Essays in Macroeconomics

by

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ABSTRACT

This dissertation consists of two parts. The first part is about understanding the mechanism behind female labor supply movement over economic development. Female labor force participation follows a U-shape pattern over per capita GDP cross nationally as well as within some countries. This paper questions if this pattern can be explained through sectoral, uneven technological movements both at market and at home. For that I develop a general equilibrium model with married couples and home production. I defined multiple sectors both at home and in the market. And by feeding the model with uneven technological growth, I observe how participation rate moves over development. My results indicate that a decrease in labor supply is mainly due to structural transformation. Meaning, a higher technology in a large sector causes prices to go up in that sector relative to other. Hence, labor allocated to this sector will decrease. Assuming this sector has a big market share, it will decrease the labor supply. Also, I found that the increase in female labor supply is mostly because of movement from home to market as a result of a higher technological growth in the market. The second part is about developing a methodology to verify and compute the existence of recursive equilibrium in dynamic economies with capital accumulation and elastic labor supply. The method I develop stems from the multi-step monotone mapping methodology which is based on monotone operators and solving a fixed point problem at each step. The methodology is not only useful for verifying and computing the recursive competitive equilibrium, but also useful for obtaining intra- and inter-temporal comparative dynamics. I provide robust intra-temporal comparative statics about how consumption and leisure decisions change in response to changes in capital stock and inverse marginal utility of consumption. I also provide inter-temporal equilibrium comparative dynamics about how recursive equilibrium consumption and investment respond to changes in discount factor and

production externality. Different from intra-temporal comparative statics, these are not robust as they only apply to a subclass of equilibrium where investment level is monotone.

ACKNOWLEDGMENTS

Pursuing a doctorate degree is more of a life story than anything. When you commit to invest six years of your life into a degree, that requires lots of sacrifices as well as taking many risks. When I first came to the US, I never thought I was going to go through all the things I went through. As most of other PhD doctorate students, I was expecting the best. Yet, life happens and at the end of the day the most important thing is your takeaway from whatever happened. I am just grateful that at the end of this journey, my takeaway is not only a degree but also lifelong friendship of great people, many life lessons as well as a much stronger personality.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

LITERATURE REVIEW AND EMPIRICAL EVIDENCE ON FEMALE LABOR SUPPLY

1.1 Introduction

The relationship between economic growth and female labor supply has been one of the fundamental questions in economics literature. As there is not a one sided causality between these two factors, we see the studies on this relationship have been divided into two.

A great deal of literature has been invested upon how female labor supply affects the economic growth both theoretically and empirically. Theoretical studies focus on developing a new model to explain the affects of female labor supply on economic growth as the exogenous growth model could not capture the structural changes in labor force. In an exogenous growth model with production, the factors of production are capital and labor. In these models, labor is an exogenous variable which is measured by the population size and the population size does not change drastically over time. Hence, capital accumulation is the only source of growth in most exogenous models such as Solow (1956). After the introduction of endogenous growth models, we have been introduced to human capital which is an endogenous variable and captures several other intangible assets of individuals such as knowledge, experience, education level, health status etc. With this third input factor, economists could explain how education, for instance, can increase the economic growth even the population size does not change (Barro and Sala-i Martin (1995)). This major step in theoretical literature lead to many empirical studies which help economists to understand the effect of female labor supply on growth.

With the help of endogenous growth models, many empirical studies focus on certain changes in women's life such as education or health to understand their changing labor supply decisions as well as its impact on economic growth. In this sense; Knowles, Lorgelly and Owen develop a neoclassical growth model with gender specific education in production technology. Their theoretical and empirical findings states the educational gender gap creates an obstacle in economic growth and increasing female education promotes labor productivity (Knowles *et al.* (2002)). Dollar and Gatti (1999) confirm the similar findings in developing countries. Low investments on women's education and health is a bad choice for economic growth. Moreover, Guner *et al.* (2011) show how individual taxation raises the lfp of married women. Klasen (2000), Klasen (2003) and Klasen and Lamanna (2009) are some of other empirical studies about this strand of literature. Hence, the positive effects of higher female labor supply on economic growth have been largely studied.

On the other hand, there are also some studies which claim that gender inequality contributes to the economic development of a country. In his study on semi-industrialized export-oriented countries, Seguino finds that GDP growth and gender wage inequality has a positive relationship. The reason behind this controversial result is that in these countries, employers are faced with international competition so they have limited bargaining power on their wages. As the majority of workers are female, they are paid little for their productivity. Seguino (2000) claims the sources of growth in these economies are technological imports and investment. Gender inequality stimulates investment hence causes economic growth. Lastly, Barro and Sala-i Martin (1995) find educating women has a negative effect on economic growth. However, their estimations have been criticized a lot due to inadequate consideration of collinearity and endogeneity problems by Dollar and Gatti (1999) and Knowles *et al.* (2002). In fact, the main problem in their paper is using GDP growth rates instead of GDP per capita as dependent varable. As we already know from Solow (1956), poor countries in general have higher growth rates as opposed to richer countries. Hence, using GDP growth as the endogenous variable causes them to misinterpret their results and they falsely conclude that low female education leads to higher growth. To sum up, although there are some studies which talk about negative effects of female labor supply on economic growth, they have been criticized a lot due to several technical problems. Today, most economists agree that increasing female labor supply positively affects economic growth of a country. Hence, we see several papers in economics literature which have been searching for new policies to increase female labor supply. Health reforms, education, maternity or paternity leaves and tax reforms are a few of these family-oriented policies which aim to increase labor supply of women because it will eventually contribute to economic development.

Although the positive impact of labor supply on economic growth has been established in economic history, the disagreement about the effects of economic growth on female labor supply is ongoing. Theoretically, one part of the literature claims that economic growth will eliminate the discrimination in the economy and any sort of gender gap will disappear. This line of thought implies economic growth increases female labor supply. Becker (2010) states that discrimination does not prevail in a competitive market in the long run as optimal decision making requires agents to take actions which maximize their profit or utility rather than taking actions according to their prejudices against one group. This is knows as "Modernization of Neoclassical Approach". He also adds that a persistence discrimination against a particular group can be explained by introducing a preference for discrimination into objective function. Hence, in a competitive market, if there is a discrimination, this must be a preference, market friction, otherwise any discrimination will disappear as explained above.

Becker's insights about discrimination leads to several studies about this topic. Some of these studies concentrate on the harmful effects of discrimination on the economy whereas the others focus on the policy side of the story. Dollar and Gatti (1999) suggest that market expansion eliminate all sorts of market imperfections one of which is women-specific investments. Similarly, Weber (1978) and Durkheim (1964) note that gender-discriminatory practices prevents a healthy market mechanism. On the other hand, a vast majority of studies focus on the policy aspect of economic development and its effects on women. Higher economic development means better health provision for women. Considering women are most likely to die in childbearing years than men due to pregnancy and giving birth, Jayachandran and Lleras-Muney (2009) find that a sudden drop in maternal mortality thanks to an effective public policy leads to a rise in life expectancy of girls as well as a convergence in the education level of boys and girls. Hence, these and many other studies support modernization of neoclassical approach and provide several aspects of this idea. However, as mentioned above, the positive impact of economic development on female labor supply is not set in stone.

While the modernization of neoclassical approach assumes a positive effect of economic development on female labor supply, or women's empowerment in general terms, a vast literature claims a convex impact. The idea goes back to Boserup (1970) who argues that there is a convex relationship between female labor supply over the course of economic development. Growth initially causes a fall in female labor supply and then a rise as the economy continues to grow. He explains this phenomena in three stages: The first stage is observed in low income or developing countries which have large agricultural sector. In these economies, most women work in family farms as agricultural workers and contribute into production apart from home production. As they work at home, their fertility rate is high. In agricultural societies, women give many births as the number of children raises the family income by participating in agricultural production. As households mostly include extended family members such as grandparents, the small children are taken care of by them. Hence, women as well as adult children have time to work at the family enterprises. Many African countries such as Sierra Lione, Kenya can be counted in this stage. In the second stage, female labor supply starts falling as the country steps into industrialization stage. Reduction of agricultural sector and labor mobility as a result of mechanization makes it more difficult for women to combine market and house work. Especially, when it is combined with the required physical strength to use machines in early industrialization and stigma towards working women outside of their family enterprises, the demand for female workers decreases. Third stage is when female labor supply rises in developed countries due mostly to the mechanisms known from modernization of neoclassical approach. Rise of the service sector and consequently the rising demand for mental skills rather than motor skills increases the demand for female workers. Many developed countries such as United States, Britain and France can be counted in this stage.

1.1.1 U-Shape in Cross Country Data

Figure 1.1 summarizes the relationship between female labor force participation (LFP, hereafter) and gross domestic product (GDP, hereafter) per capita in 2012. The data is collected from World Development Indicators. Both female LFP and GDP per capita are 5-year averages from 1965 to 2012 to be consistent with the previous literature. Finally, each dot on the figure displays a country. The quadratic fitted line displays a clear U-shaped trend over economic development which is statistically significant. This U-shaped trend persists itself over the decades up until 1970s and it is always statistically significant.

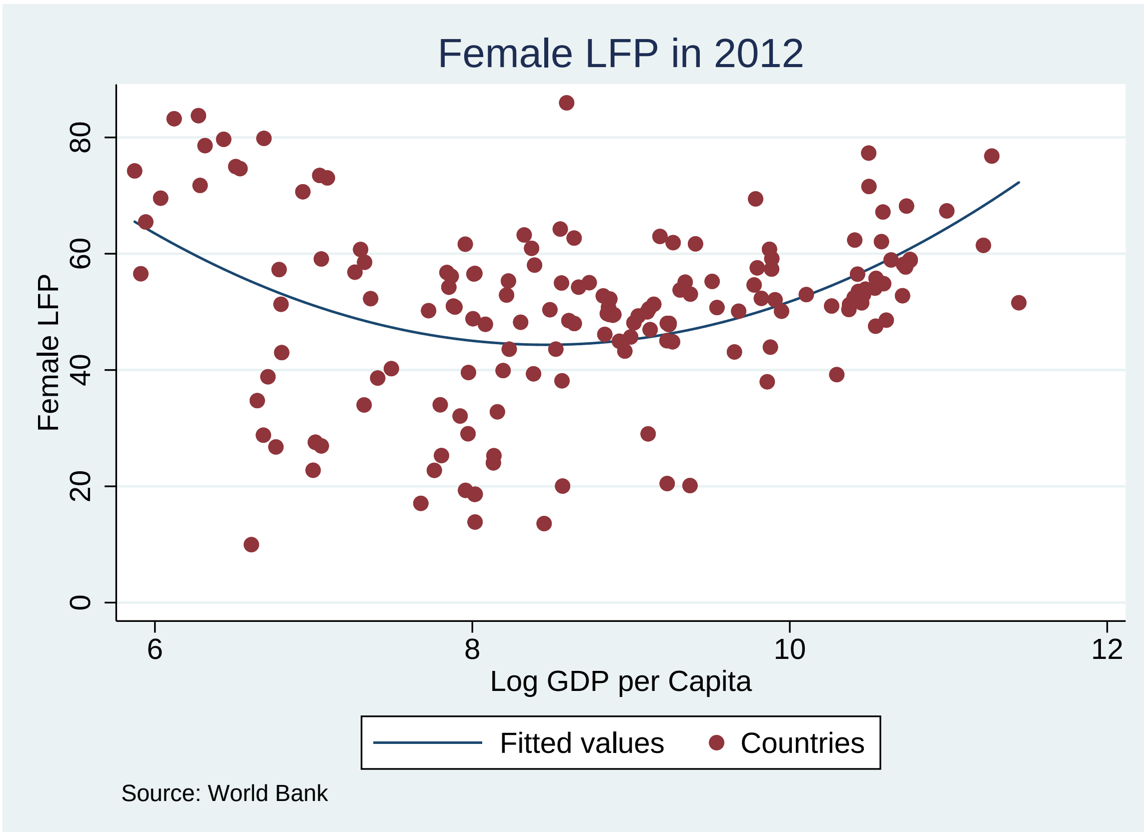

Figure 1.1: Female LFP in 2012

Several other empirical studies also proves the existence of U-shaped. Olivetti (2013) shows the existence of U-shaped labor force participation over 16 selected developed countries. She developed a panel data set for 16 high-income countries that contains comparable data on LFP for the population aged 15 and over for the period 1890 through 2005. These high-income countries includes: Australia, Belgium, Canada, Denmark, France, Finland, Germany, Ireland, Italy, Netherlands, Norway, Portugal, Spain, Sweden, United Kingdom and the United States. The data is from

1890 to 2005 and after 1900, it is at 5-year intervals. I replicate this analysis and confirm that U-shape female LFP is persistent for these countries as well. Figure 1.2 displays the replication results.

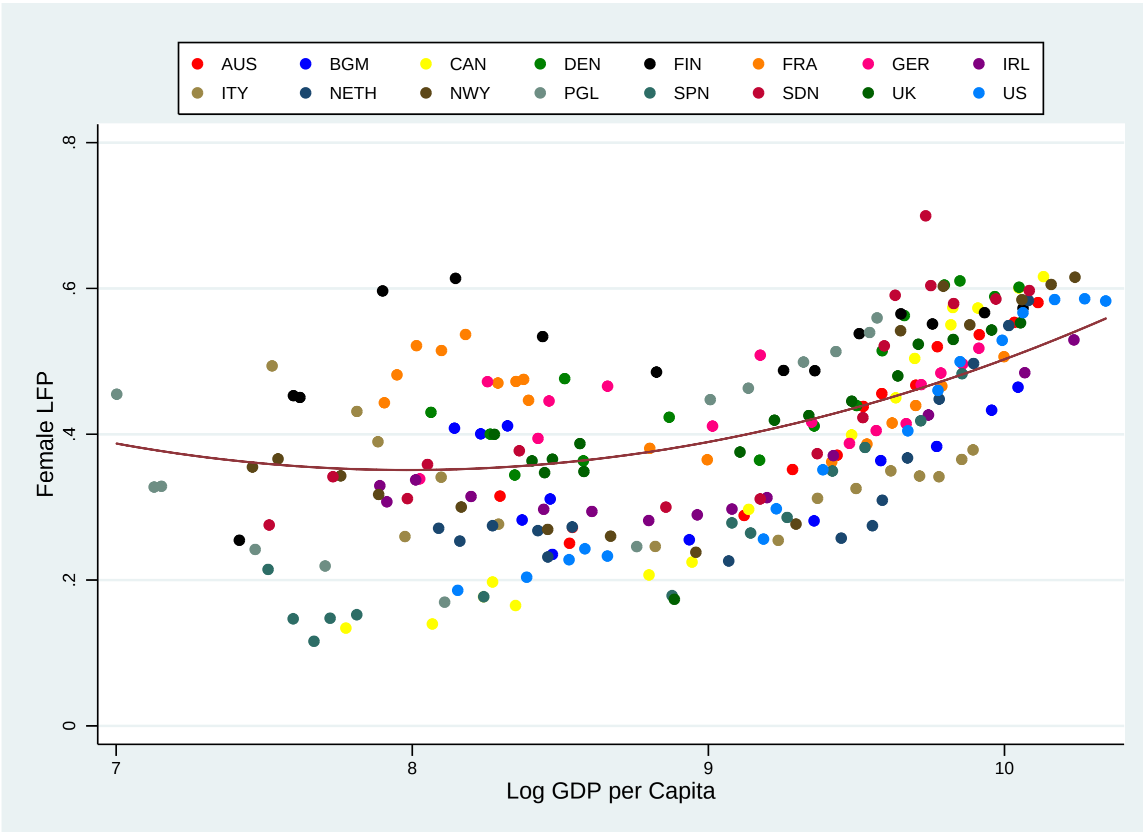

Figure 1.2: Female LFP and Economic Development, 1890-2005

Olivetti (2013) also notes that when the early OECD economies are not included, the U-shaped LFP is more muted. The reason behind this is two fold. One reason is there was a stigma towards working women in early industrialization countries and the second reason is that women have more comparative advantage in type of jobs at which they can use their mental power instead of physical power. Olivetti describes this phenomenon as women's dislike towards 'brawn jobs' as opposed to brain jobs.

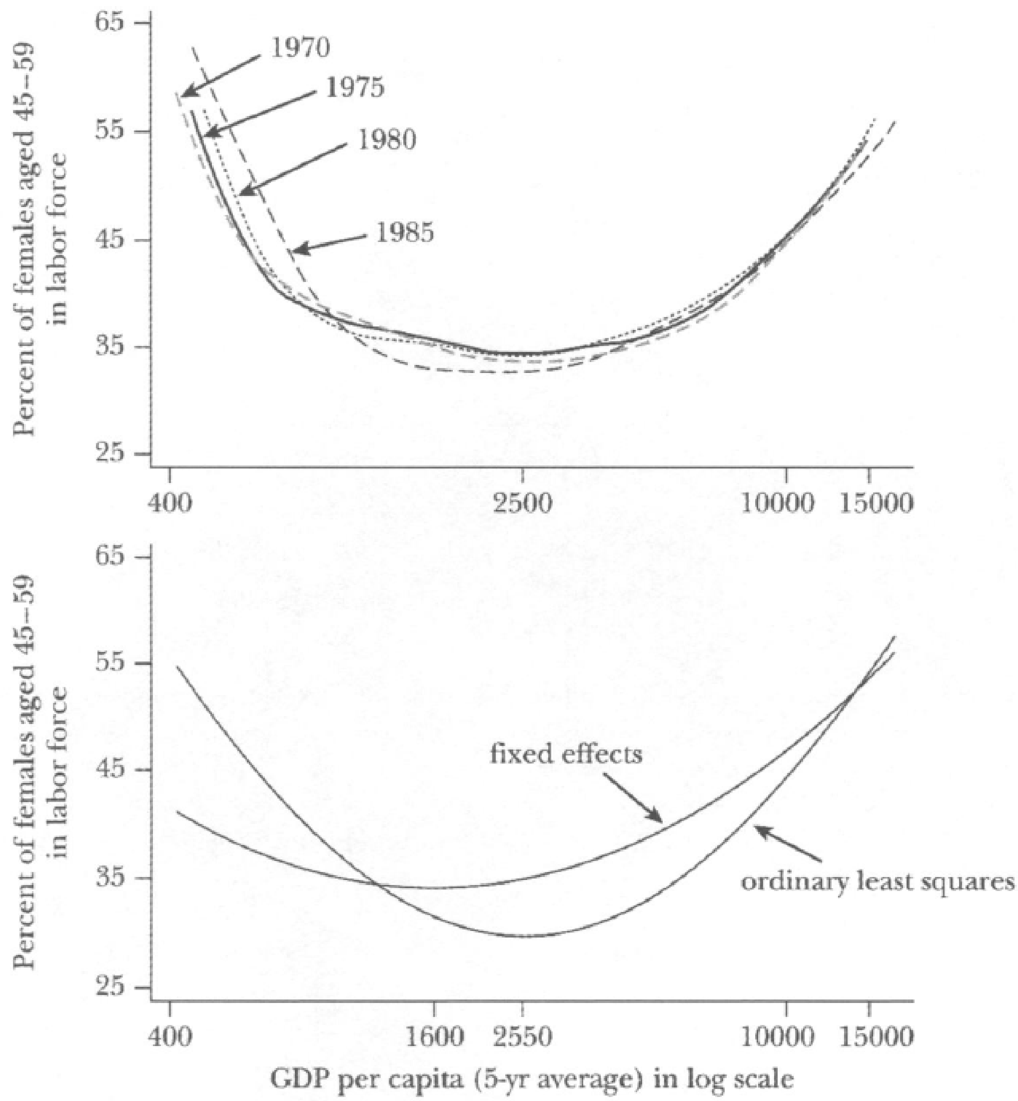

Figure 1.3: Mammen and Paxson, 2000

There is further evidence that the relationship between labor force participation and economic development is U-shaped in several cross-country studies as well. Mammen and Paxson (2000) examine 90 countries from 1970s to 1980s and found that labor force participation displays U-shaped over economic development. Their graph is in figure 1.3. Luci (2009) uses a larger panel data set to eliminate the possible endogene-

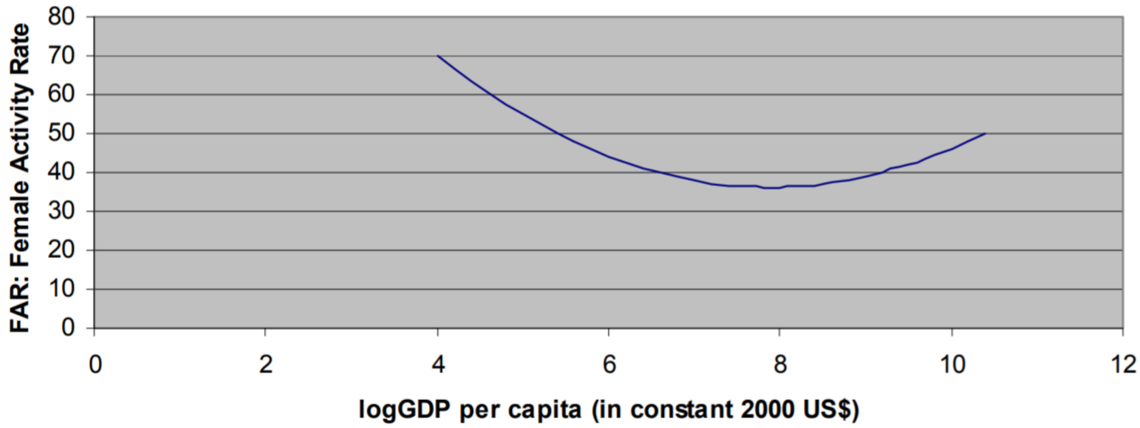

Figure 1.4: Luci, 2009

ity between economic growth and female labor force participation. She conducts her research on 184 countries from 1965 to 2005 and performed system GMM estimation. As a result, she also confirms the U-shaped participation rate over economic development. The depiction of her results can be seen in figure 1.4. Lastly, Olivetti (2013) shows additional evidence by examining 16 developed countries and confirms the U-shaped labor force participation. She also discusses whether a 'country's transition to a modern path of economic development' affect the labor force participation.

1.1.2 U-shape in Time Series

Goldin (1990) argues that in the late nineteenth and early twentieth centuries, married women's labor force participation rate was U-shaped over economic development in the United States. Although official statistics cannot fully capture the U-shaped, Goldin (1990) shows that 1890 participation rate was as high as 1940 rate and it reached a trough around 1920s. Since 1920s, the female participation rate has been rising. This rise was fast initially but later it slowed down over time.

The U-shaped female labor force participation is a claim as striking as it seems but the reason behind this U-shaped is more appealing. Goldin talks about three biases that pre-1940 data has [1]: Change in definition of labor force, change in the locus of production in the economy (Production moves from home to market.) and omission of workers in the census count (Goldin (1990)).

Before 1940, individuals are included in the labor force under the gainful worker concept. An individual is gainfully employed if she reports herself as a paid worker. After 1940, labor force is defined to be economically active population as in International Labor Organization (ILO, henceforth) construction. According to this definition, labor force includes all individuals working for pay, unpaid family workers and unemployed people looking for jobs during the survey week. Once Goldin corrects the definition of labor force before the 1940 levels, she finds that labor force participation of married women does not remarkably change.

The second bias is the change in the locus of production in the economy. According to this, as the main locus of production shifts from home to market, more and more women started looking for jobs in the market. This line of thought essentially focuses on the transition of women we were out of labor force into labor force as unemployed. Goldin questions if this might be the root of U-shaped labor force participation. However, her calculations did not reveal a remarkable change in the shape of labor force participation of married women.

The third bias is the most serious one pre-1940 data has. Especially in the late 19th century and early 20th century, large fraction of working women were reluctant to report their employment or census takers were biased against reporting womens employment although they were engaging in unpaid employment within their families excluding homemaking activities. Conk (1980) provides evidence that most census

[1]Section 2, Economic Development and the Life Cycle of Work

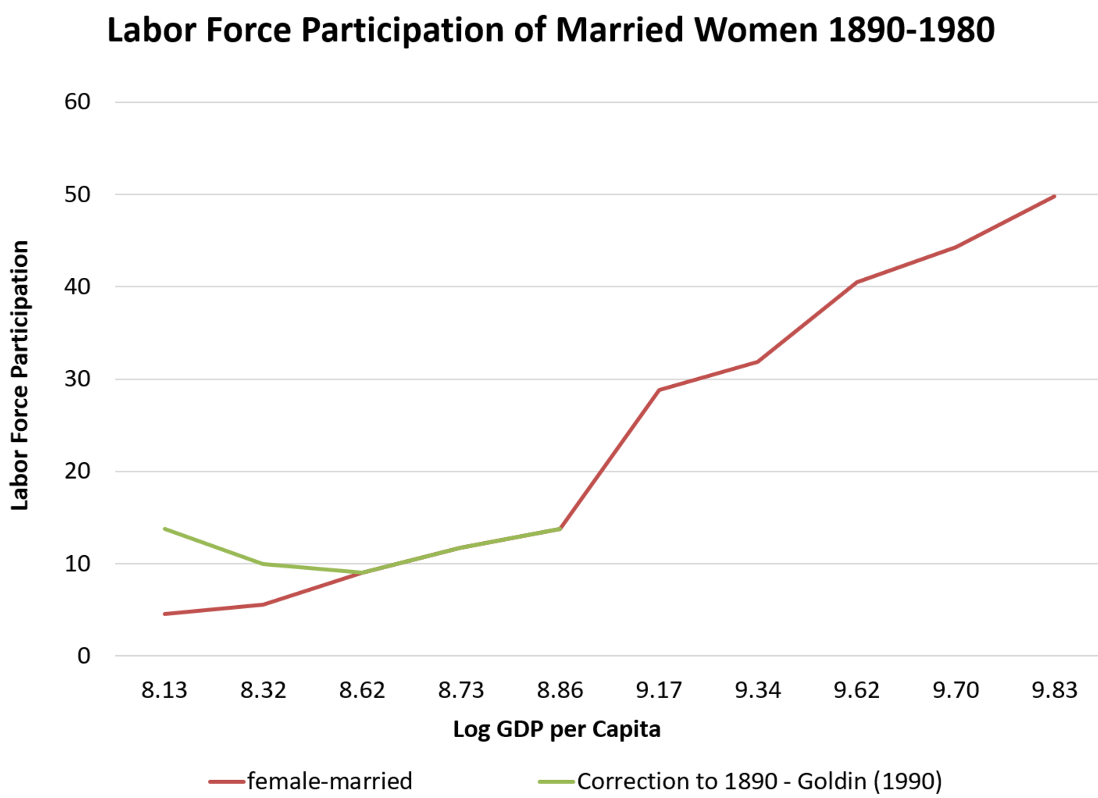

Figure 1.5: Labor Force Participation of Married Women and Economic Development, 1890-1980, US Population Census 1960 and 2010. Mitchell(1998) for GDP per Capita

takers mostly presumed that married and adult women are unemployed and altered data if occupations were unusual and atypical of female jobs. There are especially three occupations where people tent to misreport their employment status: Boarding-house keepers, agricultural workers and manufacturing workers. When Goldin (1990) adjustments of these workers and added to the labor force data, she found that LFP of women in 1890 was equal to or above 1940 participation rate of white women. Figure 1.5 displays this correction along with the original data points. As it is seen, the labor force participation of married women is, in fact, U-shaped between the years 1890 and 1940.

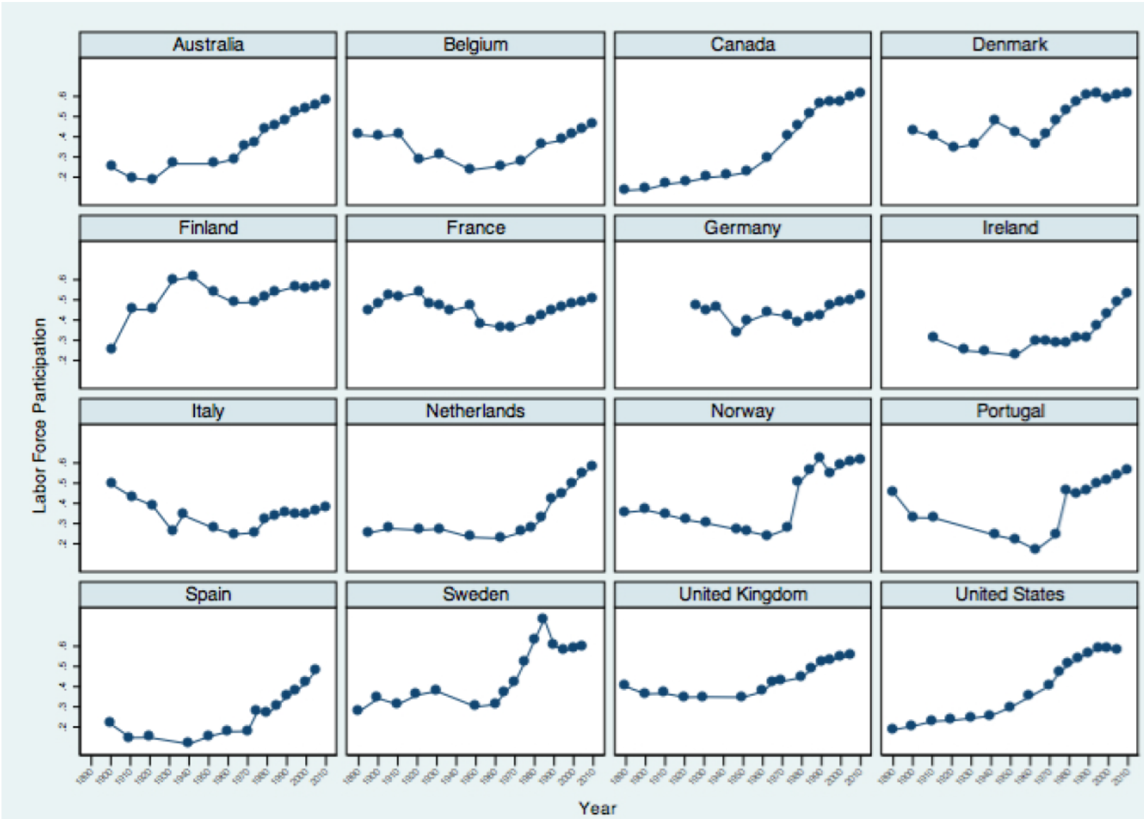

Figure 1.6: Trends in female labor force participation, 1890-2005. Olivetti (2013)

In the light of Goldin's work, Olivetti (2013) shows that 16 developed countries female labor force participation over economic development between 1890 and 2005. Apart from Canada and the United States, most countries show U-shaped female LFP although the U-shaped is more muted in some cases. Canada and the US displays an increasing trend but as mentioned earlier, US case is somewhat problematic. Figure 1.6 displays these countries.

Recent literature shows that many developing countries explain the decrease in their female LFP over economic development with U-shaped trend. Figure 1.7 displays the movement of female LFP between 1955 and 2000 in Turkey. Female LFP declined 72 percent in 1955 to about 26 percent in 2000. Tansel (2002) explains this decrease by linking it to U-shaped LFP. She asserts that there will be an upturn

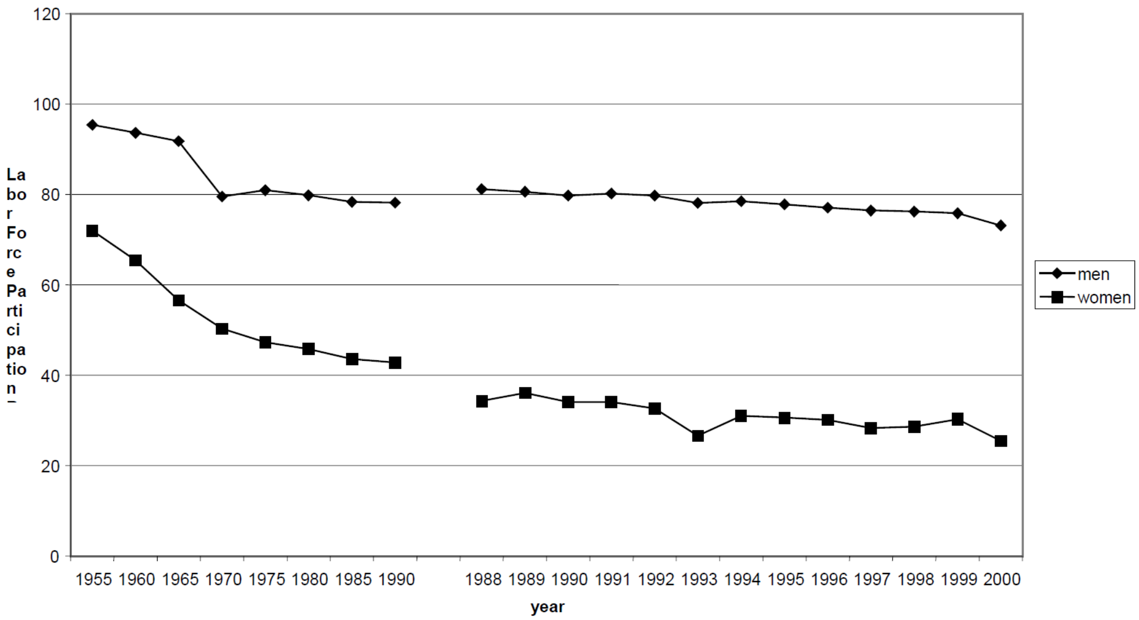

Figure 1.7: Female labor force participation in Turkey, 1955-2000. Tansel (2002)

in female LFP in the upcoming decades because the decline in the female LFP has slowed down significantly.

Lastly, 1.8 state that structural transformation pushes a lot of women out of agricultural sector while the growing mechanization in agriculture and manufacturing limit the opportunity for these women to find jobs again because of their low education and skill level. They conclude this is the main reason behind India's decreasing female LFP in the recent years.

1.1.3 Conclusion

The movement of female labor force participation over economic development has divided the literature into two. While one part of the literature follows Becker (2010) and claims an increasing trend, the other part follows Boserup (1970) and claims a U-shaped. Although it has not received as attention as Becker's increasing

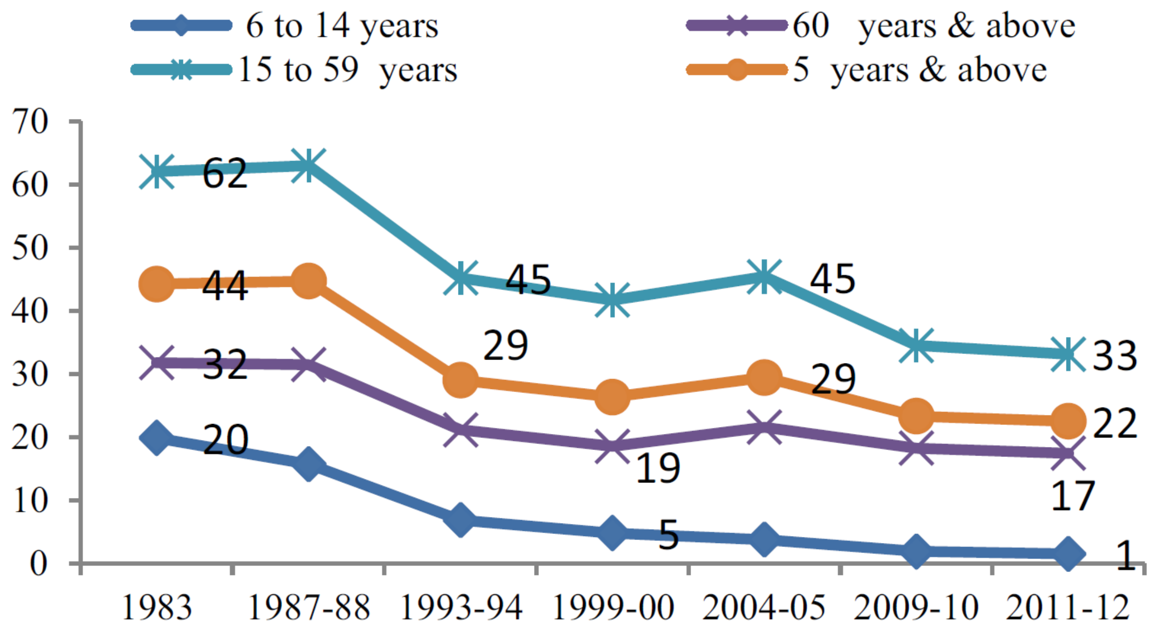

Figure 1.8: Female Labor Force Participation, 1983-2012. Mehrotra and Parida (2017)

LFP trend, Boserup's U-shaped female LFP has been widely studied and proved in many empirical studies. Both cross country and time series analysis show that U-shaped female LFP is an important phenomena which deserves to be studied more to understand the relation between economic development and female labor supply.

Chapter 2

STRUCTURAL TRANSFORMATION, MARKETIZATION AND FEMALE LABOR SUPPLY

2.1 Introduction

The causal relationship between female labor force participation (LFP, hereafter) and economic development have been one of the fundamental questions in economic history. Although the economists largely agree on the positive effect of female LFP on economic development, the disagreement about the impact of economic development on female LFP is ongoing. While Becker's Modernization Neoclassical Approach assumes a positive effect of economic development on female LFP, there is a vast amount of literature which claims a curvilinear impact since Boserup (1970). Growth initially reduces female LFP and then increases as the economy continues to grow. This U-shaped structure of female LFP over economic development is the main topic of this paper.

Figure 2.1 displays the 161 countries' female LFP and logarithmic GDP per capita paired together with the quadratic fitted line which is U-shaped. The data is taken from Word Development Indicators (WDI, hereafter). The data points are only from 2014 data but the same shape prevails in almost every decade since 1980.

The U-shaped female labor force participation has two stages. The first stage is the decreasing part which is generally observed in low income or developing countries. According to Boserup, most women work on family farms as agricultural workers and contribute into production apart from home production. Since they work at home, their fertility rate is high. In agricultural societies, women have high reproduction

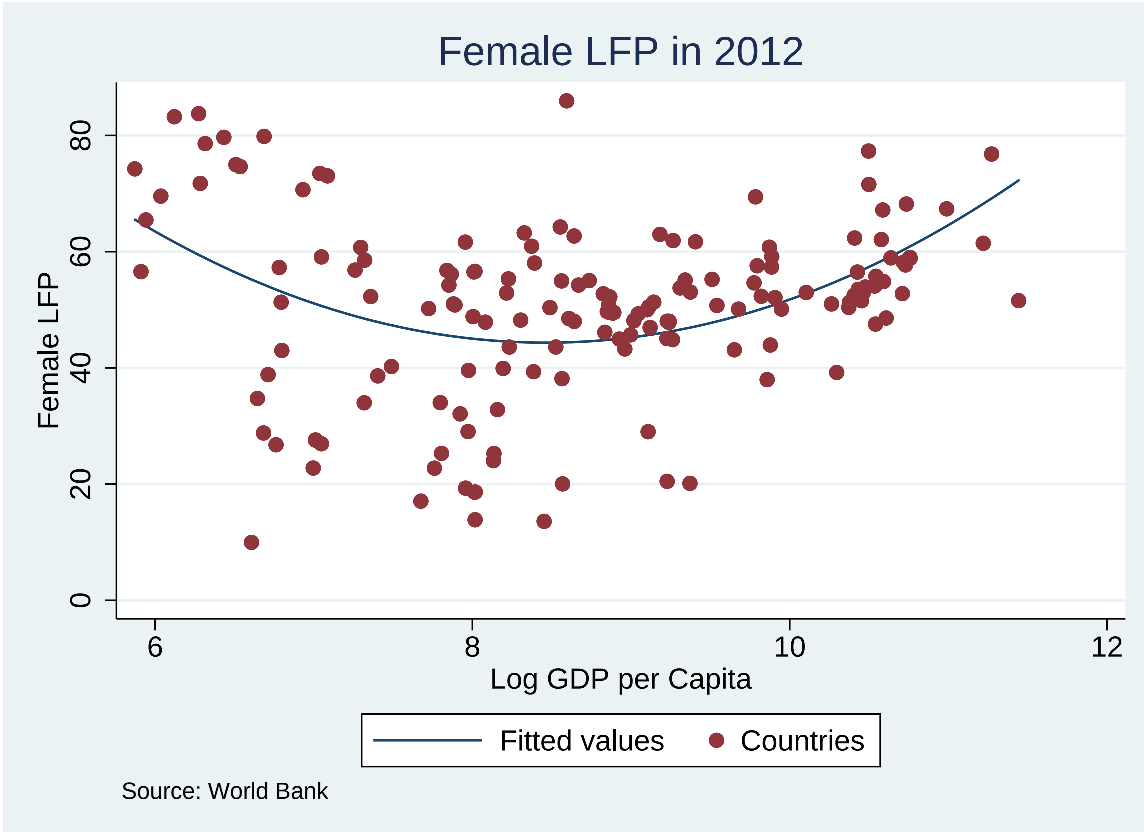

Figure 2.1: Female LFP in 2012

rates because more children raise the family income since they can participate in agricultural production. Many households include extended family members such as grandparents. Since the small children are taken care of by them, all the other family members have time to work at the family enterprises. Figure 2.2 provides a detailed overview of Figure 2.1. I define countries whose agricultural production share in its GDP is in the top 20 percent of the world agricultural production share distribution. Figure 2.2 displays that all those countries accumulate on the left side of the graph where the fitted LFP decreases over economic development.

Female LFP begins to fall as the country steps into the industrialization stage. Reduction of agricultural sector and labor mobility as a result of mechanization makes

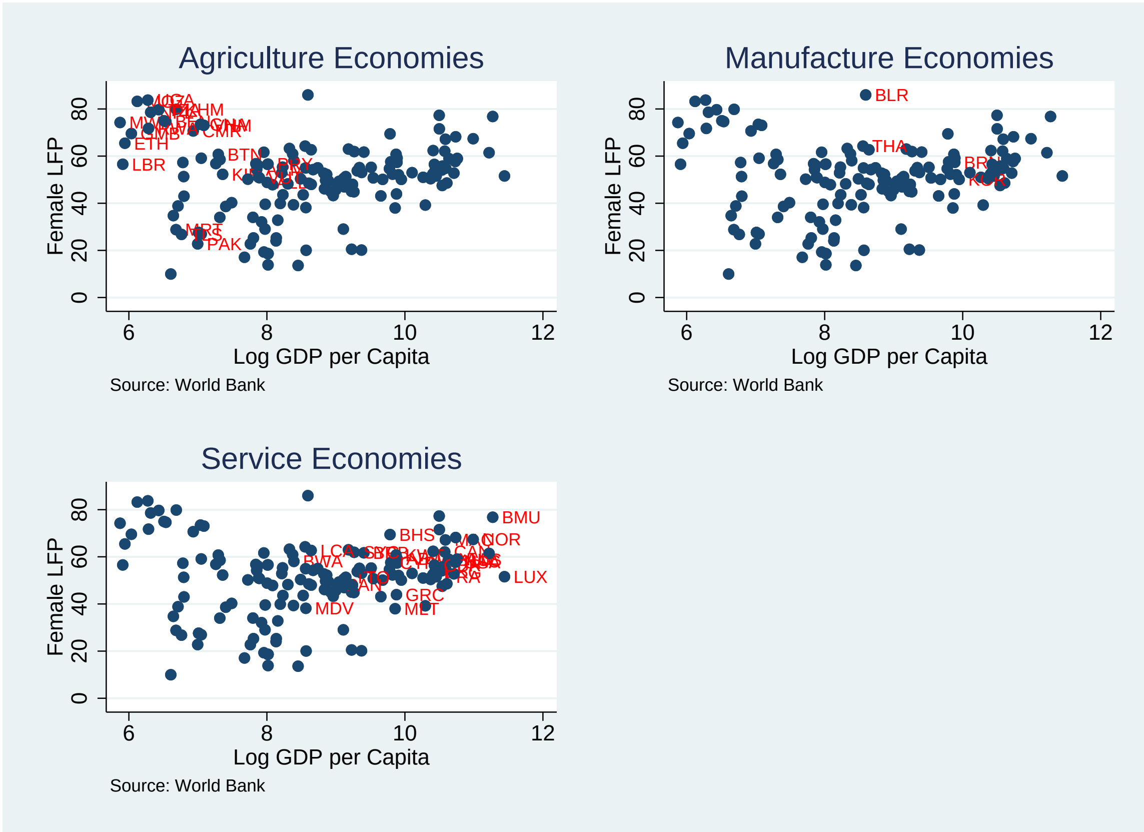

Figure 2.2: Sectoral Distribution of Countries

it more difficult for women to combine market and home work. The demand for female workers decreases when it is combined with the required physical strength to use machines in early industrialization and stigma towards working women in these societies. Ngai and Olivetti (2015) define this period with a lot of "brawn" jobs versus 'brain' jobs, clashing women's lack of comparative advantage on these brawn jobs. The graph on the top right corner at Figure 2.2 shows the countries whose manufacture production share is in the top 20 percent of the world distribution. They accumulate on the mid section of the graph.

With further advancement in technology and rise of the service economy, female LFP starts rising again in the second stage. This stage is when female LFP rises in developed countries due mostly to the mechanisms known from modernization of neoclassical approach. Rise of the service sector and consequently the rising demand for mental skills rather than motor skills increases the demand for female workers. Many developed countries such as United States, Britain and France can be counted in this stage. The graph on the bottom left corner at Figure 2.2 shows the countries whose service production share is in the top 20 percent of the world distribution. They accumulate on the right side of the graph.

Hence, U-shape female LFP over economic development basically talks about changing labor supply decisions of women as well as labor demand decisions of employers given women's natural skill set and how these skill sets are demanded conditional on the changes in the economy. Ngai and Olivetti (2015) propose a theoretical framework to create the U-shape in their '*very preliminary*' paper. However, their study do not go beyond defining a model. This study completes this attempt by extending their theoretical framework and as well as providing quantitative analysis.

For this purpose, I focus on three forces which together can alter the movement of female labor force participation. First one is the transition of women from home to market. Late nineteenth century was different from the twentieth century in two aspects. First, there was a stigma towards married women working outside. Second, women were mostly working as agriculture workers both in family enterprises and market excluding house chorus. With the industrial revolution and decreasing prejudice towards married women working in the market, more and more women start to move from home to market as workers. This transition will be named as "*marketization*" in the paper following Ngai and Petrongolo (2014). However, the marketization does not necessarily increase the working hours of married women in every sector equally which takes us to the second force.

The second force is the transition of workers between different sectors. Women have comparative advantage over brain jobs as opposed to brawn jobs so their working hours increase with the expansion of the sectors which they have comparative advantage. If the expansion is in the sector in which they do not have comparative advantage, then their working hours decreased. Hence, it creates a reallocation of labor within sectors depending on their skill set. This process will be called "*structural transformation*" following Herrendorf *et al.* (2013).

The third force is the allocation between work and leisure time. This is a choice for women to engage in production overall or just having some leisure hours. Both the marketization and structural transformation forces are challenged/propagated by this force. A higher technology may cause women to supply more labor to the market which is the result of marketization. This move is propagated by leisure-work choice since women are also willing to sacrifice their leisure time in addition to their home working hours. On the other hand, if the technological growth is in a sector which they do not have comparative advantage, then they decrease their labor allocation even more to increase not only home production but also leisure. This force is called "*Leisure-work choice*" in this paper.

Related Literature

One strand of the related literature is the change in female labor supply over economic development. This paper supports a curvilinear shape of female lfp over economic development hich was first claimed by Boserup (1970) and supported by many other studies such as Goldin (1990), Luci (2009), Olivetti (2013).

This paper is also closely related to Ngai and Olivetti (2015) and Ngai and Petrongolo (2014) in terms of analyzing the changes labor supply between sectors. In that sense, my model is very similar to the models in these two papers. Similar framework has also been used by many other papers such as Akbulut (2011), Ngai and Pissarides (2008).

Therefore, this paper tries to generate U-shaped female labor force with these three forces. The point of this paper is to mechanically and quantitatively show that marketization, structural transformation and leisure-work choices are able to create U-shaped female labor force participation. Section 2.2 introduces the data. Section 2.3 provides empirical evidence. Section 2.4 presents the model. Section 2.5 supplies calibration of the model. Section 2.6 states the calibration results. Finally, Section 2.7 concludes.

2.2 Data

The data on female labor force participation, GDP and sectoral value added shares used in the Section 2.1 are taken from World Development Indicators (WDI, hereafter). Labor force participation is defined as the proportion of economically active female population to the entire female population who are 15 and higher. GDP is defined as the sum of gross value added by all resident producers in the economy plus any product taxes and minus any subsidies not included in the value of the products. Value added is the net output of a sector after adding up all outputs and subtracting intermediate inputs. It is calculated without making deductions for depreciation of fabricated assets or depletion and degradation of natural resources. The origin of value added is determined by the International Standard Industrial Classification (ISIC), revision 3. Agriculture corresponds to ISIC divisions 1-5 and includes forestry, hunting, and fishing, as well as cultivation of crops and livestock production. Manufacturing refers to industries belonging to ISIC divisions 15-37. Services correspond to ISIC divisions 50-99 and they include value added in wholesale and retail trade (including hotels and restaurants), transport, and government, financial, professional, and personal services such as education, health care, and real estate services."[1]

For calibration of model parameters, I use US data from Integrated Public Use Microdata Series (IPUMS-USA, hereafter). The data includes people with their age, sex, marital status, their employment and labor force status, their occupation and information about which industry and occupation they work. I only used married women observations because single women almost act like men in the data. The data is between 1965 and 2012. Observations before 2000 is decennial while it is annual after 2000. It must be because of American Community Surveys designed to replace the Census long-form after 2000. I calculate sectoral labor force statistics of both genders.

The individuals are between 16 and 65 and they are all married (spouse present). They are all employed. I include both the full and part time employed individuals because the scope of this study is not related to job type.

I categorize industries documented in "IND1950" into three broad sectors. This categorization is similar to WDI categorization: agriculture, manufacture and service. The classification for each sector is made by using IND1950 variable. Agriculture sector includes *Agriculture, Forestry and Fishing*. Manufacturing sector includes *Mining, Construction* and *Manufacturing* goods. Service sector includes *Transportation, Communication, and Other Utilities*, *Wholesale and Retail Trade*, *Finance, Insurance, and Real Estate*, *Business and Repair Services*, *Personal services*, *Entertainment and Recreation Services*, *Professional and Related Services* and *Public Administration*.

I also calculate sectoral labor allocation at home for both genders. In this sense, I used Multinational Time Use Survey (MTUS) for the United States [2]. It is composed

[1] All of the definitions are taken from WDI.

[2] This document presents results drawn from the Multinational Time Use Study (MTUS), but the

of diaries of people from each country. Each diary includes how many minutes an individual spends on a particular activity such as working, travel, sleeping or leisure time. Each row in the dataset represents for one person's 24 hours.

The data is from 1965 to 2012. I only use the diaries which are good quality (badcase=0). The individuals are again between 16 and 65 and they are all married. All the individuals are either part-time or full-time employed.

Before proceeding further, it is important to define market, home production and leisure. Market production is the time spent on the production of goods and services. Individuals derive utility from the output by consuming or selling it. Also, the time spent is paid. Home production is defined as the time spent on the production of goods and services. Different from leisure, individuals derive utility from the output of the production but not the time spent on production and home production output can be 'marketized' unlike leisure. From here, it is already clear that leisure consists of activities which individuals derive utility from the time spent on that activity and they cannot be marketized (Ngai and Pissarides (2008)). Given these, the classification of MTUS activities based on these definitions is as follows.

The market work time includes paid work as the main or second job that takes place in the market. I do not include travel to work time as part of the market work. Sectoral categorization in the market are made according to IPUMS categorization although the the items listed for each industry are not entirely the same. I use the variable *occup* for this classification and I try to choose the similar categories with IND1950 in IPUMS-USA for each sector.

interpretation of this data and other views expressed in this text are those of the author. This text does not necessarily represent the views of the MTUS team or any agency which has contributed data to the MTUS archive. The author bears full responsibility for all errors and omissions in the interpretation of the MTUS data.

Home production time includes paid work that takes place at home as well as other activities that are not paid. The reason behind adding unpaid activities is all of these activities would be a paid job in the market if not done at home. Hence, I think including them as part of home production is significant. In this sense, gardening and fishing is in home agriculture [3]. Manufacturing includes cooking, home and vehicle maintenance and knitting [4].Finally, service sector includes all the other house chores such as cleaning, child care, shopping or pet care [5]. I agree that some of these activities such as cooking, fishing or knitting can be counted as leisure activity as some people like to do these as hobby. In my analyses, I choose to ignore this fact as it is hard to distinguish it from the data.

Lastly, leisure time includes activities such as party with friends, social events, sports and exercises [6]. I exclude sleeping time from my classifications.

2.3 Evidence

The regression is a cross country analysis which includes GDP per capita and labor force participation for both genders in 2012. The female data includes 161 countries, whereas the male data includes 35 countries. The aim of this analysis is to formally prove the existence of U-shape female LFP over economic development for female workers as well as compare it with their male counterparts.

I run the following regression equation for both genders separately to tests if the labor force participation of women actually follows a U-shape:

$$flfp_t = \beta_1 + \beta_2 gdppc_i + \beta_3 gdppc_i^2$$

[3] main46

[4] main18, main22, main54

[5] main19, main20, main21, main24, main25, main27, main28, main29, main30, main31, main32, main47, main66, main67

[6] main17, main33-38, main40, main42-45, main48, main50-52, main55-50

Table 2.1, column (2) displays a statistically significant U-shape pattern for female labor force participation. On the other hand, column (3) does not display a statistically significant U-shape pattern even on the 10 percent *p value*.

Table 2.1: LFP and Economic Development, 2012

(1)	(2) Female	(3) Male
Log GDP per capita	-53.31***	-14.83
	(9.86)	(10.70)
Log GDP per capita squared	3.15***	0.81
	(0.55)	(0.63)
Constant	269.99***	139.75***
	(43.31)	(43.95)
N	161	35
R^2	0.18	0.16

* $p < .1$, ** $p < .05$, *** $p < .01$

Source: World Bank

A similar analysis on a panel dataset of 16 selected developed countries is done by Olivetti (2013)[7]. As the time period spans a wide range of years (1890-2015), labor force participation may depend on certain country-specific characteristics. Hence, she also adds country and year fixed effects. Results indicate that U-shaped female labor

[7]The countries are Australia, Belgium, Canada, Denmark, Finland, France, Germany, Ireland, Italy, Netherlands, Norway, Portugal, Spain, Sweden, United Kingdom and United States

force participation is statistically significant even with 1 percent p_value. On the other hand it is statistically insignificant for male workers (Olivetti (2013)).

2.4 Static Model

Motivated by the facts represented in the previous section, this section presents a general equilibrium model for a multi-sector economy to describe the male and female market and home hours. The proposed model in this section is static model with no capital. The only factor of production is labor hours and sector-specific labor productivity.

It should be noted that the proposed model solely relies on between-sectors forces to deliver gender-specific labor allocation trends. Hence uneven technological growth together with gender specific parameters will be the source of the labor supply decisions. In this sense, having a static model will be enough to generate a U-shaped LFP. Considering the fact that there is evidence about U-shaped female lfp using cross-sectional data as well as time series data (Goldin (1990)), the static version is a simple but a good way to begin the analysis.

A more sophisticated version of this model is to add capital and make the model dynamic to analyze life-cycle behavior of individuals. The dynamic version will also be important to attribute a source to labor productivity growth such as capital intensity or TFP. Moreover, we could also analyze how skill formation such as female education can change their life time choices when confronted with uneven technological growth. In the appendix, I describe the dynamic version of it with a note of the difference between the static and dynamic version. The readers can directly jump to Section B to see the dynamic model. For now, I will continue with describing static model.

2.4.1 Setup and Primitives

There are 6 sectors to allocate labor: Agriculture (a), manufacturing (m) and service (s) sectors in the market and at home. The set of sectors is denoted by $j \in J = \{a, m, s\}$

Households consist of one male ($\tilde{m}$) and one female ($\tilde{f}$) partners. I will denote the set of genders as $g \in G = \{\tilde{m}, \tilde{f}\}$. Both of them allocate resources between home and market sectors and engage in income pooling. Male and female individuals differ by the wage they receive in the market.

2.4.2 Production Technology

For simplicity, production technology is assumed to be as follows:

$$F(A_j, H_j) = A_j H_j \tag{2.1}$$

where A_j is the market production technology for sector j and H_j is the labor aggregator which is assumed to be as follows:

$$H_j = [\xi_j H_{\tilde{f}j}^{\eta} + (1 - \xi_j) H_{\tilde{m}j}^{\eta}]^{\frac{1}{\eta}} \quad \forall j \in J \tag{2.2}$$

where ξ_j is the share of female labor time on the production of market good j. Economically speaking, this can be attributed to comparative advantage of female workers in sector j. If $\xi_j > \xi_k$, this means women have comparative advantage in sector j as opposed to sector k. Also, $H_{\tilde{f}j}$ is the female hours in the production of market j-good, whereas $H_{\tilde{m}j}$ is the male hours in the production of market j-good. The elasticity of substitution between male and female working hours is denoted by $\frac{1}{1-\eta}$.

Representative firm in each sector j solves the following profit maximization problem at each period:

$$\max_{H_{mj},H_{fj}} p_j F(A_j, H_j) - w_{mj}H_{mj} - w_{fj}H_{fj} \tag{2.3}$$

2.4.3 Preferences

There is a representative household, which is made up of a male and female members. The household lives for an infinite number of periods and derives utility from consumption and leisure. The preferences are given by:

$$U(C, L) \tag{2.4}$$

which C represents the consumption of composite good and L is total leisure time. The utility function U is given by:

$$U(C, L) = \mu ln(C) + (1 - \mu)ln(L) \tag{2.5}$$

where μ is the weight of the consumption good in the utility function.

The composite goods L and leisure C are assumed to take the following forms:

$$L = (\xi_l L_f^{\eta_l} + (1 - \xi_l)L_m^{\eta_l})^{\frac{1}{\eta_l}} \tag{2.6}$$

$$C = \left[\sum_{j \in J} \kappa_j C_j^{\theta}\right]^{\frac{1}{\theta}} \tag{2.7}$$

where L_m is the leisure of the male whereas L_f is the leisure of the female.

C_j is the sector j composite good. I use κ_j as share parameters. For instance, $\kappa_a \in [0, 1]$ stands for the share of agricultural composite good, C_a on household utility. Similarly, $\kappa_m \in [0, 1]$ stands for share of manufactured composite good, C_m and $\kappa_s \in [0, 1]$ stands for share of service composite good, C_s. Lastly, $\frac{1}{1-\theta}$ stands for elasticity of substitution between different composite commodities.

Composite commodities are CES function of goods produced both at home and in the market for each sector. For each $j \in J = \{a, m, s\}$;

$$C_j = [\psi_j c_j^{\epsilon} + (1 - \psi_j)\hat{c}_j^{\epsilon}]^{\frac{1}{\epsilon}} \qquad \forall j \in J \tag{2.8}$$

where c_j is the market good consumption, $\hat{c}_j$ is the home good consumption. It is assumed both goods are substitutes by CES aggregator. ψ_j is the share of market good in the composite consumption good and $\frac{1}{1-\epsilon}$ is the elasticity of substitution between home and market good.

It is assumed that home good is produced with the following technology:

$$\begin{aligned} \hat{c}_j &= \hat{A}_j \hat{H}_j \qquad \forall j \in J \\ &= \hat{A}_j [\hat{\xi}_j \hat{H}_{\tilde{f}j}^{\eta} + (1 - \hat{\xi}_j) \hat{H}_{\tilde{m}j}^{\eta}]^{\frac{1}{\eta}} \end{aligned} \tag{2.9}$$

where $\hat{A}_j$ is the home technology, $\hat{\xi}_j$ is the share of female time, $\hat{H}_{\tilde{f}}$, in the production of home good and $(1 - \hat{\xi}_j)$ is the share of male time, $\hat{H}_{\tilde{m}j}$. The elasticity of substitution between male and female working hours is denoted by $\frac{1}{1-\eta}$.

In this economy, men and women are endowed with one unit of time. Each of them allocates this unit of time to home production, market production and leisure.

The time allocation constraint for the representative male is given by:

$$\bar{H}_{\tilde{m}} = \sum_j H_{\tilde{m}j} + \sum_j \hat{H}_{\tilde{m}j} + L_{\tilde{m}} \tag{2.10}$$

The time allocation constraint for the representative female is given by:

$$\bar{H}_{\tilde{f}} = \sum_j H_{\tilde{f}j} + \sum_j \hat{H}_{\tilde{f}j} + L_{\tilde{m}} \tag{2.11}$$

$H_{\tilde{m}}$ and $H_{\tilde{f}}$ are male and female time in the production of market good, respectively.

Households spend their resources on consumption on market goods. They receive income from market work.

Household's budget constraint is as follows:

$$\sum_j p_j c_j = \sum_{g\in G}\sum_{j\in J} w_{gj} H_{gj} \tag{2.12}$$

Given these, representative household's sequential problem is as follows:

$$\max_{\{\{c_j,\hat{c}_j,H_{gj},\hat{H}_{gj}\}_{\{g,j\}}\}} U(C,L) \tag{2.13}$$

$$\text{subject to} \quad \sum_j p_j c_j = \sum_g \sum_j w_{gj} H_{gj} \tag{2.14}$$

$$\hat{c}_j = \hat{A}_j[\hat{\xi}_j \hat{H}^{\eta}_{\tilde{f}j} + (1-\hat{\xi}_j)\hat{H}^{\eta}_{\tilde{m}j}]^{\frac{1}{\eta}} \quad \forall j \in J \tag{2.15}$$

$$C_j = [\psi_j c_j^{\epsilon} + (1-\psi_j)\hat{c}_j^{\epsilon}]^{\frac{1}{\epsilon}} \quad \forall j \in J \tag{2.16}$$

$$\bar{H}_g = H_{gj} + \hat{H}_{gj} + L_g \quad \forall g \in G \tag{2.17}$$

$$L = (\xi_l L^{\eta_l}_{\tilde{f}} + (1-\xi_l) L^{\eta_l}_{\tilde{m}})^{\frac{1}{\eta_l}} \tag{2.18}$$

Thus, household consume and work. They consume all the home production as well as market agriculture and service goods.

2.4.4 Characterization of Equilibrium

A competitive equilibrium in this economy is a collection of allocations $\{c_j, \hat{c_j}, H_{gj}, \hat{H}_{gj}, L_g\}_{g\in G, j\in J}$ and prices $[p_j, w_{mj}, w_{fj}]$ such that given values of the state variables $(A_j, \hat{A}_j)$ and given prices,

(i) The allocations maximize firm's profit (2.3)

(ii) The allocations maximize household's utility (2.13)

(iii) All markets clear

The solution methodology consists of two steps. First, I solve for the optimal time allocations across home and market for each sector. And then, I solve for the optimal time allocations across different sectors. As the model is a Pareto optimal model, by

Second Welfare Theorem, I solve social planners problem. The reader may refer to section A for detailed solution steps.

First order conditions to problem (2.13) yields the following important equilibrium condition:

$$\alpha_j \left(\frac{H_{fj}}{H_{mj}} \right)^{\eta - 1} = x_j \tag{2.19}$$

where $x_j = \frac{w_{\tilde{f}j}}{w_{\tilde{m}j}}$ is the gender wage ratio and $\alpha_j = \frac{\xi_j}{1-\xi_j}$ captures the comparative advantage of women, i.e. if α_j is close to 1 in sector $j \in J = \{a, m, s$

$rbrace$, women have comparative advantage in sector j whereas if it is close to 0, men have comparative advantage in that sector.

Assuming free mobility across sectors implies equal marginal rate of technical substitution:

$$\frac{w_{\tilde{f}j}}{w_{\tilde{m}j}} = \frac{w_{\tilde{f}}}{w_{\tilde{m}}} = x \tag{2.20}$$

Hence, equation (2.19) becomes:

$$\alpha_j \left(\frac{H_{fj}}{H_{mj}} \right)^{\eta - 1} = x \tag{2.21}$$

According to this equation, equilibrium wage ratio can only increase if there is a rise in relative female labor supply ($\frac{H_{fj}}{H_{mj}}$) or a rise in female specific parameter ξ_j which represents their comparative advantage in sector j. This rise is typically interpreted as a gender-biased demand shift due to various factors such as a change in social norms on behalf of women (Goldin (2006)), female friendly technological change or reduced distortions in the allocation of gender talents (Ngai and Petrongolo (2014)).

Similarly,

$$\hat{\alpha}_j \left(\frac{\hat{H}_{\tilde{f}j}}{\hat{H}_{\tilde{m}j}} \right)^{\eta - 1} = x \tag{2.22}$$

$$\alpha_l \left(\frac{L_{\tilde{f}}}{L_{\tilde{m}}} \right)^{\eta_l - 1} = x \tag{2.23}$$

The second important solution yields what percentage of total work hours is allocated to home and market production for each gender. Let N_{gj} be the optimal labor allocation of gender g in sector j.

$$N_{gj} = H_{gj} + \hat{H}_{gj} \quad \forall j, g \tag{2.24}$$

Given this, the optimal share of time allocated to market and home production is as follows:

$$H_{\tilde{f}j} = \frac{N_{\tilde{f}j}}{1 + B_j} \tag{2.25}$$

$$H_{\tilde{m}j} = \frac{N_{\tilde{m}j}}{1 + B_j} \tag{2.26}$$

$$\hat{H}_{\tilde{f}j} = \frac{N_{\tilde{f}j} B_j}{1 + B_j} \tag{2.27}$$

$$\hat{H}_{\tilde{m}jt} = \frac{N_{\tilde{m}j} B_j}{1 + B_j} \tag{2.28}$$

where $B_j = \{\frac{\psi_j}{1-\psi_j} \left(\frac{A_j}{\hat{A}_j}\right)^{\epsilon}\}^{\frac{1}{\epsilon-1}}$.

It should be noted that this results depends on the assumption that $\xi_j = \hat{\xi}_j$. Economically speaking, this means women do not have any comparative advantage between market and home but they do so between sectors. Hence, I assume individuals have comparative advantage in producing the goods independent of where the good is produced. This is not a very strong assumption considering the model is intended to catch the labor movements between home and market as a result of technological change.

By construction, ϵ is the substitution parameter between home and market j-good. If they are good substitutes ($\epsilon \in (0, 1]$), a relative increase in market technology will increase the share of time allocated to market production for both genders. Similarly if the relative technology increases at home, this time share of home time allocation will increase.

Define

$$N_j = [\xi_j N_{\tilde{f}j}^{\eta} + (1-\xi_j) N_{\tilde{m}j}^{\eta}]^{\frac{1}{\eta}} \tag{2.29}$$

Given these, we can define market and home consumption in each sector as a function of N_j. Moreover, this will let us define the aggregate consumption as a function of N_j.

$$c_j = A_j \frac{1}{1+B_j} N_j \tag{2.30}$$

$$\hat{c}_j = \hat{A}_j \frac{B_j}{1+B_j} N_j \tag{2.31}$$

$$C_j = E_j N_j \tag{2.32}$$

where $E_j = \left[\psi_j A_j^{\epsilon} \left(\frac{1}{1+B_j}\right)^{\epsilon} + (1-\psi_j)\hat{A}_j^{\epsilon}\left(\frac{B_j}{1+B_j}\right)^{\epsilon}\right]^{\frac{1}{\epsilon}}$.

This takes us to one of the important implications of this model.

$$\frac{c_j}{\hat{c}_j} = \left[\frac{A_j}{\hat{A}_j}\frac{\psi}{1-\psi}\right]^{\frac{1}{1-\epsilon}} \tag{2.33}$$

where $\epsilon < 1$ by construction. Equation (2.33) is the marketization equation of j-good given $j \in J = \{a, m, s\}$. It says a higher productivity growth in market leads to higher consumption of market j-goods compared to home j-goods. The pace of this shift from home to market depends on the share of market j-good ψ as well as the elasticity of substitution parameter between market and home goods ϵ. For that, if the share of market j-good is bigger (ψ close to 1) and market and home goods are good substitutes ($\epsilon \in [0,1]$) then the marketization will be faster.

This leads us to derive marketization equation for labor supply. Using equations (2.25) for female labors at home and in the market,

$$\frac{H_{\tilde{f}j}}{\hat{H}_{\tilde{f}j}} = \frac{1}{B_j} = \left[\frac{\psi_j}{1-\psi_j}\left(\frac{A_j}{\hat{A}_j}\right)^{\epsilon}\right]^{\frac{1}{1-\epsilon}} \tag{2.34}$$

Hence, an uneven technological growth in the market compared to home shifts labors from home to market. The pace if shift is affected by the market share ψ and elasticity of substitution between parameter ϵ. As before, I assume home and market goods as good substitutes so $\epsilon \in [0, 1]$.

Total time allocation $\bar{H}_g$ is allocated across work and leisure as mentioned above: $\bar{H}_g = \sum_j N_{gj} + L_g$ for both genders $g = \{\tilde{f}, \tilde{m}\}$. Given this, optimization across sectors yields the following result between work and leisure:

$$\alpha_j \left(\frac{N_{\tilde{f}j}}{N_{\tilde{m}j}} \right)^{\eta - 1} = \alpha_l \left(\frac{L_{\tilde{f}}}{L_{\tilde{m}}} \right)^{\eta_l - 1} \tag{2.35}$$

where $\alpha_j = \frac{\xi_j}{1-\xi_j}$ and $\alpha_l = \frac{\xi_l}{1-\xi_l}$. This ratio is equal to the gender wage ratio $x = \frac{w_{\tilde{f}}}{w_{\tilde{m}}}$ as before. This equation tells us that relative work is proportional to relative leisure hours.

The relative working time in sector j for women depends on their competitive advantage and the wage ratio and the size of the market:

$$\frac{N_{\tilde{f}j}}{N_{fk}} = \frac{G_{\tilde{f}}(x, j)}{G_{\tilde{f}}(x, k)} \tag{2.36}$$

where $G_f(x, j) = \left(\xi_j^{\frac{\theta}{\eta}} (\kappa_j E_j^{\theta}) F(x, j)^{\frac{\theta}{\eta} - 1} \right)^{\frac{1}{1-\theta}}$ and
$F(x, k) = 1 + \alpha_j^{\frac{1}{\eta - 1}} x^{\frac{\eta}{1-\eta}}$.

Equation (2.36) denotes the structural transformation of labor from one sector to the other. This movement depends on technological growth (A_j and $\hat{A}_j$), productivity (x) and comparative advantage (α_j) of women as well as the share of the composite commodity (κ_j). The pace of movement is affected by elasticity of substitution parameter (η) between the labors as well as the composite commodities (θ).

The following equation is the one which shows the relationship between leisure choice and work time. This result is important because compared to other relevant models in the literature, this model can answer the question of what women do if they do not work either at home or in the market. Hence, in case of a very high technological growth in the market, the model will not be able to produce 100 percent labor force participation.

$$\frac{N_{\tilde{f}j}}{L_{\tilde{f}}} = \frac{\mu}{1-\mu}\kappa_j E_j^{\theta}\frac{N_j^{\theta}}{C^{\theta}}\frac{F(x,l)}{F(x,j)} \tag{2.37}$$

When summed over the sectors j, equation (2.37) comes down to following:

$$\frac{\bar{N}_{\tilde{f}j}}{L_{\tilde{f}}} = \frac{\mu}{1-\mu}\frac{F(x,l)}{\sum_j F(x.j)} \tag{2.38}$$

where

$$\bar{N}_{\tilde{f}j} = \sum_j N_{\tilde{f}j} \tag{2.39}$$

.

Hence, women's decisions over leisure and work is positively dependent upon their comparative advantage in leisure as well as the elasticity of substitution parameter between male and female leisure. Given that, having a higher preference over leisure (μ) also plays a positive role on leisure choice.

Female labor force participation is defined as the ratio of total market work to total time allocation. Hence,

$$lfp_{\tilde{f}} = \frac{\sum_j H_{\tilde{f}j}}{\bar{H}f} \tag{2.40}$$

Using total work time N_{fj} and $\bar{H}_{fj}$, this equation can be expressed as follows:

$$lfp_{\tilde{f}} = \sum_j \frac{H_{\tilde{f}j}}{N_{\tilde{f}j}}\left(\frac{\bar{N}_{\tilde{f}j}}{N_{\tilde{f}j}} + \frac{L_{\tilde{f}}}{N_{\tilde{f}j}}\right)^{-1} \tag{2.41}$$

Without the last term, the model is identical to Ngai and Olivetti 2013. However adding leisure to this model is important as it generalizes the model and gives an incentive to women not to work either at home or in the market. Moreover, this is also important if we consider under-counting of women who works at family enterprises in the beginning of 19th century. In the US, many women were not counted as employed although they were working in family enterprises as unpaid workers. Adding those women back into labor force actually creates the U-shaped labor force participation. This model without leisure does not grasp that fact. If we include both home and market labor hours as part of the labor force, this model without leisure will give us 100 percent labor force participation which is not realistic.

Now, if we examine each term in the labor force participation one by one, it will give us a clear understanding about the roots of U-shape. By equation (2.25), $\frac{H_{\tilde{f}j}}{N_{\tilde{f}j}} = \frac{1}{1+B_j}$. Here is the proposition for that.

Proposition 2.4.1 *If there is a higher technological growth in the market relative to home technology, B_j will decrease given $\epsilon \in [0,1]$. Then, $\frac{1}{1+B_j}$ will increase.*

Proof 2.4.1.1 *By equation (2.34),*

$$\frac{1}{1+B_j} = \frac{1}{1+\left[\frac{\psi_j}{1-\psi_j}\left(\frac{A_j}{\hat{A}_j}\right)^{\epsilon}\right]^{\frac{1}{\epsilon-1}}}$$

According to this, higher relative technological growth in sector j compared to home technology, will decrease the denominator as $\epsilon \in [0,1]$. Decrease in denominator increase the entire term. Now, the pace of this decrease will depend on how big the market share of sector j. If it is big, then the pace of increase will be higher as a result of relative technological growth.

Note that, this increase represents labors transition from home to market which is marketization. A higher market share in a particular sector will push female workers from home to market then the relative technology improves.

By equation (2.36) and (2.39), $\frac{\bar{N}_{\tilde{f}j}}{N_{\tilde{f}j}} = \frac{\sum_j G_{\tilde{f}}(x,j)}{G_{\tilde{f}(x,j)}}$. The following proposition states the movement of this term, when there is a technological change in sector j.

Proposition 2.4.2 *If relative technological growth is higher in sector j compared to sector k, given these two goods are poor substitutes ($\theta < 0$) and the market shares of sector j is big enough ($\psi_j > \psi_k \quad \forall k \neq j$), it will decrease labor allocated to sector j.*

Proof 2.4.2.1 *The proof of this statement relies on the log linearization of the term $\frac{\bar{N}_{\tilde{f}j}}{N_{\tilde{f}j}}$ around the steady state. As the proposition is based on relative technology levels $\gamma_j = \frac{A_j}{\hat{A}_j}$ rather than individual technology levels.*

Define $\pi_j = \xi_j^{\frac{\theta}{\eta}} \kappa_j (1-\psi)^{\frac{\theta-\eta}{\eta}} \hat{A}_j^{\theta} F(x,j)^{\frac{\theta-\eta}{\eta}}$.

Given this,

$$log(\frac{\bar{N}_{\tilde{f}j}}{N_{\tilde{f}j}}) = \frac{1}{1-\theta}\left[log\left(\sum_j \pi_j \left(\frac{B_j}{1+B_j}\right)^{\frac{\theta(\epsilon-1)}{\epsilon}}\right) - log\left(\pi_j \left(\frac{B_j}{1+B_j}\right)^{\frac{\theta(\epsilon-1)}{\epsilon}}\right)\right]$$

Let $\tilde{y} = \frac{y-y^*}{y^*}$ for any variable y. Also, let $(\gamma_a^*, \gamma_m^*, \gamma_s^*)$ be the equilibrium relative technologies. Taking the first order Taylor approximation leads to the following equation:

$$\begin{aligned}\tilde{\frac{\bar{N}_{\tilde{f}j}}{N_{\tilde{f}j}}} = \frac{1}{1-\theta}&\left[\sum_j \frac{\pi_j \left(\frac{B_j}{1+B_j}\right)^{\frac{\theta(\epsilon-1)}{\epsilon}}}{\sum_j \pi_j \left(\frac{B_j^*}{1+B_j}\right)^{\frac{\theta(\epsilon-1)}{\epsilon}}} \theta \frac{B_j^{1-\epsilon}}{1+B_j}\frac{\psi_j}{1-\psi_j}\gamma_j^{\epsilon}\tilde{\gamma}_j\right] \\ &- \frac{1}{1-\theta}\left[\theta\frac{B_j^{1-\epsilon}}{1+B_j}\frac{\psi_j}{1-\psi_j}\gamma_j^{\epsilon}\tilde{\gamma}_j\right] \end{aligned} \tag{2.42}$$

First of all, if there was only one sector at home and in the market, equation (2.42) would be equal to zero as first and the second term would be equal to each other. Hence with one sector only, we would not be able to grasp structural transformation.

Now if the relative technology in sector j is higher than sector k, this will decrease the second term as $\theta < 0$. Together with the minus sign, it will be an increase. The same decrease will happen in the same amount in the first term for sector j. However, as the technology k is also growing, this will decrease the first term even more. Given the share of market j is big enough, the decrease in the first term will be dominated by the increase in the second term. Hence, $\frac{\bar{N}_{\tilde{f}j}}{N_{\tilde{f}j}}$ will increase.

Hence, if relative technological growth is higher in sector j compared to sector k, given these two goods are poor substitutes ($\theta < 0$) and sector j has a higher market share , it will decrease labor allocated to sector j. This is the implication of structural transformation in the literature. Higher technological growth in sector j will increase the relative prices in this sector. It will decrease the labor allocated to this sector j towards the sector whose relative prices are lower.

Finally, by equation (2.37), higher relative technological growth in sector j leads to decrease in relative leisure hours compared to total work hours given $\eta > \theta$. This movement represents labors' movement between leisure-work.

Proposition 2.4.3 *Higher relative technological growth in sector j leads to decrease in relative leisure hours compared to total work hours given κ_j is big enough.*

Proof 2.4.3.1 *By log linearization of equation (2.37) is as follows:*

$$log\left(\frac{N_j}{L}\right) = log(\Lambda_1) + \theta log(E_j) + log(\frac{N_j^\theta}{C^\theta})$$

where $\Lambda_1 = \frac{\mu}{1-\mu}\kappa_j \frac{F(x,l)}{F(x,j)}$.

The first order Taylor approximation of $log(E_j)$ is as follows:

$$\begin{aligned}\frac{\tilde{E}_j}{E_j} &= \Lambda_2 \\ &+ \frac{1-\epsilon}{\epsilon}\left[log\left(1+\frac{1}{B_j^*}\right) + \frac{1}{1+\frac{1}{B_j^*}}(1-\epsilon)\epsilon\left(\frac{\psi_j}{1-\psi_j}\gamma_j^\epsilon\right)^{1-\epsilon}\frac{\tilde{\gamma_j}}{\gamma_j}\right]\end{aligned}$$

where $\Lambda_2 = \frac{1}{\epsilon}log\left((1-\psi_j)\hat{A}_j^\epsilon\right)$. *According to this equation, a technological growth in sector j, will increase E. This increase affects $\frac{N_{\tilde{f}j}}{L_f}$ as a decrease as θ is negative (goods in different sectors are poor substitutes).*

Here is the Taylor approximation of $\frac{C^\theta}{N_j^\theta}$. Define $CN = \frac{C^\theta}{N_j^\theta}$.

$$\begin{aligned}\frac{\tilde{CN}}{CN} &= \frac{1}{CN^*} \\ &\quad - \sum_j \left(\kappa_j \theta E_j^{\theta-1} \frac{1}{\epsilon} \left[(1-\psi_j)\hat{A}_j^\epsilon \left(\frac{1}{1+\frac{1}{B_j}} \right)^{\epsilon-1} \right]^{\frac{1}{\epsilon}} \frac{1}{B_j^2} \right) \\ &\quad \left(\times \left(\frac{\psi_j}{1-\psi_j} \right)^{\frac{1}{\epsilon-1}} \gamma_j^{\frac{\epsilon}{\epsilon-1}} \frac{\tilde{\gamma}_j}{\gamma_j} + \frac{\partial \frac{N_{\tilde{f}j}}{N_{\tilde{f}k}}}{\partial \gamma_j} \right)\end{aligned}$$

Given $\theta < 0$, the second term decreases as technology rises in sector j. Also, we already know from the structural transformation proposition that $\frac{N_{\tilde{f}j}}{N_{\tilde{f}k}}$ decreases. Overall, the direction of the change will depend on the share of composite commodities κ_j as well as the market share as established in Proposition 4.2.1*. Overall, if the structural transformation dominates, the $\frac{N_j^\theta}{C^\theta}$ increases. Otherwise, it will decrease.*

Therefore, the analytical implication of the previous three proposition is as follows: Decrease in female lfp is mostly originated by structural transformation. If this force can dominate people's leisure-work incentives and marketization, then lfp will decrease. Increase in female lfp is originated by marketization and leisure-work choices. If this force dominates structural transformation, then female lfp will increase.

2.5 Calibration

In this section, I quantitatively assess the importance of marketization, structural transformation and leisure-work movements in accounting for the observed movements in married women's labor force participation over economic development in Figure 2.1.

In a nutshell, I calibrate model parameters based on US data between 1965-2012. The current calibration requires me to have certain data points such as sectoral share of total market and home hours with their gender components. There are in total

20 parameters in the model. Among these 6 of them are taken from the data or existing literature. One of the technology parameters is normalized to 1. Hence, 13 of them are calibrated. Table 2.2 shows the parameters taken from the existing literature whereas Table 2.3 shows the calibrated parameters. The explanation for each of these parameter choice is given in Section 2.5.2 and 2.5.3.

Table 2.2: Parameters from Literature

Parameters	Values	Source
ϵ	0.6	Aguiar et.al (2012)
θ	-5.76	Various estimates
μ	0.6	Ngai and Petrongolo (2016)
κ_a	0.02	Herrendorf et.al (2013a)
κ_s	0.17	Herrendorf et.al (2013a)
κ_s	0.81	Herrendorf et.al (2013a)

Table 2.3: Calibrated Parameters

Parameters		Value
Relative time endowment	$\frac{\bar{H}_{\tilde{m}}}{\bar{H}_{\tilde{f}}}$	1.94
Elas of subs btw male and female hrs	η	-2.3
Elas of subs btw male and female leis	η_l	2.5
Agr female mrkt share	ξ_a	0.85
Man female mrkt share	ξ_m	0.16
Ser female mrkt share	ξ_s	0.97
Share of agr market good	ψ_a	0.02
Share of man market good	ψ_m	0.003
Share of ser market good	ψ_s	0.08
6 Technology parameters		

2.5.1 Data Targets

The labor force participation of women are related to changes in relevant data moments between the start and end of the sample period. The list of the 13 moments are given in Table 2.4 and my procedure to use them as well as more detailed empirical findings are explained in Section 2.5.2.

Sector shares are obtained from IPUMS-USA and is adjusted for the demographics as explained in Section 2.2. Sectoral categorization are also already explained in Section 2.2 both for IPUMS-USA and MTUS.

Table 2.5 displays the market shares of each sector among total hours worked from IPUMS-USA. First four rows show the distribution of total market hours between three different sectors. I calculate total market hours by multiplying number of hours worked in a week (hrswork2) with number of weeks worked in the previous year

Table 2.4: Moments

Moments	Data	Model
Change in wage ratio	36.46	36.46
Wage ratio	0.50	0.50
Change in male-female hours ratio	0.30	.30
Change in male-female leisure hours ratio	-0.24	-0.24
Male to female agr ratio ($\frac{H_{\tilde{m}a}}{H_{\tilde{f}a}}$)	1.96	1.96
Male to female man ratio ($\frac{H_{\tilde{m}m}}{H_{\tilde{f}m}}$)	5.46	5.49
Male to female ser ratio ($\frac{H_{\tilde{m}s}}{H_{\tilde{f}s}}$)	1.10	1.10
Female agr market to home ratio ($\frac{H_{\tilde{f}a}}{\hat{H}_{\tilde{f}a}}$)	0.37	0.34
Female man market to home ratio ($\frac{H_{\tilde{f}m}}{\hat{H}_{\tilde{f}m}}$)	1.33	1.31
Female ser market to home ratio ($\frac{H_{\tilde{f}s}}{\hat{H}_{\tilde{f}s}}$)	0.5	0.5
Ratio of agr income to man income (Y_a/Y_m)	1.0e-04	0.10
Ratio of man income to ser income (Y_m/Y_s)	0.16	0.16
Ratio of agr income to ser income (Y_a/Y_s)	1.6e-05	0.64

(wkswork2) for each individual and sum them up to find the annual total working hours. Combined with the industrial classification defined in section 2.2, I calculated the total working hours in each industry and found the shares. According to this, we see an expansion in the service sector, while there is a shrinkage in agricultural and manufacturing sector. Same trend prevails when we look at these shares by gender except the agriculture share for female workers increase.

Similarly, I calculate the sectoral shares of home production as well as its gender components using time use surveys. The classification of each sector is explained in section 2.2. I sum annual sectoral hours and leisure time up to find the total hours at

Table 2.5: Sectoral Shares in the Market

Year	Shares		
Year	Agriculture	Manufacture	Service
1965	3.42	37.57	59
2012	2.55	20.5	76.94
	Female		
Year	Agriculture	Manufacture	Service
1965	0.55	24.88	74.57
2012	1.28	9.49	89.23
	Male		
Year	Agriculture	Manufacture	Service
1965	4.68	43.1	52.22
2012	3.55	29.1	67.4
Source:IPUMS-USA			

home. I exclude sleeping time from my calculations. Table 2.6 displays the industrial shares of home hours as well as shares for each gender. As expected, service share constitutes the biggest share and it is dominated by women. Moreover, agricultural and manufacturing sectoral shares are decreasing over the years while service shares are increasing.

Table 2.7 displays the fraction of market and home hours both for female and male workers. According to this, there is a decrease in both home and market hours for female workers where as male home hours increase.

Data on market shares and work fraction in tables 2.5, 2.6 and 2.7 can be combined to characterize the full time allocation for each gender. Let M_g be the total market

Table 2.6: Sectoral Shares at Home

Year	Shares		
Year	Agriculture	Manufacture	Service
1965	1.09	25.76	73.14
2012	0.78	22.83	76.38
	Female		
Year	Agriculture	Manufacture	Service
1965	0.97	11.04	87.99
2012	0.37	8.28	91.35
	Male		
Year	Agriculture	Manufacture	Service
1965	1.19	37.95	60.85
2012	1.12	34.93	63.94
Source:IPUMS-USA			

Table 2.7: Market and Home Fraction of Total Hours

Year	Market Fraction		Home Fraction	
	Female	Male	Female	Male
1965	32.82	38.79	55.45	45.54
2012	26.81	30.57	51.7	46.9
Source:MTUS				

work of gender g and $\hat{M}_g$ be the total house work of gender g. Also, let s_{gj} and $\hat{s}_{gj}$ be the share of market work for each gender g and each sector j at market and home respectively. These shares are calculated as the ratio of sectoral annual working hours to the total annual working hours for each gender. Hence $\sum_j s_{gj} = 1 \quad \forall g \in G = \{\tilde{f}, \tilde{m}\}$.

$$\frac{H_{gj}}{\bar{H}_g} = \frac{M_g}{\bar{H}_g} s_{gj} \quad \forall s_{gj}, g \in \{\tilde{f}, \tilde{m}\}, \quad j \in \{a, m, s\}$$

$$\frac{\hat{H}_{gj}}{\bar{H}_g} = \frac{\hat{M}_g}{\bar{H}_g} \hat{s}_{gj} \quad \forall \hat{s}_{gj}, g \in \{\tilde{f}, \tilde{m}\}, \quad j \in \{a, m, s\}$$

$$\frac{L_g}{\bar{H}g} = 1 - \frac{M_g + \hat{M}_g}{\bar{H}_g} \quad \forall g$$

The complete time allocation is on Table 2.8. According to this, we observe a tremendous increase in home service hours for females as opposed to males. Female workers market hours also decrease as they allocate more of their time to home production and leisure.

Lastly, table 2.9 refers to gender wage ratio from 1965 to 2012. We observe that the gender wage ratio fluctuates over the years while following an increasing trend.

2.5.2 Calibration Procedure

Aguiar *et al.* (2011) analyze business cycle version of Benhabib et al. (2011) model to analyze how the forgone market hours are allocated across home production activities using ATUS data. According to their paper, the model version excludes sleep, eating, and personal care as leisure activities, an elasticity of approximately 2 2.5 produces a reallocation of market hours to home hours in the model that matches the actual behavior of households in the data at business cycle frequencies. When

Table 2.8: Complete Time Allocation

Female							
	Home			Market			
Time	Agr.	Man.	Ser.	Agr.	Man.	Ser.	Leisure
1965	0.54	6.12	48.79	1.72	5.85	25.25	11.73
2012	0.19	4.28	91.35	0.28	4.75	21.78	21.49
Male							
	Home			Market			
Time	Agr.	Man.	Ser.	Agr.	Man.	Ser.	Leisure
1965	0.54	17.28	27.71	10.39	9.85	18.55	15.67
2012	0.53	16.38	29.99	1.09	10.26	19.22	22.52
Source:Time Use Surveys and IPUMS-USA							

Table 2.9: Wage Ratio

Year	Wage Ratio
1965	49.45
2012	72
Source:IPUMS-USA	

we include these activities as leisure time activities, then the parameter is estimated to be around 3.5-4. As my model excludes sleep, eating, and personal care from the leisure time activities, I decide to use 2.5. Hence $2.5 = \frac{1}{1-\epsilon}$ which leads $\epsilon = 0.6$.

Ngai and Pissarides (2008) interpret $-\frac{1}{1-\theta}$ as the own price elasticity in a model without a home production because relative prices are inversely related to relative TFP levels. However, with home production, the estimated own price elasticity should be higher than $-\frac{1}{1-\theta}$ in absolute value because some market-produced goods have good home-produced substitutes. Given this, Falvey and Gemmell (1996) estimate income and price elasticity of service demand using Phase IV ICP data set which includes 60 countries. They find that the own price elasticity is -0.3 (Table 3, column 1). They compare their findings with Summers (1986) which estimates the same elasticity as -0.06 which is close to Blundell *et al.* (1993), -0.07. Lastly, Herrendorf *et al.* (2013) suggest if the technology is specified as value-added production function, then consistency dictates that each argument in the utility function is the value-added components of the utility function. This approach is called *the consumption value-added approach* and this is the aproach I take in my model too. For this approach. they estimated the elasticity of substitution between different composite goods equal to 0.002 (Table 3 at Herrendorf *et al.* (2013)) . Hence, the elasticity of substitution seems to vary between 0 and 0.3 for $\frac{1}{1-\theta}$. This means the $\theta \in [-\inf, -2.33]$. In my analysis, I used $\theta = -5.76$ as it seems to give better results.

Labor productivity growth for market agricultural sector is taken from US Department of Agriculture (USDA). I defined labor productivity as the ratio of total output to total number of labors by taking 2005 as the base year. I calculate the average growth rate from 1948 to 1998 as 1948 is the earliest data available. The reason I change the ending year to 1998 is to span $50 - year$ time period. My calculations yield 4.16 percent growth in market agricultural sector. Similarly, labor productivity growth for market manufacturing and service sectors are taken from Bureau of Labor Statistics (BLS) delivering 3.11 percent growth in manufacturing sector whereas 2.42 percent growth in service sector.

Data for the home productivity data is taken from Bridgman (2016). He calculates annual home production output and productivity levels from 1929 to 2010. As he takes the home production as one single entity, he naturally calculates one productivity growth. Hence, I used the same labor productivity level for three sectors which is 1.96 percent.

Lastly, I borrowed the values for sectoral shares of composite goods in aggregate consumption from Herrendorf *et al.* (2013). According to them, $\kappa_a = 0.02$, $\kappa_m = 0.17$ and $\kappa_s = 0.81$. The parameter μ which represents the household's preferences over consumption and leisure is chosen to be 0.6 due to Ngai and Petrongolo (2014).

2.5.3 *Calibrated Parameters*

The relative time endowment $\frac{\bar{H}_{\tilde{m}}}{\bar{H}_{\tilde{f}}}$ is equated to be 1. In previus calculations, I follow Ngai and Petrongolo (2014) and find it through service employment share. The service share in the model can be expressed as follows:

$$s = \frac{\hat{s}_{\tilde{m}s}\frac{M_{\tilde{m}}}{\bar{H}_{\tilde{m}}}\frac{\bar{H}_{\tilde{m}}}{\bar{H}_{\tilde{f}}} + \hat{s}_{\tilde{f}s}\frac{\bar{H}_{\tilde{m}}}{\bar{H}_{\tilde{f}}}}{\frac{M_{\tilde{m}}}{\bar{H}_{\tilde{f}}}\frac{\bar{H}_{\tilde{f}}}{\bar{H}_{\tilde{m}}} + \frac{M_{\tilde{f}}}{\bar{H}_{\tilde{f}}}} \tag{2.43}$$

From here, the relative time endowment can be expressed as follows:

$$\frac{\bar{H}_{\tilde{m}}}{\bar{H}_{\tilde{f}}} = \frac{(s_{\tilde{f}s} - s_s)\frac{M_{\tilde{f}}}{\bar{H}_{\tilde{f}}}}{(s_{\tilde{m}s} - s_s)\frac{M_{\tilde{m}}}{\bar{H}_{\tilde{m}}}} \tag{2.44}$$

The implied ratio is 1.94. This is rather higher than what it is usually expected. Hence, I decide to go with 1 although it is not playing a prominent role on the shape of female LFP.

Using equations (2.22) and (2.23), the elasticity parameters η and η_l are set to match the changes in log wage ratio and the changes in hours ration between beginning and the end of the period.

$$\eta = 1 - \left[\frac{log\left(\frac{\frac{H_{\tilde{m}j}}{H_{\tilde{f}j}}^{t=0}}{\frac{H_{\tilde{m}j}}{H_{\tilde{f}j}}^{t=T}}\right)}{log(\frac{x^0}{X^T})} \right]^{-1}$$

As a result, $\eta = -0.2$ and $\eta_l = 2.5$. $\eta < 0$ means female and male labor supply are complements. This results contradicts with papers close to my model. However, the complementarity of male and female labor supply is not new in the literature. As for the $\eta_l > 0$, it implies that male and female leisure is substitutes which is rather controversial. Many studies show that couples take retirement or volunteering decisions together and prefer to do these together which implies male and female leisure time are complements (Goux *et al.* (2014), Rotolo and Wilson (2006), Connelly and Kimmel (2009), Gustman and Steinmeier (2000)). On the other hand, in the added worker effect literature, studies shows a substitution between male and female labor supply when women tend to increase their labor supply when the husband lose his job (Lundberg (1985), Gruber and Cullen (1996)). These findings implies that husband and wife leisure times are substitutes. My calculations are in line with this literature.

Given these values, the gender parameters ξ_j, $\hat{\xi}_j$ and ξ_l are also calculated using equations (2.21) and (2.23). At this stage, I equate $\xi = \hat{\xi}_j \quad \forall j$ to be consistent with my assumptions in section 2.4. Given ξs, I calculate the comparative advantage parameters α_j using $\alpha_j = \frac{\xi_j}{1-\xi_j}$. The calculated values for ξ_j are 0.85, 0.16 and 0.97

respectively for a, m, s. Hence, women have higher comparative advantage in service sector in the market where as lowest comparative advantage in manufacture sector.

Lastly, by equation 2.34, I calculate the market shares of each goods which are $\psi_j's$.

$$\frac{\psi_j}{1-\psi_j} = \frac{\left(\frac{\hat{H}_{\tilde{f}j}}{H_{\tilde{f}j}}\right)^{\epsilon-1}}{\left(\frac{\hat{A}_j}{A_j}\right)^{\epsilon}}$$

To be comparable with the relevant literature, I enrich my model in Section 2.4 by introducing TFP growth rate to gender specific variables ξ_j for service sector and manufacturing sector as in Ngai and Petrongolo (2014). This change increases the labor force participation on the right a little more. This result is expected as as women have higher comparative advantage compared to men, the demand for female workers increase and that increases their labor force participation. However, as this change in the labor force participation is not drastic, I did not include this in my original analysis.

2.6 Results

As it is established in the previous sections, the movement of female labor force participation depends on 3 important factors: Marketization, structural transformation and leisure-work choices. If the structural transformation dominates the other two factors, this will induce a decrease while if marketization and leisure-work choice dominates structural transformation, this induces an increase in the female labor supply.

In the following exercise, I introduce different frowth rates to the model sectors to analyze under what circumstances, the model produces increasing and decreasing trend. It should be noted that, for this experimental part, I do not change the calibrated parameters. Hence, the market share of sectors or the composite commodities are always the same. Also, the calibration already induces a higher comparative advantage for service sector for women while a lower comparative advantage in agricultural sector. Given these values, I think testing how powerful the model is to create increasing and decreasing trends is important.

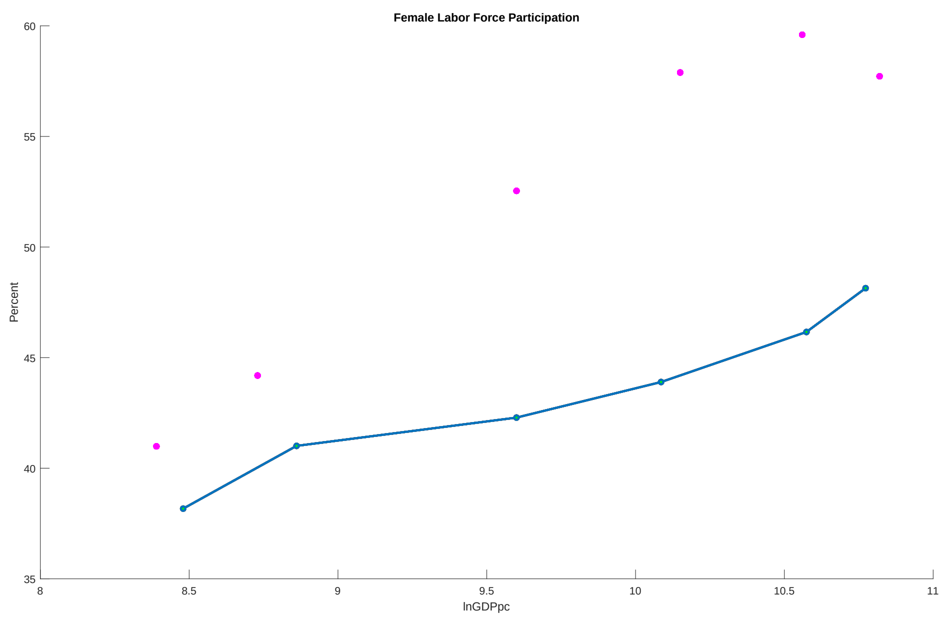

Figure 2.3: Female LFP between 1965-2012

Figure 2.3 displays how female lfp for the US moves with the labor productivity growth rates in the data. Dotted line shows the US data for female LFP whereas the solid line is the model outcome. According to data, agricultural technology grows at the rate 3.19, manufacturing grows at the rate 3.11 and lastly service sector grows at the rate of 2.42. Home technology is assumed to be growing at the rate of 1.94. Given these rates, the labor force participation follows an increasing trend and it actually increase by 26 percent between 1965 and 2012. On the other hand, female

LFP increases by 40 percent between these years. Although there is a lot of room to improve, the model is strong enough to produce an increasing trend.

This result is in line with the literature which argues the rise of service economy and increasing female labor supply. With the developing service economy, more and more women joins the labor supply through marketization or increases their labor allocation by forgoing their leisure.

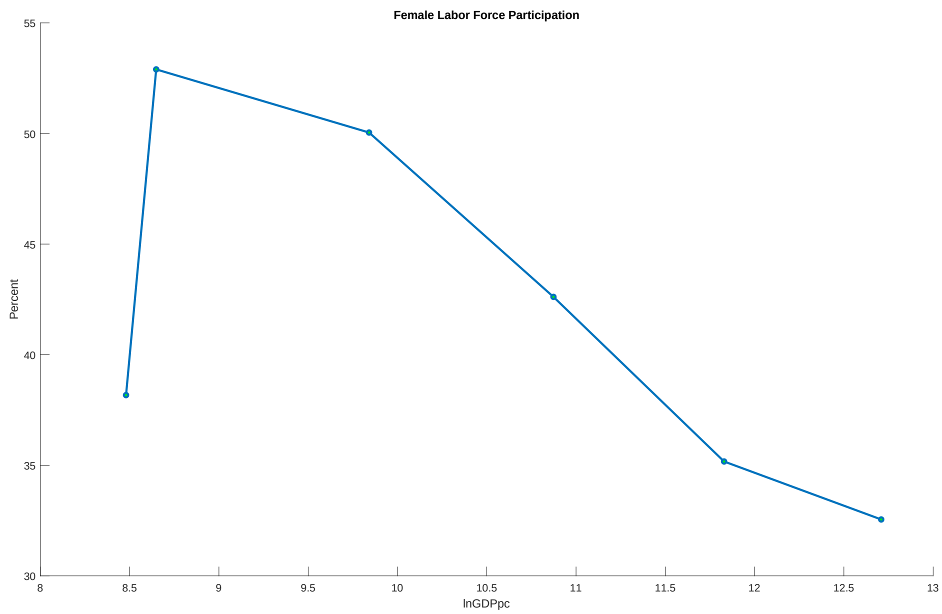

Figure 2.4: Female LFP between 1965-2012

Furthermore, I investigate the decreasing part of the female labor supply by introducing 30 percent growth rate for agriculture production, 13 percent growth rate for manufacturing and 2 percent growth rate for service sector. The home technology growth is still 1.94 for all the sectors. Given these, high technology growth in agriculture should lead a decrease in labor force participation as the structural transformation will dominate. Figure 2.4 displays my result about this experiment.

The first jump in Figure 2.4 is because the growth rates are introduced after the first period (decade). According to this, high technology growth in agriculture combined with low manufacturing and service sector leads to a decrease in labor supply. This means the structural transformation dominates the other two important factors. This result is also in line with the literature which argues structural transformation in the early industrial revolution causes a decrease in female labor supply.

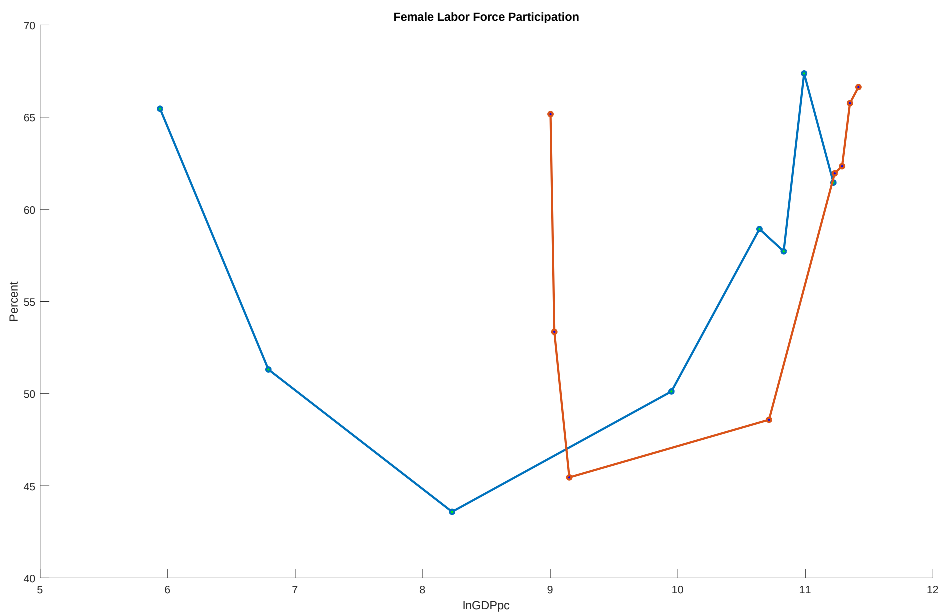

Figure 2.5: Female LFP - Selected Countries

Given these important experimental results, I test if the model can generate the U-shape trend observed in Figure 2.1. I choose 9 countries which are very close to fitted line. Next, I introduce their productivity levels to my model. Finally, I analyze if the given productivity levels together with calibrated parameters above can generate a U-shape female lfp. Figure 2.5 displays the results.

2.7 Conclusion

This paper develops a general equilibrium model to generate the U-shaped female labor force participation. Theoretical results suggest that a rise in the service sector increases the female LFP as it drives people from home to market work. On the other hand, a higher technological growth in agricultural sector compared to manufacturing and service sector can cause a decrease in female LFP as it will decrease the demand for labor. Quantitative results confirm the findings although there is room for improvements.

Chapter 3

COMPARING EQUILIBRIA IN DYNAMIC MODELS WITH NON-SEPARABLE UTILITIES AND INDETERMINACIES

3.1 Introduction

Equilibrium comparative statics has been a focal question in dynamic general equilibrium theory. To conduct comparative statics, one has to prove existence of equilibrium. Per existence and computation of equilibria, if the welfare theorems hold i.e the model is Pareto optimal, we use methods such as Negishi (1960) to verify the existence and then, apply some fixed point algorithms such as Scarf algorithm or global Newton method to compute the equilibria. However, when it comes to non Pareto optimal (equivalently nonoptimal) economies such as economies with distortionary taxation policies, production externalities and various monetary policies; usual methods do not generally work as the welfare theorems fail.

Great deal of work has been devoted to find a new methodology for proving and computing the existence of equilibrium in nonoptimal recursive equilibrium theory. For example, Coleman II (1997), Greenwood and Huffman (1995) apply monotone mapping methodology to prove existence on environments with elastic labor supply[1]. The methodology refers to a monotone operator whose fixed points define the recursive equilibrium. It is a very powerful tool for this environment but given the conditions on preferences and technology, it is quite hard to apply the proposed methodology to

[1] The methodology is developed by Lucas Jr and Stokey (1985) and Coleman (1991)

various important applied general equilibrium problems (Datta *et al.* (2002)).[23] For example, the models with monetary policies such as Cooley and Hansen (1989), Cole *et al.* (1998) are ones of these environments. In this paper, I extend the monotone mapping model in such a way that it will be applicable to a larger set of applied environments.

Another issue in these nonoptimal economies is that indeterminacy (multiple equilibria) is inherent in many of these dynamic models. Since Kareken and Wallace (1981), economists determine the conditions at which the economy has indeterminacy of equilibria. For example, Benhabib and Farmer (1994) examine the properties of one sector growth model with increasing returns. For increasing returns, they consider two organizational structure one of which is aggregate input externalities while the other is monopolistic competition. Under commonly used parameters in business cycle literature, they show that there is indeterminacy of equilibria. Similarly, Nourry *et al.* (2013) investigates the role of consumption tax on aggregate (in)stability.[4] For

[2] In Coleman II (1997), the production function is assumed to be $F(x, n, \frac{K}{N}, s)$ where x and n are individual firm's capital and labor decisions and K/N is the ratio of aggregate capital to aggregate labor supply and s is the productivity shock. This leads wage rate and rental rate to be a function of K/N which is quite restrictive. In my paper, I exclude this assumption and use a more general form of a production function.

[3] In Greenwood and Huffman (1995), the utility function is assumed to be in the following form: $u(c, l) = h(c - g(l))$ where $h(.)$ is monotone increasing and $g(l)$ is concave function. This is to say that the marginal rate of substitution of consumption to leisure is independent of consumption. As I find it restrictive, I define a more general form of utility function to span a greater applied literature.

[4] Consumption taxation has two effect on household decisions. One of which is consumption tax decreases the relative price of leisure. This leads to lower consumption and labor supply. This is what is called as intratemporal substitution effect and causes aggregate instability. On the other hand, higher consumption tax makes consumption good more expensive. Hence, household may increase the labor supply to smooth consumption. This mechanism is named as intratemporal income effect. Nourry et.al. are mainly interested in examining these two opposing effects of consumption tax on

that,they examine a one sector growth models with nonlinear consumption tax and with a specification of preferences. This specification entails them to span varying degrees of income effect depending on the choice of parameters [5]. They find that for a big set of parameters, consumption taxation can be a source of indeterminacy in most of the OECD countries. Other important papers on this are Bennett and Farmer (2000), Miao and Santos (2005), Nishimura and Venditti (2002) and some observationally equivalent models in such as Liu and Wang (2014) which talks about credit constraints. Hence, indeterminacy is very common phenomena in most of the well known applied models.

Awareness of indeterminacy in a model is important for two reasons. First, from a practical point of view, calibration and estimation results are difficult to interpret. In order to fully understand how the economy behaves in response to a policy change for instance, we need to be aware of which exact equilibrium we are talking about. If the economist is not aware that there is multiplicity, then she will not know that depending on the parameter space she chooses, her results are going to vary. Hence, the results are going to be 'parameter-biased'. For example, assume an economist has

aggregate instability. As intratemporal income effect has a stabilizing effect, they want to see how varying degrees of income effect can dominate the instabilizing effect of intratemporal substitution effect.

[5]The specification in question is a form of utility function.

$$u(c,l) = \frac{\left[c - \frac{(1-l)^{1+\chi}}{1+\chi} c^{\gamma}\right]^{1-\theta} - 1}{1-\theta}$$

This utility is the one which is derived in Jaimovich (2008) to encompass varying degrees of income effect. With this kind of utility function, we will be able to generalize our results to a wide range of cases from no-income effect to constant income effect. In fact when $\gamma = 0$ it belongs to no income effect which we know from Greenwood *et al.* (1988) whereas when $\gamma = 1$, it belongs to large income effect with endogenous growth which we know from King *et al.* (1988).

a growth model with multiple equilibria of which she is not aware. With the parameter space she uses, let's say there are three different equilibria but she is only aware of the least equilibrium of those three. Normally, when you have multiple equilibria, comparing equilibrium points would be meaningful if you compare the equilibriums in the same class. For instance, comparing the least point with the new least equilibrium is meaningful but comparing the least point with the greatest new equilibrium is not. Hence, once she perturbs some of the parameters, if she luckily jumps to the least point of the new set of equilibria, she can make some meaningful comparisons with these two points. On the other hand, if she jumps to the greatest point of the new equilibria, whatever conclusion she reaches will not mean anything. Hence, awareness of indeterminacy is crucial. While going through the indeterminacy papers in the literature, I realize that almost all the papers look for the set of parameters that create indeterminacy. They do not offer how to prove or compute the existence. However, in order to derive valuable implications, one should know not only one set of parameters which create multiplicity, but also compute the whole equilibria so that no matter what the parameter set is, she could have some idea about how the model behaves. Focusing our attention to recursive competitive economies, Coleman II (1997), Greenwood and Huffman (1995) prove the existence of equilibria. However, as mentioned in the footnote 2 and 3 the conditions on preferences and production technologies are so strong that it is hard to apply the methods on various applied problems. Besides, none of the models offer a method to compute the equilibria. Hence, the first contribution of this paper is to provide a methodology for proving and computing the recursive equilibrium in case of indeterminacy with weaker conditions on preferences and technology.

Second, from a theoretical point of view, presence of multiple equilibria generates complications for equilibrium comparisons. When an economist has multiple equilibria, she should know which particular equilibrium she is analyzing, as each equilibrium may behave differently in case of a perturbation of a parameter of the model. When dynamic economies were Pareto optimal and agents were homogeneous; convexity of preferred sets, smoothness of indifference curves with standard Inada conditions for interiority led us to unique recursive equilibrium. In that case, we could derive the local comparative statics predictions by applying implicit function theorem or lattice programming techniques.

However, in nonpareto optimal economies which I will be working in this paper, I do not have any smoothness so implicit function theorem is not applicable. Moreover, usual lattice programming techniques do not also work here because the multiplicity simply deteriorates the single crossing property of the objective function. Hence, we cannot use lattice programming for equilibrium comparative statics. This is what leads us to seek for a new robust comparative statics methodology in case of multiplicity. Keeping our attention to recursive environments, there are several studies on these such as Acemoglu and Jensen (2015) andMirman *et al.* (2008). All provide powerful tools to deal with this issue. Especially, if the model is a dynamic economy with inelastic labor supply, small capital externalities with no labor externality, we can obtain robust comparative statics predictions. However, they are also far away from providing a solution if we have elastic labor supply, labor externalities or large capital externalities which are also commonly used in applied economics literature. Therefore, the second contribution of this paper is to provide a methodology for these kind of economies.

Hence, in this paper I apply a new method to verify and compute the existence of recursive equilibrium in dynamic economies with capital accumulation and elastic labor supply. I also provide the robust equilibrium comparative statics in various sub-classes of equilibria as well as comparative dynamics in a particular subclass of equilibria i.e recursive equilibria with monotone investment process. The method is multi-step monotone mapping methodology which is based on operator equations defined in partially ordered sets and characterizing a particular subclass of recursive equilibrium over both individual and aggregate state variables at each step of computation.[6]

I apply this method to the class of models which can be categorized as "dynamic models with complementarities". Economies with externalities/nonconvexities in production, public policy such as distortional taxation, monopolistic competition and monetary economies can be cast into this framework. All of these models are also non-pareto optimal economies which is another reason of why classical methods for computing equilibria and providing comparative statics predictions do not work. The reason for that is the classical methods used in Pareto optimal case benefits from the welfare theorems. When the economy is nonoptimal, none of the welfare theorems do not work. For more interested readers, I also talk about the usual methods used in Pareto optimal case in section 3.1.1 and 3.1.2.

Hence, when we have Pareto optimal economy, social welfare theorems play an important role in both computing existence of equilibrium and comparative statics. However, in a nonpareto optimal economy, none of these methods work because welfare theorems are literally useless. Considering the fact that the class of nonoptimal economies includes a very large portion of recursive competitive equilibrium models studied extensively in the literature, it is important to have a theoretical framework for both existence and comparative statics.

[6]Monotone mapping approach was pioneered in the seminal works of Coleman (1991), Coleman II (1997), Coleman II (2000). Multi-step monotone mapping methodology was first developed by Datta *et al.* (2002).

Having a theoretical framework is basically essential for two reasons. From the theoretical point of view, it fills in a significant gap in the literature. The literature on existence of recursive equilibria in nonpareto optimal economies is still weak as almost all the papers, to my knowledge, are based on numerical solutions or the ones which proves existence offer strong assumptions (Coleman II (1997), Greenwood and Huffman (1995)). Similarly, comparative statics literature (Acemoglu and Jensen (2015) and Mirman *et al.* (2008) etc.) does not cover very important models such as the ones with elastic labor supply, or labor externalities or large capital externalities. From the practical point of view, it gives the economist a broader perspective about the economy she modeled. With a well-defined theoretical framework, it will be easier to interpret whether there is multiplicity, or how the economy behaves in response to a parameter change etc. Hence, providing a theoretical framework have both theoretical and practical benefits.

The basic idea behind this multi-step monotone mapping methodology is that as the welfare theorems do not work, I turn the problem into a fixed point problem to talk about existence and do comparative statics. In the first step, I define two 'side conditions' for consumption and leisure decisions. These conditions can be obtained by Lagrange equation for household utility maximization problem. These side conditions are important in the sense that they help me to find intratemporal consumption and leisure decisions through a fixed point problem. Note that the consumption and leisure equations in this step are not recursive equilibrium consumption and leisure decisions. They are basically functions of aggregate state variable, inverse marginal utility and exogenous shock parameter but they will help me to find the recursive equilibrium consumption and leisure decisions in the second step. Furthermore, first step calculations help me to see if there is any multiplicity due to production externality. Once I determine the multiplicity, I provide robust comparative statistics for the least and greatest consumption and leisure decisions. The equilibrium comparative statics I obtain in this step will be robust in any subclasses of equilibria. To summarize, the first step computation characterizes all the necessary and sufficient conditions to verify the existence of recursive equilibrium for any given aggregate state.

In the second step, I carry these intratemporal equations into a dynamic environment by simply introducing the evolution of aggregate state variable. It is this step where I start to use household's Euler equation to find recursive equilibrium consumption, investment and leisure decisions. As I do in the first step, I again solve a fixed point problem to find recursive competitive consumption and leisure decisions. In the second step, I will mainly be interested in a particular subclass of equilibrium at which the investment is monotone in current capital stock. Hence, equilibrium comparative dynamics will be specific to this subclass.

3.1.1 Proving and Computing Equilibrium in PO Economies

To motivate the contribution of this paper more, It is important to mention the usual methodologies used in case of Pareto optimality.

As mentioned above, in case of Pareto optimality, verifying and computing existence and getting some comparative statics predictions could be done by usual methods. For example, consider a one sector growth model with finite number of infinitely lived, homogeneous households. Using Negishi's approach, we could prove the existence of equilibrium as the solutions to social planning problem because we already know that the economy is Pareto optimal and welfare theorems hold. Set up a social planner's problem with strictly positive welfare weights for each individual. With the help of welfare weights, the set of all Pareto optimal solutions could be defined. By first welfare theorem we know that the competitive equilibrium must be

Pareto optimal. Then, the question of finding competitive equilibrium out of this set comes down to applying second welfare theorem. As nicely explained by Kehoe (1991), a price-allocation pair is equilibrium if and only if there exists a strictly positive positive vector of welfare weights such that proposed allocation solves the social planner's problem and the price vector is the corresponding vector of Lagrange multipliers with zero-sum transfers. Negishi (1960) proves the existence of such welfare weights. Hence, this method would take care of the existence of recursive competitive equilibrium if we had Pareto optimal economy.

When it comes to computing equilibrium, we would need some additional algorithms. Uzawa (1962) states that any algorithm which computes the equilibrium of an arbitrary equilibrium in terms of excess demand functions should also compute the fixed points of an arbitrary mapping of a simplex into itself. Based on this idea, Scarf (1967) Scarf and Hansen (1973) Scarf (1982) developed such an algorithm which has been used to find the equilibrium of an economy via a fixed point problem. His idea is based on Sperner's Lemma: Divide a simplex S into subsimplices. Assign every vertex of these simplices a label from $\{1, 2, ..., n\}$ where $n < \infty$ in such a way that a vertex v on the boundary of S is labeled with i for which $v_i = 0$. For finding the fixed point, start in the corner of a S where there is a subsimplex labeled with $\{2, ..., n\}$. Move from this subsimplex to another which have the same labels. When the next subsimplex we move have the label 1, the algorithm stops. Scarf argues that the algorithm must terminate with a subsimplex whose vertices have all the labels $\{1, 2, ..., n\}$ because the there are finite number of subdivisions. The point where the algorithm stops is the fixed point a.k.a. the equilibrium of the economy. More precisely, after finding all the Pareto optimal allocations and proving the existence by Negishi method, if we apply this algorithm to a price simplex, the price where the algorithm stops would be our competitive equilibrium price system.

3.1.2 Comparative Statics in Pareto Optimal Economies

Lastly, comparative statics predictions could also be obtained in case of Pareto optimality. For a global comparative statics prediction, we could use Lattice programming. For example, assume it is a growth model which social planner chooses investment level and we are interested how investment behaves in response to changes in discount parameter which is commonly denoted by β in most of the models. Assuming that the model satisfies all the necessary conditions for the existence of a value function, we check if the value function is supermodular in investment and satisfies increasing differences in investment and discount factor. If both conditions hold, we say, by Veinott (1992) or Topkis (1978), the investment is monotone in β. On the other hand, if local predictions were sought, we could use implicit function theorem on first order conditions or envelope theorem on value functions. As mentioned above, there are certain conditions to be satisfied for these methods such as convexity of preferred sets, smoothness of the functions, convexity of the objective function and interiority so we should check them. In general, when we have Pareto optimality, comparative statics results could be obtained through usual methods.

The rest of the paper is as follows. In the next section, I set up the basic one sector growth model with elastic labor supply. In Section 3.3, conditions for existence of recursive equilibrium is provided and proved. In Section 3.4, I characterize the theory of robust comparative statics. Finally, section 3.5 concludes.

3.2 Model

3.2.1 Setup and Primitives

The model is a dynamic general equilibrium model with production externalities and complementarities. The production function is adapted from Benhabib and

Farmer (1994). In its reduced-form, production structure covers many models in the literature with multiple sequential equilibria such as the models with distortionary taxes, externalities and monopolistic competition.

Time is discrete and indexed by $t \in T = \{0, 1, 2, ...\}$. The economy has a continuum of identical infinitely-lived household/firm agents with nonseparable preferences within each period. The preferences are over consumption and leisure and time separable over lifetime streams of these goods. I assume that each household is endowed with one unit of time and enters into a period with an individual capital stock $k \in \mathbb{K}$ where $\mathbb{K} \subset \mathbb{R}_+$ is a compact order interval and convex subset of $\mathbb{R}_+$. Households own the firms as well as both factors of production which are capital and labor. They rent these factors to competitive firms and enjoy the income out of these rents.

The firm technology $F(k, n, K, N, \theta)$ is constant returns to scale in private inputs of capital k and labor n, and also is subject to positive externalities due to aggregate capital K and labor N. The productivity shock $\theta \in \Theta$ is exogenous and follows a first order Markov process with stationary transition matrix $\chi(\theta, d\theta^{'})$. For simplicity, assume that the state space for the shock Θ is a finite set. Also, throughout the paper I will denote the next period variables with prime $.'$.

To elaborate more on production externalities, one can interpret capital external effects as knowledge. When a firm creates a new knowledge via R&D activities, it will have a positive external effect on the production possibilities of other firms. The reason is that knowledge cannot be perfectly kept as a secret due to spillover effects [7]. On the other hand, labor external effects can be attributed to the thick market externalities. In boom times, for instance, there are many firms and labors in the market which is described as thick market. In a thick market, as workers exert more

[7]Production externality is first introduced by Romer (1986). Production externality has been one of the fundamental pillars in endogenous growth literature after Romer's seminal paper.

search effort to find a job, the employment rate will rise. On the other hand, it will be more easy for a firm to match with a worker so the production costs will fall. As a result, more workers in the market lead to positive externality on the production of individual firms.

For each period t and state (k, θ), households choose how much to consume and save, as well as how they allocate their one unit of time between leisure l and work n. Households' lifetime preferences are time seperable, there is constant discounting at rate $\beta \in (0, 1)$, with period preferences represented by $u(c_t, l_t)$ where $(c, l) \in R_+ \times [0, 1]$. Therefore, lifetime preferences for a typical household are given by:

$$U(c, l) = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \right\} \tag{3.1}$$

where the expectation operator comes from the probability structure of the stochastic productivity shock induced by the Markov process $\chi(\theta, d\theta')$.

I make the following baseline assumption on preferences and technologies throughout the paper:

Assumption 1 (i) The period utility function is supermodular, bounded above, C^2, strictly increasing, strictly concave in (c, l) satisfies the following Inada conditions:

$$\lim_{c \to 0} u_1(c, l) = \infty \tag{3.2}$$

$$\lim_{c \to \infty} u_1(c, l) = 0 \tag{3.3}$$

$$\lim_{l \to 0} u_2(c, l) = \infty \tag{3.4}$$

$$\lim_{l \to 1} u_2(c, l) > 0 \tag{3.5}$$

(ii) Further, we assume that

$$\lim_{l \to 1} u_1(c, l) > 0 \text{ for all } c > 0$$

(iii) The production function $F(k, n, K, N, \theta) = f(k, n, \theta) \cdot e(K, N)$ is jointly C^2, strictly increasing and supermodular in all arguments, such that private returns f :

$\mathbb{K} \times [0,1] \times \Theta \to \mathbb{R}_+$ is constant returns to scale (CRS, hereafter) and strictly concave in (k, n), and social externality function $e : \mathbb{K} \times [0,1] \to \mathbb{R}_+$ is strictly increasing in both arguments. We also assume $F(k, n, K, N, \theta)$ satisfies the following Inada conditions for all $(K, N, \theta) > 0$:

$$\begin{aligned} \lim_{k \to 0} F_1(k, n, K, N, \theta) &= \infty \text{ for all } n > 0 \\ \lim_{k \to \infty} F_1(k, n, K, N, \theta) &= 0 \text{ for all } n > 0 \\ \lim_{n \to 0} F_2(k, n, K, N, \theta) &= \infty \text{ for all } k > 0 \end{aligned}$$

(iv) Moreover, we assume that

$$\lim_{n \to 0} f_n(k, n, \theta) e(k, n) = 0$$

First, by Assumption $1(i)$, as the utility function is increasing in l and concave, labor supply curve is upward sloping. Second, Assumption $1(ii)$ and Inada condition (3.5) imply that when individual decide not to work, the wage rate is still positive. Therefore, labor supply curve has a positive intercept. Lastly, Assumption $1(iv)$ guarantees that for $0 < n < 1$ which is sufficiently small, labor demand curve is initially below the labor supply curve and that there will be even number of intersections between labor supply and labor demand curves. Hence, there will be Euler equation branching whenever there is multiplicity of equilibria. This assumption is important to prove continuity of labor supply which I do in Lemma 3.3.2.

Assumption 2 Second partial order derivatives of the utility function satisfy the following conditions:

$$\frac{u_{11}}{u_1} \leq \frac{u_{21}}{u_2}, \frac{u_{22}}{u_2} \leq \frac{u_{12}}{u_1} \tag{3.6}$$

which is normality assumption. It means that marginal rate of substitution $\frac{u_2}{u_1}(c, l)$ is increasing in c and decreasing in l.

3.2.2 *Firm's Problem*

We now construct the firm's problem in a recursive equilibrium. Firms solve a static profit maximization problem. Representative firm maximizes profit by choosing capital k and labor demand n. Define the firm's profit function $\Pi : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{K} \times [0,1] \times \Theta \to \mathbb{R}_+$ such that

$$\Pi(R, w, K, N, \theta) = \max_{k \in \mathbb{K}, n \in [0,1]} F(k, n, K, N, \theta) - R(K, N, \theta)k - w(K, N, \theta)n$$

where $R = R(K, N, \theta)$ and $w = w(K, N, \theta)$.

Under Assumption $1(ii)$, the production function is continuous. Hence the objective function is continuous whereas the constraint set is nonempty and compact. Then, by Weierstrass Theorem, there exists an optimal solution to this maximization problem.

By definition of recursive competitive equilibrium, the equilibrium factor prices along recursive equilibrium path is defined where $k = K, n = N$ for each θ. Hence, the equilibrium prices are as follows:

$$R(K, N, \theta) = f_1(K, N, \theta) \cdot e(K, N) \tag{3.7}$$

$$w(K, N, \theta) = f_2(K, N, \theta) \cdot e(K, N) \tag{3.8}$$

Hence, given any law of motion for aggregate state variables, such as $K' = g(K, \theta)$, as well as a continuous function for equilibrium labor supply $N(K, \theta)$ one can derive sequential equilibrium factor prices given any set of initial conditions $(K_0, \theta_0) >> 0$.

3.2.3 Household's Problem

Assume $N(K,\theta)$ is continuous function of aggregate state variables by zero profit condition under CRS such that $\forall(K,\theta)$, where $K > 0$, $N(K,\theta) \in [0,1]$. Then the household's income process each period is as follows:

$$y(k,n,K,N,\theta) = R(K,N(K,\theta),\theta)k + w(K,N(K,\theta),\theta)n + (1-\delta)k + \Pi$$

Here δ stands for the depreciation rate of the capital.

I define the household's budget set as follows:

$$\Gamma(s;N) = \{(c,l)|\ c + k' \leq y(k,n,K,N,\theta),\ c \geq 0,\ k' \in \mathbb{K},\ l \in [0,1]\} \tag{3.9}$$

where I use $'$ to indicate the variables next period.

Under assumption $1(iii)$, because the production function is strictly concave and continuous, the budget constraint set is non-empty, compact, convex and continuous.

We now construct a dynamic programming representation of the household's sequential problem. The state space for household's dynamic program is denoted by $s = (k,K,\theta) \in \mathbb{S} = \mathbb{K} \times \mathbb{K} \times \Theta$. Give $\mathbb{S}$ its componentwise partial order. Then, the minimal state space for the economy is a subspace of $\mathbb{S}$. I denote this minimal state space as $S = D \times \Theta$ where D is the diagonal of $\mathbb{K} \times \mathbb{K}$:

$$S = D \times \Theta = \{(K,\theta) \in \mathbb{S} \mid s = (K,K,\theta), (K,K) \in D, K \in \mathbb{K}\}$$

Define $B^f(S)$ to be the subset of the space of bounded functions that satisfies the household budget constraint on a minimal state space $s \in \mathbb{S}$:

$$\begin{aligned} B^f(S) = \{c(K,\theta), N(K,\theta) \quad | \quad & N(K,\theta) \in (0,1] \text{ when } K > 0, K = (K,K) \in D, \\ & 0 \leq c(K,\theta) \leq y(K,\theta;N)\} \end{aligned} \tag{3.10}$$

Note that, in minimal state space $N(K,\theta) = n(K,K,\theta)$ and as the time endowment of individual is 1, $n = 1 - l$ as well as the aggregate labor supply $N = 1 - L$

where L stands for the aggregate leisure time. Denote $y(K, N(K,\theta), K, N(K,\theta), \theta) = y(K,\theta; N)$ for notational ease. Therefore, the household assumes the following law of motion for aggregate capital stock K:

$$K' = g(K,\theta) = y(K,\theta;N) - c(K,\theta) \text{ such that } (c(K,\theta), N(K,\theta)) \in B^f(S) \quad (3.11)$$

Now, given the initial state variables $(k_0, K_0, \theta) >> 0$, households calculate equilibrium prices by using $(c(K,\theta), N(K,\theta))$ from $(k_0, K_0) >> 0$. The state variables for the household's problem are $s = (k, K, \theta) \in \mathbb{S}$ Then, the representative household solves the following dynamic problem: given $(c, N) \in B^f(S)$ with $K \in \mathbb{K}_* = \mathbb{K} - \{0\}$, $\theta \in \Theta$, we have $g > 0$, $N > 0$, the household's value function $V : \mathbb{K} \times \mathbb{K}_* \times \Theta \times B^f(\mathbb{S}) \to \mathbb{R}$ satisfies the following parameterized Bellman equation:

$$V(s; c, N) = \max_{c,l}\{u(c,l) + \beta \sum_{\theta' \in \Theta} V(y(k,n,K,N,\theta) - c, g(K,\theta), \theta; c(g,\theta), l(g,\theta))\chi(\theta, d\theta^{'})\} \quad (3.12)$$

where $c \in [0, y(k,n,K,N,\theta)]$ and $l \in [0,1]$ and $n = 1 - l$.

Under Assumption 1(i) and (iii), given any aggregate policy function $g(K,\theta)$, and so $(c(K,\theta), N(K,\theta))$ and a transition function $\chi(\theta,\theta')$, there exists a unique value function that satisfies (3.12). Moreover, the value function is strictly increasing, differentiable and strictly concave in k. Further, it is already mentioned that household budget set (3.9) is compact and convex. Then, in the light of Coleman (1991), I can now define a monotone operator which maps into itself and the fixed point of this operator will become the recursive competitive equilibrium of the economy. Thus, the pair $(c^*(k,\theta), N^*(k,\theta))$ is a recursive equilibrium if and only if (c^*, N^*) solves the following recursive equilibrium functional equation when $k = K$, $\theta \in \Theta$.

$$\begin{aligned} A^*(c^*(k,\theta), N^*(k,\theta))(k,k,\theta) &= (c^*(k,k,\theta;c^*,l^*), n^*(k,k,\theta;c^*,l^*)) \quad (3.13) \\ &= (c^*, n^*)(k,\theta)\ ,\ k = K > 0,\ \forall\theta \in \Theta \\ &= 0, \text{ else} \end{aligned}$$

where $n^*(k,k,\theta;c^*,N^*) = 1 - l^*(k,k,\theta;c^*,N^*)$ and $N^*(k,\theta) = 1 - L^*(k,\theta)$ in equilibrium.

Formally, I define the recursive equilibrium as follows:

Definition 3.2.1 *A minimal state space recursive equilibrium is pair of functions* $(c^*(k,\theta), N^*(k,\theta)) \in B^f(\mathbb{S})$ *such that each of them are strictly positive when* $k > 0$ *for all* $\theta \in \Theta$ *and the value function* $v^*(k,k,\theta;c^*,l^*)$ *that solves (3.12) if they satisfy the followings:*

(i) *given* $g^*(k,\theta) = (y^* - c^*)(k,\theta;N^*)$ *and aggregate labor supply* $N^*(k,\theta)$*, when* $K = k > 0$ *for all* $\theta \in \Theta$*, the functions* $c(s;c^*,N^*), l(s;c^*,N^*)$ *are optimal solutions of the dynamic program in (3.12) with associated value function* $V(s;c^*(k,\theta),N^*(k,\theta))$

(ii) *the optimal solutions* $(c(s;c^*,N^*), l(s;c^*,N^*)) = (c^*(k,\theta), l^*(k,\theta)) \in B^f(S)$ *solve the necessary and sufficient fixed point problem in (3.13) if* $k = K > 0$ *and*

(iii) $(c(s;c^*,l^*), l(s;c^*,l^*)) = 0$*, else.*

In the light of Stokey (1989) Theorem 9.8, because $\mathbb{K} \in \mathbb{R}_+$ is a closed,convex subset, Θ is a countable set under $\sigma - algebra$, the budget set $\Gamma(s;N)$ in (3.9) is non-empty, compact-valued, convex and continuous, utility function is bounded, strictly concave and continuous under assumption $1(i)$, $\beta \in (0,1)$, the value function is strictly concave. Hence, by Mirman and Zilcha (1975) Lemma 1, it is at least once continuously differentiable in k.

Therefore, I can characterize the recursive equilibrium functional equation in (3.13) by writing the Euler equation as well as the first order conditions governing consumption and leisure decisions along the recursive equilibrium paths. Hence, the optimal consumption $c^* = c^*(k,k,\theta;c^*,N^*)$ and leisure decision $l^* = l^*(k,k,\theta;c^*,N^*)$ in any recursive equilibrium satisfy the following Euler equation given $K = k > 0$ and $\theta \in \Theta$. For simplicity, I denote $c^* = c^*(k,k,\theta)$ and $c^*(g^*,g^*,\theta';g^*(k,\theta))$ means consumption next period given the law of motion for aggregate state variable $g^*(k,\theta)$. Same for the leisure decision.

$$
\begin{aligned}
Z^*(k,k,\theta;c^*(k,\theta),N^*(k,\theta)) = -u_1(c^*,l^*) \\
+\beta\sum_{\theta'\in\Theta} u_1(c^*(g^*,g^*,\theta';g^*(k,\theta)),l^*(g^*,g^*,\theta^{'};g^*(k,\theta))) \\
\times f_1(g^*(k,\theta),(1-l^*(g^*,g^*,\theta';g^*(k,\theta))),\theta') \\
\times e(g^*(k,\theta),1-l^*(g^*(k,\theta)))\times\chi(\theta,\theta')
\end{aligned}
\tag{3.14}
$$

Although it will be mathematically more clear in the next sections, it is important to talk about the idea of multiplicity at this point. Normally, if this was a unique equilibrium economy, I would expect Z^* function to be monotonic with respect to state variable k. Hence, the equilibrium investment level would be defined at the point where $Z^*=0$. However, given the existence of externality, above Z^* function loses its monotonicity property and because of that, the function Z^* may be equal to zero at either finitely many times of within a continuum of points. Hence, I say that there are multiple of equilibria.

Rather than working on this Euler equation to prove the existence and computing equilibria, I start with the following two conditions which can be derived from the dual formulation of the problem which is the Lagrange equation. These are the relationships between consumption and leisure with Lagrange multiplier $(\frac{1}{m})$ of the dual formulation.

$$u_1(c,l) = \frac{1}{m} \tag{3.15}$$

$$u_2(c,l) = \frac{1}{m}w \tag{3.16}$$

Equation (3.15) is obtained by taking the first order condition of Lagrange equation with respect to c and equation (3.16) is obtained by taking the first order condition with respect to l. Using (3.15) and (3.16), the equilibrium relationship between the marginal rate of substitution and wage is given as follows:

$$\frac{u_2(c,l)}{u_1(c,l)} = w(K,N,K,N,\theta) \tag{3.17}$$

Equation (3.17) shows that how many unit of time a consumer is willing to work more to increase her consumption one more unit.

Equation (3.15) and (3.17) are basically the two side conditions I will be using to define contingent recursive equilibrium of consumption and labor decisions as a function of aggregate state variable, inverse marginal utility and productivity shock. It is also this step, where I define all the necessary and sufficient conditions to verify the existence of recursive equilibrium at any given aggregate state as well as providing robust equilibrium comparative statics. Section 3.3.1 basically puts these ideas into a mathematical structure.

3.3 Existence of Equilibrium

3.3.1 Solving Side Conditions

This section constructs the contingent decisions for consumption and leisure. The equilibrium is *contingent* in the sense that they depend on the aggregate state variables which are (K,θ) and inverse marginal utility of consumption m.

Additionally, this section also provides comparative statics on contingent consumption and leisure decisions. The comparative statics predictions are robust in the sense that they are valid in every subclass of equilibria.

The analysis starts with constructing leisure decision under the following assumption:

Assumption 3 $F(k, n, K, N, \theta)$ is such that $w(K, N, \theta) = f_2(K, N, \theta)e(K, N)$ is

(i) *(Small Labor Externality)* decreasing in N

(ii) *(Large Labor Externality)* strictly increasing in N in equilibrium.

In equilibrium. there are two driving forces behind the movement of wage rate in response to changes in labor supply in the economy. One of which is the concavity of the production function and the other is the size of the externality in the economy. One can see it from the following equation which is basically the derivative of wage rate in response to labor supply:

$$F_{22}(K, N, K, N, \theta) = f_{22}(K, N, \theta)e(K, N) + f_2(K, N, \theta)e_2(K, N) \tag{3.18}$$

where the first term captures the curvature of the production function while the second term captures the size of the externality. Now, under Assumption 1(iii), when Assumption 3(i) holds, the concavity of the production technology dominates the effect of the labor externality. Thus, I define it as small labor externality. On the other hand, if Assumption 3(ii) holds, the second term in equation (3.18) outweighs the first term. Hence, I define it as large labor externality.

In a candidate recursive equilibrium, $K = k$ and $L = l$. Under small externality assumption, take equation (3.17) and define the following mapping:

$$Z_1^l(l, K, C, \theta) = \frac{u_2(C, l)}{u_1(C, l)} - w(K, 1 - l, \theta) \tag{3.19}$$

Then, the leisure choice can be defined as

$$\begin{aligned} l_1^*(K, C, \theta) &= \{l | Z_1^l = 0, C > 0\} \\ &= 0 \text{ if } C = 0 \end{aligned}$$

When large externality is assumed, Z_1^l mapping above would be insufficient to use for computing equilibrium and providing comparative statics predictions. The reason is the mapping will lose its monotonic structure under large externality. Hence, I define another mapping which lets me compute off-path leisure choice and obtain comparative statics predictions for the least and greatest leisure decisions. Thus, when $K = k$ but $L \neq l$, define

$$Z_2^l(l, K, C, L, \theta) = \frac{u_2(C, l)}{u_1(C, l)} - f_2(K, 1 - l)e(K, 1 - L) \tag{3.20}$$

Then,

$$\begin{aligned} l_2^*(K, C, L, \theta) &= \{l | Z_2^l = 0, C > 0\} \\ &= 0 \text{ if } C = 0 \end{aligned}$$

Lemma 3.3.1 ***(i)*** *Under Assumption* 1, 2 *and* 3(i)*, when* $l = L$*, there exist* $l_1^*(K, C, \theta)$ *that is unique and continuous* (K, C, θ) *and* $l_1^*(K, C, \theta)$ *is increasing in* C *and decreasing in* K*.*

(ii) *Under Assumption* 1, 2, 3(ii)*,* $l_2^*(K, C, L, \theta)$ *is well defined, increasing in* (L, C) *and decreasing in* K*.*

(iii) *For fixed* (K, C)*, when* $l \neq L$*,* $l_2^*(L; K, C, \theta)$ *is an increasing transformation of* $[0, 1] \to [0, 1]$

(iv) *Under Assumption* 1, 2, 3(ii)*, if* $\Psi(K, C, \theta)$ *is the set of fixed points of* $l_2^*(L; K, C, \theta)$*,* $\Psi(K, C, \theta)$ *is a complete lattice with selections* $\wedge\Psi(K, C, \theta) = l_\wedge(K, C, \theta)$ *and* $\vee\Psi(K, C, \theta) = l_\vee(K, C, \theta)$ *increasing in* C *and decreasing in* K*.*

Proof 3.3.0.1 ***(i)*** *Consider equation (3.19). First I show there exists a well-defined* $l_1^*(K, C, \theta)$*. By Assumption* 2*, the first term is decreasing in* l*. Further, by assumption* 3(i)*,* $w(K, 1-l, \theta)$ *is strictly increasing in* l*. Then,* $Z^l(l, K, C, \theta)$ *is strictly decreasing in* l*.*

Also, by Assumption 1(i)*, utility function satisfy Inada conditions so*

$$\lim_{l \to 0} Z^l(l, K, C, \theta) = \infty$$

By Assumption $1(iii)$*, production function satisfies Inada condition: as* n *goes to* 0 *(which equivalently says* l *goes to* 1*), marginal product of labor* $w(K, 1-l, \theta)$ *goes to* ∞*. Moreover, as* n *goes to* 0*,* $\frac{u_2(C,l)}{u_1(C,l)}$ *is finite. Then,*

$$\lim_{l \to 1} Z^l(l, K, C, \theta) = -\infty$$

Given continuity assumptions on utility function and prices, there exist a unique $l_1^*(K, C, \theta)$ *which makes* $Z^l(l, K, C, \theta) = 0$*.*

Further, as the utility and production functions are assumed to be continuous under assumptions $1(i)$ *and* $1(iii)$*,* $l_1^*(K, C, \theta)$ *is continuous.*

Now, I need to show that $l^*(K, C, \theta)$ *is increasing in* C *and decreasing in* K*. By assumption 2,* $Z^l(l, K, c, \theta)$ *is increasing in* C *so for any* $C' \geq C$ *where* $C, C' \in [0, y(k, n, K, N, \theta)]$*,* $Z^l(l_1(K, C, \theta), K, C', \theta) > Z^l(l_1(K, C, \theta), K, C, \theta) = 0$*. As* $Z^l(l, K, C, \theta)$ *is also strictly decreasing in* l*,* $l_1^*(K, C', \theta) \geq l_1^*(K, C, \theta)$*. Thus,* $l_1^*(K, C, \theta)$ *is increasing in* C*.*

Further, by assumption $1(iii)$*, production function is increasing in all arguments so wage rate is increasing in* K*. Thus,* $Z^l(l, K, C, \theta)$ *is decreasing in* K*. Together with decreasing property of* $Z^l(l, K, C, \theta)$*, we conclude that* $l_1^*(K, C, \theta)$ *is decreasing in* K*.*

(ii) *Consider equation (3.20). The first term of* Z^l *is decreasing in* l *by Assumption 2 and the second term is increasing in* l *by concavity assumption of* $f(k, n)$*. Thus,* Z^l *is decreasing in* l*. It also satisfies the necessary Inada conditions by assumptions* $1(i)$ *and* $1(iii)$ *so there exists a unique* $l_2^*(K, C, L, \theta)$*. Hence,* $l^*(K, C, L, \theta)$ *is well-defined.*

Z^l is increasing in L as $e(K,N)$ *is strictly increasing in both arguments by assumption* $1(iii)$*. Then, for every $L' \geq L$ where $L', L \in [0,1]$, $Z^l(l_2^*(K,C,L,\theta),K,C,L',\theta) \geq Z^l(l_2^*(K,C,L,\theta),K,C,L,\theta) = 0$. As Z^l is decreasing in l, $l_2^*(K,C,L',\theta) \geq l_2^*(K,C,L,\theta)$. Hence, it is increasing in L.*

Similarly, by assumption 2*, Z^l is increasing in C. Hence, $l_2^*(c,K,L,\theta)$ is increasing in C.*

Lastly, by Assumption $1(iii)$*, as Z^l is decreasing in K, so is $l_2^*(K,C,L,\theta)$.*

(iii) *For fixed (K,C), $l^*(L;K,C,\theta)$ is defined as follows:*

$$l^*(L;K,C,\theta) = l_2^*(K,C,L,\theta) \text{ such that } Z^l(l,K,C,L,\theta) = 0$$

Note that $l^*(L;K,C,\theta)$ maps into itself; otherwise $Z^l(l,c,K,L,\theta)$ cannot be zero.

Now, I need to show that $l^*(L;K,C,\theta)$ is increasing in L. Take $L^{'} \geq L$ for every $L^{'}$ and $L \in [0,1]$. By definition of $l^*(L;K,C,\theta)$, $Z^l(l^*,K,C,L,\theta) = 0$. As $e(K,N)$ *is strictly increasing in N by assumption* $1(iii)$*, $Z^l(l,K,C,L,\theta)$ is strictly increasing in L. Hence, $Z^l(l^*(L;KC\theta),c,K,L^{'},\theta) \geq 0$. I have already shown that $Z^l(l,c,K,L,\theta)$ is decreasing in l so $l^*(L^{'};c,K,\theta) \geq l^*(L;c,K,\theta)$.*

Thus, $l^*(L;K,C,\theta)$ is increasing in L and it maps from $[0,1]$ to itself.

(iv) *$l^*(L;K,C,\theta)$ is increasing transformation in L and it maps from $[0,1]$ to itself. Also, $[0,1]$ is a complete lattice so by Tarski's fixed point theorem (Tarski (1955)), there exists a complete lattice of fixed points of $l^*(L;K,C,\theta)$ which I call as $\Psi(K,C,\theta)$. Further, I will have least and greatest leisure choices $l_\wedge(K,C,\theta)$ and $l_\vee(K,C,\theta)$ such that*

$$\begin{aligned}\wedge\Psi(K,C,\theta) &= l_\wedge(K,C,\theta)\\ \vee\Psi(K,C,\theta) &= l_\vee(K,C,\theta)\end{aligned}$$

which are well-defined by Tarski's theorem.

I also show that both of these selections are increasing in C and decreasing in K. As both proofs are very similar to each other, I will only show for $\wedge\Psi^*(K, C, \theta)$.

Consider $Z^l(l, K, C, L, \theta)$ in equation (3.20). By Assumption 2, $Z^l(l, c, K, L, \theta)$ is increasing in C so $l^*(L; K, C, \theta)$ is increasing in C. Similarly, as $Z^l(l, K, C, L, \theta)$ is decreasing in K by assumption $1(iii)$, so is $l^*(L; K, C, \theta)$. Thus, $l^*(L; K, C, \theta)$ is increasing in C and decreasing in K. Equivalently, it is increasing in $(-K, C)$.

Now, $l^*(L; K, C, \theta)$ is in $[0, 1] \times \mathbb{K} \times [0, y(K, N, \theta)] \times \Theta$. $[0, 1]$ is a complete lattice and $\mathbb{K}$ and $[0, y(K, N, \theta)]$ are partially ordered sets. $l^*(L; K, C, \theta)$ is nonempty, subcomplete sublattice of L for each (K, C, θ) and by previous paragraph $l(L; K, C, \theta)$ is increasing in $(L, -K, C)$. Then, by Veinott's Fixed Point Comparative Statics Theorem (Chapter 4, Theorem 14) (Veinott (1992))), $\wedge\Psi(K, C, \theta)$ is increasing in $(-K, C)$. Equivalently speaking, $\wedge\Psi(K, C, \theta) = l^*_\wedge(K, C, \theta)$ is decreasing in K and decreasing in C.

I can further show these selections are continuous. This is a necessary step in the rest of the paper.

Lemma 3.3.2 *Under Assumptions $1, 2$ and $3(i)$ or $3(ii)$, $\vee\Psi(K, C, \theta) = l^*_\vee(K, C, \theta)$ and $\wedge\Psi(K, C, \theta) = l^*_\wedge(K, C, \theta)$ are continuous ordered solutions in (K, C).*

Proof 3.3.0.2 *I prove that $l^*_\wedge(K, m, \theta)$ is continuous. By Raines and Stockman (2010), Proposition 4, utility function satisfies necessary Inada and concavity conditions. Also, equilibrium labor demand condition satisfies the following by Assumption $1(iv)$.*

$$\lim_{N \to 0} f_2(K, N, \theta) e(K, N) = 0 \tag{3.21}$$

Now I am going to apply local Implicit Function Theorem on this Z^l function. As the utility function and production function is assumed to be $C2$ in Assumption 0, Z^l is

C1 function. Moreover, both $(0,1)$ *and* $(0,\bar{k})$ *are open subset of Banach space and* $\mathbb{R}$ *is another subset of Banach space. I know that by definition,*

$$Z(l_{\wedge}(c^*,K^*,\theta),c^*,K^*,l_{\wedge}(c^*,K^*,\theta),\theta)=0 \tag{3.22}$$

The implicit function theorem implies that

$$\frac{\partial l^*_{\wedge}(K,C,\theta)}{\partial w}=\frac{u_1^2}{u_{22}u_1-u_{12}u_2}<0 \tag{3.23}$$

which implies that $\frac{\partial n^*_{\wedge}(K,C,\theta)}{\partial w}>0$*. Hence, Frisch labor supply curve is upward sloping.*

Also, at $l=1$ *(equivalently* $n=0$*) with* $c>0$*, the implied* w *is strictly positive because* $u_1(c,1)>0$ *and* $u_2(c,1)>0$*. Hence, by equation (3.21), we know that for* N *sufficiently small, labor supply curve is above labor demand curve. Thus, the first transversal crossing of labor supply by labor demand is from below.*

Since $\lim_{N\to 1} f_2(K,N,\theta)e(K,N)<\infty$*, labor demand curve is eventually below labor supply curve. Hence, there is at least two crossings and the last one is from above. As we also have supermodularity of utility function, by Proposition* 5 *at Raines and Stockman (2010), Euler equation branching* [8] *exists so we can write* $l^*_{\wedge}(K,m,\theta)$ *as continuous function. Same argument applies for* $l^*_{\vee}(K,m,\theta)$*.*

Hence, I obtain the leisure choices as a function of (K,C,θ). Next step is to compute the 'contingent' consumption choices. For that, I need the following assumption:

Assumption 4 $u(c,l)$ satisfies the following condition:

$$\frac{u_{11}}{u_{12}}<\frac{\frac{u_{21}}{u_2}-\frac{u_{11}}{u_1}}{\frac{u_{22}}{u_2}-\frac{u_{12}}{u_1}+F_{22}(k,n,K,N,\theta)\frac{u_1}{u_2}} \tag{3.24}$$

[8] According to Stockman and Raines, in a typical dynamic model, the dynamics are described with a differential equation such as $\dot{x}=f(x)$. However, if we have Euler equation branching, the dynamics are described by a differential inclusion $\dot{x}\in H(x)$ where H is a set-valued function. Generally, $H(x)=\{f(x),g(x)\}$ where f and g are continuous functions (referred as 'branches').(Raines and Stockman (2010))

As I have already seen in the previous lemma, when the wage rate is increasing in n, I may have multiplicity of equilibria. In that case, I have analyzed the characteristics of the least and greatest selections of leisure choice. Hence, I am going analyze the characteristics of consumption decision by taking these selections into consideration. As for each different leisure choices, individual's consumption decision changes, in the following analysis, I denote individual consumption decision as $c_\wedge(.)$ when the leisure choice is $l_\wedge(c, K, \theta)$ and as $c_\vee(.)$ when the leisure choice is $l_\vee(c, K, \theta)$.

Equilibrium consumption decisions is defined by equation (3.15). For the least selection of leisure choice, the following is going to be the case:

$$u_c(c, l^*_\wedge(K, C, \theta)) - \frac{1}{m} = 0 \text{ if } c > 0 \tag{3.25}$$

where $\frac{1}{m} > 0$ as inverse marginal utility operator $m > 0$.

Now, define

$$Z^c(c, K, C, m, \theta) = u_1(c, l^*_\wedge(K, C, \theta)) - \frac{1}{m} \tag{3.26}$$

I have the following lemma:

Lemma 3.3.3 *Given Assumptions* 0*,* 1*,* 3(ii) *and* 4*, when* $c = C$*, there exists a unique* $c^*(K, m, \theta)$ *for both the least and greatest equilibrium leisure choice selection. Further, each equilibrium consumption choice is decreasing in* K *and increasing in* m*.*

Proof 3.3.0.3 *The proof is for* $l^*_\wedge(K, C, \theta)$*. The proofs for* $l^*_1(K, C, \theta)$ *under assumption* 3(i) *and* $l^*_\vee(K, C, \theta)$ *is very similar.*

Now, consider equation 3.26 when $C = c$*. Recall that* $l^*_\wedge(K, C, \theta)$ *is increasing in* C *by Lemma 3.3.1 so by assumptions* 1(i) *and* 4*,* $Z^c(K, c, m, \theta)$ *is strictly decreasing in* c*. Also, as* $u_c(c, l^*(K, c, \theta))$ *satisfies Inada conditions by Assumption* 1(i) *so is* $\hat{Z}^c(K, c, m, \theta)$*. Hence, there exist a unique* $c^*_\wedge(K, m, \theta)$*.*

Now, I need to prove that $c^*_\wedge(K, m, \theta)$ *is decreasing in* K *and increasing in* m*. Remember that I find in Lemma 3.3.1 that* $l^*_\wedge(K, C, \theta)$ *is decreasing in* K*. Hence, higher* K *leads to lower leisure choice which implies lower* $Z^c(K, c, m, \theta)$ *by supermodularity of utility function. So, higher* K *brings about lower* $c^*_\wedge(K, m, \theta)$*. Similarly, as* $Z^c(K, c, m, \theta)$ *is strictly increasing in* m*,* $c^*_\wedge(K, m, \theta)$ *is increasing in* m*.*

Lastly, I will prove the continuity of these selections.

Lemma 3.3.4 *Given Assumptions* 1*,* 2 3 *and* 4*,* $c^*_\wedge(K, m, \theta)$ *and* $c^*_\vee(K, m, \theta)$ *is continuous in* (K, m)*.*

Proof 3.3.0.4 *By Lemma 3.3.1, leisure choice is continuous. I also know that by assumption* $1(i)$*, the utility function is continuous. Consider equation 3.26 again. Now, take a convergent series such that* $(K^n, m^n) \to (K, m)$*. As the utility function is continuous by assumption* $1(i)$ *and leisure choice is continuous by Lemma 3.3.1,* $C^*_\wedge(K, m, \theta)$ *is continuous in* (K, m)

I have $c^*_\wedge(K, m, \theta)$ and $l^*_\wedge(K, c, \theta)$. We can plug contingent consumption decisions into leisure decision to get contingent leisure choice as a function of (K, m, θ). Same argument applies for the greatest selection as well. Hence, $l^*_\wedge(K, m, \theta) = l^*_\wedge(K, c(K, m, \theta), \theta)$.

As $c^*_\wedge(K, m, \theta)$ is increasing in m and $l^*_\wedge(K, c, \theta)$ is increasing in c, $l^*_\wedge(K, m, \theta)$ is increasing in m.

Similarly, as $c^*_\wedge(K, m, \theta)$ and $l^*_\wedge(K, c, \theta)$ are decreasing in K and $c^*_\wedge(K, c, \theta)$ is decreasing in K, $l^*_\wedge(K, m, \theta)$ is decreasing in K.

3.3.2 Fixed Point Decomposition

Before proceeding with constructing an increasing fixed point decomposition, I now describe the mathematical structure of it. The idea of an increasing fixed point decomposition was introduced in Datta *et al.* (2017). Let $(G(S), \geq), (H_1(S_1), \geq), (H_2(S_2), \geq)$ be three Banach spaces where S is a subspace of S_i for each $i = 1, 2$. Assume

$$A^* : G(S) \to G(S)$$

is the fixed point problem I am trying to solve.

Definition 3.3.1 *An operator $A : H_1(S_1) \times H_2(S_2) \to H_1(S_1)$ is a* ***fixed point decomposition*** *of $A^*(h)(s)$ if the mapping A satisfies the following conditions:*

(i) *The partial map $A(h_1; h_2(s_2)(s_1)$ has a nonempty fixed point correspondence $\Psi_A(h_2(s_2))(s_1) \subset H_1(S_1)$. for each $(h_2) \in H_2(S_2)$,*

(ii) *There exist a selection $B(h_2)(s_2) \in \Psi_A(h_2(s_2))(s_1)$ such that the mapping $B(h_2)(s)$ when restricted to the subspace $S \in \mathbb{S}$ has a nonempty set of fixed points Ψ_B and each fixed point $h_2^* \in \Psi_B$ satisfies*

$$\begin{aligned} B(h_2^*(s)) &= h_2^*(s) = A^*(g^*)(s) \\ &= g^*(s) \in \Psi_A(s) \subset G^*(S) \end{aligned}$$

Definition 3.3.2 *Let $H_i(S_i), G^*(S)$ are all partial ordered sets. An operator $A(h_1(s_1), h_2(s_2))$ is an* ***increasing fixed point decomposition*** *of $A^*(g^*)$ at (g^*) if*

(i) *A is a parameterized fixed point decomposition of $A^*(g)(s)$ at $(g^*(s)) \in G^*$*

(ii) *$A(h_1, h_2)$ is jointly increasing on $H_1 \times H_2$ endowed with its product order for each $(s_1, s_2) \in S \times S$*

(iii) *The partial map $B(h_2)(s) \in \Psi_A(h_2(s))$ is a jointly increasing selection $(h_2(s)) \in G^*$ in $\Psi_A(h_2(s))(s)$ for $s \in S$.*

Definition 3.3.3 *$A(h_1(s_1), h_2(s_2))$ is an* ***order continuous fixed point decomposition*** *of $A^*(g^*)$ at (g^*) if*

(i) *$A(h_1(s_1), h_2(s_2))$ is jointly order continuous in $(h_1(s_1), h_2(s_2))$.*

(ii) *$B(h_2)(s)$ is an order continuous selection in $\Psi_A(h_2(s))$ in h_2 for each $s \in S$.*

3.3.3 Constructing an Increasing Fixed Point Decomposition

I now use these results on the equilibrium side conditions to solve for recursive equilibrium. Using the side conditions, we can write the equilibrium evolution of the individual and aggregate state variables (k, K) when $k = K$ in terms of the inverse marginal utility of consumption m defined in equation (3.15). To do this, using $[1 - l^*(K, m, \theta)] = n^*(K, m, \theta) = n^* = N^*(K, m, \theta)$, equilibrium income can be as follows:

$$y = R(K, n^*, \theta)K + w(K, n^*, \theta)n^* + (1 - \delta)K + \Pi = F(K, n^*, \theta; n^*). \tag{3.27}$$

where $F(K, n^*, \theta; n^*) = F(K, n^*(K, m, \theta), K, n^*(K, m, \theta), \theta)$. We can then update next period's state given today's inverse marginal utility m by using the period budget constraint of household in equilibrium: i.e., using the least solution for leisure, we have the following law of motion on K:

$$g^{\wedge}(K, m, \theta) = F(K, n^*_{\wedge}(K, m, \theta), \theta; n^*) - c^*(K, m, \theta) = K' \tag{3.28}$$

where $n^*_{\wedge}(K, m, \theta) = 1 - l^*_{\wedge}(K, m, \theta)$.

We have the following useful lemma under either Assumption 3(i) or 3(ii):

Lemma 3.3.5 *Let assumptions* $1, 2$ *and* $3(i)$ *or* $3(ii)$ *hold. Then, in any recursive equilibrium* $(c^*, N^*)(s)$ *when* $K = k > 0$*, recursive equilibrium investment* $g^{\wedge}(K, m, \theta)$ *is increasing in* k *for each* (m, θ) *and decreasing in* m *for each* (K, θ)*.*

Proof 3.3.0.5 *By Lemma 3.3.1,* $l^*_{\wedge}(K, m, \theta)$ *is increasing in* m *and decreasing in* K*. Hence,* $n^*_{\wedge}(K, m, \theta)$ *is decreasing in* m *and increasing in* K*. By Lemma 3.3.3,* $c^*(K, m, \theta)$ *is decreasing in* K *and increasing in* m*.*

Then, if we increase K*, first term in equation (3.28) will increase, while the second term will decrease. Therefore,* $g^{\wedge}(K, m, \theta)$ *is increasing in* K*.*

Further, if we increase m, the first term will decrease, while the second term to increase. Therefore, $g^{\wedge}(K, m, \theta)$ is decreasing in m.

Next, we look at the return on capital in equilibrium. To study the movements of the return on capital in terms of k, we introduce a new state variable, say $\hat{k}$, and decompose the comparative static into movements of k into "increasing" and "decreasing" terms, and study the behavior of the return on capital along the restriction that $k = \hat{k}$. Along those lines, define using the least equilibrium labor solution:

$$R(k, \hat{k}, m; n^*_{\wedge}(k, m, \theta)) = f_1(k, n^*_{\wedge}(\hat{k}, m, \theta), \theta) e(\hat{k}, n^*_{\wedge}(\hat{k}, m, \theta)) \tag{3.29}$$

We have the following lemma.

Lemma 3.3.6 *Under Assumption $1, 2$ and $3(i)$ or $3(ii)$, $R(k, \hat{k}, m; n^*_{\wedge}(k, m, \theta))$ is decreasing in k and m, and increasing in $\hat{k}$.*

Proof 3.3.0.6 *Consider equation (3.29). The function f is the only term which depends on k and by concavity of f in Assumption $1(iii)$, R is decreasing in k.*

Now I am going to show that R is decreasing in m. By Lemma (3.3.1), it is known that $l^*_{\wedge}(k, m, \theta)$ is increasing in m and $n^*_{\wedge}(k, m, \theta) = 1 - l^*_{\wedge}(k, m, \theta)$ so $n^*_{\wedge}(k, m, \theta)$ is decreasing in m. Hence, by supermodularity of f under Assumption $1(iii)$, lower n^*, leads to a lower f_1 and as e is increasing in its first arguments under Assumption $1(iii)$, lower n leads to lower e. Combining these two statements yields R is decreasing in m.

Finally, for R being increasing in $\hat{k}$, note by Assumption $1(iii)$, e and f_1 is increasing (as f is supermodular), and $n^*_{\wedge}$ is increasing in $\hat{k}$ by 3.3.1. Combining all of these facts gives us the result that R is increasing in $\hat{k}$.

We are now ready to define a two step monotone map method to construct recursive equilibrium on a minimal state space. To do this, first define the following mapping which is based on the household's Euler equation when $K = k$:

$$\begin{aligned}
& u_1(c^*_\wedge(k,m,\theta), l^*_\wedge(k,m,\theta)) \\
= \ & \beta \sum_{\theta' \in \Theta} \{u_1(c^*_\wedge(g^\wedge(k,m,\theta), m(g^\wedge(k,m,\theta),\theta),\theta), l^*_\wedge(g^\wedge(k,m,\theta), m(g^\wedge(k,m,\theta),\theta),\theta)) \\
\cdot \ & f_1(g^\wedge(k,m,\theta), n^*_\wedge(g^\wedge(k,m,\theta), m(g^\wedge(k,m,\theta),\theta),\theta),\theta) \\
\cdot \ & e(g^\wedge(k,m,\theta), n^*_\wedge(g^\wedge(k,m,\theta), m(g^\wedge(k,m,\theta),\theta)) \cdot \chi(\theta,\theta')\}
\end{aligned} \tag{3.30}$$

where $l^*_\wedge(k,m,\theta) = 1 - n^*_\wedge(k,m,\theta)$.

Normally, any investment function which satisfies above equation can be said to be one of the recursive equilibrium. Finding those recursive equilibrium would be easy if both sides of the above Euler equation was monotonic in k and m under Assumption 1. As this is not the case, to find the set of equilibria which satisfies the Euler equation (3.30), I construct an increasing fixed point decomposition for the recursive equilibrium functional equation defined in (3.13).

Hence, define the following two function spaces for m to use in the increasing fixed point decomposition:

$$\begin{aligned}
M_1(S) &= \{m_1(k,\theta) \mid 0 \le m_1(k,\theta) \le u_1^{-1}(F(k,m,\theta), l(F(k,m,\theta),k,\theta)) \\
& \quad \text{and } m \text{ is increasing in } k \text{ st. } g^\wedge(K,m,\theta) \text{ is increasing in } K\} \\
M_2(S) &= \{m_2(k,\theta) \mid 0 \le m_2(k) \le \infty \ \}
\end{aligned}$$

While using a fixed point decomposition, we need to be clear on where the recursive equilibrium exists for the actual operator $A^*(g)(s)$ defined on (3.13). For any decomposition of recursive equilibrium operators in the space $\mathbb{M}_1(S) \times \mathbb{M}_2(S)$, where aggregate state variables are elements of $\mathbb{M}_2$, the resulting recursive equilibrium will exist in the following space:

$$G^*(S;\mathbb{M}_2) = \quad \{m(k,\hat{k},\theta)| \text{ for each } \hat{k},\ m(k;\hat{k},\theta) \in \mathbb{M}_1; \quad \text{for fixed } k \text{ and each } \theta,\ m(k,\hat{k},\theta) \in \mathbb{M}_2(S),\ k=\hat{k}\} \tag{3.31}$$

Hence, as mentioned above, recursive equilibrium is defined on a subclass where investment ($g^\wedge(K,m,\theta)$) is increasing in K. Given these definitions, I note the following useful lemma:

Lemma 3.3.7 *Under Assumptions* 1 *and* 2*,*

(i) $\mathbb{M}_1(S)$ *is a complete lattice under pointwise partial order.*

(ii) $\mathbb{M}_2(S)$ *is a complete lattice under pointwise partial order.*

(iii) $G^*(S;\mathbb{M}_2)$ *is a complete lattice under pointwise partial order.*

Proof 3.3.0.7 ***(i)*** *To see* $\mathbb{M}_1(S)$ *is a complete lattice, take an arbitrary subset* $M_1 \subset \mathbb{M}_1(S)$*. Since the pointwise inf and sup operations on the elements of* M_1 *preserve pointwise bounds, we have* $0 \leq \wedge M_1 \leq u_1^{-1}(F(k,m,\theta), l(F(k,m,\theta),k,\theta))$ *and* $0 \leq \vee M_1 \leq u_1^{-1}(F(k,m,\theta), l(F(k,m,\theta),k,\theta))$*. Also, as for each* $m_1 \in M_1 \subset \mathbb{M}_1$*,* m_1 *is increasing in* k *such that* $(y-c)(k,m,\theta)$ *is increasing in* k *for each* θ*, pointwise inf and sup operations preserve the monotonicity property. Hence, both* $\wedge M_1$ *(respectively* $\vee M_1$*) is increasing in* k *such that* $(y-c)(k,\wedge m,\theta)$ *is increasing in* k *for each* θ *as well. Then,* $\wedge M_1 \in \mathbb{M}_1$ *and* $\vee M_1 \in \mathbb{M}_1$*. Thus,* $\mathbb{M}_1(S)$ *is a complete lattice.*

(ii) *Similarly, take an arbitrary set* $M_2 \subset \mathbb{M}_2$*. Pointwise infs are bounded by* 0 *from below and pointwise sups are bounded trivially by* $+\infty$ *from above. Hence,* $\mathbb{M}_2$ *is a complete lattice.*

(iii) *I next show that* $G^*(S;\mathbb{M}_2)$ *is a complete lattice under pointwise partial order. As I did previously, take an arbitrary* $G \in G^*(S;\mathbb{M}_2)$*. Since each* $m(k,\hat{k},\theta) \in G$ *is* *equicontinuous for each $\hat{k}$ (i.e., $m(k,\hat{k},\theta) \in \mathbb{M}_1$ for each $\hat{k}$ and θ), this property is preserved under pointwise inf and sup operators as mentioned in part (i). Similarly, for each (k,θ), $m(k,\hat{k},\theta) \in [0,\infty]$. Therefore, $\wedge G$ and $\vee G \in G^*$. Thus, $G^*(S;\mathbb{M}_2)$ is a complete lattice.*

Finally, using (3.30), we construct our operator as follows: when $K = \hat{k} = k > 0$, $m_1 > 0$, using equation (3.15), substitute $\frac{1}{m}$ for marginal utility of consumption in the original Euler equation (3.30), decompose the fixed point problem into one involving three unknown functions (m_1, m_2, m_3): i.e., define

$$\begin{aligned} &Z^m(x,k,\hat{k},m_1,m_2,m_3) = \frac{1}{x} \\ &- \beta \sum_{\theta' \in \Theta} \{ \frac{f_1(g^\wedge(k,m_3,\theta), n^*_\wedge(g^\wedge(\hat{k},m_2,\theta), m_1(g^\wedge(\hat{k},m_3,\theta)),\theta'),\theta')}{m_1(g^\wedge(k,x,\theta),\theta')} \\ &\cdot e(g^\wedge(\hat{k},m_2,\theta), n^*_\wedge(g^\wedge(\hat{k},m_2,\theta), m_1(g^\wedge(\hat{k},m_3,\theta),\theta'),\theta')) \cdot \chi(\theta,\theta')\} \end{aligned}$$

equivalently,

$$\begin{aligned} Z^m(x,k,\hat{k},\theta,m_1,m_2,m_3) = \frac{1}{x} \\ -\beta \sum_{\theta' \in \Theta} \left\{ \frac{R(g,g',m_1)}{m_1(g^\wedge(k,x,\theta),\theta')} \cdot \chi(\theta,\theta') \right\} \end{aligned} \tag{3.32}$$

where $R(g,g',m_1)$ is

$$\begin{gathered} R(g^\wedge(k,m_3,\theta), g^\wedge(\hat{k},m_2,\theta), m_1(g^\wedge(\hat{k},m_3,\theta),\theta'); \\ n^*_\wedge(g^\wedge(\hat{k},m_2,\theta), m_1(g^\wedge(\hat{k},m_3,\theta),\theta'),\theta')) \end{gathered}$$

We first prove a useful lemma about this equation.

Lemma 3.3.8 *Under Assumption 1, 2, and either Assumption 3(i) or 3(ii), Z^m is decreasing in (x, m_3), and increasing in (k, m_1, m_2).*

Proof 3.3.0.8 *Throughout the proof, consider equation (3.32)*

The first term is decreasing in x. In the second term, $g^\wedge$ is decreasing in x. As m_1 is increasing in K, the whole term is increasing in x. Due to the negative sign in front of the second term, we can conclude that Z^m is decreasing in x.

Next, if m_3 increases, $g^\wedge(k, m_3, \theta)$ decreases by Lemma 3.3.5. I know that rental rate is decreasing in its first and third arguments by Lemma 3.3.6. Hence, rental rate rises. Combining with the negative sign in front of the second term of equation Z^m, Z^m is decreasing in m_3.

Next, by Lemma 3.3.5, $g^\wedge(k, m_3, \theta)$ is increasing in k. Hence, higher k leads to lower rental rate by Lemma 3.3.6. Also, as k increases, it leads to higher capital accumulation at individual decisions, $g^\wedge(k, x, \theta)$. Together with $m_1 \in \mathbb{M}_1$ being increasing, I conclude that m_1 increases with higher k. Combining all these facts with negative sign in front of the second term leads to higher Z^m. Hence, Z^m is increasing in k.

Higher m_1 decreases the seconds term so overall increases Z^m.

Finally, for m_2, as m_2 increases, $g^\wedge(\hat{k}, m_2, \theta)$ decreases due to Lemma 3.3.5. This decrease also brings about lower labor supply. Overall, by Lemma 3.3.6, rental rate decreases. Together with the negative sign in front of the second term of equation Z^m, Z^m is increasing in m_2.

We can now define an increasing fixed point decomposition as follows: when $k = \hat{k} > 0$, impose $x^* = m_3$ pointwise, compute the root $x^* = x^*(k, \hat{k}, \theta, m_1, m_2)$ defined implicitly in:

$$Z^m(x^*, k, \hat{k}, \theta, m_1, m_2, x^*) = 0$$

and define the operator:

$$\begin{aligned} A(m_1,m_2)(k,\hat{k},\theta) &= x^*(k,\hat{k},\theta,m_1,m_2) \text{ for } k=\hat{k}>0 \text{ and } 0<m_1\leq u_1^{-1} \\ &= 0 \text{ else} \end{aligned}$$

We seek a solution $m^* = m_1 = m_2 = g^* \in G^*(S)$, when $k = \hat{k}$, where $A(m^*, m^*)(s) = A^*(g)(s)$ is the actual recursive equilibrium operator in (3.13) and $s = (k, \hat{k}, \theta)$.

The following lemma summarizes the characteristics of the operator $A(m_1, m_2)$:

Lemma 3.3.9 *Under assumption* 1, 2 *and either assumption* 3(*i*) *or* 3(*ii*), $A(m_1, m_2)(s)$ *is an increasing fixed point decomposition of* $A^*(g)(s)$ *on* $\mathbb{M}_1(\mathbb{S}) \times \mathbb{M}_2(\mathbb{S}) = G^*(S; \mathbb{M}^2)$

Proof 3.3.0.9 *First, in equation (3.32), we have* $m_3 = x^* = A(m_1, m_2)$ *for a given* m_2*:*

$$Z^m(A(m_1,m_2),k,\hat{k},\theta,m_1,m_2,A(m_1,m_2)) = \frac{1}{A(m_1,m_2)}$$
$$-\beta \sum_{\theta' \in \Theta} \left\{ \frac{R(g^\wedge(k,A(m_1,m_2),\theta), g^\wedge(\hat{k},m_2,\theta), m_1(g^\wedge(\hat{k},A(m_1,m_2)),\theta),\theta'); n_\wedge^*)}{m_1(g^\wedge(k,A(m_1,m_2),\theta),\theta')} \cdot \chi(\theta,\theta') \right\} \tag{3.33}$$

where $n_\wedge^* = n_\wedge^*(g^\wedge(\hat{k}, m_2, \theta), m_1(g^\wedge(\hat{k}, A(m_1, m_2), \theta), \theta'), \theta')$.

First, we show that $A(m_1, m_2)$ *is a fixed point decomposition of* A^* *in Definition 3.3.1. For fixed* m_2*, denote* $A(m_1; m_2)$ *as*

$$A(m_1;m_2)(s) = \Psi_A(m_2(s))(s) \tag{3.34}$$

We first show for fixed $m_2(\hat{k}, \theta) \in \mathbb{M}_2$, $A(m_1; m_2)$ *is increasing in* m_1*. By Lemma (3.3.8),* $Z^m(x, k, \hat{k}, \theta, m_1, m_2, x)$ *is decreasing in* x *for each* $(k, \hat{k}, \theta, m_1, m_2)$ *and increasing in* m_1 *for each* $(k, \hat{k}, \theta, m_2)$*; therefore, if we take any* $m_1, m_1' \in \mathbb{M}_1$ *such that* $m_1 \geq m_1'$*, as* Z^m *is increasing in* m_1, $0 = Z^m(x^*, s, m_1, x^*; m_2) \geq Z^m(x^*, s, m_1', x^*; m_2)$.

Then, $Z^m(x^*, s, m_1', x^*; m_2) = 0$ *only if* $x^*(s, m_1; m_2)/geqx^*(s, m_1'; m_2)$ *because* Z^m *is decreasing in* x*. Then,* x^* *is increasing in* m_1 *which is equivalently saying that* $A(m_1; m_2)$ *is increasing in* m_1 *due to the definition of the operator* A *in (3.33).*

By the same token, one can show that $A(m_1, m_2)$ *is increasing in* m_2 *for fixed* m_1*.*

We next show $A(m_1; m_2) \in \mathbb{M}_1$*. Recall from Lemma 3.3.8, for* Z^m *is decreasing in* x *and increasing in* k*. Take* $k_1 \geq k_0$*. Then,* $A(m_1; m_2)(k_1, \theta) \geq A(m_1; m_2)(k_0, \theta)$*. Hence, the first term of equation (3.32) is*

$$\frac{1}{A(m_1; m_2)(k_1, \theta)} \leq \frac{1}{A(m_1; m_2)(k_0, \theta)} \tag{3.35}$$

As Z^m *is increasing in* k*, it requires the second term to be smaller at* k_1 *because* R *is decreasing in its first and third argument by Lemma 3.3.6 and* m_1 *is an isotone function.*

$$g^{\wedge}(k, A(m_1; m_2)(k_1, \theta), \theta) \geq g^{\wedge}(k, A(m_1; m_2)(k_0, \theta), \theta) \tag{3.36}$$

Thus, $A(m_1; m_2)$ *is increasing in* k *such that* $g^{\wedge}(k, m, \theta)$ *is increasing in* k*. Hence,* $A(m_1; m_2)(s) \in \mathbb{M}_1$*. As* $A(m_1; m_2)(s) \in \mathbb{M}_1$ *is isotone in* $\mathbb{M}_1(S)$ *where* $\mathbb{M}_1(S)$ *is a complete lattice, by Tarski's fixed point theorem (Tarski (1955)),* $\Psi_A(m_2(s))(s)$ *is a nonempty complete lattice in* $\mathbb{M}_1(S)$*. As the state space used for* $\mathbb{M}_1$ *and* $\mathbb{M}_2$ *are the same, the second condition of fixed point decomposition definition is automatically satisfied. Denote* $B_\wedge(m_2)(s) = \wedge\Psi_A(m_2(s))(s)$ *and* $B_\vee(m_2)(s) = \vee\Psi_A(m_2(s))(s)$*.*

$$\begin{aligned} B(m_2^*(s)) &= m_2^*(s) = A^*(g^*)(s) \\ &= g^*(s) \in \Psi_A(s) \subset G^*(S) \end{aligned}$$

Hence, $A(m_1, m_2)$ *is a fixed point decomposition.*

Now I am going to show it is an increasing fixed point decomposition by checking the conditions in Definition 3.3.2.

It is already shown that it is a fixed point decomposition. Now, I show $A(m_1, m_2)$ *is jointly increasing on* $\mathbb{M}_1 \times \mathbb{M}_2$*. To see that* $A(m_1, m_2)$ *is jointly increasing in* (m_1, m_2)*,* *notice that Z^m is jointly increasing in (m_1, m_2) and decreasing in x. Then, following the same steps I took while proving A is increasing in m_1 for fixed m_2 and vice versa, we can conclude that $A(m_1; m_2)$ is jointly increasing in (m_1, m_2) on $\mathbb{M}_1 \times \mathbb{M}_2$.*

Further, as $\Psi_A(m_2(s))(s)$ is an increasing function on $\mathbb{M}_2$, by Veinott's fixed point theorem (Veinott (1992)), $\wedge\Psi_A(m_2(s))(s)$ and $\vee\Psi_A(m_2(s))(s)$ are increasing selections. As we denote $B_\wedge(m_2)(s) = \wedge\Psi_A(m_2(s))(s)$ and $B_\vee(m_2)(s) = \vee\Psi_A(m_2(s))(s)$, one can conclude the partial map $B(m_2)$ is an increasing selection. Hence, by Definition 3.3.2, A is an increasing fixed point decomposition.

In addition to this, if I prove $A(m_1, m_2)$ is order continuous, it will guarantee that upward (downward) iterations from lower (upper) solutions will be order closed under sup (inf) operations. Therefore, I will be able to provide the existence of continuous and computable comparative statics.

Lemma 3.3.10 *Under assumption* 1 *and* 2*, $A(m_1, m_2)(s)$ is an order continuous fixed point decomposition of $A^*(g)(s)$ on $\mathbb{M}_1(\mathbb{S}) \times \mathbb{M}_2(\mathbb{S}) = G^*(\mathbb{S}; \mathbb{M}^2)$.*

Proof 3.3.0.10 *According to the definition 3.3.3, I first check whether A is jointly order continuous. According to (Gierz* et al. *(2003): Page 162, Lemma II-2.8),this can be shown by proving $A(m_1; m_2)$ and $A(m_2; m_1)$ are order continuous respectively.*

Given m_2, Z^m equation looks like as follows:

$$
\begin{aligned}
&Z^m(A(m_1; m_2), k, \hat{k}, \theta, m_1, m_2, A(m_1; m_2)) = \frac{1}{A(m_1; m_2)} \\
&-\beta \sum_{\theta' \in \Theta} \left\{ \frac{R(g^\wedge(k, A(m_1; m_2), \theta), g^\wedge(\hat{k}, m_2, \theta), m_1(g^\wedge(\hat{k}, A(m_1; m_2)), \theta), \theta'); n^*_\wedge)}{m_1(g^\wedge(k, A(m_1; m_2), \theta), \theta')} \cdot \chi(\theta, \theta') \right\}
\end{aligned}
\tag{3.37}
$$

where $n^*_\wedge = n^*_\wedge(g^\wedge(\hat{k}, m_2, \theta), m_1(g^\wedge(\hat{k}, A(m_1; m_2), \theta), \theta'), \theta')$

Following the definition of order continuity, take an arbitrary increasing countable chain $\{m_1^n(s)\} \to m_1(s)$. *Hence,* $\vee\{m_1^n\} = m_1$ *exists. Under the assumption* $1(iii)$, *production function is* C^2 *so the rental rate is continuous. Further,* $g(k, A(m_1; m_2), \theta)$ *is continuous because consumption function is proved to be continuous in Lemma 3.3.4. Lastly,* $\frac{1}{A(m_1;m_2)}$ *is also continuous. Therefore, because* $\{m_1^n(s)\} \to m_1(s)$, *this leads to the following:*

$$Z^m(A(m_1^n; m_2), k, \hat{k}, \theta, m_1^n, m_2, A(m_1^n; m_2)) \to$$
$$Z^m(A(m_1; m_2), k, \hat{k}, \theta, m_1, m_2, A(m_1; m_2))$$

which implies that $A(m_1^n; m_2) \to A(m_1; m_2)$. *As it is already proved in Lemma 3.3.9 that* $A(m_1; m_2)$ *is increasing in* m_1, $A(m_1^n; m_2))$ *is an increasing countable chain. Hence,* $\vee A(m_1^n; m_2))$ *exists. Now, the following holds:*

$$A(\vee m_1^n; m_2) = A(m_1; m_2) = \vee A(m_1^n; m_2)$$

Hence, $A(m_1; m_2)$ *is order continuous.*

Now, I prove $A(m_2; m_1))$ *is order continuous. Similar to previous proof, because rental rate and next period capital stock function is continuous and additionally* $m_1(g^\wedge(k, A(m_1; m_2), \theta), \theta')$ *is also continuous,*

$$Z^m(A(m_2^n; m_1), k, \hat{k}, \theta, m_2^n, m_1, A(m_2^n; m_1)) \to$$
$$Z^m(A(m_2; m_1), k, \hat{k}, \theta, m_1, m_2, A(m_2; m_1))$$

which implies that $A(m_2^n; m_1) \to A(m_2; m_1)$. *As it is already proved in Lemma 3.3.9 that* $A(m_2; m_1)$ *is increasing in* m_2, $A(m_2^n; m_1))$ *is an increasing countable chain. Hence,* $\vee A(m_2^n; m_1))$ *exists. Now, the following holds:*

$$A(\vee m_2^n; m_1) = A(m_2; m_1) = \vee A(m_2^n; m_1)$$

Hence, $A(m_2; m_1)$ *is order continuous.*

Therefore, $A(m_2, m_1)$ is jointly order continuous.

Now, I prove $B(m_2)(s)$ is an order continuous selection in $\Psi_A(m_2(s))$ in m_2 for each $s \in S$. I prove it for the least fixed point selection $B_\wedge(m_2)(s) = \wedge\Psi_A(m_2(s))(s)$. The proof for the greatest selection $B_\vee(m_2)(s) = \vee\Psi_A(m_2(s))(s)$ is similar.

Fix $s \in S$ and the number of iteration n. Define the least fixed point map $\Delta_n : \mathbb{M}_2 \to \mathbb{M}_2$ as follows:

$$\begin{aligned} \Delta_n(m_2)(A^n(\wedge\mathbb{M}_1))(s) &= A^n(\wedge\mathbb{M}_1; m_2)(s) \\ \Delta(m_2)(s) &= \sup_n\{\Delta_n(A^n(\wedge\mathbb{M}_1))(s), n \in \mathbb{N}\} \end{aligned}$$

Now, by induction I show that Δ_n is order continuous on $\mathbb{M}_2$. Moreover, as $n \to \infty$, $\Delta_n(m_2)(s) \to \Delta(m_2)(s) = fix(m_2)(s)$ where $fix(m_2)$ is the least fixed point map on $\mathbb{M}_2$ and order continuous on $\mathbb{M}_2$ for each $s \in S$.

Note that $\Delta_0(m_2)(\wedge\mathbb{M}_1) = A^0(\wedge\mathbb{M}_1; m_2) = 1$ is a constant function by. Hence, order continuous. By induction, assume that $\Delta_n(A^n(\wedge\mathbb{M}_1))$ is order continuous. As the operator $A(m_1; m_2) : \mathbb{M}_1 \to \mathbb{M}_1$, I have:

$$\begin{aligned} \Delta_{n+1}(m_2)(A^{n+1}(\wedge\mathbb{M}_1)) &= A(A^n(\wedge\mathbb{M}_1)) \\ &= A(\Delta_n(m_2)(A^n(\wedge\mathbb{M}_1))(s)) \\ &= eval(A, \Delta_n(m_2)(A^n(\wedge\mathbb{M}_1))(s) \end{aligned}$$

I already assume by induction that $\Delta_n(A^n(\wedge\mathbb{M}_1))$ is order continuous and further, assume A is an order continuous function. Moreover, the ordered function spaces are assumed to be posets. Hence, order continuity coincides with continuous lattices [9]*. Therefore, Scott's corollary* 3.4 *on evaluation maps is applicable here (Scott and Strachey (1971)). The evaluation map is order continuous. Further, by Lemma II-2.9 at Gierz* et al. *(2003), composition of order continuous maps are order continuous.*

[9] The detailed explanation is at Datta *et al.* (2017)

For each mapping $A^n(m_1;m_2):\mathbb{M}_1\to\mathbb{M}_1$, iterations on n generate increasing chains for fixed m_2 because $A(m_1;m_2)$ is order preserving:

$$\wedge\,\mathbb{M}_1\leq A(\wedge\mathbb{M}_1;m_2)\leq A^2(\wedge\mathbb{M}_1;m_2)\leq ...\leq A^n(\wedge\mathbb{M}_1;m_2)\leq ...$$

which implies by definition of Δ,

$$\Delta_0(\wedge\mathbb{M}_1)\leq\Delta_1(m_2)(A(\wedge\mathbb{M}_1))\leq ...\leq\Delta_n(m_2)(A^n(\wedge\mathbb{M}_1))\leq ...$$

which is the set $\{\Delta_n(A^n(\wedge\mathbb{M}_1))(s),n\in\mathbb{N}\}$. This set is a countable chain. Then,

$$\begin{aligned}\Delta(m_2)(s) &= \sup_n\{\Delta_n(A^n(\wedge\mathbb{M}_1))(s),n\in\mathbb{N}\}\\ &= \vee\{A^n(\wedge\mathbb{M}_1,m_2)(s),n\in\mathbb{N}\}\\ &= fix(m_2)(s)\end{aligned}$$

where $fix(m_2)$ is the least fixed point mapping. The existence of such a fixed point is based upon the Least Fixed Point Theorem for Scott Continuous Functions because $\mathbb{M}_1$ is a poset. Thus, $fix(m_2)$ is order continuous in m_2 for each $s\in S$. Here, $fix(m_2)$ is equivalent to the lowest selection $B_\wedge(m_2,t)(s)$ in my formulation.

3.3.4 Existence of Recursive Equilibrium

Now that I have an order continuous increasing fixed point decomposition of $A^*(g)(s)$, I can prove the existence of recursive equilibrium.

Theorem 3.3.1 *Under Assumptions $1,2,3(i)$,*

(i) *There exists a complete lattice of fixed points of the recursive equilibrium operator $A^*(g)(s)$ in (3.13) in $G^*(\mathbb{S};\mathbb{M}^2)$*

(ii) *This least recursive equilibrium $c^*(s)\in G^*(S;\mathbb{M}_2)$ and the greatest recursive equilibrium investment level $g^*(s)=(y^*-c^*)(s)$ can be computed by successive approximations as follows: $\sup_n(y^*-c^*)(k,B^n(\vee\mathbb{M}_2),\theta)\searrow g^*(s)=y^*-\inf_n c^*(k,B^n(\vee\mathbb{M}_2),\theta)$,* *where* $\inf_n c^*(k, B^n(\vee\mathbb{M}_2), \theta) = c^*(K, \Psi_B, \theta)$ *with* $g^*(s)$ *increasing jointly in* (k, k) *for each* $\theta \in \Theta$.

(iii) *As* $\theta \in \Theta$ *is iid and follows a Markov process, recursive equilibrium investment* $g^*(s)$ *is jointly monotone increasing in* $s = (k, K, \theta)$ *when* $K = k$.

Proof 3.3.0.11 ***(i)*** *First, by Lemma 3.3.9,* $A(m_1, m_2)$ *is a well-defined increasing fixed point decomposition so its fixed points correspond with the fixed points of* $A^*(g)(s)$ *in equation (3.13). Second, by Lemma 3.3.7(iii),* $G^*(S; \mathbb{M}_2)$ *is a complete lattice. Third,* $A^* : G^*(S; \mathbb{M}_2) \to G^*(S; \mathbb{M}_2)$. *Hence by Tarski's Fixed Point Theorem, there exists a complete lattice of fixed points of the recursive equilibrium operator* $A^*(g)(s)$ *(Tarski (1955)).*

(ii) *As shown in Lemma 3.3.4, continuity of consumption function in* m *leads* $c^*(k, B^n(\wedge\mathbb{M}_2), \theta) \to c^*(k, \wedge\mathbb{M}_2), \theta)$ *as* $B^n(\wedge\mathbb{M}_2) \to B(\wedge\mathbb{M}_2)$. *By Lemma 3.3.3, as* c^* *is increasing in* m *and by Lemma 3.3.10,* $B(m_2)$ *is inf-preserving (i.e. order continuity) so* $\inf_n c^*(k, B^n(\wedge\mathbb{M}_2), \theta) = c^*(K, \Psi_B, \theta)$ *and this maximize the investment level.*

The fact that $g^*(k, K, \theta) = (y^* - c^*)(k, K, \theta)$ *is increasing jointly in* (k, K) *for each* θ *follows from the fact that* $c^*(s)$ *is falling in* k *when* $K = k$ *for each* θ.

(iii) *In order to prove this claim, I only need to prove if* g^* *is monotone increasing in* θ *because it has already been proven in part* (ii) *that* g^* *is monotone increasing in* k.

With iid shocks, if I define

$$\begin{aligned} \mathbb{M}_2^m(\mathbb{S}) = \{ & m_2(k,\theta) |\ 0 \leq m_2(k,\theta) \leq \infty \text{ and} \\ & m_2 \text{ is increasing in } \theta \text{ such that } g^*(k,\theta) \ = (y^* - c^*)(k,\theta) \text{is increasing in } \theta \} \end{aligned}$$

$A(m_1, m_2)$ *transforms into the space* $\mathbb{M}_1 \times \mathbb{M}_2^m$ *so both* $c^*(k, m_1(k, m_2(k, \theta), \theta))$ *will be increasing in* θ *because* c^* *is proved to be increasing in* m *in Lemma 3.3.3. Also,*

$g^*(k, m_1(k, m_2(k, \theta), \theta))$ will also be increasing in θ by assumption on $\mathbb{M}_2^m(\mathbb{S})$. As a function is said to be jointly monotone if it is monotone in all its argument, it means recursive equilibrium investment $g^*(s)$ is jointly monotone increasing in $s = (k, K, \theta)$ when $K = k$.[10]

3.4 Comparing Recursive Equilibrium

I now provide some robust recursive equilibrium comparative statics. I begin with how changes in discount rates affects the capital accumulation. After that, I show how perturbations in externality affects capital accumulation and consumption.

First, I define partial orders on discount factor and externality. For discount factors, let $\beta \in (0,1) = \mathbb{B}$ such that $\mathbb{B}$ is endowed with the standard partial order on $\mathbb{E}$ which is the Euclidean space.

For externality, define $\mathbb{F} = \{F|\ F$ satisfies Assumptions 1 and 3$(i)\}$. As Datta *et al.* (2017) do in their paper, I categorize the class of technologies into 'gradient admissible production functions', F_m which from equidifferentiable collections in the C^1 uniform topology.

For $x = (k, n, K, N, \theta) \in \mathbb{P} = \mathbb{K} \times [0,1] \times \mathbb{K} \times [0,1] \times \Theta$, I parameterize subclasses of production functions that have the gradients (respectively, diagonal elements) sat-

[10]Note that $B(\vee\mathbb{M}_2) < \infty$ for $\vee\mathbb{M}_2 = \infty$ by the definition of $\mathbb{M}_1$ and $\mathbb{M}_2$. We know $B(\infty)\ in \Psi_A(\infty) = A(m_1; \infty)$ and in lemma 3.3.9, I already show that $A(m_1; m_2) \in \mathbb{M}_1$ which is bounded by

$$\frac{1}{u_1(c(k, A(m_1; m_2), \theta), l(c(k, A(m_1; m_2), \theta), k, \theta))}$$

from above. Now, $A(m_1; m_2)$ can never be ∞ as

$$\frac{1}{u_1(c(k, A(m_1; m_2), \theta), l(c(k, A(m_1; m_2), \theta), k, \theta))}$$

is bounded from above by budget constraint of the household. Hence, $A(m_1; m_2) < \infty$. Then any selection from the fixed points of this function would be finite as well. Hence, $B(\infty) < \infty$.

isfying pointwise bounds relative to a C^1 function $b_L(x)$ (respectively, diagonal bound $b_G(x)$) where $b_L(x)$ and $b_G(x)$ have the following properties:

(i) $b_L : \mathbb{P} \to \mathbb{R}^*_+$ is an extended real valued function such that each b_L is

(a) locally Lipschitz continuous $\forall x \in \mathbb{P}$ with $b_L(x) < \infty$ for $x\ in \mathbb{P}$, $x \neq 0$,

(b) $b_L(k, n, K, N, \theta) = 0$ if any element of the vector (k, n, K, N, θ) equal to zero.

(c) satisfies the following Inada conditions:

$$\lim_{k\to 0} b_L(x) \to \infty, (n, K, N, \theta) >> 0; \; \lim_{k\to\infty} b_L(x) \to \infty, (n, K, N, \theta) >> 0$$

(ii) $b_G : \mathbb{P} \to \mathbb{R}^*_-$ is an extended real valued function such that each b_G is

(a) locally Lipschitz continuous $\forall c \in \mathbb{P}$ with $b_G(x) < -\infty$ for $x \in \mathbb{P}$, $x \neq 0$

(b) $b_G(k, n, K, N, \theta) = 0$ if any element of the vector (k, n, K, N, θ) equal to zero.

(c) satisfies the following Inada conditions:

$$\lim_{k\to 0} b_G(x) \to -\infty, (n, K, N, \theta) >> 0; \; \lim_{k\to\infty} b_G(x) \to \infty, (n, K, N, \theta) >> 0$$

Now, define a binary relation $\rhd$ on $\mathbb{F}$ as follows: for F^1 and F^2 in $\mathbb{F}$ such that:

$$F^1 \rhd F^2 if F^1(x) - F^2(x) \text{ is increasing in each component of } x$$

Here, $\rhd$ defines an equivalence class for production function in such a way that each equivalence class includes production functions which have same marginal products in each component. For simplicity, I apply the relation $\rhd$ to a particular subsets of production functions $\mathbb{F}_m$ which is the set of gradient admissible production functions. In particular, I apply it to the x_i-projections of the elements of F_m. Therefore, $\rhd$ generates a partial order on production function and we can generate comparative statics across elements of x by pointwise comparative statics analysis.

Now define

$$\mathbb{F}^i = \{\hat{F} \mid \hat{F}(x_i; x_{-i}) \in \mathbb{F}, x_i \in proj_i(\mathbb{P})\}$$

and consider the following equidifferentiable subclass of $\mathbb{F}^i$for $m(x) = (b_L, b_G)(x)$

$$F^i_m = \{F \in \mathbb{F}^i \mid \ \triangledown (\hat{F}(x_i)) \leq b_L(x_i), \triangledown^2 \hat{F}(x_i) \leq b_G(x_i), \forall x_i \in proj_i(\mathbb{P})\}$$

where I suppress the notation x_{-i} in F ans use $F(x_i)$ to imply that comparison will be in argument x_i.

As $\mathbb{P}$ is compact, the elements $\hat{F} \in \mathbb{F}^i_m$ are pointwise bounded. Also, since the technology is assumed to be continuous here, each $\mathbb{F}^i_m$ is compact (and convex) subset of $\mathbb{F}$ under C^1 uniform topology.

Lemma 3.4.1 *For any $m(x) = (b_L, b_G)(x)$, $(\mathbb{F}^i_m, \rhd)$ is a CCPO (i.e. Chain Complete Partial Order).*

Proof 3.4.0.1 *The proof is exactly the same with the proof of Lemma 16, Page 51 at (Datta* et al. *(2017)).*

Now that the partial orders are introduced, I can prove comparative statics part. I begin with a theorem on robust comparative statics on the discount rate and finish with comparative statics in the production externality.

Theorem 3.4.1 ***(Capital Deepening in Discount Rates)*** *Let $\beta_1 \geq \beta_2$ which $\beta_1, \beta_2 \in (0,1) = \mathbb{B}$. Under Assumptions $1, 2, 3(i)$, for the least recursive equilibrium consumption $c^*(s;\beta) \in G^*(S;\mathbb{M}_2)$, we have the following:*

(i) *$c^*(s;\beta_1) \leq c^*(s;\beta_2)$ and upper semicontinuous in (k, k, β) for each $\theta \in \Theta$ with implied recursive equilibrium investment $g^*(s;\beta) = y^*(s;\beta) - c^*(s;\beta)$ satisfying $g^*(s;\beta_1) \geq g^*(s;\beta_2)$ and lower semicontinuous in (k, k, β) for each θ.*

(ii) *The recursive equilibrium comparison is preserved under successive approximations for each $s \in S$,*

$$\begin{aligned} &\inf_n c^*(k, A^n(\wedge \mathbb{M}^*_1(m_2)), \theta; \beta_1) \searrow c^*(k, A(\wedge \mathbb{M}^*_1(m_2)), \theta; \beta_1) \\ \leq &\inf_n c^*(k, A^n(\wedge \mathbb{M}^*_1(m_2)), \theta; \beta_2) \searrow c^*(k, A(\wedge \mathbb{M}^*_1(m_2)), \theta; \beta_2) \end{aligned}$$

with $g^* = y^*(s) - c^*(s;\beta)$*.*

Proof 3.4.0.2 ***(i)*** *For simplicity, let's rewrite equation (3.32) when* $x^*(s; m_1, m_2, \beta) = A(m_1, m_2; \beta)$ *for a given* m_2*:*

$$Z^m(A(m_1,m_2),k,\hat{k},\theta,m_1,m_2,A(m_1,m_2)) = \frac{1}{A(m_1,m_2)}$$
$$-\beta \sum_{\theta' \in \Theta} \left\{ \frac{R(g^\wedge(k,A(m_1,m_2),\theta),g^\wedge(\hat{k},m_2,\theta),m_1(g^\wedge(\hat{k},A(m_1,m_2)),\theta),\theta');n^*_\wedge)}{m_1(g^\wedge(k,A(m_1,m_2),\theta),\theta')} \cdot \chi(\theta,\theta') \right\}$$

where $n^*_\wedge = n^*_\wedge(g^\wedge(\hat{k},m_2,\theta),m_1(g^\wedge(\hat{k},A(m_1,m_2),\theta),\theta'),\theta')$

As the second term is increasing in β*,* Z^m *is decreasing in* β *due to negative sign in front of the second term. By Assumption 1,* Z^m *is continuously decreasing in* x *so* $A(m_1, m_2; \beta)$ *is continuously decreasing in* β*. This is equivalently saying that* $c^*(s;\beta)$ *is decreasing in* β*. Then, the implied investment* $g^*(s;\beta) = y^*(s) - c^*(s;\beta)$ *is increasing in* β*.*

Further,

$$y^*(s) - c^*(k, B^n(\wedge\mathbb{M}_2)(s;\beta),\theta) = B^n_g(s;\beta)$$

the evaluation map for $B^n_g(s;\beta)$ *is jointly order continuous on* $G(S;\mathbb{M}_2) \times \mathbb{K} \times \mathbb{K} \times \mathbb{B}$ *because* $eval(B^n_g, k, k, \beta)$ *is continuous in its order topology, and increasing jointly in* (B^n_g, k, k, β)*. This implies* $\vee B^n_g(s;\beta)$ *is lower semicontinuous in* (k, k, β) *for each* θ*. Hence,* c^* *is upper semicontinuous in* (k, k, β) *for each* θ*.*

(ii) *By part (ii),*

$$A^n(m_1, m_2; \beta_1) \leq A^n(m_1, m_2; \beta_1)$$

By order continuity of $A(m_1, m_2)$ *in each argument and continuity of consumption function,*

$$\begin{aligned} &\inf_n c^*(k, A^n(\wedge\mathbb{M}^*_1(m_2)),\theta;\beta_1) \searrow c^*(k, A(\wedge\mathbb{M}^*_1(m_2)),\theta;\beta_1) \\ \leq &\inf_n c^*(k, A^n(\wedge\mathbb{M}^*_1(m_2)),\theta;\beta_2) \searrow c^*(k, A(\wedge\mathbb{M}^*_1(m_2)),\theta;\beta_2) \end{aligned}$$

The conclusion is quite intuitive. Since higher discount rates can be interpreted as households being more patient, they prefer to consume less and save more for the future.

As the production externality is one of the distinguished characteristics of this economy, I find it useful to have a comparative statics analysis on it as well. The theorem is as follow:

Theorem 3.4.2 ***(Comparing Recursive Equilibrium under Ordered Changes in the Externality)*** *Let* $F^1 = f \cdot e^1 \geq F^2 = f \cdot e^2$ *for* $F^1, F^2 \in \mathbb{F}$. *Then, under Assumption* $1, 2, 3(i)$,

(i) *The least recursive equilibrium* $c^*(s; e) \in G^*(S; \mathbb{M}_2)$ *has implied greatest recursive equilibrium investment* $g^\wedge(s; e) = (y^* - c^*)(s; e)$ *such that* $g^\wedge(s; e^1) \geq g^\wedge(s; e^2)$.

(ii) *This recursive equilibrium comparison is reserved under successive approximations such that*

$$\sup_n y^*(s; e^1) - c^*(k, B^n(\wedge\mathbb{M}_2)(s; e^1), \theta) \searrow \sup_n y^*(s; e^2) - c^*(k, B^n(\wedge\mathbb{M}_2)(s; e^2), \theta)$$

Proof 3.4.0.3 ***(i)*** *Note that* $e^1 \geq e^2$ *should be interpreted as* e^1 *is higher in* $(\mathbb{F}^i_m, \rhd)$ *for* $i = \{K, N\}$ *for fixed bounding function* m.

To be more precise about the comparative statics for perturbations of e, *I need to analyze its effects on wage rate and rental rate separately. Throughout the proof, wage rate effect is denoted by* $\hat{e}$, *and rental rate effect is denoted by* e. *The total effect will occur when* $e = \hat{e}$.

Consider equation 3.19. As Z^l_1 *is decreasing in* $\hat{e}$, *the mapping* $l^*(L; K, m, \theta; \hat{e})$ *at Lemma 3.3.1 is decreasing in* $\hat{e}$. *Hence, when* $\hat{e}^1 \geq \hat{e}^2$, *the least leisure selection satisfies the following:*

$$l^*_\wedge(K, m, \theta; \hat{e}^1) \leq l^*_\wedge(K, m, \theta; \hat{e}^2)$$

which is to say $l^*_\wedge$ is decreasing in $\hat{e}$. Then, the following holds by the fact that $n^*_\wedge = 1 - l^*_\wedge$:

$$n^*_\wedge(K, m, \theta; \hat{e}^1) \geq n^*_\wedge(K, m, \theta; \hat{e}^2) \tag{3.38}$$

When $k = K > 0 \forall \theta \in \Theta$, $m_1 \in \mathbb{M}_1$ and $m_2 \in \mathbb{M}_2$, write the fixed point decomposition as follows:

$$\begin{aligned}
&Z^m(x^*, k, \hat{k}, m_1, m_2, x^*; F) = \frac{1}{x^*} \\
&- \beta \sum_{\theta' \in \Theta} \{ \frac{f_1(g^\wedge(k, x^*, \theta; e), n^*_\wedge(g^\wedge(\hat{k}, m_2, \theta; \hat{e}), m_1(g^\wedge(\hat{k}, x^*, \theta; e)), \theta'), \theta')}{m_1(g^\wedge(k, x^*, \theta; e), \theta')} \\
&\cdot e(g^\wedge(\hat{k}, m_2, \theta; e), n^*_\wedge(g^\wedge(\hat{k}, m_2, \theta; \hat{e}), m_1(g^\wedge(\hat{k}, x^*, \theta; e), \theta'), \theta')) \cdot \chi(\theta, \theta')\} = 0
\end{aligned}$$

Denote the dependence of current income on changes in $\hat{e}$ in any period:

$$y^*(k, m, \theta; \hat{e}) = f(k, n^*_\wedge(k, m, \theta; \hat{e})) \cdot \hat{e}(k, n^*_\wedge(k, m, \theta; \hat{e}))$$

where $y^*(k, m, \theta; \hat{e})$ is increasing in $\hat{e}$ as $l^*_\wedge$ is decreasing in $\hat{e}$ and both f and e are increasing functions in their arguments (Assumption 1(iii)).

Now, consider the Z^m equation. Under Assumption 1(iii) and equation (3.38), the second term is decreasing in $\hat{e}$. This implies that $x^* = x^*(s, m_1, m_2; \hat{e}, e) = A(m_1, m_2)$ is increasing in $\hat{e}$ as Z^m is falling in x^* under Lemma 3.3.8. This is equivalently saying that $c^*(k, A(m_1, m_2), \theta)(s; \hat{e}, e))$ is increasing in $\hat{e}$.

Then, let's hold $y^*(k, A(m_1, m_2), \theta; \hat{e}) - c^*(k, A(m_1, m_2))(s; \hat{e}, e)$ constant. As the first term in Z^m is decreasing in $\hat{e}$, $g^\wedge(k, A(m_1, m_2), \theta; \hat{e}, e)$ should be increasing in $\hat{e}$. Therefore, for fixed e, both consumption and investment are increasing in $\hat{e}$.

Now I will hold the wage rate effect constant and examine the rental rate effect. Hence, hold $\hat{e}$ constant and assume $e^1 \geq e^2$. Now, as $f_1 \cdot e^1 \geq f_1 \cdot e^{21}$, the second term *in Z^m is increasing in e. Hence, Z^m is decreasing in e which implies $x^*(s, m_1, m_2; \hat{e}, e)$ is decreasing in e. Equivalently, $c^*(k, A(m_1, m_2), \theta)(s; \hat{e}, e))$ is decreasing in e. Therefore, implied investment decision satisfies the following:*

$$\begin{aligned}
\gamma(s, \hat{e}, e^1) \quad &= y^*(k, m, \theta; \hat{e}) - c^*(k, A(m_1, m_2), \theta)(s; \hat{e}, e^1)) \\
&\geq y^*(k, m, \theta; \hat{e}) - c^*(k, A(m_1, m_2), \theta)(s; \hat{e}, e^2)) \\
&= \gamma(s, \hat{e}, e^2)
\end{aligned}$$

Then, when $e = \hat{e}$ and $e^1 \geq e^2$, we have the operator $A(m_1, m_2)$ is such that

$$\gamma(s, e) = y^*(k, m, \theta; e) - A(m_1, m_2)(s; e, e) \tag{3.39}$$

increasing in e.

Here is the definition of greatest recursive equilibrium investment when we use the least consumption equilibrium:

$$B_g(s; e) = y^*(s; e) - c^*(k, B(\wedge \mathbb{M}_2)(s; e), \theta) \tag{3.40}$$

Then, by the order continuity of $B(m_1)$ under Lemma 3.3.10 and continuity of consumption under 3.3.4,

$$\begin{aligned}
\sup_n B_g^n(s; e^1) \quad &= \quad \gamma^*(s; e^1) \\
&= \quad g^\wedge(s; e^1) \\
&= \quad (y^* - c^*)(s; e^1) \\
&\geq \quad (y^* - c^*)(s; e^2) \\
&= \quad g^\wedge(s; e^2) \\
&= \quad \gamma^*(s; e^2)
\end{aligned}$$

Hence, the greatest recursive equilibrium investment is increasing in e while the behavior of consumption is indeterminate.

(ii) *By the argument in part (i),*

$$\begin{aligned} B_g^n(s;e^1) &= y^*(s;e^1) - c^*(k, B^n(\wedge\mathbb{M}_2)(s;e^1), \theta) \\ &\geq y^*(s;e^2) - c^*(k, B^n(\wedge\mathbb{M}_2)(s;e^2), \theta) \\ &= B_g^n(s;e^2) \end{aligned}$$

By order continuity of $A(m_1, m_2)$ in each argument,

$$\begin{aligned} \sup_n B_g^n(s;e^1) &= \sup_n y^*(s;e^1) - c^*(k, B^n(\wedge\mathbb{M}_2)(s;e^1), \theta) \\ &= g^{\wedge}(s;e^1) \\ &\searrow g^{\wedge}(s;e^2) \\ &= \sup_n y^*(s;e^2) - c^*(k, B^n(\wedge\mathbb{M}_2)(s;e^2), \theta) \\ &= \sup_n B_g^n(s;e^2) \end{aligned}$$

Then, when the production externality rises, it effects both wage rates and rental rates and overall effect depends on the dominance of these two channels. As it is seen from the above Lemma, higher wages increase both consumption and investment decisions of individuals. This is basically pure income effect which is to say that with higher income, individuals both raise their consumption and their saving to smooth that high level of consumption over the course of their life. On other hand, higher rental rates make individuals more inclined to save more and consume less which can be named as the price effect of higher externality. Overall, individuals prefer to save more in response to higher production externality whereas their consumption decisions is indeterminate due to two opposing forces mentioned above.

3.5 Conclusion

As a result, I extend the multi-step monotone mapping methodology to work with models with elastic labor supply, nonseperable preferences and production externalities. With this new extension, I succeed in proving the existence of recursive equilibrium and producing comparative statics predictions on deep parameters of the model.

To show the reader that the theory developed here is applicable to a wide range of models which have been extensively used in the applied literature, in my next draft, I plan to apply this methodology to some well-known models such as economies with CRRA preferences or Cobb-Douglas production function. Hence, I can use Romer-type economies with elastic labor supply and production externalities or Cass-Solow growth model with regressive tax and lump-sum transfers.

3.6 Applications

3.6.1 Monopolistic Competition

Monopolistic competition model described by Dixit and Stiglitz (1977) is a special case of my economy.

The preferences of individuals are as follows:

$$u(c,l) = \frac{[cv(l)]^{(1-\sigma)} - 1}{1-\sigma}$$

where $\sigma > 0$ but $\sigma \neq 1$. Moreover, $v(l)$ is nonnegative, strictly increasing and concave.

In this model, there is an intermediate and final good sector. There is monopolistic competition in intermediate good sector, whereas there is perfect competition in final good sector.

There is a continuum of intermediate goods $Y(i)$ where $i \in [0,1]$. Production technology for the final good is constant returns to scale:

$$Y = \left(\int_0^1 Y(i)^{\lambda} di \right)^{\frac{1}{\lambda}} \tag{3.41}$$

where $\lambda \in (0,1)$. Hence, the final good producer solves the following maximization problem:

$$\Pi = \max_{Y(i)} \{Y - \int_0^1 P(i)Y(i)di\}$$

Note that as the production technology is constant returns to scale, profits will be zero. Fist order condition yields the following demand function for intermediate goods:

$$Y(i) = P(i)^{\frac{1}{\lambda-1}} Y \tag{3.42}$$

As the intermediate good sector is monopolistic, in the light of Benhabib and Farmer (1994), I assume the following production function which displays increasing returns to scale:

$$Y(i) = K(i)^{a+c} \cdot N(i)^{b+d} \text{ where } a+b+c+d > 1 \tag{3.43}$$

Then the production function of the i'th intermediate good producer is:

$$\Pi(i) = P(i)Y(i) - wN(i) - rK(i)$$

Plugging $P(i)$ from equation (3.42) yields the following:

$$\Pi(i) = \left(\frac{Y(i)}{Y}\right)^{\lambda-1} Y(i) - wN(i) - rK(i)$$

Now the profit maximization problem for intermediate producer is as follows:

$$\Pi(i) = \max_{K(i),N(i)} Y^{1-\lambda} N(i)^{\lambda(b+d)} K(i)^{\lambda(a+c)} - wN(i) - rK(i)$$

Note that the function will be concave as long as $\lambda(a+b+c+d) \leq 1$. First order conditions give the following equations for prices:

$$\lambda(a+c)\frac{Y(i)P(i)}{K(i)} = r$$

$$\lambda(b+d)\frac{Y(i)P(i)}{N(i)} = w$$

To make it similar to the economy described here, I assume symmery among intermediate goods which leads the following:

$$N(i) = N,\ K(i) = K,\ P(i) = \bar{P}$$

As final good sector is competitive, I know:

$$\Pi = Y - \int_0^1 \bar{P}Y(i)di = 0 \tag{3.44}$$

This implies $\bar{P} = P(i) = 1$. Now, with the symmetry assumption, the aggregate final good function is

$$Y = K^{a+c}N^{b+d} \tag{3.45}$$

To make it similar to the production technology described in this paper, we can define the aggregate output technology as

$$Y = k^a n^b K^c N^d \tag{3.46}$$

where $K^cN^d = e(K,N)$ displays the production externality which is defined in detail at section 3.2. Moreover, $k^a n^b = f(k,n)$ stands for the individual production technology of each firm. Equation (3.46) is similar to equation (3.45) as in equilibrium $k = K$ and $n = N$. For simplicity, assume that there is no production shock θ.

Lastly, if we add up factor payments and profits of the intermediate sector, it will add up to total output. This is also in line with my definition of income in the original model.

$$\begin{aligned}
\int_0^1 [\Pi(i) + wN(i) + rK(i)]di &= \Pi + wN + rK \\
&= Y^{1-\lambda} \int_0^1 N(i)^{\lambda(b+d)} K(i)^{\lambda(a+c)} \\
&= Y
\end{aligned}$$

If $b+d<1$, the wage rate is decreasing in N in equilibrium. This is in line with assumption $3(i)$ which is the small externality case. In this case, there exists a unique and continuous $l_1^*(K,C)$ as stated in Lemma 3.3.1(i).

On the other hand, if $b+d>1$, the wage rate is increasing in N as in assumption $3(ii)$. In this case, there is multiplicity as pointed out by Lemma 3.3.1(iii).

To construct the fixed point decomposition for recursive equilibrium on minimal state space (e.i. $K=k$), take $m_1(k)\in M_1(k)$ and $m_2(k)\in M_2(k)$ and define the following mapping using the household's Euler equation:

$$\begin{aligned}Z(x,k,K,m_1,m_2,m_3) = & \frac{1}{x}\\ & -\beta\frac{1}{m_1(g(k,x))}ag(k,m_3)^{a-1}n(g(K,m_2),m_1(g(K,m_3)))^b\\ & \cdot g(K,m_2)^c N(g(K,m_2),m_1(g(K,m_3)))\end{aligned}$$

By Lemma 3.3.8, Z is decreasing in x and m_3 and increasing in k, m_1, m_2.

Impose $x=m_3$ pointwise and define $x^*=x^*(k,K,m_1,m_2)$ such that

$$Z(x^*,k,K,m_1,m_2)=0$$

Now, define the following operator:

$$\begin{aligned}A(m_1,m_2)(k,K,\theta) &= x^*(k,K,\theta,m_1,m_2)\text{ for }k=K>0\text{ and }0<m_1\leq u_1^{-1}\\ &= 0\text{ else}\end{aligned}$$

Under Lemma 3.3.9 and Lemma 3.3.10, A is order continuous increasing fixed point decomposition.

Under Theorem 3.3.1, there exists multiple recursive competitive investments $g^*(k,k)$. Further, the least and greatest investment decisions are increasing under minimal state space.

Under Theorem 3.4.1, higher discount rates causes the least and greatest consumption choices to decrease and the least and greatest investment choices to increase.

Under Theorem 3.4.2, higher production externality causes the least and greatest saving decision to rise.

3.6.2 Santos Economy

My model can also generalize Santos (2002) Economy, with elastic labor supply and decreasing tax rate on capital returns.

Consider a Cass-Solow growth model with regressive taxation on capital returns and elastic labor supply. There are continuum of households and they get utility from consumption and leisure. Households are endowed with 1 unit of time each period. They decide their consumption and how much time to allocate on leisure. They receive labor income and capital returns as well as profits as they own the firms. Their income are taxed and they also received lump-sum transfers.

Hence, the period utility function for each household is as follows:

$$u(c,l) = \frac{[c \cdot v(l)]^{1-\sigma} - 1}{1-\sigma}$$

where $v(l)$ is strictly increasing and concave function and $\sigma \neq 1$ and $\sigma > 0$.

The budget constraint is as follows:

$$c + k' = [w(K,N,K,N)(1-l) + r(K,N,K,N)k + \Pi(K,N)](1-\tau(K)) + T(K)$$

Assumption: $\tau : \mathbb{K} \to [0,1)$ is decreasing and $T(K) = \tau(K) \cdot y$ is the lump-sum transfers.

Lastly, we have the usual capital evolution process where I assume depreciation rate is 1 for simplicity.

Given this information, the side conditions are as follows:

$$\frac{u_2(c,l)}{u_1(c,l)} = \tfrac{1}{m} w(K,N,K,N)(1-\tau(K))$$

$$u_1(c,l) = \tfrac{1}{m}$$

The firms in this economy has the following technology:

$$F(k,n,K,N) = f(k,n)e(K,N) = k^a n^b K^c N^d$$

where $a + b = 1$.

Market clearing prices as a result of profit maximization is as follows:

$$\begin{aligned} r(K,N,K,N) = &\ aK^{a+b-1}N^{c+d} \\ w(K,N,K,N) = &\ bK^{a+b}N^{b+d-1} \end{aligned}$$

To make this model similar to the model introduced in this paper (e.i. one sector growth model with production externality and no taxes), I should define the production externality as follows:

$$e'(K,N) = K^c N^d (1-\tau(K))$$

Hence, the new production function $F'(k,n,K,N) = F(k,n,K,N) \cdot (1-\tau(K))$

This leads new price levels would be:

$$\begin{aligned} r'(K,N,K,N) = &\ a(1-\tau(K))K^{a+b-1}N^{c+d} \\ w'(K,N,K,N) = &\ b(1-\tau(K))K^{a+b}N^{b+d-1} \end{aligned}$$

New profit function:

$$\begin{aligned} \Pi'(K,N) &= F'(k,n,K,N) - r'(K,N,K,N)K - w'(K,N,K,N)N \\ &= \Pi(K,N)(1-\tau(K)) \end{aligned}$$

Lastly, the individual income is as follows:

$$
\begin{aligned}
y'(K,N,K,N) &= r'(K,N,K,N)K + w'(K,N,K,N)N + \Pi'(K,N) \\
&= r(K,N,K,N)K(1-\tau(K)) + w(K,N,K,N)N(1-\tau(K)) \\
&+ \Pi(K,N)(1-\tau(K)) \\
&= y(1-\tau(K))
\end{aligned}
$$

To construct the fixed point decomposition for recursive equilibrium on minimal state space (e.i. K=k), take $m_1(k) \in M_1(k)$ and $m_2(k) \in M_2(k)$ and define the following mapping using the household's Euler equation:

$$
\begin{aligned}
Z(x,k,K,m_1,m_2) = &\frac{1}{x} \\
&-\beta\frac{1}{m_1(g(k,x))} ag(k,m_3)^{a-1} n(g(K,m_2), m_1(g(K,m_3)))^b \\
&\cdot g(K,m_2)^c N(g(K,m_2), m_1(g(K,m_3)))(1-\tau(g(K,m_2)))
\end{aligned}
$$

By Lemma 3.3.8, Z is decreasing in x and m_3 and increasing in k, m_1, m_2.

Impose $x = m_3$ pointwise and define $x^* = x^*(k,K,m_1,m_2)$ such that

$$Z(x^*, k, K, m_1, m_2) = 0$$

Now, define the following operator:

$$
\begin{aligned}
A(m_1,m_2)(k,K,\theta) &= x^*(k,K,\theta,m_1,m_2) \text{ for } k = K > 0 \text{ and } 0 < m_1 \leq u_1^{-1} \\
&= 0 \text{ else}
\end{aligned}
$$

Under Lemma 3.3.9 and Lemma 3.3.10, A is order continuous increasing fixed point decomposition.

Under Theorem 3.3.1, there exists multiple recursive competitive investments $g^*(k,k)$. Further, the least and greatest investment decisions are increasing under minimal state space.

Under Theorem 3.4.1, higher discount rates causes the least and greatest consumption choices to decrease and the least and greatest investment choices to increase.

Under Theorem 3.4.2, higher production externality causes the least and greatest saving decision to rise.

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APPENDIX A

SOCIAL PLANNER'S PROBLEM OF STATIC MODEL

$$\max_{\{\{H_{gj},\hat{H}_{gj},\}_{\{g,j\}}\}} (\mu ln(C) + (1-\mu)ln(L)) \tag{A.1}$$

$$\text{subject to} \quad C = \left[\sum_{j\in J} jC_j^{\theta}\right]^{\frac{1}{\theta}} \tag{A.2}$$

$$L = (\xi_l L_{\tilde{f}}^{\eta_l} + (1-\xi_l)L_{\tilde{m}}^{\eta_l})^{\frac{1}{\eta_l}} \tag{A.3}$$

$$C_j = [\psi_j c_j^{\epsilon} + (1-\psi_j)\hat{c}_j^{\epsilon}]^{\frac{1}{\epsilon}} \quad \forall j \in J \tag{A.4}$$

$$c_j = A_j H_j \quad \forall j \in J \tag{A.5}$$

$$\hat{c}_j = \hat{A}_j \hat{H}_j \quad \forall j \in J \tag{A.6}$$

$$H_j = [\xi_j H_{\tilde{f}j}^{\eta} + (1-\xi_j)H_{\tilde{f}j}^{\eta}]^{\frac{1}{\eta}} \tag{A.7}$$

$$\hat{c}_j = \hat{A}_j \hat{H}_j \quad \forall j \in J \tag{A.8}$$

$$\hat{H}_j = [\xi_j \hat{H}_{\tilde{f}j}^{\eta} + (1-\xi_j)\hat{H}_{\tilde{f}j}^{\eta}]^{\frac{1}{\eta}} \tag{A.9}$$

$$\bar{H}_g = H_{gj} + \hat{H}_{gj} + L_g \quad \forall g \in G \tag{A.10}$$

To solve this problem, we first optimize across home and market. Then, we will optimize across different types of composite goods.
Step 1: Optimize across home and market
Let N_{gj} be the optimal labor allocation of gender g in sector j. Hence,

$$N_{gj} = H_{gj} + \hat{H}_{gj} \quad \forall j, g \tag{A.11}$$

This is basically maximizing C_{jt} for a given j subject to production functions and time constraints.

Now, we will maximize (A.4) subject to (A.11) and the production functions (A.7) and (A.9). This optimization gives us individual labor supply choices as a function of $N_{\tilde{f}j}$ and $N_{\tilde{m}j}$.

$$\max_{\{H_{gj},\hat{H}_{gj}\}_{g\in\tilde{f},\tilde{m}}} [\psi_j c_j^{\epsilon} + (1-\psi_j)\hat{c}_j^{\epsilon}]^{\frac{1}{\epsilon}} \tag{A.12}$$

$$\text{subject to} \quad c_j = A_j[\xi_j H_{\tilde{f}j}^{\eta} + (1-\xi_j)H_{\tilde{m}j}^{\eta}]^{\frac{1}{\eta}} \tag{A.13}$$

$$\hat{c}_j = \hat{A}_j[\hat{\xi}_j \hat{H}_{\tilde{f}j}^{\eta} + (1-\hat{\xi}_j)\hat{H}_{\tilde{m}j}^{\eta}]^{\frac{1}{\eta}} \tag{A.14}$$

$$N_{\tilde{f}j} = H_{\tilde{f}j} + \hat{H}_{\tilde{f}j} \tag{A.15}$$

$$N_{\tilde{m}j} = H_{\tilde{m}j} + \hat{H}_{\tilde{m}j} \tag{A.16}$$

First order conditions yields the following important condition about the market and home allocation in sector j:

$$\alpha_j \frac{H_{\tilde{f}j}}{H_{\tilde{m}j}}^{\eta-1} = \hat{\alpha}_j \frac{\hat{H}_{\tilde{f}j}}{\hat{H}_{\tilde{m}j}}^{\eta-1} = x \tag{A.17}$$

where x is ratio of lagrange multipliers. It is also equal to gender wage ratio $\frac{w_f}{w_m}$ in decentralized economy. This is the condition for the assumption of free labor mobility between home and market sector.

Together with the time constraints, equation (A.17) gives us a relationship between $H_{\tilde{f}j}$ and $H_{\tilde{m}j}$ but as the first order conditions are nonlinear, I could not obtain optimal levels exactly. Here is the relationship:

$$H_{\tilde{f}j} = \frac{N_{\tilde{f}j} H_{\tilde{m}j}}{z_j N_{\tilde{m}j} + (1 - z_a) H_{\tilde{m}j}} \tag{A.18}$$

where $z_j = (\frac{\alpha_j}{\hat{\alpha}\ _j})^{\frac{1}{1-\eta}}$

A.0.1 CASE 1: $\xi_j = \hat{\xi}_j$

Economically speaking, this means women do not have any comparative advantage between market and home but they do so between different industries. With this assumption, equation (A.18 gives:

$$H_{\tilde{m}j} = H_{\tilde{f}j} \frac{N_{\tilde{m}j}}{N_{\tilde{f}j}} \tag{A.19}$$

Combining this with the first order conditions (foc) of Step 1's optimization problem (i.e. Ratio of the foc of $H_{\tilde{f}j}$ to $\hat{H}_{\tilde{f}j}$) yields:

$$H_{\tilde{f}j} = \frac{N_{\tilde{f}j}}{1 + B_j} \tag{A.20}$$

$$H_{\tilde{m}j} = \frac{N_{\tilde{m}j}}{1 + B_j} \tag{A.21}$$

$$\hat{H}_{\tilde{f}j} = \frac{N_{\tilde{f}j} B_j}{1 + B_j} \tag{A.22}$$

$$\hat{H}_{\tilde{m}jt} = \frac{N_{\tilde{m}j} B_j}{1 + B_j} \tag{A.23}$$

where $B_j = \{\frac{\psi_j}{1-\psi_j} \left(\frac{A_j}{\hat{A}_j}\right)^{\epsilon}\}^{\frac{1}{\epsilon-1}}$.

Given these solutions, we can redefine the following variables in terms of $N_{\tilde{f}j}$ and $N_{\tilde{m}j}$ as they will be necessary in the next step:

$$c_j = A_j \frac{1}{1 + B_j} N_j \tag{A.24}$$

$$\hat{c}_j = \hat{A}_j \frac{B_j}{1 + B_j} N_j \tag{A.25}$$

$$C_j = E_j N_j \tag{A.26}$$

where

$$N_j = [\xi_j N_{\tilde{f}j}^{\eta} + (1-\xi_j) N_{\tilde{m}j}^{\eta}]^{\frac{1}{\eta}} \tag{A.27}$$

and $E_j = \left[\psi_j A_j^{\epsilon} \left(\frac{1}{1+B_j}\right)^{\epsilon} + (1-\psi_j)\hat{A}_j^{\epsilon}\left(\frac{B_j}{1+B_j}\right)^{\epsilon}\right]^{\frac{1}{\epsilon}}$.

Given this, we move onto the second step where we will find the optimal levels of $N_{\tilde{f}j}$ and $N_{\tilde{m}j}$.

Step 2: Optimizing across different sectors

$$\max_{\{N_g j\}_{g,j}} (\mu ln(C) + (1-\mu) ln(L)) \tag{A.28}$$

$$\text{subject to} \quad C = \left[\sum_{j\in J} \kappa_j C_j^{\theta}\right]^{\frac{1}{\theta}} \tag{A.29}$$

$$L = (\xi_l L_f^{\eta_l} + (1-\xi_l) L_m^{\eta_l})^{\frac{1}{\eta_l}} \tag{A.30}$$

$$C_j = E_j N_j \quad \forall j \tag{A.31}$$

$$\bar{H}_g = \sum_j N_{gj} + L_g \quad \forall g \in \tilde{f}, \tilde{m} \tag{A.32}$$

Here are the first order conditions:

$$foc(N_{\tilde{f}j}): \quad \mu\kappa_j E_j^{\theta} \xi_j \frac{1}{C^{\theta}} (N_j)^{\theta-\eta} N_{\tilde{f}j}^{\eta-1} = (1-\mu)\xi_l \frac{1}{L^{\eta_l}} L_{\tilde{f}}^{\eta_l - 1} \tag{A.33}$$

$$foc(N_{\tilde{m}j}): \quad \mu\kappa_j E_j^{\theta} (1-\xi_j) \frac{1}{C^{\theta}} (N_j)^{\theta-\eta} N_{\tilde{m}j}^{\eta-1} = (1-\mu)(1-\xi_l) \frac{1}{L^{\eta_l}} L_{\tilde{m}}^{\eta_l - 1}$$

First order conditions for $N_{\tilde{f}j}$ and $N_{\tilde{m}j}$ yield the following useful equation:

$$\alpha_j \left(\frac{N_{\tilde{f}j}}{N_{\tilde{m}j}}\right)^{\eta-1} = \alpha_l \left(\frac{L_{\tilde{f}}}{L_{\tilde{m}}}\right)^{\eta_l - 1} \tag{A.34}$$

where $\alpha_j = \frac{\xi_j}{1-\xi_j}$ and $\alpha_l = \frac{\xi_l}{1-\xi_l}$. In the decentralized economy, this ratio is equal to the gender wage ratio $x = \frac{w_{\tilde{f}}}{w_{\tilde{m}}}$.

From first order conditions of $N_{\tilde{f}j}$ and $N_{\tilde{f}k}$, we get the following:

$$\frac{N_j}{N_k} = \left(\frac{N_{\tilde{f}j}}{N_{fk}}\right)^{\frac{\eta-1}{\eta-\theta}} \left(\frac{\kappa_j E_j^{\theta} \xi_j}{\kappa_k E_k^{\theta} \xi_k}\right)^{\frac{1}{\eta-\theta}} \tag{A.35}$$

So, although we could not derive explicit solutions for $N_{\tilde{f}j}$ and $N_{\tilde{f}j}$, we can still derive and explicit solution for labor force participation which is the main variable of interest.

Recall N_j from (A.27). Taking the ratios $\frac{N_j}{N_k}$ provides the following equation:

$$\frac{N_{\tilde{f}j}}{N_{fk}} = \left(\frac{F(x,k)}{F(x,j)}\frac{\xi_k}{\xi_j}\right)^{\frac{1}{\eta}}\left(\frac{N_j}{N_k}\right) \tag{A.36}$$

where $F(x,k) = 1 + \alpha_j^{\frac{1}{\eta-1}} x^{\frac{\eta}{1-\eta}}$.

Combining equations (A.35) and (A.36) give labor supply allocation of women across sectors:

$$\frac{N_{\tilde{f}j}}{N_{fk}} = \frac{G_{\tilde{f}}(x,j)}{G_{\tilde{f}}(x,k)} \tag{A.37}$$

where $G_f(x,j) = \left(\xi_j^{\frac{\theta}{\eta}}(\kappa_j E_j^{\theta})F(x,j)^{\frac{\theta}{\eta}-1}\right)^{\frac{1}{1-\theta}}$.

Define

$$\bar{N}_{\tilde{f}j} = \sum_j N_{\tilde{f}j} \tag{A.38}$$

.

The equation (A.33) also yields a relationship between leisure and work given equations (A.37), (A.38) and (A.31).

$$\frac{N_{\tilde{f}j}}{L_{\tilde{f}}} = \frac{\mu}{1-\mu}\kappa_j E_j^{\theta}\frac{N_j^{\theta}}{C^{\theta}}\frac{F(x,l)}{F(x,j)} \tag{A.39}$$

When we some over the sectors j, equation (A.39) comes down to following:

$$\frac{\sum_j N_{\tilde{f}j}}{L_{\tilde{f}}} = \frac{\mu}{1-\mu}F(x,l)\sum_j \frac{\kappa_j}{F(x.j)} \tag{A.40}$$

Labor force participation is defines as follows:

$$\begin{aligned} lfp_{\tilde{f}} &= \frac{\sum_j H_{\tilde{f}j}}{\bar{H}\tilde{f}} \\ &= \sum_j \frac{H_{\tilde{f}j}}{N_{\tilde{f}j}}\frac{N_{\tilde{f}j}}{\bar{H}_{\tilde{f}}} \\ &= \sum_j \frac{H_{\tilde{f}j}}{N_{\tilde{f}j}}\frac{N_{\tilde{f}j}}{\bar{N}_{\tilde{f}j}+L_{\tilde{f}}} \\ &= \sum_j \frac{H_{\tilde{f}j}}{N_{\tilde{f}j}}\left(\frac{\bar{N}_{\tilde{f}j}}{N_{\tilde{f}j}}+\frac{L_{\tilde{f}}}{N_{\tilde{f}j}}\right)^{-1} \end{aligned} \tag{A.41}$$

APPENDIX B

SOCIAL PLANNER'S PROBLEM OF DYNAMIC MODEL

$$\max_{\{\{H_{gjt},\hat{H}_{gjt},K_{t+1}\}_{\{g,j\}}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t(\mu ln(C_t) + (1-\mu)ln(L_t)) \tag{B.1}$$

$$\text{subject to} \quad C_t = \left[\sum_{j \in J} jC_{jt}^{\theta}\right]^{\frac{1}{\theta}} \tag{B.2}$$

$$L_t = (\xi_l L_{ft}^{\eta_l} + (1-\xi_l)L_{mt}^{\eta_l})^{\frac{1}{\eta_l}} \tag{B.3}$$

$$C_{jt} = [\psi_j c_{jt}^{\epsilon} + (1-\psi_j)\hat{c}_{jt}^{\epsilon}]^{\frac{1}{\epsilon}} \quad \forall j \in J \tag{B.4}$$

$$c_{jt} = A_{jt}H_{jt}^{1-\alpha} \quad \forall j \in J \ \{m\} \tag{B.5}$$

$$c_{mt} = A_{mt}K_t^{\alpha}H_{mt}^{1-\alpha} - K_{t+1} \tag{B.6}$$

$$\hat{c}_{jt} = \hat{A}_{jt}\hat{H}_{jt} \quad \forall j \in J \tag{B.7}$$

$$\bar{H}_{gt} = H_{gjt} + \hat{H}_{gjt} + L_{gt} \quad \forall g \in G \tag{B.8}$$

To solve this problem, we first optimize across home and market. Then, we will optimize across different types of composite goods.
Step 1: Optimize across home and market
Let N_{gjt} be the optimal labor allocation of gender g in sector j. Hence,

$$N_{gjt} = H_{gjt} + \hat{H}_{gjt} \quad \forall j, g, t \tag{B.9}$$

This is basically maximizing C_{jt} for a given j subject to production functions and time constraints.

Now, we will maximize (B.4) subject to (B.9) and the production functions (B.5) and (B.7) given the definition of labor aggregators at equations (2.9) and (2.2). This optimization gives us individual labor supply choices as a function of $N_{\tilde{f}jt}$ and $N_{\tilde{m}jt}$ for $j \in J \ \{m\}$.

$$\max_{\{H_{gjt},\hat{H}_{gjt}\}_{g\in\tilde{f},\tilde{m}}} [\psi_j c_{jt}^{\epsilon} + (1-\psi_j)\hat{c}_{jt}^{\epsilon}]^{\frac{1}{\epsilon}} \tag{B.10}$$

$$\text{subject to} \quad c_{jt} = A_{jt}H_{jt} \tag{B.11}$$

$$\hat{c}_{jt} = \hat{A}_{jt}\hat{H}_{jt} \tag{B.12}$$

$$N_{\tilde{f}jt} = H_{\tilde{f}jt} + \hat{H}_{\tilde{f}jt} \tag{B.13}$$

$$N_{\tilde{m}jt} = H_{\tilde{m}jt} + \hat{H}_{\tilde{m}jt} \tag{B.14}$$

For $j = m$, the problem would be as follows:

$$\max_{\{H_{gmt},\}_{g\in\tilde{f},\tilde{m}}} c_{mt} \tag{B.15}$$

$$\text{subject to} \quad c_{mt} = A_{mt}K_{mt}^{1-\alpha}H_{mt}^{1-\alpha} \tag{B.16}$$

$$N_{\tilde{f}jt} = H_{\tilde{f}jt} + \hat{H}_{\tilde{f}jt} \tag{B.17}$$

$$N_{\tilde{m}jt} = H_{\tilde{m}jt} + \hat{H}_{\tilde{m}jt} \tag{B.18}$$

First order conditions yields two important conditions about the market and home allocation in sector j:

$$\alpha_j^{\frac{1}{1-\eta}} \frac{H_{\tilde{f}jt}}{H_{\tilde{m}jt}} = \hat{\alpha}_j^{\frac{1}{1-\eta}} \frac{\hat{H}_{\tilde{f}jt}}{\hat{H}_{\tilde{m}jt}} = x \tag{B.19}$$

where x is ratio of lagrange multipliers. It is also equal to gender wage ratio $\frac{w_{\tilde{f}}}{w_{\tilde{m}}}$ in decentralized economy. This is the condition for the assumption of free labor mobility between home and market sector.

Together with the time constraints, equation (B.19) gives us a relationship between H_{fjt} and H_{mjt} but as the first order conditions are nonlinear, I could not obtain optimal levels exactly. Here is the relationship:

$$H_{\tilde{f}jt} = \frac{N_{\tilde{f}jt} H_{\tilde{m}jt}}{z_j N_{\tilde{m}jt} + (1 - z_a) H_{\tilde{m}jt}} \tag{B.20}$$

where $z_j = (\frac{\alpha_j}{\hat{\alpha}\ _j})^{\frac{1}{1-\eta}}$

Step 2: After obtaining the individual labor supply choices as a function of $N_{\tilde{f}jt}$ and $N_{\tilde{m}jt}$, we will now maximize utility function to find optimal $N_{\tilde{f}jt}$ and $N_{\tilde{m}jt}$ as well as capital and leisure choices given production functions and time constraints.

$$\max_{\{\{N_{gjt}, K_{t+1}\}_{\{g,j\}}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (\mu ln(C_t) + (1-\mu) ln(L_t)) \tag{B.21}$$

$$\text{subject to} \quad C_t = \left[\sum_{j \in J} j C_{jt}^{\theta} \right]^{\frac{1}{\theta}} \tag{B.22}$$

$$L_t = (\xi_l L_{ft}^{\eta_l} + (1 - \xi_l) L_{mt}^{\eta_l})^{\frac{1}{\eta_l}} \tag{B.23}$$

$$C_{jt} = [\psi_j c_{jt}^{\epsilon} + (1 - \psi_j) \hat{c}_{jt}^{\epsilon}]^{\frac{1}{\epsilon}} \quad \forall j \in J \tag{B.24}$$

$$c_{jt} = A_{jt} H(N_{\tilde{f}jt}, N_{\tilde{m}jt})_{jt}^{1-\alpha} \quad \forall j / J \ \{m\} \tag{B.25}$$

$$c_{mt} = A_{mt} K_t^{\alpha} H_{mt}^{1-\alpha} - K_{t+1} \tag{B.26}$$

$$\hat{c}_{jt} = \hat{A}_{jt} \hat{H}(N_{\tilde{f}jt}, N_{\tilde{m}jt})_{jt} \quad \forall j \in J \tag{B.27}$$

$$\bar{H}_{gt} = N_{gjt} + L_{gt} \quad \forall g \in G \tag{B.28}$$

B.0.1 CASE 1: $\xi_j = \hat{\xi}_j$

Economically speaking, this means women do not have any comparative advantage between market and home but they do so between different industries. With this assumption, equation (B.20 gives:

$$H_{\tilde{m}jt} = H_{\tilde{f}jt} \frac{N_{\tilde{m}jt}}{N_{\tilde{f}jt}} \tag{B.29}$$

Combining this with first order conditions of Step 1's optimization problem yields:

$$H_{\tilde{f}jt} = \frac{N_{\tilde{f}jt}}{1+B} \tag{B.30}$$

$$H_{\tilde{m}jt} = \frac{N_{\tilde{m}jt}}{1+B} \tag{B.31}$$

$$\hat{H}_{\tilde{f}jt} = \frac{N_{\tilde{f}jt}B}{1+B} \tag{B.32}$$

$$\hat{H}_{mjt} = \frac{N_{\tilde{m}jt}B}{1+B} \tag{B.33}$$

where $B = \left((1-\alpha)\frac{\psi_j}{1-\psi_j}\left(\frac{A_{jt}}{\hat{A}_{jt}}\right)^{\epsilon}\right)^{\frac{1}{(1-\alpha)\epsilon-1}}$

Now that we have obtained these results, I can maximize the utility function as in Step 2. For that, I will first derive composite consumption as functions of $N_{\tilde{f}jt}$ and $N_{\tilde{m}jt}$

$$\forall j \neq m: \quad C_{jt} = E_{jt}N_{jt} \tag{B.34}$$

where $E_{jt} = \left[\psi_j\left(\frac{A_{jt}}{1+B}\right)^{\epsilon} + (1-\psi_j)\left(\frac{B\hat{A}_{jt}}{1+B}\right)^{\epsilon}\right]^{\frac{1}{\epsilon}}$ and

$$N_{jt} = \left[\xi_j N_{\tilde{f}jt}^{\eta} + (1-\xi_j)N_{\tilde{f}jt}^{\eta}\right]^{\frac{1}{\eta}} \tag{B.35}$$

$$j = m: \quad C_{mt} = E_{mt}K_t^{\alpha}N_{mt}^{1-\alpha} - K_{t+1} \tag{B.36}$$

where $E_{mt} = \frac{A_{mt}}{1+B}$

Also, time constraints will become:

$$\bar{H}_{\tilde{f}t} = N_{\tilde{f}at} + H_{\tilde{f}mt} + N_{\tilde{f}st} + L_{\tilde{f}t} \tag{B.37}$$

$$\bar{H}_{\tilde{m}t} = N_{\tilde{m}at} + H_{\tilde{m}mt} + N_{\tilde{m}st} + L_{\tilde{m}t} \tag{B.38}$$

Now, maximize Utility function B.1 subject to (B.3), (B.34), (B.36) and time constraints.

Euler equation will be as follows:

$$\frac{C_{mt}^{\theta-1}}{C_t^{\theta}} = \beta\frac{C_{mt+1}^{\theta-1}}{C_{t+1}^{\theta}}\alpha E_{mt+1}K_{t+1}^{\alpha-1}N_{mt}^{1-\alpha} \tag{B.39}$$

And here are the firs order conditions for female workers:

$$foc(N_{fjt})\forall j \neq m: \quad \mu j E_{jt}^{\theta}\xi_j\frac{1}{C_t^{\theta}}N_{jt}^{\theta-\eta}N_{\tilde{f}jt}^{\eta-1} = (1-\mu)\xi_l\frac{L_{ft}^{\eta_l-1}}{L_t^{\eta}}$$

$$foc(N_{fmt}) \text{ for } j = m: \quad \mu m\xi_j\frac{1}{C_t^{\theta}}C_{mt}^{\theta-1}D_{mt}K_{mt}^{\alpha}N_{mt}^{1-\alpha-\eta}N_{\tilde{f}mt}^{\eta-1} = (1-\mu)\xi_l\frac{L_{ft}^{\eta_l-1}}{L_t^{\eta}}$$

$$\frac{K}{N_m} = (\beta \alpha E_m)^{\frac{1}{1-\alpha}} \tag{B.40}$$

Recall equation (B.35) for both sector $j \neq m$ and sector m. That gives us the following useful equation:

$$\left(\frac{N_j}{N_m}\right)^{\eta} = \frac{\xi_j \left(1 + x^{\frac{\eta}{1-\eta}} \alpha_j^{\frac{1}{1-\eta}}\right)}{\xi_m \left(1 + x^{\frac{\eta}{1-\eta}} \alpha_m^{\frac{1}{1-\eta}}\right)} \left(\frac{N_{fj}}{N_{fm}}\right)^{\eta} \tag{B.41}$$

where $x = \frac{w_{\tilde{f}}}{w_{\tilde{m}}}$ and $\alpha_j = \frac{\xi_j}{1-\xi_j}$ in the decentralized economy.

Denote $F(x, j) = \xi_j \left(1 + x^{\frac{\eta}{1-\eta}} \alpha_j^{\frac{1}{1-\eta}}\right) \quad \forall j \in J$

Using first order conditions together with equation (B.41) give us the following:

$$\frac{N_{\tilde{f}j}}{N_{\tilde{f}m}} = \left[\frac{j}{m} \frac{\xi_j}{\xi_m} \frac{E_j^{\theta}}{(\beta\alpha)^{\frac{\alpha+\theta-1}{1-\alpha}} E_m^{\frac{\theta}{1-\alpha}} (E_m^{2-\alpha} - 1)^{\theta-1}} \left(\frac{F(x,j)}{F(x,m)}\right)^{\frac{\theta-\eta}{\eta}}\right]^{\frac{1}{1-\theta}} = \frac{G(x,j)}{G(x,m)} \tag{B.42}$$

Now, given this result and time constraint for female workers, I can actually calculate the main variable interest of this paper: labor force participation.

$$\bar{N}_{\tilde{f}} = \sum_j N_{\tilde{f}j} \tag{B.43}$$

$$lfp_{\tilde{f}} = \sum_j \frac{H_{\tilde{f}j}}{N_{\tilde{f}j}} = \sum_j \frac{H_{\tilde{f}j}}{N_{\tilde{f}j}} \frac{N_{\tilde{f}j}}{\bar{N}_f} \tag{B.44}$$

Using (B.30) and (B.42) and (B.43) yields the following:

$$lfp_{\tilde{f}} = \sum_j \frac{1}{1+B_j} \frac{G(x,j)}{\sum_j G(x,j)} \tag{B.45}$$

is the labor force participation rate for women which is the variable of interest of this paper. In the main text of the paper, I will write my comments on this variable.