideration as a modest contribution to this well-developed area of quantum mechanics and partial differential equations. Further relations of the quadratic invariants with the solution of the initial value problem are discussed in the forthcoming paper [198].

The paper is organized as follows. In Section 2 we review several exactly solvable models of the damped and generalized oscillators in quantum mechanics. Some of these "exotic" oscillators with variable quadratic Hamiltonians appear to be missing, and/or are just recently introduced, in the available literature. The corresponding Green functions are found in terms of elementary functions. The dynamical invariants and quadratic energy-related operators are discussed in Sections 3 and 4. The last section is concerned with an application to the Cauchy initial value problems. The classical equations of motion for the expectation values of the position operator for the quantum oscillators under consideration are derived in Appendix A. The Heisenberg uncertainty relation and linear dynamic invariants are revisited, respectively, in Appendices B and C. Solutions of a required differential equation are given in Appendix D to make our presentation is as self-contained as possible.

4.2 Some Integrable Quadratic Hamiltonians

Quantum systems with the Hamiltonians (4.11) are called the generalized harmonic oscillators [15], [62], [94], [125], [218], [219]. In this paper we concentrate, among others, on the following variable Hamiltonians: the Caldirola-Kanai Hamiltonian of the quantum damped oscillator

[1] A complete bibliography on classical and quantum generalized harmonic oscillators, their invariants, group-theoretical methods and applications is very extensive. Only case of the damped oscillators in [60] includes about 600 references!

[22], [60], [107], [209] and some of its natural modifications, a modified oscillator introduced by Meiler, Cordero-Soto and Suslov [143], [52], the quantum damped oscillator of Chruściński and Jurkowski [46] in the coordinate and momentum representations and a quantum-modified parametric oscillator which is believed to be new. The Green functions are derived in a united way.

The Caldirola-Kanai Hamiltonian

A model of the quantum damped oscillator with a variable Hamiltonian of the form

$$H = \frac{\omega_0}{2}\left(e^{-2\lambda t}\ p^2 + e^{2\lambda t}\ x^2\right) \tag{4.13}$$

is called the Caldirola-Kanai model [12], [22], [60], [107], [209]. Nowadays it is a standard way of adding friction to the quantum harmonic oscillator. The Green function is given by

$$G(x,y,t) = \sqrt{\frac{\omega e^{\lambda t}}{2\pi i \omega_0 \sin \omega t}}\ e^{i\left(\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2\right)}, \quad \omega = \sqrt{\omega_0^2 - \lambda^2} > 0, \tag{4.14}$$

where

$$\alpha(t) = \frac{\omega \cos \omega t - \lambda \sin \omega t}{2\omega_0 \sin \omega t} e^{2\lambda t}, \tag{4.15}$$

$$\beta(t) = -\frac{\omega}{\omega_0 \sin \omega t} e^{\lambda t}, \tag{4.16}$$

$$\gamma(t) = \frac{\omega \cos \omega t + \lambda \sin \omega t}{2\omega_0 \sin \omega t}. \tag{4.17}$$

This popular model had been studied in detail by many authors from different viewpoints; see, for example, [3], [20], [25], [26], [35], [40], [41], [63], [108], [110], [111], [114], [116], [123], [156], [159], [163], [181], [199], [200], [208], [220] and references therein, a detailed bibliography can be found in [60], [209].

A Modified Caldirola-Kanai Hamiltonian

In this paper, we would like to consider another version of the quantum damped oscillator with variable Hamiltonian of the form

$$H = \frac{\omega_0}{2}\left(e^{-2\lambda t}\ p^2 + e^{2\lambda t}\ x^2\right) - \lambda\left(px + xp\right). \tag{4.18}$$

The Green functions in (4.14) has

$$\begin{aligned}\alpha(t) &= \frac{\omega\cos\omega t+\lambda\sin\omega t}{2\omega_0\sin\omega t}e^{2\lambda t}, && (4.19)\\ \beta(t) &= -\frac{\omega}{\omega_0\sin\omega t}e^{\lambda t}, && (4.20)\\ \gamma(t) &= \frac{\omega\cos\omega t-\lambda\sin\omega t}{2\omega_0\sin\omega t}. && (4.21)\end{aligned}$$

This can be derived directly from equations (4.2)–(4.8) following Refs. [49] and [50].

The Ehrenfest theorem for both Caldirola-Kanai models has the same form

$$\frac{d^2}{dt^2}\langle x\rangle+2\lambda\frac{d}{dt}\langle x\rangle+\omega_0^2\langle x\rangle=0, \tag{4.22}$$

which coincides with the classical equation of motion for a damped oscillator [13], [120]. Details are provided in Appendix A.

The United Model

The following non-self-adjoint Hamiltonian:

$$H=\frac{\omega_0}{2}\left(e^{-2\lambda t}\,p^2+e^{2\lambda t}\,x^2\right)-\mu xp \tag{4.23}$$

coincides with the original Caldirola-Kanai model when $\mu=0$ and the Hamiltonian is self-adjoint. Another special case $\lambda=0$ corresponds to the quantum damped oscillator discussed in [50] as an example of a simple quantum system with the non-self-adjoint Hamiltonian. (This is an alternative way to introduce dissipation of energy to the quantum harmonic oscillator.) Combining both cases we refer to (4.23) as the united Hamiltonian.

The Green function is given by

$$G(x,y,t)=\sqrt{\frac{\omega e^{(\lambda-\mu)t}}{2\pi i\omega_0\sin\omega t}}\;e^{i\left(\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2\right)}, \tag{4.24}$$

where

$$\begin{aligned}\alpha(t) &= \frac{\omega\cos\omega t+(\mu-\lambda)\sin\omega t}{2\omega_0\sin\omega t}e^{2\lambda t}, && (4.25)\\ \beta(t) &= -\frac{\omega}{\omega_0\sin\omega t}e^{\lambda t}, && (4.26)\\ \gamma(t) &= \frac{\omega\cos\omega t+(\lambda-\mu)\sin\omega t}{2\omega_0\sin\omega t} && (4.27)\end{aligned}$$

with $\omega=\sqrt{\omega_0^2-(\lambda-\mu)^2}>0.$

In this case the Ehrenfest theorem takes the form:

$$\frac{d^2}{dt^2}\langle x\rangle+\ 2(\lambda+\mu)\frac{d}{dt}\langle x\rangle+\left(\omega_0^2+4\lambda\mu\right)\langle x\rangle=0. \tag{4.28}$$

It is derived in Appendix A and the Heisenberg uncertainty relation is discussed in Appendix B.

A Modified Oscillator

The one-dimensional Hamiltonian of a modified oscillator introduced by Meiler, Cordero-Soto and Suslov [143], [52] has the form

$$\begin{aligned} H &= (\cos t\ p+\sin t\ x)^2 \qquad (4.29)\\ &= \cos^2 t\ p^2+\sin^2 t\ x^2+\sin t\cos t\ \ (px+xp)\\ &= \frac{1}{2}\left(p^2+x^2\right)+\frac{1}{2}\cos 2t\ \left(p^2-x^2\right)+\frac{1}{2}\sin 2t\ \ (px+xp)\,. \end{aligned}$$

(A physical interpretation of this Hamiltonian from the viewpoint of quantum dynamical invariants will be discussed in Section 4.) The Green function is given in terms of trigonometric and hyperbolic functions as follows

$$\begin{aligned} G(x,y,t) &= \frac{1}{\sqrt{2\pi i\,(\cos t\sinh t+\sin t\cosh t)}} \qquad (4.30)\\ &\quad\times\exp\left(\frac{\left(x^2-y^2\right)\sin t\sinh t+2xy-\left(x^2+y^2\right)\cos t\cosh t}{2i\,(\cos t\sinh t+\sin t\cosh t)}\right). \end{aligned}$$

More details can be found in [143], [52]. The corresponding Ehrenfest theorem, namely,

$$\frac{d^2}{dt^2}\langle x\rangle+\ 2\tan t\frac{d}{dt}\langle x\rangle-2\langle x\rangle=0, \tag{4.31}$$

is derived in Appendix A.

The Modified Damped Oscillator

The time-dependent Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}=H(t)\,\psi \tag{4.32}$$

with the variable quadratic Hamiltonian of the form

$$H=\frac{p^{2}}{2m\cosh^{2}(\lambda t)}+\frac{m\omega_{0}^{2}}{2}\cosh^{2}(\lambda t)\ x^{2},\quad p=\frac{\hbar}{i}\frac{\partial}{\partial x} \tag{4.33}$$

has been recently considered by Chruściński and Jurkowski [46] as a model of the quantum damped oscillator; see also [151].

In this case the characteristic equation (4.6) takes the form

$$\mu''+2\lambda\tanh(\lambda t)\,\mu'+\omega_{0}^{2}\mu=0. \tag{4.34}$$

The particular solution is given by

$$\mu(t)=\frac{\hbar}{m\omega}\frac{\sin(\omega t)}{\cosh(\lambda t)},\qquad \omega=\sqrt{\omega_{0}^{2}-\lambda^{2}}>0 \tag{4.35}$$

and the corresponding propagator can be presented as follows

$$G(x,y,t)=\sqrt{\frac{m\omega\cosh(\lambda t)}{2\pi i\hbar\sin(\omega t)}}\,e^{i\left(\alpha(t)x^{2}+\beta(t)xy+\gamma(t)y^{2}\right)}, \tag{4.36}$$

where

$$\alpha(t)=\frac{m\cosh(\lambda t)}{2\hbar\sin(\omega t)}\left(\omega\cos(\omega t)\cosh(\lambda t)-\lambda\sin(\omega t)\sinh(\lambda t)\right), \tag{4.37}$$

$$\beta(t)=-\frac{m\omega\cosh(\lambda t)}{2\hbar\sin(\omega t)}, \tag{4.38}$$

$$\gamma(t)=\frac{m\omega\cos(\omega t)}{2\hbar\sin(\omega t)}. \tag{4.39}$$

(We somewhat simplify the original propagator found in [46]; see also [115].) This Green function can be independently derived from our equations (4.3)–(4.5) with the help of the following elementary antiderivative:

$$\begin{aligned}&\left(\frac{\lambda\cos(\omega t+\delta)\sinh(\lambda t)+\omega\sin(\omega t+\delta)\cosh(\lambda t)}{\omega\cos(\omega t+\delta)\cosh(\lambda t)-\lambda\sin(\omega t+\delta)\sinh(\lambda t)}\right)' \\ &=\frac{\omega\omega_{0}^{2}\cosh^{2}(\lambda t)}{\left(\omega\cos(\omega t+\delta)\cosh(\lambda t)-\lambda\sin(\omega t+\delta)\sinh(\lambda t)\right)^{2}}.\end{aligned} \tag{4.40}$$

Further details are left to the reader.

Special cases are as follows: when $\lambda=0$, one recovers the standard propagator for the linear harmonic oscillator [81], and $\omega_0=0$ gives a pure damping case [115]:

$$G(x,y,t)=\sqrt{\frac{m\lambda}{2\pi i\hbar\tanh(\omega t)}}\exp\left(\frac{im\lambda\,(x-y)^{2}}{2\hbar\tanh(\omega t)}\right). \tag{4.41}$$

In the limit $\lambda \to 0$ formula (4.41) reproduces the propagator for a free particle [81].

The Ehrenfest theorem for the quantum damped oscillator of Chruściński and Jurkowski coincides with our characteristic equation (4.34); see Appendix A for more details.

It is worth adding that in the momentum representation, when $p \leftrightarrow x$, a rescaled Hamiltonian (4.33) ($\hbar = m\omega_0 = 1$) takes the form

$$H = \frac{\omega_0}{2}\left(\cosh^2(\lambda t)\ p^2 + \frac{x^2}{\cosh^2(\lambda t)}\right). \tag{4.42}$$

The corresponding characteristic equation

$$\mu'' - 2\lambda\tanh(\lambda t)\,\mu' + \omega_0^2\mu = 0 \tag{4.43}$$

has a required elementary solution

$$\mu = \frac{1}{\omega_0}\left(\lambda\cos(\omega t)\sinh(\lambda t) + \omega\sin(\omega t)\cosh(\lambda t)\right) \tag{4.44}$$

with $\mu'(0) = 2a(0) = \omega_0$ and

$$\mu \to \frac{1}{2\omega_0}e^{\lambda t}\left(\lambda\cos(\omega t) + \omega\sin(\omega t)\right) \tag{4.45}$$

as $t \to \infty$. The Green function is given by formula (4.2) with the following coefficients:

$$\alpha(t) = \frac{\omega_0\cos(\omega t)}{2\cosh(\lambda t)\left(\lambda\cos(\omega t)\sinh(\lambda t) + \omega\sin(\omega t)\cosh(\lambda t)\right)}, \tag{4.46}$$

$$\beta(t) = -\frac{\omega_0}{\lambda\cos(\omega t)\sinh(\lambda t) + \omega\sin(\omega t)\cosh(\lambda t)}, \tag{4.47}$$

$$\gamma(t) = \frac{\omega_0\left(\omega\cos(\omega t)\cosh(\lambda t) - \lambda\sin(\omega t)\sinh(\lambda t)\right)}{2\omega\left(\lambda\cos(\omega t)\sinh(\lambda t) + \omega\sin(\omega t)\cosh(\lambda t)\right)}. \tag{4.48}$$

The details are left to the reader.

A Modified Parametric Oscillator

In a similar fashion we consider the following Hamiltonian:

$$H = \frac{\omega}{2}\left(\tanh^2(\lambda t + \delta)\ p^2 + \coth^2(\lambda t + \delta)\ x^2\right) + \frac{\lambda}{\sinh(2\lambda t + 2\delta)}(px + xp) \qquad (\delta \neq 0), \tag{4.49}$$

which seems to be missing in the available literature. The corresponding characteristic equation:

$$\mu'' - \frac{4\lambda}{\sinh(2\lambda t + 2\delta)}\mu' + \left(\omega^2 + \frac{2\lambda^2}{\sinh^2(\lambda t + \delta)}\right)\mu = 0 \tag{4.50}$$

has an elementary solution of the form:

$$\mu = \sin(\omega t)\,\frac{\tanh(\lambda t + \delta)}{\coth\delta}. \tag{4.51}$$

In the limit $t \to \infty$, $\mu \to \sin(\omega t)\tanh\delta$.

The Green function can be found as follows

$$G(x,y,t) = \sqrt{\frac{\coth\delta}{2\pi i \sin(\omega t)\tanh(\lambda t + \delta)}}\; e^{i\left(\alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2\right)}, \tag{4.52}$$

where

$$\alpha(t) = \frac{1}{2}\cot(\omega t)\coth^2(\lambda t + \delta), \tag{4.53}$$

$$\beta(t) = -\frac{\coth\delta}{\sin(\omega t)}\coth(\lambda t + \delta), \tag{4.54}$$

$$\gamma(t) = \frac{1}{2}\cot(\omega t)\coth^2\delta. \tag{4.55}$$

The Ehrenfest theorem coincides with the characteristic equation (4.50). One should interchange $a \leftrightarrow b$ and $d \to -d$ in the momentum representation [52]. The corresponding solutions can be found with the help of the substitution $\delta \to \delta + i\pi/2$. The trigonometric cases, when $\lambda \to i\lambda$, $\delta \to i\delta$ and $\omega \to -\omega$, are left to the reader.

Parametric Oscillators

In conclusion a somewhat related quantum parametric oscillator:

$$H = \frac{1}{2}\left(p^2 + \left(\omega^2 + \frac{2\lambda^2}{\cosh^2(\lambda t)}\right)x^2\right), \tag{4.56}$$

when

$$\mu'' + \left(\omega^2 + \frac{2\lambda^2}{\cosh^2(\lambda t)}\right)\mu = 0 \tag{4.57}$$

and

$$\mu = \frac{\lambda\cos(\omega t)\sinh(\lambda t) + \omega\sin(\omega t)\cosh(\lambda t)}{\left(\omega^2 + \lambda^2\right)\cosh(\lambda t)}, \tag{4.58}$$

has the Green function (4.2) with the following coefficients:

$$\alpha(t) = \frac{\left(\omega^2+\lambda^2\cosh^{-2}(\lambda t)\right)\cos(\omega t)-\lambda\omega\tanh(\lambda t)\sin(\omega t)}{2(\omega\sin(\omega t)+\lambda\tanh(\lambda t)\cos(\omega t))}, \tag{4.59}$$

$$\beta(t) = -\frac{\omega^2+\lambda^2}{\omega\sin(\omega t)+\lambda\tanh(\lambda t)\cos(\omega t)}, \tag{4.60}$$

$$\gamma(t) = \frac{\left(\omega^2+\lambda^2\right)(\omega\cos(\omega t)-\lambda\tanh(\lambda t)\sin(\omega t))}{2\omega(\omega\sin(\omega t)+\lambda\tanh(\lambda t)\cos(\omega t))}. \tag{4.61}$$

The Green function for the parametric oscillator in general:

$$H = \frac{1}{2}\left(p^2+\omega^2(t)\,x^2\right) \tag{4.62}$$

can be found, for example, in Ref. [122]. (The time-dependent quantum oscillator was thoroughly examined by Husimi [100], [101] and later many authors had treated different aspects of the problem; see [65], [140], [138], [139], [164], [165], [168], [169], [185] and [189]; a detailed bibliography is given in Ref. [23].)

4.3 Expectation Values of Quadratic Operators

We start from a convenient differentiation formula.

Lemma 2 *Let*

$$H = a(t)\,p^2+b(t)\,x^2+d(t)\,(px+xp), \tag{4.63}$$

$$O = A(t)\,p^2+B(t)\,x^2+C(t)\,(px+xp) \tag{4.64}$$

and

$$\langle O\rangle = \langle\psi, O\psi\rangle = \int_{-\infty}^{\infty}\psi^* O\psi\,dx, \qquad i\frac{\partial\psi}{\partial t} = H\psi \tag{4.65}$$

(we use the star for complex conjugate). Then

$$\begin{aligned}\frac{d}{dt}\langle O\rangle &= \left(\frac{dA}{dt}+4(aC-dA)\right)\langle p^2\rangle \\ &\quad+\left(\frac{dB}{dt}+4(dB-bC)\right)\langle x^2\rangle \\ &\quad+\left(\frac{dC}{dt}+2(aB-bA)\right)\langle px+xp\rangle.\end{aligned} \tag{4.66}$$

Proof. The time derivative of the expectation value can be written as [121], [145], [183]:

$$\frac{d}{dt}\langle O\rangle = \left\langle \frac{\partial O}{\partial t}\right\rangle + \frac{1}{i}\langle [O,H]\rangle, \tag{4.67}$$

where $[O,H] = OH - HO$ (we freely interchange differentiation and integration throughout the paper, it can be justified for certain classes of solutions [133], [160], [166], [211]). One should make use of the standard commutator properties, including familiar identities

$$\begin{aligned} &\left[x^2, p^2\right] = 2i(px+xp), \qquad \left[x, p^2\right] = 2ip, \qquad \left[x^2, p\right] = 2ix, \\ &\left[px+xp, p^2\right] = 4ip^2, \qquad \left[x^2, px+xp\right] = 4ix^2, \end{aligned} \tag{4.68}$$

in order to complete the proof. ■

Quantum systems with the self-adjoint Hamiltonians (4.63) are called the generalized harmonic oscillators [15], [62], [94], [125], [218], [219]. At the same time one has to deal with non-self-adjoint Hamiltonians in the theory of dissipative quantum systems (see, for example, [50], [60], [116], [204], [209] and references therein) or when using separation of variables in an accelerating frame of reference for a charged particle moving in an uniform time-dependent magnetic field [49]. An extension to the case of non-self-adjoint Hamiltonians is as follows.

Lemma 3 *If*

$$H = a(t)p^2 + b(t)x^2 + c(t)px + d(t)xp, \tag{4.69}$$

$$O = A(t)p^2 + B(t)x^2 + C(t)px + D(t)xp, \tag{4.70}$$

then

$$\begin{aligned} \frac{d}{dt}\langle O\rangle \;=\; & \left(\frac{dA}{dt} + 2a(C+D) - (3c+d)A\right)\left\langle p^2\right\rangle \\ & + \left(\frac{dB}{dt} - 2b(C+D) + (c+3d)B\right)\left\langle x^2\right\rangle \\ & + \left(\frac{dC}{dt} + 2(aB-bA) - (c-d)C\right)\langle px\rangle \\ & + \left(\frac{dD}{dt} + 2(aB-bA) - (c-d)D\right)\langle xp\rangle . \end{aligned} \tag{4.71}$$

Proof. One should use

$$\frac{d}{dt}\langle O\rangle = \left\langle \frac{\partial O}{\partial t}\right\rangle + \frac{1}{i}\left\langle OH - H^{\dagger}O\right\rangle, \tag{4.72}$$

where $H^{\dagger}$ is the Hermitian adjoint of the Hamiltonian operator H. Our formula is a simple extension of the well-known expression [121], [145], [183] to the case of a non-self-adjoint Hamiltonian [50]. Standard commutator evaluations complete the proof. ■

Polynomial operators of the higher orders in x and p can be differentiated in a similar fashion. An analog of the product rule is given in [198]. The details are left to the reader.

4.4 Energy Operators and Quadratic Invariants

In the case of the time-independent Hamiltonian, one gets

$$\frac{d}{dt}\langle H\rangle = 0 \tag{4.73}$$

by (4.67). The law of conservation of energy states that

$$E = \langle H\rangle = constant. \tag{4.74}$$

In general one has to construct quantum integrals of motion, or dynamical invariants, that are different from the variable Hamiltonian (see, for example, [132], [218], [219]; linear case is dealt with in [63], [65], [140], [139] and Appendix C).

Energy Operators

A familiar definition is in order (see, for example, [65], [140]).

Definition 1 *We call the quadratic operator (4.64) an energy operator E, or a quadratic (dynamical) invariant, if*

$$\frac{d}{dt}\langle E\rangle = 0 \tag{4.75}$$

for the corresponding variable Hamiltonian (4.63).

By Lemma 2 the coefficients of an energy operator,

$$E = A(t)\,p^2 + B(t)\,x^2 + C(t)\,(px + xp)\,, \tag{4.76}$$

must satisfy the system of ordinary differential equations:

$$\frac{dA}{dt} + 4\left(a\left(t\right)C - d\left(t\right)A\right) = 0, \tag{4.77}$$

$$\frac{dB}{dt} + 4\left(d\left(t\right)B - b\left(t\right)C\right) = 0, \tag{4.78}$$

$$\frac{dC}{dt} + 2\left(a\left(t\right)B - b\left(t\right)A\right) = 0. \tag{4.79}$$

In general a unique solution of this system with respect to arbitrary initial conditions $A_0 = A\left(0\right)$, $B_0 = B\left(0\right)$, $C_0 = C\left(0\right)$ [97] determines a three-parameter family of the quadratic invariants (4.76). Special cases, when solutions can be found explicitly, are of the most practical importance.

In this section we find the simplest energy operators for all quadratic models under consideration as follows:

$$E = \frac{\omega_0}{2}\left(e^{-2\lambda t}\ p^2 + e^{2\lambda t}\ x^2\right) + \frac{\lambda}{2}\left(px + xp\right), \tag{4.80}$$

$$E = \frac{\omega_0}{2}\left(e^{-2\lambda t}\ p^2 + e^{2\lambda t}\ x^2\right) - \frac{\lambda}{2}\left(px + xp\right), \tag{4.81}$$

$$E = \frac{1}{2}\cos 2t\ \left(p^2 - x^2\right) + \frac{1}{2}\sin 2t\ \left(px + px\right), \tag{4.82}$$

$$E = \tanh^2\left(\lambda t + \delta\right)\ p^2 + \coth^2\left(\lambda t + \delta\right)\ x^2 \tag{4.83}$$

for the Caldirola-Kanai Hamiltonian (4.13) [199], the modified Caldirola-Kanai Hamiltonian (4.18), the modified oscillator of Meiler, Cordero-Soto and Suslov (4.29) and for the modified parametric oscillator (4.49), respectively. Their coefficients solve the corresponding systems (4.77)–(4.79) for special initial data.

An energy operator for the united model (4.23) is given by

$$E = \frac{\omega_0}{2}e^{\mu t}\left(e^{-2\lambda t}\ p^2 + e^{2\lambda t}\ x^2\right) + \frac{1}{2}\left(\lambda - \mu\right)e^{\mu t}\left(px + xp\right). \tag{4.84}$$

One should use Lemma 3; verification is left to the reader. Finally an energy operator for the quantum damped oscillator of Chruściński and Jurkowski with a rescaled Hamiltonian (4.179) is given by expression (4.180). A general case of the variable quadratic Hamiltonian is discussed in Theorem 1.

Classical Hamiltonian of the generalized harmonic oscillator can be transformed into the Hamiltonian of a parametric oscillator [15], [94], [161], [219]. All quadratic invariants of the quantum parametric oscillator (4.62) can be found as follows [129], [130], [131], [132]. The corresponding system,

$$A' + 2C = 0, \tag{4.85}$$

$$B' - 2\omega^2(t)\, C = 0, \tag{4.86}$$

$$C' + B - \omega^2(t)\, A = 0, \tag{4.87}$$

is integrated by the substitution $A = \kappa^2$. Then $C = -\kappa\kappa'$, $B = \kappa\kappa'' + (\kappa')^2 + \omega^2(t)\,\kappa^2$ and equation (4.86) becomes

$$\begin{aligned} \left(\kappa\kappa'' + \left(\kappa'\right)^2 + \omega^2(t)\,\kappa^2\right)' + 2\omega^2(t)\,\kappa\kappa' &= 0, \\ \kappa\left(\kappa'' + \omega^2(t)\,\kappa\right)' + 3\kappa'\left(\kappa'' + \omega^2(t)\,\kappa\right) &= 0 \end{aligned}$$

or with an integrating factor:

$$\frac{d}{dt}\left(\kappa^3\left(\kappa'' + \omega^2(t)\,\kappa\right)\right) = 0 \tag{4.88}$$

(see [132] and [127]). Thus

$$\kappa'' + \omega^2(t)\,\kappa = \frac{c_0}{\kappa^3} \qquad (c_0 = 0, 1) \tag{4.89}$$

and a general solution of the system (4.85)–(4.87) is given by

$$A = \kappa^2, \quad B = \left(\kappa'\right)^2 + \frac{c_0}{\kappa^2}, \quad C = -\kappa\kappa' \tag{4.90}$$

in terms of solutions of the nonlinear equation (4.89), which is called Ermakov's equation, when $c_0 = 1$ [72] (see also, [126], [131], [167] and [185]). Thus the quadratic integrals of motion can be presented in the form [132]:

$$E = \left(\kappa p - \kappa' x\right)^2 + \frac{c_0}{\kappa^2} x^2 \tag{4.91}$$

for any given solution of the Ermakov equation (4.89). This quantum invariant is an analog of the Ermakov–Lewis integral of motion for the classical parametric oscillator [72], [129], [130], [131], [201].

In general if two linearly independent solutions of the classical parametric oscillator equation are available:

$$u'' + \omega^2(t)\, u = 0, \qquad v'' + \omega^2(t)\, v = 0, \tag{4.92}$$

then solutions of the nonlinear Ermakov equation:

$$\kappa'' + \omega^2(t)\, \kappa = \frac{1}{\kappa^3} \tag{4.93}$$

are given by

$$\kappa = \left(Au^2 + 2Buv + Cv^2\right)^{1/2} \tag{4.94}$$

(so-called Pinney's solution [167], [70], [126], [131], [161]), where the constants A, B and C are related according to $AC - B^2 = 1/W^2$ with W being the constant Wronskian of the two linearly independent solutions.

For example, in the case of the simple harmonic oscillator with $\omega(t) = 1$, there are two elementary solutions:

$$\kappa = 1 \quad (c_0 = 1), \qquad \kappa = \cos t \quad (c_0 = 0) \tag{4.95}$$

and the energy operators are given by

$$H = \frac{1}{2}\left(p^2 + x^2\right), \tag{4.96}$$

$$E = (\cos t\ p + \sin t\ x)^2 . \tag{4.97}$$

It provides a somewhat better understanding of the nature of the Hamiltonian discussed by Meiler, Cordero-Soto and Suslov [143] — this operator plays a role of the simplest time-dependent quadratic integral of motion for the linear harmonic oscillator.

In a similar fashion the dynamical invariants of the parametric oscillator (4.56) are given by the expression (4.91) with $c_0 \neq 0$. In the Pinney solution (4.94) one can choose

$$u = \frac{\omega \cos(\omega t)\cosh(\lambda t) - \lambda \sin(\omega t)\sinh(\lambda t)}{\cosh(\lambda t)}, \tag{4.98}$$

$$v = \frac{\omega \sin(\omega t)\cosh(\lambda t) + \lambda \cos(\omega t)\sinh(\lambda t)}{\cosh(\lambda t)} \tag{4.99}$$

as two linearly independent solutions of the classical equation of motion (4.57) with $W(u,v) = \omega\left(\omega^2+\lambda^2\right)$. If $A=C$ and $B=0$, then

$$\kappa = \left(\omega^2 + \lambda^2 \tanh^2(\lambda t)\right)^{1/2} \tag{4.100}$$

is a particular solution of the corresponding Ermakov equation:

$$\kappa'' + \left(\omega^2 + \frac{2\lambda^2}{\cosh^2(\lambda t)}\right)\kappa = \frac{\omega^2\left(\lambda^2+\omega^2\right)^2}{\kappa^3}. \tag{4.101}$$

The simplest positive energy integral for our parametric oscillator (4.56) is given by

$$\begin{aligned} E = {} & \left(\omega^2+\lambda^2\tanh^2(\lambda t)\right) p^2 + \lambda^3 \frac{\sinh(\lambda t)}{\cosh^3(\lambda t)} \, (px+xp) \qquad (4.102) \\ & + \frac{\lambda^6 \sinh^2(\lambda t) + \omega^2\left(\lambda^2+\omega^2\right)^2 \cosh^6(\lambda t)}{\cosh^6(\lambda t)\left(\omega^2+\lambda^2\tanh^2(\lambda t)\right)} \, x^2. \end{aligned}$$

Another possibility is to take a general solution of (4.57) with $c_0 = 0$.

An Extension to General Quadratic Hamiltonians

We consider the following generalization of the Lewis–Riesenfeld invariant (4.91) (see also [125], [219]).

Theorem 4 *The dynamical invariants for the general quadratic Hamiltonian (4.69) are given by*

$$E = \frac{1}{\mu_1}\left(\kappa\, p - \frac{1}{2a}\frac{d\kappa}{dt}\, x\right)^2 + \frac{C_0}{\mu_2 \kappa^2}\, x^2, \tag{4.103}$$

where C_0 is a constant,

$$\mu_1 = \exp\left(-\int_0^t (3c+d)\, ds\right), \quad \mu_2 = \exp\left(\int_0^t (c+3d)\, ds\right), \tag{4.104}$$

and κ satisfies the auxiliary nonlinear equation:

$$k\frac{d}{dt}\left(k\frac{d\kappa}{dt}\right)+4abk^2\kappa=\frac{C_0}{\kappa^3}, \tag{4.105}$$

where

$$k=\frac{1}{2a}\exp\left(2\int_0^t(c+d)\ ds\right). \tag{4.106}$$

(For the self-adjoint Hamiltonians $c=d$.)

The case, $a=1/2$, $b=\omega^2(t)/2$ and $c=d=0$, corresponds to the original invariant (4.91).

Proof. By Lemma 3 in order to find quadratic invariants of the form

$$E=Ap^2+Bx^2+Cpx+Dxp \tag{4.107}$$

we have to solve the following system of ordinary differential equations:

$$\begin{aligned}
\frac{dA}{dt}+2a(C+D)-(3c+d)A &= 0, && (4.108)\\
\frac{dB}{dt}-2b(C+D)+(c+3d)B &= 0, && (4.109)\\
\frac{dC}{dt}+2(aB-bA)-(c-d)C &= 0, && (4.110)\\
\frac{dD}{dt}+2(aB-bA)-(c-d)D &= 0, && (4.111)
\end{aligned}$$

say, for arbitrary analytic coefficients $a(t)$, $b(t)$, $c(t)$ and $d(t)$. The substitution $C=C_1+D_1$, $D=C_1-D_1$ allows one to transform the last two equations:

$$\frac{dC_1}{dt}+2(aB-bA)-(c-d)C_1=0, \tag{4.112}$$

$$\frac{dD_1}{dt}=(c-d)D_1,\quad D_1=\text{constant}\ \exp\left(\int_0^t(c-d)\ ds\right). \tag{4.113}$$

Then

$$Cpx+Dxp=C_1(px+xp)+D_1(px-xp)$$

and, in view of the canonical commutation relation, the coefficient D_1 can be eliminated from the consideration as belonging to the linear invariants (see appendix C).

Introducing integrating factors into (4.108), (4.109) and (4.112), we get

$$\frac{d}{dt}(\mu_1 A) + 4a\mu_1 C_1 = 0, \qquad \frac{\mu_1'}{\mu_1} = -3c - d, \tag{4.114}$$

$$\frac{d}{dt}(\mu_2 B) - 4b\mu_2 C_1 = 0, \qquad \frac{\mu_2'}{\mu_2} = c + 3d, \tag{4.115}$$

$$\frac{d}{dt}(\mu_3 C_1) + 2\mu_3 (aB - bA) = 0, \qquad \frac{\mu_3'}{\mu_3} = -c + d \tag{4.116}$$

with $\mu_3^2 = \mu_1\mu_2$. After the substitution

$$\widetilde{A} = \mu_1 A, \qquad \widetilde{B} = \mu_2 B, \qquad \widetilde{C} = \mu_3 C_1, \tag{4.117}$$

the system takes the form

$$\frac{d\widetilde{A}}{dt} + 4a\sqrt{\frac{\mu_1}{\mu_2}}\,\widetilde{C} = 0, \tag{4.118}$$

$$\frac{d\widetilde{B}}{dt} - 4b\sqrt{\frac{\mu_2}{\mu_1}}\,\widetilde{C} = 0, \tag{4.119}$$

$$\frac{d\widetilde{C}}{dt} + 2\left(a\sqrt{\frac{\mu_1}{\mu_2}}\,\widetilde{B} - b\sqrt{\frac{\mu_2}{\mu_1}}\,\widetilde{A}\right) = 0. \tag{4.120}$$

Introducing a “proper time”:

$$\tau = \int_0^t 2a\sqrt{\frac{\mu_1}{\mu_2}}\,ds, \tag{4.121}$$

we finally obtain:

$$\frac{d\widetilde{A}}{d\tau} + 2\widetilde{C} = 0, \tag{4.122}$$

$$\frac{d\widetilde{B}}{d\tau} - 2\omega^2(\tau)\widetilde{C} = 0, \tag{4.123}$$

$$\frac{d\widetilde{C}}{d\tau} + \widetilde{B} - \omega^2(\tau)\widetilde{A} = 0, \quad \omega^2(\tau) = \frac{b\mu_2}{a\mu_1}, \tag{4.124}$$

which is identical to the original Lewis–Riesenfeld system (4.85)–(4.87) (positivity of ω^2 is not required). The solution is given by

$$\widetilde{A} = \kappa^2, \quad \widetilde{B} = \left(\frac{d\kappa}{d\tau}\right)^2 + \frac{C_0}{\kappa^2}, \quad \widetilde{C} = -\kappa\frac{d\kappa}{d\tau}, \tag{4.125}$$

where κ satisfies the Ermakov equation:

$$\frac{d^2\kappa}{d\tau^2} + \omega^2(\tau)\,\kappa = \frac{C_0}{\kappa^3}, \quad \omega^2(\tau) = \frac{b\mu_2}{a\mu_1}, \tag{4.126}$$

with respect to the new time (4.121). In view of

$$\frac{d}{d\tau} = k\frac{d}{dt}, \qquad k = \frac{1}{2a}\exp\left(2\int_0^t (c+d)\ ds\right), \tag{4.127}$$

the Ermakov equation (4.126) is transformed into our auxiliary equation (4.105). A back substitution results in the dynamical invariant (4.103) when the square is completed. ■

Lemma 5 *The dynamical invariant (4.103) can be represented in more symmetric form*

$$\begin{aligned} E &= \left(\left(\mu\ p - \frac{1}{2a}\left(\frac{d\mu}{dt} - (c+d)\,\mu\right)\ x\right)^2 + \frac{C_0}{\mu^2}\ x^2\right) \\ &\quad \times \exp\left(\int_0^t (c-d)\ ds\right), \end{aligned} \tag{4.128}$$

where C_0 is a constant and μ is a solution of the following auxiliary equation:

$$\mu'' - \frac{a'}{a}\mu' + \left(4ab + \left(\frac{a'}{a} - c - d\right)(c+d) - c' - d'\right)\mu = C_0\frac{(2a)^2}{\mu^3}. \tag{4.129}$$

Proof. Use the substitution

$$\kappa = \mu \exp\left(-\int_0^t (c+d)\ ds\right) \tag{4.130}$$

in (4.103) and (4.105). A somewhat different proof is given in [198]. ■

The corresponding classical invariant is discussed, for example, in Refs. [201] and [219]. (Compare also our expression (4.128) with the one given in the last paper for the self-adjoint case; we give a detailed proof for the non-self-adjoint Hamiltonians and emphasize connection with the Ermakov equation.)

It is worth noting, in conclusion, that, if μ_1 and μ_2 are two linearly independent solutions of the linear equation:

$$\mu'' - \frac{a'}{a}\mu' + \left(4ab + \left(\frac{a'}{a} - c - d\right)(c+d) - c' - d'\right)\mu = 0, \tag{4.131}$$

the general solution of the nonlinear auxiliary equation (4.129) is given by

$$\mu = \left(A\mu_1^2 + 2B\mu_1\mu_2 + C\mu_2^2\right)^{1/2}, \tag{4.132}$$

where the constants A, B and C are related according to

$$AC - B^2 = C_0\frac{(2a)^2}{W^2(\mu_1, \mu_2)} \tag{4.133}$$

with $W(\mu_1,\mu_2) = \mu_1\mu_2' - \mu_1'\mu_2 = constant\ (2a)$ being the Wronskian of the two linearly independent solutions. This is a simple extension of Pinney's solution (4.94); our equations (4.129) and (4.131) form the generalized Ermakov system [70], [161]. Further generalization of the superposition formula (4.132)–(4.133) is discussed in Ref. [198]. (If $C_0 \neq 0$, the substitution $\mu \to C_0^{1/4}\mu$ reduces equation (4.129) to a similar form with $C_0 = 1$.) Special case of the time-dependent damped harmonic oscillator, when $a = e^{-F(t)}/2$, $b = \omega^2(t)\, e^{F(t)}/2$, $F(t) = \int_0^t f(s)\ ds$ and $c = d = 0$, is discussed in [123], [124].

An Example

The simplest energy operators have been already discussed in section 4.1 for all models of quantum oscillators under consideration. In order to demonstrate how the general approach works we discuss the united Hamiltonian (4.23), when $a = (\omega_0/2)\, e^{-2\lambda t}$, $b = (\omega_0/2)\, e^{2\lambda t}$ and $c = 0$, $d = -\mu$. A direct calculation shows that the function

$$\kappa = \sqrt{\frac{\omega_0}{2}} e^{-\lambda t} \tag{4.134}$$

satisfies the following equation

$$\kappa'' + 2\lambda\,\kappa' + \omega_0^2\kappa = \left(\frac{\omega_0\omega}{2}\right)^2 \frac{e^{-4\lambda t}}{\kappa^3}, \quad \omega^2 = \omega_0^2 - (\lambda-\mu)^2 > 0, \tag{4.135}$$

which corresponds to the nonlinear auxiliary equation (4.129) with $C_0 = \omega^2/4$. The quadratic invariant (4.128) simplifies to the previously found expression (4.84). Solution (4.132) can be used for the most general case. Details are left to the reader.

Factorization of the Dynamical Invariant

Following Ref. [50] the energy operator (4.128) can be presented in the standard harmonic oscillator form:

$$E = \frac{\omega(t)}{2}\left(\widehat{a}(t)\,\widehat{a}^{\dagger}(t) + \widehat{a}^{\dagger}(t)\,\widehat{a}(t)\right), \tag{4.136}$$

where

$$\omega(t) = \omega_0 \exp\left(\int_0^t (c-d)\ ds\right), \qquad \omega_0 = 2\sqrt{C_0} > 0, \tag{4.137}$$

$$\widehat{a}(t) = \left(\frac{\sqrt{\omega_0}}{2\mu} - i\frac{\mu' - (c+d)\,\mu}{2a\sqrt{\omega_0}}\right)x + \frac{\mu}{\sqrt{\omega_0}}\frac{\partial}{\partial x}, \tag{4.138}$$

$$\widehat{a}^{\dagger}(t) = \left(\frac{\sqrt{\omega_0}}{2\mu} + i\frac{\mu' - (c+d)\,\mu}{2a\sqrt{\omega_0}}\right)x - \frac{\mu}{\sqrt{\omega_0}}\frac{\partial}{\partial x}, \tag{4.139}$$

and μ is a solution of the nonlinear auxiliary equation (4.129). Here the time-dependent annihilation $\widehat{a}(t)$ and creation $\widehat{a}^{\dagger}(t)$ operators satisfy the usual commutation relation:

$$\widehat{a}(t)\,\widehat{a}^{\dagger}(t) - \widehat{a}^{\dagger}(t)\,\widehat{a}(t) = 1. \tag{4.140}$$

The oscillator-type spectrum and the corresponding time-dependent eigenfunctions of the dynamical invariant E can be obtain now in a standard way by using the Heisenberg–Weyl algebra of the rasing and lowering operators (a "second quantization" [132], the Fock states). Explicit solution of the Cauchy initial value problem in terms of the quadratic invariant eigenfunction expansion is found in Ref. [198]. In addition the n-dimensional oscillator wave functions form a basis of the irreducible unitary representation of the Lie algebra of the noncompact group $SU(1,1)$ corresponding to the discrete positive series $\mathscr{D}_{+}^{j}$ (see [143], [157] and [188]). Our operators (4.138)–(4.139) allow one to extend these group-theoretical properties to the general dynamical invariant (4.136). We shall further elaborate on these connections elsewhere.

4.5 Application to the Cauchy Initial Value Problems

Explicit solution of the initial value problem in terms of eigenfunctions of the general quadratic invariant is given in Ref. [198]. Here we formulate the following uniqueness result.

Lemma 6 *Suppose that the expectation value*

$$\langle H_0 \rangle = \langle \psi, H_0 \psi \rangle \geq 0 \tag{4.141}$$

for a positive quadratic operator

$$H_0 = f(t)\,(\alpha(t)\,p + \beta(t)\,x)^2 + g(t)\,x^2 \qquad (f(t) \geq 0,\ g(t) > 0) \tag{4.142}$$

($\alpha(t)$ and $\beta(t)$ are real-valued functions) vanishes for all $t \in [0,T)$:

$$\langle H_0 \rangle = \langle H_0 \rangle(t) = \langle H_0 \rangle(0) = 0, \tag{4.143}$$

when $\psi(x,0) = 0$ almost everywhere. Then the corresponding Cauchy initial value problem

$$i\frac{\partial \psi}{\partial t} = H\psi, \qquad \psi(x,0) = \varphi(x) \tag{4.144}$$

may have only one solution, when $x\psi(x,t) \in L^2(\mathbb{R})$ (if $H_0 = g(t)\,I$, where $I = id$ is the identity operator, $\psi \in L^2(\mathbb{R})$).

Here it is not assumed that H_0 is the quantum integral of motion when $\frac{d}{dt}\langle H_0\rangle \equiv 0$.

Proof. If there are two solutions:

$$i\frac{\partial \psi_1}{\partial t} = H\psi_1, \qquad i\frac{\partial \psi_2}{\partial t} = H\psi_2$$

with the same initial condition $\psi_1(x,0) = \psi_2(x,0) = \varphi(x)$, then by the superposition principle the function $\psi = \psi_1 - \psi_2$ is also a solution with respect to the zero initial data $\psi(x,0) = \varphi(x) - \varphi(x) = 0$. By the hypothesis of the lemma

$$\langle \psi, H_0\psi\rangle = f(t)\langle(\alpha p + \beta x)\psi, (\alpha p + \beta x)\psi\rangle + g(t)\langle x\psi, x\psi\rangle = 0$$

for all $t \in [0,T)$. Therefore $x\psi(x,t) = x(\psi_1(x,t) - \psi_2(x,t)) = 0$ and $\psi_1(x,t) = \psi_2(x,t)$ almost everywhere for all $t > 0$ by the axiom of the inner product in $L^2(\mathbb{R})$. ■

In order to apply this lemma to the variable Hamiltonians one has to identify the corresponding positive operators H_0 and establish their required uniqueness dynamics properties with respect to the zero initial data. In addition to the simplest available dynamical invariant (4.219), it is worth exploring other (quadratic) possibilities. The authors believe that it is interesting and may be important on its own. For example, our approach gives an opportunity to determine a complete time-evolution of the standard deviations (4.227)–(4.228) for each of the generalized harmonic oscillators under consideration. The details will be discussed elsewhere.

The Caldirola-Kanai Hamiltonian

The required operators are given by

$$H = H_0 = \frac{\omega_0}{2}\left(e^{-2\lambda t}\, p^2 + e^{2\lambda t}\, x^2\right), \tag{4.145}$$

$$L = \frac{\partial H}{\partial t} = \lambda\omega_0\left(-e^{-2\lambda t}\, p^2 + e^{2\lambda t}\, x^2\right), \tag{4.146}$$

$$E = \frac{\omega_0}{2}\left(e^{-2\lambda t}\, p^2 + e^{2\lambda t}\, x^2\right) + \frac{\lambda}{2}(px + xp), \quad \frac{d}{dt}\langle E\rangle = 0. \tag{4.147}$$

By (4.67)

$$\frac{d}{dt}\langle H\rangle = \left\langle \frac{\partial H}{\partial t}\right\rangle = \langle L\rangle. \tag{4.148}$$

Applying formula (4.66) one gets

$$\begin{aligned}\frac{d}{dt}\langle L\rangle \quad &= \quad 2\lambda^2\omega_0\left(e^{-2\lambda t}\,\left\langle p^2\right\rangle + e^{2\lambda t}\,\left\langle x^2\right\rangle\right) \\ &\quad +2\lambda\omega_0^2\langle px + xp\rangle\end{aligned} \tag{4.149}$$

and

$$\frac{d}{dt}\langle L\rangle + 4\omega^2\langle H\rangle = 4\omega_0^2\langle E\rangle_0 \tag{4.150}$$

with the help of (4.145) and (4.147).

In view of (4.148) and (4.150) the dynamics of the Hamiltonian expectation value $\langle H\rangle$ is governed by the following second-order differential equation

$$\frac{d^2}{dt^2}\langle H\rangle + 4\omega^2\langle H\rangle = 4\omega_0^2\langle E\rangle_0 \tag{4.151}$$

with the unique solution given by

$$\langle H\rangle = \frac{\omega^2\langle H\rangle_0 - \omega_0^2\langle E\rangle_0}{\omega^2}\cos(2\omega t) + \frac{1}{2\omega}\left\langle\frac{\partial H}{\partial t}\right\rangle_0 \sin(2\omega t) + \frac{\omega_0^2}{\omega^2}\langle E\rangle_0 . \tag{4.152}$$

The hypotheses of Lemma 6 are satisfied. Our solution allows to determine a complete time-evolution of the expectation values of the operators p^2, x^2 and $px+xp$. Further details are left to the reader.

The Modified Caldirola-Kanai Hamiltonian

The required operators are

$$H = \frac{\omega_0}{2}\left(e^{-2\lambda t}\, p^2 + e^{2\lambda t}\, x^2\right) - \lambda\,(px+xp), \tag{4.153}$$

$$L = \frac{\partial H}{\partial t} = \lambda\,\omega_0\left(-e^{-2\lambda t}\, p^2 + e^{2\lambda t}\, x^2\right) = \frac{\partial H_0}{\partial t}, \tag{4.154}$$

$$E = \frac{\omega_0}{2}\left(e^{-2\lambda t}\, p^2 + e^{2\lambda t}\, x^2\right) - \frac{\lambda}{2}(px+xp). \tag{4.155}$$

We consider the expectation value $\langle H_0\rangle$ of the positive operator

$$H_0 = \frac{\omega_0}{2}\left(e^{-2\lambda t}\, p^2 + e^{2\lambda t}\, x^2\right). \tag{4.156}$$

In this case $H = 2E - H_0$, and

$$\frac{d}{dt}\langle H\rangle = \left\langle\frac{\partial H}{\partial t}\right\rangle = \langle L\rangle = -\frac{d}{dt}\langle H_0\rangle, \tag{4.157}$$

$$\frac{d}{dt}\langle L\rangle = 4\omega^2\langle H_0\rangle - 4\omega_0^2\langle E\rangle_0, \tag{4.158}$$

which results in the differential equation (4.151) with the explicit solution

$$\langle H_0\rangle = \frac{\omega^2\langle H_0\rangle_0 - \omega_0^2\langle E\rangle_0}{\omega^2}\cos(2\omega t) - \frac{1}{2\omega}\left\langle\frac{\partial H_0}{\partial t}\right\rangle_0 \sin(2\omega t) + \frac{\omega_0^2}{\omega^2}\langle E\rangle_0 \tag{4.159}$$

of the initial value problem. The hypotheses of the lemma are satisfied.

The related operators can be conveniently extended as follows

$$H_0 = \frac{\omega_0}{2} e^{\mu t} \left(e^{-2\lambda t}\ p^2 + e^{2\lambda t}\ x^2 \right), \tag{4.160}$$

$$L = e^{\mu t} \left(-e^{-2\lambda t}\ p^2 + e^{2\lambda t}\ x^2 \right), \tag{4.161}$$

$$M = e^{\mu t} (px + xp) \tag{4.162}$$

and

$$\begin{aligned} E &= H_0(t) + \frac{1}{2}(\lambda - \mu) M(t) \\ &= \frac{\omega_0}{2} e^{\mu t} \left(e^{-2\lambda t}\ p^2 + e^{2\lambda t}\ x^2 \right) + \frac{1}{2}(\lambda - \mu) e^{\mu t} (px + xp). \end{aligned} \tag{4.163}$$

Then by Lemma 3

$$\frac{d}{dt} \langle M \rangle = -2\omega_0 \langle L \rangle, \tag{4.164}$$

$$\frac{d}{dt} \langle H_0 \rangle = \omega_0 (\lambda - \mu) \langle L \rangle, \tag{4.165}$$

$$\frac{d}{dt} \langle E \rangle = 0 \tag{4.166}$$

and

$$\frac{d}{dt} \langle L \rangle = 4 \frac{\lambda - \mu}{\omega_0} \langle H_0 \rangle + 2\omega_0 \langle M \rangle. \tag{4.167}$$

In terms of the energy operator

$$\frac{d}{dt} \langle L \rangle + \frac{4\omega^2}{(\lambda - \mu)\,\omega_0} \langle H_0 \rangle = \frac{4\omega_0}{\lambda - \mu} \langle E \rangle \tag{4.168}$$

and as a result

$$\frac{d^2}{dt^2} \langle H_0 \rangle + 4\omega^2 \langle H_0 \rangle = 4\omega_0^2 \langle E \rangle_0, \quad \omega = \sqrt{\omega_0^2 - (\lambda - \mu)^2} > 0 \tag{4.169}$$

with the unique solution of the initial value problem given by

$$\begin{aligned} \langle H_0 \rangle &= \frac{\omega^2 \langle H_0 \rangle_0 - \omega_0^2 \langle E \rangle_0}{\omega^2} \cos(2\omega t) \\ &\quad + \frac{1}{2}(\lambda - \mu) \frac{\omega_0}{\omega} \langle L \rangle_0 \sin(2\omega t) + \frac{\omega_0^2}{\omega^2} \langle E \rangle_0. \end{aligned} \tag{4.170}$$

The hypotheses of Lemma 6 are satisfied.

The Modified Oscillator

The required operators are

$$\begin{aligned} H &= (\cos t\ p + \sin t\ x)^2 && (4.171) \\ &= \cos^2 t\ p^2 + \sin^2 t\ x^2 + \sin t \cos t\ (px + xp) \\ &= \frac{1}{2}\left(p^2 + x^2\right) + \frac{1}{2}\cos 2t\ \left(p^2 - x^2\right) + \frac{1}{2}\sin 2t\ (px + px) \\ &= H_0 + E(t), \end{aligned}$$

where

$$H_0 = \frac{1}{2}\left(p^2 + x^2\right), \tag{4.172}$$

$$E = E(t) = \frac{1}{2}\cos 2t\ \left(p^2 - x^2\right) + \frac{1}{2}\sin 2t\ (px + px) \tag{4.173}$$

and

$$L = \frac{\partial H}{\partial t} = \frac{\partial E}{\partial t} = -\sin 2t\ \left(p^2 - x^2\right) + \cos 2t\ (px + px). \tag{4.174}$$

Here

$$\frac{d}{dt}\langle H_0\rangle = \frac{d}{dt}\langle H\rangle = \left\langle \frac{\partial H}{\partial t}\right\rangle = \left\langle \frac{\partial E}{\partial t}\right\rangle = \langle L\rangle \tag{4.175}$$

and

$$\frac{d}{dt}\langle L\rangle = 4\langle H_0\rangle. \tag{4.176}$$

The expectation value $\langle H_0\rangle$ satisfies the following differential equation

$$\frac{d^2}{dt^2}\langle H_0\rangle = 4\langle H_0\rangle \tag{4.177}$$

with the explicit solution

$$\langle H_0\rangle = \langle H_0\rangle_0 \cosh(2t) + \frac{1}{2}\langle L\rangle_0 \sinh(2t). \tag{4.178}$$

The hypotheses of Lemma 6 are satisfied.

The Modified Damped Oscillator

Let $\hbar = m\omega_0 = 1$ in the Hamiltonian (4.33):

$$H = \frac{\omega_0}{2}\left(\frac{p^2}{\cosh^2(\lambda t)} + \cosh^2(\lambda t)\ x^2\right) \tag{4.179}$$

without loss of generality. The corresponding energy operator can be found as follows

$$E=\frac{\omega_0}{2\cosh^2(\lambda t)}p^2+\frac{\omega_0^2\sinh^2(\lambda t)+\omega^2}{2\omega_0}x^2 \tag{4.180}$$
$$+\frac{\lambda}{2}\tanh(\lambda t)(px+xp), \qquad \frac{d}{dt}\langle E\rangle=0,$$

in view of (4.77)–(4.79) (one should replace $A\leftrightarrow B, C\rightarrow -C$ in the momentum representation).

Introducing the following complementary operators

$$H_0 = \frac{p^2}{\cosh^2(\lambda t)}+\cosh^2(\lambda t)\ x^2, \tag{4.181}$$
$$L = \frac{p^2}{\cosh^2(\lambda t)}-\cosh^2(\lambda t)\ x^2, \tag{4.182}$$
$$M = px+xp, \tag{4.183}$$

we get

$$\frac{d}{dt}\langle H_0\rangle = -2\lambda\tanh(\lambda t)\langle L\rangle, \tag{4.184}$$
$$\frac{d}{dt}\langle L\rangle = -2\lambda\tanh(\lambda t)\langle H_0\rangle-2\omega_0\langle M\rangle, \tag{4.185}$$
$$\frac{d}{dt}\langle M\rangle = 2\omega_0\langle L\rangle. \tag{4.186}$$

Then

$$E = \frac{\omega_0}{2}\left(1-\frac{\lambda^2}{2\omega_0^2\cosh^2(\lambda t)}\right)H_0+\frac{\lambda^2}{4\omega_0\cosh^2(\lambda t)}L \tag{4.187}$$
$$+\frac{\lambda}{2}\tanh(\lambda t)M$$

and, eliminating $\langle M\rangle$ and $\langle L\rangle$ from the system, one gets:

$$\frac{d^2}{dt^2}\langle H_0\rangle-\frac{4\lambda}{\sinh(2\lambda t)}\frac{d}{dt}\langle H_0\rangle+2\left(2\omega^2+\frac{\lambda^2}{\cosh^2(\lambda t)}\right)\langle H_0\rangle=8\omega_0\langle E\rangle_0. \tag{4.188}$$

The required initial conditions:

$$\left(\frac{d}{dt}\langle H_0\rangle\right)_0=0, \qquad \left(\coth(\lambda t)\frac{d}{dt}\langle H_0\rangle\right)_0=-2\lambda\langle L\rangle_0 \tag{4.189}$$

follow from (4.184). The unique explicit solution is given by

$$\langle H_0\rangle = -\lambda\frac{\lambda^2\langle E\rangle_0+\omega_0\omega^2\langle L\rangle_0}{\omega_0\omega^2\left(2\omega^2+\lambda^2\right)} \tag{4.190}$$
$$\times\left(2\omega\tanh(\lambda t)\sin(2\omega t)+\lambda\left(1+\tanh^2(\lambda t)\right)\cos(2\omega t)\right)$$
$$+2\langle E\rangle_0\frac{\omega_0}{\omega^2}\left(1-\frac{\lambda^2}{2\omega_0^2\cosh^2(\lambda t)}\right)$$

(see appendix D). The hypotheses of Lemma 6 are satisfied.

In the case (4.49), the energy operator (4.83) is a positive operator:

$$\langle E\rangle = \tanh^2(\lambda t+\delta)\ \left\langle p^2\right\rangle + \coth^2(\lambda t+\delta)\ \left\langle x^2\right\rangle = \langle E\rangle_0 > 0. \tag{4.191}$$

The related operators are

$$L = \tanh^2(\lambda t+\delta)\ p^2 - \coth^2(\lambda t+\delta)\ x^2, \tag{4.192}$$

$$M = px + xp, \tag{4.193}$$

$$H = \frac{\omega}{2} E + \frac{\lambda}{\sinh(2\lambda t + 2\delta)} M \tag{4.194}$$

with

$$\frac{d}{dt}\langle L\rangle = -2\omega\langle M\rangle, \qquad \frac{d}{dt}\langle M\rangle = -2\omega\langle L\rangle. \tag{4.195}$$

From here

$$\frac{d^2}{dt^2}\langle L\rangle + 4\omega^2\langle L\rangle = 0, \qquad \frac{d^2}{dt^2}\langle M\rangle + 4\omega^2\langle M\rangle = 0, \tag{4.196}$$

which determines the time-evolution of the expectation values.

Parametric Oscillators

In general the Lewis–Riesenfeld quadratic invariant (4.91) for the parametric oscillator (4.62) is obviously a positive operator for real-valued solutions of the Ermakov equation (4.89) that satisfies the conditions of our lemma.

General Quadratic Hamiltonian

In the case of Hamiltonian (4.69) applying formula (4.71) to the operators, $O = \left\{p^2, x^2, px + xp\right\}$, one obtains [50]:

$$\frac{d}{dt}\begin{pmatrix} \left\langle p^2\right\rangle \\ \left\langle x^2\right\rangle \\ \langle px+xp\rangle \end{pmatrix} = \begin{pmatrix} -3c(t)-d(t) & 0 & -2b(t) \\ 0 & c(t)+3d(t) & 2a(t) \\ 4a(t) & -4b(t) & -c(t)+d(t) \end{pmatrix} \begin{pmatrix} \left\langle p^2\right\rangle \\ \left\langle x^2\right\rangle \\ \langle px+xp\rangle \end{pmatrix}. \tag{4.197}$$

This system has a unique solution for suitable coefficients [97], which allows one to apply Lemma 6, say, for the positive operator x^2. Our Theorem 1 provides another choice of positive operators. On the second thought a positive integral (4.221) determines time-evolution of the squared norm and guarantees uniqueness in $L^2(\mathbb{R})$. Details are left to the reader.

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4.6 Appendix A: The Ehrenfest Theorems

Application of formula (4.67) to the position x and momentum p operators allows one to derive the Ehrenfest theorem [69], [145], [183] for the models of oscillators under consideration.

For the Caldirola-Kanai Hamiltonian (4.13) one gets

$$\frac{d}{dt}\langle x\rangle = \omega_0 e^{-2\lambda t}\langle p\rangle, \qquad \frac{d}{dt}\langle p\rangle = -\omega_0 e^{2\lambda t}\langle x\rangle. \tag{4.198}$$

Elimination of the expectation value $\langle p\rangle$ from this system results in the classical equation of motion for a damped oscillator [13], [120]:

$$\frac{d^2}{dt^2}\langle x\rangle + 2\lambda\frac{d}{dt}\langle x\rangle + \omega_0^2\langle x\rangle = 0. \tag{4.199}$$

For the modified Caldirola-Kanai Hamiltonian (4.18) the system

$$\frac{d}{dt}\langle x\rangle = \omega_0 e^{-2\lambda t}\langle p\rangle - 2\lambda\langle x\rangle, \qquad \frac{d}{dt}\langle p\rangle = -\omega_0 e^{2\lambda t}\langle x\rangle + 2\lambda\langle p\rangle \tag{4.200}$$

gives the same classical equation.

In the case of the united model (4.23) one should use the differentiation formula (4.72). Then

$$\frac{d}{dt}\langle x\rangle = \omega_0 e^{-2\lambda t}\langle p\rangle - 2\mu\langle x\rangle, \qquad \frac{d}{dt}\langle p\rangle = -\omega_0 e^{2\lambda t}\langle x\rangle \tag{4.201}$$

and the second order equations are given by

$$\frac{d^2}{dt^2}\langle x\rangle + 2(\lambda+\mu)\frac{d}{dt}\langle x\rangle + \left(\omega_0^2 + 4\lambda\mu\right)\langle x\rangle = 0, \tag{4.202}$$

$$\frac{d^2}{dt^2}\langle p\rangle + 2(\mu-\lambda)\frac{d}{dt}\langle p\rangle + \omega_0^2\langle p\rangle = 0. \tag{4.203}$$

The general solutions are

$$\langle x\rangle = Ae^{-(\lambda+\mu)t}\sin(\omega t+\delta), \tag{4.204}$$

$$\langle p\rangle = Be^{(\lambda-\mu)t}\sin(\omega t+\gamma), \tag{4.205}$$

where $\omega = \sqrt{\omega_0^2 - (\lambda-\mu)^2} > 0$.

In a similar fashion for a modified oscillator with the Hamiltonian (4.29) we obtain

$$\frac{d}{dt}\langle x\rangle = 2\cos^2 t\ \langle p\rangle + 2\sin t\cos t\ \langle x\rangle, \tag{4.206}$$

$$\frac{d}{dt}\langle p\rangle = -2\sin^2 t\ \langle x\rangle - 2\sin t\cos t\ \langle p\rangle. \tag{4.207}$$

Then

$$\frac{d^2}{dt^2}\langle x\rangle + 2\tan t\frac{d}{dt}\langle x\rangle - 2\langle x\rangle = 0, \tag{4.208}$$

which coincides with the characteristic equation (4.6) in this case [52].

In the case of the damped oscillator of Chruściński and Jurkowski one obtains

$$\frac{d}{dt}\langle x\rangle = \frac{\langle p\rangle}{m\cosh^2(\lambda t)}, \tag{4.209}$$

$$\frac{d}{dt}\langle p\rangle = -m\omega_0^2\cosh^2(\lambda t)\langle x\rangle. \tag{4.210}$$

The Ehrenfest theorems coincide with the Newtonian equations of motion [46]:

$$\frac{d^2}{dt^2}\langle x\rangle + 2\lambda\tanh(\lambda t)\frac{d}{dt}\langle x\rangle + \omega_0^2\langle x\rangle = 0, \tag{4.211}$$

$$\frac{d^2}{dt^2}\langle p\rangle - 2\lambda\tanh(\lambda t)\frac{d}{dt}\langle p\rangle + \omega_0^2\langle p\rangle = 0 \tag{4.212}$$

with the general solutions given by

$$\langle x \rangle = A \frac{\sin(\omega t + \delta)}{\cosh(\lambda t)}, \qquad \omega = \sqrt{\omega_0^2 - \lambda^2} > 0, \tag{4.213}$$

$$\langle p \rangle = B \left(\lambda \cos(\omega t + \delta) \sinh(\lambda t) + \omega \sin(\omega t + \delta) \cosh(\lambda t) \right), \tag{4.214}$$

respectively. It is worth noting that both equations (4.199) and (4.211) give the same frequency of oscillations for the damped motion; see [46] for more details.

Combining all models together for the general quadratic Hamiltonian (4.69):

$$\frac{d}{dt} \langle x \rangle = 2a(t) \ \langle p \rangle + 2d(t) \ \langle x \rangle, \quad \frac{d}{dt} \langle p \rangle = -2b(t) \ \langle x \rangle - 2c(t) \ \langle p \rangle \tag{4.215}$$

with the help of (4.72). The Newtonian-type equation of motion for the expectation values has the form

$$\frac{d^2}{dt^2} \langle x \rangle - \tau(t) \frac{d}{dt} \langle x \rangle + 4\sigma(t) \langle x \rangle = 0 \tag{4.216}$$

with

$$\tau(t) = \frac{a'}{a} - 2c + 2d, \qquad \sigma(t) = ab - cd + \frac{d}{2} \left(\frac{a'}{a} - \frac{d'}{d} \right). \tag{4.217}$$

In order to explain a connection with the characteristic equation (4.6)–(4.7) we temporarily replace $c \rightarrow c_0$ and $d \rightarrow d_0$ in the original Hamiltonian (4.1). Then it takes the standard form (4.69), if $c_0 = c + d$ and $d_0 = c$. Using the new notations in (4.6)–(4.7) we find

$$\tau - \tau_0 = 4(d - c), \quad \sigma - \sigma_0 = \frac{a}{2} \left(\frac{c - d}{a} \right)'. \tag{4.218}$$

Therefore our characteristic equation (4.6) coincides with the corresponding Ehrenfest theorem (4.216) only in the case of self-adjoint Hamiltonians, when $c = d$ (or $c_0 = 2d_0$). The united model shows that these equations are different otherwise.

4.7 Appendix B: The Heisenberg Uncertainty Relation Revisited

A detailed review with an extensive list of references is given in Refs. [64] and [155] (see also [141]). We only discuss the Heisenberg uncertainty relation for the position x and momentum $p = -i\partial/\partial x$ operators (in the units of $\hbar$) in the case of the general quadratic Hamiltonian (4.69). By our Lemma 3 the simplest integral of motion is given by

$$E_0 = \exp \left(\int_0^t (c(\tau) - d(\tau)) \ d\tau \right) \ (px - xp) \tag{4.219}$$

with

$$[x,p] = xp - px = i. \tag{4.220}$$

This implies the following time evolution:

$$\langle \psi, \psi \rangle = \exp\left(\int_0^t \left(d\left(\tau\right) - c\left(\tau\right)\right)\, d\tau\right) \langle \psi, \psi \rangle_0 \tag{4.221}$$

of the squared norm of the wave functions.

With the expectation values

$$\overline{x} = \frac{\langle x \rangle}{\langle 1 \rangle} = \frac{\langle \psi, x\psi \rangle}{\langle \psi, \psi \rangle}, \qquad \overline{p} = \frac{\langle p \rangle}{\langle 1 \rangle} = \frac{\langle \psi, p\psi \rangle}{\langle \psi, \psi \rangle} \tag{4.222}$$

and the operators

$$\Delta x = x - \overline{x}, \qquad \Delta p = p - \overline{p} \tag{4.223}$$

let us consider

$$\begin{aligned} 0 &\leq \langle \left(\Delta x + i\lambda \Delta p\right)\psi, \left(\Delta x + i\lambda \Delta p\right)\psi \rangle \\ &= \langle \psi, \left(\Delta x - i\lambda \Delta p\right)\left(\Delta x + i\lambda \Delta p\right)\psi \rangle \\ &= \left\langle \left(\Delta x\right)^2 \right\rangle - \lambda \langle 1 \rangle + \lambda^2 \left\langle \left(\Delta p\right)^2 \right\rangle \end{aligned} \tag{4.224}$$

for a real parameter λ. Here we have used the operator identity

$$\left(\Delta x - i\lambda \Delta p\right)\left(\Delta x + i\lambda \Delta p\right) = \left(\Delta x\right)^2 - \lambda + \lambda^2 \left(\Delta p\right)^2. \tag{4.225}$$

Then one gets

$$\left\langle \left(\Delta p\right)^2 \right\rangle \left\langle \left(\Delta x\right)^2 \right\rangle \geq \frac{1}{4} \langle 1 \rangle^2 = \frac{1}{4} \exp\left(2\int_0^t \left(d\left(\tau\right) - c\left(\tau\right)\right)\, d\tau\right), \tag{4.226}$$

if $\langle 1 \rangle_0 = \langle \psi, \psi \rangle_0 = 1$. For the standard deviations:

$$\left(\delta p\right)^2 = \frac{\left\langle \left(\Delta p\right)^2 \right\rangle}{\langle 1 \rangle} = \overline{\left(p^2\right)} - \left(\overline{p}\right)^2, \tag{4.227}$$

$$\left(\delta x\right)^2 = \frac{\left\langle \left(\Delta x\right)^2 \right\rangle}{\langle 1 \rangle} = \overline{\left(x^2\right)} - \left(\overline{x}\right)^2, \tag{4.228}$$

we finally obtain

$$\delta p \ \delta x \geq \frac{1}{2} \tag{4.229}$$

in the units of $\hbar$. It is worth noting that Eq. (4.229) is derived, in fact, for any operators x and p with the commutator (4.220) — the structure of the quadratic Hamiltonian (4.69) has only been used in the norm (4.221). Time-evolution of the standard deviations (4.227)–(4.228) will be discussed elsewhere.

4.8 Appendix C: Linear Integrals of Motion: The Dodonov–Malkin–Man'ko–Trifonov Invariants

All invariants of the form

$$P = A(t)\,p + B(t)\,x + C(t) \tag{4.230}$$

for the general quadratic Hamiltonian (4.69) can be found as follows (see, for example, [63], [65], [140], [139] and references therein). Use of the differentiation formula (4.72) results in the following system:

$$\begin{aligned} \frac{dA}{dt} &= 2c(t)A - 2a(t)B, && (4.231)\\ \frac{dB}{dt} &= 2b(t)A - 2d(t)B, && (4.232)\\ \frac{dC}{dt} &= (c(t) - d(t))\,C. && (4.233) \end{aligned}$$

The last equation is explicitly integrated and elimination of B and A from (4.231) and (4.232), respectively, gives the second-order equations:

$$\begin{aligned} A'' - \left(\frac{a'}{a} + 2c - 2d\right)A' + 4\left(ab - cd + \frac{c}{2}\left(\frac{a'}{a} - \frac{c'}{c}\right)\right)A &= 0, && (4.234)\\ B'' - \left(\frac{b'}{b} + 2c - 2d\right)B' + 4\left(ab - cd - \frac{d}{2}\left(\frac{b'}{b} - \frac{d'}{d}\right)\right)B &= 0. && (4.235) \end{aligned}$$

The first is equivalent to our characteristic equation (4.6)–(4.7) and coincides with the Ehrenfest theorem (4.216)–(4.217) when $c \leftrightarrow d$.

Thus the linear quantum invariants are given by

$$P = A(t)\,p + \frac{2c(t)A(t) - A'(t)}{2a(t)}x + C_0 \exp\left(\int_0^t (c(\tau) - d(\tau))\,d\tau\right), \tag{4.236}$$

where $A(t)$ is a general solution of equation (4.234) depending upon two parameters and C_0 is the third constant. Our Theorem 1 gives a similar description of the quadratic invariants in terms of solutions of the auxiliary equation (4.105). Relations between linear and quadratic invariants are analyzed in [198]. Group-theoretical applications are discussed in [42], [62], [65], [140], [63], [132], [139] and elsewhere.

4.9 Appendix D: An Elementary Differential Equation

The nonhomogeneous differential equation of the form

$$y'' - \frac{4\lambda}{\sinh(2\lambda t + 2\gamma)} y' + \left(\omega^2 + \frac{2\lambda^2}{\cosh^2(\lambda t + \gamma)} \right) y = 1 \tag{4.237}$$

(ω, λ and γ are some parameters) has the following general solution:

$$y = C_1 y_1(t) + C_2 y_2(t) + Y(t), \tag{4.238}$$

where C_1 and C_2 are constants,

$$y_1 = \omega \tanh(\lambda t + \gamma) \cos(\omega t) - \lambda \left(1 + \tanh^2(\lambda t + \gamma)\right) \sin(\omega t), \tag{4.239}$$

$$y_2 = \omega \tanh(\lambda t + \gamma) \sin(\omega t) + \lambda \left(1 + \tanh^2(\lambda t + \gamma)\right) \cos(\omega t) \tag{4.240}$$

are the fundamental solutions of the corresponding homogeneous equation with the Wronskian given by

$$W(y_1, y_2) = \omega \left(\omega^2 + 4\lambda^2\right) \tanh^2(\lambda t + \gamma), \tag{4.241}$$

and

$$Y = \frac{1}{\omega^2} \left(1 - \frac{2\lambda^2}{\left(\omega^2 + 4\lambda^2\right) \cosh^2(\lambda t + \gamma)} \right) \tag{4.242}$$

is a particular solution of the nonhomogeneous equation.

One can also verify that functions:

$$z_1 = \omega \cos(\omega t) - \lambda \coth(\lambda t + \gamma) \sin(\omega t), \tag{4.243}$$

$$z_2 = \omega \sin(\omega t) + \lambda \coth(\lambda t + \gamma) \cos(\omega t) \tag{4.244}$$

with $W(z_1, z_2) = \omega \left(\omega^2 + \lambda^2\right)$ are fundamental solutions of the following equation:

$$z'' + \left(\omega^2 - \frac{2\lambda^2}{\sinh^2(\lambda t + \gamma)} \right) z = 0 \tag{4.245}$$

and then carry out the substitution $y = z \tanh(\lambda t + \gamma)$. Details are left to the reader. The particular solution of the nonhomogeneous equation can be found by the variation of parameters and/or verified by the substitution. Review of other integrable second-order differential equations is given in [72].

Chapter 5

Characteristic Equation and the Gauge Transformation

citation: R. Cordero-Soto, *The Gauge Transformation and Reduced Characteristic Equation of a Quadratic Time-Dependent Schrödinger Equation*, under preparation.

5.1 Introduction

In [49], the authors study and solve the time-dependent Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = H(t)\,\psi \tag{5.1}$$

where

$$H = -a(t)\frac{\partial^2}{\partial x^2} + b(t)x^2 - i\left(c(t)x\frac{\partial}{\partial x} + d(t)\right) \tag{5.2}$$

and where $a(t)$, $b(t)$, $c(t)$, and $d(t)$ are real-valued functions of time t only; see Refs. [49], [52], [50],[122], [134], [143], [194], [195], and [196] for a general approach and currently known explicit solutions. The solution is given by

$$\psi(x,t) = \int_{-\infty}^{\infty} G(x,y,t)\ \psi_0(y)\ dy \tag{5.3}$$

where the Green's function, or particular solution is given by

$$G(x,y,t) = \frac{1}{\sqrt{2\pi i\mu(t)}}\ e^{i\left(\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2\right)}. \tag{5.4}$$

The time-dependent functions are found via a substitution method that reduces (5.2) to a system of differential equations (see [49]). This system is explicitly integrable up to the function $\mu(t)$ which satisfies the following so-called characteristic equation

$$\mu'' - \tau(t)\mu' + 4\sigma(t)\mu = 0 \tag{5.5}$$

with

$$\tau(t) = \frac{a'}{a} - 2c + 4d, \qquad \sigma(t) = ab - cd + d^2 + \frac{d}{2}\left(\frac{a'}{a} - \frac{d'}{d}\right). \tag{5.6}$$

This equation must be solved subject to the initial data

$$\mu(0) = 0, \qquad \mu'(0) = 2a(0) \neq 0 \tag{5.7}$$

in order to satisfy the initial condition for the corresponding Green's function. While the Green's function (5.4) is explicit up to integration, explicit solutions of the characteristic equation (5.5) are rare. In this paper we find the solution of (5.1) by solving a reduced characteristic equation. The reduced characteristic equation is obtained by using a gauge-like transformation lemma following the work of [50]. The gauge transformation itself provides a simplified form of the Green's function. Furthermore, we use a similar transformation to generate a general quadratic dynamical invariant (see [51]) of (5.2) that is explicit up the solution of the very same reduced characteristic equation. We conclude by using use the gauge transformation to obtain a uniqueness result for the solution in Schwartz Space of a general quantum dot model or damped model in a simple fashion.

5.2 Transformation Lemmas

Lemma 7 *Let $\widetilde{\psi}(x,t)$, with $\widetilde{\psi}(x,0)$ in Schwartz space, solve the following time-dependent Schrödinger equation:*

$$i\frac{\partial\widetilde{\psi}}{\partial t}=\widetilde{H}\widetilde{\psi}, \tag{5.8}$$

where

$$\widetilde{H}=-a(t)\frac{\partial^2}{\partial x^2}+b(t)x^2-ic(t)x\frac{\partial}{\partial x}. \tag{5.9}$$

Then

$$\psi(x,t)=\widetilde{\psi}(x,t)\exp\left(-\int_0^t d(s)\ ds\right) \tag{5.10}$$

solves (5.1)-(5.2) for

$$\psi(x,0)=\widetilde{\psi}(x,0). \tag{5.11}$$

Proof. Let $\psi(x,t)=\widetilde{\psi}(x,t)\exp\left(-\int_0^t d(s)\ ds\right)$ and assume $\widetilde{\psi}(x,t)$ solves (5.8)-(5.9), where $\widetilde{\psi}(x,0)$ is in Schwartz space. We differentiate $\psi(x,t)$ with respect to time:

$$i\frac{\partial\psi}{\partial t}=i\frac{\partial\widetilde{\psi}}{\partial t}\exp\left(-\int_0^t d(s)\ ds\right)-id(t)\widetilde{\psi}(x,t)\exp\left(-\int_0^t d(s)\ ds\right). \tag{5.12}$$

For H given by (5.2) and $\widetilde{H}$ given by (5.9), we have

$$H=\widetilde{H}-id(t), \tag{5.13}$$

and

$$i\frac{\partial\psi}{\partial t}=\widetilde{H}\left[\widetilde{\psi}\right]\exp\left(-\int_0^t d(s)\ ds\right)-id(t)\,\psi. \tag{5.14}$$

Since

$$\widetilde{H}\left[\widetilde{\psi}\right]\exp\left(-\int_0^t d(s)\ ds\right)=\widetilde{H}\left[\widetilde{\psi}\exp\left(-\int_0^t d(s)\ ds\right)\right]=\widetilde{H}\left[\psi\right], \tag{5.15}$$

we have that

$$i\frac{\partial\psi}{\partial t}=\widetilde{H}\left[\psi\right]-id(t)\,\psi=H\psi. \tag{5.16}$$

■

By the method of [49] for $d=0$ we can find $\widetilde{\psi}(x,t)$: We simply generate the Green's function for $\widetilde{\psi}(x,t)$ by substituting $d=0$ in (5.3). It is worth noting that we now have a *reduced characteristic equation* given by

$$\mu''-\widetilde{\tau}(t)\,\mu'+4\widetilde{\sigma}(t)\,\mu=0, \tag{5.17}$$

where

$$\widetilde{\tau}(t)=\frac{a'}{a}-2c, \tag{5.18}$$

$$\widetilde{\sigma}(t)=ab \tag{5.19}$$

and initial conditions are given by (5.7).

The Schwartz requirement on the initial condition is necessary to show that (5.3) is in fact the solution of (5.1)-(5.2) since we can justify the interchanging of the time-derivative and integral operators. In particular, we note that

$$\left|\frac{\partial}{\partial t}G(x,y,t)\,\psi_0(y)\right|=\left|\frac{\partial}{\partial t}\left[A(t)\ e^{i\left(\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2\right)}\psi_0(y)\right]\right|. \tag{5.20}$$

Here,

$$A(t)=\frac{1}{\sqrt{2\pi i\mu(t)}}. \tag{5.21}$$

Thus, (5.20) reduces to

$$\left|\left(\frac{\partial A}{\partial t}+Ai\frac{\partial S}{\partial t}\right)\psi_0(y)\right|, \tag{5.22}$$

where

$$S(x,y,t)=\alpha(t)\,x^2+\beta(t)\,xy+\gamma(t)\,y^2. \tag{5.23}$$

Since $\psi_0(y)$ is in Schwartz space, (5.22) is also in Schwartz space. It follows that the time-derivative operator can be exchanged with the integral (see [1]).

We state the following extension Corollary:

Corollary 8 *Let $\widetilde{\psi}(x,t)$, with $\widetilde{\psi}(x,0)$ uniquely solve (5.8)-(5.9). Then 5.10 uniquely solves (5.1)-(5.2) for (5.11).*

5.3 Quantum Integrals

We say that a quadratic operator

$$E = A(t)\,p^2 + B(t)\,x^2 + C(t)\,(px + xp) \tag{5.24}$$

is a quadratic dynamical invariant of (5.2) if

$$\frac{d\langle E\rangle}{dt} = 0 \tag{5.25}$$

for (5.2). In [51] it is shown that

$$E_1 = \left[up - \frac{(u' - cu)\,x}{2a}\right]^2 \exp\left(\int_0^t -c(s)\,ds\right) \tag{5.26}$$

is an invariant for

$$H_1 = a(t)\,p^2 + b(t)\,x^2 + c(t)\,xp \tag{5.27}$$

where $u(t)$ satisfies the following second-order differential equation:

$$u'' - \frac{a'}{a}u' + \left[4ab + \left(\frac{a'}{a} - c\right)c - c'\right]u = 0. \tag{5.28}$$

But (5.28) can be simplified by using the substitution $u = \mu \exp\left(\int_0^t c(s)\,ds\right)$. In doing so, we find that (5.26) can be rewritten as

$$E_1 = \left[\mu p - \frac{\mu' x}{2a}\right]^2 \exp\left(\int_0^t c(s)\,ds\right) \tag{5.29}$$

where μ is given by the reduced characteristic equation (5.17). We thus obtain the invariant for (5.2)

Lemma 9 *Let*

$$E = \left[\mu p - \frac{\mu' x}{2a}\right]^2 \exp\left(\int_0^t [2d(s) + c(s)]\,ds\right). \tag{5.30}$$

Then (5.30) is a dynamical invariant of (5.2).

Proof. *We note that (5.2) and (5.30) can be rewritten as*

$$E = \exp\left(\int_0^t 2d(s)\,ds\right) E_1 \tag{5.31}$$

and

$$H = H_1 - id \tag{5.32}$$

where E_1 and H_1 are given by (5.29) and (5.27) respectively. The reader should note that here $\langle Q\rangle = \int \psi^* Q\psi dx$ where ψ solves the Schrödinger equation for (5.2). Furthermore, by the transformation lemma, we have that $\psi = \widetilde{\psi}\exp\left(-\int_0^t d(s)\ ds\right)$ where $\widetilde{\psi}$ solves the Schrödinger equation for H_1. Thus, we have that

$$\left\langle \exp\left(\int_0^t 2d(s)\,ds\right) Q\right\rangle = \int \psi^* \left[\exp\left(\int_0^t 2d(s)\,ds\right) Q\right] \psi dx = \int \widetilde{\psi}^* Q\widetilde{\psi} dx. \tag{5.33}$$

This then shows that

$$\frac{d\langle E\rangle}{dt} = \frac{d}{dt}\left[\int \widetilde{\psi}^* E_1 \widetilde{\psi} dx\right] = 0 \tag{5.34}$$

since E_1 is an invariant of H_1. ■

5.4 Quantum Dot Model

Essentially, a quantum dot is a small box that contains electrons. The box is coupled via tunnel barriers to a source and drain reservoir (see [96], [71]) with which particles can be exchanged. When the size of this so-called box is comparable to the wavelength of the electrons that occupy it, the energy spectrum is discrete, resembling atoms. This is why quantum dots are artificial atoms in a sense. Vladimiro Mujica has suggested that the following model is of use to the theory of Semiconductor quantum dots:

$$H = a(t)\,p^2 + b(t)\,x^2 - id(t) \tag{5.35}$$

Uniqueness

We wish to obtain uniqueness of solutions of (5.1) for (5.35) in Schwartz Space. We follow the approach of quantum integrals in ([51]) to first prove the uniqueness for the following Hamiltonian:

$$H_0 = a(t)\,p^2 + b(t)\,x^2. \tag{5.36}$$

In particular, we will show that for (5.36),

$$\langle H_0 \rangle = 0 \text{ when } \psi(x,0) = 0. \tag{5.37}$$

We first recall that

$$\langle Q \rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\ Q[\psi(x,t)]\ dx \tag{5.38}$$

Since, we have that ψ is in Schwartz space (see the Fourier Transform on $\mathbb{R}$ in [192]), it follows that

$$\langle H_0 \rangle = a(t)\left\langle p^2 \right\rangle + b(t)\left\langle x^2 \right\rangle\ < \infty. \tag{5.39}$$

Thus, to prove (5.37) , we will show that

$$\left\langle p^2 \right\rangle = \left\langle x^2 \right\rangle = 0 \text{ when } \psi(x,0) = 0. \tag{5.40}$$

Again, since ψ is in Schwartz space, we have that

$$\frac{d}{dt}\langle Q \rangle = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(\psi^*(x,t)\ Q[\psi(x,t)]\ \right)dx = \frac{1}{i}\left\langle QH - H^{\dagger}Q \right\rangle \tag{5.41}$$

for $Q = p, x, px, xp, p^2$and x^2.

Given (5.41) we have the following ODE system:

$$\begin{aligned} \frac{d}{dt}\left\langle p^2 \right\rangle &= -2b(t)\langle px + xp \rangle \\ \frac{d}{dt}\left\langle x^2 \right\rangle &= 2a(t)\langle px + xp \rangle \\ \frac{d}{dt}\langle px + xp \rangle &= 4a(t)\left\langle p^2 \right\rangle - 4b(t)\left\langle x^2 \right\rangle. \end{aligned} \tag{5.42}$$

If $\psi(x,0) = 0$, then

$$\begin{aligned} \left\langle p^2 \right\rangle_0 &= 0 \\ \left\langle x^2 \right\rangle_0 &= 0 \\ \langle px + xp \rangle_0 &= 0. \end{aligned} \tag{5.43}$$

According to the general theory of homogeneous linear systems of ODE's, we have that

$$\begin{aligned} \left\langle p^2 \right\rangle &= 0 \\ \left\langle x^2 \right\rangle &= 0 \\ \langle px + xp \rangle &= 0. \end{aligned} \tag{5.44}$$

Thus, we have shown that (5.40) holds, thereby proving (5.37). We then use the following (see [51]) lemma:

Lemma 10 *Suppose that the expectation value*

$$\langle H_0 \rangle = \langle \psi, H_0 \psi \rangle \geq 0 \tag{5.45}$$

for a positive quadratic operator

$$H_0 = f(t)(\alpha(t)p + \beta(t)x)^2 + g(t)x^2 \qquad (f(t) \geq 0,\ g(t) > 0) \tag{5.46}$$

($\alpha(t)$ and $\beta(t)$ are real-valued functions) vanishes for all $t \in [0, T)$:

$$\langle H_0 \rangle = \langle H_0 \rangle (t) = \langle H_0 \rangle (0) = 0, \tag{5.47}$$

when $\psi(x, 0) = 0$ almost everywhere. Then the corresponding Cauchy initial value problem

$$i\frac{\partial \psi}{\partial t} = H\psi, \qquad \psi(x, 0) = \varphi(x) \tag{5.48}$$

may have only one solution in Schwartz space.

Since we have proven (5.37), we have that H_0 satisfies this lemma. Thus we use Corollary 8 to extend the uniqueness in Schwartz space to Hamiltonian (5.35).

5.5 Future Work

Almost Self-Adjoint or Almost Symmetric

In this report, the author has often studied Hamiltonians that are not Self-adjoint (by self-adjoint we formally mean symmetric). While some of these are not self-adjoint, they seem to be "almost self-adjoint" in the sense that they lack a term that would enable the symmetry. For example, the Hamiltonians (3.10) and (3.11) only require an extra term to be symmetric. Specifically, H_1 and H_2 are not symmetric but

$$H_1 - \lambda xp \tag{5.49}$$

and

$$H_2 - \lambda px \tag{5.50}$$

are symmetric. While this does not necessarily shed light into what an appropriate and useful definition of "almost symmetric" would be, it seems that it would be helpful to use Hamiltonians that can be written as a linear shift of a symmetric one, for example (3.93):

$$H = \frac{\omega_0}{2}\left(p^2 + x^2\right) - \lambda xp = H_0 - i\frac{\lambda}{2}.$$

More on Dynamical Invariants

The author is currently working on finding a dynamical invariant for the following Hamiltonian:

$$H = a(t)\, p^2 + b(t)\, x^2 + c(t)\, px + d(t)\, xp + f(t)\, x + g(t)\, p. \tag{5.51}$$

It is the author's belief that the invariant will be related to the invariant of

$$H = a(t)\, p^2 + b(t)\, x^2 + c(t)\, px + d(t)\, xp \tag{5.52}$$

in a fashion similar to that of (5.2)and (5.27).

Nonlinear Mimicking

It has been suggested by Dr. Carlos Castillo-Chávez that the Hamiltonian (1.10) can be used to study Nonlinear problems by using time-dependent coefficients as "mimic" functions, in the sense that such functions should mimic the mathematical, and more importantly, the physical behavior of the nonlinear term. The author has reason to believe that existence of appropriate mimic functions would depend solely on the physical scenario that is to be modeled. The author is currently looking into the literature to find any papers that have attempted similar ideas.

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