

Toward Enumerating the Chains of Maximum Length of Cambrian and m -eralized

Cambrian Lattices

by

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ABSTRACT

The Cambrian lattice corresponding to a Coxeter element c of \mathcal{A}_n , denoted $\text{Camb}(c)$, is the subposet of \mathcal{A}_n induced by the c -sortable elements, and the m -eralized Cambrian lattice corresponding to c , denoted $\text{Camb}^m(c)$, is defined as a subposet of the braid group accompanied with the right weak ordering induced by the c -sortable elements under certain conditions. Both of these families generalize the well-studied Tamari lattice \mathcal{T}_n first introduced by D. Tamari in 1962. S. Fishel and L. Nelson enumerated the chains of maximum length of Tamari lattices.

In this dissertation, I study the chains of maximum length of the Cambrian and m -eralized Cambrian lattices, precisely, I enumerate these chains in terms of other objects, and then find formulas for the number of these chains for all m -eralized Cambrian lattices of \mathcal{A}_1 , \mathcal{A}_2 , \mathcal{A}_3 , and \mathcal{A}_4 . Furthermore, I give an alternative proof for the number of chains of maximum length of the Tamari lattice \mathcal{T}_n , and provide conjectures and corollaries for the number of these chains for all m -eralized Cambrian lattices of \mathcal{A}_5 .

To my parents

To my wife

To my friends in this world

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Chapter 1

INTRODUCTION

[Reiner et al. (2016)]

In [Reading (2006)], where Cambrian lattices were first introduced, N. Reading defined the family of Cambrian lattices for the symmetric group \mathfrak{S}_n as quotients of the right weak order on \mathfrak{S}_n with respect to certain lattice congruences. In the same paper, Reading also gave a combinatorial realization of the Cambrian lattices in terms of triangulations of an $(n + 2)$ -gon under a certain cover relation via diagonal flips.

In [Reading (2007a)], Reading introduced the definition of Coxeter-sorting and Coxeter-sortable elements of a Coxeter group. After that, the Coxeter-sortable elements played a crucial role in studying Cambrian lattices; in [Reading (2007b)], Reading showed that these elements are the minimal congruence-class representatives of the Cambrian congruence of the right weak ordering on \mathfrak{S}_n so that the Cambrian lattices can be expressed as the right weak orders on Coxeter-sortable elements.

The m -eralized Cambrian lattices were first introduced by S. Stump, H. Thomas, and N. Williams, see [Stump *et al.* (2015)], as subposets of the right weak order on the braid group induced by Coxeter-sortable elements under certain conditions. The m -eralized Cambrian lattices generalize the Cambrian lattices, precisely, the Cambrian lattices are the m -eralized Cambrian lattices with $m = 1$.

The Cambrian and m -eralized Cambrian lattices are open areas to study, both of these families generalize the well-studied Tamari lattice first introduced by D. Tamari

in 1962 as the poset of all binary bracketings on a set of $n + 1$ objects under a cover relation based on the associativity rule in one direction so that $(ab)c \leq a(bc)$, see [Tamari (1962)].

Finding a formula for the number of chains of maximum length of both Cambrian and m -eralized Cambrian lattices is still unsolved, except for Tamari lattices where a formula is found by S. Fishel and L. Nelson, see [Fishel and Nelson (2014)]. In this dissertation, we study the chains of maximum length of these lattices. In particular, we enumerate these chains in terms of other objects, and then give an alternative proof of the formula for the number of chains of maximum length of Tamari lattices. Additionally, we find formulas for the number of chains of maximum length of the m -eralized Cambrian lattices of \mathcal{A}_1 , \mathcal{A}_2 , \mathcal{A}_3 , and \mathcal{A}_4 , for all $m \in \mathbb{Z}^+$. Furthermore, we provide conjectures and corollaries for the number of chains of maximum length of all m -eralized lattices of \mathcal{A}_5 .

This dissertation is organized as follows: In Chapter 2, we give basic background information concerning the concepts: partially ordered sets, lattices, partitions, compositions, Young diagrams, Young tableaux, standard Young tableaux, shifted tableaux, and standard shifted tableaux, that are necessary for the understanding of this dissertation. Then we introduce the definition of Coxeter systems, and study the symmetric group as a Coxeter group, denoted \mathcal{A}_n , where we present the right weak ordering, reduced words, and reduced decompositions of elements in \mathcal{A}_n . At the end of the chapter, we briefly recall the Catalan objects and introduce one of these objects.

In Chapter 3, we begin by introducing the definitions of Coxeter elements, the

half-infinite word of a Coxeter element, and the Coxeter-sortable elements in \mathcal{A}_n . Then we define Cambrian lattices of \mathcal{A}_n via the Coxeter-sortable elements, and introduce some of their properties. We also introduce the Cambrian lattices via the triangulations of specific polygons, and investigate the relation between these two characterizations of Cambrian lattices. After that, we define two maps, ρ^h and ρ^v , on the set of Coxeter elements of \mathcal{A}_n , and introduce some of their properties. These two maps appear in studying the duality and the isomorphism classes of Cambrian lattices, enumerating the self-dual Cambrian lattices, and enumerating the distinct non-isomorphic Cambrian lattices of \mathcal{A}_n . In this chapter we also define the poset of compositions, and define an equivalence relation on it. We prove that this poset forms a self-dual lattice, and study some of its other properties. The poset of compositions plays an important role in studying the chains of maximum length of both Cambrian and m -eralized Cambrian lattices. In particular, in this chapter we find a relation between the chains of maximum length of a Cambrian lattice and the equivalence class of the composition corresponding to its maximum element.

In Chapter 4, we briefly review the hook-length formula introduced by M. Thrall in [Thrall (1952)], and study the commutation classes of permutations. Then we give some basic background concerning the heap of a reduced word in \mathcal{A}_n , the lattice words, and dominant permutations. At the end of the chapter, we provide an alternative proof of the number of chains of maximum length of Tamari lattices. Moreover, we enumerate the chains of maximum length of Cambrian lattices via the linear extensions of the heaps corresponding to their maximum elements.

In Chapter 5, we present the definition of m -eralized Cambrian lattices of \mathcal{A}_n , and introduce the m -eralized Tamari lattice as a special case. Then we use the maps ρ^h

and ρ^v , defined in Chapter 3, to study the duality and isomorphism classes of these lattices. We also study the chains of maximum length of this family of lattices, and find a relation between the chains of maximum length of an m -eralized Cambrian lattice and the equivalence class of the composition corresponding to its maximum element. Furthermore, we define the super-heap of a composition, and prove that the number of chains of maximum length of an m -eralized Cambrian lattice is equal to the number of linear extensions of the super-heap corresponding to its maximum element. After that, we use these results to find formulas for the number of chains of maximum length of all m -eralized Cambrian lattices of \mathcal{A}_1 , \mathcal{A}_2 , \mathcal{A}_3 , and \mathcal{A}_4 , for all $m \in \mathbb{Z}^+$. Moreover, we collect other results concerning the super-heaps of the maximum elements of these lattices.

In the epilogue, Chapter 6, we outline plans for future research. We also provide conjectures and corollaries about formulas for the number of chains of maximum length of the m -eralized Cambrian lattices of \mathcal{A}_5 .

Chapter 2

PRELIMINARIES

In this chapter we introduce the basic definitions and terminologies that we need in this dissertation. The outline of this chapter goes as follows: In Section 2.1, we recall the definitions of partially ordered sets and lattices. We also recall some related terminology and cover some examples. In Section 2.2, we briefly study partitions and compositions. In Section 2.3, we present the definition of Young tableaux and shifted Young tableaux. We also discuss some related terminologies and basic examples. In Section 2.4, we study the Coxeter systems and present some properties and examples. In Section 2.5 we briefly introduce the Catalan numbers and an example of Catalan objects.

2.1 Partially Ordered Sets and Lattices

The definitions in this section are obtained from [Brualdi (2010)], [Grätzer (2003)], and [Stanley (2012)].

A *partially ordered set* P , a *poset* for short, is a set together with a binary relation, denoted \leq , such that for all $x, y, z \in P$, the following conditions hold:

1. *Reflexivity*: $x \leq x$.
2. *Antisymmetry*: $x \leq y$ and $y \leq x$ implies that $x = y$.
3. *Transitivity*: $x \leq y$ and $y \leq z$ implies that $x \leq z$.

We say that two elements x and y of the poset P are *comparable* if $x \leq y$ or $y \leq x$. Otherwise, x and y are said to be *incomparable*. We use the notation $y \geq x$ to mean $x \leq y$, and we use the notation $x < y$ to mean that $x \leq y$ with $x \neq y$. A *weak subposet* of P is a subset Q of P together with a partial ordering such that for all $x, y \in Q$, if $x \leq y$ in Q , then $x \leq y$ in P . If Q is a weak subposet of P with $P = Q$ as sets, then P is called a *refinement* of Q . If Q is a subset of P such that for all $x, y \in Q$; $x \leq y$ in P if and only if $x \leq y$ in Q , then Q is called an *induced subposet* of P . By a *subposet* of P we mean an induced subposet of P . If $x \leq y$, then the set $\{z \in P : x \leq z \leq y\}$ is called a *closed interval* of P and denoted by $[x, y]$. Similarly the set $\{z \in P : x < z < y\}$ is called an *open interval* of P and denoted by (x, y) . Note that the number of closed intervals of P is equal to the number of pairs x and y in P such that $x \leq y$.

A subset C of P is called a *chain*, or *totally ordered*, or *linearly ordered*, if any two elements of C are comparable. The *length* of a chain $C : x_1 < x_2 < \cdots < x_n$ is equal to $n - 1$. A chain C of P is said to be *saturated* if there is no $v \in P - C$ such that $C \cup \{v\}$ is a chain with $u < v < w$ for some $u, w \in C$, and a saturated chain is called *maximal* if it is not a proper subposet of another chain. We say that y *covers* x and write $x \triangleleft y$, or $y \triangleright x$, if $x < y$ and there is no z such that $x < z < y$. An element x is said to be *maximal* if there is no y such that $x < y$, and x is said to be *minimal* if there is no y such that $y < x$. We say P has a $\hat{0}$ if there exists an element $\hat{0} \in P$ satisfying $\hat{0} \leq x$ for all $x \in P$, and we say P has a $\hat{1}$ if there exists an element $\hat{1} \in P$ satisfying $x \leq \hat{1}$ for all $x \in P$. We call the elements $\hat{0}$ and $\hat{1}$, if they exist, the *minimum* and *maximum* elements of P respectively.

Two posets P and Q are *isomorphic*, denoted by $P \cong Q$, if there exists a bijection

$\phi : P \rightarrow Q$ such that:

$$x \leq y \text{ in } P \iff \phi(x) \leq \phi(y) \text{ in } Q.$$

The *Hasse diagram* of a finite poset is the graph whose vertices are the elements of P and the edges are determined by the cover relations so that if $x \lessdot y$, then we put x below y and draw an edge between them. When $x \lessdot y$, we say that x is a *child* of y , and y is a *parent* of x .

Example 2.1.1. For a positive integer $n \in \mathbb{Z}^+$. The set of all positive integer divisors of n forms a poset, denoted D_n , where the partial ordering \leq on D_n is defined as

$$i \leq j \text{ if and only if } i \text{ divides } j.$$

Let $n = 12$. Then $D_{12} = \{1, 2, 3, 4, 6, 12\}$, where $1 \lessdot 2$, $1 \lessdot 3$, $2 \lessdot 4$, $2 \lessdot 6$, $3 \lessdot 6$, $4 \lessdot 12$, and $6 \lessdot 12$. The Hasse diagram of D_{12} is shown in Figure 2.1.

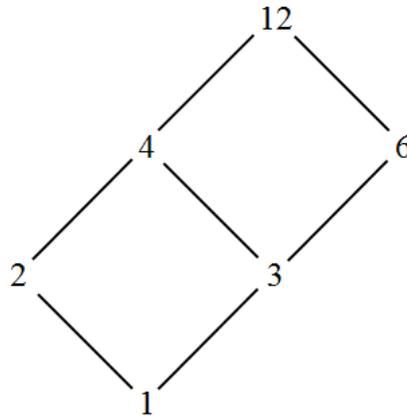


Figure 2.1: The Hasse Diagram of D_{12}

The *dual* of a poset P is the poset P^* on the same set of vertices of P , that is $P = P^*$ as sets, such that for all $x, y \in P$, $x \leq y$ in P if and only if $y \leq x$ in P^* .

The poset P is said to be *self-dual* if P is isomorphic to its dual P^* . In general, two posets P and Q are *anti-isomorphic* if there exists a bijection $\phi : P \rightarrow Q$ such that:

$$x \leq y \text{ in } P \iff \phi(y) \leq \phi(x) \text{ in } Q.$$

Example 2.1.2. *The poset D_n discussed in Example 2.1.1 is a self-dual poset.*

For $x, y \in P$, if $x \leq z$ and $y \leq z$ for some $z \in P$, then z is called an *upper bound* of x and y . If $x \geq z$ and $y \geq z$ for some $z \in P$, then z is called a *lower bound* of x and y . A *least upper bound*, also called *supremum*, of x and y in P is an upper bound $u \in P$ of x and y such that any upper bound v of x and y satisfies $u \leq v$. When a least upper bound of x and y exists it is denoted by $x \vee y$. A *greatest lower bound*, also called an *infimum*, of x and y in P is a lower bound $u \in P$ of x and y such that any lower bound v of x and y satisfies $v \leq u$. When a greatest lower bound of x and y exists we denote it by $x \wedge y$. A poset P is said to be a *lattice* if for all $x, y \in P$, $x \vee y$ and $x \wedge y$ exist. An *atom* of a finite lattice is an element that covers $\hat{0}$, and a *coatom* is an element that is covered by $\hat{1}$.

Example 2.1.3. *Consider the two posets whose corresponding Hasse diagrams are as shown in Figure 2.2. The poset corresponding to the Hasse diagram to the left is not a lattice since b and c don't have a supremum, also d and e don't have an infimum. On the other hand, the poset corresponding to the Hasse diagram to the right is a lattice since every pair of elements has an infimum and a supremum.*

Example 2.1.4. *Consider the two posets whose corresponding Hasse diagrams are as shown in Figure 2.3. Every two elements of the poset corresponding to the Hasse diagram to the left has infimum but not every two have a supremum. On the other hand, every two elements of the poset corresponding to the Hasse diagram to the right have a supremum but not every two have an infimum.*

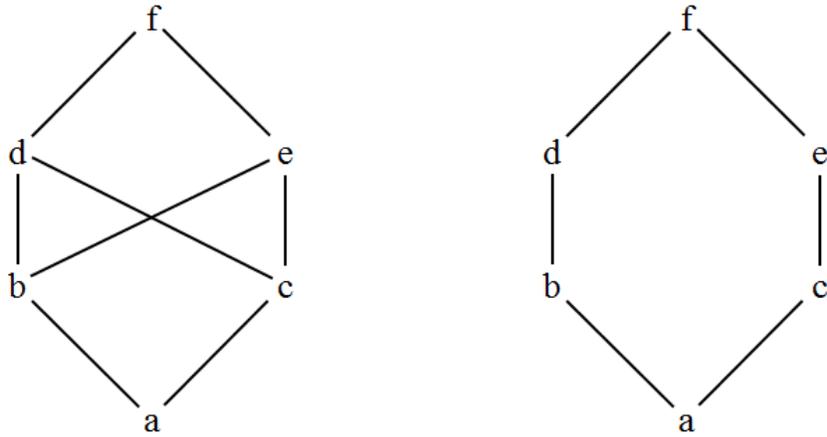


Figure 2.2: The Hasse Diagrams of Two Posets

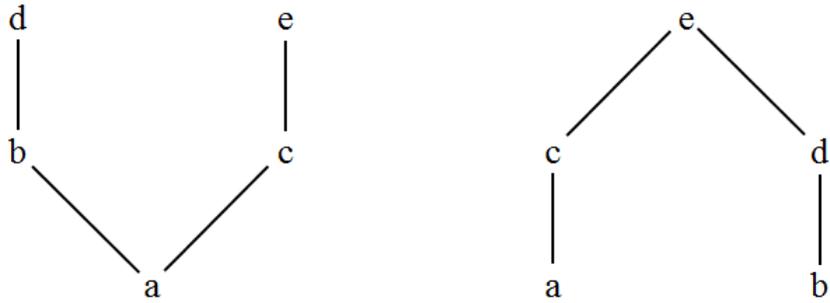


Figure 2.3: The Hasse Diagrams of Two Posets

Definition 2.1.5. A lattice homomorphism is a map ϕ between two lattices, say $\phi : L_1 \rightarrow L_2$, such that for all $x, y \in L_1$;

1. $\phi(x \vee y) = \phi(x) \vee \phi(y)$, and
2. $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$.

A lattice isomorphism is a bijective lattice homomorphism.

2.2 Partitions and Compositions

In this section we present the definition of a partition, the conjugate of a partition, and a composition. We also present the definition of Young diagram, the hook and the hook-length of a cell in a Young diagram. The definitions in this section are obtained from [Lam *et al.* (2013)], [Macdonald (2015)], and [Stanley (2012)].

A *partition* is any finite or infinite sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell, \dots)$ of weakly decreasing non-negative integers containing only finitely many non-zero entries. If λ and μ are two partitions that differ only by a list of zeros at the end, then they are defined to be the same partition. The non-zero entries of λ are called the *parts* of λ , the number of parts is called the *length* of λ , denoted by $\ell(\lambda)$, and the sum of the parts is called the *size* of the partition λ and is denoted by $|\lambda|$. If λ is of size m , then we say that λ partitions m and denote this by $\lambda \vdash m$. The set of all partitions of m is denoted by \mathcal{P}_m and the set of all partitions is denoted by \mathcal{P} . Every partition is represented by a *Young diagram*, where the Young diagram of a partition λ consists of *boxes*, also called *cells*, that are arranged in left justified rows so that the top row consists of λ_1 boxes, the second row consists of λ_2 boxes, and so on so that the bottom row consists of λ_ℓ boxes. The *conjugate* of a partition λ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_{\lambda_1})$, where $\lambda'_i = |\{j : \lambda_j \geq i\}|$. The Young diagram of λ' can be constructed by reflecting the Young diagram of λ about the main diagonal. Equivalently, if the Young diagram of λ is given, then λ'_i is equal to the number of boxes in the i^{th} column of the Young diagram of λ . For example, if $\lambda = (5, 3, 2, 1)$, then $\lambda' = (4, 3, 2, 1, 1)$. We note that λ and its conjugate λ' have the same size, $\ell(\lambda') = \lambda_1$, $\ell(\lambda) = \lambda'_1$, and $\lambda'' = \lambda$. Figure 2.4 shows the Young diagrams of $\lambda = (5, 3, 2, 1)$ and its conjugate λ' , where the Young diagram of λ appears to the left and the Young diagram of λ' appears to the right.

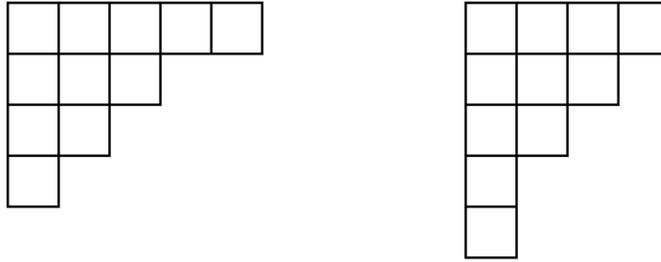


Figure 2.4: The Young Diagrams of $\lambda = (5, 3, 2, 1)$ to the Left and of λ' to the Right

A Young diagram of λ can also be represented as a set of ordered pairs as follows:

$$\text{dg}(\lambda) = \{(i, j) : i \in [\ell(\lambda)], \text{ and } j \in [\lambda_i]\},$$

where the ordered pair (i, j) represents the cell that lies in the i^{th} row and j^{th} column in the Young diagram of λ . If (i, j) is a cell in the Young diagram of a partition λ , then the *hook* of (i, j) consists of cells in row i to the right of (i, j) , the cells in column j below (i, j) , and the cell (i, j) itself. The *hook length* of the cell (i, j) is the number of cells in the hook of (i, j) and is denoted by $\text{hook}_\lambda(i, j)$. In particular;

$$\text{hook}_\lambda(i, j) = \lambda_i + \lambda'_j - i - j + 1.$$

The hook of the cell $(1, 1)$ is called the *main hook* of λ and the hook length of the main hook is called the *main hook length* of λ . For example, if $\lambda = (5, 3, 2)$, then $\text{hook}_\lambda(2, 1) = 4$ and the main hook length of λ is equal to 7.

A *composition* of a positive integer n is a sequence of positive integers (a_1, a_2, \dots, a_k) such that $\sum_{i=1}^k a_i = n$. Every partition of n is a composition of n but not vice versa as the order of the parts of a composition is important while the parts of any partition are already arranged in a weakly decreasing order from left to right. As in partitions, the number of parts in a composition α is called the *length* of α and is denoted by $\ell(\alpha)$, and the sum of the parts is called the *size* of the composition α and is denoted

by $|\alpha|$. The set of all compositions of a positive integer n is denoted by \mathfrak{C}_n , and we denote by \mathfrak{C} the set of all compositions. That is $\mathfrak{C} = \cup_{n \in \mathbb{Z}^+} \mathfrak{C}_n$.

2.3 Young Tableaux and Shifted Young Tableaux

In this section we present the definitions of Young's lattice, a skew diagram, a Young tableau, and a shifted tableau. We also introduce some examples and basic properties. The definitions in the section are obtained from [Lam *et al.* (2013)], [Macdonald (2015)], [Stanley (1999)], and [Stanley (2012)].

Let λ and μ be two partitions, not necessarily of the same size, and let $\text{dg}(\lambda)$ and $\text{dg}(\mu)$ be the Young diagrams of λ and μ respectively. We say that

$$\lambda \subseteq \mu \text{ if and only if } \text{dg}(\lambda) \subseteq \text{dg}(\mu).$$

In other words, $\lambda \subseteq \mu$ if and only if $\lambda_i \leq \mu_i$ for all $i \geq 1$. This is a partial order relation on \mathcal{P} , called *inclusion*. In particular, the set of partitions \mathcal{P} with this partial ordering forms a lattice known as *Young's lattice*. We note that μ covers λ if $\lambda \subseteq \mu$ and $|\lambda| = |\mu| - 1$.

Suppose that $\lambda \subseteq \mu$, then the *skew partition* $\theta = \mu/\lambda$ represents the cells that are in $\text{dg}(\mu)/\text{dg}(\lambda)$, that is, the cells of $\text{dg}(\mu)$ except the cells of $\text{dg}(\lambda)$. The corresponding diagram of a skew partition is called a *skew diagram*. A *path* in a skew diagram θ is a sequence of cells x_0, x_1, \dots, x_s so that x_i and x_{i+1} have a common side for all $i = 0, 1, 2, \dots, s - 1$. A subset ϑ of a skew partition θ is said to be *connected* if for any two cells of ϑ there is a path of cells from one to the other. The maximal connected subsets of θ are called the *connected components* of θ . The conjugate of a skew diagram $\theta = \mu/\lambda$ is $\theta' = \mu'/\lambda'$. The skew diagram θ with $|\theta| = m$ is called

a *horizontal m -strip* if there is at most one cell in each column of θ , and is called a *vertical m -strip* if there is at most one cell in each row of θ .

Example 2.3.1. Consider the partitions $\mu = (4, 3, 2, 1)$ and $\lambda = (3, 2, 1)$ whose corresponding Young diagrams are as shown in Figure 2.5. The skew diagram of

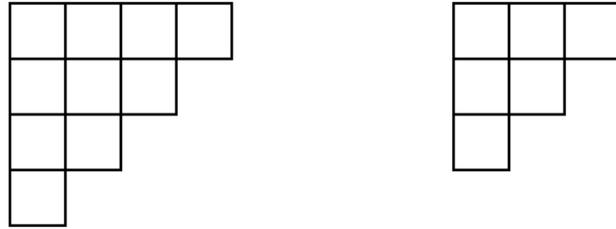


Figure 2.5: The Young Diagrams of $\mu = (4, 3, 2, 1)$ to the Left and of $\lambda = (3, 2, 1)$ to the Right

$\theta = \mu/\lambda$, as shown in Figure 2.6, is obtained from the Young diagram of μ by removing the cells that form the Young diagram of λ . The skew diagram θ is a horizontal and a vertical strip simultaneously.

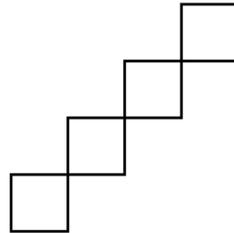


Figure 2.6: The Skew Diagram of $\theta = \mu/\lambda$, Where $\mu = (4, 3, 2, 1)$ and $\lambda = (3, 2, 1)$

Example 2.3.2. Consider the partitions $\mu = (4, 2, 1)$, and $\lambda = (2, 1)$. The Young diagrams corresponding to these partitions are shown in Figure 2.7. The skew diagram of $\theta = \mu/\lambda$, as shown in Figure 2.8, is a horizontal strip but not a vertical strip.

A *column-strict tableau*, simply a *tableau*, also called a *semistandard tableau*, T is

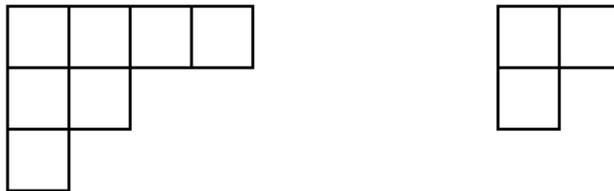


Figure 2.7: The Young Diagrams of $\mu = (4, 2, 1)$ to the Left and $\lambda = (2, 1)$ to the Right

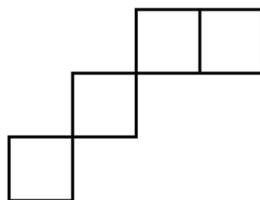


Figure 2.8: The Skew Diagram of $\theta = \mu/\lambda$, Where $\mu = (4, 2, 1)$ and $\lambda = (2, 1)$

a sequence of partitions

$$T : \lambda = \lambda^{(0)} \subset \lambda^{(1)} \subset \dots \subset \lambda^{(t)} = \mu,$$

such that for every $i \in [t]$ the skew diagram $\theta^{(i)} = \lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip. If for every $i \in [t]$ we number each square of the skew diagram $\theta^{(i)} = \lambda^{(i)}/\lambda^{(i-1)}$ by the number i , then we can think of a tableau as a numbered Young diagram so that the numbers increase weakly from left to right and increase strictly down each column. This explains the name column-strict tableau. The skew diagram $\theta = \mu/\lambda$ is called the *shape* of the tableau T . The *weight* of the tableau T is the sequence $(|\theta^{(1)}|, |\theta^{(2)}|, \dots, |\theta^{(t)}|)$. A *standard tableau* is a tableau of weight $(1, 1, \dots, 1)$. A standard tableau of shape λ refers to a saturated chain in the interval $[\phi, \lambda]$ of the Young lattice. Let $(\phi = \lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(m)} = \lambda)$ be a saturated chain of the Young lattice. Then the standard tableau corresponding to it can be formed by inserting i in the box representing $\lambda^{(i)}/\lambda^{(i-1)}$.

Example 2.3.3. Consider the tableau T that appears in Figure 2.9. Since the integers are weakly increasing along rows and strictly increasing along columns, the tableau T is a semistandard tableau of shape $(5, 3, 2, 1)$ and of weight $(2, 2, 1, 1, 2, 1, 1, 1)$.

1	1	2	3	4
2	5	5		
7	6			
8				

Figure 2.9: A Tableau of Shape $(5, 3, 2, 1)$ and Weight $(2, 2, 1, 1, 2, 1, 1, 1)$

Example 2.3.4. Consider the tableau T that appears in Figure 2.10. The tableau T is a standard tableau of shape $(5, 3, 2, 1)$.

1	3	5	10	11
2	4	6		
7	8			
9				

Figure 2.10: A Standard Tableau of Shape $(5, 3, 2, 1)$

For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of distinct parts, the *shifted diagram* of shape λ is the diagram obtained from the Young diagram of λ by shifting the i^{th} row $i - 1$ steps to the right for every $i \in [\ell]$. A *shifted tableau* of shape λ is a filling of the shifted diagram of λ by positive integers such that the integers are weakly increasing along rows and strictly increasing along columns. A shifted tableau of shape λ is said

to be a *standard shifted tableau* if all the entries are from the set $\{1, 2, 3, \dots, |\lambda|\}$ with no repetitions.

Example 2.3.5. Consider the partition $\lambda = (4, 2, 1)$. Figure 2.11 shows the Young diagram of λ to the left, and the shifted Young diagram of λ to the right.

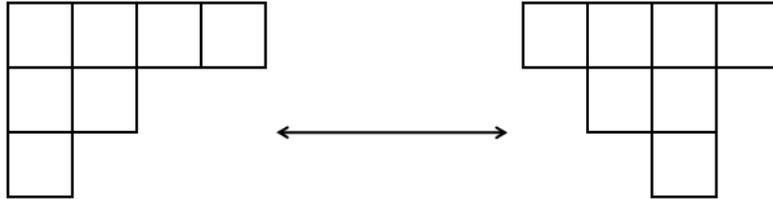


Figure 2.11: The Young and Shifted Diagrams of $\lambda = (4, 2, 1)$

Example 2.3.6. Consider the partition $\lambda = (4, 2, 1)$. Figure 2.12 shows a standard shifted tableau of shape λ .

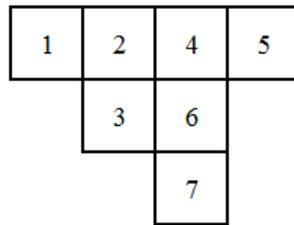


Figure 2.12: A Standard Shifted Tableau of Shape $\lambda = (4, 2, 1)$

2.4 Coxeter Systems

In this section we study the Coxeter groups and their corresponding Coxeter diagrams. The definitions in this section are obtained from [Humphreys (1990)], [Reading (2004)], [Reading (2007a)], and [Stanley (1984)].

A *Coxeter group* is a group W generated by the elements of a set $S \subseteq W$ such that $|s| = 2$ and $2 \leq |st| \leq \infty$ for all $s, t \in S$ with $s \neq t$, where for $x \in W$, $|x|$ denotes the order of x in the group W . By $|st| = \infty$ we mean that no relation of the form $(st)^m = 1$ with $m \in \mathbb{Z}^+$ holds. We call the elements of S *simple reflections*, and their conjugates are called *reflections*. A *Coxeter System* is a pair (W, S) , where W is a Coxeter group and S is a set of generators of W that satisfies the conditions above. The *rank* of W is defined as the size of S , namely $\text{rank}(W) = |S|$. A Coxeter group might have more than one set of generators and so we take the term W is a Coxeter group to mean that there is a distinguished set $S \subseteq W$ such that (W, S) is a Coxeter system. Every Coxeter system is represented by a *Coxeter diagram* which is a diagram with the vertex set S so that there is an edge between $s, t \in S$ labeled $|st|$ when $|st| \geq 3$. We leave the edges labeled by 3 unlabeled as it appears more often. Every element $w \in W$ can be written as a *word* in the alphabet of S , which corresponds to a *decomposition* of w in terms of elements of S , for instance, if w can be decomposed as $w = a_{i_1} a_{i_2} \cdots a_{i_k}$, where $a_{i_j} \in S$, for all $j \in [k]$, then the word corresponding to this decomposition is $i_1 i_2 \cdots i_k$. Note that $i_1 i_2 \cdots i_k$ is a word in $[n]$, where $n = |S|$, and that these expressions need not be unique. The *length* of a word of w is the number of letters appearing in the word. A word of w with minimal-length is called a *reduced word* of w , and the corresponding decomposition is called a *reduced decomposition* of w . The length of a reduced word of w is called the *length* of w and denoted by $\ell(w)$. One can see that $\ell(ws) = \ell(w) \pm 1$ and $\ell(sw) = \ell(w) \pm 1$, for all $w \in W$ and all $s \in S$. Also; $\ell(ws) < \ell(w)$ if and only if there is a reduced decomposition of w that ends with s , and similarly $\ell(sw) < \ell(w)$ if and only if there is a reduced decomposition of w that starts with s . For any Coxeter group W we can define a partial order, called *right weak order*, on W as the transitive closure of the cover relation $u \prec v$ when $v = us$ for some $s \in S$ with $\ell(u) < \ell(v)$. This

shows that $u < v$ if u appears as a prefix of a word of v ; that is, $u < v$ if there exist $\tau_1, \tau_2, \dots, \tau_j \in S$ such that $v = u\tau_1\tau_2 \cdots \tau_j$ with $\ell(v) = \ell(u) + j$. The right weak order is also called the *right Bruhat order*. For simplicity we denote by W both the Coxeter group W as well as the partially ordered set W . Furthermore, we use the terms word and decomposition interchangeably when it is clear from the context. For a subset K of S , the subgroup of W generated by K is denoted by W_K and the corresponding Coxeter system is denoted by (W_K, K) . The elements of W_K form a lower interval of the Coxeter group W under the right weak order.

Example 2.4.1. Consider the symmetric group \mathfrak{S}_n^\pm of all odd permutations, also called signed permutations, of $\pm[n]$ under composition of functions as the group operation; that is, the group of permutations π such that $\pi(-i) = -\pi(i)$ for all $i \in [n]$. Let $S := \{b_0, b_1, b_2, \dots, b_{n-1}\}$, where $b_0 = (1, -1)$, and $b_i = (i, i+1)(-i, -i-1)$ for all $i \in [n-1]$. Then S generates the group \mathfrak{S}_n^\pm with $|b_i| = 2$ for all $i \in [n-1] \cup \{0\}$, $|b_0b_1| = 4$, $|b_ib_{i+1}| = 3$ for all $i \in [n-2]$, and $|b_ib_j| = 2$ whenever $|i-j| > 1$. This makes \mathfrak{S}_n^\pm a Coxeter group with the set of simple reflections S . This Coxeter group is denoted by \mathcal{B}_n , and its Coxeter diagram is shown below, where every simple reflection b_i is represented by the integer i for simplicity.

$$0 \overset{4}{\text{---}} 1 \text{---} 2 \text{---} \dots \text{---} (n-1)$$

In order to study the right weak ordering on \mathcal{B}_n , we assume that all permutations are written in one-line representation. Let $\pi = \pi_{-n}\pi_{-n+1} \cdots \pi_{-1}\pi_1 \cdots \pi_{n-1}\pi_n$ be a permutation in \mathcal{B}_n written in one-line notation, where $\pi_i = \pi(i)$ for all $i \in [n]$. Since πb_0 interchanges π_{-1} and π_1 in the one-line notation of π , and since πb_i interchanges the entries π_i and π_{i+1} as well as the entries π_{-i} and π_{-i-1} , then the cover relation of the right weak order over \mathcal{B}_n is the transposition of the entries π_1 and π_{-1} or the entries π_i and π_{i+1} as well as π_{-i} and π_{-i-1} of a permutation π written in one-line notation,

where going up means putting the transposed entries out of numerical order, and going down means putting them into numerical order. The unique minimum element of \mathcal{B}_n is $\hat{0} = (-n) \cdots (-2) (-1) (1) (2) \cdots (n)$, and the unique maximum element of \mathcal{B}_n is $\hat{1} = (-1) (-2) \cdots (-n) (n) \cdots (2) (1)$, where both $\hat{0}$ and $\hat{1}$ are written in one-line notation. This example is introduced to give an example of a Coxeter group, but we will not consider this type of Coxeter groups furthermore in this dissertation.

2.4.1 The Coxeter Group \mathcal{A}_n

In this subsection we study the Coxeter group \mathcal{A}_n , which is the type we study for the Cambrian lattices.

Consider the symmetric group \mathfrak{S}_{n+1} of all permutations of $[n+1]$ with the composition as the operation of the group, and let S be the set of adjacent transpositions $s_i = (i, i+1)$, for $i = 1, 2, \dots, n$. The elements of S generate the group \mathfrak{S}_{n+1} with $|s_i| = 2$ and $|s_i s_j| \geq 2$, for all $i, j \in [n]$ with $i \neq j$. This makes the group \mathfrak{S}_{n+1} a Coxeter group where the simple reflections are the adjacent transpositions. This Coxeter group is denoted by \mathcal{A}_n . We use the notation \mathcal{A}_n instead of \mathfrak{S}_{n+1} in order to emphasize that the simple reflections are the simple transpositions and the partial order is the right weak order. Since $|s_i s_j| = 2$ when $|i-j| > 1$, and $|s_i s_j| = 3$ otherwise, the corresponding Coxeter diagram is a path with n vertices, namely s_1, s_2, \dots, s_n , and $n-1$ edges such that there is an edge between s_i and s_j when $|i-j| = 1$, where all of these edges are unlabeled as $|s_i s_j| = 3$ whenever s_i and s_j are connected by an edge. In particular, the Coxeter diagram of \mathcal{A}_n is shown below, where the number i represents the simple reflection s_i .

$$1 \text{ --- } 2 \text{ --- } 3 \text{ --- } \dots \text{ --- } n$$

In order to study the right weak ordering on \mathcal{A}_n , let us assume that all permutations of \mathcal{A}_n are expressed using the one-line notation, and let $\pi = \pi_1\pi_2\cdots\pi_i\pi_{i+1}\cdots\pi_{n+1}$ be any permutation in \mathcal{A}_n . Then πs_i is the permutation obtained from π by interchanging π_i and π_{i+1} in the one-line notation of π , that is $\pi s_i = \pi_1\pi_2\cdots\pi_{i+1}\pi_i\cdots\pi_{n+1}$. This shows that the cover relation of the right weak order over \mathcal{A}_n is the transposition of adjacent entries of a permutation written in one-line notation, where going up means putting the adjacent entries out of numerical order, and going down means putting them into numerical order. The unique minimum element of \mathcal{A}_n is $\hat{0} = 1\,2\,3\cdots(n+1)$, and the unique maximum element is $\hat{1} = (n+1)\,n\,(n-1)\cdots 1$, where both $\hat{0}$ and $\hat{1}$ are here written in one-line notation. The elements $\hat{0}$ and $\hat{1}$ of \mathcal{A}_n are denoted by id and w_0 respectively.

Note that for any $\pi = \pi_1\pi_2\cdots\pi_i\pi_{i+1}\cdots\pi_{n+1} \in \mathcal{A}_n$, represented in one-line notation,

$$w_0\pi = (n+2-\pi_1)(n+2-\pi_2)\cdots(n+2-\pi_{n+1}), \text{ and}$$

$$\pi w_0 = \pi_{n+1}\pi_n\cdots\pi_1.$$

In fact, the maps $\pi \mapsto w_0\pi$, and $\pi \mapsto \pi w_0$ on \mathcal{A}_n are automorphisms. Figure 2.13 shows the Hasse diagram of \mathcal{A}_3 .

A reduced decomposition of $\pi \in \mathcal{A}_n$ has the form $s_{i_1}s_{i_2}\cdots s_{i_{\ell(\pi)}}$, where each s_{i_j} is an adjacent transposition of the form $(i_j, i_j + 1)$ for some $i_j \in [n]$ so that $\pi = s_{i_1}s_{i_2}\cdots s_{i_{\ell(\pi)}}$.

The following theorem and corollary show how to compute the length of a permutation written in one-line notation.

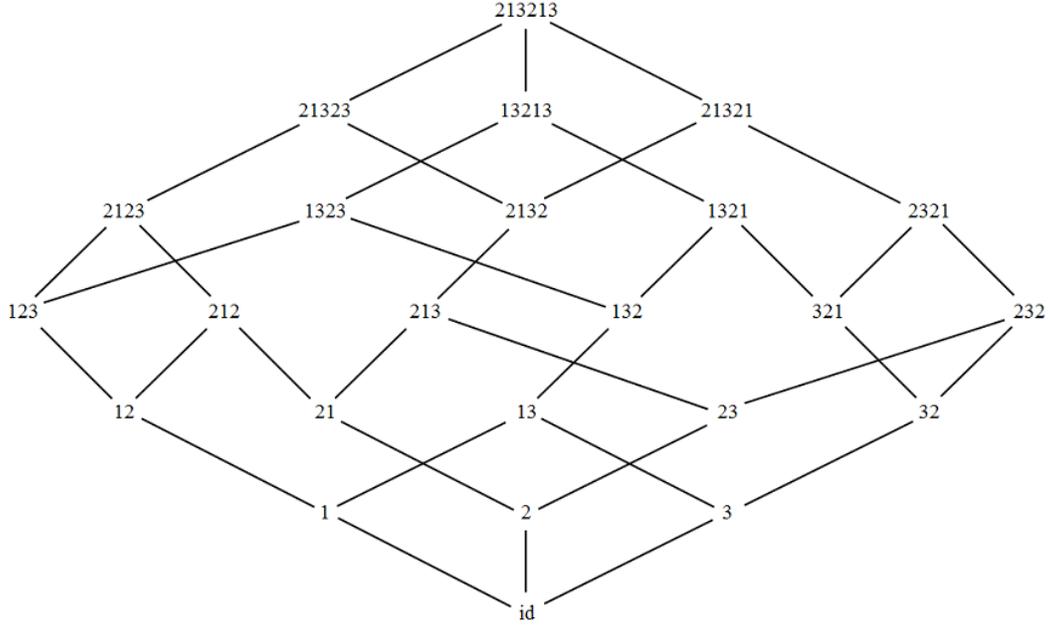


Figure 2.13: The Hasse Diagram of \mathcal{A}_3

Theorem 2.4.2. [Stanley (1984)] For $w = a_1a_2 \cdots a_{n+1} \in \mathfrak{S}_{n+1}$ written in one-line notation, we have

$$\ell(w) = \sum_{i=1}^{n+1} r_i(w),$$

where

$$r_i(w) = \text{card}\{j : j < i \text{ and } a_j > a_i\}, 1 \leq i \leq n + 1.$$

Corollary 2.4.3. [Stanley (1984)] For $w = a_1a_2 \cdots a_{n+1} \in \mathfrak{S}_{n+1}$ written in one-line notation, we have

$$\ell(w) = \sum_{i=1}^{n+1} l_i(w),$$

where

$$l_i(w) = \text{card}\{j : j > i \text{ and } a_j < a_i\}, 1 \leq i \leq n + 1.$$

The following theorem shows how the reduced decompositions of a permutation π are related to each others.

Theorem 2.4.4. [Bourbaki (1982)] *Any two reduced decompositions for a fixed permutation π can be obtained from each another by the use of braid moves, where the braid moves are*

- *Short-braid move C_1 : $s_i s_j = s_j s_i$, when $|i - j| > 1$.*
- *Long-braid move C_2 : $s_i s_j s_i = s_j s_i s_j$, when $|i - j| = 1$.*

Define the relation “ \sim_{C_1} ” on the set of decompositions of elements of \mathfrak{S}_n so that if d_1 and d_2 are such two decompositions, we say that: $d_1 \sim_{C_1} d_2$ when d_2 can be obtained from d_1 by applying a sequence of short-braid moves.

Proposition 2.4.5. [Stembridge (1997)] *The relation \sim_{C_1} defined above is an equivalence relation.*

2.5 The Catalan Objects

In this section we briefly recall the Catalan numbers and Catalan objects. Some suggested references for this section are: [Brualdi (2010)], [Cameron (1994)], [Koshy (2009)], and [Stanley (1999)].

We start with the following definition.

Definition 2.5.1. [Brualdi (2010)] *The Catalan sequence is the sequence*

$$C_0, C_1, \dots, C_n, \dots,$$

where

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

The number C_n in Definition 2.5.1 above is called the n^{th} -Catalan number. The Catalan numbers also satisfy the following recursive relation:

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i}, \text{ where } n \in \mathbb{Z}^+ \text{ and } C_0 = 1.$$

The following is a list of the first ten Catalan numbers: $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $C_5 = 42$, $C_6 = 132$, $C_7 = 429$, $C_8 = 1430$, $C_9 = 4862$.

The Catalan numbers were first studied by L. Euler while attempting to count the triangulations of a convex polygon. These numbers take their name after the French mathematician E. Catalan. The Catalan numbers are involved in many counting problems, see [Stanley (1999)], and [Stanley (2015)]. The objects counted by the Catalan numbers are called *Catalan objects*. The following proposition introduces one of the Catalan objects.

Proposition 2.5.2. [Stanley (1999)] *The Catalan number C_n counts the number of ways to triangulate a convex $(n + 2)$ -gon into n triangles by drawing $n - 1$ diagonals, no two of which intersect in their interior.*

Example 2.5.3. *Figure 2.14 shows the $C_3 = 5$ triangulations of a pentagon.*

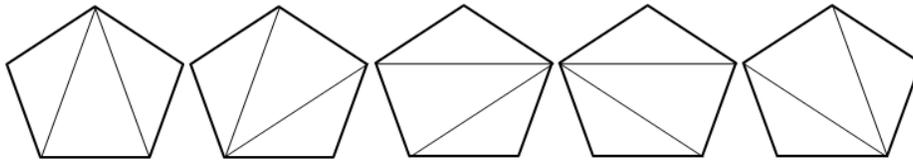


Figure 2.14: The Triangulations of a Pentagon

Chapter 3

CAMBRIAN LATTICES

In this chapter we study the Cambrian lattices of \mathcal{A}_n and introduce some of our results. The outline of this chapter goes as follows: in Section 3.1, we introduce the definition of Cambrian lattices via Coxeter-sortable elements of \mathcal{A}_n . In Section 3.2, we study a characterization of the definition of Cambrian lattices via triangulations of convex polygons. In Section 3.3, we define the poset of compositions, which we need to enumerate the chains of maximum length of Cambrian and m -eralized Cambrian lattices in later sections, and study some of its basic properties. In Section 3.4, we present some well-known facts about Cambrian lattices beside some of our results. In Section 3.5, we define the vertical and horizontal maps as well as the symmetric Coxeter elements, and study some of their properties. In Section 3.6, we study the isomorphisms among Cambrian lattices, and then enumerate the distinct non-isomorphic Cambrian lattices of \mathcal{A}_n . In Section 3.7, we study the duality between Cambrian lattices, and introduce the relation between symmetric Coxeter elements and self-dual Cambrian lattices. We also enumerate the distinct non-isomorphic self-dual Cambrian lattice of \mathcal{A}_n and check other results.

3.1 Cambrian Lattices via c -sortable Elements of \mathcal{A}_n

In this section we give background information about the Coxeter and Coxeter-sortable elements of \mathcal{A}_n , then we introduce the definition of Cambrian lattices via the Coxeter-sortable elements of \mathcal{A}_n . We also present some main examples and facts. The definitions in this section are obtained from [Coxeter (1951)], [Reading (2007a)], [Reading (2007b)], and [Grätzer and Wehrung (2016)].

A *Coxeter element* c of a Coxeter group W is an element with a reduced decomposition of the form $c = c_1 c_2 \cdots c_n$, where $S = \{c_1, c_2, \dots, c_n\}$ is the set of simple reflections generating W . For a fixed Coxeter element $c = c_1 c_2 \cdots c_n$, the *half-infinite word* of c is defined as:

$$c^\infty = c_1 c_2 \cdots c_n | c_1 c_2 \cdots c_n | c_1 c_2 \cdots c_n | \cdots .$$

The vertical bars “|” are called *dividers*, and when a subword of c^∞ is interpreted as an expression for an element of W the dividers are ignored. A *c-sorting word* of $w \in W$ is the lexicographically first subword, by position, of c^∞ that is a reduced word of w , and its corresponding decomposition is called a *c-sorting decomposition* of w . Every *c-sorting decomposition* can be expressed by a sequence of subsets of S such that every subset represents the sequence of letters of the *c-sorting decomposition* that appear between two consecutive dividers. An element $w \in W$ is said to be *c-sortable* if the sequence corresponding to its *c-sorting decomposition* is weakly decreasing under inclusion. The identity element is defined to be *c-sortable* for any Coxeter element c .

Recall that we use the notation \mathcal{A}_n to refer to the Coxeter group \mathfrak{S}_{n+1} as a partially ordered set, ordered by the right weak ordering, so that the simple reflections are the adjacent transpositions.

Example 3.1.1. Consider the Coxeter group \mathcal{A}_4 and the Coxeter element $c = s_1 s_2 s_3 s_4$ of \mathcal{A}_4 . Let $w = s_1 s_4 s_3 s_4$. The following are the reduced decompositions of w ,

$$\begin{array}{llll} w_1 = s_1 s_4 s_3 s_4 & w_2 = s_4 s_1 s_3 s_4 & w_3 = s_4 s_3 s_1 s_4 & w_4 = s_4 s_3 s_4 s_1 \\ w_5 = s_1 s_3 s_4 s_3 & w_6 = s_3 s_1 s_4 s_3 & w_7 = s_3 s_4 s_1 s_3 & w_8 = s_3 s_4 s_3 s_1. \end{array}$$

The c -sorting of each of these decompositions is as follows:

- the c -sorting of $w_1 = s_1s_4s_3s_4$ is $\mathbf{s}_1s_2s_3\mathbf{s}_4|s_1s_2\mathbf{s}_3\mathbf{s}_4|s_1s_2s_3s_4$,
- the c -sorting of $w_2 = s_4s_1s_3s_4$ is $s_1s_2s_3\mathbf{s}_4|\mathbf{s}_1s_2\mathbf{s}_3\mathbf{s}_4|s_1s_2s_3s_4$,
- the c -sorting of $w_3 = s_4s_3s_1s_4$ is $s_1s_2s_3\mathbf{s}_4|s_1s_2\mathbf{s}_3s_4|\mathbf{s}_1s_2s_3\mathbf{s}_4$,
- the c -sorting of $w_4 = s_4s_3s_4s_1$ is $s_1s_2s_3\mathbf{s}_4|s_1s_2\mathbf{s}_3\mathbf{s}_4|\mathbf{s}_1s_2s_3s_4$,
- the c -sorting of $w_5 = s_1s_3s_4s_3$ is $\mathbf{s}_1s_2\mathbf{s}_3\mathbf{s}_4|s_1s_2\mathbf{s}_3s_4|s_1s_2s_3s_4$,
- the c -sorting of $w_6 = s_3s_1s_4s_3$ is $s_1s_2\mathbf{s}_3s_4|\mathbf{s}_1s_2s_3\mathbf{s}_4|s_1s_2\mathbf{s}_3s_4$,
- the c -sorting of $w_7 = s_3s_4s_1s_3$ is $s_1s_2\mathbf{s}_3\mathbf{s}_4|\mathbf{s}_1s_2\mathbf{s}_3s_4|s_1s_2s_3s_4$,
- the c -sorting of $w_8 = s_3s_4s_3s_1$ is $s_1s_2\mathbf{s}_3\mathbf{s}_4|s_1s_2\mathbf{s}_3s_4|\mathbf{s}_1s_2s_3s_4$.

This shows that the decomposition $w_5 = s_1s_3s_4s_3$ is the c -sorting decomposition of w and that w is c -sortable. In general, not all permutations have Coxeter-sortable decompositions, for example; the permutation $v = s_2s_1$ has no c -sorting decomposition. In fact, every permutation has at most one c -sortable decomposition, for any Coxeter element c , see [Reading (2007b)] for more details.

We will sometimes represent a permutation in \mathcal{A}_n by its decomposition or the corresponding word, and sometimes by its one-line notation; we will indicate which notation we are using when needed. The following algorithm shows how to find the c -sorting decomposition of an element $w \in \mathcal{A}_n$.

Algorithm 3.1.2. Given $w \in \mathcal{A}_n$ and a Coxeter element $c = c_1c_2 \cdots c_n$ of \mathcal{A}_n . Consider the half-infinite word of c :

$$c^\infty = c_1c_2 \cdots c_n | c_1c_2 \cdots c_n | c_1c_2 \cdots c_n | \cdots .$$

If w is the identity of \mathcal{A}_n , then we are done as the identity element is c -sortable by definition. If w is not the identity, apply the following steps:

Step 1: find $\ell(w)$ using Theorem 2.4.2, then set $j := 1$, and $w' := id$.

Step 2: Do while ($w \neq id$)

If $\ell(c_jw) < \ell(w)$, then set $w := c_jw$, $w' := w'c_j$, and $j := (j \bmod n) + 1$

Else set $j := (j \bmod n) + 1$

Loop

Step 3: the resulting decomposition w' in Step 2 is the c -sorting decomposition of w .

Proof. We run an induction on $\ell = \ell(w)$. If $\ell = 1$, then $w = s_i$, for some $i \in [n]$. As s_i is the c -sorting decomposition of w and it is the decomposition obtained from applying Algorithm 3.1.2, the base step is checked. Suppose Algorithm 3.1.2 is valid for $\ell - 1$, and let w be of length ℓ . Let $j = \min \{i : \ell(c_iw) = \ell(w) - 1\}$, and set $v = wc_j$. Then $\ell(v) = \ell - 1$. Assume that v' is the decomposition obtained from applying Algorithm 3.1.2 on v . Then, by the inductive step, v' is the c -sorting decomposition of v . The decomposition obtained from applying Algorithm 3.1.2 on w is $w' = v'c_j$. As v' is c -sorting, w' is c -sorting as well, and this completes the proof. \square

The following example illustrates how Algorithm 3.1.2 works.

Example 3.1.3. Consider the Coxeter group \mathcal{A}_3 , and consider the Coxeter element $c = s_3s_1s_2$ of \mathcal{A}_3 , where

$$c^\infty = s_3s_1s_2|s_3s_1s_2|\cdots.$$

Let $w = 3214$ written in one-line notation, and set $w' = id$. By Theorem 2.4.2, $\ell(w) = 3$. Since $s_3w = 4213$ with $\ell(s_3w) = 4 \not< \ell(w)$, then we go to the next element in c^∞ , namely s_1 . As $s_1w = 3124$ and $\ell(s_1w) = 2 < \ell(w)$, then we set $w = s_1w$ and $w' = s_1$, and then we go to the next element in c^∞ , namely s_2 . As $s_2w = 2134$ and $\ell(s_2w) = 1 < \ell(w)$, then we set $w = s_2w$ and $w' = s_1s_2$. By applying the same procedure using the next element, namely s_3 , in c^∞ , we get $s_3w = 2143$ with $\ell(s_3w) = 2 \not< \ell(w)$. Thus we go to the next element in c^∞ , namely s_1 . Since $s_1w = id$, then we set $w' = s_1s_2s_1$ and stop. This shows that $w' = s_1s_2s_1$ is the c -sorting decomposition of w and that w is c -sortable.

We call the Coxeter element $c = s_1s_2s_3 \dots s_n$ of \mathcal{A}_n the *Tamari Coxeter element* of \mathcal{A}_n . In the following example we apply Algorithm 3.1.2 to find the Tamari Coxeter-sorting decomposition of the maximum element w_0 of \mathcal{A}_n .

Example 3.1.4. Let c be the Tamari Coxeter element of \mathcal{A}_n , and let w_0 be the maximum element of \mathcal{A}_n . Consider the half-infinite word c^∞

$$c^\infty = s_1s_2 \cdots s_n|s_1s_2 \cdots s_n|s_1s_2 \cdots s_n|\cdots$$

The one-line notation of w_0 is $(n+1)n(n-1) \cdots 21$. By Theorem 2.4.2, $\ell(w_0) = \binom{n+1}{2}$. We follow Algorithm 3.1.2, so we first set $w'_0 = id$, and start by the first transposition appearing in c^∞ from the left, namely by s_1 . Since $s_1w_0 = (n+1)n(n-1) \cdots 12$, and $\ell(s_1w_0) = \ell(w_0) - 1$, then we set $w'_0 = s_1$. Similarly $s_2s_1w_0$ decreases the length of s_1w_0 by one, so we set $w' = s_1s_2$. We keep applying the left multiplication

by the adjacent transpositions s_i in the same order they appear in c^∞ , namely by s_i from $i = 3$ to $i = n$, and each time this decreases the length of the resulting permutation by one. At the end of the first block of c^∞ we get $s_n s_{n-1} \cdots s_2 s_1 w_0 = n(n-1) \cdots 21(n+1)$, where at this stage $w'_0 = s_1 s_2 \cdots s_n$. Note that the restriction of the permutation $s_n s_{n-1} \cdots s_2 s_1 w_0$ on the set $[n]$ is the maximum element of \mathcal{A}_{n-1} . So by applying the same process as above, namely by multiplying the resulting permutation from the left by s_i from $i = 1$ to $i = (n-1)$ in the same order they appear in the second block of c^∞ , the length of the resulting permutation each time is decreased by one, while multiplying the resulting permutation by the last element of the second block of c^∞ , namely by s_n , increases the length of the resulting permutation by one. That is; after finishing the second block of c^∞ , we get $w'_0 = s_1 s_2 \cdots s_n s_1 s_2 \cdots s_{n-1}$, and $s_1 s_2 \cdots s_n s_1 s_2 \cdots s_{n-1} w_0 = (n-1)(n-2) \cdots 21n(n+1)$. Again, the restriction of the resulting permutation on $[n-1]$ is the maximum element of \mathcal{A}_{n-2} , so we continue with this process until we get at the end the c -sorting decomposition $w'_0 = s_1 s_2 s_3 \cdots s_n s_1 s_2 s_3 \cdots s_{n-1} \cdots s_1 s_2 s_1$ of w_0 , which shows that w_0 is c -sortable.

Proposition 3.1.5. [Reading (2007a)] *The maximum element w_0 of \mathcal{A}_n is c -sortable for every Coxeter element c of \mathcal{A}_n .*

For any Coxeter element c of \mathcal{A}_n , we denote the c -sorting decomposition of w_0 by $w_0(c)$.

An *orientation* of a Coxeter group W is obtained from its Coxeter diagram by replacing each edge by a single directed edge in either direction. When an edge between two elements s and t in the Coxeter diagram of W is replaced by a single directed edge from s to t , in symbols $s \longrightarrow t$, we say s *precedes* t . Similarly, if the edge is replaced by a single directed edge from t to s , we say t *precedes* s .

Example 3.1.6. *The following are all orientations of the Coxeter diagram of \mathcal{A}_3 :*

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 4,$$

$$1 \rightarrow 2 \rightarrow 3 \leftarrow 4,$$

$$1 \rightarrow 2 \leftarrow 3 \rightarrow 4,$$

$$1 \leftarrow 2 \rightarrow 3 \rightarrow 4,$$

$$1 \rightarrow 2 \leftarrow 3 \leftarrow 4,$$

$$1 \leftarrow 2 \rightarrow 3 \leftarrow 4,$$

$$1 \leftarrow 2 \leftarrow 3 \rightarrow 4,$$

and

$$1 \leftarrow 2 \leftarrow 3 \leftarrow 4.$$

Recall the relation “ \sim_{C_1} ” defined on the set of decompositions of elements of \mathfrak{S}_n where for any two decompositions d_1 and d_2 , we say that $d_1 \sim_{C_1} d_2$ if and only if d_2 can be obtained from d_1 by applying a sequence of short-braid moves. The following proposition introduces the relation between the C_1 -classes of the Coxeter elements of \mathcal{A}_n and the orientations of the Coxeter diagram of \mathcal{A}_n .

Proposition 3.1.7. [Reading (2006)] *There is a bijection between the set of orientations of the Coxeter diagram of \mathcal{A}_n and the set $\{[c]_{\sim_{C_1}} : c \text{ is a Coxeter element of } \mathcal{A}_n\}$.*

Proof. Let A be the set $\{[c]_{\sim_{C_1}} : c \text{ is a Coxeter element of } \mathcal{A}_n\}$, and let B be the set of orientations of the Coxeter diagram of \mathcal{A}_n . Pick a Coxeter element $c = s_{i_1} s_{i_2} \cdots s_{i_n}$ of \mathcal{A}_n , and consider the class $[c]_{\sim_{C_1}} \in A$. Let \mathcal{O}^c be the orientation obtained from the diagram of \mathcal{A}_n by replacing the edge connecting s_i and s_{i+1} by the single directed edge from s_i toward s_{i+1} whenever s_i precedes s_{i+1} in the reduced word of c , and

by the single directed edge from s_{i+1} toward s_i whenever s_{i+1} precedes s_i in the reduced word of c . The orientation \mathcal{O}^c is unique, and for any Coxeter element c' with $c \sim_{C_1} c'$, $\mathcal{O}^c = \mathcal{O}^{c'}$. Thus the map $\phi : A \rightarrow B$ such that $\phi([c]_{\sim_{C_1}}) = \mathcal{O}^c$ is well defined. We claim that ϕ is bijective. Let $[c]_{\sim_{C_1}}$ and $[c']_{\sim_{C_1}}$ be two classes in A with $\phi([c]_{\sim_{C_1}}) = \phi([c']_{\sim_{C_1}})$. Then $\mathcal{O}^c = \mathcal{O}^{c'}$, which means that for all $i \in [n]$, s_i precedes s_{i+1} in c if and only if s_i precedes s_{i+1} in c' . This implies that c can be obtained from c' by applying a sequence of short-braid moves, that is $c \sim_{C_1} c'$. Thus $[c]_{\sim_{C_1}} = [c']_{\sim_{C_1}}$ and hence ϕ is injective. In order to prove that ϕ is surjective, let \mathcal{O} be an orientation in B . Define the element $c = s_{j_1} s_{j_2} \cdots s_{k_1} s_{k_2} \cdots$ so that s_{j_i} represents the simple reflection that appears in the orientation \mathcal{O} such that $s_{j_{i-1}} \leftarrow s_{j_i}$ where $j_1 > j_2 > \cdots$, and the simple reflections $\{s_{i_k}\}$ are the remaining simple reflections ordered so that $k_1 < k_2 < \cdots$. The element c is a Coxeter element of \mathcal{A}_n with $\mathcal{O}^c = \mathcal{O}$. That is $\phi([c]_{\sim_{C_1}}) = \mathcal{O}$ which means that the map ϕ is surjective. \square

The proof of Proposition 3.1.7 above shows that if $c \sim_{C_1} c'$, then $\mathcal{O}^c = \mathcal{O}^{c'}$. When $c \sim_{C_1} c'$, we say that c and c' are of the same *type*. Now we introduce the definition of Cambrian lattice corresponding to a coxeter element of \mathcal{A}_n via the c -sortable elements.

Definition 3.1.8. [Reading (2007b)] *Let c be a Coxeter element of \mathcal{A}_n . The Cambrian Lattice corresponding to c , denoted $\text{Camb}(c)$, is defined as the subposet of \mathcal{A}_n induced by the c -sortable elements of \mathcal{A}_n .*

Example 3.1.9. *Consider the Coxeter group \mathcal{A}_3 and the Coxeter element $c = s_3 s_1 s_2$. The c -sortable elements of \mathcal{A}_3 are the elements $e, s_1, s_2, s_3, s_1 s_2, s_3 s_1, s_3 s_2, s_1 s_2 s_1, s_3 s_1 s_2, s_3 s_2 s_3, s_3 s_1 s_2 s_1, s_3 s_1 s_2 s_3, s_3 s_1 s_2 s_3 s_1, s_3 s_1 s_2 s_3 s_1 s_2$. The covering relations on these elements goes as follows:*

$$e \lessdot s_1, e \lessdot s_2, e \lessdot s_3, s_1 \lessdot s_1 s_2, s_1 \lessdot s_3 s_1, s_2 \lessdot s_1 s_2 s_1, s_2 \lessdot s_3 s_2 s_3, s_3 \lessdot s_3 s_1,$$

$s_3 \prec s_3s_2$, $s_1s_2 \prec s_1s_2s_1$, $s_1s_2 \prec s_3s_1s_2s_3$, $s_3s_1 \prec s_3s_1s_2$, $s_3s_2 \prec s_3s_1s_2s_1$, $s_3s_2 \prec s_3s_2s_3$,
 $s_1s_2s_1 \prec s_3s_1s_2s_3s_1s_2$, $s_3s_1s_2 \prec s_3s_1s_2s_1$, $s_3s_1s_2 \prec s_3s_1s_2s_3$, $s_3s_2s_3 \prec s_3s_1s_2s_3s_1s_2$,
 $s_3s_1s_2s_1 \prec s_3s_1s_2s_3s_1$, $s_3s_1s_2s_3 \prec s_3s_1s_2s_3s_1$, $s_3s_1s_2s_3s_1 \prec s_3s_1s_2s_3s_1s_2$.

The Hasse diagram of the Cambrian lattice $\text{Camb}(c)$ is shown in Figure 3.1.

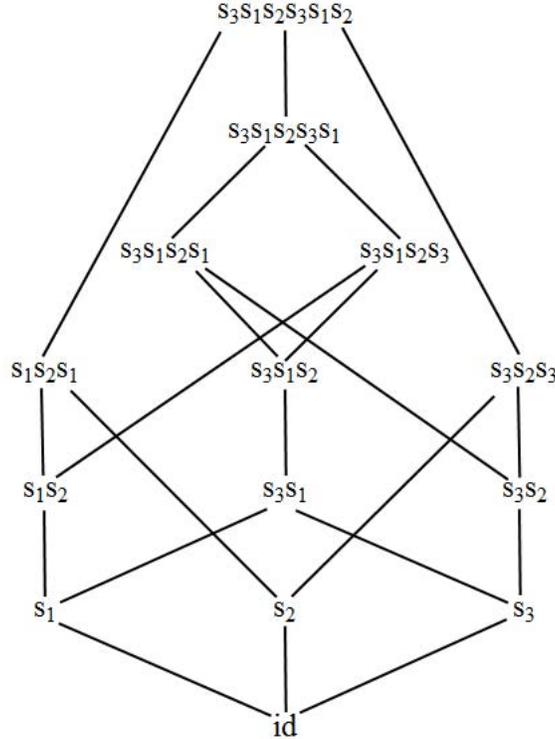


Figure 3.1: The Hasse Diagram of $\text{Camb}(s_3s_1s_2)$

Definition 3.1.8 says that every Coxeter element of \mathcal{A}_n defines a Cambrian lattice of \mathcal{A}_n . The following definition introduces the Tamari lattice as a special case of Cambrian lattices.

Definition 3.1.10. [Reading (2007b)] *The Tamari lattice \mathcal{T}_n is the Cambrian lattice corresponding to the Tamari Coxeter element of \mathcal{A}_n .*

Example 3.1.11. Consider the Coxeter group \mathcal{A}_2 and its Tamari Coxeter element $c = s_1s_2$. The c -sortable elements of \mathcal{A}_2 are e, s_1, s_2, s_1s_2 , and $s_1s_2s_1$, where

$$e \lessdot s_1 \lessdot s_1s_2 \lessdot s_1s_2s_1, \text{ and}$$

$$e \lessdot s_2 \lessdot s_1 \lessdot s_1s_2s_1.$$

The Hasse diagram of the Tamari lattice \mathcal{T}_2 is shown in Figure 3.2.

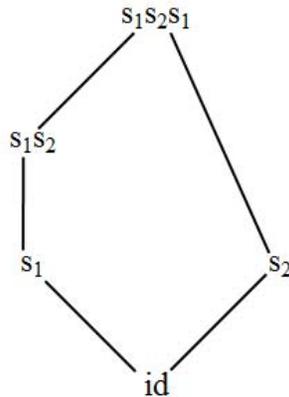


Figure 3.2: The Hasse Diagram of $\text{Camb}(s_1s_2)$

Example 3.1.12. Consider the Coxeter group \mathcal{A}_3 and its Tamari Coxeter element $c = s_1s_2s_3$. The c -sortable elements of \mathcal{A}_3 are the elements $e, s_1, s_2, s_3, s_1s_2, s_1s_3, s_2s_3, s_1s_2s_1, s_1s_2s_3, s_2s_3s_2, s_1s_2s_3s_1, s_1s_2s_3s_2, s_1s_2s_3s_1s_2, s_1s_2s_3s_1s_2s_1$. The Hasse diagram of \mathcal{T}_3 is shown in Figure 3.3.

3.2 Cambrian Lattices via Triangulation

In this section we introduce a different representation of Cambrian lattices via triangulations of convex polygons. This representation, combined with the representation discussed in the previous section, shall be used later to study the duality and isomorphism classes of Cambrian lattices, also to verify some results and to provide

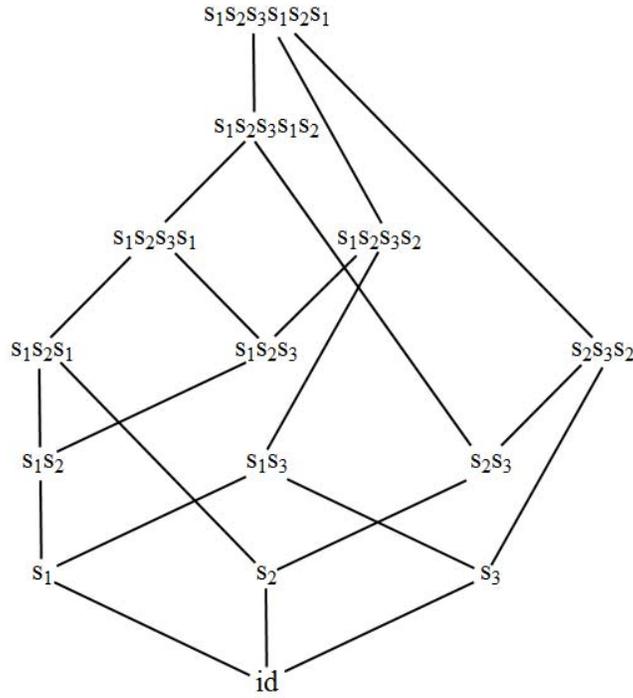


Figure 3.3: The Hasse Diagram of $\text{Camb}(s_1s_2s_3)$

alternative proofs of known facts. The definitions in this section are mainly obtained from [Reading (2006)], [Hohlweg and Lange (2007)], and [Reading (2012)].

Let c be a Coxeter element of \mathcal{A}_n , and let \mathcal{O}^c be its corresponding orientation. A *polygon* Q^c corresponding to the Coxeter element c is defined as a convex polygon consisting of the vertices $\{0, 1, 2, \dots, n, n+1, n+2\}$ so that the vertices 0 and $n+2$ lie on a horizontal line, called the *main horizontal line* of the polygon Q^c , where the vertex 0 is on the left side and the vertex $n+2$ is on the right side. The other vertices are classified into two categories; the category of upper vertices and the category of the lower vertices, where the *upper vertices* are the vertices that appear above the main horizontal line, and the *lower vertices* are the vertices that appear under the main horizontal line. For each $b \in [2, n]$, if s_b precedes s_{b-1} in c , then the vertex b is

classified as an upper vertex, and if s_{b-1} precedes s_b in c , then the vertex b is classified as a lower vertex. The vertex 1 is classified to be in the same category as the vertex 2, and the vertex $n + 1$ is classified to be in the same category as the vertex n . The vertices $0, 1, 2, \dots, n + 1, n + 2$ of Q^c appear in an increasing order from left to right such that for each $b \in [1, n + 2]$, the vertex $b - 1$ lies to the left side of the vertical line passing through the vertex b regardless of whether b is an upper vertex or a lower vertex. We denote by \mathcal{Q}_n the set of all polygons corresponding to the Coxeter elements of \mathcal{A}_n .

The following example illustrates how to construct a polygons Q corresponding to a Coxeter element.

Example 3.2.1. For the Coxeter element $c = s_1 s_3 s_4 s_2$ of \mathcal{A}_4 , the orientation \mathcal{O}^c corresponding to c is

$$s_1 \longrightarrow s_2 \longleftarrow s_3 \longrightarrow s_4.$$

Since $s_1 \longrightarrow s_2$, and $s_3 \longrightarrow s_4$, then 2 and 4 are lower vertices. Since $s_2 \longleftarrow s_3$, then 3 is an upper vertex. As 2 and 4 are lower vertices, the vertices 1 and 5 are lower as well. The polygon Q^c corresponding to c is shown in Figure 3.4.

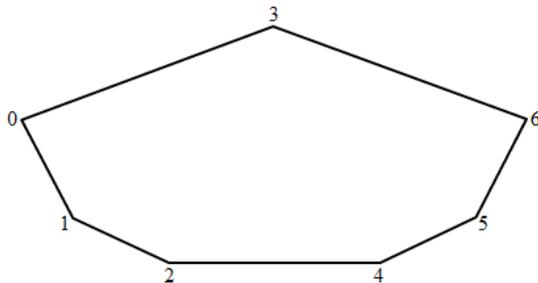


Figure 3.4: A Polygon Corresponding to the Coxeter Element $c = s_1 s_3 s_4 s_2$

A *triangulation* of a convex polygon Q with $n + 3$ vertices is defined as a collection of $n + 1$ diagonals with endpoints from the vertices $\{0, 1, 2, \dots, n + 2\}$ such that no

two different diagonals intersect except maybe at an endpoint. The operation of removing a diagonal from a triangulation of a polygon to create a quadrilateral and then inserting the other diagonal of the quadrilateral to create a new triangulation is called a *diagonal flip*. The diagonal with end points i and j is denoted by \overline{ij} . By assuming that the right horizontal direction refers to the positive x -axis and the up vertical direction refers to the positive y -axis, the *slope* of a diagonal refers to the usual slope.

Example 3.2.2. Consider the triangulated polygon shown in Figure 3.5. In order to

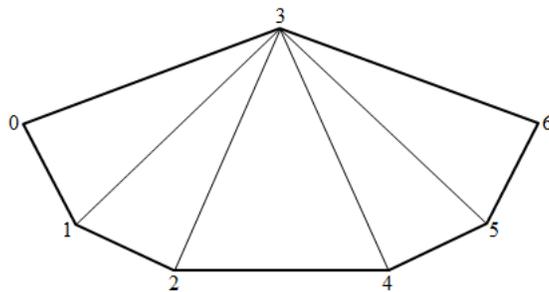


Figure 3.5: A Triangulated Polygon

apply a diagonal flip on the diagonal $\overline{35}$, we first remove the diagonal $\overline{35}$ to construct the quadrilateral with vertices 3, 4, 5, 6, then we insert the diagonal $\overline{46}$. This process is shown in Figure 3.6.

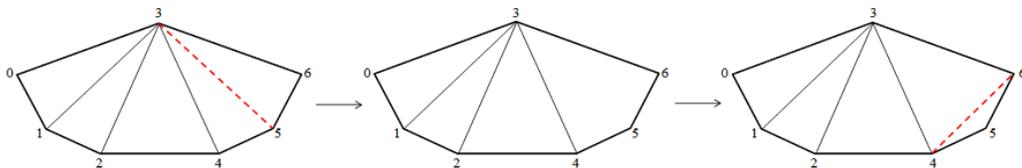


Figure 3.6: Diagonal Flip

Let c be a Coxeter element of \mathcal{A}_n , and let \mathfrak{T}_c denote the set of all triangulations of the polygon Q^c . Given two triangulations $T_1, T_2 \in \mathfrak{T}_c$ that differ only by one diagonal

flip, the diagonal in T_1 that is not a diagonal in T_2 is denoted by $T_1 - T_2$, and similarly the diagonal in T_2 that is not a diagonal in T_1 is denoted by $T_2 - T_1$.

Define the relation “ \leq ” on \mathfrak{T}_c as the transitive closure of the following cover relation: For two triangulations $T_1, T_2 \in \mathfrak{T}_c$ that differ only by one diagonal flip, we define

$$T_1 \leq T_2 \text{ if and only if } \text{slope}(T_1 - T_2) < \text{slope}(T_2 - T_1).$$

Proposition 3.2.3. [Reading (2006)] *For any Coxeter element c , the relation \leq defined above on the set \mathfrak{T}_c is a partial ordering.*

Fix a Coxeter element c of \mathcal{A}_n , and let Q^c be its corresponding polygon. We define a map, η_c , that assigns to each permutation $\pi \in \mathcal{A}_n$ a triangulation, $\eta_c(\pi)$, of Q^c recursively using the following algorithm.

Algorithm 3.2.4. [Reading (2006)] *(From permutation to triangulation algorithm).* We start with a permutation $\pi = \pi_1\pi_2\pi_3 \cdots \pi_{n+1}$ in \mathcal{A}_n , written in one-line notation, and a polygon $Q^c \in \mathcal{Q}_n$ corresponding to a Coxeter element c of \mathcal{A}_n .

Step 1: Let $\lambda_0(\pi)$ be the path from 0 to $n + 2$ passing through all the vertices under the main horizontal line of Q^c such that the path passes through these vertices in order from left to right.

Step 2: For each i from 1 to $n + 1$, define $\lambda_i(\pi)$ recursively from $\lambda_{i-1}(\pi)$ such that if π_i is under the main horizontal line of Q^c , then delete the vertex π_i from the list of vertices visited by the path $\lambda_{i-1}(\pi)$. If π_i is above the main horizontal line of Q^c , then add the vertex π_i to the list of vertices visited by the path $\lambda_{i-1}(\pi)$. The resulting triangulation of Q^c at the last step is the triangulation corresponding to the

permutation π and we denote it by $\eta_c(\pi)$.

The following example illustrates how Algorithm 3.2.4 works.

Example 3.2.5. Consider the permutation $\pi = 426315$, written in one-line notation, and the Coxeter element $c = s_3s_5s_4s_1s_2$ of \mathcal{A}_5 . The orientation \mathcal{O}^c corresponding to c is

$$s_1 \longrightarrow s_2 \longleftarrow s_3 \longrightarrow s_4 \longleftarrow s_5,$$

which shows that the set of lower vertices is $\{1, 2, 4\}$ and the set of upper vertices is $\{3, 5, 6\}$. The polygon Q^c is shown in Figure 3.7. The steps of applying Algorithm

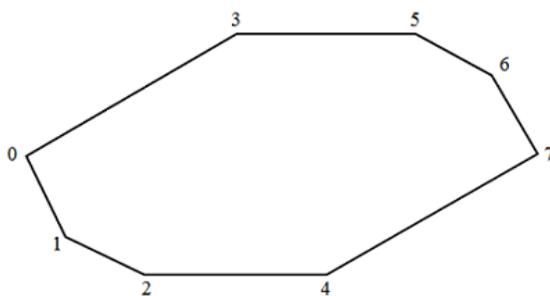


Figure 3.7: The Polygon Q^c , Where $c = s_5s_3s_1s_2s_4$

3.2.4 in order to construct the triangulation of Q^c corresponding to π are shown in Figure 3.8. The triangulation $\eta_c(\pi)$ of Q^c is shown in Figure 3.9.

For a Coxeter element c of \mathcal{A}_n , the following theorem presents the relation between the triangulations of the polygon Q^c and the c -sortable elements of \mathcal{A}_n .

Theorem 3.2.6. [Reading (2006)] Let c be a Coxeter element of \mathcal{A}_n . Then there is a one-to-one correspondence between the set of triangulations of Q^c and the set of c -sortable elements of \mathcal{A}_n .

Proposition 3.2.7. [Reading (2006)] Let x and y be two c -sortable elements of \mathcal{A}_n . Then $x \prec y$ in $\text{Camb}(c)$ if and only if $\eta_c(x)$ and $\eta_c(y)$ differ only by one diagonal flip so that the slope of $\eta_c(x) - \eta_c(y)$ is less than the slope of $\eta_c(y) - \eta_c(x)$.

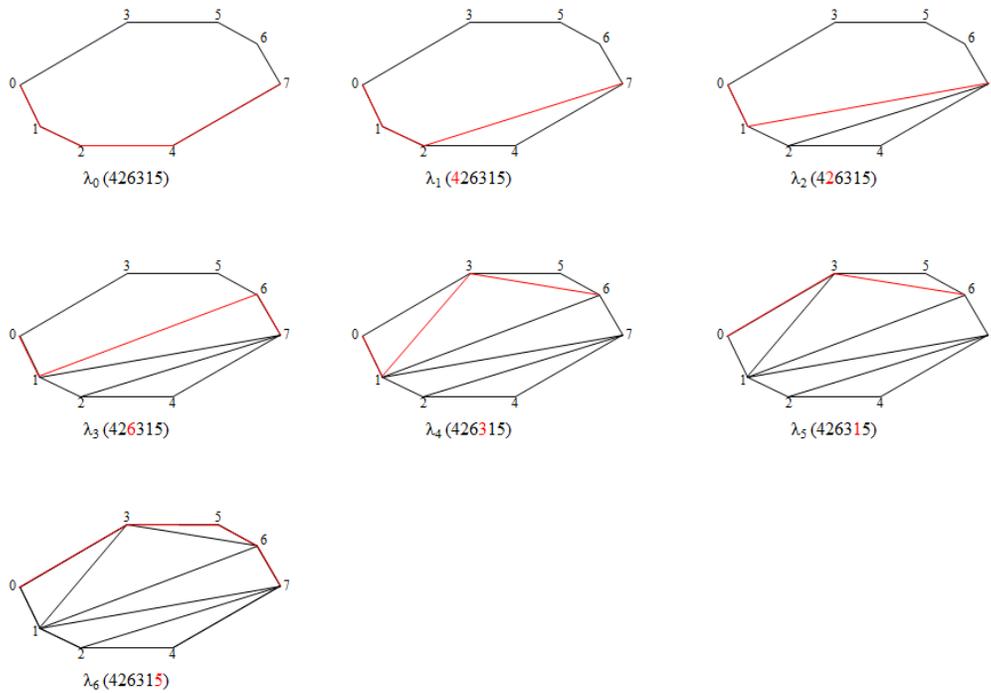


Figure 3.8: The Triangulation Steps Corresponding to $\pi = 426315$

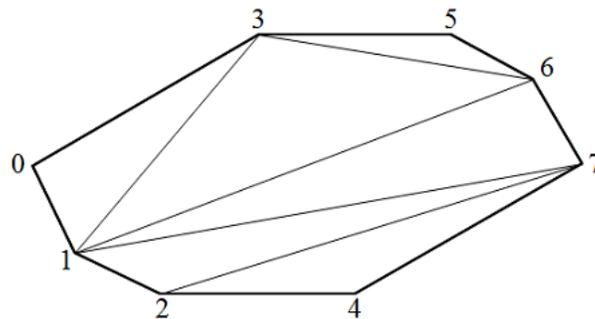


Figure 3.9: The Triangulation Corresponding to the Permutation $\pi = 426315$

The following theorem gives a different representation of Cambrian lattices in terms of triangulations of polygons.

Theorem 3.2.8. [Reading (2006)] *For a Coxeter element c of \mathcal{A}_n , the Cambrian lattice $\text{Camb}(c)$ is isomorphic to the partial order on triangulations of Q^c whose cover relations are diagonal flips, where going up in the cover relation corresponds to increasing the slope of the diagonal, and going down in the cover relation corresponds to decreasing the slope of the diagonal.*

The two representations of Cambrian lattices, via Coxeter-sortable elements of \mathcal{A}_n and via triangulations of polygons, shall be used in Section 3.6 and Section 3.7 to study the isomorphism classes and duality of Cambrian lattices, also in Section 3.4 we use them to provide alternative proofs of some known facts.

In Example 3.1.9, we introduced the Cambrian lattice $\text{Camb}(s_3s_1s_2)$ in terms of the $s_3s_1s_2$ -sortable elements of \mathcal{A}_3 , and in the following example we introduce the same Cambrian lattice in terms of triangulations of a polygon.

Example 3.2.9. [Reading (2012)] *The orientation \mathcal{O}^c corresponding to the Coxeter element $c = s_3s_1s_2$ of \mathcal{A}_3 is*

$$s_1 \longrightarrow s_2 \longleftarrow s_3,$$

so that the set of lower vertices is $\{1, 2\}$, and the set of upper vertices is $\{3, 4\}$. The polygon Q^c is shown in Figure 3.10, and the Hasse diagram of $\text{Camb}(c)$ in terms of triangulations of Q^c is shown in Figure 3.11.

According to the representation of Cambrian lattices via triangulations, the Tamari lattice \mathcal{T}_n can be seen as the poset of triangulations of the polygon with vertices $0, 1, 2, \dots, n + 2$ such that the vertices 0 and $n + 2$ are the end points of the main horizontal diagonal, and the other vertices all are lower vertices.

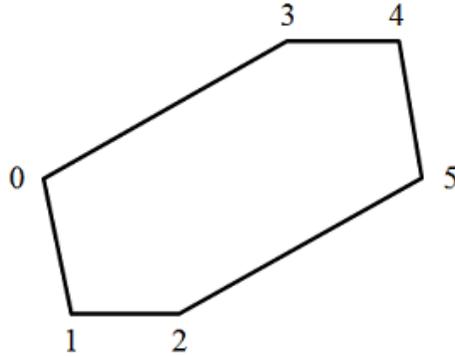


Figure 3.10: The Polygon $Q^{s_3s_1s_2}$

3.3 The Poset of Compositions

In this section we define an equivalence relation and a partial ordering on the set \mathfrak{C}_n of compositions of a positive integer $n \in \mathbb{Z}^+$, and prove that the poset $\hat{\mathfrak{C}}_n$ is a self-dual lattice. This section will be used in the next section and Chapter 5 to study the chains of maximum length of both Cambrian and m -eralized Cambrian lattices. In particular, we find a relation between the chains of maximum length of Cambrian and m -eralized Cambrian lattices, and the size of the class of the composition corresponding to their maximum elements.

Recall that a composition of a positive integer n is a sequence of positive integers (a_1, a_2, \dots, a_k) such that $\sum_{i=1}^k a_i = n$. The length of a composition α , denoted $\ell(\alpha)$, is the number of parts in α , and the sum of the parts of α is called the size of α and denoted by $|\alpha|$. The set of all compositions of a positive integer n is denoted by \mathfrak{C}_n . We aim to define an equivalence relation and a partial ordering on \mathfrak{C}_n , we start with the following two definitions.

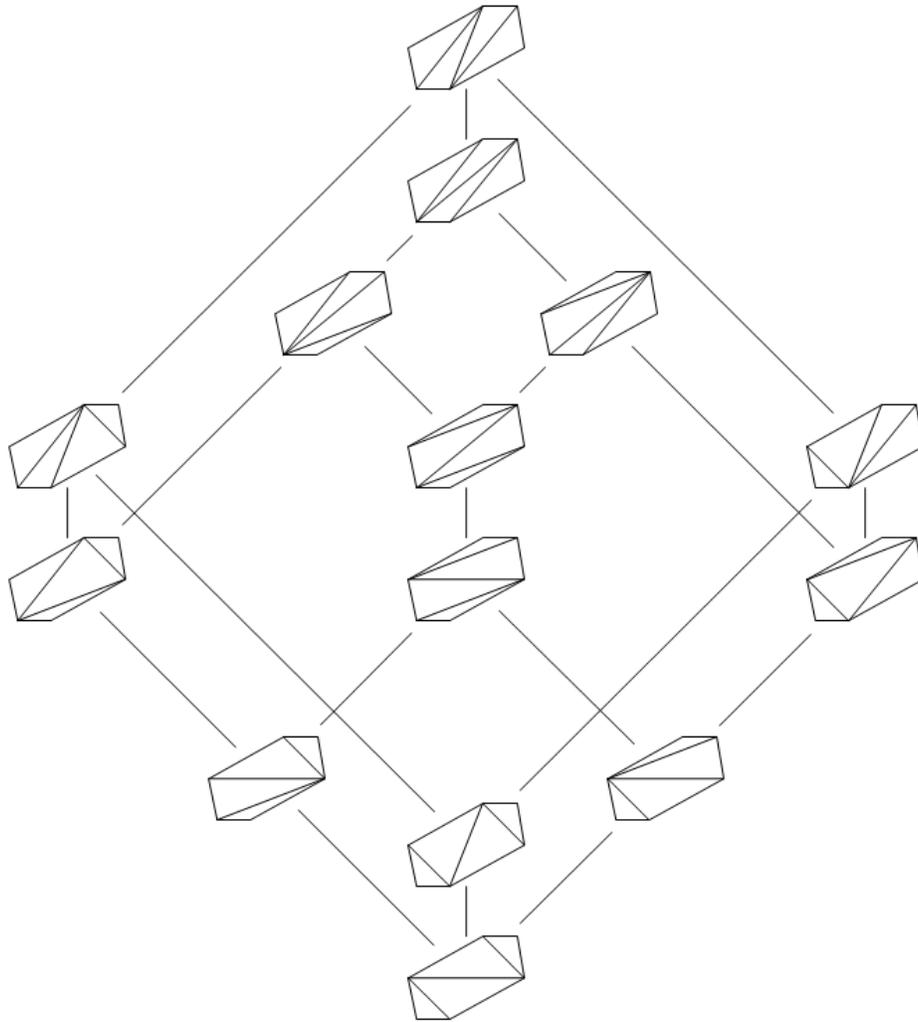


Figure 3.11: The Cambrian Lattice $\text{Camb}(s_3s_1s_2)$

Definition 3.3.1. For a composition $\alpha = (a_1, a_2, \dots, a_k)$, two adjacent parts a_i and a_{i+1} of α are said to be commutant if $|a_i - a_{i+1}| > 1$. Otherwise, they are called noncommutant.

Definition 3.3.2. A commuting move on compositions is the operation we apply on a composition to obtain another composition by commuting two commutant parts.

Example 3.3.3. For the composition $\alpha = (4, 5, 2, 1)$ of $n = 12$, the parts 5 and 2 are commutant while the parts 1 and 2 are noncommutant. The resulting composition obtained from α by applying a commuting move on the parts 5 and 2 is $\beta = (4, 2, 5, 1)$.

For $n \in \mathbb{Z}^+$, define the relation “ \simeq ” on the set of compositions \mathfrak{C}_n as follows: For $\alpha, \beta \in \mathfrak{C}_n$; $\alpha \simeq \beta$ if and only if β can be obtained from α by applying a sequence of commuting moves on α .

Proposition 3.3.4. The relation \simeq is an equivalence relation on the set \mathfrak{C}_n .

Proof. Let $\alpha, \beta, \gamma \in \mathfrak{C}_n$. As α is obtained from itself by applying no commuting moves, $\alpha \simeq \alpha$ and hence \simeq is reflexive. Suppose that $\alpha \simeq \beta$. Then β is obtained from α by applying a sequence of commuting moves. By applying the sequence of moves in reverse, we obtain α back from β which shows that $\beta \simeq \alpha$. That is, \simeq is symmetric. Finally assume that $\alpha \simeq \beta$, and $\beta \simeq \gamma$. Since $\alpha \simeq \beta$, then β can be obtained from α by applying a sequence of commuting moves. Similarly, $\beta \simeq \gamma$ means that γ can be obtained from β by applying a sequence of commuting moves. Apply the commuting moves from α to β , then from β to γ , we get γ from α which shows that $\alpha \simeq \gamma$. Thus \simeq is transitive, and hence \simeq is an equivalence relation. \square

Note that compositions in the same class have the same length and parts.

Define the relation “ \preceq ” on \mathfrak{C}_n so that for $\alpha = (a_1, a_2, \dots, a_k), \beta = (b_1, b_2, \dots, b_l) \in \mathfrak{C}_n$, $\alpha \preceq \beta$ if and only if

1. $\alpha \simeq \beta$, and
2. $\sum_{i=1}^k a_i \cdot 10^i \leq \sum_{i=1}^l b_i \cdot 10^i$.

Proposition 3.3.5. *The relation \preceq on \mathfrak{C}_n is a partial order relation.*

Proof. Let $\alpha = (a_1, a_2, \dots, a_k)$, $\beta = (b_1, b_2, \dots, b_l)$, and $\gamma = (c_1, c_2, \dots, c_m)$ be any three compositions of \mathfrak{C}_n . As the relation \succeq is reflexive and $\sum_{i=1}^k a_i \cdot 10^i$ depends only on the components of α , the relation \preceq is reflexive. To check the antisymmetry of \preceq , suppose that $\alpha \preceq \beta$ and $\beta \preceq \alpha$. This implies that $\alpha \succeq \beta$ and thus α and β have the same length, that is $k = l$. Now $\alpha \preceq \beta$ implies that $\sum_{i=1}^k a_i \cdot 10^i \leq \sum_{i=1}^k b_i \cdot 10^i$, and on the other hand; $\beta \preceq \alpha$ implies that $\sum_{i=1}^k b_i \cdot 10^i \leq \sum_{i=1}^k a_i \cdot 10^i$. Therefore,

$$\sum_{i=1}^k a_i \cdot 10^i \leq \sum_{i=1}^k b_i \cdot 10^i \leq \sum_{i=1}^k a_i \cdot 10^i,$$

and hence $\sum_{i=1}^k a_i \cdot 10^i = \sum_{i=1}^k b_i \cdot 10^i$. Since $\alpha \succeq \beta$, then $\alpha = \beta$. This checks that \preceq is antisymmetric. Finally, suppose that $\alpha \preceq \beta$, and $\beta \preceq \gamma$. Then $\alpha \succeq \beta$ and $\beta \succeq \gamma$ with $\sum_{i=1}^k a_i \cdot 10^i \leq \sum_{i=1}^k b_i \cdot 10^i$ and $\sum_{i=1}^k b_i \cdot 10^i \leq \sum_{i=1}^m c_i \cdot 10^i$. This implies that $\alpha \succeq \gamma$ with $\sum_{i=1}^k a_i \cdot 10^i \leq \sum_{i=1}^m c_i \cdot 10^i$. Therefore, $\alpha \preceq \gamma$, and hence \preceq is transitive. \square

Proposition 3.3.6. *Every class of \mathfrak{C}_n / \preceq forms a saturated chain of $(\mathfrak{C}_n, \preceq)$. Thus, \mathfrak{C}_n is a disjoint union of chains.*

Proof. Pick a class $[\alpha]_{\preceq}$ of \mathfrak{C}_n / \preceq . In order to show that the class $[\alpha]_{\preceq}$ is a chain of $(\mathfrak{C}_n, \preceq)$, let $\beta, \gamma \in [\alpha]_{\preceq}$. Then $\beta \succeq \gamma$ and hence β and γ are of the same length and parts. Assume that $\beta = (b_1, b_2, \dots, b_l)$, and $\gamma = (c_1, c_2, \dots, c_l)$. Then either $\sum_{i=1}^l b_i \cdot 10^i \leq \sum_{i=1}^l c_i \cdot 10^i$, or $\sum_{i=1}^l b_i \cdot 10^i \geq \sum_{i=1}^l c_i \cdot 10^i$. That is, either $\beta \preceq \gamma$ or $\beta \succeq \gamma$. Since β and γ were chosen arbitrarily, then any two compositions in the class $[\alpha]_{\preceq}$ are comparable and hence the class $[\alpha]_{\preceq}$ is a chain of \mathfrak{C}_n . Let δ be a composition in \mathfrak{C}_n with $\beta \prec \delta \prec \gamma$, for some $\beta, \gamma \in [\alpha]_{\preceq}$. Then $\beta \succeq \delta$ and hence $\delta \succeq \alpha$. This implies that $\delta \in [\alpha]_{\preceq}$ and thus the class $[\alpha]_{\preceq}$ is a saturated chain as desired. \square

Example 3.3.7. *Figure 3.12 shows the Hasse diagram of the class $[(4, 2, 1, 1)]_{\preceq}$.*

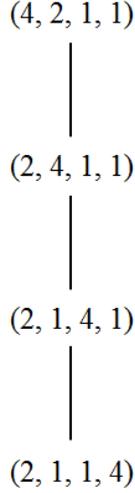


Figure 3.12: The Hasse Diagram of $[(4, 2, 1, 1)]_{\simeq}$

By Proposition 3.3.6, for every class $[\alpha]_{\simeq}$ of \mathfrak{C}_n/\simeq , there are two unique compositions $\hat{0}_\alpha$ and $\hat{1}_\alpha$ such that $\hat{0}_\alpha \preceq \beta \preceq \hat{1}_\alpha$ for all $\beta \in [\alpha]_{\simeq}$. We call the compositions $\hat{0}_\alpha$ and $\hat{1}_\alpha$ the *minimum* and *maximum* elements of the class $[\alpha]_{\simeq}$ respectively.

Proposition 3.3.8. *The poset $(\hat{\mathfrak{C}}_n, \preceq)$ is a self-dual lattice, where $\hat{\mathfrak{C}}_n = \mathfrak{C}_n \cup \{\hat{0}, \hat{1}\}$.*

Proof. Let $\alpha, \beta \in \hat{\mathfrak{C}}_n$. We first show that α and β have a supremum in $\hat{\mathfrak{C}}_n$. If $\hat{1} \in \{\alpha, \beta\}$, then $\alpha \vee \beta = \hat{1}$ and we are done. Suppose that $\hat{1} \notin \{\alpha, \beta\}$. There are two cases to study:

Case 1: $\alpha \simeq \beta$. Then $[\alpha]_{\simeq} = [\beta]_{\simeq}$. By Proposition 3.3.6, $[\alpha]_{\simeq}$ is a chain and hence either $\alpha \preceq \beta$ or $\beta \preceq \alpha$. Without loss of generality assume that $\alpha \preceq \beta$, then $\alpha \vee \beta = \beta$ and we are done.

Case 2: $\alpha \not\simeq \beta$. As $\alpha \preceq \hat{1}$, and $\beta \preceq \hat{1}$, then $\hat{1}$ is an upper bound of α and β . We claim that $\alpha \vee \beta = \hat{1}$. Suppose not, then there must be $\gamma \in \mathfrak{C}_n$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$, which implies that $\alpha \simeq \gamma$ and $\beta \simeq \gamma$. Since the relation \simeq is an equivalence relation, then $\alpha \simeq \beta$ which is a contradiction. Thus $\alpha \vee \beta = \hat{1}$.

This shows that every pair of elements in $\hat{\mathfrak{C}}_n$ has a supremum. The proof that α and

β have an infimum is similar.

Now, by Proposition 3.3.6, every class of \mathfrak{C}_n/\simeq is a saturated chain. Since every chain is self-dual, and as the Hasse diagram of $\hat{\mathfrak{C}}_n$ is constructed from the Hasse diagram of \mathfrak{C}_n by attaching $\hat{0}$ and $\hat{1}$ to the bottom and the top respectively with the property that $\hat{0}$ is covered by $\hat{0}_\alpha$, and $\hat{1}$ covers $\hat{0}_\alpha$, for all $\alpha \in \mathfrak{C}_n$, the lattice $(\hat{\mathfrak{C}}_n, \preceq)$ is self-dual and this completes the proof. \square

As a consequence of Proposition 3.3.8, we have the following two corollaries.

Corollary 3.3.9. *Let $\alpha \in \mathfrak{C}_n$. Then*

1. α is the minimum element of its class if and only if α is an atom of $\hat{\mathfrak{C}}_n$,
2. α is the maximum element of its class if and only if α is a coatom of $\hat{\mathfrak{C}}_n$.

Proof. Consider the class $[\alpha]_{\simeq}$ of \mathfrak{C}_n/\simeq . α is the minimum of its class implies that $\alpha \preceq \beta$ for all $\beta \in \mathfrak{C}_n$ with $\beta \in [\alpha]_{\simeq}$. If $\gamma \in \mathfrak{C}_n$ and $\gamma \preceq \alpha$, then $\gamma \in [\alpha]_{\simeq}$, and hence $\alpha \preceq \gamma$. That is $\alpha = \gamma$. Since $\hat{0} \preceq \alpha$, then the argument above shows that α covers $\hat{0}$, and thus α is an atom of $\hat{\mathfrak{C}}_n$. Suppose that α is an atom of $\hat{\mathfrak{C}}_n$, then $\alpha \in \mathfrak{C}_n$. Let $\beta \in \mathfrak{C}_n$ with $\beta \simeq \alpha$. Then either $\alpha \preceq \beta$ or $\beta \preceq \alpha$. If $\alpha \preceq \beta$, then we are done. If $\beta \preceq \alpha$, then as $\hat{0} \preceq \beta$, we get $\hat{0} \preceq \beta \preceq \alpha$. Since α is an atom of $\hat{\mathfrak{C}}_n$, then $\beta = \alpha$. This checks the other direction of the first statement. The second statement follows by the first statement and Proposition 3.3.8. \square

Corollary 3.3.10. *For any $n \in \mathbb{Z}^+$, the following are equal:*

1. the number of atoms of the poset $(\hat{\mathfrak{C}}_n, \preceq)$,
2. the number of coatoms of the poset $(\hat{\mathfrak{C}}_n, \preceq)$,
3. the number of equivalence classes of \mathfrak{C}_n/\simeq .

Proof. Let $\mathfrak{M} = \{[\alpha]_{\preceq} : \alpha \in \mathfrak{C}_n\}$, $\mathfrak{N} = \{\hat{0}_\alpha : [\alpha]_{\preceq} \in \mathfrak{M}\}$, and $\mathfrak{D} = \{\hat{1}_\alpha : [\alpha]_{\preceq} \in \mathfrak{M}\}$. As \mathfrak{C}_n is finite, the sets \mathfrak{M} , \mathfrak{N} , and \mathfrak{D} are finite. The maps from each class $[\alpha]_{\preceq}$ to its minimum and maximum elements $\hat{0}_\alpha$ and $\hat{1}_\alpha$ respectively are bijections. Thus $|\mathfrak{M}| = |\mathfrak{N}| = |\mathfrak{D}|$. By Corollary 3.3.9, the number of atoms of $\hat{\mathfrak{C}}_n$ is equal to the size of \mathfrak{N} , and the number of coatoms of $\hat{\mathfrak{C}}_n$ is equal to the size of \mathfrak{D} . That is, the number of atoms of $(\hat{\mathfrak{C}}_n, \preceq)$, the number of coatoms of $(\hat{\mathfrak{C}}_n, \preceq)$, and the number of equivalent classes of \mathfrak{C}_n / \simeq are equal. \square

3.4 Some Basic Facts and Results about Cambrian Lattices

In this section we present some known facts about Cambrian lattices; we provide our own proofs for these facts as we couldn't find explicit references either stating or proving them. We also state and prove some of our results that will be used in coming sections and chapters.

Recall that two Coxeter elements c and c' of \mathcal{A}_n are said to be of the same type if $c \sim_{C_1} c'$. In the following proposition we enumerate the different types of Coxeter elements of \mathcal{A}_n .

Proposition 3.4.1. *The number of different types of Coxeter elements of \mathcal{A}_n is equal to 2^{n-1} .*

Proof. The number of different types of Coxeter elements of \mathcal{A}_n is equal to the number of different classes $[c]_{\sim_{C_1}}$, where c is a Coxeter element of \mathcal{A}_n . By Proposition 3.1.7, this number is equal to the number of orientations of the Coxeter diagram of \mathcal{A}_n . The Coxeter diagram of \mathcal{A}_n is

$$1 \text{ --- } 2 \text{ --- } 3 \text{ --- } \dots \text{ --- } n,$$

which consists of n vertices and $n-1$ edges. The orientations of this Coxeter diagram

are obtained by replacing each edge by a single directed edge. Since every edge can be replaced by either a left directed or a right directed edge, then by the multiplication principle of counting there are 2^{n-1} different orientations. That is, the number of different types of Coxeter elements is 2^{n-1} . \square

Corollary 3.4.2. *The number of distinct Cambrian lattices of \mathcal{A}_n is equal to 2^{n-1} .*

Proof. Since there is a bijection between the Cambrian lattices obtained from \mathcal{A}_n and the set of orientations of the Coxeter elements, and as the number of such orientations is equal to the number of types of Coxeter elements of \mathcal{A}_n , then by Proposition 3.4.1, the number of Cambrian lattices of \mathcal{A}_n is equal to 2^{n-1} . \square

Not all Cambrian lattices counted in Corollary 3.4.2 are non-isomorphic. In Proposition 3.6.4 we enumerate the distinct non-isomorphic Cambrian lattices of \mathcal{A}_n . The following proposition is well-known, we provide our own proof as we could not find a reference for it.

Proposition 3.4.3. *For any Coxeter element c of \mathcal{A}_n , the maximum length of chains of the Cambrian lattice $\text{Camb}(c)$ is equal to $\binom{n+1}{2}$.*

Proof. Fix a positive integer $n \in \mathbb{Z}^+$, and let c be a Coxeter element of \mathcal{A}_n . The maximum element w_0 of \mathcal{A}_n has the one-line representation

$$w_0 = (n+1) \ n \ (n-1) \ \cdots \ 2 \ 1.$$

By Proposition 3.1.5, w_0 has a c -sorting decomposition. By Corollary 2.4.3, $\ell(w_0) = n + (n-1) + (n-2) + \cdots + 2 + 1$, and hence $\ell(w_0) = \binom{n+1}{2}$. This means that the c -sorting decomposition of w_0 has the length $\binom{n+1}{2}$. Set $\ell = \binom{n+1}{2}$ and let

$$w_0(c) = c_1 c_2 \cdots c_\ell$$

be the c -sorting decomposition of w_0 . Since the Cambrian lattice $\text{Camb}(c)$ is the subposet of \mathcal{A}_n induced by the c -sortable elements of \mathcal{A}_n and since the chain

$$id \leq c_1 \leq c_1c_2 \leq c_1c_2c_3 \leq \cdots \leq c_1c_2 \cdots c_\ell = w_0(c)$$

is a chain of c -sortable elements of \mathcal{A}_n , then the chain above is a chain in the Cambrian lattice $\text{Camb}(c)$. As this chain starts by the minimum element of $\text{Camb}(c)$, namely id , and ends by the maximum element of $\text{Camb}(c)$, namely $w_0(c)$, and since $\ell(c_1c_2 \cdots c_{i+1}) = \ell(c_1c_2 \cdots c_i) + 1$ for all $i \in [\ell - 1]$, this chain is saturated and has the maximum length of chains in $\text{Camb}(c)$. That is, the maximum length of chains in $\text{Camb}(c)$ is equal to $\binom{n+1}{2}$. \square

The following well-known corollary follows directly from Proposition 3.4.3 as the Tamari lattice is the Cambrian lattice of \mathcal{A}_n corresponding to the Tamari Coxeter element. However; it was checked differently by D. Knuth, see [Wells (1973)], and by G. Markowsky, see [Markowsky (1992)].

Corollary 3.4.4. [Wells (1973)] *The maximum length of chains in Tamari lattice \mathcal{T}_n is equal to $\binom{n+1}{2}$.*

For the following known proposition, we provide a proof based on triangulations of polygons.

Proposition 3.4.5. [Reading and Speyer (2009)] *For any Coxeter element c of \mathcal{A}_n , the size of the Cambrian lattice $\text{Camb}(c)$ is equal to the n^{th} -Catalan number C_n .*

Proof. Let c be a Coxeter element of \mathcal{A}_n , and consider the polygon \mathcal{Q}^c corresponding to c . The polygon \mathcal{Q}^c consists of $n + 2$ vertices, and by Proposition 2.5.2, the number of triangulations of \mathcal{Q}^c is equal to the Catalan number C_n . By Theorem 3.2.8, the triangulations of \mathcal{Q}^c are the vertices of the Cambrian lattice $\text{Camb}(c)$. Thus, the size of the Cambrian lattice $\text{Camb}(c)$ is equal to C_n . \square

Corollary 3.4.6. *For any Coxeter element c of \mathcal{A}_n , the number of c -sortable elements of \mathcal{A}_n is equal to the n^{th} -Catalan number C_n .*

Proof. This corollary follows directly by Definition 3.1.8 of Cambrian lattices and Proposition 3.4.5. □

It is well-known that Cambrian lattices are regular lattices, see [Reading (2006)] or [Reading and Speyer (2009)]. We provide a proof for this fact based on the triangulation-definition of Cambrian lattice.

Theorem 3.4.7. [Reading and Speyer (2009)] *All Cambrian lattices of \mathcal{A}_n are n -regular.*

Proof. Let c be a Coxeter element of \mathcal{A}_n and consider the polygon Q^c corresponding to c . By Theorem 3.2.8, $\text{Camb}(c)$ is the set \mathfrak{T}_c under the diagonal flip ordering. Pick an element α in $\text{Camb}(c)$. Then α represents a triangulation T^α of Q^c . The polygon Q^c consists of $n + 3$ vertices and n diagonals; Let $\{d_1, d_2, \dots, d_n\}$ be the set of these diagonals. For each i , a triangulation T_i^α of Q^c can be constructed from T^α by removing the diagonal d_i to obtain a quadrilateral, then inserting the other possible diagonal of the quadrilateral. The triangulations $T_1^\alpha, T_2^\alpha, \dots, T_n^\alpha$ are all different and also different than the original triangulation T^α . Based on the diagonal flip ordering each of these triangulations might cover T^α or be covered by T^α . That is, the degree of the vertex T^α in the Hasse diagram of $\text{Camb}(c)$ is equal to n . Since α was chosen arbitrarily, $\text{Camb}(c)$ is n -regular. □

Corollary 3.4.8. [Reading and Speyer (2009)] *The Tamari lattice \mathcal{T}_n is n -regular.*

Proof. It follows directly from Theorem 3.4.7 since the Tamari lattice \mathcal{T}_n is a Cambrian lattice of \mathcal{A}_n . □

Every decomposition $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ of an element in \mathcal{A}_n can be embedded into the poset of compositions $(\mathfrak{C}_m, \preceq)$, where $m = \sum_{j=1}^{\ell} i_j$, as the composition $(i_1, i_2, \dots, i_\ell)$ which we call the *composition corresponding* to the decomposition w . Conversely, every composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$ can be embedded into the set of decompositions of elements of \mathcal{A}_n , where $n = \max \{\alpha_i : i \in [\ell]\}$, as the decomposition $s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_\ell}$ which we call the *decomposition corresponding* to the composition α . This shows a one-to-one correspondence between the set of compositions and the set of decompositions of elements in $\bigcup_{n \in \mathbb{Z}^+} \mathcal{A}_n$, so that we can use the two terms, compositions and decomposition, interchangeably when needed. We mean by $[w_0(c)]_{\preceq}$ the equivalence class of the c -sorting decomposition $w_0(c)$ of the maximum element w_0 of \mathcal{A}_n .

Proposition 3.4.9. *For any Coxeter element c of \mathcal{A}_n , the number of chains of maximum length of $\text{Camb}(c)$ is equal to the size of the class $[w_0(c)]_{\preceq}$.*

Proof. Let c be a Coxeter element of \mathcal{A}_n , and let $w_0(c)$ be the c -sorting decomposition of the maximum element w_0 of \mathcal{A}_n . In Proposition 3.4.3 we showed that the length of chains of maximum length is equal to $\binom{n+1}{2}$, which is equal to the length of w_0 . Every chain of maximum length has the form

$$id \triangleleft c_1 \triangleleft c_1c_2 \triangleleft c_1c_2c_3 \triangleleft \cdots \triangleleft c_1c_2c_3 \cdots c_\ell = w_0(c),$$

where c_i is an adjacent transposition of \mathcal{A}_n , and $\ell = \ell(w_0)$. Based on the right weak covering definition, and the fact that applying a long-braid move on a c -sorting decomposition makes it non c -sorted, every maximum-length chain in $\text{Camb}(c)$ represents an element in $[w_0(c)]_{\preceq}$. On the other hand, every decomposition obtained from $[w_0(c)]_{\preceq}$ is a reduced decomposition of w_0 and also represents a chain of maximum length of $\text{Camb}(c)$. Therefore the number of chains of maximum length of $\text{Camb}(c)$ is equal to the size of the class $[w_0(c)]_{\preceq}$ as desired. \square

Example 3.4.10. Consider the Coxeter element $c = s_3s_1s_2$ of \mathcal{A}_3 . The c -sorting decomposition of w_0 is $w_0(c) = s_3s_1s_2s_3s_1s_2$, and Figure 3.1 shows the Hasse diagram of $\text{Camb}(c)$. The chains of maximum length of $\text{Camb}(c)$ are:

$$id \triangleleft s_1 \triangleleft s_3s_1 \triangleleft s_3s_1s_2 \triangleleft s_3s_1s_2s_1 \triangleleft s_3s_1s_2s_3s_1 \triangleleft s_3s_1s_2s_3s_1s_2,$$

$$id \triangleleft s_1 \triangleleft s_3s_1 \triangleleft s_3s_1s_2 \triangleleft s_3s_1s_2s_3 \triangleleft s_3s_1s_2s_3s_1 \triangleleft s_3s_1s_2s_3s_1s_2,$$

$$id \triangleleft s_3 \triangleleft s_3s_1 \triangleleft s_3s_1s_2 \triangleleft s_3s_1s_2s_1 \triangleleft s_3s_1s_2s_3s_1 \triangleleft s_3s_1s_2s_3s_1s_2,$$

and

$$id \triangleleft s_3 \triangleleft s_3s_1 \triangleleft s_3s_1s_2 \triangleleft s_3s_1s_2s_3 \triangleleft s_3s_1s_2s_3s_1 \triangleleft s_3s_1s_2s_3s_1s_2.$$

On the other hand, the class $[w_0(c)]_{\leq}$ consists of the compositions $(3, 1, 2, 3, 1, 2)$, $(1, 3, 2, 3, 1, 2)$, $(3, 1, 2, 1, 3, 2)$, and $(1, 3, 2, 1, 3, 2)$.

3.5 The Horizontal and Vertical Reflection Maps

In this section we define the maps ρ^h and ρ^v on the set of Coxeter elements of \mathcal{A}_n , and study some of their properties. These two maps play an important role in studying the duality and isomorphism between Cambrian lattices in the next two sections. Additionally, we define the horizontal and vertical reflection maps \mathfrak{R}^h and \mathfrak{R}^v on the set of polygons \mathcal{Q} , we study some of their properties and study the relation between the maps ρ^h and \mathfrak{R}^h as well as the relation between the maps ρ^v and \mathfrak{R}^v . We also define the symmetric Coxeter elements that shall mainly be used in studying the self-dual Cambrian lattices in the next section.

Every Coxeter element c of \mathcal{A}_n can be written on the form

$$c = s_{i_1}s_{i_2} \cdots s_{i_a} s_1 s_{j_1}s_{j_2} \cdots s_{j_b},$$

where s_{i_k} 's are the upper vertices of c ordered such that $i_1 > i_2 > \cdots > i_a$, and s_{j_k} 's are the lower vertices of c ordered such that $j_1 < j_2 < \cdots < j_b$. We call this form the *standard representation* of the Coxeter element c . The existence and uniqueness of the standard representation of a Coxeter element can be checked by straightforward induction on n .

Fix a positive integer $n \in \mathbb{Z}^+$, we define the map ρ^h on the set of Coxeter elements of \mathcal{A}_n so that if $c = s_{i_1}s_{i_2}\cdots s_{i_a}s_1s_{j_1}s_{j_2}\cdots s_{j_b}$ is a Coxeter element of \mathcal{A}_n written in its standard representation, then

$$\rho^h(c) = s_{n+2-i_a}\cdots s_{n+2-i_2}s_{n+2-i_1}s_1s_{n+2-j_b}\cdots s_{n+2-j_2}s_{n+2-j_1}.$$

Example 3.5.1. Consider the Coxeter element $c = s_1s_2s_6s_4s_3s_5$ of \mathcal{A}_6 . The upper vertices of c are s_4 and s_6 , while the lower vertices are s_2, s_3 , and s_5 . Thus, the standard representation of c is $s_6s_4s_1s_2s_3s_5$, and hence $\rho^h(c) = s_4s_2s_1s_3s_5s_6$.

Lemma 3.5.2. For any Coxeter element c of \mathcal{A}_n with $n \geq 2$, if s_n is an upper vertex of c , then s_2 is an upper vertex of $\rho^h(c)$.

Proof. Let $n \geq 2$, and let c be a Coxeter element of \mathcal{A}_n . Suppose that s_n is an upper vertex of c . Then s_n lies to the left of s_1 in the standard representation of c and thus s_2 lies to the left of s_1 in $\rho^h(c)$. That is s_2 precedes s_1 in $\rho^h(c)$ and so s_2 is an upper vertex of $\rho^h(c)$. \square

Lemma 3.5.3. For any Coxeter element of \mathcal{A}_n with $n \geq 2$, if s_n is a lower vertex of c , then s_2 is a lower vertex of $\rho^h(c)$.

Proof. Let $n \geq 2$ and c be a Coxeter element of \mathcal{A}_n . If s_n is a lower vertex of c , then s_n lies to the right of s_1 in the standard representation of c . Hence s_2 lies to the right of s_1 in $\rho^h(c)$. That is, s_1 precedes s_2 in $\rho^h(c)$ and thus s_2 is a lower vertex of $\rho^h(c)$. \square

We use Lemma 3.5.2 and Lemma 3.5.3 to verify the following proposition.

Proposition 3.5.4. *For a Coxeter element c of \mathcal{A}_n , the orientation $\mathcal{O}^{\rho^h(c)}$ is obtained from the orientation \mathcal{O}^c by reading its directed edges backward so that the first directed edge of $\mathcal{O}^{\rho^h(c)}$ is the last directed edge of \mathcal{O}^c , the second directed edge of $\mathcal{O}^{\rho^h(c)}$ is the second-to-last directed edge of \mathcal{O}^c , and so on so that the last directed edge of $\mathcal{O}^{\rho^h(c)}$ is the first directed edge of \mathcal{O}^c .*

Proof. We run an induction on n . In order to avoid the trivial case, we assume that $n \geq 2$.

Base step: $n = 2$. The Coxeter elements of \mathcal{A}_2 are $c_1 = s_1s_2$ and $c_2 = s_2s_1$. Since $\rho^h(c_1) = c_1$ and $\rho^h(c_2) = c_2$ with $\mathcal{O}^{c_1} = \mathcal{O}^{\rho^h(c_1)}$ and $\mathcal{O}^{c_2} = \mathcal{O}^{\rho^h(c_2)}$, the assumption is valid for $n = 2$.

Inductive step: We assume that the assumption holds for $n - 1$, and we want to prove that it is valid for n . There are two cases to study:

Case 1: s_n is an upper vertex of c . Then $c = s_n c'$, where c' is a Coxeter element of \mathcal{A}_{n-1} , and $(n - 1) \leftarrow n$ is the last edge to the right in \mathcal{O}^c . By Lemma 3.5.2, s_2 is an upper vertex of $\rho^h(c)$ and hence $1 \leftarrow 2$ is the first edge from the left of $\mathcal{O}^{\rho^h(c)}$. By the inductive step, the assumption holds for c' and hence the orientation $\mathcal{O}^{\rho^h(c')}$ is obtained from the orientation $\mathcal{O}^{c'}$ by reading the directed edges backward. Attaching the edge $1 \leftarrow 2$ to the left of the orientation $\mathcal{O}^{\rho^h(c')}$ shows that the orientation $\mathcal{O}^{\rho^h(c)}$ is obtained from the orientation \mathcal{O}^c by reading the directed edges backward as desired.

Case 2: s_n is a lower vertex of c . Then $c = c' s_n$, where c' is a Coxeter element of \mathcal{A}_{n-1} , and $(n - 1) \rightarrow n$ is the last edge in \mathcal{O}^c to the right. By Lemma 3.5.3, s_2 is a lower vertex of $\rho^h(c)$ and thus $1 \rightarrow 2$ is the first edge from the left of $\mathcal{O}^{\rho^h(c)}$. By the inductive step, the orientation $\mathcal{O}^{\rho^h(c')}$ is obtained from the orientation $\mathcal{O}^{c'}$ by reading its directed edges backward. Attaching the edge $1 \rightarrow 2$ to the left of the orientation

$\mathcal{O}^{\rho^h(c')}$ gives us the orientation $\mathcal{O}^{\rho^h(c)}$ which is obtained from the orientation \mathcal{O}^c by reading the directed edges backward. This completes the proof. \square

Example 3.5.5. Consider the Coxeter element $c = s_6s_3s_2s_1s_4s_5$ of \mathcal{A}_6 , written in the standard representation. Then $\rho^h(c) = s_6s_5s_2s_1s_3s_4$. The orientation \mathcal{O}^c is

$$1 \xleftarrow{\text{red}} 2 \xleftarrow{\text{blue}} 3 \xrightarrow{\text{green}} 4 \xrightarrow{\text{yellow}} 5 \xleftarrow{\text{black}} 6,$$

and the orientation $\mathcal{O}^{\rho^h(c)}$ is

$$1 \xleftarrow{\text{black}} 2 \xrightarrow{\text{yellow}} 3 \xrightarrow{\text{green}} 4 \xleftarrow{\text{blue}} 5 \xleftarrow{\text{red}} 6.$$

Note that $\mathcal{O}^{\rho^h(c)}$ is obtained from \mathcal{O}^c by reading the directed edges backward.

The following corollary follows by Proposition 3.5.4.

Corollary 3.5.6. For any Coxeter element c of \mathcal{A}_n , the number of upper (lower) vertices of c is equal to the number of upper (lower) vertices of $\rho^h(c)$.

Recall that for any two decompositions v_1 and v_2 of elements in \mathcal{A}_n with corresponding decompositions α_1 and α_2 respectively, we say that $v_1 \simeq v_2$ if $\alpha_1 \simeq \alpha_2$.

Definition 3.5.7. A symmetric Coxeter element of \mathcal{A}_n is a Coxeter element c such that $c \simeq \rho^h(c)$.

Example 3.5.8. The Coxeter element $c = s_5s_3s_1s_6s_4s_2$ is symmetric as $c \simeq \rho^h(c)$, while the Coxeter element $c' = s_3s_4s_5s_1s_2s_6$ is not symmetric as $c' \not\simeq \rho^h(c')$.

Example 3.5.9. All Tamari Coxeter elements are symmetric.

In the next section we study the relation between the symmetric Coxeter elements and the self-dual Cambrian lattices of \mathcal{A}_n . In the following Proposition we enumerate the symmetric Coxeter elements of \mathcal{A}_n .

Proposition 3.5.10. *The number of non \simeq -equivalent symmetric Coxeter elements of \mathcal{A}_n is equal to*

$$2^{\lfloor \frac{n}{2} \rfloor}.$$

Proof. A Coxeter element c of \mathcal{A}_n is symmetric if $c \simeq \rho^h(c)$. We use Proposition 3.5.4 so that we study two cases based on whether n is odd or even.

Case 1: n is odd. The center of the Coxeter diagram of \mathcal{A}_n is the vertex $\frac{n+1}{2}$. Break the Coxeter diagram of \mathcal{A}_n into two parts, left and right, where the left part consists of the vertices $\{1, 2, \dots, \frac{n+1}{2}\}$, and the right part consists of the vertices $\{\frac{n+1}{2}, \frac{n+3}{2}, \dots, n\}$. $c \simeq \rho^h(c)$ implies that the orientation \mathcal{O}^c is symmetric about the center $\frac{n+1}{2}$. In other words, the right part of this orientation is a reflected copy of the left part. So, in order to count such orientations, we only need to count the orientations of the left part, which is equal to $2^{\frac{n-1}{2}}$.

Case 2: n is even. The two central vertices of the Coxeter diagram of \mathcal{A}_n are the vertices $\frac{n}{2}$ and $\frac{n}{2} + 1$. Break the Coxeter diagram of \mathcal{A}_n into three parts, left, middle, and right, where the left part consists of the vertices $\{1, 2, \dots, \frac{n}{2}\}$, the middle part consists of the two vertices $\{\frac{n}{2}, \frac{n}{2} + 1\}$ and the right part consists of the vertices $\{\frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n\}$. $c \simeq \rho^h(c)$ implies that the orientation \mathcal{O}^c is symmetric about the directed edge connecting the vertices $\frac{n}{2}$ and $\frac{n}{2} + 1$. That is, the right part of the orientation \mathcal{O}_n is a reflected copy of the left part. Thus, in order to find the number of such orientations, we only need to count the number of orientations of the left part as well as the middle part. This is equal to $2^{\frac{n}{2}}$.

In both cases the number of orientations is equal to $2^{\lfloor \frac{n}{2} \rfloor}$ and this is the number of symmetric Coxeter elements of \mathcal{A}_n as desired. \square

Define the map ρ^v on the set of Coxeter elements of \mathcal{A}_n so that if $c = s_{i_1} s_{i_2} \cdots s_{i_n}$ is a Coxeter element of \mathcal{A}_n , then

$$\rho^v(c) = s_{i_n} s_{i_{n-1}} \cdots s_{i_1}.$$

In the following proposition we introduce some properties of the maps ρ^h and ρ^v that follows directly from their definitions.

Proposition 3.5.11. *Let ρ^v and ρ^h be the maps defined above. Then*

1. ρ^h and ρ^v are involutions,
2. $\rho^h \circ \rho^v = \rho^v \circ \rho^h$,
3. the composite $\rho^h \circ \rho^v$, hence $\rho^v \circ \rho^h$, is an involution.

Proposition 3.5.12. *For any Coxeter element c of \mathcal{A}_n , the orientation $\mathcal{O}^{\rho^v(c)}$ is obtained from the orientation \mathcal{O}^c by reversing each directed edge in \mathcal{O}^c .*

Proof. Let c be a Coxeter element of \mathcal{A}_n . The definition of $\rho^v(c)$ shows that every upper vertex of c is a lower vertex of $\rho^v(c)$ and vice versa. Thus, the orientation $\mathcal{O}^{\rho^v(c)}$ is obtained from the orientation \mathcal{O}^c by reversing each directed edges in \mathcal{O}^c . \square

The following corollary follows by Proposition 3.5.12.

Corollary 3.5.13. *For any Coxeter element c of \mathcal{A}_n , the number of upper (lower) vertices of c is equal to the number of lower (upper) vertices of $\rho^v(c)$.*

By combining Corollary 3.5.6 and Corollary 3.5.13, we get the following two corollaries.

Corollary 3.5.14. *For any Coxeter element c of \mathcal{A}_n , the number of upper (lower) vertices of $\rho^h(c)$ is equal to the number of lower (upper) vertices of $\rho^v(c)$.*

Proposition 3.5.15. *For a Coxeter element c of \mathcal{A}_n , the orientation $\mathcal{O}^{\rho^h \circ \rho^v(c)}$ is obtained from the orientation \mathcal{O}^c by reversing the direction of each edge in \mathcal{O}^c and then reading them backward.*

Proof. The proof follows directly by Proposition 3.5.12 and Proposition 3.5.4. \square

Example 3.5.16. *Consider the Coxeter element $c = s_6 s_3 s_2 s_1 s_4 s_5$ of \mathcal{A}_6 , written in the standard representation. Then $\rho^h \circ \rho^v(c) = s_4 s_3 s_1 s_2 s_5 s_6$. The orientation \mathcal{O}^c is*

$$1 \leftarrow 2 \leftarrow 3 \rightarrow 4 \rightarrow 5 \leftarrow 6,$$

and the orientation $\mathcal{O}^{\rho^h \circ \rho^v(c)}$ is

$$1 \rightarrow 2 \leftarrow 3 \leftarrow 4 \rightarrow 5 \rightarrow 6.$$

As a consequence of Proposition 3.5.15, we get the following corollary.

Corollary 3.5.17. *For any Coxeter element c of \mathcal{A}_n , the number of upper (lower) vertices of c is equal to the number of lower (upper) vertices of $\rho^h \circ \rho^v(c)$.*

Proposition 3.5.18. *Let c and c' be two Coxeter elements of \mathcal{A}_n . Then*

$$c \preceq c' \text{ if and only if } \rho^h(c) \preceq \rho^h(c').$$

Proof. If $c \preceq c'$, then c and c' have the same standard representation. Thus $\rho^h(c) = \rho^h(c')$, and hence $\rho^h(c) \preceq \rho^h(c')$. The other direction holds by the same argument and by Proposition 3.5.11 as ρ^h is an involution. \square

Proposition 3.5.19. *Let c and c' be two Coxeter elements of \mathcal{A}_n . Then*

$$c \preceq c' \text{ if and only if } \rho^v(c) \preceq \rho^v(c').$$

Proof. For all $j \in [n-1]$, s_{i_j} and $s_{i_{j+1}}$ commute in c if and only if they commute in $\rho^v(c)$. That is, if c' is obtained from c by applying a sequence of commuting moves, then $\rho^v(c')$ is obtained from $\rho^v(c)$ by the same sequence. As ρ^v is an involution, the converse holds as well. \square

Corollary 3.5.20. *Let c and c' be two Coxeter elements of \mathcal{A}_n . Then*

$$c \simeq c' \text{ if and only if } \rho^h \circ \rho^v(c) \simeq \rho^h \circ \rho^v(c').$$

Proof. It follows directly by Proposition 3.5.18 and Proposition 3.5.19. □

Proposition 3.5.21. *If c is a symmetric Coxeter element of \mathcal{A}_n , then $\rho^h(c)$, $\rho^v(c)$, and $\rho^h \circ \rho^v(c)$ are symmetric.*

Proof. Suppose that c is a symmetric Coxeter element of \mathcal{A}_n . Then by definition $c \simeq \rho^h(c)$. Now,

$$\begin{aligned} c \simeq \rho^h(c) &\Rightarrow \rho^h(c) \simeq \rho^h(\rho^h(c)) \quad (\text{Proposition 3.5.18}) \\ &\Rightarrow \rho^h(c) \text{ is symmetric.} \end{aligned}$$

And,

$$\begin{aligned} c \simeq \rho^h(c) &\Rightarrow \rho^v(c) \simeq \rho^v(\rho^h(c)) \quad (\text{Proposition 3.5.19}) \\ &\Rightarrow \rho^v(c) \simeq \rho^h(\rho^v(c)) \quad (\text{Proposition 3.5.11}) \\ &\Rightarrow \rho^v(c) \text{ is symmetric.} \end{aligned}$$

Also,

$$\begin{aligned} c \simeq \rho^h(c) &\Rightarrow \rho^h \circ \rho^v(c) \simeq \rho^h \circ \rho^v(\rho^h(c)) \quad (\text{Corollary 3.5.20}) \\ &\Rightarrow \rho^h \circ \rho^v(c) \simeq \rho^h(\rho^v \circ \rho^h(c)) \\ &\Rightarrow \rho^h \circ \rho^v(c) \simeq \rho^h(\rho^h \circ \rho^v(c)) \quad (\text{Proposition 3.5.11}) \\ &\Rightarrow \rho^h \circ \rho^v(c) \text{ is symmetric.} \end{aligned}$$

This shows that $\rho^h(c)$, $\rho^v(c)$, and $\rho^h \circ \rho^v(c)$ all are symmetric. □

Define the maps \mathfrak{R}^h and \mathfrak{R}^v on the set of polygons \mathcal{Q}_n corresponding to the Coxeter elements of \mathcal{A}_n so that for $Q \in \mathcal{Q}_n$, $\mathfrak{R}^h(Q)$ is the polygon Q' obtained from Q by

reflecting Q about the vertical line through the vertex 0 and then relabeling its vertices so that $Q' \in \mathcal{Q}_n$, and $\mathfrak{R}^v(Q)$ is the polygon Q' obtained from Q by reflecting Q about its main horizontal diagonal and relabeling its vertices so that $Q' \in \mathcal{Q}_n$. The maps \mathfrak{R}^h and \mathfrak{R}^v are called the *horizontal reflection* and the *vertical reflection* respectively. Note that for all $Q \in \mathcal{Q}_n$, $\mathfrak{R}^h(Q) \in \mathcal{Q}_n$ and $\mathfrak{R}^v(Q) \in \mathcal{Q}_n$. Furthermore, we have the following properties of \mathfrak{R}^h and \mathfrak{R}^v that are straightforward to check.

Proposition 3.5.22. *Let \mathfrak{R}^h and \mathfrak{R}^v be the maps defined above. Then*

1. \mathfrak{R}^h and \mathfrak{R}^v are involutions,
2. $\mathfrak{R}^h \circ \mathfrak{R}^v = \mathfrak{R}^v \circ \mathfrak{R}^h$,
3. the composite $\mathfrak{R}^h \circ \mathfrak{R}^v$, hence $\mathfrak{R}^v \circ \mathfrak{R}^h$, is an involution.

The following proposition shows the relation between the maps ρ^h and \mathfrak{R}^h .

Proposition 3.5.23. *For any Coxeter element c of \mathcal{A}_n ,*

$$\mathfrak{R}^h(Q^c) = Q^{\rho^h(c)}.$$

Proof. Let c be a Coxeter element of \mathcal{A}_n . Applying a horizontal reflection on Q^c puts the vertices in the opposite order they should appear in a polygon corresponding to a Coxeter element. In particular, 0 appears to the right, $n+2$ appears to the left, and if $i < j$, then j appears to the left of the vertical line passing through i . In order to fix this, we use the transformation map $f : i \mapsto n+2-i$ on the set of vertices. On the other hand, by Proposition 3.5.4, the orientation corresponding to $\rho^h(c)$ is obtained from the orientation corresponding to c by reading the directed edges backward from right to left, which means that s_i is an upper vertex of c if and only if s_{n+2-i} is an upper vertex of $\rho^h(c)$ for all $i \in \{2, 3, \dots, n\}$. That is, $Q^{\rho^h(c)}$ appears the same as $\mathfrak{R}^h(Q^c)$. This shows that $\mathfrak{R}^h(Q^c) = Q^{\rho^h(c)}$ and completes the proof. \square

The following proposition shows the relation between the maps ρ^v and \mathfrak{R}^v .

Proposition 3.5.24. *For any Coxeter element c of \mathcal{A}_n ,*

$$\mathfrak{R}^v(Q^c) = Q^{\rho^v(c)}.$$

Proof. Let c be a Coxeter element of \mathcal{A}_n . Then c can be obtained from its corresponding polygon Q^c by reading the vertices $\{1, 2, \dots, n\}$ in the same order they appear in Q^c counterclockwise starting from the vertex $n+2$ so that if $c = \dots s_i s_j \dots$, then the vertex i is read before the vertex j in Q^c . Now, the reading of the Coxeter element corresponding to the polygon $\mathfrak{R}^v(Q^c)$ appears as reading the vertices $\{0, 1, 2, \dots, n\}$ in the same order they appear in Q^c clockwise starting from the vertex $n+2$, which looks exactly as reading the Coxeter element c backward. This shows that the Coxeter element corresponding to the polygon $\mathfrak{R}^v(Q^c)$ is $\rho^v(c)$ meaning that $\mathfrak{R}^v(Q^c) = Q^{\rho^v(c)}$. The proposition is checked. \square

By combining Proposition 3.5.23 and Proposition 3.5.24, we obtain the following corollary.

Corollary 3.5.25. *For any Coxeter element c of \mathcal{A}_n ,*

$$\mathfrak{R}^h \circ \mathfrak{R}^v(Q^c) = Q^{\rho^h \circ \rho^v(c)}.$$

The maps \mathfrak{R}^h and \mathfrak{R}^v can naturally be extended over the set of triangulations \mathfrak{T} of polygons in \mathcal{Q} , the following two propositions provide some properties that are straightforward to check.

Proposition 3.5.26. *On the set of triangulations \mathfrak{T} ,*

1. \mathfrak{R}^h and \mathfrak{R}^v are involutions,
2. $\mathfrak{R}^h \circ \mathfrak{R}^v = \mathfrak{R}^v \circ \mathfrak{R}^h$,
3. the composite $\mathfrak{R}^h \circ \mathfrak{R}^v$, thus $\mathfrak{R}^v \circ \mathfrak{R}^h$, is an involution.

Proposition 3.5.27. *Let c be a Coxeter element of \mathcal{A}_n , and let d be a diagonal in a triangulation T of Q^c . Then*

1. $\text{slope}(d)$ in $T = -\text{slope}(d)$ in $\mathfrak{R}^v(T)$,
2. $\text{slope}(d)$ in $T = -\text{slope}(d)$ in $\mathfrak{R}^h(T)$,
3. $\text{slope}(d)$ in $\mathfrak{R}^v(T) = \text{slope}(d)$ in $\mathfrak{R}^h(T)$,
4. $\text{slope}(d)$ in $T = \text{slope}(d)$ in $\mathfrak{R}^h \circ \mathfrak{R}^v(T)$.

Lemma 3.5.28. *Let c be a Coxeter element of \mathcal{A}_n , and let T_1 and T_2 be two triangulations of the polygon Q^c . Then*

$$T_1 \triangleleft T_2 \text{ if and only if } \mathfrak{R}^v(T_2) \triangleleft \mathfrak{R}^v(T_1),$$

where \triangleleft is the diagonal flip cover relation.

Proof. By the definition of the diagonal flip cover relation on triangulations, $T_1 \triangleleft T_2$ if and only if T_1 and T_2 differ only by one diagonal flip with $\text{slope}(T_1 - T_2) < \text{slope}(T_2 - T_1)$. Since the vertical reflection neither removes nor adds diagonals to triangulations, then T_1 and T_2 differ by only one diagonal flip if and only if $\mathfrak{R}^v(T_1)$ and $\mathfrak{R}^v(T_2)$ differ by only one diagonal flip. On the other hand, by using Proposition 3.5.27, we get

$$\begin{aligned} \text{slope}_{Q^c}(T_1 - T_2) < \text{slope}_{Q^c}(T_2 - T_1) & \text{ iff } -\text{slope}_{Q^c}(T_1 - T_2) > -\text{slope}_{Q^c}(T_2 - T_1) \\ & \text{ iff } \text{slope}_{\mathfrak{R}^v(Q^c)}(T_1 - T_2) > \text{slope}_{\mathfrak{R}^v(Q^c)}(T_2 - T_1). \end{aligned}$$

This shows that $T_1 \triangleleft T_2$ if and only if $\mathfrak{R}^v(T_2) \triangleleft \mathfrak{R}^v(T_1)$ and checks the lemma. \square

Corollary 3.5.29. *Let c be a Coxeter element of \mathcal{A}_n , and let T_1 and T_2 be two triangulations of the polygon Q^c . Then*

$$T_1 \leq T_2 \text{ if and only if } \mathfrak{R}^v(T_2) \leq \mathfrak{R}^v(T_1),$$

where \leq is the diagonal flip ordering.

Proof. Follows directly by Lemma 3.5.28. \square

Lemma 3.5.30. *Let c be a Coxeter element of \mathcal{A}_n , and let T_1 and T_2 be two triangulations of the polygon Q^c . Then*

$$T_1 \triangleleft T_2 \text{ if and only if } \mathfrak{R}^h(T_2) \triangleleft \mathfrak{R}^h(T_1),$$

where \triangleleft is the cover relation of the diagonal flip ordering.

Proof. As in the proof of Lemma 3.5.28, the horizontal reflection neither removes nor adds diagonals to triangulations, thus T_1 and T_2 differ only by one diagonal flip if and only if $\mathfrak{R}^h(T_1)$ and $\mathfrak{R}^h(T_2)$ differ only by one diagonal flip. By Proposition 3.5.27,

$$\begin{aligned} \text{slope}_{Q^c}(T_1 - T_2) < \text{slope}_{Q^c}(T_2 - T_1) &\text{ iff } -\text{slope}_{Q^c}(T_1 - T_2) > -\text{slope}_{Q^c}(T_2 - T_1) \\ &\text{ iff } \text{slope}_{\mathfrak{R}^h(Q^c)}(T_1 - T_2) > \text{slope}_{\mathfrak{R}^h(Q^c)}(T_2 - T_1). \end{aligned}$$

That is $T_1 \triangleleft T_2$ if and only if $\mathfrak{R}^h(T_2) \triangleleft \mathfrak{R}^h(T_1)$ as desired. \square

Corollary 3.5.31. *Let c be a Coxeter element of \mathcal{A}_n , and let T_1 and T_2 be two triangulations of the polygon Q_c . Then*

$$T_1 \leq T_2 \text{ if and only if } \mathfrak{R}^h(T_2) \leq \mathfrak{R}^h(T_1),$$

where \leq is the diagonal flip ordering.

Proof. Follows directly by Lemma 3.5.30. \square

3.6 Isomorphic Cambrian Lattices

In the section we study the isomorphism classes of Cambrian lattices, and study the necessary conditions that Coxeter elements must satisfy so that their corresponding Cambrian lattices are isomorphic. We also enumerate the distinct non-isomorphic Cambrian lattices of \mathcal{A}_n .

We start with the following lemma.

Lemma 3.6.1. *For any Coxeter element c of \mathcal{A}_n ,*

$$\text{Camb}(c) \cong \text{Camb}(\rho^h \circ \rho^v(c)).$$

Proof. Let c be a Coxeter element and let Q^c be its corresponding polygon. By Proposition 3.5.24, $\mathfrak{R}^v(Q^c) = Q^{\rho^v(c)}$, and by Corollary 3.5.25, $\mathfrak{R}^h \circ \mathfrak{R}^v(Q^c) = Q^{\rho^h \circ \rho^v(c)}$. Let \mathfrak{T}^c , $\mathfrak{T}^{\rho^v(c)}$, and $\mathfrak{T}^{\rho^h \circ \rho^v(c)}$ be the sets of triangulations of Q^c , $Q^{\rho^v(c)}$, and $Q^{\rho^h \circ \rho^v(c)}$ respectively. Then, by Proposition 3.5.22, $\mathfrak{R}^h \circ \mathfrak{R}^v$ is a bijection from \mathfrak{T}^c onto $\mathfrak{T}^{\rho^h \circ \rho^v(c)}$. Furthermore,

$$\begin{aligned} T_1 \leq_{\mathfrak{T}^c} T_2 & \text{ iff } \mathfrak{R}^v(T_2) \leq_{\mathfrak{T}^{\rho^v(c)}} \mathfrak{R}^v(T_1) \quad (\text{by Corollary 3.5.29}) \\ & \text{ iff } \mathfrak{R}^h \circ \mathfrak{R}^v(T_1) \leq_{\mathfrak{T}^{\rho^h \circ \rho^v(c)}} \mathfrak{R}^h \circ \mathfrak{R}^v(T_2) \quad (\text{by Corollary 3.5.31}). \end{aligned}$$

This shows that the map $\mathfrak{R}^h \circ \mathfrak{R}^v$ is an isomorphism from \mathfrak{T}^c onto $\mathfrak{T}^{\rho^h \circ \rho^v(c)}$. By Theorem 3.2.8, the Cambrian lattices $\text{Camb}(c)$ and $\text{Camb}(\rho^h \circ \rho^v(c))$ are the posets \mathfrak{T}^c and $\mathfrak{T}^{\rho^h \circ \rho^v(c)}$ respectively. Hence $\text{Camb}(c)$ and $\text{Camb}(\rho^h \circ \rho^v(c))$ are isomorphic. \square

The previous lemma has the following corollary.

Corollary 3.6.2. *For any Coxeter element c of \mathcal{A}_n ,*

$$\text{Camb}(\rho^v(c)) \cong \text{Camb}(\rho^h(c)).$$

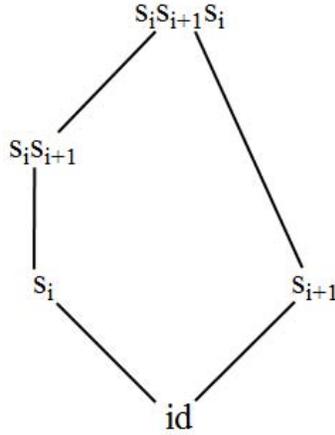


Figure 3.13: The Hasse Diagram of the Interval $[id, s_i \vee s_{i+1}]$ in $\text{Camb}(c)$, where s_i precedes s_{i+1} in c

Proof. Let c be a Coxeter element of \mathcal{A}_n . By Lemma 3.6.1, $\text{Camb}(\rho^v(c)) \cong \text{Camb}(\rho^h \circ \rho^v(\rho^v(c)))$, and thus $\text{Camb}(\rho^v(c)) \cong \text{Camb}(\rho^h(\rho^v \circ \rho^v(c)))$. By Proposition 3.5.11, ρ^v is an involution, hence $\rho^v \circ \rho^v(c) = c$. This shows that $\text{Camb}(\rho^v(c)) \cong \text{Camb}(\rho^h(c))$. \square

For any Coxeter element c , and adjacent transpositions s_i and s_{i+1} of \mathcal{A}_n , s_i precedes s_{i+1} in c if and only if $s_i \vee s_{i+1} = s_i s_{i+1} s_i$ in $\text{Camb}(c)$, where the Hasse diagram of the interval $[id, s_i \vee s_{i+1}]$ is shown in Figure 3.13. Similarly, s_{i+1} precedes s_i in c if and only if $s_i \vee s_{i+1} = s_{i+1} s_i s_{i+1}$ in $\text{Camb}(c)$ so that the Hasse diagram of the interval $[id, s_i \vee s_{i+1}]$ is shown in Figure 3.14. On the other hand, when $|i - j| > 1$, the Hasse diagram of the interval $[id, s_i \vee s_j]$ in $\text{Camb}(c)$ is shown in Figure 3.15. We use these ideas to prove the following theorem, where some ingredients of the proof were assembled by N. Reading in [Reading (2006)].

Theorem 3.6.3. *Let c and c' be two Coxeter elements. Then*

$$\text{Camb}(c') \cong \text{Camb}(c) \text{ if and only if } c' \preceq c \text{ or } c' \preceq \rho^h \circ \rho^v(c).$$

Proof. If $c' \preceq c$, then in the proof of Proposition 3.1.7, we checked that c and c' give the same orientation for the Coxeter diagram of \mathcal{A}_n . Thus $Q^c = Q^{c'}$, and hence

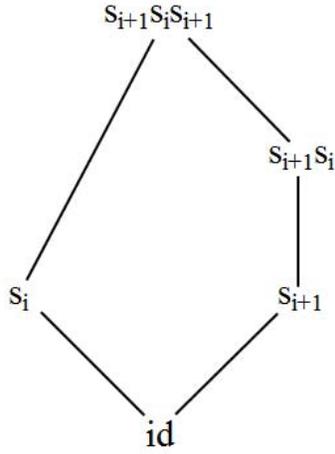


Figure 3.14: The Hasse Diagram of the Interval $[id, s_i \vee s_{i+1}]$ in $\text{Camb}(c)$, where s_{i+1} precedes s_i in c

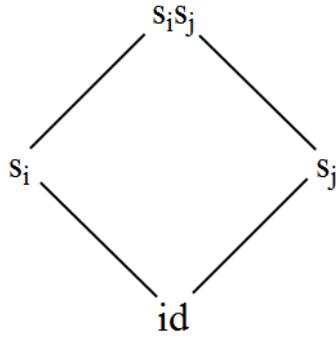


Figure 3.15: The Hasse Diagram of Interval $[id, s_i \vee s_j]$ in $\text{Camb}(c)$, When $|i - j| > 1$ in c

$\text{Camb}(c') \cong \text{Camb}(c)$. Also, if $c' = \rho^h \circ \rho^v(c)$, then by Lemma 3.6.1, $\text{Camb}(c') \cong \text{Camb}(c)$. This checks one direction. In order to prove the other direction, suppose that $\text{Camb}(c') \cong \text{Camb}(c)$, and let ϕ be an isomorphism from $\text{Camb}(c')$ onto $\text{Camb}(c)$. Then ϕ is a bijection on the set of adjacent transpositions of \mathcal{A}_n so that $\phi(s_i \vee s_j) = \phi(s_i) \vee \phi(s_j)$. This implies that $s_i s_j = s_j s_i$ if and only if $\phi(s_i)\phi(s_j) = \phi(s_j)\phi(s_i)$, for all $i, j \in [n]$. Since the orientation $\mathcal{O}^{c'}$ is determined by the intervals $[id, s_i \vee s_{i+1}]$ in $\text{Camb}(c')$, and the orientation \mathcal{O}^c is determined by the intervals $[id, \phi(s_i) \vee \phi(s_{i+1})]$

in $\text{Camb}(c)$, then ϕ can be seen as a bijection from the set of vertices of $\mathcal{O}^{c'}$ onto the set of vertices of \mathcal{O}^c that preserves the directed edges.

Claim 1: $\phi(s_1) = s_1$ or $\phi(s_1) = s_n$.

proof of claim: Suppose on the contrary that $\phi(s_1) = s_j$ for some $s_j \notin \{s_1, s_n\}$. Since s_{j-1} and s_j don't commute, $\phi^{-1}(s_{j-1})$ and $\phi^{-1}(s_j) = s_1$ don't commute as well, and thus $\phi^{-1}(s_{j-1}) = s_2$. Similarly, as s_{j+1} and s_j don't commute, $\phi^{-1}(s_{j+1})$ and $\phi^{-1}(s_j) = s_1$ don't commute, and hence $\phi^{-1}(s_{j+1}) = s_2$. This is a contradiction as ϕ is a bijection on the set of adjacent transpositions. Therefore, $\phi(s_1) = s_1$ or $\phi(s_1) = s_n$.

Claim 2: If $\phi(s_1) = s_1$, then $\phi(s_i) = s_i$ for all $i \in [n]$.

proof of claim: Suppose not and let $k = \min\{j : \phi(s_j) \neq s_j, j \in [n]\}$. Then $k \geq 2$ with $\phi(s_{k-1}) = s_{k-1}$, and $\phi(s_k) = s_m$ for some $m > k$. This implies that s_k and s_{k-1} don't commute while $\phi(s_k)$ and $\phi(s_{k-1})$ commute, which is a contradiction. Thus $\phi(s_i) = s_i$ for all $i \in [n]$ as desired.

Claim 3: If $\phi(s_1) = s_n$, then $\phi(s_i) = s_{n+1-i}$ for all $i \in [n]$.

proof of claim: The proof is very similar to the proof of Claim 2 above.

The claims above show that the only bijections from the set of vertices of $\mathcal{O}^{c'}$ onto the set of vertices of \mathcal{O}^c that preserves the directed edges are the identity map and the map $s_i \mapsto s_{n+1-i}$ such that s_i precedes s_{i+1} in c' if and only if s_{n-i} precedes s_{n-i+1} in c . That is, by Proposition 3.1.7 and Proposition 3.5.15, either $c' \simeq c$ or $c' \simeq \rho^h \circ \rho^v(c)$. The theorem is checked. \square

Two Cambrian lattices of \mathcal{A}_n are said to be *distinct* if they correspond to different

orientations of the Coxeter diagram of \mathcal{A}_n . Define the map $\nu : \mathbb{Z}^+ \rightarrow \{0, 1\}$ such that:

$$\nu(m) = m \bmod 2.$$

In the following proposition we enumerate the distinct non-isomorphic Cambrian lattices of \mathcal{A}_n .

Proposition 3.6.4. *The number of distinct non-isomorphic Cambrian lattices of \mathcal{A}_n is equal to*

$$2^{n-2} + 2^{\frac{n-3}{2}} \nu(n).$$

Proof. By Corollary 3.4.2, the number of Cambrian lattices is equal to 2^{n-1} , where this number is equal to the number of orientations of the Coxeter diagram of \mathcal{A}_n . In particular, there is a bijection between the set \mathcal{O}_n of orientations of the Coxeter diagram of \mathcal{A}_n and the set of distinct, but possibly isomorphic, Cambrian lattices. The set \mathcal{O}_n can be partitioned into two sets A and B , where A consists of the orientations \mathcal{O}^c such that $\mathcal{O}^c \neq \mathcal{O}^{\rho^v \circ \rho^h(c)}$, while B consists of the orientations \mathcal{O}^c such that $\mathcal{O}^c = \mathcal{O}^{\rho^v \circ \rho^h(c)}$.

Claim: $|B| = 2^{\frac{n-1}{2}} \nu(n)$.

Proof of the claim: If n is odd, then the number of edges, which is equal to $n - 1$, of the Coxeter diagram of \mathcal{A}_n is even. For any orientation $\mathcal{O} \in \mathcal{O}_n$, \mathcal{O} can be cut into two parts, \mathcal{O}^- and \mathcal{O}^+ , where \mathcal{O}^- is the part of \mathcal{O} consisting of the vertices $\{1, 2, \dots, \frac{n+1}{2}\}$, and \mathcal{O}^+ is the part of \mathcal{O} consisting of the vertices $\{\frac{n+1}{2}, \frac{n+1}{2}+1, \dots, n\}$. Now, if $\mathcal{O} \in B$, then by Proposition 3.5.15, the orientation \mathcal{O} is completely determined by \mathcal{O}^- . The number of such orientations is equal to $2^{\frac{n-1}{2}}$, hence $|B| = 2^{\frac{n-1}{2}}$. If n is even, then by the same argument used above we see that $|B| = 0$. In both cases, $|B| = 2^{\frac{n-1}{2}} \nu(n)$. The claim is checked.

Now let N be the number of distinct non-isomorphic Cambrian lattices of \mathcal{A}_n . Then;

$$\begin{aligned}
N &= \frac{1}{2}|A| + |B| \\
&= \frac{1}{2}(|A| + 2|B|) \\
&= \frac{1}{2}(|\mathcal{O}_n| + |B|) \\
&= \frac{1}{2}\left(2^{n-1} + 2^{\frac{n-1}{2}}\nu(n)\right) \\
&= 2^{n-2} + 2^{\frac{n-3}{2}}\nu(n).
\end{aligned}$$

□

3.7 The Duality of Cambrian Lattices

In this section we study the duality between Cambrian lattices, and check the relation between the self-dual Cambrian lattices and the symmetric Coxeter elements of \mathcal{A}_n . Additionally, we enumerate the distinct non-isomorphic self-dual Cambrian lattices of \mathcal{A}_n .

First, recall that two posets P and Q are *anti-isomorphic* if there exists a bijection $\phi : P \rightarrow Q$ such that:

$$x \leq y \text{ in } P \iff \phi(y) \leq \phi(x) \text{ in } Q.$$

We start with the following two lemmas.

Lemma 3.7.1. *For any Coxeter element c of \mathcal{A}_n ,*

Camb(c) and Camb($\rho^v(c)$) are anti-isomorphic.

Proof. Let c be a Coxeter element of \mathcal{A}_n , and let Q^c and $Q^{\rho^v(c)}$ be the polygons corresponding to the Coxeter elements c and $\rho^v(c)$ respectively. Also let \mathfrak{T}^c and $\mathfrak{T}^{\rho^v(c)}$ be the sets of triangulations of Q^c and $Q^{\rho^v(c)}$ respectively. By Theorem 3.2.8,

the Cambrian lattice $\text{Camb}(c)$ is the set \mathfrak{T}^c under the diagonal flip ordering, and similarly $\text{Camb}(\rho^v(c))$ is the set $\mathfrak{T}^{\rho^v(c)}$ under the diagonal flip ordering. Consider the map \mathfrak{R}^v . By Proposition 3.5.24, $\mathfrak{R}^v(Q^c) = Q^{\rho^v(c)}$, and by Proposition 3.5.26, \mathfrak{R}^v is a bijection from \mathfrak{T}^c onto $\mathfrak{T}^{\rho^v(c)}$. On the other hand, by Corollary 3.5.29, $T_1 \leq T_2$ in \mathfrak{T}^c if and only if $\mathfrak{R}^v(T_2) \leq \mathfrak{R}^v(T_1)$ in $\mathfrak{T}^{\rho^v(c)}$. This shows that \mathfrak{R}^v is an anti-isomorphism from \mathfrak{T}^c onto $\mathfrak{T}^{\rho^v(c)}$, and hence $\text{Camb}(c)$ and $\text{Camb}(\rho^v(c))$ are anti-isomorphic. \square

Lemma 3.7.2. *For any Coxeter element c of \mathcal{A}_n ,*

$\text{Camb}(c)$ and $\text{Camb}(\rho^h(c))$ are anti-isomorphic.

Proof. Let c be a Coxeter element of \mathcal{A}_n . We use arguments similar to those used in the proof of Lemma 3.7.1, so we let Q^c and $Q^{\rho^h(c)}$ be the polygons corresponding to the Coxeter elements c and $\rho^h(c)$ respectively. By Proposition 3.5.23, $\mathfrak{R}^h(Q^c) = Q^{\rho^h(c)}$, and by Proposition 3.5.26, \mathfrak{R}^h is a bijection from \mathfrak{T}^c onto $\mathfrak{T}^{\rho^h(c)}$, where \mathfrak{T}^c and $\mathfrak{T}^{\rho^h(c)}$ are the sets of triangulations of Q^c and $Q^{\rho^h(c)}$ respectively. Furthermore, by Corollary 3.5.31, $T_1 \leq T_2$ in \mathfrak{T}^c if and only if $\mathfrak{R}^h(T_2) \leq \mathfrak{R}^h(T_1)$ in $\mathfrak{T}^{\rho^h(c)}$. This shows that \mathfrak{R}^h is an anti-isomorphism. As, by Theorem 3.2.8, the Cambrian lattice $\text{Camb}(c)$ is the set \mathfrak{T}^c under the diagonal flip ordering, and $\text{Camb}(\rho^h(c))$ is the set $\mathfrak{T}^{\rho^h(c)}$ under the diagonal flip ordering, then $\text{Camb}(c)$ and $\text{Camb}(\rho^h(c))$ are anti-isomorphic as desired. \square

Theorem 3.7.3. *Let c and c' be two Coxeter elements of \mathcal{A}_n . Then*

$\text{Camb}(c') \cong \text{dual}(\text{Camb}(c))$ if and only if $c' \simeq \rho^v(c)$ or $c' \simeq \rho^h(c)$.

Proof. If $c' \simeq \rho^v(c)$, then by Lemma 3.7.1, $\text{Camb}(c)$ and $\text{Camb}(c')$ are anti-isomorphic and hence $\text{Camb}(c') \cong \text{dual}(\text{Camb}(c))$. Similarly, if $c' \simeq \rho^h(c)$, then by Lemma 3.7.2, $\text{Camb}(c)$ and $\text{Camb}(c')$ are anti-isomorphic and thus $\text{Camb}(c') \cong \text{dual}(\text{Camb}(c))$. This checks the first direction. In order to check the other direction, suppose that $\text{Camb}(c') \cong \text{dual}(\text{Camb}(c))$. Then by Lemma 3.7.1, $\text{Camb}(c') \cong \text{Camb}(\rho^v(c))$, and

hence by Theorem 3.6.3, $c' \simeq \rho^v(c)$ or $c' \simeq \rho^h \circ \rho^v(\rho^v(c))$. By Proposition 3.5.11, this implies that $c' \simeq \rho^v(c)$ or $c' \simeq \rho^h(c)$. The theorem is checked. \square

Corollary 3.7.4. *Let c be a Coxeter element of \mathcal{A}_n . Then*

$\text{Camb}(c)$ is self-dual if and only if c is symmetric.

Proof. It follows directly from Theorem 3.7.3 above as the case $c \simeq \rho^v(c)$ is impossible. \square

Corollary 3.7.5. *If c is a symmetric Coxeter element of \mathcal{A}_n , then*

$$\text{Camb}(c) \cong \text{dual}(\text{Camb}(c)) \cong \text{Camb}(\rho^v(c)) \cong \text{dual}(\text{Camb}(\rho^v(c))).$$

Proof. Suppose that c is symmetric, then $c \simeq \rho^h(c)$, and hence by Corollary 3.7.4, $\text{Camb}(c)$ is self-dual. That is $\text{Camb}(c) \cong \text{dual}(\text{Camb}(c))$. By Lemma 3.6.1,

$$\begin{aligned} \text{Camb}(c) &\cong \text{Camb}(\rho^h \circ \rho^v(c)) \\ &\cong \text{Camb}(\rho^v \circ \rho^h(c)) \text{ (By Proposition 3.5.11)} \\ &\cong \text{Camb}(\rho^v(c)). \end{aligned}$$

This shows that $\text{dual}(\text{Camb}(c)) \cong \text{dual}(\text{Camb}(\rho^v(c)))$. By Proposition 3.5.21, $\rho^v(c)$ is symmetric, and then, by Corollary 3.7.4, $\text{Camb}(\rho^v(c)) \cong \text{dual}(\text{Camb}(\rho^v(c)))$. The corollary is checked. \square

Since the Tamari Coxeter element is symmetric, Corollary 3.7.4 gives an alternative proof of the following well-known fact about the Tamari lattices.

Corollary 3.7.6. [Hong *et al.* (2001)] *The Tamari Lattices \mathcal{T}_n are self-dual.*

In the following proposition we enumerate the distinct non-isomorphic self-dual Cambrian lattices of \mathcal{A}_n .

Proposition 3.7.7. *The number of distinct non-isomorphic self-dual Cambrian lattices of \mathcal{A}_n is equal to*

$$2^{\lfloor \frac{n}{2} \rfloor - 1}.$$

Proof. By Corollary 3.7.4, a Cambrian lattice $\text{Camb}(c)$ is self-dual if and only if c is symmetric. We checked in Proposition 3.5.21 that if c is symmetric, then $\rho^v(c)$ is symmetric as well. Note that, by Proposition 3.5.11, if c is symmetric, then $\rho^h \circ \rho^v(c) = \rho^v(c)$. Thus, by Theorem 3.6.3, for two distinct self-dual Cambrian lattices $\text{Camb}(c)$ and $\text{Camb}(c')$; $\text{Camb}(c') \cong \text{Camb}(c)$ if and only if $c' \simeq \rho^v(c)$. Let S be the set of non \simeq -equivalent symmetric Coxeter elements of \mathcal{A}_n . Then S can be partitioned into two sets S_1 and S_2 , where S_1 consists of the Coxeter elements in S with s_1 preceding s_2 , and S_2 consists of the Coxeter elements in S with s_2 preceding s_1 . Note that $\rho^v(c) \in S_2$ for all $c \in S_1$ and vice versa. This shows that ρ^v is a bijection from S_1 onto S_2 , and hence $|S_1| = |S_2|$. That is $|S_1| = \frac{1}{2}|S|$. As $c \not\simeq \rho^v(c)$, for all Coxeter element c of \mathcal{A}_n , the number of non-isomorphic self-dual Cambrian lattices is equal to $|S_1|$, equivalently $|S_2|$. By Proposition 3.5.10, $|S| = 2^{\lfloor \frac{n}{2} \rfloor}$, and hence $|S_1| = 2^{\lfloor \frac{n}{2} \rfloor - 1}$. This completes the proof. \square

Chapter 4

COUNTING THE CHAINS OF MAXIMUM LENGTH OF TAMARI LATTICES

One of the fundamental problems in combinatorics is to enumerate the maximal chains of a poset. The chains of maximum length of the Tamari lattice \mathcal{T}_n were studied by S. Fishel and L. Nelson, see [Fishel and Nelson (2014)], where they enumerated these chains by studying the Young diagrams corresponding to partitions of staircase shape. In this chapter, we mainly introduce an alternative proof of the number of chains of maximum length of \mathcal{T}_n , and collect results on equinumerous objects. Most of the definitions and facts in this chapter are obtained from [Edelman (1995)].

The outline of this chapter goes as follows: In Section 4.1, we review some basic background that we need throughout the chapter such as the hook-length formula, the commutation classes of a permutation, and the type of a word of a permutation. In Section 4.2, we present the definition of linear extensions of a poset, and study the heap of a reduced word of a permutation in \mathcal{A}_n . In Section 4.3, we introduce the left-right maxima of a permutation, then we study the left-right words and the lattice words of permutations. In Section 4.4, we give an alternative proof of the number of chains of maximum length of \mathcal{T}_n , and open the door toward finding formulas for the number of chains of maximum length of other Cambrian lattices of \mathcal{A}_n .

4.1 Some Basic Background

In this section we introduce the hook-length formula which enumerates the standard shifted tableaux of some shapes, then we study the commutativity classes of permutations in \mathfrak{S}_n , the type of a word of a permutation, and some other terminolo-

gies. We also present some main facts and examples.

We start with the following theorem.

Theorem 4.1.1. [Thrall (1952)] *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ be a partition of distinct parts with $\lambda_1 > \lambda_2 > \dots > \lambda_\ell > 0$. Then the number of standard shifted tableaux of shape λ is equal to*

$$\frac{|\lambda|!}{\lambda_1! \lambda_2! \cdots \lambda_\ell!} \cdot \frac{\Delta(\lambda)}{\nabla(\lambda)}, \quad (4.1)$$

where

$$\Delta(\lambda) = \prod_{i>j} (\lambda_j - \lambda_i),$$

and

$$\nabla(\lambda) = \prod_{i>j} (\lambda_j + \lambda_i).$$

The Formula (4.1) appearing in Theorem 4.1.1 above is called the *hook-length formula*.

A partition of shape $\lambda = (n, n-1, \dots, 2, 1)$ is called a partition of *staircase shape*. The following corollary gives the number of standard shifted tableaux of such partitions.

Corollary 4.1.2. [Thrall (1952)] *Let $\lambda = (n, n-1, n-2, \dots, 1)$. Then the number of standard shifted tableaux of shape λ is equal to*

$$\binom{n}{2}! \frac{(n-2)!(n-3)! \cdots 1!}{(2n-5)!(2n-3)! \cdots 1!}.$$

Now we introduce the commutation classes of permutations in \mathfrak{S}_n . For a permu-

tation $\pi \in \mathfrak{S}_n$, let $\mathcal{R}(\pi)$ denote the set of all reduced words of π , and let

$$\mathcal{R}_n = \bigcup_{\pi \in \mathfrak{S}_n} \mathcal{R}(\pi).$$

If $\alpha = \alpha_1\alpha_2 \cdots \alpha_\ell$ and $\beta = \beta_1\beta_2 \cdots \beta_\ell$ are two words in $\mathcal{R}(\pi)$, where $\ell = \ell(\pi)$, and for all $i, j \in [\ell]$, α_i and β_j represent the adjacent transpositions s_{α_i} and s_{β_j} of \mathfrak{S}_n respectively, we say that α is less than β *lexicographically*, and write $\alpha \prec \beta$, if when i is the smallest integer such that $\alpha_i \neq \beta_i$, then α_i is less than β_i . In symbols;

$$\alpha \prec \beta \text{ if and only if } \alpha_i < \beta_i, \text{ where } i = \min\{j : \alpha_j \neq \beta_j\}.$$

The poset $(\mathcal{R}(\pi), \preceq)$ is totally ordered. We denote the unique minimum element of the poset $(\mathcal{R}(\pi), \preceq)$ by $\omega(\pi)$ and call it the *lexicographically first* word of π , or for simplicity the *lex-first* word of π . For example, for $\pi = 1432 \in \mathfrak{S}_4$, written in one-line notation, $\omega(\pi) = 232$.

By Theorem 2.4.4, a reduced word of π can be obtained from another reduced word of π by applying a sequence of short and long braid moves. By Proposition 2.4.5, the relation \sim_{C_1} is an equivalence relation. Thus, \sim_{C_1} partitions $\mathcal{R}(\pi)$ into equivalence classes, say $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_k$, so that

$$\mathcal{R}(\pi) = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_k,$$

where each class is called a *commutativity class* of π . A permutation π is said to be *fully commutative* if $\mathcal{R}(\pi)$ consists of a single class. That is; the reduced words of π are obtained from each other by applying a sequence of short-braid moves only. The fully commutative permutations were first introduced by J. Stembridge in [Stembridge (1996)]. For more details about commutation classes and fully commutative permutations, see [Shi (2005)], [Hanusa and Jones (2010)], and [Biagioli *et al.* (2015)].

We end this section by presenting the following definitions and facts.

Definition 4.1.3. *Given a permutation $\pi \in \mathfrak{S}_n$, define $\pi|_k$ to be the permutation in \mathfrak{S}_k such that the numbers $1, 2, 3, \dots, k$ appear in the one-line notation of $\pi|_k$ in the same order they appear in the one-line notation of π .*

For instance, if $\pi = 83147265$, then $\pi|_5 = 31425$.

Definition 4.1.4. [Edelman (1995)] *Define the mapping γ from \mathfrak{S}_n into \mathcal{R}_n so that for $\pi \in \mathfrak{S}_n$:*

$$\gamma(\pi) = \begin{cases} \gamma(\pi|_{n-1}) & : \pi^{-1}(n) = n \\ \gamma(\pi|_{n-1})(n-1)(n-2)\cdots\pi^{-1}(n) & : \pi^{-1}(n) \neq n, \end{cases}$$

where $\gamma(\pi|_1) := id$.

We call the sequence of integers $(n-1)(n-2)\cdots\pi^{-1}(n)$ obtained from applying every recursive step in the definition of γ a *block*. We number the blocks from left to right so that the first block is the leftmost block, and the last block is the rightmost block. Note that resulting word is a concatenation of these blocks.

Example 4.1.5. *Consider the permutation $\pi = 24135 \in \mathfrak{S}_5$ written in one-line notation. Since $\pi^{-1}(5) = 5$, then $\gamma(\pi) = \gamma(\pi|_4)$, where $\pi|_4 = 2413$. As $\pi|_4^{-1}(4) = 2$, then $\gamma(\pi|_4) = \gamma(\pi|_3)32$ with $\pi|_3 = 213$. Similarly, $\pi|_3^{-1}(3) = 3$ implies that $\gamma(\pi|_3) = \gamma(\pi|_2)$ with $\pi|_2 = 21$. Finally, $\pi|_2^{-1}(2) = 1$ implies that $\gamma(\pi|_2) = \gamma(\pi|_1)1$, but $\gamma(\pi|_1) = id$, thus $\pi|_2 = 1$. By combining the steps above together, we get*

$$\begin{aligned} \gamma(\pi) &= \gamma(\pi|_4) \\ &= \gamma(\pi|_3)32 \\ &= \gamma(\pi|_2)32 \\ &= \gamma(\pi|_1)132 \\ &= 132. \end{aligned}$$

Theorem 4.1.6. [Edelman (1995)] For any permutation $\pi \in \mathfrak{S}_n$,

$$\gamma(\pi) = \omega(\pi).$$

Example 4.1.7. By Theorem 4.1.6 and Example 4.1.5, the word $v = 132$ is the lex-first word of the permutation $\pi = 24135 \in \mathfrak{S}_5$, written in one-line notation.

Definition 4.1.8. [Edelman (1995)] For a permutation $\pi \in \mathfrak{S}_n$, and a word $v = v_1v_2 \cdots v_\ell \in \mathcal{R}(\pi)$, the type of the word v , denoted $\text{type}(v)$, is defined as:

$$\text{type}(v) = (m_1, m_2, \dots, m_{n-1}),$$

where m_i is equal to the number of times i appears in the word v .

For instance, if $v = 3123$ is a reduced word in \mathcal{R}_5 , then $m_1 = 1$, $m_2 = 1$, $m_3 = 2$, and $m_4 = 0$. That is $\text{type}(v) = (1, 1, 2, 0)$.

The integer m_i appearing in Definition 4.1.8 above is called the *multiplicity* of i in v . We write $m_i(v)$ instead of m_i in order to refer to the word v when needed.

4.2 The Heap of a Reduced Word in \mathcal{A}_n

In this section we introduce the heaps corresponding to reduced words of permutations in \mathcal{A}_n . Some suggested references for this section are [Viennot (1989)], [Stembridge (1996)], [Brualdi (2010)], and [Biagioli *et al.* (2015)].

Let π be a permutation in \mathcal{A}_n , and let $w = i_1i_2 \cdots i_\ell$ be a reduced word of π . Consider the set $[\ell] = \{1, 2, 3, \dots, \ell\}$, where $\ell = \ell(\pi)$, and define the partial ordering \sqsubseteq on $[\ell]$ as the transitive closure of the binary relation:

$$j \sqsubseteq k \text{ if and only if } j < k \text{ as integers, and } |s_{i_j}s_{i_k}| \neq 2.$$

The labeled poset $P_w = ([\ell], \sqsubseteq, w)$ is called the *heap* of the reduced word w . Note that the definition of the heap P_w depends on the reduced word w not on the permutation π itself.

Example 4.2.1. Consider the reduced word $w = 1234123$ corresponding to the permutation $\pi = 34521 \in \mathfrak{S}_5$, written in one-line notation. $[\ell(w)] = [7]$, with

$$1 \sqsubset 2, 5, 6$$

$$2 \sqsubset 3, 5, 6, 7$$

$$3 \sqsubset 4, 6, 7$$

$$4 \sqsubset 7$$

$$5 \sqsubset 6$$

and

$$6 \sqsubset 7.$$

The heap P_w is the transitive closure of these relations, where the Hasse diagram of P_w is shown in Figure 4.1.

Example 4.2.2. Consider the maximum element w_0 and the Tamari Coxeter element c of \mathcal{A}_3 . Example 3.1.4 shows that the c -sorting word of w_0 is $w_0(c) = 123121$. The Hasse-diagram of the heap $P_{w_0(c)}$ is shown in Figure 4.2.

Let \leq_1 and \leq_2 be two partial orders on a set P . The partially ordered set (P, \leq_2) is said to be an *extension* of the partially ordered set (P, \leq_1) if (P, \leq_1) is a weak subposet of (P, \leq_2) . If (P, \leq_2) is totally ordered, then it is called a *linear extension* of (P, \leq_1) . A linear extension of a finite poset (P, \leq_P) can be understood as a bijection ψ from (P, \leq_P) onto $([|P|], \leq)$, where “ \leq ” is the usual order on the set of integers \mathbb{Z} , such that for all $p_1, p_2 \in P$, $p_1 \leq_P p_2$ implies that $\psi(p_1) \leq \psi(p_2)$.

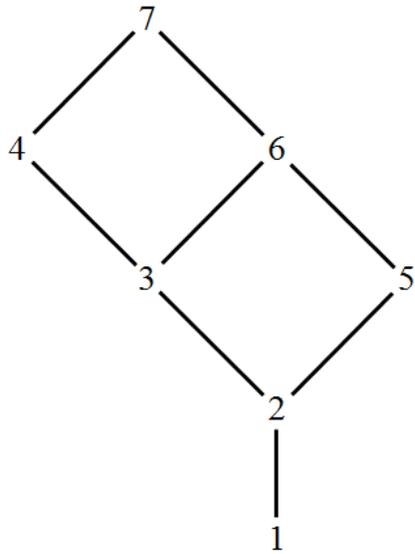


Figure 4.1: The Hasse Diagram of the Heap $P_{1234123}$

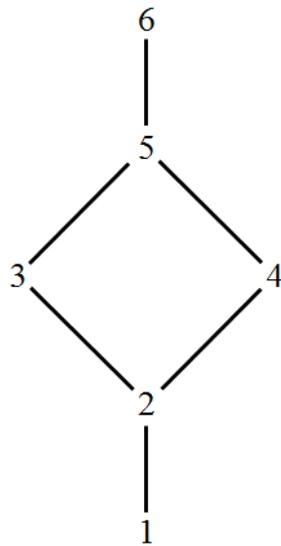


Figure 4.2: The Hasse Diagram of the Heap $P_{w_0(c)}$, Where c is the Tamari Coxeter Element of \mathcal{A}_3

Example 4.2.3. Let $X = \{a, b, c\}$, and consider the poset $(\mathcal{P}(X), \subseteq)$, where $\mathcal{P}(X)$ is the power set of X consisting of all subsets of X , and \subseteq is the containment partial order. Figure 4.3 shows the Hasse diagram of $(\mathcal{P}(X), \subseteq)$, and Figure 4.4 shows the

Hasse diagram of a linear extension of $(\mathcal{P}(X), \subseteq)$, which can also be interpreted via the map $\psi : (\mathcal{P}(X), \subseteq) \rightarrow ([|\mathcal{P}(X)|], \leq)$ such that $\psi(\{\}) = 1$, $\psi(\{a\}) = 2$, $\psi(\{b\}) = 3$, $\psi(\{c\}) = 4$, $\psi(\{a, b\}) = 5$, $\psi(\{a, c\}) = 6$, $\psi(\{b, c\}) = 7$, and $\psi(\{a, b, c\}) = 8$.

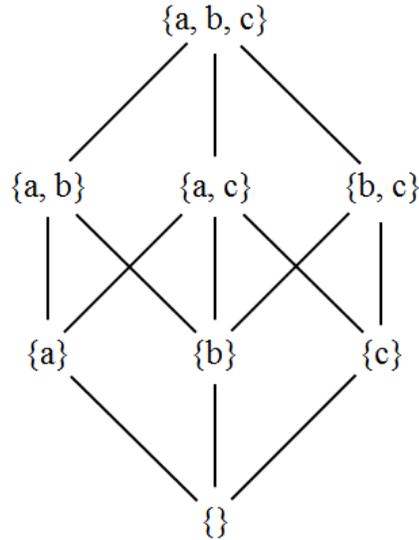


Figure 4.3: The Hasse Diagram of $(\mathcal{P}(X), \subseteq)$, Where $X = \{a, b, c\}$

The following theorem is known to the community, see for example [Brualdi (2010)].

Theorem 4.2.4. *Let (P, \leq) be a finite partially ordered set. Then there is a linear extension of (P, \leq) .*

Theorem 4.2.5. [Stembridge (1996)] *For any reduced word w of a permutation in \mathcal{A}_n , the number of linear extensions of the heap P_w is equal to the number of reduced words obtained from w by applying sequences of short-braid moves on w .*

Example 4.2.6. *Consider the heap discussed in Example 4.2.1. The linear extensions of the heap P_w are:*

$$1 < 2 < 5 < 3 < 6 < 4 < 7,$$

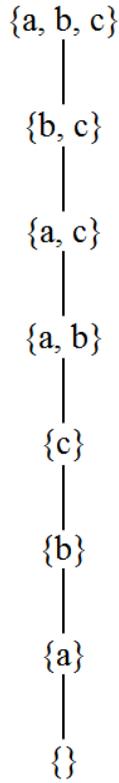


Figure 4.4: The Hasse Diagram of a Linear Extension of $(\mathcal{P}(X), \subseteq)$, Where $X = \{a, b, c\}$

$$1 < 2 < 5 < 3 < 4 < 6 < 7,$$

$$1 < 2 < 3 < 5 < 6 < 4 < 7,$$

$$1 < 2 < 3 < 5 < 4 < 6 < 7,$$

and

$$1 < 2 < 3 < 4 < 5 < 6 < 7.$$

On the other hand, the reduced words obtained from w by applying sequences of short-braid moves are: 1234123, 1231423, 1213423, 1231243, and 1213243. That is, the number of linear extensions of the heap P_w is equal to the number of reduced words obtained from w by applying only short-braid moves.

4.3 Lattice Words

In this section we introduce left-right words, lattice words, and dominant permutations. We also present some examples and known facts. Beside the references appearing in this section, we suggest the following references: [Myers and Wilf (2008)] and [Khovanova and Lewis (2013)].

Definition 4.3.1. [Gessel and Stanley (1995)] *Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$, written in one-line notation. A left-right maximum, simply lr-max, of π is a number π_i such that $\pi_i > \pi_j$, for all $j \in \{1, 2, \dots, i-1\}$.*

Example 4.3.2. *For the permutation $\pi = 521643 \in \mathfrak{S}_5$, written in one-line notation, the numbers $\pi_1 = 5$ and $\pi_4 = 6$ are the left-right maxima of π .*

Definition 4.3.3. [Edelman (1995)] *For a permutation $\pi \in \mathfrak{S}_n$, a reduced word $w = w_1w_2\cdots w_\ell$ of π is called a left-right word, simply lr-word, of π if for all $k \in \{1, 2, \dots, \ell\}$, the transposition s_{w_k} moves a lr-max to the left in the permutation $s_{w_1}s_{w_2}\cdots s_{w_{k-1}}$ when multiplied from the right.*

Example 4.3.4. *For the permutation $\pi = 3421 \in \mathfrak{S}_5$, written in one-line notation, the word $w = 12312 \in \mathcal{R}(\pi)$ is a lr-word. This can be checked as the one-line notation of the identity permutation in \mathfrak{S}_5 is $id = 1234$ so that $s_1 = id s_1 = 2134$ moves the lr-max number 2 of id to the left. Furthermore, $s_1s_2 = 2314$ moves the lr-max number 3 of s_1 to the left, and also $s_1s_2s_3 = 2341$ moves the lr-max number 4 of s_1s_2 to the left. Similarly $s_1s_2s_3s_1 = 3241$ moves the lr-max number 3 of $s_1s_2s_3$ to the left, and $s_1s_2s_3s_1s_2 = 3421$ moves the lr-max number 4 of $s_1s_2s_3s_1$ to the left.*

The following proposition shows which reduced words of a permutation are lr-words.

Proposition 4.3.5. [Edelman (1995)] *For a permutation $\pi \in \mathfrak{S}_n$, the C_1 -class of $\omega(\pi)$ is exactly the set of lr -words of π .*

Definition 4.3.6. [Lehmer (1960)] *For a permutation $\pi \in \mathfrak{S}_n$, the Lehmer code, or code for simplicity, of π , denoted $c(\pi)$, is defined as:*

$$c(\pi) = (c_1, c_2, \dots, c_n),$$

where

$$c_i = |\{j > i : \pi(j) < \pi(i)\}|.$$

Definition 4.3.7. [Manivel (2001)] *A permutation $\pi \in \mathfrak{S}_n$ with Lehmer code $c(\pi) = (c_1, c_2, \dots, c_n)$ is said to be dominant if*

$$c_1 \geq c_2 \geq \dots \geq c_n.$$

The definition above says that a permutation $\pi \in \mathfrak{S}_n$ is dominant if its code forms a partition.

Definition 4.3.8. [Edelman (1995)] *For permutation $\pi \in \mathfrak{S}_n$, we call the partition obtained from the code, $c(\pi)$, of π by arranging the entries in a weak decreasing order the type of π and denote this by $\text{type}(\pi)$.*

Based on Definition 4.3.8 above, when a permutation π is dominant, the terms code and type are the same, and we can use the two terms interchangeably. Note that the type in Definition 4.1.8 is defined on words, while the type in Definition 4.3.8 is defined on permutations.

Definition 4.3.9. [Edelman (1995)] *For a permutation $\pi \in \mathfrak{S}_n$, a reduced word $v = v_1 v_2 \dots v_\ell \in \mathcal{R}(\pi)$ is said to be a lattice word of π if for all $i \in [n - 2]$ and every $k \in [\ell]$, the number of i 's is at least as large as the number of $i + 1$'s appearing in $v_1 v_2 \dots v_k$.*

Theorem 4.3.10. [Edelman (1995)] *Let $\pi \in \mathfrak{S}_n$ and let $v \in \mathcal{R}(\pi)$ with type $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t)$. Then v is a lattice word of π if and only if all the following conditions are satisfied:*

1. $0 \leq \lambda_i - \lambda_{i+1} \leq 1$, for all $1 \leq i \leq t - 1$,
2. π is a dominant permutation of type λ ,
3. v is a *lr*-word.

Example 4.3.11. *Consider the permutation $\pi = 3241 \in \mathfrak{S}_4$, written in one-line notation, and consider the word $v = 1231 \in \mathcal{R}(\pi)$. The code of the permutation π is $c(\pi) = (2, 1, 1)$, thus π is dominant. Also, the type of the word v is $\text{type}(v) = (2, 1, 1)$. By applying the recursive relation appearing in Definition 4.1.4, we find that $\gamma(\pi) = 1213$. Now, by Theorem 4.1.6, $\omega(\pi) = \gamma(\pi)$ and hence $\omega(\pi) = 1213$. Furthermore, the word v can be obtained from the word $\omega(\pi)$ by applying a sequence of short-braid moves. Therefore, by Proposition 4.3.5, v is a *lr*-word of π . This shows that the word v satisfies the three conditions of Theorem 4.3.10 and hence v is a lattice word of π .*

Theorem 4.3.12. [Edelman (1995)] *Let π be a dominant permutation of type $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ with*

$$0 \leq \lambda_i - \lambda_{i+1} \leq 1, \text{ for all } 1 \leq i \leq t - 1.$$

Then the number of lattice words of π is equal to the number of shifted tableaux of shape λ' , where λ' is the conjugate partition of λ .

4.4 Enumerating the Chains of Maximum Length in \mathcal{T}_n

In this section we introduce some of our results. Mainly, we enumerate the chains of maximum length of the Tamari lattice \mathcal{T}_n and collect results on equinumerous objects. We also enumerate the chains of maximum length of other Cambrian lattices

via the linear extensions of some posets.

We start with the following proposition.

Proposition 4.4.1. [Edelman (1995)] *For the maximum permutation $w_0 \in \mathfrak{S}_n$,*

$$\gamma(w_0) = 1 \ 21 \ 321 \ \cdots \ (n-1)(n-2) \cdots 21.$$

Proof. Note first that $\pi|_n = \pi$ for any permutation π and thus $w_0|_n = w_0$. Let us assume that $n > 1$. We claim that $\gamma(w_0|_k) = \gamma(w_0|_{k-1})(k-1)(k-2) \cdots 1$ for all $2 \leq k \leq n$. Let $k = n$. As $w_0^{-1}(n) = 1$, then $\gamma(w_0|_n) = \gamma(w_0|_{n-1})(n-1)(n-2) \cdots 1$ which checks the first step. Now $w_0|_{n-1}$ is a permutation of \mathfrak{S}_{n-1} . In particular, $w_0|_{n-1}$ is the maximum element of \mathfrak{S}_{n-1} . Applying the same argument above on $\gamma(w_0|_{n-1})$ as an element of \mathfrak{S}_{n-1} yields that $\gamma(w_0|_{n-1}) = \gamma(w_0|_{n-2})(n-2)(n-3) \cdots 1$ so that $\gamma(w_0) = \gamma(w_0|_{n-2})(n-2)(n-3) \cdots 1(n-1)(n-2) \cdots 1$. Continue with this process; at the end we have $\gamma(w_0|_2) = \gamma(w_0|_1)1$. As $\gamma(w_0|_1) = id$, the process ends here. This shows that $\gamma(w_0) = 1 \ 21 \ 321 \ \cdots \ (n-1)(n-2) \cdots 21$. \square

In the following proposition we discuss the type of $\omega(w_0)$.

Proposition 4.4.2. *For the maximum permutation $w_0 \in \mathfrak{S}_n$,*

$$\text{type}(\omega(w_0)) = (n-1, n-2, \dots, 2, 1).$$

Proof. By Theorem 4.1.6, $\omega(w_0) = \gamma(w_0)$, and by Proposition 4.4.1,

$$\gamma(w_0) = 1 \ 21 \ 321 \ \cdots \ (n-1)(n-2) \cdots 21.$$

Thus $\omega(w_0) = 1 \ 21 \ 321 \ \cdots \ (n-1)(n-2) \cdots 21$. Since i appears in $n-i$ blocks for all $1 \leq i \leq n-1$, then $m_i = n-i$ for all $i \in [n-1]$. That is $\text{type}(\omega(w_0)) = (n-1, n-2, \dots, 2, 1)$ as desired. \square

Proposition 4.4.3. *If u and v are two words in \mathcal{R}_n with $u \simeq v$, then*

$$\text{type}(u) = \text{type}(v).$$

Proof. As $u \simeq v$, u can be obtained from v by a sequence of short-braid moves. This reorders the numbers appearing in the words u and v but doesn't change the multiplicity of each integer i for all $1 \leq i \leq n - 1$. Thus $m_i(u) = m_i(v)$ for all $i \in [n - 1]$, and hence $\text{type}(u) = \text{type}(v)$. \square

In the following proposition we discuss the type of w_0 .

Proposition 4.4.4. *The permutation $w_0 \in \mathfrak{S}_n$ is dominant, and*

$$\text{type}(w_0) = (n - 1, n - 2, \dots, 2, 1).$$

Proof. The one-line notation of $w_0 \in \mathfrak{S}_n$ is $w_0 = n (n - 1) (n - 2) \cdots 2 1$. This shows that $w_0(i) > w_0(j)$, for all $1 \leq i < j \leq n$. That is;

$$\begin{aligned} c_i &= |\{j > i : w_0(j) < w_0(i)\}| \\ &= |\{i + 1, i + 2, \dots, n\}| \\ &= n - i. \end{aligned}$$

Thus

$$c(w_0) = (n - 1, n - 2, \dots, 2, 1, 0)$$

which is a partition. This shows that w_0 is dominant with type

$$\text{type}(w_0) = (n - 1, n - 2, \dots, 2, 1).$$

\square

The previous two propositions have the following corollaries.

Corollary 4.4.5. For $w_0 \in \mathfrak{S}_n$,

$$\text{type}(w_0) = \text{type}(\omega(w_0)).$$

Proof. This follows from Proposition 4.4.4 and Proposition 4.4.2. \square

Corollary 4.4.6. For $w_0 \in \mathfrak{S}_n$,

$$\text{type}(w_0) = \text{type}(v), \text{ for all reduced words } v \simeq \omega(w_0).$$

Proof. Let $v \simeq \omega(w_0)$. Then

$$\begin{aligned} \text{type}(v) &= \text{type}(\omega(w_0)) \text{ (Proposition 4.4.3)} \\ &= \text{type}(w_0) \text{ (Corollary 4.4.5)} \end{aligned}$$

\square

Corollary 4.4.7. For $w_0 \in \mathfrak{S}_n$, the lex-first word $\omega(w_0)$ is a lattice word.

Proof. By Proposition 4.4.4, w_0 is a dominant permutation of type

$$\lambda = (n-1, n-2, \dots, 2, 1).$$

By Corollary 4.4.5, $\text{type}(\omega(w_0)) = \text{type}(w_0)$, which shows that the second condition of Theorem 4.3.10 holds. Since $\lambda_i - \lambda_{i+1} = 1$, for all $1 \leq i \leq n-2$, the first condition of Theorem 4.3.10 also holds. The third condition of Theorem 4.3.10 follows by Proposition 4.3.5. Thus, the word $\omega(w_0)$ is a lattice word of w_0 as desired. \square

The following lemma introduces a relation between the lattice words of w_0 and the \simeq -class of $\omega(w_0)$.

Lemma 4.4.8. For the permutation $w_0 \in \mathfrak{S}_n$, the number of words $v \in \mathcal{R}_n$ such that $v \simeq \omega(w_0)$ is equal to the number of lattice words of w_0 .

Proof. Let $C(w_0)$ be the set of all $v \in \mathcal{R}_n$ with $v \succeq \omega(w_0)$, and let $LW(w_0)$ be the set of all lattice words of w_0 . By the definition of $C(w_0)$, $\omega(w_0) \in C(w_0)$, and by Corollary 4.4.7, $\omega(w_0) \in LW(w_0)$; that is, both $C(w_0)$ and $LW(w_0)$ are non-empty sets. Let $v \in LW(w_0)$. By Proposition 4.4.4, the permutation w_0 is dominant. Theorem 4.3.10 shows that v is a lr-word of w_0 , and so, by Proposition 4.3.5, we conclude that $v \succeq \omega(w_0)$. This implies that $v \in C(w_0)$ so that $LW(w_0) \subseteq C(w_0)$, and hence $|LW(w_0)| \leq |C(w_0)|$. On the other hand, let $v \in C(w_0)$. Then by Corollary 4.4.6, $\text{type}(v) = \text{type}(w_0)$, and by Proposition 4.3.5, v is a lr-word. Since w_0 is a dominant permutation of the same type as v , then by Theorem 4.3.10, $v \in LW(w_0)$. This shows that $C(w_0) \subseteq LW(w_0)$. Thus $|C(w_0)| \leq |LW(w_0)|$ and hence $|C(w_0)| = |LW(w_0)|$. The lemma is checked. \square

In the following theorem we introduce an alternative proof for the number of the chains of maximum length of the Tamari lattice \mathcal{T}_n .

Theorem 4.4.9. *The number of chains of maximum length of the Tamari lattice \mathcal{T}_n is equal to*

$$\binom{n}{2}! \frac{(n-2)!(n-3)! \cdots 1!}{(2n-5)!(2n-3)! \cdots 1!}.$$

Proof. Consider the group \mathcal{A}_n . Let $LW(w_0)$ be the set of all lattice words of $w_0 \in \mathcal{A}_n$, and let N be the number of chains of maximum length of \mathcal{T}_n . Consider the Tamari Coxeter element c of \mathcal{A}_n . Example 3.1.4 shows that the word $\omega(w_0)$ is the c -sorting word of w_0 . That is $\omega(w_0) = w_0(c)$. Now

$$\begin{aligned} N &= |[w_0(c)]_{\succeq}| \quad (\text{by Proposition 3.4.9}) \\ &= |[\omega(w_0)]_{\succeq}| \\ &= |LW(w_0)| \quad (\text{by Lemma 4.4.8}) \end{aligned}$$

By Proposition 4.4.4, the type of w_0 is the partition $\lambda = (n, n-1, n-2, \dots, 1)$. By

Theorem 4.3.12, $|LW(w_0)|$ is equal to the number of standard shifted tableaux of shape λ' , where λ' is the conjugate partition of λ . Since λ is symmetric, $\lambda' = \lambda$. By Corollary 4.1.2, the number of standard shifted tableaux of shape $(n, n-1, n-2, \dots, 1)$ is equal to

$$\binom{n}{2}! \frac{(n-2)!(n-3)! \cdots 1!}{(2n-5)!(2n-3)! \cdots 1!}.$$

This show that

$$N = \binom{n}{2}! \frac{(n-2)!(n-3)! \cdots 1!}{(2n-5)!(2n-3)! \cdots 1!},$$

and completes the proof. \square

The previous theorem has the following two corollaries.

Corollary 4.4.10. *Let w_0 be the maximum element of \mathcal{A}_n , and let c be the Tamari Coxeter element of \mathcal{A}_n , and $w_0(c)$ is the c -sorting word of w_0 . Then the following are equal:*

1. *the number of chains of maximum length of the Tamari lattice \mathcal{T}_n ,*
2. *the number of standard shifted tableaux of shape $(n, n-1, n-2, \dots, 1)$,*
3. *the number of lattice words of w_0 ,*
4. *the size of the class $[w_0(c)]_{\leq}$,*
5. *the size of the class $[\omega(w_0)]_{\leq}$,*
6. *the size of the class $[\gamma(w_0)]_{\leq}$,*
7. *the number of linear extensions of the heap $P_{w_0(c)}$,*
8. $\binom{n}{2}! \frac{(n-2)!(n-3)! \cdots 1!}{(2n-5)!(2n-3)! \cdots 1!}.$

Proof. The corollary is a consequence of Theorem 4.4.9, Corollary 4.1.2, Proposition 3.4.9, Proposition 4.4.1, Theorem 4.1.6, Theorem 4.2.5, and Lemma 4.4.8. \square

Corollary 4.4.11. *Let $c' = s_n s_{n-1} s_{n-2} \cdots s_2 s_1$ be a Coxeter element of \mathcal{A}_n , and $w_0(c')$ is the c' -sorting word of the maximum element w_0 . Then the following are equal:*

1. *the number of chains of maximum length of $\text{Camb}(c')$,*
2. *the size of the class $[w_0(c')]_{\simeq}$,*
3. *the number of linear extensions of the heap $P_{w_0(c')}$,*
4. $\binom{n}{2}! \frac{(n-2)!(n-3)! \cdots 1!}{(2n-5)!(2n-3)! \cdots 1!}$.

Proof. Since $c' \simeq \rho^h(c')$, then c' is symmetric. As $c' = \rho^v(c)$, where c is the Tamari Coxeter element of \mathcal{A}_n , then, by Corollary 3.7.5, $\text{Camb}(c') \cong \text{Camb}(c) = \mathcal{T}_n$. The corollary follows by Theorem 4.4.9, Proposition 3.4.9, and Theorem 4.2.5. \square

We end this section by the following proposition which is a step toward finding formulas for the number of chains of maximum length of other Cambrian lattices of \mathcal{A}_n .

Proposition 4.4.12. *Let c be a Coxeter element of \mathcal{A}_n . Then the following are equal:*

1. *the number of chains of maximum length of the Cambrian lattice $\text{Camb}(c)$,*
2. *the size of the class $[w_0(c)]_{\simeq}$,*
3. *the number of linear extensions of the heap $P_{w_0(c)}$.*

Proof. The proof follows by Proposition 3.4.9, and Theorem 4.2.5. \square

COUNTING THE CHAINS OF MAXIMUM LENGTH OF M -ERALIZED
CAMBRIAN LATTICES

The m -eralized Cambrian lattices were first introduced by S. Stump, H. Thomas, and N. Williams, see [Stump *et al.* (2015)], and finding a general formula for the number of chains of maximum length of this family of lattices is still a mysterious problem. In this chapter we enumerate the chains of maximum length of the m -eralized Cambrian lattices of \mathcal{A}_n in terms of other objects, and find formulas for the number of these chains for the cases when $n = 1, 2, 3$, and 4 .

The outline of this chapter goes as follows: In Section 5.1, we introduce the m -eralized Cambrian lattices, and study the chains of maximum length in this family of lattices. In Section 5.2, we study the duality and isomorphism classes of m -eralized Cambrian lattices. In Section 5.3, we define the super-heap of a composition, and prove that the number of chains of maximum length of any m -eralized Cambrian lattice is equal to the number of linear extensions of the super-heap of the composition corresponding to its maximum element. In Section 5.4, we find formulas for the number of chains of maximum length for all m -eralized Cambrian lattices of \mathcal{A}_1 , \mathcal{A}_2 , \mathcal{A}_3 , and \mathcal{A}_4 .

5.1 The m -eralized Cambrian Lattices and their Chains of Maximum Length

In this section we introduce the definitions of m -eralized Cambrian lattices and m -eralized Tamari lattices of \mathcal{A}_n . We also study the chains of maximum length in these lattices, and find a relation between this type of chains of an m -eralized Cambrian

lattice and the equivalence class of the composition corresponding to its maximum element. The definitions in the chapter are from [Adams (2004)], and [Stump *et al.* (2015)].

Consider the Coxeter group \mathcal{A}_n and its set of generators S . The *Braid group* corresponding to the Coxeter system (\mathcal{A}_n, S) , denoted $B(\mathcal{A}_n)$, is the group generated by the simple transpositions of \mathcal{A}_n such that $s_i s_j = s_j s_i$, when $|i - j| > 1$, and $s_i s_j s_i = s_j s_i s_j$, when $|i - j| = 1$. In symbols,

$$B(\mathcal{A}_n) = \langle S : s_i s_j = s_j s_i \text{ when } |i - j| > 1, s_i s_j s_i = s_j s_i s_j \text{ when } |i - j| = 1 \rangle.$$

Note that $s_i^2 \neq id$, for all $i \in [n]$. As before, the *length* of an element $w \in B(\mathcal{A}_n)$, denoted $\ell(w)$, is the length of the shortest decomposition of w in terms of the generators in S . Any decomposition of w with length equal to $\ell(w)$ is called a *reduced* decomposition. If c is a Coxeter element of \mathcal{A}_n , then the *c-sorting* decomposition of w is the lexicographically first subword of c^∞ which is a reduced decomposition of w . Every *c-sorting* decomposition can be expressed by a sequence of subsets of S such that every subset represents the sequence of letters of the *c-sorting* decomposition that appear between two consecutive dividers. The element $w \in B(\mathcal{A}_n)$ is said to be *c-sortable* if the sequence corresponding to its *c-sorting* decomposition is weakly decreasing under inclusion.

Let $m, n \in \mathbb{Z}^+$ be positive integers, and let c be a Coxeter element of \mathcal{A}_n . Consider the right weak order on $B(\mathcal{A}_n)$ so that $u < v$ in $B(\mathcal{A}_n)$ if there exist $\tau_1, \tau_2, \dots, \tau_j \in S$ such that $v = u\tau_1\tau_2 \cdots \tau_j$ with $\ell(v) = \ell(u) + j$. Define the element $w_0^{(m)}(c)$ such that

$$w_0^{(m)}(c) = \begin{cases} c^{\frac{(n+1)m}{2}} & : \text{ when } m \text{ is even} \\ c^{\frac{(n+1)(m-1)}{2}} w_0(c) & : \text{ when } m \text{ is odd,} \end{cases}$$

where $w_0(c)$ is the c -sorting decomposition of the maximum element w_0 of \mathcal{A}_n . Note that $w_0^{(m)}(c) \in B(\mathcal{A}_n)$. The m -eralized Cambrian lattice, denoted $\text{Camb}^m(c)$, is defined as the subposet $[id, w_0^{(m)}(c)]$ of the right weak order on the braid group $B(\mathcal{A}_n)$ induced by the c -sortable elements. The m -eralized Tamari lattice, denoted \mathcal{T}_n^m , is the m -eralized Cambrian lattice corresponding to the Tamari Coxeter element $c = s_1 s_2 s_3 \cdots s_n$ of \mathcal{A}_n .

The m -eralized Cambrian lattices generalize the Cambrian lattices. In particular, for any Coxeter element c of \mathcal{A}_n , $\text{Camb}(c) = \text{Camb}^1(c)$, and the m -eralized Tamari lattice \mathcal{T}_n^m generalize the Tamari lattice \mathcal{T}_n so that $\mathcal{T}_n = \mathcal{T}_n^1$.

Example 5.1.1. For the Coxeter element $c = s_1 s_2$ of \mathcal{A}_2 , Figure 5.1 shows the Hasse diagram of $\text{Camb}^{(2)}(c)$.

In the following proposition we study the maximum length of chains in the m -eralized Cambrian lattices.

Proposition 5.1.2. For $m, n \in \mathbb{Z}^+$ and any Coxeter element c of \mathcal{A}_n , the maximum length of chains in $\text{Camb}^m(c)$ is equal to $\binom{n+1}{2} m$.

Proof. Let $m, n \in \mathbb{Z}^+$ and let c be a Coxeter element of \mathcal{A}_n . We study the following two cases.

Case 1: m is even. In this case, $w_0^{(m)}(c) = c^{\frac{(n+1)m}{2}}$, and thus $w_0^{(m)}(c)$ is of length $\ell = n \binom{(n+1)m}{2} = \binom{n+1}{2} m$. Assume that $w_0^{(m)}(c) = c_1 c_2 \cdots c_\ell$, where c_i is an adjacent transposition for all $i \in [\ell]$, is the c -sorting decomposition of $w_0^{(m)}(c)$. Based on the right weak ordering on $\text{Camb}^m(c)$, the chain

$$id < c_1 < c_1 c_2 < c_1 c_2 c_3 < \cdots < c_1 c_2 \cdots c_\ell = w_0^{(m)}(c)$$

is a saturated chain of length $\binom{n+1}{2} m$. Since $c_1 c_2 \cdots c_\ell$ is a reduced c -sorting decomposition, then $c_1 c_2 \cdots c_i$ is a reduced c -sorting decomposition, for all $i \in [\ell]$. As the

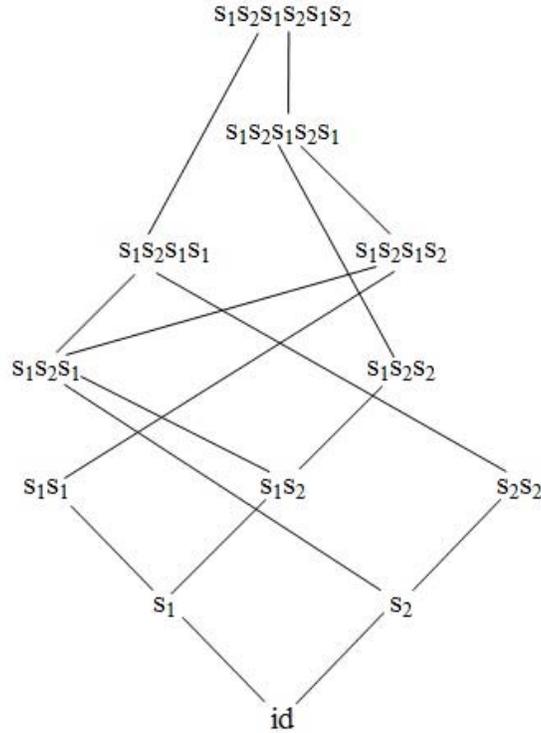


Figure 5.1: The Hasse Diagram of $\text{Camb}^2(s_1 s_2)$

lengths of any two consecutive elements in the chain differ by one, $\binom{n+1}{2} m$ is the maximum length that any chain in $\text{Camb}^m(c)$ might attain.

Case 2: m is odd. In this case, $w_0^{(m)}(c) = c^{\frac{(n+1)(m-1)}{2}} w_0(c)$, where $w_0(c)$ is the c -sorting decomposition of the maximum element w_0 of \mathcal{A}_n . By Corollary 2.4.3, $\ell(w_0) = \binom{n+1}{2}$. Thus, the length of $w_0^{(m)}(c)$ is equal to $\ell = n \binom{(n+1)(m-1)}{2} + \binom{n+1}{2} = \binom{n+1}{2} m$. By assuming that $w_0^{(m)}(c) = c_1 c_2 \cdots c_\ell$, where c_i is an adjacent transposition for all $i \in [\ell]$, is the c -sorting decomposition of $w_0^{(m)}(c)$, then as in Case 1, the chain

$$id < c_1 < c_1 c_2 < c_1 c_2 c_3 < \cdots < c_1 c_2 \cdots c_\ell = w_0^{(m)}(c)$$

is a saturated chain of length $\binom{n+1}{2} m$ with the property that every element $c_1 c_2 \cdots c_i$ is a reduced c -sorting decomposition, for all $i \in [\ell]$, and the lengths of any two consecutive elements in it differ by one. Thus $\binom{n+1}{2} m$ is the maximum length of any

chain in $\text{Camb}^m(c)$. This checks Case 2 and completes the proof. \square

The following theorem illustrates the relation between the chains of maximum length of an m -eralized Cambrian lattice and the equivalence class of the composition corresponding to its maximum element.

Theorem 5.1.3. *For $m, n \in \mathbb{Z}^+$ and any Coxeter element c of \mathcal{A}_n , the number of chains of maximum length of the m -eralized Cambrian lattice $\text{Camb}^m(c)$ is equal to the size of the class $[w_0^{(m)}(c)]_{\leq}$.*

Proof. Let c be a Coxeter element of \mathcal{A}_n , and let $w_0^{(m)}(c)$ be the c -sorting decomposition of the maximum element of $\text{Camb}^m(c)$. By Proposition 5.1.2, the maximum length of chains in $\text{Camb}^m(c)$ is $\ell = \binom{n+1}{2} m$, and hence every chain of maximum length in $\text{Camb}^m(c)$ can be written as

$$id \leq c_1 \leq c_1 c_2 \leq c_1 c_2 c_3 \leq \cdots \leq c_1 c_2 \cdots c_\ell = w_0^{(m)}(c),$$

where c_i is an adjacent transposition in \mathcal{A}_n for all $i \in [\ell]$. Based on the right weak covering, and as applying a long-braid move on a c -sorting decomposition makes it non- c -sorting, every maximum-length chain in $\text{Camb}^m(c)$ represents an element in $[w_0^{(m)}(c)]_{\leq}$. On the other hand, every decomposition obtained from $[w_0^{(m)}(c)]_{\leq}$ is a reduced decomposition of $w_0^{(m)}(c)$ and represents a chain of maximum length of $\text{Camb}^m(c)$. Therefore the number of chains of maximum length of $\text{Camb}^m(c)$ is equal to the size of the class $[w_0^{(m)}(c)]_{\leq}$. \square

As a consequence of Theorem 5.1.3, we have the following corollary.

Corollary 5.1.4. *For $m, n \in \mathbb{Z}^+$, the number of chains of maximum length of \mathcal{T}_n^m is equal to the size of the class $[w_0^{(m)}(c)]_{\leq}$, where c is the Tamari Coxeter element of \mathcal{A}_n .*

Example 5.1.5. Consider the Tamari Coxeter element $c = s_1s_2s_3$ of \mathcal{A}_3 . $w_0^{(2)}(c) = s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2s_3$. Figure 5.2 shows the Hasse diagram of the chains of maximum length in $\text{Camb}^{(2)}(c)$, where these chains are:

$$\begin{aligned} id &< s_1 < s_1s_2 < s_1s_2s_1 < s_1s_2s_3s_1 < s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_1 < s_1s_2s_3s_1s_2s_3s_1 \\ &< s_1s_2s_3s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3s_1s_2s_1 < s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1 \\ &< s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2s_3, \end{aligned}$$

$$\begin{aligned} id &< s_1 < s_1s_2 < s_1s_2s_3 < s_1s_2s_3s_1 < s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_1 < s_1s_2s_3s_1s_2s_3s_1 \\ &< s_1s_2s_3s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3s_1s_2s_1 < s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1 \\ &< s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2s_3, \end{aligned}$$

$$\begin{aligned} id &< s_1 < s_1s_2 < s_1s_2s_1 < s_1s_2s_3s_1 < s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3 < s_1s_2s_3s_1s_2s_3s_1 \\ &< s_1s_2s_3s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3s_1s_2s_1 < s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1 \\ &< s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2s_3, \end{aligned}$$

$$\begin{aligned} id &< s_1 < s_1s_2 < s_1s_2s_3 < s_1s_2s_3s_1 < s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3 < s_1s_2s_3s_1s_2s_3s_1 \\ &< s_1s_2s_3s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3s_1s_2s_1 < s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1 \\ &< s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2s_3, \end{aligned}$$

$$\begin{aligned} id &< s_1 < s_1s_2 < s_1s_2s_1 < s_1s_2s_3s_1 < s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_1 < s_1s_2s_3s_1s_2s_3s_1 \\ &< s_1s_2s_3s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3s_1s_2s_3 < s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1 \\ &< s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2s_3, \end{aligned}$$

$$\begin{aligned} id &< s_1 < s_1s_2 < s_1s_2s_3 < s_1s_2s_3s_1 < s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_1 < s_1s_2s_3s_1s_2s_3s_1 \\ &< s_1s_2s_3s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3s_1s_2s_3 < s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1 \\ &< s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2s_3, \end{aligned}$$

$$\begin{aligned}
id &< s_1 < s_1s_2 < s_1s_2s_1 < s_1s_2s_3s_1 < s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3 < s_1s_2s_3s_1s_2s_3s_1 \\
&< s_1s_2s_3s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3s_1s_2s_3 < s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1 \\
&< s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2s_3,
\end{aligned}$$

and

$$\begin{aligned}
id &< s_1 < s_1s_2 < s_1s_2s_3 < s_1s_2s_3s_1 < s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3 < s_1s_2s_3s_1s_2s_3s_1 \\
&< s_1s_2s_3s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3s_1s_2s_3 < s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1 \\
&< s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2 < s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2s_3.
\end{aligned}$$

On the other hand, the class $[w_0^{(2)}(c)]_{\leq}$ consists of the compositions:

$$\begin{aligned}
&(1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3), (1, 2, 1, 3, 2, 3, 1, 2, 3, 1, 2, 3), (1, 2, 3, 1, 2, 1, 3, 2, 3, 1, 2, 3), \\
&(1, 2, 1, 3, 2, 1, 3, 2, 3, 1, 2, 3), (1, 2, 3, 1, 2, 3, 1, 2, 1, 3, 2, 3), (1, 2, 1, 3, 2, 3, 1, 2, 1, 3, 2, 3), \\
&(1, 2, 3, 1, 2, 1, 3, 2, 1, 3, 2, 3), \text{ and } (1, 2, 1, 3, 2, 1, 3, 2, 1, 3, 2, 3).
\end{aligned}$$

5.2 The Duality and Isomorphism Classes of m -eralized Cambrian Lattices

In this section we study the duality and isomorphism classes of m -eralized Cambrian lattices. We also enumerate the distinct non-isomorphic m -eralized Cambrian lattices, the self-dual m -eralized Cambrian lattices, and the distinct non-isomorphic self-dual m -eralized Cambrian lattices of \mathcal{A}_n .

We start with the following lemma, where the case when $m = 1$ was verified in [Reading (2006)].

Lemma 5.2.1. *Let $m, n \in \mathbb{Z}^+$, and let c and c' be two Coxeter elements of \mathcal{A}_n . Then*

$$\text{Camb}^m(c') \cong \text{Camb}^m(c) \Rightarrow c' \simeq c \text{ or } c' \simeq \rho^h \circ \rho^v(c).$$

Proof. For any Coxeter element c'' of \mathcal{A}_n , every adjacent transposition $s_i \in \mathcal{A}_n$ is c'' -sortable, thus $s_i \in \text{Camb}^m(c'')$ for all $i \in [n]$. Note that s_i precedes s_{i+1} in c''

$[id, s_i \vee s_{i+1}]$ appears as Figure 5.3. Otherwise, the edge is directed toward s_i . Now, suppose that $\text{Camb}^m(c') \cong \text{Camb}^m(c)$, and let φ be an isomorphism from $\text{Camb}^m(c')$ onto $\text{Camb}^m(c)$. As the adjacent transpositions are the atoms of both $\text{Camb}^m(c')$ and $\text{Camb}^m(c)$, the map φ is a bijection on the set of adjacent transpositions of \mathcal{A}_n so that $\varphi(s_i \vee s_j) = \varphi(s_i) \vee \varphi(s_j)$. This implies that s_i and s_j commute if and only if $\varphi(s_i)$ and $\varphi(s_j)$ commute, for all $i, j \in [n]$. Therefore, φ can be seen as a bijection from the set of vertices of $\mathcal{O}^{c'}$ onto the set of vertices of \mathcal{O}^c that preserves the directed edges so that for all $i \in [n]$, if $\varphi(s_i) = s_j$, then $\varphi(s_{i+1}) = s_{j-1}$ or $\varphi(s_{i+1}) = s_{j+1}$.

Claim 1: $\varphi(s_1) = s_1$ or $\varphi(s_1) = s_n$.

proof of claim: Suppose on the contrary that $\varphi(s_1) = s_j$ for some $j \notin \{1, n\}$. As s_{j-1} and s_j don't commute, $\varphi^{-1}(s_{j-1})$ and $\varphi^{-1}(s_j) = s_1$ don't commute. Thus $\varphi^{-1}(s_{j-1}) = s_2$. Similarly, since s_{j+1} and s_j don't commute, then $\varphi^{-1}(s_{j+1})$ and $\varphi^{-1}(s_j) = s_1$ don't commute, and hence $\varphi^{-1}(s_{j+1}) = s_2$. This contradicts that φ is a bijection on the set of adjacent transpositions. Therefore, $\varphi(s_1) = s_1$ or $\varphi(s_1) = s_n$ as desired.

Claim 2: If $\varphi(s_1) = s_1$, then $\varphi(s_i) = s_i$ for all $i \in [n]$.

proof of claim: Suppose not and let $k = \min\{j : \varphi(s_j) \neq s_j, j \in [n]\}$. Then $k \geq 2$ so that $\varphi(s_{k-1}) = s_{k-1}$ and $\varphi(s_k) = s_m$, for some $m > k$. This implies that s_k and s_{k-1} don't commute while $\varphi(s_k)$ and $\varphi(s_{k-1})$ commute, which is a contradiction. Thus $\varphi(s_i) = s_i$ for all $i \in [n]$ as desired.

Claim 3: If $\varphi(s_1) = s_n$, then $\varphi(s_i) = s_{n+1-i}$ for all $i \in [n]$.

proof of claim: The proof is very similar to the proof of Claim 2 above.

By Claim 1, Claim 2, and Claim 3, the identity map and the map $s_i \mapsto s_{n+1-i}$, where

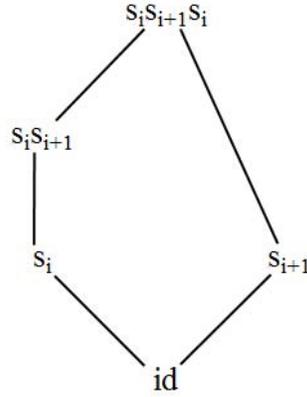


Figure 5.3: The Hasse Diagram of the Interval $[id, s_i \vee s_{i+1}]$ in $\text{Camb}^m(c)$, Where s_i precedes s_{i+1} in c

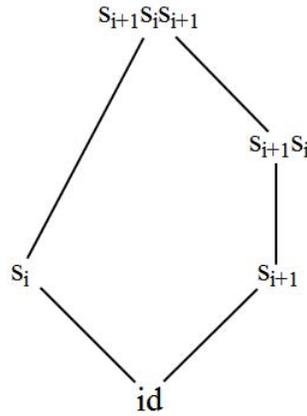


Figure 5.4: The Hasse Diagram of the Interval $[id, s_i \vee s_{i+1}]$ in $\text{Camb}^m(c)$, Where s_{i+1} precedes s_i in c

s_i precedes s_{j+1} in c' if and only if s_{n-i} precedes s_{n-i+1} in c are the only bijections from the set of vertices of $\mathcal{O}^{c'}$ onto the set of vertices of \mathcal{O}^c that preserve the directed edges. By Proposition 3.1.7 and Proposition 3.5.15, this implies that either $c' \simeq c$ or $c' \simeq \rho^h \circ \rho^v(c)$. \square

Conjecture 5.2.2. *Let $m, n \in \mathbb{Z}^+$, and let c and c' be two Coxeter elements of \mathcal{A}_n . Then*

$$c' \simeq \rho^v(c) \text{ or } c' \simeq \rho^h(c) \Rightarrow \text{Camb}^m(c) \cong \text{dual}(\text{Camb}^m(c')).$$

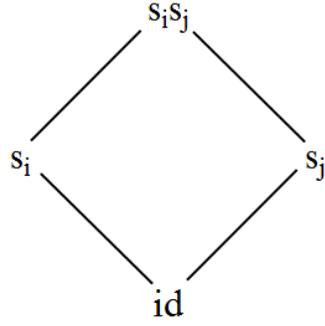


Figure 5.5: The Hasse Diagram of the Interval $[id, s_i \vee s_j]$ in $\text{Camb}^m(c)$, Where $|i - j| > 1$ in c

Lemma 5.2.3. *Suppose Conjecture 5.2.2 holds, let $m, n \in \mathbb{Z}^+$, and let c and c' be two Coxeter elements of \mathcal{A}_n . Then*

$$\text{Camb}^m(c) \cong \text{dual}(\text{Camb}^m(c')) \Rightarrow c' \preceq \rho^v(c) \text{ or } c' \preceq \rho^h(c).$$

Proof. If $\text{Camb}^m(c) \cong \text{dual}(\text{Camb}^m(c'))$, then by Conjecture 5.2.2, $\text{Camb}^m(\rho^v(c)) \cong \text{Camb}^m(c')$. This implies, by Lemma 5.2.1, that $c' \preceq \rho^v(c)$ or $c' \preceq \rho^h \circ \rho^v(\rho^v(c))$. By Proposition 3.5.11, $c' \preceq \rho^v(c)$ or $c' \preceq \rho^h(c)$ as desired. \square

The following theorem completely classifies the duality of m -eralized Cambrian lattices.

Theorem 5.2.4. *Suppose Conjecture 5.2.2 holds, let $m, n \in \mathbb{Z}^+$, and let c and c' be two Coxeter elements of \mathcal{A}_n . Then*

$$\text{Camb}^m(c) \cong \text{dual}(\text{Camb}^m(c')) \text{ if and only if } c' \preceq \rho^v(c) \text{ or } c' \preceq \rho^h(c).$$

Proof. It follows by Lemma 5.2.3, and Conjecture 5.2.2. \square

The previous theorem has the following corollary which illustrates the relation between the symmetric Coxeter elements and the self-dual m -eralized Cambrian lattices of \mathcal{A}_n .

Corollary 5.2.5. *Suppose Conjecture 5.2.2 holds, let $m, n \in \mathbb{Z}^+$, and let c be a Coxeter element of \mathcal{A}_n . Then*

$\text{Camb}^m(c)$ *is self-dual if and only if c is symmetric.*

Proof. It follows by Theorem 5.2.4 as the case $c \simeq \rho^v(c)$ is impossible. □

Since the Tamari Coxeter element is symmetric, then by Corollary 5.2.5, the m -eralized Tamari lattice \mathcal{T}_n^m is self-dual. The following corollary enumerates the self-dual m -eralized Cambrian lattices of \mathcal{A}_n .

Corollary 5.2.6. *Suppose Conjecture 5.2.2 holds, let $m, n \in \mathbb{Z}^+$. Then the number of self-dual m -eralized Cambrian lattices of \mathcal{A}_n is equal to*

$$2^{\lfloor \frac{n}{2} \rfloor}.$$

Proof. This follows by Corollary 5.2.5 and Proposition 3.5.10. □

The following theorem completely classifies the isomorphism classes of m -eralized Cambrian lattices.

Theorem 5.2.7. *Suppose Conjecture 5.2.2 holds, let $m, n \in \mathbb{Z}^+$, and let c and c' be two Coxeter elements of \mathcal{A}_n . Then*

$\text{Camb}^m(c) \cong \text{Camb}^m(c')$ *if and only if $c' \simeq c$ or $c' \simeq \rho^h \circ \rho^v(c)$.*

Proof. Suppose that $c' \simeq c$ or $c' \simeq \rho^h \circ \rho^v(c)$, then by Proposition 3.5.11, $c' \simeq \rho^h(\rho^h(c))$ or $c' \simeq \rho^v(\rho^h(c))$. By Conjecture 5.2.2, $\text{Camb}^m(\rho^h(c)) \cong \text{dual}(\text{Camb}^m(c'))$, and then $\text{Camb}^m(c) \cong \text{Camb}^m(c')$. The other direction is checked by Lemma 5.2.1. □

Two m -eralized Cambrian lattices of \mathcal{A}_n are said to be *distinct* if they correspond to different orientations of the Coxeter diagram of \mathcal{A}_n . Recall the function ν defined in Chapter 3 on the set of positive integers \mathbb{Z}^+ so that:

$$\nu(m) = m \bmod 2.$$

In the following proposition we enumerate the distinct non-isomorphic m -eralized Cambrian lattices of \mathcal{A}_n .

Proposition 5.2.8. *Suppose Conjecture 5.2.2 holds, let $m, n \in \mathbb{Z}^+$. Then the number of distinct non-isomorphic m -eralized Cambrian lattices of \mathcal{A}_n is equal to*

$$2^{n-2} + 2^{\frac{n-3}{2}} \nu(n).$$

Proof. It follows by Theorem 5.2.7, and similar arguments to those used to verify Proposition 3.6.4. □

In the following proposition we enumerate the distinct non-isomorphic self-dual m -eralized Cambrian lattices of \mathcal{A}_n .

Proposition 5.2.9. *Suppose Conjecture 5.2.2 holds, let $m, n \in \mathbb{Z}^+$. Then the number of distinct non-isomorphic self-dual m -eralized Cambrian lattices of \mathcal{A}_n is equal to*

$$2^{\lfloor \frac{n}{2} \rfloor - 1}.$$

Proof. It follows by Corollary 5.2.5, and similar arguments to those used to verify Proposition 3.7.7. □

5.3 The Super-heap of Compositions

In this section we define the super-heap of a composition which generalizes the heap discussed in Section 4.2. Then we prove that the number of chains of maximum length of an m -eralized Cambrian lattice is equal to the number of linear extensions of the super-heap of the composition corresponding to its maximum element.

Let $n \in \mathbb{Z}^+$ be a positive integer, and let $\alpha \in \mathfrak{C}_n$ be a composition of n so that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$, where $\ell = \ell(\alpha)$. Consider the set $[\ell] = \{1, 2, 3, \dots, \ell\}$ and define

the partial ordering \sqsubseteq on $[\ell]$ as the transitive closure of the binary relation:

$$i \sqsubseteq j \text{ if and only if } i < j \text{ as integers with } |\alpha_i - \alpha_j| \in \{0, 1\}.$$

The poset $\mathfrak{H}_\alpha = (\ell(\alpha), \sqsubseteq)$ is called the *super-heap* of the composition α .

Example 5.3.1. Consider the composition $\alpha = (1, 2, 3, 1, 2, 1, 4) \in \mathfrak{C}_{14}$. Then $[\ell(\alpha)] = [7]$, with

$$1 \sqsubseteq 2, 4, 5, 6$$

$$2 \sqsubseteq 3, 4, 5, 6$$

$$3 \sqsubseteq 5, 7$$

$$4 \sqsubseteq 5, 6$$

and

$$5 \sqsubseteq 6.$$

The super-heap \mathfrak{H}_α is the transitive closure of these relations, and the Hasse diagram of \mathfrak{H}_α is shown in Figure 5.6.

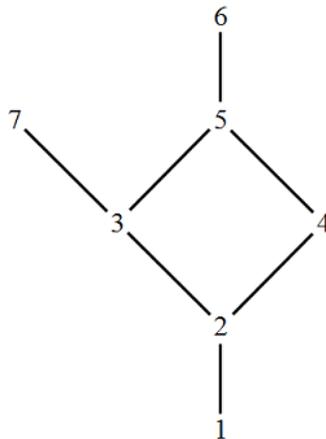


Figure 5.6: The Hasse Diagram of the Super-heap $\mathfrak{H}_{(1,2,3,1,2,1,4)}$

For a Coxeter element c of \mathcal{A}_n , every element in $\text{Camb}^m(c)$ can be represented by the composition corresponding to its c -sorting decomposition. For instance, if x is an element in $\text{Camb}^m(c)$ with the c -sorting decomposition $x = s_{i_1} s_{i_2} \cdots s_{i_k}$, then the composition corresponding to x is $\alpha_x = (i_1, i_2, \dots, i_k)$. We denote by α_m^c the composition corresponding to the maximum element $w_0^{(m)}(c)$ of $\text{Camb}^m(c)$, and denote by \mathfrak{H}_m^c the super-heap $\mathfrak{H}_{\alpha_m^c}$. Let $\mathcal{L}(\mathfrak{H}_m^c)$ be the set of all linear extensions of the super-heap \mathfrak{H}_m^c . The following theorem illustrates the relation between the number of chains of maximum length in m -eralized Cambrian lattices and the number of the linear extensions of the super-heap corresponding to their maximum elements.

Theorem 5.3.2. *For $m, n \in \mathbb{Z}^+$ and any Coxeter element c of \mathcal{A}_n , the number of chains of maximum length of $\text{Camb}^m(c)$ is equal to the number of linear extensions of the super-heap \mathfrak{H}_m^c .*

Proof. Let $m, n \in \mathbb{Z}^+$ be positive integers and let c be a Coxeter element of \mathcal{A}_n . We start by claiming that $|\llbracket \alpha \rrbracket_{\prec} = |\mathcal{L}(\mathfrak{H}_{\alpha})|$, for any composition α . To check this claim, assume that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\ell})$, where $\ell = \ell(\alpha)$. Let $\phi : \mathfrak{H}_{\alpha} \rightarrow [\ell]$ be defined such that $\phi(i) = i$, for all $i \in \mathfrak{H}_{\alpha}$. Then ϕ is a linear extension of the super-heap \mathfrak{H}_{α} with $\phi(1) \leq \phi(2) \leq \phi(3) \leq \cdots \leq \phi(\ell)$ in $[\ell]$. Let $\alpha' \in \llbracket \alpha \rrbracket_{\prec}$ be a composition obtained from α by applying a single commuting move, say $\alpha' = (\alpha_1, \alpha_2, \dots, \alpha_{j-1}, \alpha_{j+1}, \alpha_j, \alpha_{j+2}, \dots, \alpha_{\ell})$, then $|\alpha_j - \alpha_{j+1}| > 2$. As α_j and α_{j+1} commute, j and $j+1$ are incomparable in the super-heap \mathfrak{H}_{α} . Define the map $\sigma : \mathfrak{H}_{\alpha} \rightarrow [\ell]$ such that

$$\sigma(i) = \begin{cases} i & : i \neq j, j+1 \\ j+1 & : i = j \\ j & : i = j+1 \end{cases}$$

Then σ is another linear extension of the super-heap \mathfrak{H}_{α} with $\sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(j-1) \leq \sigma(j+1) \leq \sigma(j) \leq \cdots \leq \sigma(\ell)$ in $[\ell]$. This illustrates that the lin-

ear extensions obtained from different compositions in $[\alpha]_{\leq}$ are different, and hence $|[\alpha]_{\leq}| \leq |\mathcal{L}(\mathfrak{H}_\alpha)|$.

We check the other inequality by induction on the length ℓ . If $\ell = 1$, then $|[\alpha]_{\leq}| = |\mathcal{L}(\mathfrak{H}_\alpha)| = 1$ which checks the base step. For the inductive step, assume that ς is a linear extension of \mathfrak{H}_α with $\varsigma(\ell) = i$ for some $i \in [\ell]$. This implies that $k \not\leq \ell$ in \mathfrak{H}_α , hence $|\alpha_k - \alpha_\ell| > 2$, for all $k \geq i + 1$. Therefore α_ℓ commutes with α_k for all $k \in \{i + 1, i + 2, \dots, \ell - 1\}$. Let $\alpha'' = (\alpha_1, \alpha_2, \dots, \alpha_{\ell-1})$. Then $\mathfrak{H}_{\alpha''}$ is the subposet of \mathfrak{H}_α induced by the vertices $\{1, 2, 3, \dots, \ell - 1\}$. Define the map $\varsigma' : \mathfrak{H}_{\alpha''} \rightarrow [\ell - 1]$ such that:

$$\varsigma'(x) = \begin{cases} \varsigma(x) & : \varsigma(x) < i \\ \varsigma(x) - 1 & : \varsigma(x) > i \end{cases}$$

Then ς' is a linear extension of the heap $\mathfrak{H}_{\alpha''}$. By the inductive step, ς' corresponds to a composition $\alpha''' \in [\alpha'']_{\leq}$. As α_ℓ commutes with α_k for all $k \in \{i + 1, i + 2, \dots, \ell - 1\}$, ς corresponds to a composition in $[\alpha]_{\leq}$ and hence $|[\alpha]_{\leq}| \geq |\mathcal{L}(\mathfrak{H}_\alpha)|$. Thus $|[\alpha]_{\leq}| = |\mathcal{L}(\mathfrak{H}_\alpha)|$ and this checks the claim.

Now, by Theorem 5.1.3, the number of chains of maximum length of $\text{Camb}^m(c)$ is equal to the size of the class $[w_0^{(m)}(c)]_{\leq}$, and by the claim above, $|[w_0^{(m)}(c)]_{\leq}| = |\mathcal{L}(\mathfrak{H}_m^c)|$. Therefore, the number of chains of maximum length of $\text{Camb}^m(c)$ is equal to the number of linear extensions of the super-heap \mathfrak{H}_m^c as desired. \square

As a consequence of Theorem 5.1.3, Theorem 5.2.4, Theorem 5.2.7, and Theorem 5.3.2, we have the following Corollary.

Corollary 5.3.3. *Suppose Conjecture 5.2.2 holds, let $n, m \in \mathbb{Z}^+$ be positive integers, and let c be a Coxeter element of \mathcal{A}_n . Then the following are equal:*

1. *the number of chains of maximum length of $\text{Camb}^m(c)$,*
2. *the size of the class $[w_0^{(m)}(c)]_{\succeq}$,*
3. *the number of linear extensions of the super-heap \mathfrak{H}_m^c ,*
4. *the number of chains of maximum length of $\text{Camb}^m(\rho^h(c))$,*
5. *the size of the class $[w_0^{(m)}(\rho^h(c))]_{\succeq}$,*
6. *the number of linear extensions of the super-heap $\mathfrak{H}_m^{\rho^h(c)}$,*
7. *the number of chains of maximum length of $\text{Camb}^m(\rho^v(c))$,*
8. *the size of the class $[w_0^{(m)}(\rho^v(c))]_{\succeq}$,*
9. *the number of linear extensions of the super-heap $\mathfrak{H}_m^{\rho^v(c)}$,*
10. *the number of chains of maximum length of $\text{Camb}^m(\rho^v \circ \rho^h(c))$,*
11. *the size of the class $[w_0^{(m)}(\rho^v \circ \rho^h(c))]_{\succeq}$,*
12. *the number of linear extensions of the super-heap $\mathfrak{H}_m^{\rho^v \circ \rho^h(c)}$.*

5.4 Enumerating the Chains of Maximum Length of the m -eralized Cambrian

Lattices of \mathcal{A}_n for $n = 1, 2, 3$, and 4

In this section we find formulas for the number of chains of maximum length for all m -eralized Cambrian lattices of \mathcal{A}_n when $n = 1, 2, 3$, and 4. We break this section into two subsections so that in Subsection 5.4.1, we study the cases when $n = 1, 2, 3$, and in Subsection 5.4.2, we study the case when $n = 4$. Throughout these two subsections, we also state and prove some other new results.

5.4.1 *Enumerating the Chains of Maximum Length of the m -eralized Cambrian Lattices of \mathcal{A}_n for $n = 1, 2$, and 3*

In this subsection, we enumerate the chains of maximum length of the m -eralized Cambrian lattices of \mathcal{A}_n when $n = 1, 2, 3$, and collect some results concerning the linear extensions of the super-heaps corresponding to the maximum elements of these lattices.

Proposition 5.4.1. *For any Coxeter element c of \mathcal{A}_1 or \mathcal{A}_2 , and for any $m \in \mathbb{Z}^+$, the m -eralized Cambrian lattice $\text{Camb}^m(c)$ has a unique chain of maximum length.*

Proof. For \mathcal{A}_1 , there is a unique Coxeter element, namely $c = s_1$. Also for \mathcal{A}_2 , there are two Coxeter elements, namely, $c_1 = s_1s_2$ and $c_2 = s_2s_1$. Since $|[w_0^{(m)}(c)]_{\leq}| = |[w_0^{(m)}(c_1)]_{\leq}| = |[w_0^{(m)}(c_2)]_{\leq}| = 1$, then by Theorem 5.1.3, every m -eralized Cambrian lattice of \mathcal{A}_1 and \mathcal{A}_2 has a single chain of maximum length. \square

The previous proposition has the following corollary.

Corollary 5.4.2. *For any $m \in \mathbb{Z}^+$, there is a unique chain of maximum length of $\mathcal{T}_1^{(m)}$, and a unique chain of maximum length of $\mathcal{T}_2^{(m)}$.*

As a consequence of Proposition 5.4.1 and Theorem 5.3.2, we have the following Corollary.

Corollary 5.4.3. *For any Coxeter element c of \mathcal{A}_1 or \mathcal{A}_2 , and for any $m \in \mathbb{Z}^+$, there is a unique linear extension of the super-heap \mathfrak{H}_m^c .*

Since for any Coxeter elements c and c' of \mathcal{A}_n with $c \simeq c'$, a permutation $\pi \in \mathcal{A}_n$ is c -sortable if and only if π is c' -sortable, then for the case when $n = 3$, the distinct m -eralized Cambrian lattices of \mathcal{A}_3 correspond to the Coxeter elements: $c_1 = s_1s_2s_3$, $c_2 = s_3s_2s_1$, $c_3 = s_3s_1s_2$, and $c_4 = s_2s_1s_3$. In the following propositions and corollaries,

we enumerate the chains of maximum length of the m -eralized Cambrian lattices corresponding to these Coxeter elements.

Proposition 5.4.4. *For the Coxeter element $c_1 = s_1 s_2 s_3$, and for any $m \in \mathbb{Z}^+$, the number of chains of maximum length of $\text{Camb}^m(c_1)$ is equal to 2^{2m-1} .*

Proof. Suppose that m is even. Then $w_0^{(m)}(c_1) = c_1^{2m}$. By Theorem 5.1.3, the number of chains of maximum length of $\text{Camb}^m(c_1)$ is equal to the size of the class $[w_0^{(m)}(c_1)]_{\leq}$. Since s_1 and s_3 commute while neither s_1 and s_2 nor s_2 and s_3 commute, then $w_0^{(m)}(c_1)$ can be seen as:

$$w_0^{(m)}(c_1) = s_1 s_2 \mid s_3 s_1 \mid s_2 \mid s_3 s_1 \mid s_2 \mid s_3 s_1 \mid \cdots \mid s_3 s_1 \mid s_2 s_3, \quad (5.1)$$

where the vertical dividers are used here in order to show which components commute. There are $2m$ s_1 's in $w_0^{(m)}(c_1)$ and (5.1) shows that the number of blocks in $w_0^{(m)}(c_1)$ of the form “ $s_3 s_1$ ” is equal to $2m - 1$. Since $[s_3 s_1]_{\leq}$ is a doubleton class, $[[w_0^{(m)}(c_1)]_{\leq}] = 2^{m-1}$.

Suppose that m is odd. Then $w_0^{(m)}(c_1) = c_1^{2m-2} w_0(c_1)$. By Example 3.1.4, $w_0(c_1) = s_1 s_2 s_3 s_1 s_2 s_1$. Based on the commuting components of $w_0^{(m)}(c_1)$, $w_0^{(m)}(c_1)$ can be seen as:

$$w_0^{(m)}(c_1) = s_1 s_2 \mid s_3 s_1 \mid s_2 \mid s_3 s_1 \mid s_2 \mid s_3 s_1 \mid \cdots \mid s_3 s_1 \mid s_2 s_1. \quad (5.2)$$

In $w_0^{(m)}(c_1)$, s_1 can be found $2m + 1$ times, and (5.2) shows that the dividers cover all of them except the first one from the left and the first one from the right. Thus $[[w_0^{(m)}(c_1)]_{\leq}] = 2^{2m-1}$ and by Theorem 5.1.3, this is equal the number of chains of maximum length of $\text{Camb}^m(c_1)$. That is, in both cases, the number of chains of maximum length of $\text{Camb}^m(c_1)$ is equal to 2^{2m-1} . \square

Note that the m -eralized Cambrian lattice studied in Proposition 5.4.4 is the m -Tamari lattice \mathcal{T}_3^m .

Corollary 5.4.5. *For the Coxeter element $c_1 = s_1s_2s_3$, and for any $m \in \mathbb{Z}^+$, the following are equal:*

1. *the size of the class $[w_0^{(m)}(c_1)]_{\simeq}$,*
2. *the number of the linear extensions of the super-heap $\mathfrak{H}_m^{c_1}$,*
3. 2^{2m-1} .

Proof. It follows directly by Theorem 5.1.3, Proposition 5.4.4 and Theorem 5.3.2. \square

Example 5.4.6. *Consider the Tamari Coxeter element $c = s_1s_2s_3$ of \mathcal{A}_3 . Then $w_0^{(2)}(c) = s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2s_3$, and $\alpha_2^c = (1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3)$. $[\ell(\alpha_2^c)] = [12]$, with*

$$1 \sqsubset 2, 4, 5, 7, 8, 10, 11$$

$$2 \sqsubset 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$$

$$3 \sqsubset 5, 6, 8, 9, 11, 12$$

$$4 \sqsubset 5, 7, 8, 10, 11$$

$$5 \sqsubset 6, 7, 8, 9, 10, 11, 12$$

$$6 \sqsubset 8, 9, 11, 12$$

$$7 \sqsubset 8, 10, 11$$

$$8 \sqsubset 9, 10, 11, 12$$

$$9 \sqsubset 11, 12$$

$$10 \sqsubset 11$$

and

$$11 \sqsubset 12.$$

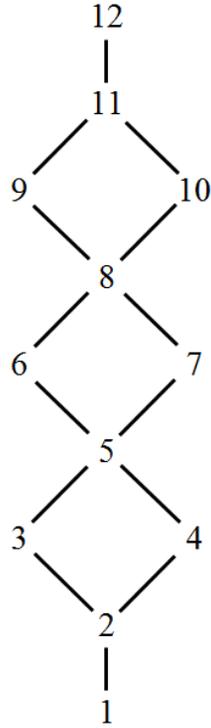


Figure 5.7: The Hasse Diagram of the Super-heap \mathfrak{H}_2^c

The super-heap \mathfrak{H}_2^c is the transitive closure of these relations, and the Hasse diagram of \mathfrak{H}_2^c is shown in Figure 5.7. The linear extensions of \mathfrak{H}_2^c are:

$$1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9 < 10 < 11 < 12$$

$$1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 10 < 9 < 11 < 12$$

$$1 < 2 < 3 < 4 < 5 < 7 < 6 < 8 < 9 < 10 < 11 < 12$$

$$1 < 2 < 4 < 3 < 5 < 6 < 7 < 8 < 9 < 10 < 11 < 12$$

$$1 < 2 < 3 < 4 < 5 < 7 < 6 < 8 < 10 < 9 < 11 < 12$$

$$1 < 2 < 4 < 3 < 5 < 6 < 7 < 8 < 10 < 9 < 11 < 12$$

$$1 < 2 < 4 < 3 < 5 < 7 < 6 < 8 < 9 < 10 < 11 < 12$$

and

$$1 < 2 < 4 < 3 < 5 < 7 < 6 < 8 < 10 < 9 < 11 < 12.$$

Example 5.1.5 also shows the chains of maximum length of $\text{Camb}^2(c)$, and the elements of the class $[\alpha_2^c]_{\leq}$.

Since $s_3s_2s_1 = \rho^v(s_1s_2s_3)$, then by Theorem 5.1.3, Proposition 5.4.4, and Corollary 5.3.3, we have the following corollary.

Corollary 5.4.7. *Suppose Conjecture 5.2.2 holds, then for the Coxeter element $c_2 = s_3s_2s_1$, and for any $m \in \mathbb{Z}^+$, the following are equal:*

1. the size of the class $[w_0^{(m)}(c_2)]_{\leq}$,
2. the number of chains of maximum length of $\text{Camb}^m(c_2)$,
3. the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_2}$,
4. 2^{2m-1} .

Proposition 5.4.8. *For the Coxeter element $c_3 = s_3s_1s_2$, and for any $m \in \mathbb{Z}^+$, the number of chains of maximum length of $\text{Camb}^m(c_3)$ is equal to 2^{2m} .*

Proof. If m is even, then $w_0^{(m)}(c_3) = c_3^{2m}$. If m is odd, then $w_0^{(m)}(c_3) = c_3^{2m-2}w_0(c_3)$, where, by Algorithm 3.1.2, $w_0(c_3) = c_3^2$ which implies that $w_0^{(m)}(c_3) = c_3^{2m}$ as well. By Theorem 5.1.3, in order to find the number of chains of maximum length of $\text{Camb}^m(c_3)$, we find the size of the class $[w_0^{(m)}(c_3)]_{\leq}$. By writing $w_0^{(m)}(c_3)$ as

$$w_0^{(m)}(c_3) = |s_3s_1| s_2 |s_3s_1| s_2 |s_3s_1| s_2 \cdots |s_3s_1| s_2,$$

we see that the dividers cover all of the $2m$ s_1 's appearing in $w_0^{(m)}(c_3)$. As $[[s_3s_1]_{\leq}] = 2$, by the multiplication principal of counting we get that $[[w_0^{(m)}(c_3)]_{\leq}] = 2^m$ which is the number of chains of maximum length of $\text{Camb}^m(c_3)$ as desired. \square

Corollary 5.4.9. *For the Coxeter element $c_3 = s_3s_1s_2$, and for any $m \in \mathbb{Z}^+$, the following are equal:*

1. *the size of the class $[w_0^{(m)}(c_3)]_{\succeq}$,*
2. *the number of the linear extensions of the super-heap $\mathfrak{H}_m^{c_3}$,*
3. 2^{2m} .

Proof. It follows by Theorem 5.1.3, Proposition 5.4.8 and Theorem 5.3.2. □

As $s_2s_1s_3 = \rho^v(s_3s_1s_2)$, by Theorem 5.1.3, Proposition 5.4.8 and Corollary 5.3.3, we have the following corollary.

Corollary 5.4.10. *Suppose Conjecture 5.2.2 holds, then for the Coxeter element $c_4 = s_2s_1s_3$, and for any $m \in \mathbb{Z}^+$, the following are equal:*

1. *the number of chains of maximum length of $\text{Camb}^m(c_4)$,*
2. *the size of the class $[w_0^{(m)}(c_4)]_{\succeq}$,*
3. *the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_4}$,*
4. 2^{2m} .

Note that the previous corollary can also be verified without Conjecture 5.2.2 by applying very similar steps to those used to verify Proposition 5.4.8.

5.4.2 Enumerating the Chains of Maximum Length of the m -eralized Cambrian Lattices of \mathcal{A}_4

In this subsection we enumerate the chains of maximum length of the m -eralized Cambrian lattices of \mathcal{A}_4 , and collect results concerning the linear extensions of the

super-heaps corresponding to the maximum elements of these lattices.

The distinct m -eralized Cambrian lattices of \mathcal{A}_4 corresponds to the Coxeter elements: $c_1 = s_1s_2s_3s_4$, $c_2 = s_4s_3s_2s_1$, $c_3 = s_4s_1s_2s_3$, $c_4 = s_4s_3s_1s_2$, $c_5 = s_3s_1s_2s_4$, $c_6 = s_4s_2s_1s_3$, $c_7 = s_2s_1s_3s_4$, and $c_8 = s_3s_2s_1s_4$.

We start with the following theorem.

Theorem 5.4.11. *For the Coxeter element $c_1 = s_1s_2s_3s_4$ and for any $m \in \mathbb{Z}^+$, the number of chains of maximum length of $\text{Camb}^m(c_1)$ is equal to*

$$\frac{2-\sqrt{2}+(22+18\sqrt{2})\nu(m)}{4}(3+2\sqrt{2})^{\frac{5}{2}(m-\nu(m))} + \frac{2+\sqrt{2}+(22-18\sqrt{2})\nu(m)}{4}(3-2\sqrt{2})^{\frac{5}{2}(m-\nu(m))}.$$

Proof. There are two cases to study.

Case 1: m is even. Then $w_0^{(m)}(c_1) = c_1^{\frac{5}{2}m}$. By Theorem 5.1.3, the number of chains of maximum length of $\text{Camb}^m(c_1)$ is equal to the size of the class $[w_0^{(m)}(c_1)]_{\leq}$. Let us study the size of $[c_1^k]_{\leq}$, for any $k \in \mathbb{Z}^+$. If $w = c_1^k$, then w can be seen as k blocks divided by vertical bars as follows

$$w = \overbrace{s_1s_2s_3s_4}^{b_k} \mid \overbrace{s_1s_2s_3s_4}^{b_{k-1}} \mid \overbrace{s_1s_2s_3s_4}^{b_{k-2}} \mid \cdots \mid \overbrace{s_1s_2s_3s_4}^{b_2} \mid \overbrace{s_1s_2s_3s_4}^{b_1}.$$

The first block from the right is denoted by b_1 , the block to the left of b_1 is denoted by b_2 , and so on so that the last block is denoted b_k . Let $A_k = b_k b_{k-1} \cdots b_2 b_1$ and $a_k = |[A_k]_{\leq}|$.

Claim 1: $a_k = 6a_{k-1} - a_{k-2}$

proof of claim: $A_k = b_k A_{k-1}$, and every element of $[A_{k-1}]_{\leq}$ starts from the left by the component s_1s_2 . We count the elements in $[A_k]_{\leq}$ in two steps:

Step 1: As $[b_k s_1 s_2]_{\leq} = \{b_k s_1 s_2, s_1 s_2 s_3 s_1 s_4 s_2, s_1 s_2 s_1 s_3 s_4 s_2, s_1 s_2 s_3 s_1 s_2 s_4, s_1 s_2 s_1 s_3 s_2 s_4\}$, the number of elements of $[A_k]_{\leq}$ which start by initial segments from the set $[b_k s_1 s_2]_{\leq}$ is equal to $5a_{k-1}$.

Step 2: The elements of $[s_3s_4A_{k-2}]_{\subseteq}$ start on the left with $s_3s_4s_1$, $s_3s_1s_4$, $s_1s_3s_4$, $s_3s_1s_2$, or $s_1s_3s_2$. The elements that start with $\mathbf{s_1s_3s_4}$ and $\mathbf{s_1s_3s_2}$ in $[s_3s_4A_{k-2}]_{\subseteq}$ commute with $s_1s_2s_3s_1s_2s_4$, and $s_1s_2s_1s_3s_2s_4$ from $[b_ks_1s_2]_{\subseteq}$, where we use boldface symbols here to distinguish the elements of $[s_3s_4A_{k-2}]_{\subseteq}$ from the elements of $[b_ks_1s_2]_{\subseteq}$. The new elements obtained from such commutations follow the patterns $s_1s_2s_3s_1s_2\mathbf{s_1s_4s_3s_4} \cdots$, $s_1s_2s_1s_3s_2\mathbf{s_1s_4s_3s_4} \cdots$, $s_1s_2s_3s_1s_2\mathbf{s_1s_4s_3s_2} \cdots$, and $s_1s_2s_1s_3s_2\mathbf{s_1s_4s_3s_2} \cdots$. In fact these elements are exactly the elements of $[A_{k-1}]_{\subseteq}$ except the elements that appear as $b_{k-1} \cdots$. The number of these elements is equal to $a_{k-1} - a_{k-2}$.

By Step 1 and Step 2, $a_k = 5a_{k-1} + a_{k-1} - a_{k-2}$ and hence $a_k = 6a_{k-1} - a_{k-2}$ as desired. Claim 1 is checked.

In order to solve this recursive relation we need two initial conditions. By studying the classes $[A_1]_{\subseteq}$ and $[A_2]_{\subseteq}$, we find that $a_1 = 1$ and $a_2 = 5$. Thus, $a_k = \frac{2-\sqrt{2}}{4}(3+2\sqrt{2})^k + \frac{2+\sqrt{2}}{4}(3-2\sqrt{2})^k$. Since $|[w_0^{(m)}(c_1)]_{\subseteq}| = a_{\frac{5}{2}m}$, then

$$|[w_0^{(m)}(c_1)]_{\subseteq}| = \frac{2-\sqrt{2}}{4}(3+2\sqrt{2})^{\frac{5}{2}m} + \frac{2+\sqrt{2}}{4}(3-2\sqrt{2})^{\frac{5}{2}m},$$

which is equal to the number of chains of maximum length of $\text{Camb}^m(c_1)$.

Case 2: m is odd. Then $w_0^{(m)}(c_1) = c_1^{\frac{5}{2}(m-1)}w_0(c_1)$. As in Case 1, we need to study the size of $[w_0^{(m)}(c_1)]_{\subseteq}$. We start with studying the size of $[c_1^k w_0(c_1)]_{\subseteq}$ for any $k \in \mathbb{N}$. By Algorithm 3.1.2, $w_0(c_1) = s_1s_2s_3s_4s_1s_2s_3s_1s_2s_1$. If $w = c_1^k w_0(c_1)$, then w can be seen as $k+1$ blocks divided by vertical bars as follows,

$$w = \overbrace{s_1s_2s_3s_4}^{b_k} \mid \overbrace{s_1s_2s_3s_4}^{b_{k-1}} \mid \overbrace{s_1s_2s_3s_4}^{b_{k-2}} \mid \cdots \mid \overbrace{s_1s_2s_3s_4}^{b_2} \mid \overbrace{s_1s_2s_3s_4}^{b_1} \mid \overbrace{w_0(c_1)}^{b_0}.$$

We denote the block $w_0(c_1)$ by b_0 , the block to the left of b_0 is denoted by b_1 , the block to the left of b_1 is denoted by b_2 , and so on so that the first block from the left is denoted b_k . Let $A_k = b_k b_{k-1} \cdots b_2 b_1$ with $a_k = |[A_k]_{\subseteq}|$, $A_k^* = b_k b_{k-1} \cdots b_1 w_0(c_1)$

with $a_k^* = |[A_k^*]_{\subseteq}|$, and $A_k'' = b_k b_{k-1} \cdots b_1 s_1 s_2 s_3 s_1 s_2$ with $a_k'' = |[A_k'']_{\subseteq}|$.

Claim 2: $a_k'' = \frac{1}{2}(a_{k+2} - a_{k+1})$.

proof of claim: Consider the class $[A_{k+2}]_{\subseteq}$. Every element in $[A_{k+2}]_{\subseteq}$ ends from the right with $s_3 s_4$. Also s_4 from block b_2 commute with s_1 and s_2 from block b_1 . Thus $|[A_{k+2}]_{\subseteq}| = |B_k|$, where $B_k = [b_k b_{k-1} \cdots b_1 s_1 s_2 s_3 s_1 s_2 s_4]_{\subseteq}$. The elements of B_k follow the patterns $\cdots s_4$, $\cdots s_4 s_2$, or $\cdots s_4 s_1 s_2$. Then B_k can be partitioned into three sets Y_1, Y_2 , and Y_3 , where Y_1 consists of the elements of B_k that ends with s_4 , Y_2 consists of the elements of B_k that ends with $s_4 s_2$, and Y_3 consists of the elements of B_k that ends with $s_4 s_1 s_2$. As $|Y_1| = a_k''$, $|Y_2| = a_k''$, and $|Y_3| = a_{k+1}$, then

$$\begin{aligned}
a_{k+2} &= |[A_{k+2}]_{\subseteq}| \\
&= |B_k| \\
&= |Y_1| + |Y_2| + |Y_3| \\
&= a_k'' + a_k'' + a_{k+1} \\
&= 2a_k'' + a_{k+1}.
\end{aligned}$$

This shows that $a_k'' = \frac{1}{2}(a_{k+2} - a_{k+1})$ as desired. Claim 2 is checked.

Claim 3: $a_k'' = 6a_{k-1}'' - a_{k-2}''$.

proof of claim:

$$\begin{aligned}
6a_{k-1}'' - a_{k-2}'' &= 6 \left(\frac{1}{2}(a_k + a_{k-1}) \right) - \frac{1}{2}(a_{k-1} + a_{k-2}) \quad (\text{by Claim 2}) \\
&= \frac{1}{2}(6a_k + 6a_{k-1} - a_{k-1} - a_{k-2}) \\
&= \frac{1}{2}((6a_k - a_{k-1}) + (6a_{k-1} - a_{k-2})) \\
&= \frac{1}{2}(a_{k+1} + a_k) \\
&= a_k'' \quad (\text{by Claim 2}).
\end{aligned}$$

Claim 4: $a_k^* = 6a_{k-1}^* - a_{k-2}^*$.

proof of claim: Since every element in $[A_k^*]_{\subseteq}$ ends with s_1 from the right, then $a_k^* =$

a''_{k+1} , which implies that

$$\begin{aligned}
6a_k^* - a_{k-2}^* &= 6a''_{k+1} - a''_{k-1} \\
&= a''_{k+1} \quad (\text{by Claim 3}) \\
&= a_k^*.
\end{aligned}$$

The claim is checked.

$$\text{Finally, as } a_0^* = 12 \text{ and } a_1^* = 70, a_{k-1}^* = \frac{24+17\sqrt{2}}{4}(3+2\sqrt{2})^k + \frac{24-17\sqrt{2}}{4}(3-2\sqrt{2})^k.$$

Since $|[w_0^{(m)}]_{\leq}| = a_{\frac{5}{2}(m-1)}^*$, then

$$|[w_0^{(m)}(c_1)]_{\leq}| = \frac{24+17\sqrt{2}}{4}(3+2\sqrt{2})^{\frac{5}{2}(m-1)} + \frac{24-17\sqrt{2}}{4}(3-2\sqrt{2})^{\frac{5}{2}(m-1)},$$

which is, by Theorem 5.1.3, the number of chains of maximum length of $\text{Camb}^m(c_1)$.

By combining Case 1 and Case 2 together, the number of chains of maximum length of $\text{Camb}^m(c_1)$ is equal to

$$\frac{2-\sqrt{2}+(22+18\sqrt{2})\nu(m)}{4}(3+2\sqrt{2})^{\frac{5}{2}(m-\nu(m))} + \frac{2+\sqrt{2}+(22-18\sqrt{2})\nu(m)}{4}(3-2\sqrt{2})^{\frac{5}{2}(m-\nu(m))}$$

as desired. □

Corollary 5.4.12. *For the Coxeter element $c_1 = s_1s_2s_3s_4$ and for any $m \in \mathbb{Z}^+$, the following are equal:*

1. *the size of the class $[w_0^{(m)}(c_1)]_{\leq}$,*
2. *the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_1}$,*
3. $\frac{2-\sqrt{2}+(22+18\sqrt{2})\nu(m)}{4}(3+2\sqrt{2})^{\frac{5}{2}(m-\nu(m))} + \frac{2+\sqrt{2}+(22-18\sqrt{2})\nu(m)}{4}(3-2\sqrt{2})^{\frac{5}{2}(m-\nu(m))}.$

Proof. It follows by Theorem 5.1.3, Theorem 5.4.11 and Theorem 5.3.2. □

The m -eralized Cambrian lattice discussed in Theorem 5.4.11 and Corollary 5.4.12 above is the m -eralized Tamari lattice \mathcal{T}_4^m . Theorem 5.4.11 also has the following corollary.

Corollary 5.4.13. *Suppose Conjecture 5.2.2 holds, then for the Coxeter element $c_2 = s_4s_3s_2s_1$, and for any $m \in \mathbb{Z}^+$, the following are equal:*

1. *the number of chains of maximum length of $\text{Camb}^m(c_2)$,*
2. *the size of the class $[w_0^{(m)}(c_2)]_{\simeq}$,*
3. *the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_2}$,*
4. $\frac{2-\sqrt{2}+(22+18\sqrt{2})\nu(m)}{4}(3+2\sqrt{2})^{\frac{5}{2}(m-\nu(m))} + \frac{2+\sqrt{2}+(22-18\sqrt{2})\nu(m)}{4}(3-2\sqrt{2})^{\frac{5}{2}(m-\nu(m))}$.

Proof. Since $c_2 = \rho^v(c_1)$, where c_1 is as in Theorem 5.4.11, the corollary follows by Theorem 5.1.3, Theorem 5.4.11, Theorem 5.2.4, and Theorem 5.3.2.. \square

Corollary 5.4.13 above can also be proved by using very similar arguments to those used to prove Theorem 5.4.11. In the following theorem, we find a formula for the number of chains of maximum length of the m -eralized Cambrian lattice corresponding to the Coxeter element $c_3 = s_4s_1s_2s_3$.

Theorem 5.4.14. *For the Coxeter element $c_3 = s_4s_1s_2s_3$, and for any $m \in \mathbb{Z}^+$, the number of chains of maximum length of $\text{Camb}^m(c_3)$ is equal to*

$$\frac{(41+29\sqrt{2})\nu(m)}{2}(3+2\sqrt{2})^{\frac{5}{2}(m-\nu(m))} + \frac{(41-29\sqrt{2})\nu(m)}{2}(3-2\sqrt{2})^{\frac{5}{2}(m-\nu(m))}.$$

Proof. Suppose that m is even. Then $w_0^{(m)}(c_3) = c_3^{\frac{5}{2}m}$. If $w = c_3^k$, then w can be seen as k blocks divided by vertical bars as follows

$$w = s_4s_1s_2s_3 \mid s_4s_1s_2s_3 \mid s_4s_1s_2s_3 \mid \cdots \mid s_4s_1s_2s_3.$$

By shifting each vertical divider three steps to the left we get

$$w = s_4 \mid s_1 s_2 s_3 s_4 \mid s_1 s_2 s_3 s_4 \mid s_1 s_2 s_3 s_4 \mid \cdots \mid s_1 s_2 s_3.$$

Construct the element w' from w by attaching s_4 to the right end of w . That is,

$$w' = s_4 \mid \overbrace{s_1 s_2 s_3 s_4}^{b_k} \mid \overbrace{s_1 s_2 s_3 s_4}^{b_{k-1}} \mid \overbrace{s_1 s_2 s_3 s_4}^{b_{k-2}} \mid \cdots \mid \overbrace{s_1 s_2 s_3 s_4}^{b_2} \mid \overbrace{s_1 s_2 s_3 s_4}^{b_1}.$$

Since every element in $[w]_{\leq}$ ends with s_3 on the right and since $|[s_3 s_4]_{\leq}| = 1$, then $|[w']_{\leq}| = |[w]_{\leq}|$. We denote the first block of w' from the right by b_1 , the block appearing to the left of b_1 is denoted by b_2 and so on so that the last block to the right of the first vertical divider from the left is denoted by b_k . Let $A_k = b_k b_{k-1} \cdots b_2 b_1$ and $a_k = |[A_k]_{\leq}|$. Also let $A'_k = s_4 b_k b_{k-1} \cdots b_2 b_1$ and $a'_k = |[A'_k]_{\leq}|$. We checked in the proof of Theorem 5.4.11 that $a_k = 6a_{k-1} - a_{k-2}$.

Claim 1: $a'_k = \frac{1}{2}(a_{k+1} + a_k)$

proof of claim: Since every element in $[A_{k+1}]_{\leq}$ starts from the left by $s_1 s_2$, then $|[A_{k+1}]_{\leq}| = |[s_3 s_4 A_k]_{\leq}|$. Since the elements of $[A'_k]_{\leq}$ follow the patterns $s_4 s_1 s_2 \cdots$, $s_1 s_4 s_2 \cdots$, or $s_1 s_2 s_4 \cdots$, then $[A'_k]_{\leq}$ can be partitioned into three sets T'_1, T'_2 , and T'_3 , where T'_1 consists of the elements of $[A'_k]_{\leq}$ which start from the left by $s_4 s_1 s_2$, T'_2 consists of the elements of $[A'_k]_{\leq}$ which start from the left by $s_1 s_4 s_2$, and T'_3 consists of the elements of $[A'_k]_{\leq}$ which start from the left by $s_1 s_2 s_4$. Similarly, the elements of the set $[s_3 s_4 A_k]_{\leq}$ follow the patterns $s_3 s_4 s_1 s_2 \cdots$, $s_3 s_1 s_4 s_2 \cdots$, $s_1 s_3 s_4 s_2 \cdots$, $s_3 s_1 s_2 s_4 \cdots$, or $s_1 s_3 s_2 s_4 \cdots$, so that $[s_3 s_4 A_k]_{\leq}$ can be partitioned into three sets T_1, T_2 , and T_3 , where T_1 consists of the elements of $[s_3 s_4 A_k]_{\leq}$ which start from the left by $s_3 s_4 s_1 s_2$, T_2 consists of the elements of $[s_3 s_4 A_k]_{\leq}$ which start from the left by $s_3 s_1 s_4 s_2$ or $s_1 s_3 s_4 s_2$, and T_3 consists of the elements of $[s_3 s_4 A_k]_{\leq}$ which start from the left by $s_3 s_1 s_2 s_4$ or

$s_1s_3s_2s_4$. Note that $|T_1| = |T'_1| = a_k$, $|T_2| = 2|T'_2|$, and $|T_3| = 2|T'_3|$. Thus,

$$\begin{aligned}
a_{k+1} + a_k &= |[s_3s_4A_k]_{\succeq}| + |T'_1| \\
&= |T_1 \cup T_2 \cup T_3| + |T'_1| \\
&= (|T_1| + |T_2| + |T_3|) + |T'_1| \\
&= (|T'_1| + 2|T'_2| + 2|T'_3|) + |T'_1| \\
&= 2(|T'_1| + |T'_2| + |T'_3|) \\
&= 2|[A'_k]_{\succeq}| \\
&= 2a'_k.
\end{aligned}$$

Thus $a'_k = \frac{1}{2}(a_{k+1} + a_k)$, and the claim is checked.

Now, by Claim 1 above, we have

$$\begin{aligned}
6a'_{k-1} - a'_{k-2} &= 6\left(\frac{1}{2}(a_k + a_{k-1})\right) - \frac{1}{2}(a_{k-1} + a_{k-2}) \\
&= \frac{1}{2}(6a_k + 6a_{k-1} - a_{k-1} - a_{k-2}) \\
&= \frac{1}{2}((6a_k - a_{k-1}) + (6a_{k-1} - a_{k-2})) \\
&= \frac{1}{2}(a_{k+1} + a_k) \\
&= a'_k.
\end{aligned}$$

This shows that a'_k and a_k have the same recursive relation, hence $a'_k = \alpha(3 + 2\sqrt{2})^k + \beta(3 - 2\sqrt{2})^k$. Since $a_1 = 1$, and $a_2 = 5$, then by Claim 1 above $a'_2 = 17$. Also $a'_1 = 3$ as $a'_1 = |[s_4s_1s_2s_3s_4]_{\succeq}|$. Thus $\alpha = \beta = 1/2$. By Theorem 5.1.3, the number of chains of maximum length of $\text{Camb}^m(c_3)$ is equal to $|[w_0^{(m)}(c_3)]_{\succeq}|$ which is equal to $a'_{\frac{5}{2}m}$. This implies that the number of chains of maximum length of $\text{Camb}^m(c_3)$, when m is even, is equal to

$$\frac{1}{2}(3 + 2\sqrt{2})^{\frac{5}{2}m} + \frac{1}{2}(3 - 2\sqrt{2})^{\frac{5}{2}m}.$$

Suppose now that m is odd. Then $w_0^{(m)}(c_3) = c_3^{\frac{5}{2}(m-1)}w_0(c_3)$, where by Algorithm 3.1.2, $w_0(c_3) = s_4s_1s_2s_3s_4s_1s_2s_3s_1s_2$. Set $w = c_3^k w_0(c_3)$. Then w can be seen as

$$w = s_4 \mid \overbrace{s_1s_2s_3s_4}^{b_{k+1}} \mid \overbrace{s_1s_2s_3s_4}^{b_k} \mid \cdots \mid \overbrace{s_1s_2s_3s_4}^{b_1} \mid \overbrace{s_1s_2s_3s_1s_2}^{b_0}.$$

Let b_0 denote the first block of w from the right, namely $b_0 = s_1s_2s_3s_1s_2$, b_1 denote the block to the left of b_0 , and so on so that b_k is the block of w to the right of the first vertical bar from the left. Let $A_k'' = b_k b_{k-1} \dots b_1 b_0$ and $a_k'' = |[A_k'']_{\leq}|$. Also, let $A_k''' = s_4 A_k''$ and $a_k''' = |[A_k''']_{\leq}|$. By Claim 2 and Claim 3 in the proof of Theorem 5.4.11, $a_k'' = \frac{1}{2}(a_{k+2} - a_{k+1})$ and $a_k''' = 6a_{k-1}'' - a_{k-2}''$.

Claim 2: $a_k''' = \frac{1}{2}(a_{k+1}'' - a_k'')$

proof of claim: This follows exactly as the proof of Claim 1 above.

Claim 3: $a_k''' = 6a_{k-1}''' - a_{k-2}'''$

proof of claim:

$$\begin{aligned} 6a_{k-1}''' - a_{k-2}''' &= 6 \left(\frac{1}{2}(a_k'' + a_{k-1}'') \right) - \frac{1}{2}(a_{k-1}'' + a_{k-2}'') \quad (\text{by Claim 2}) \\ &= \frac{1}{2}(6a_k'' + 6a_{k-1}'' - a_{k-1}'' - a_{k-2}'') \\ &= \frac{1}{2}((6a_k'' - a_{k-1}'') + (6a_{k-1}'' - a_{k-2}'')) \\ &= \frac{1}{2}(a_{k+1}'' + a_k'') \quad \text{as } a_k'' = 6a_{k-1}'' - a_{k-2}'' \\ &= a_k'''. \end{aligned}$$

By combining Claim 2, Claim 3, the fact that $a_k'' = \frac{1}{2}(a_{k+2} - a_{k+1})$, the recursive relation of a_k , and the fact that $a_1 = 1$ and $a_2 = 5$, we get $a_1''' = 41$ and $a_2''' = 239$.

Thus

$$a_{k+1}''' = \frac{41 + 29\sqrt{2}}{2}(3 + 2\sqrt{2})^k + \frac{41 - 29\sqrt{2}}{2}(3 - 2\sqrt{2})^k.$$

Since $|[w_0^{(m)}(c_3)]_{\leq}| = a_{\frac{5}{2}(m-1)+1}'''$, then by Theorem 5.1.3, the number of chains of maximum length of $\text{Camb}^m(c_3)$ is equal to

$$\frac{41 + 29\sqrt{2}}{2}(3 + 2\sqrt{2})^{\frac{5}{2}(m-1)} + \frac{41 - 29\sqrt{2}}{2}(3 - 2\sqrt{2})^{\frac{5}{2}(m-1)}.$$

Combining the cases when m is even and when m is odd, we find that the number of chains of maximum length of $\text{Camb}^m(c_3)$ is equal to

$$\frac{(41 + 29\sqrt{2})^{\nu(m)}}{2}(3 + 2\sqrt{2})^{\frac{5}{2}(m-\nu(m))} + \frac{(41 - 29\sqrt{2})^{\nu(m)}}{2}(3 - 2\sqrt{2})^{\frac{5}{2}(m-\nu(m))}$$

as desired. \square

Theorem 5.4.14 has the following two corollaries.

Corollary 5.4.15. *For the Coxeter element $c_3 = s_4s_1s_2s_3$, and for any $m \in \mathbb{Z}^+$, the following are equal:*

1. *the size of the class $[w_0^{(m)}(c_3)]_{\succeq}$,*
2. *the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_3}$,*
3. $\frac{(41+29\sqrt{2})^{\nu(m)}}{2}(3 + 2\sqrt{2})^{\frac{5}{2}(m-\nu(m))} + \frac{(41-29\sqrt{2})^{\nu(m)}}{2}(3 - 2\sqrt{2})^{\frac{5}{2}(m-\nu(m))}$.

Proof. It follows by Theorem 5.1.3, Theorem 5.4.14 and Theorem 5.3.2. \square

Corollary 5.4.16. *Suppose Conjecture 5.2.2 holds, then for the Coxeter elements $c_4 = s_4s_3s_1s_2$, $c_7 = s_2s_1s_3s_4$, and $c_8 = s_3s_2s_1s_4$, the following are equal:*

1. *the number of chains of maximum length of $\text{Camb}^m(c_4)$,*
2. *the number of chains of maximum length of $\text{Camb}^m(c_7)$,*
3. *the number of chains of maximum length of $\text{Camb}^m(c_8)$,*
4. *the size of the class $[w_0^{(m)}(c_4)]_{\succeq}$,*
5. *the size of the class $[w_0^{(m)}(c_7)]_{\succeq}$,*
6. *the size of the class $[w_0^{(m)}(c_8)]_{\succeq}$,*
7. *the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_4}$,*

8. the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_7}$,

9. the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_8}$,

$$10. \frac{(41+29\sqrt{2})^{\nu(m)}}{2}(3+2\sqrt{2})^{\frac{5}{2}(m-\nu(m))} + \frac{(41-29\sqrt{2})^{\nu(m)}}{2}(3-2\sqrt{2})^{\frac{5}{2}(m-\nu(m))}.$$

Proof. For the Coxeter element $c_3 = s_4s_1s_2s_3$ as in Theorem 5.4.14, $c_4 = \rho^v \circ \rho^h(c_3)$, $c_7 = \rho^h(c_3)$, and $c_8 = \rho^v(c_3)$. The corollary follows by Theorem 5.1.3, Theorem 5.4.14 and Corollary 5.3.3. \square

Note that some of the quantities in Corollary 5.4.16 may be proved equal independent of Conjecture 5.2.2. In the following theorem, we find a formula for the number of chains of maximum length of the m -eralized Cambrian lattice corresponding to the Coxeter element $c_5 = s_3s_1s_2s_4$.

Theorem 5.4.17. *For the Coxeter element $c_5 = s_3s_1s_2s_4$, and for any $m \in \mathbb{Z}^+$, the number of chains of maximum length of $\text{Camb}^m(c_5)$ is equal to*

$$\frac{2+\sqrt{2}+(138+98\sqrt{2})^{\nu(m)}}{4}(3+2\sqrt{2})^{\frac{5}{2}(m-\nu(m))} + \frac{2-\sqrt{2}+(138-98\sqrt{2})^{\nu(m)}}{4}(3-2\sqrt{2})^{\frac{5}{2}(m-\nu(m))}.$$

Proof. Suppose that m is even. Then $w_0^{(m)}(c_5) = c_5^{\frac{5}{2}m}$. If $w = c_5^k$, then w can be seen as

$$w = \overbrace{s_3s_1s_2s_4}^{d_k} \mid \overbrace{s_3s_1s_2s_4}^{d_{k-1}} \mid \overbrace{s_3s_1s_2s_4}^{b_{k-2}} \mid \cdots \mid \overbrace{s_3s_1s_2s_4}^{d_2} \mid \overbrace{s_3s_1s_2s_4}^{d_1}.$$

We denote the first block of w from the right by d_1 , the block appearing to the left of d_1 is denoted by d_2 and so on so that the last block to the right of the first vertical divider from the left is denoted by d_k . Let $A_k = b_k b_{k-1} \dots b_2 b_1$ and $a_k = |[A_k]_{\succeq}|$ as exactly as in the proof of Theorem 5.4.11. Also let $R_k = d_k d_{k-1} \dots d_2 d_1$ and $r_k = |[R_k]_{\succeq}|$.

Claim 1: $r_k = a_{k+1}$

proof of claim: Consider the representative element $v = b_{k+1}b_k b_{k-1} \cdots b_2 b_1$ of the class $[A_{k+1}]_{\sphericalangle}$, which can be seen as

$$v = \overbrace{s_1 s_2 s_3 s_4}^{b_{k+1}} \mid \overbrace{s_1 s_2 s_3 s_4}^{b_k} \mid \overbrace{s_1 s_2 s_3 s_4}^{b_{k-1}} \mid \cdots \mid \overbrace{s_1 s_2 s_3 s_4}^{b_2} \mid \overbrace{s_1 s_2 s_3 s_4}^{b_1}.$$

Since every element in $[A_{k+1}]_{\sphericalangle}$ starts on the left with $s_1 s_2$ and ends on the right with $s_3 s_4$, then $|[A_{k+1}]_{\sphericalangle}| = |[v']_{\sphericalangle}|$, where v' is obtained from v by deleting $s_1 s_2$ from block b_{k+1} and deleting $s_3 s_4$ from block b_1 . That is, v' has the form:

$$v' = s_3 s_4 \mid s_1 s_2 s_3 s_4 \mid s_1 s_2 s_3 s_4 \mid \cdots \mid s_1 s_2 s_3 s_4 \mid s_1 s_2.$$

Shift each divider two steps to the right, we get

$$v' = s_3 s_4 s_1 s_2 \mid s_3 s_4 s_1 s_2 \mid s_3 s_4 s_1 s_2 \mid \cdots \mid s_3 s_4 s_1 s_2 \mid s_3 s_4 s_1 s_2. \quad (5.3)$$

Note that the number of blocks in Equation (5.3) is equal to k . Since s_1 and s_4 commute, then

$$v' = s_3 s_1 s_4 s_2 \mid s_3 s_1 s_4 s_2 \mid s_3 s_1 s_4 s_2 \mid \cdots \mid s_3 s_1 s_4 s_2 \mid s_3 s_1 s_4 s_2.$$

This shows that $v' = w$. Thus v' is a representative of the class $[R_k]_{\sphericalangle}$, and then

$$\begin{aligned} a_{k+1} &= |[A_{k+1}]_{\sphericalangle}| \\ &= |[v]_{\sphericalangle}| \\ &= |[v']_{\sphericalangle}| \\ &= |[w]_{\sphericalangle}| \\ &= |[R_k]_{\sphericalangle}| \\ &= r_k. \end{aligned}$$

The claim is checked.

Since $|[w_0^{(m)}(c_5)]_{\leq}| = r_{\frac{5}{2}m}$, then by Claim 1 above and by Theorem 5.4.11, the number of chains of maximum length of $\text{Camb}^m(c_5)$ is equal to $a_{\frac{5}{2}m+1}$, which is equal to $\frac{2-\sqrt{2}}{4}(3+2\sqrt{2})^{\frac{5}{2}m+1} + \frac{2+\sqrt{2}}{4}(3-2\sqrt{2})^{\frac{5}{2}m+1}$.

Suppose now that m is odd. Then $w_0^{(m)}(c_5) = c_5^{\frac{5}{2}(m-1)}w_0(c_5)$, where by Algorithm 3.1.2, $w_0(c_5) = s_3s_1s_2s_4s_3s_1s_2s_4s_3s_1$. If $w = c_5^k w_0(c_5)$, then w can be seen as

$$w = \overbrace{s_3s_1s_2s_4}^{d_{k+2}} \mid \overbrace{s_3s_1s_2s_4}^{d_{k+1}} \mid \overbrace{s_3s_1s_2s_4}^{d_k} \mid \cdots \mid \overbrace{s_3s_1s_2s_4}^{d_2} \mid \overbrace{s_3s_1s_2s_4}^{d_1} \mid \overbrace{s_3s_1}^{d_0} .$$

We use the same notations used for blocks in the case above, let $R'_k = d_k d_{k-1} \dots d_1 d_0$ and $r'_k = |[R'_k]_{\leq}|$, also recall the notations A_k^* and a_k^* from the proof of Theorem 5.4.11.

Claim 2: $r'_k = a_{k-1}^*$.

proof of claim: Consider the representative element $v = b_{k-1}b_{k-2}b_{k-3} \dots b_1 w_0(c_1)$ of the class $[A_{k-1}^*]_{\leq}$, where $c_1 = s_1s_2s_3s_4$, and $w_0(c_1) = s_1s_2s_3s_4s_1s_2s_3s_1s_2s_1$. Then v can be seen as:

$$v = \overbrace{s_1s_2s_3s_4}^{b_{k-1}} \mid \overbrace{s_1s_2s_3s_4}^{b_{k-2}} \mid \overbrace{s_1s_2s_3s_4}^{b_{k-3}} \mid \cdots \mid \overbrace{s_1s_2s_3s_4}^{b_1} \mid \overbrace{s_1s_2s_3s_4s_1s_2s_3s_1s_2s_1}^{b_0} .$$

Since every element in $[v]_{\leq}$ starts on the left with s_1s_2 and ends on the right with s_2s_1 , then $|[v]_{\leq}| = |[v']_{\leq}|$, where v' is obtained from v by deleting s_1s_2 from block b_{k-1} and deleting the component s_2s_1 that appears to the most right of block b_0 . That is, v' looks like:

$$v' = s_3s_4 \mid s_1s_2s_3s_4 \mid s_1s_2s_3s_4 \mid \cdots \mid s_1s_2s_3s_4 \mid s_1s_2s_3s_4s_1s_2s_3s_1s_2s_1 .$$

by shifting the dividers two steps to the right, and then adding a divider between the second and the third elements from the right, we get

$$v' = s_3s_4s_1s_2 \mid s_3s_4s_1s_2 \mid s_3s_4s_1s_2 \mid \cdots \mid s_3s_4s_1s_2 \mid s_3s_4s_1s_2 \mid s_3s_1 .$$

Since s_4 commutes with s_1 and s_2 , then

$$v' = s_3 s_1 s_2 s_4 \mid s_3 s_1 s_2 s_4 \mid s_3 s_1 s_2 s_4 \mid \cdots \mid s_3 s_1 s_2 s_4 \mid s_3 s_1 s_2 s_4 \mid s_3 s_1. \quad (5.4)$$

Note that the number of blocks in Equation (5.4) is equal to $k + 1$. This shows that v' is a representative of the class $[R'_k]_{\simeq}$. Now,

$$\begin{aligned} a_{k-1}^* &= |[A_{k-1}^*]_{\simeq}| \\ &= |[v]_{\simeq}| \\ &= |[v']_{\simeq}| \\ &= |[R'_k]_{\simeq}| \\ &= r'_k, \end{aligned}$$

which checks the claim.

Since $|[w_0^{(m)}(c_5)]| = r'_{\frac{5}{2}(m-1)+2} = a_{\frac{5}{2}(m-1)+1}^*$, then by Theorem 5.4.11, we conclude that the number of chains of maximum length of $\text{Camb}^m(c_5)$ is equal to $\frac{24+17\sqrt{2}}{4}(3 + 2\sqrt{2})^{\frac{5}{2}(m-1)+2} + \frac{24-17\sqrt{2}}{4}(3 - 2\sqrt{2})^{\frac{5}{2}(m-1)+2}$. By combining the two cases together, we find that the number of chains of maximum length of $\text{Camb}^m(c_5)$ is equal to

$$\frac{2+\sqrt{2}+(138+98\sqrt{2})\nu(m)}{4}(3 + 2\sqrt{2})^{\frac{5}{2}(m-\nu(m))} + \frac{2-\sqrt{2}+(138-98\sqrt{2})\nu(m)}{4}(3 - 2\sqrt{2})^{\frac{5}{2}(m-\nu(m))}$$

as desired. \square

Theorem 5.4.17 has the following two corollaries.

Corollary 5.4.18. *For the Coxeter element $c_5 = s_3 s_1 s_2 s_4$, and for any $m \in \mathbb{Z}^+$, the following are equal:*

1. *the size of the class $[w_0^{(m)}(c_5)]_{\simeq}$,*
2. *the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_5}$,*

$$3. \frac{2+\sqrt{2}+(138+98\sqrt{2})\nu(m)}{4}(3+2\sqrt{2})^{\frac{5}{2}(m-\nu(m))} + \frac{2-\sqrt{2}+(138-98\sqrt{2})\nu(m)}{4}(3-2\sqrt{2})^{\frac{5}{2}(m-\nu(m))}.$$

Proof. It follows by Theorem 5.1.3, Theorem 5.4.17 and Theorem 5.3.2. \square

Corollary 5.4.19. *Suppose Conjecture 5.2.2 holds, then for the Coxeter element $c_6 = s_4s_2s_1s_3$, and for any $m \in \mathbb{Z}^+$, the following are equal:*

1. *the number of chains of maximum length of $\text{Camb}^m(c_6)$,*
2. *the size of the class $[w_0^{(m)}(c_6)]_{\simeq}$,*
3. *the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_6}$,*
4. $\frac{2+\sqrt{2}+(138+98\sqrt{2})\nu(m)}{4}(3+2\sqrt{2})^{\frac{5}{2}(m-\nu(m))} + \frac{2-\sqrt{2}+(138-98\sqrt{2})\nu(m)}{4}(3-2\sqrt{2})^{\frac{5}{2}(m-\nu(m))}.$

Proof. It follows by Theorem 5.1.3, Theorem 5.4.17, Theorem 5.2.4, and Theorem 5.3.2 as $c_6 = \rho^v(c_5)$, where c_5 is as in Theorem 5.4.17. \square

Chapter 6

FUTURE WORK

I plan to keep studying the Cambrian and m -eralized Cambrian lattices, especially their chains of maximum length as well as their chains of other lengths. In Section 6.1, we provide conjectures and corollaries for the number of chains of maximum length for all m -eralized Cambrian lattices of \mathcal{A}_5 .

6.1 Enumerating the Chains of Maximum Length of the m -eralized Cambrian Lattices of \mathcal{A}_5

Our study of the m -eralized Cambrian lattices showed that working on finding recursive relations, and then formulas, is useful to enumerate the chains of maximum lengths. Beside the cases studied in Chapter 5, we have the following conjectures and corollaries for the number of chains of maximum length of the m -eralized Cambrian lattices of \mathcal{A}_5 , for all $m \in \mathbb{Z}^+$. We aim to prove these conjectures, study more cases, and hope to find a general formula for all cases.

The distinct m -eralized Cambrian lattices of \mathcal{A}_5 corresponds to the Coxeter elements: $c_1 = s_1s_2s_3s_4s_5$, $c_2 = s_5s_4s_3s_2s_1$, $c_3 = s_5s_1s_2s_3s_4$, $c_4 = s_5s_4s_3s_1s_2$, $c_5 = s_2s_1s_3s_4s_5$, $c_6 = s_4s_3s_2s_1s_5$, $c_7 = s_4s_1s_2s_3s_5$, $c_8 = s_5s_4s_2s_1s_3$, $c_9 = s_3s_1s_2s_4s_5$, $c_{10} = s_5s_3s_2s_1s_4$, $c_{11} = s_5s_4s_1s_2s_3$, $c_{12} = s_3s_2s_1s_4s_5$, $c_{13} = s_5s_3s_1s_2s_4$, $c_{14} = s_4s_2s_1s_3s_5$, $c_{15} = s_5s_2s_1s_3s_4$, and $c_{16} = s_4s_3s_1s_2s_5$.

Conjecture 6.1.1. For the Coxeter element $c_1 = s_1s_2s_3s_4s_5$ and for any $m \in \mathbb{Z}^+$, the number of chains of maximum length of $\text{Camb}^m(c_1)$ is equal to

$$\alpha(11 + 5\sqrt{5})^{3m-3\nu(m)} + \beta(11 - 5\sqrt{5})^{3m-3\nu(m)} + \frac{(-8)^m}{4},$$

where $\alpha = \frac{5-2\sqrt{5}+(2875+1290\sqrt{5})\nu(m)}{20}$, and $\beta = \frac{5+2\sqrt{5}+(2875-1290\sqrt{5})\nu(m)}{20}$.

Corollary 6.1.2. Suppose Conjecture 6.1.1 holds, then for the Coxeter element $c_1 = s_1s_2s_3s_4s_5$ and for any $m \in \mathbb{Z}^+$, the following are equal:

1. the size of the class $[w_0^{(m)}(c_1)]_{\simeq}$,
2. the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_1}$,
3. $\alpha(11 + 5\sqrt{5})^{3m-3\nu(m)} + \beta(11 - 5\sqrt{5})^{3m-3\nu(m)} + \frac{(-8)^m}{4}$,
where $\alpha = \frac{5-2\sqrt{5}+(2875+1290\sqrt{5})\nu(m)}{20}$, and $\beta = \frac{5+2\sqrt{5}+(2875-1290\sqrt{5})\nu(m)}{20}$.

Proof. It follows by Conjecture 6.1.1, Theorem 5.1.3, and Theorem 5.3.2. □

The m -eralized Cambrian lattice discussed in Conjecture 6.1.1 and Corollary 6.1.2 above is the m -eralized Tamari lattice \mathcal{T}_5^m . Note that the first two quantities in Corollary 6.1.2 are equal by Theorem 5.1.3 and Theorem 5.3.2, and don't rely on Conjecture 6.1.1.

Corollary 6.1.3. Suppose Conjecture 5.2.2 and Conjecture 6.1.1 hold, then for the Coxeter element $c_2 = s_5s_4s_3s_2s_1$ and for any $m \in \mathbb{Z}^+$, the following are equal:

1. the size of the class $[w_0^{(m)}(c_2)]_{\simeq}$,
2. the number of chains of maximum length of $\text{Camb}^m(c_2)$,
3. the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_2}$,

$$4. \alpha(11 + 5\sqrt{5})^{3m-3\nu(m)} + \beta(11 - 5\sqrt{5})^{3m-3\nu(m)} + \frac{(-8)^m}{4},$$

$$\text{where } \alpha = \frac{5-2\sqrt{5}+(2875+1290\sqrt{5})\nu(m)}{20}, \text{ and } \beta = \frac{5+2\sqrt{5}+(2875-1290\sqrt{5})\nu(m)}{20}.$$

Proof. Since $c_2 = \rho^v(c_1)$, where $c_1 = s_1s_2s_3s_4s_5$, the corollary follows by Conjecture 6.1.1, Theorem 5.1.3, Theorem 5.2.4, and Theorem 5.3.2. \square

Note that the first three quantities in Corollary 6.1.3 are equal by Theorem 5.1.3 and Theorem 5.3.2, and don't rely on Conjecture 6.1.1.

Conjecture 6.1.4. *For the Coxeter element $c_3 = s_5s_1s_2s_3s_4$ and for any $m \in \mathbb{Z}^+$, the number of chains of maximum length of $\text{Camb}^m(c_3)$ is equal to*

$$\alpha(11 + 5\sqrt{5})^{3m-3\nu(m)} + \beta(11 - 5\sqrt{5})^{3m-3\nu(m)} + \frac{(-8)^m}{2},$$

$$\text{where } \alpha = \frac{5-\sqrt{5}+(15075+6745\sqrt{5})\nu(m)}{20}, \text{ and } \beta = \frac{5+\sqrt{5}+(15075-6745\sqrt{5})\nu(m)}{20}.$$

Corollary 6.1.5. *Suppose Conjecture 6.1.4 holds, then for the Coxeter element $c_3 = s_5s_1s_2s_3s_4$ and for any $m \in \mathbb{Z}^+$, the following are equal:*

1. the size of the class $[w_0^{(m)}(c_3)]_{\succeq}$,
2. the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_3}$,
3. $\alpha(11 + 5\sqrt{5})^{3m-3\nu(m)} + \beta(11 - 5\sqrt{5})^{3m-3\nu(m)} + \frac{(-8)^m}{2}$,
where $\alpha = \frac{5-\sqrt{5}+(15075+6745\sqrt{5})\nu(m)}{20}$, and $\beta = \frac{5+\sqrt{5}+(15075-6745\sqrt{5})\nu(m)}{20}$.

Proof. It follows by Conjecture 6.1.4, Theorem 5.1.3, and Theorem 5.3.2. \square

The first two quantities in the previous corollary are equal by Theorem 5.1.3 and Theorem 5.3.2. This does not rely on Conjecture 6.1.4. The same in the following corollary, where not all equalities rely on Conjecture 5.2.2 and Conjecture 6.1.4.

Corollary 6.1.6. *Suppose Conjecture 5.2.2 and Conjecture 6.1.4 hold, then for the Coxeter elements $c_4 = s_5s_4s_3s_1s_2$, $c_5 = s_2s_1s_3s_4s_5$, $c_6 = s_4s_3s_2s_1s_5$, and for any $m \in \mathbb{Z}^+$, the following are equal:*

1. *the number of chains of maximum length of $\text{Camb}^m(c_4)$,*
2. *the number of chains of maximum length of $\text{Camb}^m(c_5)$,*
3. *the number of chains of maximum length of $\text{Camb}^m(c_6)$,*
4. *the size of the class $[w_0^{(m)}(c_4)]_{\succeq}$,*
5. *the size of the class $[w_0^{(m)}(c_5)]_{\succeq}$,*
6. *the size of the class $[w_0^{(m)}(c_6)]_{\succeq}$,*
7. *the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_4}$,*
8. *the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_5}$,*
9. *the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_6}$,*
10. $\alpha(11 + 5\sqrt{5})^{3m-3\nu(m)} + \beta(11 - 5\sqrt{5})^{3m-3\nu(m)} + \frac{(-8)^m}{2}$,
where $\alpha = \frac{5-\sqrt{5}+(15075+6745\sqrt{5})\nu(m)}{20}$, and $\beta = \frac{5+\sqrt{5}+(15075-6745\sqrt{5})\nu(m)}{20}$.

Proof. For the Coxeter elements $c_3 = s_5s_1s_2s_3s_4$, $c_4 = \rho^v \circ \rho^h(c_3)$, $c_5 = \rho^h(c_3)$, and $c_6 = \rho^v(c_3)$. The corollary follows by Conjecture 6.1.4, Theorem 5.1.3, and Corollary 5.3.3. □

Conjecture 6.1.7. *For the Coxeter element $c_7 = s_4s_1s_2s_3s_5$ and for any $m \in \mathbb{Z}^+$, the number of chains of maximum length of $\text{Camb}^m(c_7)$ is equal to*

$$\alpha(11 + 5\sqrt{5})^{3m-3\nu(m)} + \beta(11 - 5\sqrt{5})^{3m-3\nu(m)} + \frac{(-8)^m}{2},$$

where $\alpha = \frac{5+\sqrt{5}+(39475+17655\sqrt{5})\nu(m)}{20}$, and $\beta = \frac{5-\sqrt{5}+(39475-17655\sqrt{5})\nu(m)}{20}$.

Corollary 6.1.8. *Suppose Conjecture 6.1.7 holds, then for the Coxeter element $c_7 = s_4s_1s_2s_3s_5$ and for any $m \in \mathbb{Z}^+$, the following are equal:*

1. *the size of the class $[w_0^{(m)}(c_7)]_{\succeq}$,*
2. *the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_7}$,*
3. $\alpha(11 + 5\sqrt{5})^{3m-3\nu(m)} + \beta(11 - 5\sqrt{5})^{3m-3\nu(m)} + \frac{(-8)^m}{2}$,
where $\alpha = \frac{5+\sqrt{5}+(39475+17655\sqrt{5})\nu(m)}{20}$, and $\beta = \frac{5-\sqrt{5}+(39475-17655\sqrt{5})\nu(m)}{20}$.

Proof. It follows by Conjecture 6.1.7, Theorem 5.1.3, and Theorem 5.3.2. □

The first two quantities in Corollary 6.1.8 are equal by Theorem 5.1.3 and Theorem 5.3.2. This does not rely on Conjecture 6.1.7. The same for the following corollary.

Corollary 6.1.9. *Suppose Conjecture 5.2.2 and Conjecture 6.1.7 hold, then for the Coxeter elements $c_8 = s_5s_4s_2s_1s_4$, $c_9 = s_3s_1s_2s_4s_5$, $c_{10} = s_5s_3s_2s_1s_4$, and for any $m \in \mathbb{Z}^+$, the following are equal:*

1. *the number of chains of maximum length of $\text{Camb}^m(c_8)$,*
2. *the number of chains of maximum length of $\text{Camb}^m(c_9)$,*
3. *the number of chains of maximum length of $\text{Camb}^m(c_{10})$,*
4. *the size of the class $[w_0^{(m)}(c_8)]_{\succeq}$,*
5. *the size of the class $[w_0^{(m)}(c_9)]_{\succeq}$,*
6. *the size of the class $[w_0^{(m)}(c_{10})]_{\succeq}$,*
7. *the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_8}$,*
8. *the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_9}$,*

9. the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_{10}}$,

$$10. \alpha(11 + 5\sqrt{5})^{3m-3\nu(m)} + \beta(11 - 5\sqrt{5})^{3m-3\nu(m)} + \frac{(-8)^m}{2},$$

$$\text{where } \alpha = \frac{5+\sqrt{5}+(39475+17655\sqrt{5})\nu(m)}{20}, \text{ and } \beta = \frac{5-\sqrt{5}+(39475-17655\sqrt{5})\nu(m)}{20}.$$

Proof. For the Coxeter elements $c_7 = s_4s_1s_2s_3s_5$, $c_8 = \rho^v \circ \rho^h(c_7)$, $c_9 = \rho^h(c_7)$, and $c_{10} = \rho^v(c_7)$. The corollary follows by Conjecture 6.1.7, Theorem 5.1.3, and Corollary 5.3.3. \square

Conjecture 6.1.10. *For the Coxeter element $c_{11} = s_5s_4s_1s_2s_3$ and for any $m \in \mathbb{Z}^+$, the number of chains of maximum length of $\text{Camb}^m(c_{11})$ is equal to*

$$\frac{5 - \sqrt{5}}{10}(11 + 5\sqrt{5})^{3m} + \frac{5 + \sqrt{5}}{10}(11 - 5\sqrt{5})^{3m}.$$

Corollary 6.1.11. *Suppose Conjecture 6.1.10 holds, then for the Coxeter element $c_{11} = s_5s_4s_1s_2s_3$ and for any $m \in \mathbb{Z}^+$, the following are equal:*

1. the size of the class $[w_0^{(m)}(c_{11})]_{\succeq}$,
2. the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_{11}}$,
3. $\frac{5-\sqrt{5}}{10}(11 + 5\sqrt{5})^{3m} + \frac{5+\sqrt{5}}{10}(11 - 5\sqrt{5})^{3m}$.

Proof. It follows by Conjecture 6.1.10, Theorem 5.1.3, and Theorem 5.3.2. \square

The first two quantities in Corollary 6.1.11 are equal by Theorem 5.1.3 and Theorem 5.3.2. This does not rely on Conjecture 6.1.10. The same in the following corollary.

Corollary 6.1.12. *Suppose Conjecture 5.2.2 and Conjecture 6.1.10 hold, then for the Coxeter element $c_{12} = s_3s_2s_1s_4s_5$ and for any $m \in \mathbb{Z}^+$, the following are equal:*

1. the number of chains of maximum length of $\text{Camb}^m(c_{12})$,

2. the size of the class $[w_0^{(m)}(c_{12})]_{\succeq}$,
3. the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_{12}}$,
4. $\frac{5-\sqrt{5}}{10}(11+5\sqrt{5})^{3m} + \frac{5+\sqrt{5}}{10}(11-5\sqrt{5})^{3m}$.

Proof. Since $c_{12} = \rho^v(c_{11})$, where $c_{11} = s_5s_4s_1s_2s_3$, the corollary follows by Conjecture 6.1.10, Theorem 5.1.3, Theorem 5.2.4, and Theorem 5.3.2. \square

Conjecture 6.1.13. *For the Coxeter element $c_{13} = s_5s_3s_1s_2s_4$ and for any $m \in \mathbb{Z}^+$, the number of chains of maximum length of $\text{Camb}^m(c_{13})$ is equal to*

$$\frac{5+\sqrt{5}}{10}(11+5\sqrt{5})^{3m} + \frac{5-\sqrt{5}}{10}(11-5\sqrt{5})^{3m}.$$

Corollary 6.1.14. *Suppose Conjecture 6.1.13 holds, then for the Coxeter element $c_{13} = s_5s_3s_1s_2s_4$ and for any $m \in \mathbb{Z}^+$, the following are equal:*

1. the size of the class $[w_0^{(m)}(c_{13})]_{\succeq}$,
2. the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_{13}}$,
3. $\frac{5+\sqrt{5}}{10}(11-5\sqrt{5})^{3m} + \frac{5-\sqrt{5}}{10}(11+5\sqrt{5})^{3m}$.

Proof. It follows by Conjecture 6.1.13, Theorem 5.1.3, and Theorem 5.3.2. \square

The first two quantities in Corollary 6.1.14 are equal by Theorem 5.1.3 and Theorem 5.3.2. This does not rely on Conjecture 6.1.7. The same for the first three quantities in the following corollary.

Corollary 6.1.15. *Suppose Conjecture 5.2.2 and Conjecture 6.1.13 hold, then for the Coxeter element $c_{14} = s_4s_2s_1s_3s_5$ and for any $m \in \mathbb{Z}^+$, the following are equal:*

1. the number of chains of maximum length of $\text{Camb}^m(c_{14})$,
2. the size of the class $[w_0^{(m)}(c_{14})]_{\succeq}$,

3. the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_{14}}$,

$$4. \frac{5+\sqrt{5}}{10}(11-5\sqrt{5})^{3m} + \frac{5-\sqrt{5}}{10}(11+5\sqrt{5})^{3m}.$$

Proof. Since $c_{14} = \rho^v(c_{13})$, where $c_{13} = s_1s_2s_3s_4s_5$, the corollary follows by Conjecture 6.1.13, Theorem 5.1.3, Theorem 5.2.4, and Theorem 5.3.2. \square

Conjecture 6.1.16. For the Coxeter element $c_{15} = s_5s_2s_1s_3s_4$ and for any $m \in \mathbb{Z}^+$, the number of chains of maximum length of $\text{Camb}^m(c_{15})$ is equal to

$$\alpha(11+5\sqrt{5})^{3m-3\nu(m)} + \beta(11-5\sqrt{5})^{3m-3\nu(m)} + \frac{(-8)^m}{4},$$

where $\alpha = \frac{5+2\sqrt{5}+(51675+23110\sqrt{5})\nu(m)}{20}$, and $\beta = \frac{5-2\sqrt{5}+(51675-23110\sqrt{5})\nu(m)}{20}$.

Corollary 6.1.17. Suppose Conjecture 6.1.16 holds, then for the Coxeter element $c_{15} = s_5s_2s_1s_3s_4$ and for any $m \in \mathbb{Z}^+$, the following are equal:

1. the size of the class $[w_0^{(m)}(c_{15})]_{\simeq}$,

2. the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_{15}}$,

$$3. \alpha(11+5\sqrt{5})^{3m-3\nu(m)} + \beta(11-5\sqrt{5})^{3m-3\nu(m)} + \frac{(-8)^m}{4},$$

where $\alpha = \frac{5+2\sqrt{5}+(51675+23110\sqrt{5})\nu(m)}{20}$, and $\beta = \frac{5-2\sqrt{5}+(51675-23110\sqrt{5})\nu(m)}{20}$.

Proof. It follows by Conjecture 6.1.16, Theorem 5.1.3, and Theorem 5.3.2. \square

The first two quantities in Corollary 6.1.17 are equal by Theorem 5.1.3 and Theorem 5.3.2. This does not rely on Conjecture 6.1.16. The same for the first three quantities in the following corollary.

Corollary 6.1.18. Suppose Conjecture 5.2.2 and Conjecture 6.1.16 hold, then for the Coxeter element $c_{16} = s_4s_3s_1s_2s_5$ and for any $m \in \mathbb{Z}^+$, the following are equal:

1. the number of chains of maximum length of $\text{Camb}^m(c_{16})$,

2. the size of the class $[w_0^{(m)}(c_{16})]_{\simeq}$,

3. the number of linear extensions of the super-heap $\mathfrak{H}_m^{c_{16}}$,

4. $\alpha(11 + 5\sqrt{5})^{3m-3\nu(m)} + \beta(11 - 5\sqrt{5})^{3m-3\nu(m)} + \frac{(-8)^m}{4}$,

where $\alpha = \frac{5+2\sqrt{5}+(51675+23110\sqrt{5})\nu(m)}{20}$, and $\beta = \frac{5-2\sqrt{5}+(51675-23110\sqrt{5})\nu(m)}{20}$.

Proof. Since $c_{16} = \rho^v(c_{15})$, where $c_{15} = s_5s_2s_1s_3s_4$, the corollary follows by Conjecture 6.1.16, Theorem 5.1.3, Theorem 5.2.4, and Theorem 5.3.2. □

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