An Investigation into the Definitions and Development of Pedagogical Content Knowledge Among Pre-Service and Current Mathematics Teachers

by

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ABSTRACT

The principle purpose of this research was to compare two definitions and assessments of Mathematics Pedagogical Content Knowledge (PCK) and examine the development of that knowledge among pre-service and current math teachers. Seventy-eight current and future teachers took an online version of the Measures of Knowledge for Teaching (MKT) - Mathematics assessment and nine of them took the Cognitively Activating Instruction in Mathematics (COACTIV) assessment. Participants answered questions that demonstrated their understanding of students' challenges and misconceptions, ability to recognize and utilize multiple representations and methods of presenting content, and understanding of tasks and materials that they may be using for instruction. Additionally, participants indicated their college major, institution attended, years of experience, and participation in various other learning opportunities. This data was analyzed to look for changes in knowledge, first among those still in college, then among those already in the field, and finally as a whole group to look for a pattern of growth from pre-service through working in the classroom. I compared these results to the theories of learning espoused by the creators of these two tests to see which model the data supports. The results indicate that growth in PCK occurs among college students during their teacher preparation program, with much less change once a teacher enters the field. Growth was not linear, but best modeled by an s-curve, showing slow initial changes, substantial development during the 2nd and 3rd year of college, and then a leveling off during the last year of college and the first few years working in a classroom. Among current teachers' the only group that demonstrated any measurable growth were teachers who majored in a non-education field. Other factors like internships and

professional development did not show a meaningful correlation with PCK. Even though some of these models were statistically significant, they did not account for a substantial amount of the variation among individuals, indicating that personal factors and not programmatic ones may be the primary determinant of a teachers' knowledge.

TABLE OF CONTENTS

CHAPTER Page

CHAPTER Page

LIST OF TABLES

Table | Page

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Background

Teacher quality, and especially mathematics teacher quality, has been an important issue in the education community since the creation of a comprehensive education system. Accompanying each development and expansion of the education system was a call to better prepare teachers for their new expanded roles. More recently, the introduction of new standards and policies such as State Standards (Blank & Dalkilic, 1990), NCLB (Berry, Hoke, & Hirsch, 2004) and Common Core (McLaughlin, Glaab, & Carrasco, 2014) prompted educational leaders to request teachers with increased capabilities. These 'highly qualified teachers' supposedly can foster increased learning and educational success among their students (Cantrell & Scantelbury, 2011; Darling-Hammond, 2000). The requirements for being "highly qualified" have usually involved university preparation and scores on an assessment of teaching ability, but recently the search for the impact of such qualification systems has led to the development of Value Added Assessment Systems and more nuanced ways of describing and measuring teaching through raters using video tape and rubrics. There has been debate on how large that effect is, with some claiming that teacher qualities show measurable impact on student learning (Chetty, Friedman, & Rockoff, 2014; Metzler & Woessmann, 2012; Rockoff, 2004), while others contend that it may be miniscule when compared to other factors that affect student success such as socio-economic status and parental involvement (American Statistical Association, 2014).

Regarding teacher quality, researchers have reached the conclusion that pedagogical, content, and most especially pedagogical content knowledge, or PCK, do have positive effects on student success (Baumert et al., 2010; Campbell et al., 2014). The definition and description of Mathematical Pedagogical Content Knowledge has been made by several different groups with key centers in Michigan (Ball & Forzani, 2011) and Germany (Baumert et al., 2010), and both models have been tested to show a positive correlation between individual teachers' levels of PCK and student success on standardized tests. However, those descriptions have not been compared and tested to see how compatible they are with each other. There has not been a systematic effort to see if the different research centers in the field are discussing and measuring the same things. Additionally, this information about teacher knowledge has not been connected to what those teachers actually do in the classroom, nor has it had a measurable effect on how we train teachers, with teacher preparation programs requiring roughly the same education classes today as they did in 1930 (Angus, 2001) and the same mathematics classes since the 1990's. It seems evident that there is a need for more information on the definitions of pedagogical content knowledge and its development among current and future educators. Additional study of what components of PCK, such as those defined by Deborah Ball's Mathematical Knowledge for Teaching, are most helpful and how that knowledge that teachers have translates into their classroom activities would improve the preparation of future educators.

A common assumption in each these theories of PCK is that the practices and skills of effective teaching are learnable (Ball & Forzani, 2011). Pre-service programs have long been concerned with giving content knowledge to teachers that they can then pass on to their students. Understanding of student difficulties and misconceptions with the content being learned, the ability to use multiple representations of that content, and a knowledge of different instructional practices and their effectiveness in teaching different topics, are also things that future teachers need to gain and not an inherent ability that some have and others do not. To improve teacher quality, we must find teachers who already have this requisite knowledge and the skills associated with it, or teach these things to future teachers through class work and practical applications. Training highly qualified mathematics teachers, however, is a challenging process.

Teachers learn many things prior to making the decision to become a teacher, through the experience of being a student in a classroom and observing his or her own teachers. In teacher preparation programs or other in-service programs, we teacher classes that share knowledge and can inform practice. Improvement of teaching ability can also occur through the experience of teaching and reflecting on what happened. Research shows that in both situations collaboration with other teachers is a powerful tool for improving teacher pedagogical knowledge, content knowledge, and pedagogical content knowledge. All of these activities interact with each other to make research on the value of individual components difficult (Goldsmith, Doerr, & Lewis, 2014). However, identifying the strengths and weakness of different learning strategies is an important part in improving teacher training. Studying the development of teacher knowledge, and specifically pedagogical content knowledge, should begin before a teacher enters the classroom and take into account the impact that activities in and out of the classroom have on teacher learning.

Theoretical Perspective

Learning in general has been defined as a change in understanding, change in behavior, or changes in participation within a community of practice. These three points of view can all be beneficial in analyzing how and what an individual learns, and each has been researched extensively (Parise & Spillane, 2002). While there have been debates about which of these definitions is the most useful, all of them are problematic when dealing with teachers. The purpose that teachers have in learning is to impact learning done by their students (Horn, 2005) and the situations in which they try and foster learning may be different from the one in which they learned. In other words, they learn to teach in a college setting and then use that knowledge in a k-12 school. This stands in contrast to the traditional view of communities of practice as perpetuating themselves through the learning process, and the view of learning as connected to the environment in which it is gained.

Both of the main descriptions of PCK are made by authors that attempted to define a theoretical perspective that accommodates the interplay of internal and external change. Deborah Ball is considered a constructivist who believes that mathematics teachers develop knowledge for teaching through "pedagogical deliberations" (Ball, 1993). She claimed that teachers need a "bifocal perspective" to perceive the mathematics they are teaching about and the mind of the child they are teaching it to. In this framework, learning comes through reflection on what is to be taught and to whom it is to be taught. Baumert and the COACTIV group believe that teaching is a cognitive activity (Kunter et al., 2013) that is developed through specific training and is not significantly improved upon through the practical experience teachers have during their career. They emphasize their belief that teacher learning is not achieved through socialization into the profession, nor should it be studied based on the individual constructs of the knowledge teachers' gain. In their framework learning occurs in formal teacher training when learners are taught some fact that they memorize and then apply in their classrooms

Both of these frameworks seem to assume that teachers begin learning when they decide to be teachers, either by entering a training program (Kunter et al., 2013) or by reflecting on what it means to teach (Ball, 1993). However, there is evidence that teachers may gain knowledge in many different ways, oft times beginning before they are even teachers (Lortie, 1975). This learning may come through passive observation, active participation, repetition of observed behaviors and reflection on personal practices. In teacher preparation programs, future teachers are assigned professors to give them knowledge and mentor teachers to monitor their application of knowledge in the classroom (Greenberg, Pomerance, & Walsh, 2011). Once in the profession teachers may engage in professional development, collaborate with colleagues in professional learning communities, and reflect on their own classrooms to improve their instruction. Even activities unrelated to their profession, such as their personal relationships and responsibilities, may affect teachers' beliefs and attitudes about education. Unfortunately, all of these activities may be occurring without any learning that will improve teachers' abilities in the classroom taking place (Hill, 2009). Teachers participating in these activities may be learning new things, and even report that they were very useful, but there may not be a direct connection between those experiences that they teacher is having and their actions with their students.

To incorporate these ideas for this dissertation we will define teacher learning as a change in knowledge and practice (either current or in the future) that has an effect on student learning. This definition has also been used to describe transformative learning (Darling-Hammond, 2008; Mezirow, 1997). Not all knowledge gained by a teacher translates into changes in their classroom activities; however, there is some evidence (Begle, 1979; Monk, 1994) of a correlation between teacher knowledge, as measured by certification exams or number of course taken, and student achievement. This definition allows us to look for learning that occurred both during teacher training and gained through the experience of teaching. This also allows us to look for learning in several different ways. First, changes in knowledge can be measured through standardized tests and the aggregated scores that are earned on them. Secondly, changes in practice can be examined through observations made by others and through the teachers' personal descriptions of the knowledge and understanding that they have and the changes that they have made in those things.

The knowledge that is useful for teaching has been grouped into several areas, including pedagogical, content, and pedagogical-content knowledge (Shulman, 1987). Pedagogical knowledge can encompass classroom management, student psychology, lesson planning, and presentation methods. Content knowledge covers the specific skills, processes and abilities utilized in the solving of mathematical problems and explaining of the solutions, from arithmetic to the calculus. Pedagogical content knowledge deals with the intersection of these two groups and is one of the key factors in teacher effectiveness. While there is overlap in the knowledge in these groups, there is evidence that these sets of knowledge can be assessed individually, either through questioning or through observation (Baumert et al., 2010)

Teacher learning and its effect in the classroom can be examined across these areas. Pedagogical knowledge is utilized in classroom management and design, teacher-student interactions, and motivational ability (Tamri, 1988). There is evidence of mathematical content knowledge in a teachers' presentation of the material and the clarity of their explanations (Hill, Rowan, & Ball, 2005). Pedagogical content knowledge is utilized in specific lesson design, understanding and anticipation of student thinking, and clarity of presentation (Hill, Ball, & Schilling, 2008). Because the relationship between knowledge and practice is not a simple one, observations of these behaviors should be supplemented by reflections made by the teacher. These reflections could connect what they do in the classroom with their reasons for doing it. Outside observers, school administrators, and even the students in the class can also make observation of actions by teachers.

Research Perspective

Learning has been categorized as changes that occur in a multitude of different realms; knowledge, understanding, behavior, participation, values, skills, etc. Because of these different definitions, the studying of teacher learning is challenging because the purpose of their learning is to foster the learning of their students, meaning that their changed understanding or behavior must be viewed in relationship to how they changed another persons' understanding or behavior (Rowland, Turner, & Thwaites, 2014). These changes, for both teacher and student, can be brought about by many different factors, which leads to the idea that teacher learning is always situated in the environment and experiences that fostered it. Using the framework of situated cognition allows us to look at teaching as a natural increase in participation within the education community in general and of a specific classroom in a specific school (Putnam & Borko, 2000). This increase in participation is built into our education system, as nearly everyone begins as a student in a school classroom observing a teacher, and after years of preparation returns to a classroom where they are observed by a new set of students.

While this framework fits with my personal view of teacher learning, researchers with different frameworks created the two assessments that we will be using. Therefore, a hoped for outcome of this study will be to provide some empirical evidence to validate the theoretical claims of the different schools of thought. In truth, it may be that these different frameworks may provide insights into each other. Increased participation can be thought of as facilitated by the learning that the teachers experienced prior to being an educator. Viewing that training as the source of all learning is the cognitive framework advocated by Baumert (2010), who disregards the concept that the experience of teaching may significantly improve a teachers' ability to teach in the future. However, during their years of observing and participating in classrooms teachers may have constructed their own understanding of the roles and skills that are necessary to be a teacher. This view of learning fits more closely with the constructivist framework held by Ball (1993) and allows us to examine how the knowledge that was gained prior to the current classroom being observed is mediated by the new experiences they are having (Van Den Brink, 2006). In our study of teacher learning, we will try to find support for ideas from cognitivism, constructivism and situated learning. Cognitive development allows us to assess future teachers understanding and assume that it will have some impact on the classes that they will one day teach. Constructivism allows us to look at learning that took place outside of a teachers' current classroom and treat it as developed through their own reflection. Situated learning lets us look at teacher learning that is occurring from interactions with their current students, and how that knowledge that they have and are currently developing is affecting those same interactions.

The Study

Rationale

It has long been assumed that teacher knowledge is important for their success in the classroom (Barr, 1935; Dewey, 1904; Robinson, 1936). It has been much more difficult to identify what knowledge is important and how it was developed. Mathematics subject matter knowledge was viewed historically as the most important, because it would be impossible to teach something one does not know (Shulman, 1986). However direct correlations between teacher's mathematical knowledge (or proxies for this) and their student success on standardized mathematics tests have been weak at best. Pedagogical knowledge is valued, but disagreements have existed on what constitutes good pedagogy (Wood, 2001). Pedagogical content knowledge shows more value as a predictor of student success than either content or pedagogical knowledge separately, yet there are continuing disagreements about the importance of this or any single teacher characteristic in impacting learning. Additionally, mathematics PCK has been defined and tested differently by several groups, making it difficult to describe what exactly it is.

Research on teacher learning has also been limited, often involving asking teachers what classes they took or what they learned in professional development. In the past twenty years, an alternate approach has developed where teachers were tested to see what they know. From these studies, there is evidence that teachers may be learning how to teach in lots of different ways, or they may not be learning things that will improve their teaching at all. What is missing from the research is a clear delineation of what constitutes pedagogical content knowledge and evidence of how teachers are developing that knowledge. These gaps in our knowledge hinder the educational community's ability to improve the preparation of future teachers and by extension improve the learning of their future students.

Purpose

The principle objective of this study is to compare two different descriptions of PCK and the assessments that have been developed based on those definitions. A secondary objective will be to describe the development of mathematical pedagogical content knowledge from the time a future teacher enters college through the beginning of their teaching career based on the above named assessments. To describe the learning of PCK, data was collected from college students and current teacher measuring their mathematics knowledge for teaching, and this data was analyzed to find the role that time in college and experience teaching have in the growth of PCK scores. Additional statistical and descriptive analysis were run to compare the two assessments, looking for correlation between the assessments as a whole as well as individual sections of the two tests, and the role that other activities may have in affecting teacher knowledge.

CHAPTER 2

REVIEW OF THE RESEARCH

The purpose of this literature review is to describe our current understanding of both mathematics teacher knowledge and learning, and the effects that these things have on their classroom practices and their students' achievement. In the first section, I will describe three different aspects of teachers' knowledge and how that knowledge may be learned. Next I will explain how pre-service teachers may gain that knowledge, and the effect those programs have on teacher learning. We will then look at the connection that learning and knowledge have on the teachers' behaviors and their students' success. Finally, a description of two different assessments of teacher knowledge will be given as well as the role that those assessments play in defining the constructs of PCK.

Teacher Knowledge and Learning

Shulman (1987) provided a listing of seven categories of teacher knowledge that must be connected to practice. These are (a) content knowledge, (b) general pedagogical knowledge, (c) curriculum knowledge, (d) pedagogical content knowledge, (e) knowledge of learners and their characteristics, (f) knowledge of educational contexts, and (g) knowledge of educational ends, purposes, and values. In later work, he would group these differently, however these categories provide us with an avenue to separate areas of knowledge and the practices that employ and demonstrate them. Three of them, content knowledge, pedagogical knowledge, and pedagogical content knowledge, have been referred to as the core dimensions of teacher knowledge, and occupy a large amount of the literature in the field of mathematics education.

Content Knowledge

For the past thirty years, there has been significant interest in research on teacher knowledge of mathematical content, following the field's initial focus on curriculum in mathematics education, and the changes then brought on by the reform movements. Ball (2001) stated, "The claim that teachers' knowledge matters is commonsense. However, the empirical support for this fact has been surprisingly elusive." This concern over the lack of research on teacher content knowledge has been echoed by the National Mathematics Advisory Panel (2008), and more recently by Baumert (2010). The research on teacher learning of content is even more limited. Baumert believes that content knowledge is "acquired through formal training at the university level…and not picked up incidentally." While Baumert (2010) found evidence that teachers in Germany who went through rigorous pre-service mathematics training had greater knowledge than those who did not, he was unable to account for differences in knowledge that may have developed prior to the program, making his assertion questionable. He also speculated, as have others, that this evidence may not be visible in other countries (namely the United States) that lack a unified curriculum and have such varied standards at both the secondary and university levels (Schmidt, Cogan, & Houang, 2014).

A classic way to study content knowledge is to looks at the mathematics courses teachers have taken, their degrees, or their certifications, and uses this as a representation for what or how much mathematics they know. The works by Begle (1979) and Monk (1994) attempted to quantify the effects of coursework on student achievement. Both showed that the number of mathematics courses taken did not serve as clearly or as large a predictor of achievement as expected. Other work, such as the National Commission on

Teaching and America's Future Report (1996) and Ferguson's (1991) study of the Texas Examination of Current Administrators and Teachers showed some more positive correlations between the number of content courses taken or scores on certification tests and student achievement. However, even Ferguson pointed out that the certification test, which measures literacy is a poor representation of mathematical knowledge. He claimed that the resulting value of r^2 was biased and used another formula to calculate that the model accounted for 50% of the fraction of the variation in the dependent variable.

In terms of studying teacher knowledge, the studies of this type seem to suffer from three major flaws. The first is that they do not asses what teachers know, only what they have studied or scored. Secondly, the mathematical knowledge is constructed as a constant that was obtained at some point in the past and remains the same through the intervening years. This does not allow for the development of mathematical knowledge that may occur through the teaching process. Lastly, these studies cannot measure how different levels of content knowledge affect teachers' behaviors in the classroom. While two teachers may have had the same experiences in mathematics, their beliefs, attitudes, and learning derived from other settings may affect how that knowledge is exhibited in their classrooms.

In response to these studies that generalize teacher's knowledge, some researchers looked for methods that are more refined. Ball, Lubienski, and Mewborn (2001) stated, "Many researchers were convinced that teachers' knowledge of mathematics content mattered in ways that were masked by counting numbers of courses. They turned to a closer problem of mathematical knowledge rather than measuring second-order indicators of knowledge." This second type of study identified by Ball offers a more descriptive analysis of teachers' mathematical knowledge. These studies are generally qualitative in nature and more focused on specific knowledge. An example is Lampert and Ball's (1988) work on teachers understanding of place value. For this study, they interviewed pre-service teachers to determine how well they could explain the steps involved in multiplying large numbers, and correct an error in place value. The study found that "In some cases, the pre-service teachers clearly had only partial or incomplete understanding of the role of place value in multiplication." Ma (1999) performed a similar study, but found that Chinese teachers were more adept at explaining the concepts than their American counterparts. Similar studies in other content areas include rational numbers (Post, Harel, Behr, & Lesh, 1991; Tirosh, Fischbein, Graeber, & Wilson, 1999), geometry (Mayberry, 1983; Swafford, Jones, & Thornton, 1997) and proofs (Ma, 1999; Simon & Blume, 1996). A variation of this type of study was performed by Baumert et al. (2010), who collected large amounts of data on teachers' content knowledge covering several mathematical domains. This paper and pencil test "required complex mathematical argumentation or proof" and was part of a longitudinal study, which allowed the teachers tests results to be correlated to their students learning. In general, these studies found that teachers who scored higher on their assessments had students with higher scores on some other standardized assessment.

This second type of study provides a more nuanced understanding of teachers' content knowledge than the first type. They generally involve a onetime assessment of individuals focused on a specific type of mathematical knowledge. However, most do not provide a context for how the material was learned, nor the progression of its development. Several of the studies examined teachers based on predetermined criteria (restricting to only pre-service teachers or practicing teachers of a certain level), but obtained limited information on prior experiences leading to current understandings.

Recently, work has begun to appear in a third category, research on teacher learning of content knowledge. Hill and Ball (2004) used pre and post-test assessments of elementary teachers involved in a summer training program. They were able to document learning patterns, but said "Subsequent analyses should be undertaken that account for the effect on outcomes of both teacher characteristics, such as motivation, educational background, teaching methods, and institute characteristics." Liu and Thompson's work on hypothesis testing (2009) and probability (2007) involved analyzing the work done by teachers in a two-week professional development seminar. The participants were interviewed three different times, sessions were recorded on video, and written work was collected. The results of the studies seemed to indicate that teachers had many misconceptions about the mathematics that they were teaching, and that reflection allowed them to change their understandings (Liu & Thompson, 2007).

This third type of study uses multiple assessments of teacher knowledge, usually on a specific topic, thus providing more information than the second type. They also provide the context for where the material was learned, and the methods that were used in the instruction. A recent goal has also been to connect that understanding of content knowledge with classroom behaviors and student learning. Hill, Rowan, and Ball (2005) measured teacher content knowledge and student test scores in mathematics. While they did not have enough information to assess how the teachers' content knowledge changed during the study, they were able to show that increased teacher content knowledge correlated with increased student scores on standardized assessments.

Pedagogical Knowledge

Along with content knowledge, there has been significant interest in understanding teacher's pedagogical knowledge over the past twenty years. Since the early days of teacher training, preparation in "best practices" or pedagogy has been required for all teachers. This knowledge has been viewed as separate from content knowledge (Ball, 2000), but comparable across different subjects. According to the National Board for Professional Teaching Standards (1998), pedagogy is "the skills teachers use to impart the specialized knowledge/content of their subject area." and includes commitment to student learning, knowledge of teaching methods/principles, managing and monitoring student, and personal reflection. Unfortunately, there is much less written on studying how teachers develop pedagogical knowledge than there has been work done on defining what constitutes good pedagogy. From traditional vs. reform/ constructivist pedagogy (Simon, 1995; Wood, Nelson, & Warfield, 2001) to the California math wars (Klein, 2007) to ethnocentric (D'Ambrosio, 2007; Greer, Mukhopadhyay, Powell, & Nelson-Barber, 2009) and social justice (Burton, 2003; Gutstein, 2003), there is still great debate on what constitutes good pedagogies. While there has been work done examining the process of changing beliefs from one system to another, "Relatively little is known about the characteristics of such teaching itself" (Wood et al., 2001).

Recently Teaching Works (2013) at the University of Michigan developed 19 "High Leverage Practices" that are used by effective teachers in "a broad range of subjects, grade levels, and teaching contexts." These are hoped to become the basis of a common curriculum of teacher development that can be researched and revised to improve future teacher's success. The assumption is that these can be learned while in pre-service training and will allow beginning teachers to progress and become instructors that are more effective. In presenting these foundational items, the director of the program lamented the lack of research that "identified specific instructional practices that should be taught during initial teacher education" or research to indicate how teachers' best learn those practices (Ball & Forzani, 2011).

One of the early theories on how teachers learn pedagogy belongs to Berliner (1988). Based on teacher observations he created five stages in the development of pedagogical knowledge and believed that teachers moved from one stage through another because of experience and interest. Beginning as a novice, teachers then may move to become advanced beginners, competent, proficient, and possibly expert. In 1992, Kagan reviewed the recent literature on the topic, and said that the data mostly supported Berliner's claims, but indicated there was still division on what was causing teachers to move from one stage to another. Along with these observational studies there has been some work done attempting to utilize distal measures of pedagogical ability (Begle, 1979; Darling-Hammond, 2000; Monk, 1994), such as counting the number of classes in pedagogy that the teachers had taken, or the type of professional development they had received, and using these as proxies for the individual pedagogical knowledge of the teacher. Because of a belief that good pedagogy exists separate from content knowledge, other studies have examined teachers in multiple subjects by grouping together the practices they engaged in, making it difficult to determine exactly what knowledge the mathematics teachers have. In Kagan's review (1992), for example, only three of the forty studies looked specifically at mathematics teachers. Yet, effective pedagogy in one subject may not be effective in another (Tamir, 1988), which limits even further the research on how mathematics teachers learn pedagogy.

There is ample evidence that teachers enter the field with significant preconceptions of what good teaching is, developed through many years of “apprenticeship by observation” (Hammerness et al., 2005). These preconceptions are hard to change in teacher training programs, and may persist throughout a teacher’s career. Yet even with these years of preparation 62% of new teachers say they graduated from their school of education unprepared for “classroom realities” (U.S. Department of Education Presentation, 2011). Some would argue that this is proof that teachers do not gain pedagogical knowledge in their teacher training. There have been several studies of pedagogy related to pre-service teachers, but the results have been uninspiring. Vacc and Bright (1999) looked at students introduced to Cognitively Guided Instruction, or CGI in a math methods course, and found that their beliefs changed as a result but their ability to use those beliefs to improve planning was limited. Santagata, Zannoni, and Stigler (2007) had pre-service mathematics teachers look at video-taped lessons, and using a pre-test, post-test system showed that they were more effective at lesson analysis, which may assist the future teachers in being reflective on their own teaching. McGinnis et al. (2002) found that pre-service teachers participating in the Maryland Collaborative for Teacher Preparation, or MCTP, program changed their attitudes and beliefs during their time in the program. The goal of the MCTP is to prepare middle school mathematics and science teachers to be innovative instructors, and students involved reported a change in their comfort with and support of NSF funded reform curriculum materials and goals, which require different pedagogical skills than traditional curriculum. However, McGinnis et al.

(2002) did not measure how these changes in attitudes and beliefs affected actual classroom instruction, stating that the "notion that teachers' attitudes (or preferences) toward mathematics influence their teaching practice has been suggested by researchers" and that by changing attitudes and beliefs in pre-service settings instruction can be assumed to be improved.

In looking at experienced teachers there have been a few studies of pedagogical knowledge gained through professional development. The CGI professional development program (Wilson & Berne, 1999) involved teachers in understanding student thinking on various mathematics topics. Teachers who were involved in the program showed changes in their beliefs about mathematics education, though at various different levels (Franke, Carpenter, Levi, & Fennema, 2001). They also collected student data and found that those teachers who had participated in the program taught differently than those who did not, and their students had better problem solving skills. On a much smaller scale, the Algebra Study Group (Horn, 2005) demonstrated that a group of mathematics teachers working collaboratively could improve their teaching and increase students' success. It was later found that the structure of the conversations, whether they were just checking in to see what the other teachers were doing or if they were invested in analyzing other teachers' efforts to improve their own, has a large effect on the learning that develops (Horn & Little, 2010). One model that could be used for the study of teacher development of pedagogical knowledge in the process of teaching was promoted by Simon (1995, 1999). His work on teacher development experiments, where teachers present a lesson, analyze the effect of the lesson, and the present a revised form of the same lesson, developed accounts of practice that could be used to track changes in pedagogical knowledge over time. However, the method has not been used on a large sample to document this learning.

Defining Pedagogical Content Knowledge

Content and, to a lesser extent, pedagogical knowledge have been the two focal points on research into teacher knowledge for the past three decades. There are specific classes that all pre-service teachers are required to complete and professional development classes offered for current teachers. This implies that they should have concepts, skills, and goals that can be differentiated and assessed. The studies of Begle (1979) and Monk (1994) found that while content courses and education courses taken by a mathematics teacher may have some effect on student learning, "it appears that courses in undergraduate mathematics pedagogy contribute more to pupil performance gains than do courses in undergraduate mathematics" (Monk, 1994). One possible reason for this is the idea that while content courses and educational courses provide content and pedagogical knowledge, subjects that future teachers may have already been learning over years of observation, mathematics methods courses provide teachers with a more focused study of pedagogical content knowledge. It may also be that pre-service teachers, while taking mathematics content courses, view themselves as only a student and may not be focusing on how to utilize the methods demonstrated when they become a teacher. Shulman (1987) defined pedagogical content knowledge as "the blending of content and pedagogy into an understanding of how particular topics, problems or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction." (see

Figure 1) This concept has also been referred to as "Mathematical Knowledge for Teaching" (Ball, 2001) and "Craft Knowledge" (Grimmett & Mackinnon, 1992).

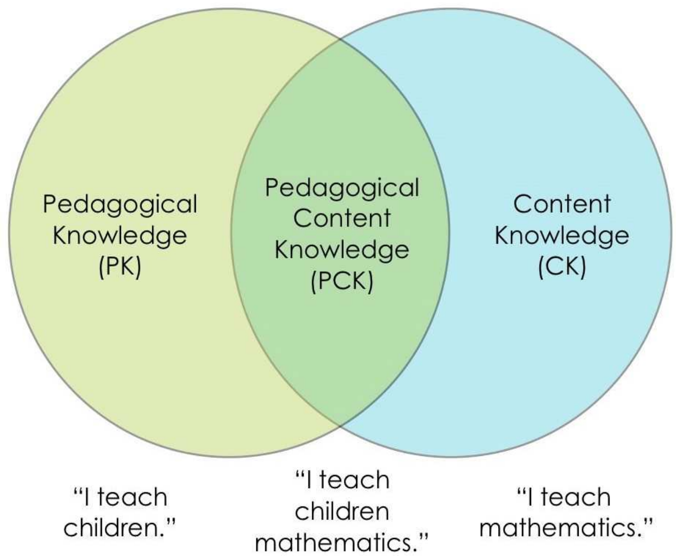

Figure 1. Intersection of Pedagogical and Content Knowledge to Describe Pedagogical Content Knowledge (Hunter, 2013).

Hill, Ball, and Schilling (2008) defined PCK as having three components; Knowledge of Curriculum, Knowledge of Content and Students, and Knowledge of Content and Teaching. They constructed this definition so that all of the areas were distinct from simple content knowledge, because "a teacher might have strong knowledge of the content itself but weak knowledge of how students learn the content or vice versa." Thus, in their construct CK and PCK are separate aspects of teacher knowledge that can be developed independently. Hill et al. (2008) then constructed their assessment, the Measures of Knowledge for Teaching Mathematics or MKT, and attempted to measure teachers' knowledge in these areas. Based on interviews with teachers and some factor analysis they believed that PCK was measurable in their multiple-choice assessment as separate from CK, but there was no "proof of concept" item that could definitively prove it.

Within each sub-section, there were multiple questions to assess teachers' knowledge; however, there may be overlap in the knowledge required. In the understanding students section participants are asked to anticipate what might cause students to have difficulty understanding a problem and look at students work to identify what caused their errors. For example, one question states that Mr. Anderson gave his student the problem $\frac{2(a+1)}{3a} + 3 - \frac{2}{3a} - \frac{6a-2}{6} =$ and one student showed the following

$$\frac{2(a+1)}{3a} + 3 - \frac{2}{3a} - \frac{6a-2}{6}$$

$$= \frac{2a+2}{3a} + 3 - \frac{2}{3a} - \frac{6a}{6} + \frac{2}{6}$$

$$= \frac{2a}{3a} + 2\frac{6}{6} + \frac{2}{6} - a$$

$$= \frac{2}{3} + 2\left(\frac{6}{6} + \frac{1}{6}\right) - a$$

$$= \frac{2}{3} + 2\frac{7}{6} - a$$

$$= \frac{4}{6} + \frac{14}{6} - a$$

$$= \frac{18}{6} - a$$

$$= \underline{\underline{3 - a}}$$

Participants are given four choices for what caused the student error,

A) This student used the distributive property incorrectly

B) This student confounded mixed fractions with factors.

C) This student forgot to cancel common factors in several places

D) This student needs to apply a more formal procedure by finding the common denominator and then adding all terms.

with the correct answer being B. The utilizing multiple representations section requires participants to look at problems and identify correct methods of solving them. One problem presents four different methods that students had used to solve $-5x + 8 = 13x - 10$. Participants had to decide whether the work provided evidence that the student reasoned correctly.

		Provides Evidence of Correct Student Reasoning	**Does Not Provide Evidence of Correct Student Reasoning**
A)	$-5x+8=13x-10$ $8=18x-10$ $18=18x$ $1=x$		
B)	$-5x+8-13x+10=13x-10-13x+10$ $-5x-13x+8+10=0$ $-18x+18-18=0-18$ $\frac{\cancel{1}}{\cancel{-18}}\cdot\frac{\cancel{-18}x}{\cancel{1}}=\frac{\cancel{-18}}{\cancel{1}}\cdot\frac{\cancel{1}}{\cancel{-18}}$ $x=1$		
C)	$-5x+8=13x-10$ $-5\cancel{x}+8=3x$ $+\cancel{5}x \quad +5x$ $\frac{8}{8}=\frac{8x}{8}$ $1=x$		
D)	$-5x+\cancel{8}=13\cancel{x}-10$ $-13x\ \cancel{+8}\quad -\cancel{13}x\ -8$ $\frac{-18x}{-18}=\frac{-18}{-18}$ $x=1$		

In this problem, students A, B and D all showed acceptable work, but student C made an error. To answer this correctly participants need to know three different methods for solving this problem, but they also need to know that you can only combine like terms. The recognition that a student incorrectly combined two terms might be a sign of understanding students just as much as it is a component of understanding methods.

A similar overlap may exist with the last section of understanding curriculum and instruction. This section asks teachers to recognized different instructional materials and the benefits or challenges they may face while using them. For example, one question refers to a teacher using a geoboard to model slope, and a description of a geoboard is included explaining that they are blocks with pins sticking out in a 1 inch grid pattern and are accompanied by rubber bands that can be stretched between the pins to create lines or polygons. In the problem, a student asks the question "Since the diagonal of one of the unit squares has length$\sqrt{2}$, does that mean you can make a line segment with slope $\sqrt{2}$ on the geoboard?" Four student responses are provided, and the participant is asked which statement gives the best insight into the question. The statements are:

Andy: Edward's right that the diagonal of the unit square has length $\sqrt{2}$, but its slope is 1.

Beth: Well, that doesn't matter. We can just turn the geoboard so that the diagonal is horizontal, and then we can see squares with side length $\sqrt{2}$.

Caitlin: Sure, but the square roots of two would just cancel. I think they always would, so you can't get $\sqrt{2}$ as a slope.

Dan: That's not right, because we can make one length of $\sqrt{2}$ and another length of 1 and use them as the rise and run.

This question is designed to test teachers' familiarity with and understanding of geoboards, as evidenced by Caitlin giving the correct answer. It may be possible to answer this question by thinking through the methods described in the answers, reasoning through the limitations of shapes on a geoboard, and identifying which is the most insightful.

In contrast to Ball's work, Baumert's group (Krauss et al., 2008) felt that Content Knowledge was a necessary precondition to developing PCK. They utilized Shulman's 1986 original definition of PCK that it "includes knowledge on how best to represent and formulate the subject to make it comprehensible to others, as well as knowledge on students' subject-specific conceptions and misconceptions." The groups then added a third component of PCK based on research of effective mathematics instruction, namely the appropriate use of tasks as a means of laying students foundations of knowledge. Thus their definition of PCK also has three areas; Knowledge of Mathematical Tasks for Learning, Understanding of Students Conceptions and Misconceptions, and Knowledge of Appropriate Mathematics-Specific Instructional Methods. They also created an assessment, known as Professional Competence of Teachers Cognitively Activating Instruction and the Development of Student's Mathematical Literacy or COACTIV, and by including questions about both content and pedagogical content were able to show that their measurement of PCK was distinct from CK.

This assessment tries to draw a more distinct difference between the categories, even though the questions they are based on may be related. One section of the test starts with the statement "Many student have difficulty accepting the definition $a^0 = 1$". A question from the Understanding Students section asks, "What might be the reasons for this? List as many as possible." The follow up question falls into the Understanding Representations section by asking participants to "outline as many ways (methods) as possible to make this definition accessible to students."

Another page begins with a review of a previous problem. It says that students were told that, “There are **S** students and **P** professors at a university. There are six students to a professor” and that the most common error that students made in representing this problem algebraically was writing “P=6S”. The understanding students question asks participants to “Please give possible reasons for this error being made – what might the students have been thinking?” This is followed by an Understanding Tasks question where teachers are asked to “Please briefly describe possible didactic interventions targeting this error.” There are also additional questions related purely to Content Knowledge, such as “Please prove that $\sqrt{2}$ is irrational” and “Prove that the base angles of an isosceles triangle are congruent.”

These two groups are noteworthy because of their attempts to study their constructs of PCK through assessments and analysis of the results, and the continued use of those assessments by both these groups and other researchers studying the topic; however, there are many other definitions of PCK. Depaepe, Verschaffel, and Kelchtermans (2013) found 51 articles that give a definition, and they identified eight different components that appear in some combination within those descriptions (see Table 1). Some have given assessments to validate their constructs, but other than Ball (2001) and Baumert (2010), most of these were on a small scale (the studies involving pre- service teachers averaged around 68 participants). The differences between these different definitions often were based on the researcher theoretical perspective about PCK, whether it was cognitively gained through specific learning activities or it was situated and developed in the classrooms that teachers work in.

Table 1

Common Components of Pedagogical Content Knowledge

	Ball – Mathematics Knowledge for Teaching	Baumert – Pedagogical Content Knowledge
Students Misconceptions and Difficulties	Knowledge of Content and Students – Anticipating student challenges in doing particular math problems and in providing justifications and explanations, and anticipating challenges due to limited background knowledge	Knowledge of typical conceptions and misconceptions of students, including adequate handling of mistakes and diagnostic competence concerning students' mathematical achievement
Instructional Strategies and Representations	Knowledge of Content and Teaching – Using multiple representations to support mathematical understanding, and using problems that vary in complexity to elicit students mathematical thinking	Explanatory knowledge for use in teaching situations, for instance concerning multiple representations of mathematical entities and flexible knowledge of appropriate reactions in critical teaching situations
Mathematics Tasks and Cognitive Demand	*Not addressed directly*	Knowledge of the cognitive and pedagogical potential of mathematical tasks and process and knowledge of selection and orchestration of tasks
Educational Ends	*Not addressed directly*	*Not addressed directly*
Curriculum and Media	Knowledge of Content and Curriculum – Knowing what instructional materials are available, what approach these materials take and how effective they are.	*Not addressed directly*
Context Knowledge	*Not addressed directly*	*Not addressed directly*
Content Knowledge	Standard Content Knowledge is a pre-requisite for PCK, but both Specialized Content Knowledge and Horizon Content Knowledge	A Profound Understanding of Fundamental Mathematics is considered to be necessary for but distinct from PCK
Pedagogical Knowledge	*Viewed as distinct from MKT*	*Viewed as distinct from PCK*

Studying Pedagogical Content Knowledge

Because of the newness of the topic, the development of an individual's pedagogical content knowledge is understudied. While it likely begins in pre-service training, Ball (2001) theorizes, "Bundles of such knowledge are built up over time by teachers as they teach the same topics to children of certain ages and by researchers as they investigate the teaching and learning of specific mathematical ideas." However, the Berliner study (1986) of novice-expert development contends that: "Experience is a necessary but not a sufficient condition for being an expert." More recently, Baumert (2010) theorized that: "a profound understanding of the subject matter taught is a necessary, but far from sufficient, precondition for providing insightful instruction," which is a sign of effective PCK. Combining these two theories would give us the claim that PCK may be developed when a person with sufficient content knowledge goes through the experience of repeatedly teaching that specific content to students at a specific mathematical level. While this theory is useful, it has been viewed as difficult to prove given the lack of knowledge researchers have on PCK among pre-service teachers and the varied experiences that they have in their university preparation (Speer, King, & Howell, 2015). There is also a proposal that PCK can be developed when an individual gains a key developmental understanding of some mathematical concept and then reflect on how that understanding could be viewed by others (Silverman & Thompson, 2008).

Some of the analysis of teachers PCK has been done by researchers who have also studied content knowledge. For example, Ma's (1999) interviews of teachers found that they had ideas about "the process of opening up and cultivating" students understanding of the multiplication algorithm, yet Post (1991) showed that many teachers could not provide pedagogically sound explanations for their methods of computations with fractions. A great amount of the early research done on the concept of PCK has come from the CGI work done at the University of Wisconsin and Ball's work (2001) on quantifying and measuring in elementary school mathematics. As mentioned previously, the CGI professional development program involved teachers working to understand the different methods that students use to solve problems. Methods of instruction were also discussed as a means for helping students change from one type of thinking to another. The goal of the program was to see if "detailed knowledge about children's thinking and problem solving might affect 'teacher's' knowledge of their own students and their planning of instruction", which they equate with PCK (Carpenter & Fennema, 1992). Ball et al. (2001) developed a framework of teacher's mathematical knowledge, identifying three mathematics areas (everyday, specialized, and knowledge for teaching) and three content areas. In both cases teachers who participated in the professional development programs scored higher on assessments of their PCK than those who did not attend, and their students scored higher on standardized assessments that the students of teachers who did not participate.

Based on the framework of teacher's mathematical knowledge, the CGI Group developed assessments to measure the different categories (Hill, 2004). These assessments were used and refined in the California Mathematics Professional Developments Institutes (Hill, 2005). Teachers in this study who participated in the professional development had higher levels of PCK than those who did not, and this measure of PCK was found to be predictive of teacher success as measured by an evaluation of their teaching by outside observers (see Figure 2) and student success on a standardized assessment. More recently, Baumert (2010) found evidence that pedagogical content knowledge as developed in teacher training is correlated with a teachers' content knowledge. This study also showed a correlation between a teacher's level of PCK and student success in standardized assessments.

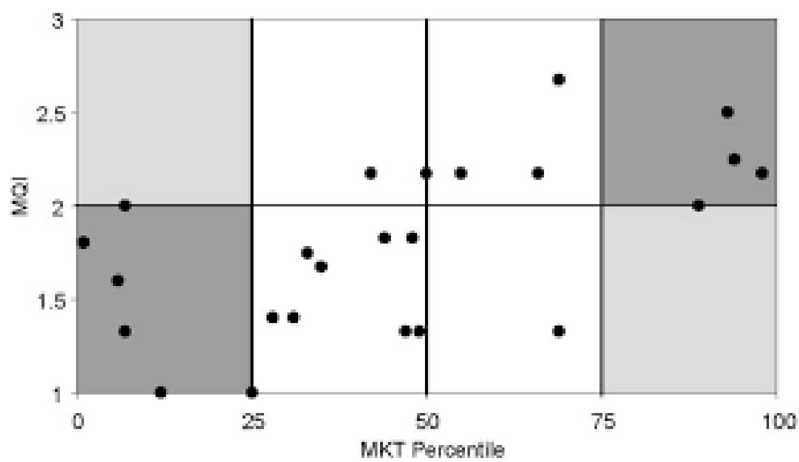

Figure 2. Comparison of Teacher Scores on the MKT Assessment and Their Scores on a Measure of Quality Instruction (MQI). From Hill, Umland, Litke & Kapitula, 2012.

While it is the most recent area of study, there does seem to be a consensus on how pedagogical content knowledge may be learned by teachers. The development of PCK seems to begin in pre-service mathematical methods classes (Baumert, 2010: Monk, 1994). Professional development once teaching has begun, through activities like CGI and collaboration with other mathematics educators, enhances it (Hill, 2005). Finally, the act of teaching itself may help refine it (Ball, 2001). This last point has been theorized by many but has not been investigated with any success. Given the changing nature of teachers' class loads (teaching different subjects from one year to the next), student populations and course requirements, it might take years to document how they learned specific ideas.

Pre-Service Preparation, Student Teaching and Learning

Teacher preparation in the United States has remained relatively constant for the past 100 years (Angus, 2001). Once someone decides to become a teacher they take university classes in the content areas they will be teaching, child psychology and development, and general pedagogical practices. Additional classes in methods of instruction in the subject specialty, or in multiple subjects for elementary school teachers, often accompany internships in K-12 classrooms. This is followed with a student teaching experience, usually once all other course work has been completed but sometimes concurrently with the final classes in the program. Unfortunately, this system has been judged by many researchers and educators as ineffective at preparing high quality teachers (Levine, 2006; NCTQ, 2014). Some possible reason for this may be that teacher preparation programs are not as selective in their admissions as other departments (with more than 82% of undergraduate programs in the US allowing students in with less than a 3.0 GPA and 75% of graduate programs not requiring applicants to have taken the GRE or provide other examples of academic ability), do not teach with the same level of rigor as other college departments (as evidence by the disproportionate percentage of pre-service teachers graduating with honors when compared with students from other fields) and are administered by professors who often lack practical experience in the settings they are preparing their students to teach in (NCTQ, 2013). Additionally, many classes for future teachers are focused on sharing knowledge with them, while this knowledge may not be easily transferable into teaching behaviors they will need in the classroom, like reflection or collaboration (Fairbanks et al., 2009).

While the value of teacher preparation programs is being questioned, many teachers view student teaching as the most valuable part of their program (Wilson, Floden, & Ferrini-Mundy, 2001). While this activity should have an effect on teacher knowledge, there is little documentation of what is actually gained from the experience (Greenberg, Pomerance, & Walsh, 2011). One reason for this gap is the perceived difficulty in observing and collecting this data (Roscoe & Butt, 2010). There are at least two different views on how student teaching experiences should be engineered, with some holding that it is a time for socialization where the student teacher imitates the practices of their mentor (Peterson & Williams, 2008) and others believing that it is a time for personal reflection and experimentation (Kimmer, 2005). Because student teachers and mentor teacher may have different views of their roles, and there is usually little effort to determine those views beforehand, individuals may be learning very different things.

The view of student teaching as a set of requirement to fulfill instead of a learning experience (Hoy & Woolfolk, 1990) may also explain why it is difficult to find evidence of learning, yet there are some positive outcomes that have been observed. The study by Boyd, Grossman, Lankford, Loeb, and Wyckoff (2009) did show a positive effect from student teaching on teacher performance, and possibly teacher learning, but only under certain conditions including having the mentor chosen for them and being closely monitored by their teacher preparation program. Most student teachers finish the process feeling more confident in their teaching ability (Awaya et al., 2003). This confidence may be manifested by an increased concern for their students' success towards the end of the experience than was expressed at the beginning. Additionally, most student teachers gain a greater understanding of the procedures and practices that occur in a school setting (Fives, Hamman, & Olivares, 2006). The gains that have been documented seem to be consisted across settings, however there is no evidence that they are long lasting or have an effect on classroom effectiveness, and none were focused on mathematics teachers (Plourde, 2002; Stockero, 2008).

Teacher Knowledge, Teacher Training and Classroom Practices

One of the accepted beliefs of teacher preparation programs is that they are giving future teachers knowledge that, based on research, will be useful to them in the classroom. Unfortunately, there is limited evidence that links what a teacher knows to what they do in the classroom. This may be because the connection between what we know and what we do is mediated by what we believe (Ng, Nichols, & Williams, 2010). Knowledge is different from belief in that beliefs do not require rigorous external validity or complete consistency (Sadeghi, & Zanjani, 2014). Some researcher hold that beliefs have a greater impact on teacher behavior than knowledge does (Williams, & Burden, 2000; Zheng, 2009). It is also unclear exactly how teachers change their beliefs (Tillema, 2000) and some have argued that some teachers may not change their beliefs regardless of the outside experience they are exposed too (Kagan, 1992). However, evidence that is more recent suggests that teachers can change their beliefs about mathematics if they are provided opportunities for both reflection and discussion (Szydlik, Szydlik, & Benson, 2003).

An additional confounding factor between teacher preparation and classroom practices is the role that in-service learning playing in affecting teacher behaviors. Throughout the world teachers continue to receive training once they enter the profession.

This may include one-time workshops, semester long courses, or ongoing professional learning communities (Walton, Nel, Muller, & Lebeloane, 2014). Some of these activities may be highly effective in changing teachers' knowledge and practices (Fernandez & Yoshida, 2012) while others may be viewed as a "complete waste of time" (Walton et al., 2014).

In-service activities vary greatly between countries. In the United States, individual school districts and schools determine what is required for teachers, either by having school staff offer the training, inviting outside organizations to provide it, or specifying what external sources are acceptable. In Japan, teachers may have complete autonomy in determining what to do to improve their practices. The lesson study model, usually implemented in professional learning communities, has a long history and has been shown to improve teachers' skills (Fernandez & Yoshida, 2012). In Germany, the regional educational authority directs in-service teacher training. While the actual trainings may be held at individual schools and teachers have some autonomy to determine what they will participate in, the topics covered and curriculum used is usually mandated.

The comparison between German and American teacher in-service training is interesting given the models that Baumert (2010) and Ball (2001) espouse regarding the development of PCK. Baumert (2010) believes that knowledge for teaching is developed entirely during pre-service training, yet in Germany teacher training is prescribed and often focused on the development of that knowledge. Ball (2001) believes that mathematical knowledge for teaching develops over time as teachers reflect on their practices and their students; however, in the U.S. there is no systematic effort to encourage teacher reflection.

It may be that both views, in addition to being their personal beliefs about learning, are critiques of the systems in which they were developed.

Teacher Knowledge and Student Achievement

Since the work at Michigan studying mathematical knowledge for teaching there have been several attempts to connect teacher knowledge with student success. The large-scale studies by Hill (2005) and Baumert (2010) have been followed by a number of smaller ones. One difference between the studies have been the grade levels they examined. Baumert (2010) found a significant correlation between German high school teachers' knowledge and their students' success, while Hill (2005) discovered similar results among 3rd grade teachers in California. Others have looked at middle school students in Texas, sixth graders in Peru, and sixth graders in the Mid-West.

The results of these studies have been mixed. For example, Ottmar, Rimm-Kaufman, Larsen, and Berry (2015) were studying teachers' use of the Responsive Classroom Approach to teaching, a standards based teaching strategy. In the course of their research, they also tested teachers' mathematical knowledge for teaching. Ottmar et al. (2015) found that teachers with higher mathematics PCK use more standards based teaching practices, and teachers' use of standards based teaching practices was correlated with higher student test scores. However, teacher knowledge did not correlate to increased student achievement, at least not in any significant way. A lack of effect was also found among fifth graders whose teachers had used an online professional development program (Dash et al., 2012)

On the other side, Campbell (2014) found that there was a significant correlation (with a large effect size) between early career teachers' knowledge and student achievement, but only when they accounted for teacher beliefs and background, and when teacher's instructional assignments were factored in. Larger effects were also found among middle school students in El Paso (Tchoshanov, 2011). Somewhere in the middle of these studies are works from other countries. Small correlations have been found for sixth grade teachers in Peru (Metzler & Woessmann, 2012) and third graders in Guatemala (Marshall & Sorto, 2012).

These mixed results from studies relating teacher knowledge and student achievement come at the same time that there has been significant development in verifying the effects of teachers in general on students. The work by Chetty, Friedman, and Rockoff (2011, 2013, 2014) shows that individual teachers do have a measurable effect on student test scores, college attendance, future earnings and several other factors, when controlling for factors outside of a teacher's control. While the effect may be small, between 1% and 14%, over the lifetime of the student, those variations may have a large impact. While it is possible that students could have a string of highly effective or highly ineffective teachers, a more likely possibility (as Chetty et al., 2013 speculates,) is that the effect of one highly effective teacher may counteract losses accumulated from having poor teachers in the past or strengthen the student again poor teaching in the future.

Assessment of Knowledge

One challenge experienced in the study of both teacher learning and student teachers is the difficulty in assessing teacher knowledge. Researchers have used a sum of mathematics and mathematical methods courses taken as a proxy to teacher knowledge (Monk, 1994). Most states in the U.S. require teachers to pass a competency test of their teaching knowledge, but there is little evidence to suggest that success on those certification tests correlate to success in the classroom (Ferguson, 1991; Ferguson & Brown, 2000; Madaus & Pullin, 1987). There has been recent work to develop valid assessments of teachers' mathematical and pedagogical knowledge and both provide evidence that increased knowledge leads to increased student achievement. The two most effective tools for measuring content and PCK is the COACTIV assessment (Baumert, 2010) and the MKT (Hill, Rowan, & Ball, 2005). COACTIV is an open-ended assessment that was developed in Germany and tested on over 200 secondary mathematics teachers. It was demonstrated that higher levels of CK and PCK as measured on the assessment were correlated with increase student achievement on standardized assessments. This research study also indicated that variations of teachers' knowledge were determined in their teacher preparation program and remained relatively fixed throughout the remainder of their teaching career.

MKT is a multiple choice assessment that correlates well with measure of teacher quality and student achievement (Hill, Umland, Litke, & Kapitula, 2012). Developed at the University of Michigan, it has been used for over ten years at sites across the U.S and has been split into elementary (K-5) and middle school (6-8) levels, with the recent introduction of a High School level (9) Algebra assessment. While the test shows overall validity, Hill (2012) demonstrated that using cut scores of the quartiles allows for useful grouping of teachers. Those in the upper quartile have significantly higher quality lessons and student achievement than those in the lower quartile, while those in the middle two quartiles showed greater variation. Both of these tests have traditionally been administered using paper and pencil with a proctor observing the test takers. There is some evidence that online versions of assessments provide equivalent results (Weigold, Weigold, & Russell, 2013).

Summary

Teachers gain knowledge in many different ways, and utilize many different types of knowledge in their classroom. This knowledge may be gained in teacher preparation programs or student teaching, but their effectiveness may be limited. It has been difficult to connect knowledge with classroom behaviors because teacher beliefs mediate what they know with what they do. However, that knowledge, and most specifically Pedagogical Content Knowledge, may have an impact on their students' achievement. Among the things that we still do not know:

How do the two main frameworks of Pedagogical Content Knowledge compare? Given the multitudes of definitions and possible components, it would be helpful to see if there is overlap between the assessments. If MKT as described by Ball et al. (2001) and PCK as defined by Baumert et al. (2010) are describing the same concept, then there should be a high level of correlation between the assessments that they have made. There should also be a correlation between individuals Content Knowledge and their PCK.

How does mathematics teacher knowledge change and develop? It is speculated that PCK begins to develop in Pre-Service training or possibly earlier, and then grows over time through teacher experience. If this is the case, then experienced teachers should have much higher levels of PCK than beginning ones because, according to Ball's (2001) beliefs, they have had more time to reflect on their experiences and improve their understanding.

However, Baumert (2010) did not find significant growth when correlated only with experience among teachers in Germany. Do the same patterns hold here in the United States? Additionally, we do not know how much mathematics PCK develops during the k-12 education that future teachers receive, or learned during content classes, methods classes, or as a part of student teaching. Are there specific experiences or practices that can be shown to improve PCK, or are at least correlated with higher scores on an assessment of PCK?

Research Questions

1. Does the average level of PCK change for cohorts of students progressing through their pre-service preparation programs and once they enter the teaching profession? If so, how much?
2. If there is a change in level of PCK by cohort, is it possible to measure the effects of coursework, internships and student teaching (for pre-service teachers) and professional development activities (for current teachers) on those changes, given that those things will be standardized for most people from the same institution? (i.e. Taking the same classes and participating in field experiences during the same semesters)
3. How closely correlated are a teachers' scores on the MKT and COACTIV assessments of PCK? Both tests have shown validity as a measure of teacher effects on student achievement, but do they have concurrent validity? While we may find a statistically significant relationship between the two overall scores by running a simple Pearson correlation, there may be relationships between different sections or items from each test.

CHAPTER 3

METHOD

Overview

This study contains two distinct but related sets of data. The first is assessment data from a wide range of pre-service and current teachers answering questions from the MKT assessment. The second is test scores and responses from pre-service and current mathematics teachers answering questions from the COACTIV assessment. These two data sets were collected individually and were correlated to look for patterns of growth and similarities between the assessments.

Table 2

Study Data Schedule

	MKT Assessment	COACTIV Assessment
Who	Current and Pre-Service Teachers	Current and Pre-Service Secondary Education Mathematics Teachers
When	January and February of 2016	March and April of 2016

Measures of Knowledge for Teaching Mathematics

Settings and Participants

In January of 2016 students majoring in mathematics education at two traditional four year teacher preparation programs (University A and University B) and a random selection of current teachers, including teachers at a larger urban high school district, a small group of charter schools, and members of the Arizona Association of Teachers of Mathematics (or AATM) were asked to participate in this research project. Students and teachers were sent an e-mail asking them to complete an online survey based on the questions from the MKT, and one participant was selected at random to receive a $50 gift card. Due to privacy concerns, the invitations to University A students had to be sent by their professors and the e-mails to AATM members were sent through their organization. Thus, it is unknown exactly how many participants were invited to take the survey, or what the response rate was. As an estimate, there were 78 e-mails sent to math education majors at University B, and nine attempted the assessment, giving us a response rate of around 11.5%. Assuming this percentage is roughly consistent among all groups, and given that 90 people took the survey we can estimate that around 800 people were invited to participate.

Instrumentation

All participants completed an online survey containing the same questions as those administer in a recent version of the MKT assessment (see Appendix A). After agreeing to participate in this research, participants began by identifying if they are a pre-service or current teacher, and that lead them to several demographic questions. For pre-service teachers, the questions were: 1) What year are you in school, 2) What institution are you attending, 3) What is your major, 4) How many semesters of internships working in k-12 classrooms have you completed, and 5) How many semesters of student teaching have you completed.

For current teachers the questions were: 1) How many years have you been teaching, 2) What institution did you attend for your teaching degree, 3) What was your bachelors' degree major and 4) What professional development activities related to mathematics education have you participated in within the last five years, with preprogrammed options of professional learning communities, site based training or university classes, and the option to indicate other PD activities.

Participants then indicated the grade level that they currently teach, or planned on teaching, either elementary (K-5) middle school (6-8) or high school (9-12). That choice determined the type and number of questions they were asked, either 19 (for elementary school), 21 (for middle school) or 22 (for high school) questions concerning. Four people took the elementary test and an additional eight took the middle school version, making them eligible for the gift card, however their data was not incorporated into this analysis of PCK. The high school MKT assessments had questions that covered four topics; anticipating student challenges, eliciting and evaluating student work, explaining and using concepts and procedures, and using examples, models and representations. For our calculations we used Depaepe (2013) categories of PCK, grouping the first two topics into understanding students. Based on prior administrations the MKT has a reliability value estimated to be between .71 and .84 (Hill et al., 2004; Ottmar et al., 2015). Prior uses of MKT had incorporated up to three different variations of the assessment; however, only one was used for all participants.

The MKT assessment is accompanied by an answer key to allow for number right scoring, using one point for every correct answer. However, to account for the possibility of participants guessing the answer to the questions a secondary method of scoring was developed using the formula of 3 points for a correct answer to a multiple choice question with four options and 1 point for a correct answer to a binary choice question, while incorrect answers were scored as -1 points and skipped questions received 0 points. The formula scoring model has been debated in the past, but may result in increased reliability of the assessment (Espinosa & Gardeazabal, 2010). This system also allows us to establish zero as the score a person with no pedagogical content knowledge would receive. Prior applications of this assessment have used both the number right (Hill & Ball, 2004) and formula scoring (Hill, Rowan, & Ball, 2005). An initial analysis of the correlation between the two systems was run to see if there was a significant difference between the two, and some analysis was run using both scores.

Statistical Analysis

Using this data alone I attempted to answer several questions. The first is what level of mathematics PCK do future mathematics teachers have when they start their teacher preparation program. It is assumed that in year one of college students already have significantly different learning histories. Those who identify themselves as future mathematics teachers have likely been successful and enjoyed high-level secondary mathematics. My first goal is to explore mean PCK scores using the MKT assessment for beginning college students planning to teach, and then to determine if the score of those beginning future mathematics teachers is different from that of those who have finished their teacher preparation program using a simple t-test. Thus, the null hypothesis is that there are no significant differences in scores on the MKT between those who begin the mathematics teacher preparation program and those who complete it. Accepting this would lead to the conclusion that PCK is not learned in university teacher preparation but is instead developed prior to the collegiate mathematics experience (Hammerness et al., 2005) or after they have begun their careers. Rejection of the null hypothesis would lead to the conclusion that some of mathematics teachers' PCK is learned during teacher training.

My second research question is designed to determine the unique contribution of collegiate experiences to future mathematics teachers' development of PCK. I did this by assessing future mathematics teachers' scores on the MKT and correlating it with the data on their year in the program, major, number of semesters spent interning in schools, and number of semesters of student teaching. Given that most of the participants in the study will have graduated from high school in Arizona, and those from other states will have met secondary education requirements similar to those enforced in Arizona, I assumed that each cohort of teachers would have started with an average PCK assessment score roughly identical to that of current first year participants. While these assumptions, the initial similarity of cohorts and comparable learning experiences in different classes, can be debated, it has been used by other researchers to allow the view of time as an independent variable without running a decades' long longitudinal study (Baumert et al., 2010; Begle, 1979).

In their first two years of the college, the majority of future mathematics teachers are required to take mathematics courses, which may not have any focus on preparing future teachers. In the later semesters, students are usually required to take classes more directly related to methods of mathematics instruction, and these courses should have a more direct effect on mathematics PCK. Thus, we might expect to see growth in the average score year by year, with students in their 4th year of college having a significantly different score, both overall and in certain sub-constructs, then students in their 1st year. For the statistical analysis, I will start by calculating a Pierson Correlation Coefficient between pre-service teachers' year in school, blocking students into whole number year groups, and their score on the MKT, both in aggregate and for the separate subcategories of the MKT assessment.

To evaluate the effect that other factors may have had on the development of PCK among future teachers I ran a multimodal linear regression using institution attending, number of years in the program, major, number of completed semesters interning, and number of completed student teaching semesters as independent variables. While all education students during their pre-service program will be taking mathematics courses and pedagogical courses, mathematics education majors may take slightly different mathematics and methods courses than mathematics majors who plan to become educators. At University A, for example, students pursuing a Bachelor's of Science degree in Mathematics through the College of Liberal Arts and Studies take one Mathematics class in Term 1, two in Term 2, three in Term 3, four in Term 4, three in Term 5, three in Term 6, and two in Term 7 (see Table 3 for sequence). Those seeking a Bachelor's of Arts degree in Secondary Education with a focus on Mathematics through the Teachers College take one mathematics class in Term 1, one in Term 2, three in Term 3, three in Term 4, two in term 5, three in Term 6, and one in Term 7 (see Table 3 for Sequence). Thus while it can be theorized that all mathematics education majors should demonstrate growth in their knowledge, those in the Mathematics Department may demonstrate different growth rates in PCK than those in the Teachers College or majoring in other fields, both on their MKT scores overall and in the subcategories. While this growth may be linear, when calculating the regression, I also graphed the Normal Probability Plots to see if some other shape better described the data.

Table 3

Mathematics and Mathematics Education Courses Required for Graduation with Bachelors of Science Degree – Mathematics (Secondary Education) vs. Bachelor of Arts Degree – Secondary Education (Mathematics) at University A

	Term 1	Term 2	Term 3	Term 4	Term 5	Term 6	Term 7
B.S. Math	MAT 270	MAT 207 MAT 271	MAT 208 MAT 272 MTE 250	MAT 300 MAT 310 MAT 342 or 343 MTE 320	MAT 371 MTE 430 MAT 274 or 275	STP 420 MAT 415 or 416 MAT or STP elective	MTE 482 MAT 443 or 445 or 440
B.A. Sec. Ed	MAT 270	MAT 271	MAT 272 MAT 208 MAT 300	MAT 207 MTE 210 MAT310 STP 420	MAT342 MAT 370 or 371	MTE 482 MAT 411 or MTE 483 MTE 250 or MAT 443 or MAT 445 or MAT 447	MTE 485

As part of my second question, I also analyzed the growth of PCK among current teacher to see how it compares to pre-service teachers. It can be speculated that mathematics teachers may continue to develop their knowledge over time, but their knowledge may remain static or they might even lose what knowledge they had. Again, Pierson Correlation Coefficients between year teaching and score on the MKT were initially calculated. While pre-service teachers are theorized to be going through similar experiences, it is hard to justify the same belief for current teachers. Differences in classrooms, professional development activities, initial teacher certification programs attended, and personal interest in learning and growth mean that the cohort concept used for pre-service teachers may have less validity for current teachers. An additional confounding factor is the fact that teacher preparation programs change their requirements over time. At University A, for example, current mathematics education majors take three methods classes, eleven mathematics classes and one statistics class. Fifteen years ago, graduates with the same degree were required to take one methods class, eight mathematics classes, one statistics class and one computer programming class (see Table 4). Thus a multimodal linear assessment will also be run using years of experience, certification program attended (traditional state university, traditional private university, non-traditional certification program), major in college and participation in various professional development activities within the past five years as independent variables and MKT score as the dependent one.

As part of my second question, I investigated if there was a general growth pattern for all involved in mathematics education, both those in teacher preparation programs and those in the field. This was accomplished by creating an adjusted years' category for those already teaching. Assuming that the average teacher took four years to complete their teacher preparation program, a first year teacher could be thought of as being in their fifth year of learning how to teach. Thus, all participants will be put on the same years' scale and a Pierson Correlation Coefficient will be calculated for years learning to teach and MKT score. Again, assuming that other factors may affect learning of PCK I also ran a multimodal linear regression using years, major, institution, internships, student teaching and professional development activities participated in within the past five years as independent variables. Again, I plotted the normal probabilities graph to allow me to investigate if another shape would better describe the data.

Table 4

Comparison of the Courses Required for B.A. in Mathematics Education by Year at University A

15-16	07-08	00-01
		CSC 101
MAT 208		
MAT 270	MAT 270	MAT 270
MAT 271	MAT 271	MAT 271
MAT 272	MAT 272	MAT 272
MAT 274 or 275		
MAT 300	MAT 300	MAT 300
MAT 310	MAT 310	MAT 310
MAT 342 or 343	MAT 342	
MAT 371	MAT 370 or 371	MAT 370 or 371
MAT 415 or 416		MAT 411
MAT 440, 443 or 445	MAT 443, 445 or 447	MAT 443, 445 or 447
	MAT 483	
MTE 250	MTE 482	MTE 482
MTE 320	MTE 494	
MTE 482		
STP 420	STP 420	STP 420

While the differences between teachers from different institutions may make finding significant patterns difficult, if there are sufficient participants all from one institution it may be possible to find correlations for that subgroup. Therefore, I ran all of the above-mentioned analyses for all participants for each institution that had a minimum number of both future and current teachers. As the only pre-service programs that were willing to participate were University A and University B, this entailed two additional analyses.

Cognitively Activating Instruction (COACTIV) Assessment

Settings and Participants

After the data from the MKT was collected, those who indicated that they are majoring in mathematics education or currently teaching mathematics and are willing to answer additional questions were invited to participated in phase two of the study. Of the ninety participants from part one, twenty indicated a willingness to answer additional questions, and all were sent an e-mail inviting them to take a longer online survey that asked questions based on the COACTIV assessment. In the e-mail participant were told that if they completed this second survey they would receive a $15 gift card. Of the twenty invited to take this assessment nine completed it. These students and teachers had already indicated in part one their answers to the demographic questions.

Instrumentation

Because those majoring in mathematics education are usually planning on teaching at the high school level, all those who participated in the MKT assessment and met that criteria were asked to take an online survey based on a recent version of the COACTIV questions designed specifically for secondary school teachers and translated into English (see Appendix B for the modified version). As all participants in this part of the study had already completed the first survey, I did not collect any additional demographic data. The survey asked participants to verify their participation in part one of the project, and then asked them 27 questions, some with multiple parts. The questions fall into four sections; methods of problem solving, mathematical explanations, student difficulties and ideas, mathematics content knowledge. Again, I used the Depaepe (2013) categories of PCK. The reliability of the test has been calculated to be .83 (Baumert et al., 2010).

The COACTIV Assessment was created with a Code Book (See Appendix C for a selection) to allow for standardized scoring. This system awards multiple points for questions that have multiple correct answers, while other questions may only be worth a maximum of 1 point. The majority of this assessment requires participants to write out their explanations and justify their answers; however, there is one question that provides four answers from which to pick. For consistency sake, this assessment was scored using the number right scoring method and a formula scoring model with 1 point for each correct answer, zero points for skipped questions, and -1 for incorrect answers.

Statistical Analysis

My final question addresses the issue of measurement of mathematics PCK. To what extant do the MKT and COACTIV assessments demonstrate concurrent validity? To accomplish this, I first ran a simple linear regression between the aggregate scores of those who took the COACTIV assessment and their scores on the MTK and calculated the Pierson product-moment correlation coefficient. I also wanted to assess the degree to which the overlapping sub-constructs in the two instruments correlate, and the extent to which the non-overlapping sub-constructs in the instruments might improve the comprehensiveness and construct validity of each. To do this I took the individual scores on the three PCK sections from each assessment and calculated the Pierson product-moment correlation coefficient.

As a follow-up to my third question, I attempted to determine if the COACTIV assessments showed a similar developmental pattern among mathematics education majors and teachers as the MKT did. To do this I calculated the Pearson Correlation Coefficient comparing years teaching and score on the COACTIV, and then compared the two correlations using Fisher's r to z transformation.

Descriptive Analysis

After the COACTIV assessments had been scored, the data was gone through a second time to see if there were patterns that emerged in the responses that are given, based on the institution that a participant is or did attend, their years of learning how to teach mathematics, and their major in college. The majority of questions on this assessment have more than one correct answer or explanation, and all responses were coded based on the Rater Codebook. I looked to see if similar students had answers with similar codes, and then tried to unpack the thinking behind those answers and the development of knowledge they were exhibiting.

CHAPTER 4

RESULTS

Overview

This Chapter presents the results of the data analysis to answer the following questions:

1. Does the average level of PCK and its components change for cohorts of students progressing through their pre-service preparation programs and once they enter the teaching profession? If so, how much?
2. Does the average level of PCK and its components change for cohorts of teachers during their teaching career? If so, how much?
3. Is there an overall progression of development of PCK from the beginning of teacher training that continues during teaching?
4. If there is a change in level of PCK by cohort, either for pre-service, current teacher, or both, is it possible to measure the effects of different activities on those changes, given that those things may be standardized for people from the same institution?
5. How closely correlated are a teachers' scores on the MKT and COACTIV assessments of PCK? Both tests have shown validity as a measure of teacher effects on student achievement, but do they have concurrent validity? While we may find a statistically significant relationship between the two overall scores by running a simple Pearson correlation, there may be relationships between different sections or items from each test.
6. Does a qualitative analysis on the responses given in the COACTIV assessment provide any insights into the levels and development of Pedagogical Content Knowledge?

Preliminary Analysis

Seventy-eight people participated in the online survey based on the MKT high school assessment, 24 pre-service teachers and 54 current educators (see Table 5). The participants ranged, in terms of years teaching, from -4 years of experience to 45, though the data was skewed left (.562) due to the high number of 4th year college students willing to participate and only one teacher with more than 35 years of experience completing the survey. The majority of participants were mathematics education (52.6%) or mathematics (25.6%) majors. Four participants majored in other education majors, with three science education majors (3.8%) and one elementary education (1.3%). The remaining math teachers (16.7%) majored in unrelated fields ranging from criminal justice to geology.

Those surveyed received their bachelors' degrees from 21 different institutions. Half indicated that they had or were currently attending University A, with an additional 9% hailing from University B. All of the other teacher preparation programs in the state were represented, as well as eleven out of state programs and two universities in other countries.

Examining the pre-service teacher data showed us that of the 24 who participated 17 attend University A and seven attend University B. University B only offers a Bachelor's of Science in Education in Mathematics through their Department of Mathematics in the College of Natural Science. Thus all of them are Mathematics Majors, while the University A students were split between Mathematics (4 students), Mathematics Education (12 Students), and Engineering (1 student). Because of the differences in programs, and the possibility of individual choice, there was great variation

Table 5

Demographic Variables for All Participants

Variable	n	Percentage	
Teaching Status			
Pre-Service	24	30.8	
Current Teachers	54	69.2	
Major			
Mathematics Education	41	52.6	
Mathematics	20	25.6	
Other Education Major	4	5.1	
Other Major	12	15.4	
Unknown	1	1.3	
Institution Attended for Teaching Degree			
University A	39	50.0	
University B	7	9.0	
Online University A	6	7.7	
University C	2	2.6	
Online University B	2	2.6	
University D	1	1.3	
Online University C	1	1.3	
Other Out of State Universities	11	14.1	
Out of Country University	3	3.8	
Unknown	6	7.7	
Pre-Service Teachers Year in Program			
1st	5	6.4	(20.8 of pre-service)
2nd	1	1.3	(4.2 of pre-service)
3rd	4	5.1	(16.7 of pre-service)
4th	14	17.9	(58.3 of pre-service)
Current Teachers Years of Experience			
1 to 5	11	14.1	(20.4 of teachers)
6 to 10	5	6.4	(9.3 of teachers)
11 to 15	13	16.7	(24.1 of teachers)
16 to 20	5	6.4	(9.3 of teachers)
21 to 25	14	17.9	(25.9 of teachers)
26 to 30	2	2.6	(3.7 of teachers)
31 to 35	3	3.8	(5.6 of teachers)
35+	1	1.3	(1.9 of teachers)

in the number of semesters students had spent working or interning in a classroom (see Table 6). The mean number of semesters spent interning was 2.625 with a standard deviation of 1.689. Additionally, seven of the student indicated that they had already completed at least one semester of student teaching

Table 6

Internship/Student Teaching Experience of Pre-Service Teachers

Variable	n
Number of Semesters Interning/Working in Schools	
0	3
1	1
2	7
3	9
4	2
5	1
8	1
Participated in Student Teaching	
Yes	7
No	17

The 54 current teachers were involved in a variety of professional development activities (see Table 7). 92% were part of a professional learning community of fellow mathematics teachers at their school. 68% had taken University classes as some point within the past 5 years. 83% engaged in professional development at their school. A total of 29 teachers, or 53.7% of the participants, indicated that they had participated in all three activities in the past five years. These numbers appear higher than those of average teachers in the United States, where in 2013 about 84% participated in professional development courses/workshops but only 47% were part of a professional learning community (Strizek, Tourkin, Erberber, & Gonzalez, 2014). 16% of teachers surveyed in the U.S. indicated that they had participated in college courses program, which is significantly lower that our participants, however the study from 2013 only counted courses taken that year, while our participants were indicating if they had taken one in the previous five years.

Table 7

Professional Development Activities of Current Teachers

Variable	n
Involved in Professional Learning Communities	
Yes	50
No	4
Taking University Classes within the past 5 years	
Yes	37
No	17
Professional Development Activities at School	
Yes	45
No	9

Measure of Knowledge for Teaching Results

Seventy-eight participants took the High School Mathematics MKT assessment. The online survey tool gave a recommended time of 30 minutes to complete the assessment; however, the survey allowed participants to take an unlimited amount of time. While seven participants took over two hours to complete it, the average participant spent 31 minutes on the survey. Using the number right scoring system gave a mean of 13.96

out of a possible 35, with a standard deviation of 9. Participants' scores ranged from 0 to 30. The formula scoring system gave a mean of 17.33 out of a possible 65, with a standard deviation of 16. In this system the scores ranged from -8 to 57 (see Appendix D). This data was not normally distributed according to the Shapiro-Wilk test (significance = .001), being mildly skewed to the right (.356). Running the correlation between the results from two scoring systems gave me an r^2 of .77 with p-value $<.001$, indicating that the two systems are closely correlated, and from the standard deviations we know that formula scoring provides for greater variation. I ran most of the analysis using both results, but since this greater variation will allow us to better examine differences between participants, I will use the formula scores for the bulk of the data analysis.

Each question on the MKT was sorted into the three components of PCK represented on the test, namely a) Understanding of Student Misconceptions and Difficulties, b) Understanding Instructional Strategies and Representations, and c) Understanding Curriculum and Media. Using formula scoring the Understanding Students section scores ranged from -3 to 24 with a mean of 8.385 and standard deviation of 7.174. Understanding Strategies ranged from -10 to 26 with a mean of 7.192 and standard deviation of 8.207. Understanding Curriculum ranged from -5 to 15 with a mean of 1.756 and standard deviation of 4.000. Again, these data sets were not normally distributed according to the Shapiro-Wilk test, all being skewed to the right (Ranging from .188 to 1.200)

COACTIV Results

Nine people who completed the MKT high school assessment also completed the COACTIV assessment. This limited number of participants may be due to the time requirements of this test. The creators of the assessment gave a recommended time of two hours to complete it; however, the survey allowed participants to take an unlimited amount of time. While one participant took over 4 hours to complete it, the average participant spent 2 hours and 21 minutes on the survey. Using the number right scoring system gave a mean of 17.89 and standard deviation of 9.12; however no maximum possible can be calculated as participants can received multiple points for each question depending on how many valid answers they put. Formula scoring was also applied to this assessment, where correct answers are worth 1 point each, incorrect -1 points and omitted questions 0. On the COACTIV assessment formula scoring gave us a mean of 10.44 and standard deviation of 15.34. Running the correlation between the two scoring systems gave an r^2 of .840 with p-value <.001. Again, because of the high correlation and greater standard deviation I will be using the formula scoring system for all of the statistical analysis.

Each question on the COACTIV assessment was sorted into the three components of PCK represented on the test, namely a) Understanding Students Misconceptions and Difficulties, b) Understanding Instructional Strategies and Representations, and c) Understanding Mathematics Tasks and Cognitive Demand. Additional questions relating to Content Knowledge were also included on the assessment. Tests for normality were not run on this data due to the limited number of samples.

Statistical Analysis

Question 1 PCK Development in Pre-Service Program.

To describe the development of PCK knowledge among Pre-Service teachers I first analyzed the scores of the 1st year college students who took the MKT assessment to see what level of PCK with which they may have entered college. A one-sample t-test was run to determine whether the scores were different than 0. While there were only five participants in their 1st year of college, and one outlier with a score of 37, the mean scores for this group (M=7.60, SD = 17.8) were not significantly different from 0 with t(4)=0.9548 and p=.3938. None of the 1st year students took the COACTIV assessment, so I was not able to run any comparisons from that data.

Next, I ran a two-sample t-test between the MKT scores of 1st year college students and those in their 4th year of college (Only one college student took the COACTIV assessment, so that data was omitted for this section). By their 4th year in the program most students have been taking math methods classes for several semesters prior, thus it was assumed that they would have higher scores than 1st year students, so this was run as a one-tailed test. An initial comparison of their scores showed that under the answers correct scoring system there was not much difference between their mean scores (13.2 to 16.1) however using the formula scoring gave us more distinct means (7.6 to 22.7) with comparable variation (Standard Deviation of 17.8 for 1st year compared to 14.6 for 4th year). Table 8 summarizes the results of the t-test.

Table 8

T-test for Equality of Means Between 1st Year and 4th Year Students MKT Scores

Test	t	Mean Difference	Std. Error Difference	p
Equal variances assumed	-1.884	-15.114	8.024	.039

With a significance of .039 we can conclude that 4th year students do have higher scores on the MKT than 1st year students. To see if this development is linear I ran a linear regression comparing the scores of all college students with their year in the program. Table 9 summarizes the results of that analysis.

Table 9

Linear Regression Results Predicting MKT Formula Score Among College Students

Measure	β	Beta	t	p
Constant	22.327		5.779	.000
Year	4.469	.344	1.719	.100
R^2	0.118			

Note. Dependent variable = MKT Formula Score

These results indicated that the Year in the Program (β=4.469, p=.100) was not a significant predictor of MKT score, and that the model only accounted for 12% of the variation in scores. As a part of the analysis, a normal plot was created (see Figure 3) comparing the residual scores with the expected probabilities. Examination of this indicated that there might be some other underlying patterns in the data. Because of this apparent curved shape, I used SPSS to test if non-linear equations might better model the data.

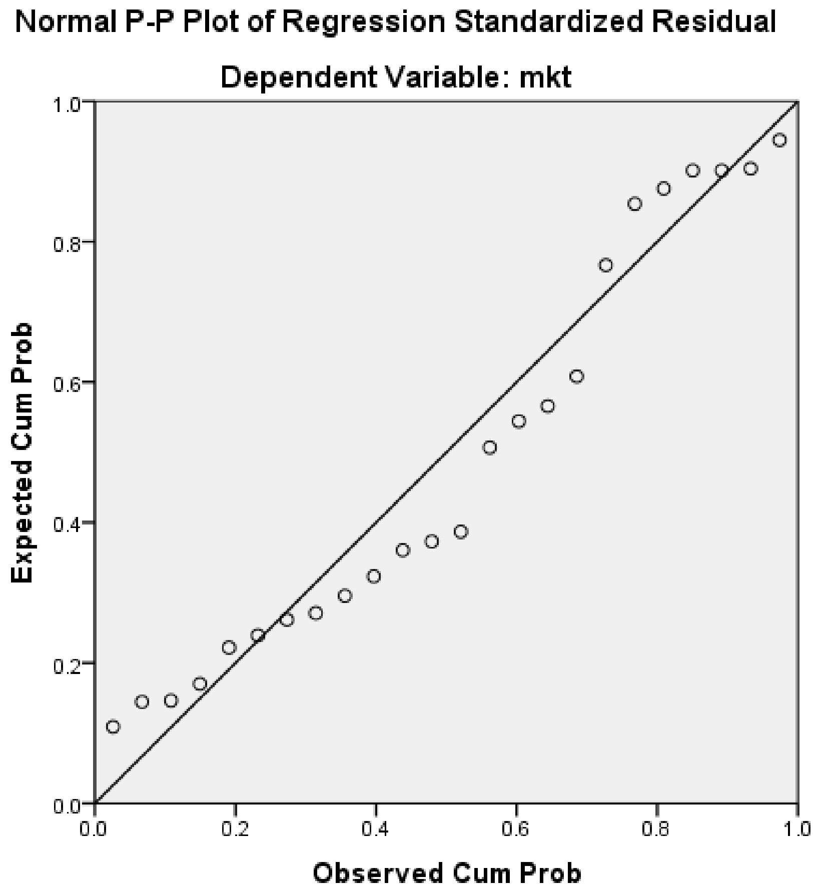

Figure 3. Comparing MKT Residual Scores With Expected Probabilities for Pre-Service Teachers.

Examining these results I found that while several models showed statistical significance, the s-curve equation comparing years and MKT scores seemed the most promising. The results are found in Table 10. While it is difficult to describe the effect that individual factors play in this equation, the model as a whole accounts for 23% of the variability in pre-service teachers MKT scores and has a statistical significance of .016. Based on the equation students in their 1st year would have an average MKT score of 0.990, 2nd year students a mean of 11.392, 3rd year a mean score of 16.709, and 4th year students an average of 19.844. Figure 4 shows the graph of the pre-service teachers' scores on the MKT and what the model would predict for them.

Table 10

S-Curve Regression Results Predicting MKT Formula Score Among College Students

Measure	β	Beta	t	p
Constant	3.729		13.616	.000
1/year	-1.332	-.486	-2.610	.016
R^2	0.236			

Note. Dependent variable = ln (MKT Formula Score + 10)

Additional t-tests were run for the scores of pre-service teachers on each of the three components tested on the MKT. 4^{th} year students did have higher average scores than 1^{st} year students on all three categories (see Table 11); however, the only category that showed a statistically significant difference using a 1-tailed t-test was Understanding Students. To see if the growth in scores from the three different sections were linear I ran a regression comparing the scores of all college students on each section of the assessment. Again, only the understanding students' linear model was significant with a p-value of .040, and this accounted for 18% of the variation among the students' scores on this section.

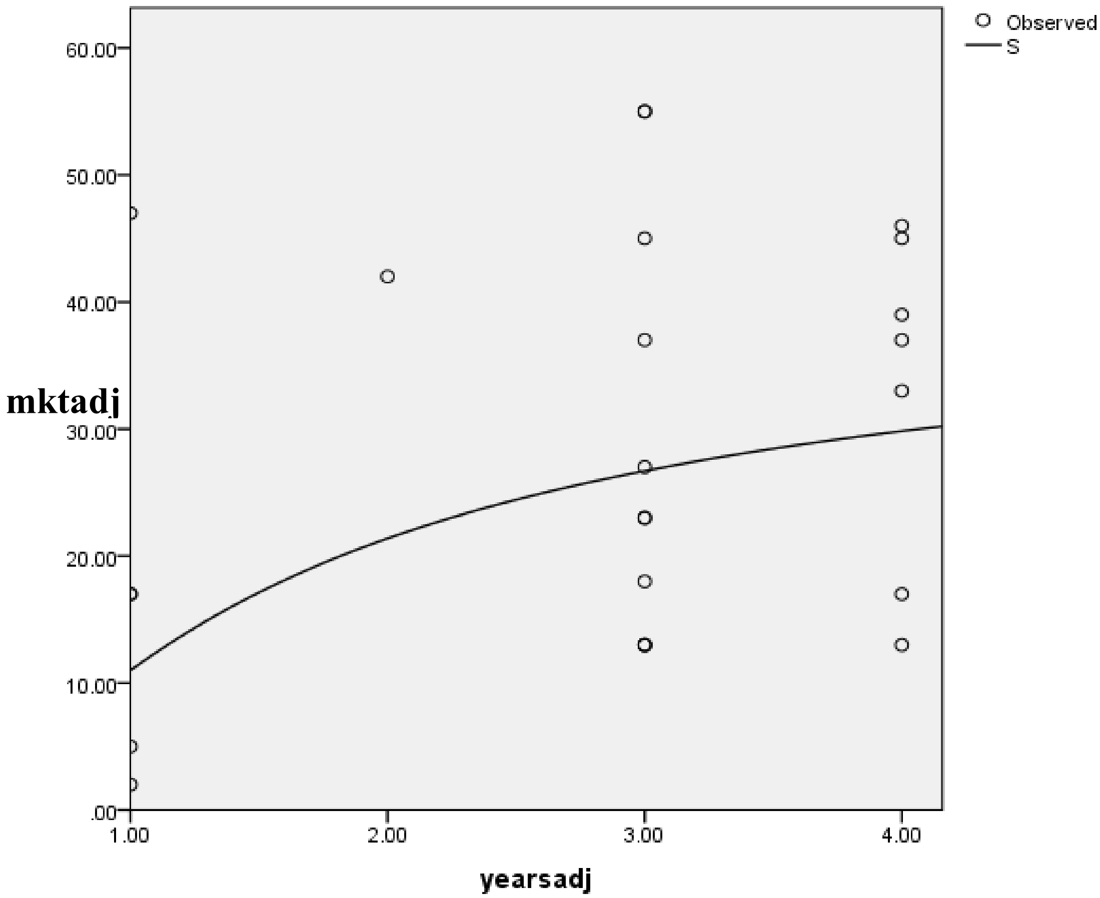

Figure 4. Graph of Pre-Service Teachers MKT Scores and S-Curve Results.

Table 11

T-test for equality of Means among 1st Year and 4th Year Students on Components of MKT

Category	1st Year Mean	4th Year Mean	Std. Error Diff.	p
Understanding Students	-.200	9.214	4.383	.023
Understanding Strategies	5.400	9.929	4.016	.138
Understanding Curriculum	2.400	3.571	2.622	.331

However, given that greater significance had previously been found using an s-curve, I ran another regression using that model. The results are found in Table 12.

Table 12

S-Curve Regression Results Predicting MKT Understanding Student Score Among College Students

Measure	β	Beta	t	p
Constant	3.261		13.812	.000
1/year	-1.263	-.508	-2.768	.011
R^2	0.258			

Note. Dependent variable = ln (MKT Formula Score + 10)

This model accounts for 26% of the variation in participants' scores on that section of the assessment. Based on the equation students in their 1st year would have an average score on this section of -2.62, 2nd year students a mean of 3.87, 3rd would average 7.12, and 4th year students would have an average score in understanding students of 9.02

Question 2 – PCK Development Among Current Teachers

To examine the development of PCK among current teachers I first ran a t-test comparing the MKT scores of pre-service teachers to current teachers. Initial analysis showed that participants still in college had a mean score of 18.4 with a standard deviation of 15.9, while current teachers had a mean of 16.9 with a standard deviation of 16.2 (see Table 13).

Table 13

T-test for Equality of Means Between Pre-Service and Current Teachers MKT Scores

Test	t	Mean Difference	Std. Error Difference	p
Equal variances assumed	.396	1.565	3.956	.694
Equal variances not assumed	.398	1.565	3.930	.692

Note. 24 Pre-service Teachers were compared with 54 Current Teachers

With a significance of .694, we cannot conclude that current teachers or future teachers have different levels of PCK. However, this lack of difference may be due to participants' major in college. I eliminated from the data set those who majored in anything other than math or math education, and reran the t-test. This did not result in a significant difference (p-value = .595). Given that all of the pre-service teachers came from two in-state traditional teacher preparation programs, I also filtered out those who received training from other places. While this analysis indicated that current teachers from those programs with similar majors had a higher level of PCK than the comparable pre-service teachers (21.4 to 18.4), the t-test again gave insignificant results (p-value = .641). Thus, even controlling for major and teacher preparation program, we cannot conclude that current teachers have a different level of PCK than pre-service teachers.

While investigating the effect of major on PCK I compared the mean MKT scores for current teachers based on college major (see Table 14). Based on this analysis it appears that those who majored in mathematics education had on average the highest level of PCK, followed closely by those who majored in a non-education field. Those

Table 14

Mean MKT Score Based on College Major of Current Teachers

Major	n	Mean	Standard Deviation
Mathematics	16	9.250	12.461
Mathematics Education	22	21.682	17.7881
Other Education	4	10.250	19.776
Non Education	11	21.636	13.478

who majored in mathematics or some other area of education had the lowest levels of PCK. A two tailed t-test showed that those differences, between mathematics and mathematics education majors and between mathematics and other majors, were significant (p-values of .022 and .021 respectively). These differences do not show up as drastically among pre-service teachers. Among college students those who are majoring in mathematics education do have on average higher MKT scores than those who major in mathematics (20.1 versus 13.2) however a two-sample t-test did not indicate that they were statistically different ($p=.4439$). Even thought there was not a significant result for pre-service teachers, I ran a linear regression for current teachers comparing years of experience with MKT scores to see if there was some development of PCK over time (see Table 15).

Table 15

Linear Regression Results Predicting MKT Formula Score for Current Teachers

Measure	β	Beta	t	p
Constant	16.541		8.437	.000
Year	-.050	-.082	-.590	.558
R^2	0.007			

Note. Dependent variable = MKT Formula Score

These results indicate that years of experience was not a significant predictor of MKT Scores (β= -.050, p= .558), and if anything had a negative impact on it. The model accounted for .7% of the variation among scores. Even though only eight teachers took the COACTIV assessment, I also ran a linear regression comparing years of experience and score on that assessment. These results indicated that years of experience also had a negative impact on COACTIV scores, and a significant one at that (β=-.899, p=.002)! Applying fisher's r to z transformation indicated that there was a significant difference between these correlations (p=0.0025 on two tailed test). Because of these confounding results (experienced teachers having less pedagogical content knowledge that new teachers), I reran the analysis on the MKT data, first controlling for major (see Table 16).

These models gave very different results. While none showed a significant relationship between years of experience and MKT score (p-values ranged from .098 to .993), the model did account for 18% of the variation among those who majored in mathematics and 16% of the variation for those who majored in a non-education field.

Table 16

Linear Regression Results Predicting MKT Formula Score for Current Teachers Separated by Major

	Measure	β	Beta	t	p
Math Ed.	Constant	24.352		3.299	.004
	Year	-.189	-.095	-.425	.676
	R^2	0.009			
Mathematics	Constant	24.953		2.678	.018
	Year	-.840	-.428	-1.774	.098
	R^2	0.184			
Other Ed.	Constant	10.133		.591	.615
	Year	.007	.007	.010	.993
	R^2	0.000			
Non Ed	Constant	14.456		2.142	.061
	Year	.479	.400	1.308	.223
	R^2	0.160			

Note. Dependent variable = MKT Formula Score

However, there was a negative effect for the mathematics majors and a positive effect for other majors.

Both Mathematics Education and Other Education Majors showed little effect (less than 1% of the variation). Because of the continuing negative result, I ran another regression to examine just those teachers who had attended the same schools that the pre-service teachers attended (see Table 17).

Table 17

Linear Regression Results Predicting MKT Formula Score for Current Teachers Who Majored in Mathematics or Mathematics Education and Attended University A or University B

Measure	β	Beta	t	p
Constant	11.631		3.579	.003
Year	.045	.094	.367	.718
R^2	0.009			

Note. Dependent variable = MKT Formula Score

This model did not have significant results (p=.718) and accounted for < 1% of the variance. However, it did show a positive relationship between years of experience and MKT scores. One possible reason for the negative relationship encountered in the previous analysis may have been that teachers from different programs received different levels of training, and those differences may have persisted throughout their career. Among pre-service teachers there was a difference in mean MKT scores between those who attended University A and those who went to University B (20.5 versus 13.3), however a two tailed t-test indicated that the difference was not significant ($p=.371$). To investigate the possibility of a significant difference among current teachers I ran a t-test comparing the MKT scores of those teachers who attended University A to those who went to any other program (see Table 18).

Table 18

T-test for Equality of Means Between University A and Non-University A Alumni MKT Scores

Test	t	Mean Difference	Std. Error Difference	p
Equal variances assumed	1.910	6.821	3.570	.060
Equal variances not assumed	1.910	6.821	3.570	.060

Those who came from University A had a mean score of 20.744 on the MKT, while those who came from other institutions had a mean of 13.923. While this difference may not be statistically significant ($p=.060$) it may account for the negative relationship between years and MKT. While University A accounted for 50% of the overall participants, only 5 out of the 20 teachers with more than 20 years of experience attended University A. Of those from University A 2 were math education majors, 1 was a math major and 2 were non-education majors. Of those not from University A 4 were math education majors, 7 were math majors, 2 majored in other education fields and 2 were non-education majors. Those with over 20 years of teaching experience from University A had a mean of 32.200, while those from other institutions had a mean of 9.733. A t-test on those participants showed that they were significantly different (p-value = .013).

Additional comparisons were run between current teachers' years of experience and their scores on the components of the MKT. None showed a significant correlation or accounted for more than 1% of the variation. However, when separating out participants by major, those with a non-education major did show positive growth in Understanding Students (β=.378, p=.116) that accounted for 25% of the variation, and Understanding Strategies (β=.157, p=.251) that accounted for 14% of the variation. Those who majored in Mathematics exhibited negative change in Understanding Students (β=-.426, p=.111) that accounted for 17% of the variation, and in Understanding Curriculum (β=-.171, p=.160) that accounted for 14% of the variation.

Question 3 – Growth of PCK From Training Through Teaching.

To investigate the possibility of growth in PCK from teacher training throughout a teaching career, I ran a linear regression comparing years with MKT scores for all participants (see Table 19). These results indicate that years is not a significant predictor of MKT scores (β =-.099, p=.545), and that this model accounts for only .5% of the variance. Even though only nine participants took the COACTIV assessment I also ran a linear regression comparing years of experience and score on that assessment. These results indicated that years of experience was not also not a significant predicator of COACTIV scores (β=-.638, p=.064) but did account for 40% of the variance. Applying fisher's r to z transformation indicated that there was not a significant difference between these correlations (p=0.1236 on two tailed test). The Normal Probability Plot was also

Table 19

Linear Regression Results Predicting MKT Formula Score for All Participants

Measure	β	Beta	t	p
Constant	18.383		7.320	.000
Year	-.099	-.070	-.608	.545
R^2	0.005			

Note. Dependent variable = MKT Formula Score

generated for the MKT data to investigate the possibility of a non-linear relationship, and appears as Figure 5.

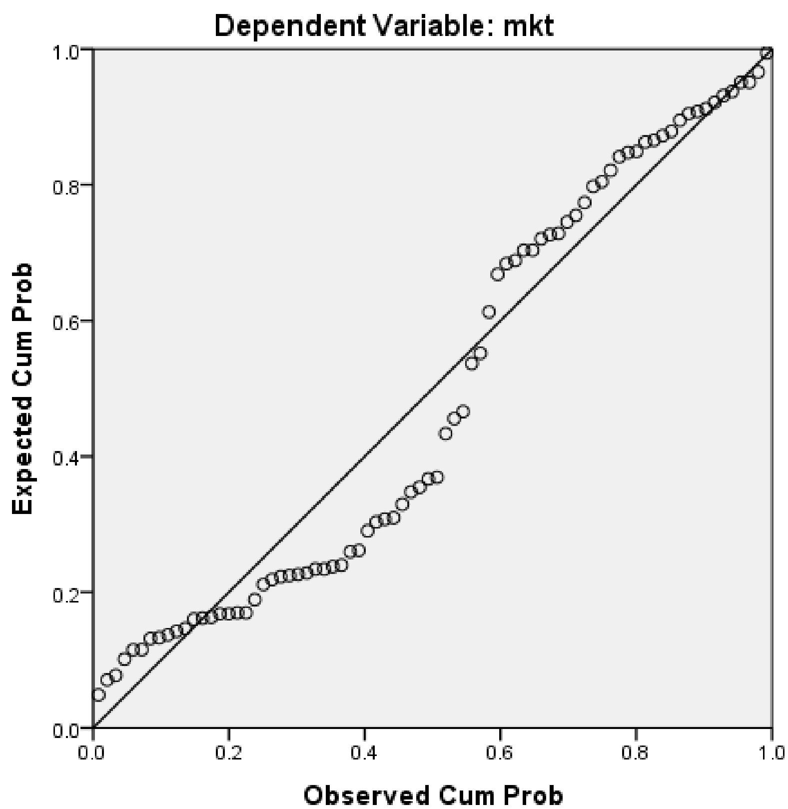

Figure 5. Comparing MKT Residual Scores With Expected Probabilities for all Participants.

Because of the curved pattern, I used SPSS to investigate non-linear equations that might better model the data. Again, the S-Curve was the most promising, however in this case it was not a significant predictor for the full data set (p-value = .158). Based on the prior analysis that showed that there might be a difference in pre-service training between teachers that are more experienced and the rest of the sample, I ran multiple analysis by constricting the sample to pre-service teachers and removing the group of older teachers in increments of 5 years of experience (see Table 20). Once I found the range that provided

Table 20

S-Curve Regression Results Predicting MKT Formula Score for Different Ranges of Participants

Years Range	n	R^2	F	Significance
Pre-Service to 5 years	35	.139	5.312	.028
Pre-Service to 8 years	**39**	**.176**	**7.883**	**.008**
Pre-Service to 10 years	40	.160	7.225	.011
Pre-Service to 15 years	53	.122	7.079	.010
Pre-Service to 20 years	58	.075	4.512	.038
Pre-Service to 25 years	72	.028	2.025	.159
Pre-Service to 30 years	74	.025	1.841	.179
Full Data Set	78	.026	2.028	.158

Note. Dependent variable = MKT Formula Score the most significant results, both in terms of variance explained by the model and the p-value, I re-ran the analysis between those two groups of data, only removing 1 year of experience at a time.

While running the regression for the range from Pre-Service to 8 years gave the greatest significance and accounted for the most variability ($R^2 = .176$, sig = .008), I chose to include the S-curve data from Pre-Service to 15 years of experience to demonstrate the growth pattern (see Table 21). The graph of the MKT scores and predicted scores are in Figure 6. This model indicates that there is significant growth in PCK knowledge during the years of teacher training, increasing from 2.4 during a college students' 1st year to 14.3 in their 4th year, with slower growth once teaching, rising from 15.4 in the first year of teaching to 17.5 in the fifth year.

Table 21

S-Curve Regression Results Predicting MKT Formula Score Among Participants From Pre-Service to 15 Years of Teaching Experience

Measure	β	Beta	t	p
Constant	3.413		27.353	.110
1/year	-.895	-.349	-2.661	.010
R^2	0.122			

Note. Dependent variable = ln (MKT Formula Score + 10)

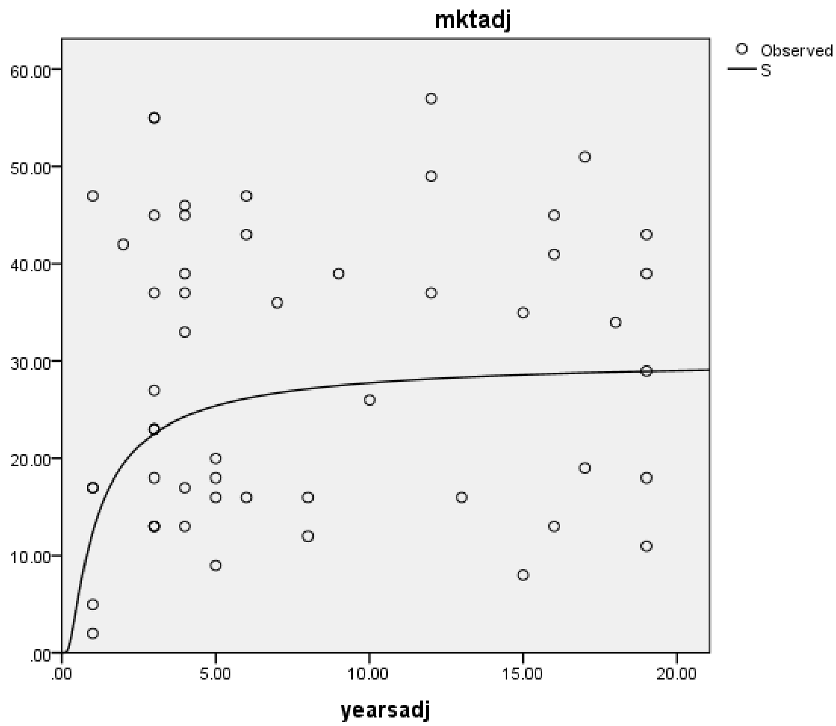

Figure 6. Graph of Participants MKT Scores and S-Curve Predicted Results.

Question 4 – Effect of Other Activities on MKT

In addition to collecting data on the year participants were in their program or had spent teaching, data on other demographics and possible learning activities they engaged in was also collected and analyzed. For pre-service teachers' things like major, internships, student teaching, and the school someone is attending may also have an effect on MKT score. To investigate this, a multiple regression was also run using those as factors (see Table 22).

Table 22

Multiple Regression Results Predicting MKT Formula Score for Pre-Service Teachers

Measure	β	Beta	t	p
Constant	34.343		2.378	.029
Year	-.581	-.045	-.118	.908
College (A or B)	-15.490	-.451	-1.304	.209
Major (Math Ed or Math)	-4.499	-.195	-.801	.428
Internships	3.331	.353	.904	.378
Student Teaching	1.707	.076	.341	.737
R^2	0.209			

Note. Dependent variable = MKT Formula Score

These results indicate that year in the program (β=-5.81, p=.908), college attended (β=-15.490, p=.209), major (β=-4.499, p=.428), internships (β=3.331, p=.378) and student teaching (β=1.707, p=.737) are not significant predictors of MKT scores. Additionally, this model only accounts for 21% of the variance in student scores.

For current teachers there was greater variation in both Bachelor's Degree Major (16 different ones were indicated by participants) and Institution attended to earn that degree (21). To simplify the calculations, Majors were classified as either: Mathematics, Mathematics Education, Other Education, Non-Education, or Unknown. For Colleges attended the options were In-State Traditional (A 4+ year bachelors' degree granting institution), Out of State Traditional, Online/Non Traditional, Out of Country, or Unknown (see Table 23).

Table 23

Multiple Regression Results Predicting MKT Formula Score for Current Teachers

Measure	β	Beta	t	p
Constant	-.990		-.072	.943
Year	.011	.007	.047	.963
College (5 Options)	3.636	.305	2.020	.049
Major (5 Options)	-.189	-.014	-.098	.922
PLC	-5.350	-.087	-.548	.586
University Classes	1.376	5.154	.267	.791
School Based PD	10.090	6.292	1.604	.115
R^2	0.115			

Note. Dependent variable = MKT Formula Score

These results indicate that years of teaching (β =.011, p=.963), major in college (β =-.189, p=.922), and participation in a professional learning community (β =-5.350, p=.586), university classes (β =.1.376, p=.791), or school based professional development (β =10.090, p=.115) are not significant predictors of MKT scores, but that the College someone attended may be (β =3.636, p=.049). Overall, this model only accounts for 11% of the variation in participants' scores.

Question 5 – Relationship Between MKT and COACTIV Assessments

Nine participants completed the MKT and COACTIV assessments. Their demographic information is located in Table 24.

Table 24

Participants with MKT and COACTIV Scores

College Attended	Years of Experience	Major
University A	12	Mathematics
University A	12	Mathematics Education
University A	8	Mathematics Education
Online University B	13	Mathematics
Out of State University A	2	Geography
University A	-1	Mathematics Education
Online University A	15	Criminal Justice
University A	12	Mathematics
Out of State University B	25	Mathematics

This group was relatively evenly split between mathematics (44%), mathematics education (33%), and non-education majors (22%). While only one pre-service teacher completed both, eight current teachers took it, ranging from having 2 to 25 years of experience, with a mean of 12.375 years. 55% of participants came from University A, with the other four participants having attended four different programs. Other than the number of pre-service participants, the demographics for this group appears similar to that of the entire sample. The MKT and COACTIV scores for these participants is located in Table 25.

Table 25

MKT and COACTIV Scores

MKT	Strat	Studen	Curricu	COACTIV	Content	Strat	Studen	Tasks
25	6	12	7	10	2	9	-3	2
31	22	10	-1	22	7	7	5	3
47	24	16	7	23	8	8	5	2
41	16	18	7	18	8	9	0	1
33	14	12	7	27	9	8	6	4
23	14	-6	15	5	0	5	0	0
33	14	16	3	9	0	3	3	3
35	22	10	3	4	1	2	2	-1
3	6	2	-5	-24	-10	-6	-7	-1

To test the relationship between the two tests a Pearson Correlation test was run between the overall MKT and COACTIV test scores. While the data from the sample is not normally distributed, our analysis of the overall MKT scores showed a slight skew, which should not disqualify it from this test (Chok, 2010). The correlation coefficient was calculated as .840 with a two-tailed significance of .005, indicating that there is a positive correlation between the scores of the participants on the two tests.

To investigate the possible relationships between the different sections of the test correlations were calculated for participants scores each section (see Table 26). Based on these results the only subsections that are correlated are the Understanding Strategy section of the MKT and the Understanding Students section of the COACTIV ($r=.771$, $p=.015$). None of the corresponding sections were significantly correlated. Understanding Students from the two tests had an r. of .426 with p of .253, and Understanding Strategy had an r of .338 with p of .373. Additionally, the Content Knowledge questions on the COACTIV assessment were strongly correlated with overall MKT scores. While this topic is not part of this study, it does validate the claim by Baumert and others that Content Knowledge and Pedagogical Content Knowledge, while separate areas, are highly correlated (Klickmann et al., 2015). Scores from the content knowledge section were also correlated with scores for all three sections on the COACTIV assessment, though not with any one section of the MKT assessment.

Table 26

Correlations Between Scores on Sections of MKT and COACTIV

	Mean	MKT	Strat	Stud	Curr	COAC	Cont	Strat	Stud	Task
MKT	30.1	1								
Strategy	15.3	.752	1							
Students	10.0	.697	.310	1						
Curriculum	4.7	.403	.089	-.161	1					
COACTIV	10.4	.840*	.572	.559	.442	1				
Content	2.8	.856**	.600	.580	.419	.979**	1			
Strategy	5.0	.745*	.338	.474	.616	.908**	.902**	1		
Students	1.2	.776*	.771*	.426	.251	.856**	.801**	.584	1	
Tasks	1.4	.438	.133	.526	.108	.762*	.660	.627	.670*	1

*. Correlation is significant at the 0.05 level (2-tailed)

**. Correlation is significant at the 0.01 level (2-tailed)

Question 6 – Descriptive Analysis of COACTIV Results

Because of the limited number of participants, it is difficult to generalize about the development of Pedagogical Content Knowledge from the COACTIV results. However, some patterns did emerge. The COACTIV assessment, because of its length, assumes that many participants will not answer every question. In the grader's codebook there are three separate codes for unanswered questions, one for "missing" which may include a dash or question mark, a second for "non-classifiable" which may include statements like "not in the mood” or “material not covered in my class”, and a third for incomplete or irrelevant answers that may imply that the reader did not understand the question. Of the 35 question that were given to participants in this study, the average number answered was 23, with three people answering 30 or more, and one only answering 7. Interestingly, the lowest score on the assessment was earned by the participant answering the second most questions (31), receiving a total of 4 points using the total correct system and earning a -24 using the formula score. By comparison, the third lowest score was earned by the participant who only answered 7 questions, receiving 7 and 5 points in the respective systems. This person gave statements like “I don’t have a great meaning for this” and “I’m not use to seeing this” on some questions but the other questions that they did answer were answered correctly and with good explanations.

The person who answered the fewest questions and earned the third lowest score indicated that they were a 4th year college student majoring in mathematics education and had already completed several semesters of internships and one semester of their student teaching. The person who answered the second most questions, but got the lowest score, indicated that earned a mathematics major in college, had been teaching for 25 years, and had taken university classes for professional development purposes within the past 5 years. This person seemed to be an outlier, and is largely responsible for the large negative relationship between years of experience and test scores. In fact, removing this person from the data analysis gives us a correlation coefficient of -.12077 that accounts for only 1.5% of the variation. Because of this it is tempting to disregard this persons’ submission as extraneous, however their scores on the MKT assessment correlate with the COACTIV scores, being one of the lowest scoring on that as well. Our earlier analysis indicated that there might be different developmental tracks based on a person's major in college, so to facilitate the analysis of the responses I divided the submissions up into three groups based on major and looked for patterns in what they did or did not answer and the types of answers they gave.

Mathematics Majors

The first question on the assessment had to do with changes to the area of a rectangle. Participants were asked a question, "How does the surface area of a square change if the side length is tripled", and then asked to provide different ways to solve and reason through the problem. Two of the three math majored provided the right answer, "The surface area increases 9 times." However, the most experienced participant, who we will refer to as Al, said only that. The math major with 13 years of experience, who we will call Bob added the following:

> When the side length of a square is tripled the surface area is multiplied times 9. This can be obtained first by using the formula: SA of a square = $6s^2$. Now if the side length is tripled, we would replace s with 3s. This gives $6(3s)^2$ or $6(9s^2)$ or $54s^2$ which is 9 time the original surface area.
>
> Also, length, area, and volume are proportionate measurements of similar figures. So if the length is tripled of any 1 dimensional measurement (i.e. length, width, height, perimeter) the other 1 dimensional measurements are tripled. Since area is in square units, whatever is done to 1-dimensional units, that dilation squared is done to all area measurements.

In the problem, a square was mentioned, but Bob referenced the formula for a cube ($6s^2$). Mathematically it is a valid justification, but one that extends the problem in unnecessary ways. The teacher with 12 years of experience, who we will call Carol, did not provide the answers at all, instead providing several methods of solving it.

> 1 you can make up a number and then triple it to calculate the change in the length,
>
> 2 you can use a variable for the side length to determine the relationship
>
> 3 you can use either method 1 or 2 to find the ratio of the change in length compared to the surface area

Carol, with the least experience (relatively), provided an answer that would be most useful in planning classroom instruction. Bob, with one more year of experience, provided a very technical answer along with the justification requested. Al, with the most experience in the group, gave an answer without any explanation or justification. This pattern seemed to hold with the majority of other problems. The most experienced teacher gave the most limited answers, and had the most wrong answers, while those with less experience explained and justified in much greater detail.

The sixth question in the assessment asked about negatives. The problem stated, "A student says "I don't understand why (-1) x (-1) = 1." Please outline as many different ways as possible of explaining this concept to the student." Al said:

> I think that the student can be shown this concept by using the general theorem that a negative times a negative is a positive.

The COACTIV Codebook counts this as a wrong answer because of the reliance on a rule, not an explanation. On the same question Carol answered, if you owe 2 friends \$7 each and they decide to forgive your debt, you have gained 14. Owing money is like a negative and taking it away is a negative but you now gained \$14.

The Codebook gives this answer 1 point, because while it is not technical or rigorous it does provide an explanation that students would understand.

Another question concerns fractions. It states, "A student calculates $1\frac{3}{4}$ divided by $\frac{1}{2}$ correctly but says that "it doesn't mean a thing" to her. Please outline one way this could be explained to her." Carol said:

> I would start with an example of a whole number divided by 1/2. If you have 6 chocolate bars and you are going to split them each in half, do you expect more or less than 6 pieces? Then I would talk about the 12 pieces we would now have. Finally we could look at splinting a recipe in half, so we only need half of the milk 1 3/4. That would be the meaning of the answer.

This answer provides for both a visual example that could help the student understand the situation and the through process behind what the answer means. On the same problem Bob said:

> Dividing is a way of cutting into equally sized pieces. So, if we have a rope that is 1 3/4 feet long and we want to cut it into 1/2 foot sections, we use division to see how many of those sections it would make. 3 and 1/.2 (1/2 foot) sections.

This also provides a visual example, and gives the answer in the same terms that the student found. Al's response to the question was:

> I would use fabric squares to illustrate the principle.

While fabric squares may be a useful tool for instruction, it is not an explanation, and the answer does not elaborate on how they would be used.

As a final example to illustrate the differences between these participants, there is a two part questions about exponents. The first says "Students frequently have difficulty accepting the definition a° = 1. What might be a reason for this? *Please list as many reasons as possible."* The follow up question asks participants to "Please briefly outline as many ways as possible to make this definition more accessible to students". Al said that student had difficulty with the definition because "Students may not understand the concept of exponents" and to make the definition more accessible "I would show examples as to how this is true." While examples may be valuable, they do not necessarily clear up confusion. Carol added the following:

> I usually show them how to reduce with whole numbers. (5*5*5)/(5*5*5)= 1/1 which is just 1 then I show them the same thing with variables. x*x/x*x = 1/1 which is just 1. Then I show them the rules of exponents a^n/a^m is just n-m.
>
> I also show them in a calculator so they believe me.

While Bob responded with:

> I feel that showing students how do divide exponents with the same base is the easiest way to get students to understand that anything to the 0 power is equal to 1

Both of these answers demonstrate an understanding of the mathematics involved in this problem and a pedagogical approach to explaining the concept to students.

All of the Mathematics majors came from different teacher preparation programs (only 1 from University A) making it difficult to compare how much PCK they gained in college. However, based on the responses it seems apparent that the teacher with the most experience has the lowest level of knowledge. The other two teachers with roughly the same level of experience had similar scores, with Bob's being a few point higher, likely because he answered a few more questions.

Mathematics Education Majors

Among the math education majors, all of whom attended University A, differences did not seem to be based on years of experience. While our pre-service participant, who we will call Ron, only answered 7 questions and earned 5 points a 12-year veteran, whom we will call Steve, answered 21 questions and only earned 4 points. The other two, with 8 and 12 years of experience (Tom and Val for this report) answered 28 and 32 questions and earned 23 and 22 points respectively. This makes it difficult to explain the variation in their answers. For example, on the area question our teachers with twelve years of experience simply gave an answer. Steve said, "Area is multiplied by 9. 3 x 3", which is mathematically correct but lacks reasoning, while Val likely assumed the problem was referencing a cube by saying "The surface are would be 27 times larger." Ron also referenced a formula for a cube but gave a correct answer:

> The surface area is 9 times more when the side length is tripled. Surface area of a square is $6s^2$ so if the side length is tripled then we have $SA = 6(3s)^2 = 6(9s^2) = 9(6s^2)$.

Tom provided the most detailed answer:

> The surface would be multiplied by 9. The easiest way to see this is to draw a square with side lengths of x and x, this gives an area of x^2. Then triple the sides to make them 3x and 3x, this gives $9x^2$ so the area is 9 times as large.

Everyone in this group gave a satisfactory answer to the negative problem. Steve said

> The first negative means the opposite of, the second number is a numeral. So the opposite of -1 is 1. The first negative means going back in time, the second number is -$1. So back one moment in time you had and extra dollar before you gave it away.

From Ron we heard:

> A negative times a negative is a positive....
>
> If it wasn't then the distributive property wouldn't work:
>
> $-1(-1+1) = -1(-1) + (-1)(1)$
>
> $-1(0) = -1 \quad + \quad -1$
>
> $0 = -2$

Val said:

> 1. If you think of negative as the word "opposite" then this is asking you for the opposite of negative one which would be positive one.
>
> 2. Because I said so.

And Tom replied:

> Use a number line to show that something like 3x2 means taking 3 steps forward with each step being 2 units for a total of 6 units of movement. -1x-1 implies taking steps in the negative direction and walking backwards which means you are really just walking in the positive direction.

On the exponent question, the pre-service teacher did not provide an answer. Tom said:

> Many students recognize exponents as multiplication and then make the assumption that a*0 cannot be 1.

And to explain it to students

> You can start with positive powers and show a pattern working back to a^0. Also, you could use multiple examples for values of a to show that it is always 1 regardless of the value of a.

Val thought the reason struggle with the zero power because:

> They associate an exponent with multiplication. They associate multiplication and Zero with the answer Zero.

and to explain it to students said

> Usually, i start with 'a to the fifth', then divide it by 'a' to result in 'a to the fourth.' Then take 'a to the fourth' and divide it by 'a' to result in 'a to the third' or cubed. I ask them to find a patter to the exponent as i divide by 'a' or 'a to the first' to get them to see that the exponent is reducing by 1. I continue on to taking 'a to the first' and dividing by 'a' and show that following the pattern, the result would be 'a to the zero' and that 'a/a' is equivalent to 1. That means that 'a to the zero' is 1. Depending on time, I continue dividing by 'a' to show how negative exponents come about.

Steve thought the reason for the problem was because:

> they see a to the zero as a times zero. Too philosophical for them. something raised to nothing. Contradictory

and to explain it said

> Use a^5/a^5 to show subtraction rules so 5-5 = 0 so a^5/a^5 = a^0 = 1 because anything divided by itself is 1 (maybe not zero) Students can follow the rule with acceptance and not have to prove it (not recommended)

Among the mathematics education majors there was significant variation, but it did not fall along lines of years of experience or program attended, but may be due to other factors beyond the scope of this study.

Non-Education Majors

Two participants in the COACTIV section did not major in an education field. One was a Geography major who has been teaching for two years, while the other is a fifteen-year veteran who studied Criminal Justice in College. In Germany, where this assessment was developed, this would not be possible because teacher certification is directly tied to having a college degree in mathematics education. Arizona, and many other states in the United States, allows for alternative certification routes based on the passing of a professional knowledge exam and completion of an educator preparation program while working in the field. Thus, these two people represent a group that may not have participated in this assessment previously, and were likely not anticipated as participants.

The responses of these two did not show any significant patterns based on years of experience. The more experienced teacher answered fewer questions than the less experienced one (15 versus 30), and earned fewer points overall (9 to 27). However, comparing problems that both answers shows the more experience teacher providing slightly more detail and explanation that the less experienced one. On the question of area, the one with two years of experience said, "The surface area increases by a factor of 9. Each dimension is tripled, so you just square the scale factor (2 dimensions = 2nd exponent)." While the one with 15 years of experience said "Since area is in units squared, the side being enlarged will be squared. If it is being tripled, it will be 9 times larger in area." Both teachers correctly answered the prompt, and gave better explanation than some of the other teachers who majored in mathematics did.

One of the questions deals with anticipating student errors. The prompt says "A group of students is given the following problem: There are S students and P professors at a university. There are 6 students to a professor. Write an equation to show the relationship between S and P. What might be a common student error?" The teacher with two years of experience gives the correct answer "6s=p" while the teacher with 15 years of experience says "6s = p. Students might switch the variables."

On the question concerning multiplying two negatives, the teacher with 2 years of experience said, "Removing a debt of $1 is the same as earning $1". The more experienced person elaborated:

> You are negating a negative.
>
> It is just like English, a double negative cancels out and makes it positive.
>
> With an even number of negative signs in multiplying, you are cancelling the negative and making it positive.

The COACTIV codebook would rate this as a higher scoring answer because even though the answers may be "superficial", they are usable explanation that may help students understand the principle being taught. Again, the differences are slight, and the less experience teacher had a higher score overall, but on the questions that they both answered the more experienced teacher did provide some slightly better explanations.

CHAPTER 5

DISCUSSION

There were two main questions posed at the outset of this study. In the following section, I will outline the findings in relation to each of the questions. These questions were:

1. Are the definitions of Pedagogical Content Knowledge as given by Baumert and Ball describing the same thing?

2. How does Pedagogical Content Knowledge develop, between Pre-Service and Current Mathematics Teachers separately and looking at the development throughout training and teaching?

Are the Definitions of PCK Made by Baumert and Ball Related?

Even though Baumert and Ball used different words to describe their components of Mathematical Pedagogical Content Knowledge, Depaepe, Verschaffel, and Kelchtermans (2013) identified two areas, Understanding of Students and Understanding of Strategies, which were conceptually identical for both. Even the third area, knowledge of curriculum for Ball (2001) and knowledge of tasks for Baumert (2010), seemed to overlap, seeing as curriculum can be broken down into tasks put into a specific order. According to both authors these descriptions of PCK came before the development of their assessments, thus the test should be a good measure of the definitions. Given that the scores of the participants on the two tests are correlated with a coefficient of .840 ($p = 0.005$), we can conclude that the tests are measuring mostly the same thing. In effect, 70% of the variation of scores on one is accounted for by the variation of scores on the other, meaning that 30% of the variation is different. Since they both claim to be testing for Mathematical Pedagogical Content Knowledge, we can conclude that yes the two definitions of PCK are the mostly same. An ideal explanation of the difference would be that while two section of each are described using similar language, the other area described differently in each is different. However, none of the scores for the individual components are significantly correlated with their counterparts. This may be due to lack of participants, the length of the COACTIV assessment which lead to a higher number of unanswered questions, or because the type of questions that were used to assess the separate type of knowledge were slightly different. Additionally, this difference could be cause by the differences in the initial construction of the test. The MKT and COACTIV were created in two different languages and were tested with teachers working with very different types of students. In the Understanding Students section, for example, issues like working with 2^{nd} language learners may be highly critical for one group but less important for the other. Thus the constructs themselves may be working with slightly different areas of knowledge.

However, knowing that these two groups are discussing pretty much the same topic should mean that the results based on one are similar to the results based on the other. Both groups had previously shown a relationship between a teacher's level of PCK and their students' achievement. Baumert's group had found that Content Knowledge was a separate but correlated area of knowledge (2010). These results support this claim, as CK from the COACTIV assessment was highly correlated with scores on the MKT (.856)

How Does PCK Develop Among College Students?

Based on the results of our one-sample t-test on first year college students MKT scores, we can conclude that in general PCK was not learned prior to our students entering college. While there was one 1^{st} year student who scored very highly, the mean was not significantly different from zero. Our s-curve model of development for both college students and all participants supports this claim by predicting MKT scores for college in their 1st year to be either 0.990 or 2.4.

The results of our two-sample t-test between first and fourth year college students allows us to conclude that PCK is learned in college. Again, the s-curve models supports this claim by predicting MKT scores of 4th year college students to be 19.844 or 14.3. While the passage of time, and with it the taking of classes related to mathematics and mathematics education and participating in internships and student teaching, only accounts for 18% of the variation in a linear model, it does provide support to Baumert's (2010) theoretical model of teacher learning. Incorporating other factors into our model, like college major, internships, and student teaching increased the percentage of variation described by our model to 20%. If we change from a linear model of growth to an s-curve, where knowledge starts out low, increases steeply for a period, and eventually levels out at some maximum, we can account for 26% of the variation with a significance of .011.

Within the MKT assessment, the section that college students showed the greatest improvement on was Understanding of Students. The growth was from a mean score of -.200 among 1st year college students to 9.214 among those in their 4th. A one-tailed t-test showed that this growth was significant with a p-value of .023. The linear regression between the MKT score for Understanding Students and years in the program for college students accounted for 14% of the variance, but was not statistically significant (p-value = .071). Students showed growth in the other two sections as well, but it was not significant, either as the change from 1st year students to 4th year or as a linear regression.

While this does show the value of teacher training programs, it also shows the current limitations of those programs in ensuring that all graduates are highly knowledgeable teachers upon entering the classroom. While the mean MKT score of 4^{th} year college students majoring in mathematics and mathematics education was 22.7, the standard deviation of 14.5 indicates that there is a 5.9% chance that a person from this group would have a score close to zero. Likewise, those teachers in this study who majored in a non-education field and received an alternative certificate had a mean score of 21.6 and standard deviation of 13.5, meaning there is only a 5.4% chance that a person from this group would have a score of zero. Thus, if the sample is representative of the larger population of mathematics teacher candidates from these institutions, roughly 1 in 20 would have shown no measurable level of PCK upon graduating.

Individual components of teacher preparation programs were not significant in our linear model, but there still may be underlying differences related to major in college and program attended. While the difference in level of PCK between Mathematics and Mathematics Education majors was not statistically significant, it may still exist but have been undiscovered because of our small sample size. Similarly, we see no significant difference between MKT scores of students at University A versus University B, however that may have also been due to sample size limitations and the limited number of programs involved in this study. We did find significant differences in these things among current teachers, so it makes sense that they likely exist among pre-service teachers as well.

Perhaps the most interesting result from this section is the fact that the best model of pre-service teacher development left 74% of the variation unexplained by the given data. Individual differences in students are three times more important in the level and growth of PCK than the programmatic differences they experience in teacher training. Thus, Baumert's (2010) theory of cognitive development has evidence supporting it, but is not the major factor in the development of PCK among college students. Other factors that could be affecting the results include differences in placements for internships, effectiveness of mentors and instructors in teaching the material, attitude about the importance of gaining this knowledge and time spent reflecting on and processing it. Without any other data, it is not possible to identify what could be the cause, but it does leave the door open for Ball's (2001) theory of constructed understanding. It could be that individual reflection on the material they are learning determined how much knowledge individual students gained, and without rigorous assessments all got moved on to the next year regardless.

How Does PCK Develop Among Current Teachers?

Our two-sampled t-test did not indicate a significant difference between MKT scores of 4th year college students preparing to enter the field and current mathematics teachers. The linear model correlating years of experience with MKT scores actually showed a negative relationship, indicating that the more experienced teachers have lower levels of PCK than younger teachers do. While it is possible that teachers gain knowledge in their pre-service programs and for the first years of teaching, and then lose some of that knowledge over time, I sought out other possible explanations. By separating teachers according to their college major, we see that the most acute decline of knowledge is among those who majored in mathematics, where this trend accounted for 18% of the variation. Among mathematics education and other education majors, the MKT scores appeared to be stagnant in relation to years of experience, while non-education alternate certification teachers showed growth that accounted for 16% of the variation. We may explain this finding by the fact that different cohorts had different requirements for graduation from college. Participants who graduated from college more than 20 years ago may have received significantly different preparation then those graduating today and thus may have started with a lower level of PCK. Given that Mathematics Education majors at University A today take three or more methods classes, while those who graduated 15 years ago may have only taken one, the more experienced teacher may have been increasing their knowledge during their career, but still have a lower score because of a lower starting point.

While it is not surprising that teachers who majored in a non-education field show growth in PCK related to their years of experience, it is somewhat surprising that their mean MKT scores were nearly identical to those who majored in mathematics education (21.7) and significantly higher than those who majored in mathematics (9.3). Running a linear regression that factored in major, school, experience and professional development opportunities did not reveal any significant connection between these things and MKT scores. Even though program attended did have a significant effect, the entire model only accounted for 11% of the variation in scores.

Because the s-curve accounted for the greatest amount of variation among college students, it makes sense to apply this model for teacher learning as a whole. When we group together pre-service teachers and those in their first 15 years of teaching (whose teacher training was somewhat similar) we see a period of steep growth during teacher preparation, slight growth during the first few years of teaching, and then a leveling off of changes to PCK. This model of growth only accounts for 12% of the variation, while the model based on only the first 8 years of teaching (where training had greater similarity) accounts for 18% of the variation. This provides some evidence to support Baumert's claims that significant learning does not occur through the experience of teaching (2010).

Again, the most interesting result of this research is that even the best model of teacher development of Pedagogical Content Knowledge does not account for over 80% of the variation among teachers involved. Teacher differences in learning during their pre-service program, classroom makeup, courses taught, outside responsibilities, individual attitude about improving their skills and time spent reflecting on their practice are some of the things that may be affecting knowledge of PCK. Ball's view of reflection as a determining factor in teaching learning may be correct but near impossible to prove because of the difficulty of an individual to assess his or her own reflections. Looking at this from a policy perspective brings into question what we require teachers to do to verify and improve their knowledge. Because our sample was drawn from public school teachers, we can assume that most were certified by the State and passed the subject certification exams. This test measures mathematical content knowledge, not PCK, and a cut score should ensure that all teachers have some base level of knowledge in their field. Yet this system has allowed for a great level of variation in measures of PCK, which is more closely connected to student achievement. Additionally, most public school districts incorporate things like Professional Learning Communities and Professional Development Classes into their salary schedules, yet there seems to be little evidence that they have an impact on a teacher's level of PCK. It may be that those activities affect teachers in other ways, but there is little evidence to support that claim (Yoon, Duncan, Lee, Scarloss, & Shapley, 2007).

What is the Best Model for Learning of PCK?

While Ball (2001) and Baumert (2010) were creating their definitions of Pedagogical Content Knowledge and the assessments to measure that type of knowledge, they based their definitions on their personal theoretical models of learning. Ball felt and likely still believes that learning comes from reflection or "pedagogical deliberation" (Ball, 1993) and may develop in little packets as teacher present a topic and reflect on how their students reacted to it. Baumert (2010) believes that learning comes from specific training, engaged in prior to entering the field, and is not developed from the experience of teaching (Kunter et al., 2013). Based on the results of this study, there is some evidence to support Baumert's claims (2010). PCK as measured on Ball's assessment (2001) wasn't significantly developed prior to teaching training, did grow during that process, and based on the cohort model didn't change much after that. One of the more accurate model of overall learning, the s-curve for pre-service through 15 years of teaching, models 1st year students growing from an average PCK score of 2.4 to around 14.3 for 4th year students, or a growth of nearly 500%. The change from the 4th year of college to the 5th year of teaching (17.5) is only 22%, and diminishes from there. Unfortunately, this model only accounts for 12% of the variation, which means that while Baumert's (2010) theory is supported, it does not seem to have much descriptive power. 88% of teachers' knowledge is explained by something else besides training. Even restricting the model to the pre-service plus the first 8 years of teaching leaves 82% unaccounted for. Individual differences in learning beyond program choices seems to be the biggest determinant in PCK scores. Thus, while Baumert's theory (2010) has statistical support, it may be that Ball's belief (2001) about reflection may be more impactful.

Baumert's theoretical model (2010) was developed in Germany, where teacher training is more regulated and professional development for current teacher has greater standardization than in the United States. Baumert himself acknowledged the importance of this when he said that while subject matter knowledge was "cross-culturally invariant", there were significant differences in the training of teachers from different countries, which affected their pedagogical content knowledge (Kleickmann et al., 2015). Thus, his model may not be a great fit here, with decentralized teacher training, limited governmental oversight and haphazard continuing education for teachers. An additional difference is that in Germany teachers must participate in a university sponsored teacher preparation program, and pass numerous exams administered by the Ministry of Education. The Alternative Certification Programs common here in the United States do not exist there. Baumert's studies (2010) never examined this type of teacher, who begins a teacher preparation program concurrent with the start of their teaching career. These teachers may be in a program disconnected from a traditional University, and may be excused from some Department of Education proficiency exams.

Ball's theory (2001), that learning is a cognitive process that occurs through reflection, is neither well supported nor strongly contradicted by the data. We have evidence that learning is occurring in teacher preparation programs, and it may be that those students who are the most reflective on what they are learning are gaining the most knowledge. There was not a question on the survey related to reflection, and it is difficult for an individual to describe how reflective they are, let alone for a researcher to measure it. Thus, it may be that reflection is the explanation for learning in College.

According to her theory, teachers with many years of experience would have had more opportunities to reflect, and more experiences to reflect on, than those who are just beginning. While she never explicitly says (at least where I have seen) that experienced teachers should have higher levels of PCK then new teachers, time, experience and intention matter in this model of learning. The data indicated that there was a negative relationship between years of experience and level of PCK. Separating this by major may provide an explanation for this. Those who majored in Mathematics Education or other Education fields showed no change over time, which may mean that they developed a certain level of PCK during teacher training and have not, in general, worked to improve it. Mathematics majors demonstrated the most significant decline, which may indicate that those who began teaching more than 15 years ago started with lower levels of PCK than students who are graduating today, and those older teachers have not improved their understanding to catch up with the beginners. The growth shown by those who majored in something other than education, because they chose to change from their original field to teaching math, implies that they may be more motivated to improve their knowledge and abilities in the classroom. These three explanations may be supported by the COACTIV data, which showed similar patterns based on major and year.

If we limit ourselves to pre-service teachers and only the first 8 or 15 years of teaching, we do see some growth according to the s-curve model. While this model estimates that a 15-year veteran teacher would top out with a mean MKT score of 19.0, this still indicates a growth of 23% from our 1st year teachers mean score of 15.4. Thus, Ball's model (2001) for teacher learning may be valid, but there is insufficient evidence from the data to support it.

CHAPTER 6

CONCLUSIONS

Learning of Mathematical Pedagogical Content Knowledge is an important topic of study for those involved in the preparation of future teacher and the further development of current teachers, along with those who evaluate educators and educational programs. In this chapter, I will outline some of the ways that this research could be used to inform improvements in teacher training. I will also address some of the limitations of this research and provide ideas for future steps in gaining knowledge in this field.

Implications for Teacher Training and Development

Most mathematics teachers gain pedagogical content knowledge during pre-service preparation. The greatest amount of growth seems to occur during the second and third years, when the teacher candidates are taking their content and methods classes and have started interning in schools but have not yet become full-time student teachers. With the goal of having better prepared teacher in the classroom, it seems that strengthening instruction in those classes would provide the greatest benefit. Baumert (Kunter et al., 2015) found that when comparing two different countries mathematics education systems, the country with the most rigorous teacher preparation program had the teachers with the highest levels of PCK. Other interventions such as student teaching while in school and professional development once someone has entered the profession may provide teachers with valuable insights into pedagogy in general or confidence in their own abilities, but it did not seem to provide participants with much growth in this area. While the area of understanding students does show significant growth during the pre-service process, understanding of strategies for instruction and understanding of curriculum/tasks did not show such significant gains. Having more rigorous classes in mathematics methods focused on these two areas might provide more immediate improvements in PCK.

An additional implication in terms of teacher development is that both and beliefs and identity matter in studying teacher learning. We know that not all groups of teachers show continuing improvement in their mathematics PCK during their service in the classroom. Those who graduated college with a non-education degree and then decided to pursue a career in teaching mathematics showed modest growth and a relatively high level of PCK. Those who majored in mathematics or mathematics education either stagnated or showed a decline in knowledge over time, with the math education majors having the highest level of PCK. It may be that those teachers who believe they learned all they needed to in college continue with the level of knowledge they had then and do not look for opportunities to improve it, while those from different field believe that they need to improve their abilities which leads to their continued growth. These beliefs may come because those who majored in the field in college identified themselves as a math teacher when they received their degree, while those from other field are still becoming math teachers when they enter the classroom. If teachers are convinced that they still have things to learn, and that the effort in gaining that knowledge will have benefits for them in the classroom, it may foster continued learning among current teachers. Professional development that attempts to show teachers how this material is valuable might be more effective than simply presenting these topics to teachers and assuming they will find the subject useful.

A final implication comes from the correlation between our two definitions of Pedagogical Content Knowledge. Both Baumert (2010) and Ball (2001) believe that

Content Knowledge of Mathematics is pre-requisite of PCK, and may be a subset. In this analysis, we can correlate the individual scores on the content knowledge questions from the COACTIV assessment with PCK scores on both the MKT and COACTIV. Thus, it would make sense that teachers who had majored in mathematics and mathematics education would have higher levels of PCK than those from other fields because of the increased number of math courses they would have taken in college. The results show that the groups with the highest levels of PCK were the math ed. majors and those from other fields, both having MKT significantly higher than those who majored in math. If PCK scores are correlated with content knowledge, and mathematics majors take more mathematics content classes than all others included in this study, how do they have such lower PCK scores that those who majored in unrelated fields like criminal justice and geography and did not take a significant number of college level math classes? It may be that the content involved in the content knowledge for teachers is not covered in the content classes taken by mathematics majors. To get a B.S. in mathematics at University A, a student begins by taking Calculus I and completes 14 total math classes to get their degree. However, most teachers at the secondary level do not teach Calculus or anything more challenging. Thus while those classes are interesting and informative, they may not be doing much to prepare future teachers to work in a high school setting, and the time spent on that material may be preventing those students from learning things that might be more useful.

Limitations of This Study

The goal of this study was to examine the development of PCK among pre-service and current teachers in Arizona. While the data collected is valuable in that regard, there are limits on what can be concluded. Arizona has at least six pre-service teacher preparation programs and at least three additional alternative certification programs. The people sampled include teachers from at least seven of these organizations, but only two were willing to have their students contacted to participate. One program indicated that they were unwilling to have any outside researcher engage in research on their students while several others expressed a possibility of their participation but later either declined to proceed or failed to return correspondence. This makes it difficult to say how representative the pre-service teachers in this study are in comparison to the rest of the state. A similar limitation on current teachers may also limit the breadth of the sample. While two districts allowed all of their mathematics teachers to be invited to participate, at least four other district decline. Thankfully, additional teachers were recruited from the membership of the Arizona Mathematics Teachers Association, but it is impossible to know if the membership of that group is demographically similar to all educators in the state. Consider that the median years of experience in the sample is 15, which is significantly higher than the median among mathematics teachers in one of the districts sampled, which is 8. This may mean that our sample skews older than most teachers in the state do, however our search for growth over time may render that difference insignificant.

As stated previously, another limitation is on the concept of cohorts. It can be assumed that pre-service teachers in the same year of their program are going through similar classes and completing similar assignments, usually with the same instructors and working in groups together. But the internships that they serve in and the learning that may occur there can be very different, both in terms of the learning experiences that are occurring with students and teaching that is being modeled by their mentor. To mediate this many pre-service programs try to give their students a range of experiences, varying the grade levels and the types of schools they are working in. For example, by the time they enter student teaching most mathematics education majors at University A have completed one semester of internship at a middle school and a second at a high school, one at a low-SES school and a second at a higher-SES location. Thus by the time they leave the program student at this University should have had similar learning experiences outside of their academic courses, but it is impossible to know how similar.

For current teachers the cohort model is even more problematic. Looking at any group of teachers in this study with a given number of years of experience will allow you to see teachers that have come from up to three different teacher preparation programs. These teachers may be at different schools, teaching different classes, working with demographically different students, and participating in different professional development activities. Given all of these limitations, the fact that the s-curve shows some growth in PCK from pre-service through 15 years of experience is somewhat amazing!

A final limitation may be researcher bias. Fifteen years ago, I graduated from one of these Universities with a Bachelor’s degree and began teaching mathematics. While I was in that program I complained about the education courses not relating to the work I wanted to do, the mathematics courses covering material that I would likely never teach, and the internships teaching me things I already knew. After I started teaching, I spoke of it in more glowing terms and would reference the challenging classes that I had taken and the opportunities I had interact with experts in the field. It is human nature to look for evidence that supports our own previously held beliefs and disregard facts that contradict ideas them. Throughout this experience, I have tried to allow the data to determine the direction results and not allow my personal beliefs to cloud my judgment. Given that the data shows that those teachers who did not participate in any traditional teacher preparation program performed as well as graduates with the same major that I have, I hope I have succeeded.

Questions for Further Study

For more effective teacher preparation, it would be useful to discover what sections within the framework of Pedagogical Content Knowledge are most valuable for teachers. We have evidence that pre-service teachers demonstrate the most growth in Understanding of Students and may improve in the other areas, but we do not know what classes or experiences are causing that growth, nor the value of that knowledge. While many of the studies of teacher knowledge and student achievement have used single scores for comparison, there may be specific components of PCK that have a greater effect. Knowing that would allow teacher educators to focus on those topics

We do not know how a teachers' knowledge displays itself in the classroom. If two teachers have different levels of PCK, are they going to plan, teach or behave differently? There is evidence that teachers with higher levels of PCK teach better lessons according to the Measures of Quality Instruction, which were graded by researchers looking at video tape of specific lessons (Hill, Umland, Litke, & Kapitula, 2012). However, they could not tell if a teacher's level of mathematical knowledge was noticeable by students, administrators or other observers. It would be useful to have secondary assessments of teacher's knowledge and behaviors from outside sources to use as a comparison between what they say about teaching mathematics and the manner in which they actually teach mathematics.

Moreover, we do not really know how big of a role teacher knowledge plays in student learning. It has been estimated that teachers account for between 1% and 14% of the variability in student improvement (American Statistical Association, 2014). If this is true, how much of the variability is described by the teachers' knowledge versus their beliefs, behaviors, or other demographic information? Most of the studies linking PCK to student achievement have been limited to end of course test results or assessments designed specifically to find the relationship, while studies of the value added by specific teachers rely on longitudinal data related to student achievement on standardized tests. It would be useful to link those two methods together to tease out the value of teacher knowledge on student success.

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APPENDIX A

MATHEMATICAL KNOWLEDGE FOR TEACHING ASSESSMENT

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CKT Mathematics for Teachers of Algebra 1: Study Sample

The Algebra 1 assessment is designed to be used with teachers who are responsible for teaching Algebra 1 content. The assessment was first piloted with approximately 300 teachers to assess basic question-level measurement characteristics. A small number of cognitive interviews were conducted and questions were subsequently revised. A final assessment was administered to ninth grade teachers of Algebra 1 in five of the six school districts participating in the Measures of Effective Teaching project.

The final assessment presented in this report includes a total of 22 questions. Most of these questions are four-option multiple choice. However, some questions are table versions with multiple items included in the table. Table questions for the Algebra 1 assessment include between 3 and 5 items.

Table 2: Algebra 1

Questions	Items	Teachers (N)	Scale Reliability
22	35	143	0.77

One way to use the Algebra 1 assessment is to create a score for each participant. This can be done by simply counting the total number of correct responses. The maximum possible score is 35.

However, in using scores from this assessment, it is critical to be aware of appropriate uses for the assessment scores. This assessment was not designed to be used as a selection or evaluation instrument for individual teachers. Because these assessments are neither secure nor designed for purposes other than research, they should not be used to support any high-stakes decisions about individual teachers or teacher candidates. The assessment is also not designed with a criterion reference. There is no score on the assessment that indicates a particular proficiency level, such as readiness to teach or a direct connection to a level of teaching quality.

While teachers participating in MET do not constitute a representative sample, the districts in the MET study included schools that served students from a wide range of backgrounds.

Because the assessment questions focus on the types of content knowledge used in teaching practice, they may also be valuable resources for use as study material for teachers and others interested in learning about CKT. These questions provide useful indicators for the types of content knowledge that prospective teachers should be developing as part of their teacher education. And they provide important guidance for practicing teachers about the kinds of content specific tasks and problems they should understand in order to teach Algebra 1 content.

CKT Mathematics for Teachers of Algebra 1: Answer Key

The Algebra 1 assessment questions are included in full at the end of this report. The answer key is provided below.

MET Study Algebra I Form Keys		
SEQ #	Item Type	Key
1	MC	B
2	MC	C
3	TABLE	Valid, Valid, Not Valid, Valid
4	MC	C
5	MC	C
6	TABLE	Not Support, Support, Support
7	MC	B
8	MC	C
9	MC	C
10	MC	B
11	TABLE	Valid, Valid, Not Valid, Valid, Not Valid
12	MC	B
13	MC	B
14	MC	C
15	TABLE	Provides, Provides, Provides, Does Not Provide, Does Not Provide
16	MC	C
17	MC	C
18	MC	B
19	MC	D
20	MC	C
21	MC	D
22	MC	C

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Content Knowledge for Teaching

Algebra 1 Assessment

This assessment is not to be used as a selection or evaluation instrument for individual teachers. Because these assessments are neither secure nor designed for purposes other than research, they should not be used to support any high-stakes decisions about individual teachers or teacher candidates.

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1. Mr. Wright asked his students to solve the equation $6 - 3(x - 5) = 24$. After reviewing his students' work, he found two interesting methods for solving the equation and asked those two students to present their methods on the board.

Brenda's method	Daniel's method
$6 - 3(x - 5) = 24$	$6 - 3(x - 5) = 24$
$6 - 3x + 15 = 24$	$-3(x - 5) = 18$
$-3x + 21 = 24$	$x - 5 = -6$
$-3x = 3$	$x = -1$
$x = -1$	

After Brenda and Daniel presented their methods, Mr. Wright's class discussed these two methods. One student, Steve, compared them and then said, "I like Daniel's method because there are less steps. However, if it is a harder division, Brenda's method would be easier." For which of the following equations would Steve be most likely to use Brenda's method?

A) $5 - 2(x - 3) = 21$

B) $10 - 7(x + 5) = 6$

C) $6 - 3(x + 1) = 9$

D) $8 + 4(x - 3) = 14$

2. A lesson in Ms. Hagerman's textbook defines the distributive property, but the exercises merely ask for its definition. To motivate her students to learn the definition, Ms. Hagerman tells them that the distributive property can often be used to simplify the evaluation of expressions.

   She wants to give her students an example that will focus their attention on how the distributive property can be useful in evaluating expressions. Of the following expressions, which would best serve her purpose?

   A) $12\times\left(\frac{3}{4}+\frac{1}{4}\right)$

   B) $18\times\left(\frac{3}{5}-\frac{1}{10}\right)$

   C) $36\times\left(\frac{5}{12}+\frac{2}{9}\right)$

   D) Each of these expressions would serve her purpose equally well.

3. During a lesson on solving multistep equations, Ms. Kane asked her students to solve the equation $-5x + 8 = 13x - 10$. While walking around the classroom looking at what the students were writing, she noticed several different strategies. For each of the following student solutions, indicate whether or not the work provides evidence that the student is reasoning correctly about this problem.

		Provides Evidence of Correct Student Reasoning	**Does Not Provide Evidence of Correct Student Reasoning**
A)	$-5x + 8 = 13x - 10$ $8 = 18x - 10$ $18 = 18x$ $1 = x$		
B)	$-5x + 8 - 13x + 10 = 13x - 10 - 13x + 10$ $-5x - 13x + 8 + 10 = 0$ $-18x + 18 - 18 = 0 - 18$ $\frac{\cancel{1}}{\cancel{-18}} \cdot \frac{\cancel{-18}x}{\cancel{1}} = \frac{\cancel{-18}}{\cancel{1}} \cdot \frac{\cancel{1}}{\cancel{-18}}$ $x = 1$		
C)	$-5x + 8 = 13x - 10$ $-5\cancel{x} + 8 = 3x$ $+\cancel{5}x \qquad +5x$ $\frac{8}{8} = \frac{8x}{8}$ $1 = x$		
D)	$-5x + \cancel{8} = 13\cancel{x} - 10$ $-13x \cancel{+8} \quad -\cancel{13}x \quad -8$ $\frac{-18x}{-18} = \frac{-18}{-18}$ $x = 1$		

4. Mr. Brownstein's class was solving the equation:

$$3(x-2)^2 = 6(x-2)(x+5).$$

Kenneth suggested dividing both sides by 3 to get:

$$(x-2)^2 = 2(x-2)(x+5).$$

Then he suggested dividing both sides by $(x-2)$, but Sandra said, "You cannot divide both sides by $(x-2)$." In response, Kenneth asked, "If you can divide both sides by 3, why can't you divide both sides by $(x-2)$?"

Of the following statements, which best explains <u>why</u> you cannot divide both sides of the equation by $(x-2)$ as Kenneth suggested?

A) You cannot cancel $(x-2)$ because it represents a real number.

B) It is better to expand the expressions on both sides of the equation first to obtain $x^2 - 4x + 4 = 2(x^2 + 3x - 10)$, and then you won't have to worry about $(x-2)$.

C) Division by zero is not defined, so you would have to consider the case of $x = 2$ separately.

D) Because x is a variable, it can vary—you may not be canceling the same amount from both sides.

5. During a unit on solving linear equations, Ms. Martino asks her students to write one question that she could use on the unit test at the end of the chapter. Joe writes the following question.

   If Joe had to solve the sentence $8y - 9 = 0$ for y, what would be the value of y ?

   Ms. Martino thinks the equation would be a good one for her students to solve. However, she decides to revise the question because she is concerned that the wording may cause some of her English Language Learner (ELL) students to answer this question incorrectly, even if they understand the mathematics involved. Of the following revisions, which one best addresses Ms. Martino's concern?

   A) When Joe solved for y in the equation $8y - 9 = 0$, what was the value of y ?

   B) Joe solved the sentence $8y - 9 = 0$ for y. What is the value of y ?

   C) What is the value of y in the equation $8y - 9 = 0$?

   D) If $8y - 9 = 0$, what would be the value of y ?

6. Before teaching a lesson on multiplying two trinomials, Ms. Ryan wants a better sense of what her students know about multiplying two binomials. She asks them to find the product $(2x+1)(x-4)$ and explain their methods. While walking around the class, she notices several different methods. For each of the following, indicate whether or not the student response provides evidence that the student has an understanding of the multiplication of binomials that would support the development of a strategy for multiplying two trinomials.

		Would Support a Strategy for Trinomials	**Would Not Support a Strategy for Trinomials**
A)	I used the FOIL method. The product of the first terms is $2x^2$, the product of the outer terms is $-8x$, the product of the inner terms is x, and the product of the last terms is -4. Then, I combined the two like terms, the $-8x$ and the x, to get $-7x$, so my final answer is $2x^2-7x-4$. F L $(2x+1)(x-4)$ I O F O I L $2x^2\ -8x\ +x\ -4$ $2x^2-7x-4$		
B)	I just multiplied like you do with regular numbers. I put the $2x+1$ on the top and the $x-4$ on the bottom. Then I multiplied each term on the bottom by each one on the top. Finally, I added up the like terms to get my answer, $2x^2-7x-4$. $2x+1$ $x-4$ $-8x\ -4$ $2x^2\ \ x$ $2x^2-7x-4$		
C)	I drew a box with two columns for the $2x$ and the 1 and two rows for the x and the -4. Then, I multiplied the terms that went with the row and column that each small box was in. Then, I wrote out what I got in each small box, combined the like terms, and got $2x^2-7x-4$. $\begin{array}{c\|c\|c\|} & 2x & 1 \\ \hline x & 2x^2 & x \\ \hline -4 & -8x & -4 \\ \hline \end{array}$ $2x^2+x-8x-4$ $2x^2-7x-4$		

7. Mr. Anderson asked his students to simplify the following algebraic expression.

$$\frac{2(a+1)}{3a} + 3 - \frac{2}{3a} - \frac{6a-2}{6} =$$

One of his students gave the incorrect solution shown below.

$$\frac{2(a+1)}{3a} + 3 - \frac{2}{3a} - \frac{6a-2}{6}$$

$$= \frac{2a+2}{3a} + 3 - \frac{2}{3a} - \frac{6a}{6} + \frac{2}{6}$$

$$= \frac{2a}{3a} + 2\frac{6}{6} + \frac{2}{6} - a$$

$$= \frac{2}{3} + 2\left(\frac{6}{6} + \frac{1}{6}\right) - a$$

$$= \frac{2}{3} + 2\frac{7}{6} - a$$

$$= \frac{4}{6} + \frac{14}{6} - a$$

$$= \frac{18}{6} - a$$

$$= \underline{\underline{3 - a}}$$

Of the following descriptions, which best characterizes what is wrong with this student's work?

A) This student used the distributive property incorrectly.

B) This student confounded mixed fractions with factors.

C) This student forgot to cancel common factors in several places.

D) This student needs to apply a more formal procedure by finding the common denominator and then adding all terms.

8. Having nearly finished a chapter on linear equations, Mr. Hassan's students seem quite proficient in generating standard formats for linear equations, using techniques for graphing linear equations, and solving systems of two linear equations. However, he is concerned that his students are applying these techniques in routine ways and tend to think only algebraically or only geometrically without reasoning fluidly in both ways.

   Mr. Hassan wants to give his students a problem that would require an understanding of the topic that goes beyond the set of procedures students have learned and that would support his students' ability to work and talk across algebraic and geometric interpretations. Of the following problems, which would best serve this dual purpose?

   A) Describe in your own words a procedure for finding the point of intersection given the equations of two lines.

   B) Find the intersection of the following two lines and graph them.

   $$y = 2x + 3$$
   $$y = 2x - 7$$

   C) Consider two linear functions, where a and b are negative.

   $$y = x + 3$$
   $$y = ax + b$$

   What can you say about the point of intersection of their graphs?

   D) Using ideas about solving systems of two linear equations, solve the following system of three equations and explain what the solution means.

   $$x + 4y + z = 0$$
   $$x - 4y + 2z = 3$$
   $$x = 4y + z$$

9. Having taught her students to factor quadratics with integer coefficients, integer roots, and a leading coefficient of 1, Ms. Quezada explained that she was going to give them a harder problem. She then asked them to solve the following.

$$3x^2 - 3x - 6 = 0$$

After a few minutes of work, the class discussed their solutions. Letitia said that x was -1 or 2 and explained, "I added $3x$ to both sides and divided by 3."

$$3x^2 - 6 = 3x$$
$$x^2 - 2 = x$$

She then continued, "The parabola's just down a little and the line's at 45 degrees, so it's just below zero and about 2 to the right. x can be -1 and 2, and those are the only possible ones."

Of the following, which best characterizes Letitia's approach to this problem?

A) Letitia's method is wrong because she should have first divided by 3 and then factored the left side of the equation.

B) Letitia's method is wrong because this is a parabola and you could graph it, but you would have to graph the original equation and look for the roots.

C) Letitia's reasoning is correct, but her method often leads to points of intersection that might be hard to determine visually.

D) Letitia's reasoning is correct, but her method requires knowledge of calculus.

10. Ms. Lang's class had been studying the concept of slopes of lines, so she asked them to consider all of the lines passing through one point and how the slopes of those lines vary. The students had used geoboards in some earlier work, so they started talking about the slopes of lines on an "infinitely extended" geoboard. (Geoboards are flat blocks of wood, roughly one foot square, with pegs laid out on a grid where rubber bands can be hooked to make lines or polygons.) The students decided that the pegs of the infinite geoboard could be thought of as the set of points with integer coordinates in the Cartesian plane.

    During the discussion, students had the following exchange.

    Yonah: On the geoboard, you can't get all of the slopes, because the geoboard points are too spread out—there are a whole bunch of lines between the ones you can make.

    Andy: I disagree. I think we can make any slope. Starting at one point, by choosing another geoboard point far enough away, we can tilt the line as much or as little as we like.

    Becky: What I was thinking was if you run a line through one geoboard point, it will always hit another one far enough out.

    Of the following concepts, which is most directly related to the mathematics underlying this discussion?

    A) Interpretation of the derivative—the derivative is the slope of the tangent line.

    B) Density of numbers on the real line—the rational numbers are dense, but not every real number is rational.

    C) The parallel postulate—given a point and a line, there is a unique line through the given point parallel to the given line.

    D) Each of these concepts is equally related to the underlying mathematics.

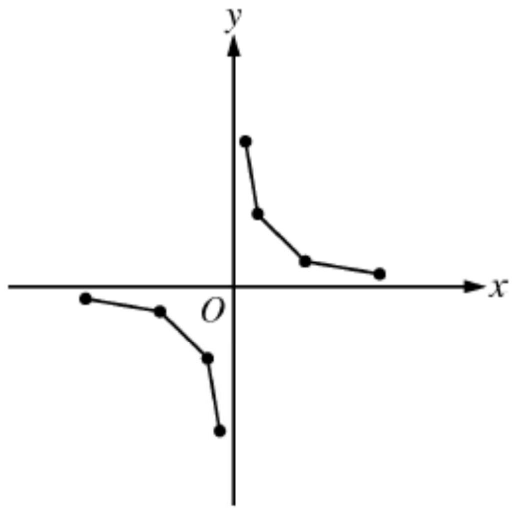
11. During a lesson on solving multistep equations, Mr. Steinbrecher asked his students to solve the equation $4(5x - 11) = 16$. While walking around the class looking at what the students were writing, he noticed several different strategies. For each of the following student solutions, indicate whether or not it is a valid strategy for solving this problem.

		Strategy Is Valid	Strategy Is Not Valid
A)	$\frac{1}{4}\cdot 4(5x-11) = 16\cdot\frac{1}{4}$ $5x-11=4$ $5x=15$ $x=3$		
B)	$4\cdot 4(5x-11) = 16\cdot 4$ $\frac{16(5x-11)}{16} = \frac{16\cdot 4}{16}$ $5x-11=4$ $+11\quad +11$ $5x=15$ $x=3$		
C)	$4(5x-11)=16$ $9x-11=16$ $+11\quad +11$ $\frac{9x}{9}=\frac{27}{9}$ $x=3$		
D)	$4(5x-11)=16$ $\frac{20x}{20}-\frac{44}{20}=\frac{16}{20}$ $+\frac{44}{20}\quad +\frac{44}{20}$ $x=\frac{60}{20}$ $x=3$		
E)	$\frac{4}{4}\left(\frac{5x}{4}-\frac{11}{4}\right)=\frac{16}{4}$ $\frac{5x}{4}-\frac{11}{4}=4$ $+\frac{11}{4}\quad +\frac{11}{4}$ $\frac{4}{5}\cdot\frac{5x}{4}=\frac{15}{4}\cdot\frac{4}{5}$ $x=3$		

12. Mr. Jakobsen's students were graphing the function below, where y is inversely proportional to x.

$$y = \frac{1}{x}$$

One of his students drew the following graph.



Mr. Jakobsen has noticed that students often draw graphs with line segments like this despite frequent reminders that the graph should be curved. To get his students to discuss this issue, he asked the class what was wrong with the drawing. Of the following student explanations, which provides the best mathematical explanation of why drawing connected line segments is inappropriate for this graph?

A) When the x changes, the graph should change at the same rate all the time and it shouldn't have corners.

B) The graph changes all the time, but it cannot have sudden changes at some of the points.

C) For any whole number, $\frac{1}{x}$ will always be a rational number and that makes it hard to draw the graph for irrational numbers.

D) The problem is that you don't have enough points. You need to include more points to make it look correct.

13. In a unit on simplifying expressions, one of Mr. Serrano's students wrote the following correct solution.

$$\frac{4(a+2)}{3a}+2-\frac{8}{3a}-\frac{6a-1}{6}$$

$$=\frac{4a+8}{3a}+2-\frac{8}{3a}-\frac{6a}{6}+\frac{1}{6}$$

$$=\frac{4a}{3a}+1\frac{6}{6}-a+\frac{1}{6}$$

$$=1\frac{1}{3}+1\frac{7}{6}-a$$

$$=1\frac{2}{6}+1\frac{7}{6}-a$$

$$=2\frac{9}{6}-a$$

$$=3\frac{3}{6}-a$$

$$=3\frac{1}{2}-a$$

Of the following descriptions, which best characterizes the student's work?

A) The student knows how to simplify expressions very well and demonstrates strategic use of standard procedures.

B) The student shows good computational skill but does not use processes efficiently.

C) The student knows how to simplify expressions very well, but in the solution the student should write all steps involved in the calculation, such as the step $\frac{8}{3a}-\frac{8}{3a}$.

D) The student should apply a more formal procedure by first finding the common denominator and then adding all terms.

14. A lesson in Ms. Taylor's textbook states the associative and commutative properties of addition. To motivate the students to learn the properties, she tells her students that the properties can often be used to simplify the evaluation of expressions.

    She wants to give her students an example that will focus their attention on how these properties can be useful in evaluating expressions. Of the following expressions, which would best serve her purpose?

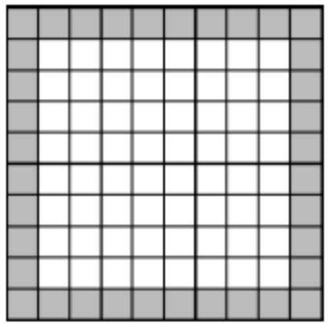
    A) $(455 + 456) + (457 + 458)$

    B) $(647 + 373) + (227 + 456)$

    C) $(551 + 775) + (49 + 225)$

    D) Each of these expressions would serve her purpose equally well.

15. Ms. Kamp asks her students to consider squares of different side lengths with only the boxes along the sides shaded as in the figure below.



She asks each student to write an expression for the number of shaded boxes in a square with a side length of n boxes and to explain why the expression gives the number of shaded boxes for any size square. For each of the following explanations, indicate whether or not it provides evidence that the student understands why the expression can be used to find the number of such shaded boxes in any square.

		Provides Evidence	**Does Not Provide Evidence**
A)	If you start on the bottom left and go to just below the top left, then start with the top left and go to just before the top right, and keep doing that, you will get 4 groups, and each group has 1 less than n, so you get $4(n-1)$.		
B)	My expression is $n+2(n-1)+(n-2)$ because the top of the square has n shaded boxes, then each of the sides has $n-1$ shaded boxes left, and then the bottom has $n-2$ shaded boxes left.		
C)	Inside the square with a side length of n boxes is a square with side length of $n-2$ boxes, so if you find the area of the two squares and subtract them, you will find the number of shaded boxes. So, I get $n^2-(n-2)^2$.		
D)	I get $4(n-2)+4$ because there are 36 boxes shaded, and when you put 10 in for the n in $4(n-2)+4$ and follow the order of operations, the answer is 36.		
E)	My expression is $2n+2(n-2)$ because if you simplify $2n+2(n-2)$ you get $2n+2n-4$, which is equal to $4n-4$, and because this doesn't depend on n, it works for any n.		

16. Ms. Quinn asked her students to solve the following quadratic equation.

$$3x^2 - 6x - 24 = 0$$

Maurice explained, "I added 24 to both sides and divided by 3."

$$3x^2 - 6x = 24$$
$$x^2 - 2x = 8$$
$$x(x-2) = 8$$

He then concluded, "The only numbers that are 2 apart and multiply to be 8 are 2 and 4, and -2 and -4, so x has to be 4 or -2." Students agreed that 4 and -2 work when you substitute them into the original equation, but they were unsure about his method.

Of the following statements, which best characterizes Maurice's approach to this problem?

A) Maurice's method is wrong because you cannot solve an equation by factoring unless one side of the equation is equal to zero.

B) Maurice's method is wrong because he should have first divided by 3 and then factored the left side of the equation.

C) Maurice's reasoning is correct, but his method often leads to an equation that cannot be solved by inspection.

D) Maurice's reasoning is correct, but his method only works for equations with real roots.

17. In the last class, Mr. Rosen's students graphed quadratics of the form $y = x^2 + c$ for various values of c and developed a rule about shifting the graph of $y = x^2$ up or down. Today Mr. Rosen asked the students to predict what would happen when they graphed $y = (x - 3)^2$ and then to graph it. Students were surprised that the graph shifted 3 units to the right rather than left or down as they had predicted. Mr. Rosen asked them to explore a little further in groups.

As he walked around the classroom, each group explained to him the strategy they were using to explore the problem. Of the following student descriptions of a strategy for exploring the problem, which is most directly related to the underlying mathematical reason for the graph's behavior?

A) We are trying to prove the rule, so each of us is graphing another one, $y = (x - 2)^2$, $y = (x - 5)^2$, $y = (x + 2)^2$, and $y = (x + 1)^2$, and then we will compare our results.

B) We are making a table like yesterday and putting x and x^2 and $(x - 3)^2$, so we can plug in different inputs and compare what the outputs are in x^2 and $(x - 3)^2$.

x	x^2	$(x-3)^2$
0		
1		

C) We decided to look at the roots of $y = x^2$ and $y = (x - 3)^2$. The vertex of $y = (x - 3)^2$ is $(3, 0)$, and the vertex of $y = x^2$ is $(0, 0)$. We are looking to see what x-values we have to put in $(x - 3)^2$ to make the outputs the same as in x^2.

D) We are going to FOIL it out so it will look more like the ones from yesterday, and then we can graph it and compare the way we did yesterday.

18. Ms. Seidel is introducing the distributive property. To motivate her students, she wants to give them an example that will focus their attention on how using the distributive property can simplify computations. In which of the following examples will the use of the distributive property most simplify the computations?

A) $12 \times 29 + 12 \times 38 =$ ___

B) $17 \times 37 + 17 \times 63 =$ ___

C) $13 \times 13 + 15 \times 15 =$ ___

D) $16 \times 24 + 16 \times 24 =$ ___

19. Ms. Christensen asked Carla to simplify the following expression.

$$\frac{10a + 4}{2a}$$

Carla wrote the following incorrect solution.

$$\frac{10\cancel{a} + \overset{2}{\cancel{4}}}{\cancel{2}\cancel{a}} = 10 + 2 = 12$$

Ms. Christensen then asked the class what was wrong with this solution. Which of the following student explanations characterizes what was most likely wrong with Carla's solution?

A) She should have written $2(5a + 2)$ in the top and then canceled the 2s in the top and the bottom.

B) She saw you can break the fraction into two fractions, but the way she simplified each fraction is wrong.

C) She divided the 4 by 2, but you cannot cancel when you have more than one thing added in the top.

D) It's not possible to cancel like this because you can only cancel factors that are the same for the top and the bottom.

20. Ms. Lindsey's textbook uses a geoboard to model slope as rise over run. (Geoboards are flat blocks of wood, roughly one foot square, with pegs laid out on a grid where rubber bands can be hooked to make lines or polygons.) As her students explored different slopes they could make on the geoboard, Edward asked, "Since the diagonal of one of the unit squares has length $\sqrt{2}$, does that mean you can make a line segment with slope $\sqrt{2}$ on the geoboard?" When Ms. Lindsey asked the class whether they thought this could be done, the following exchange occurred.

Andy:	Edward's right that the diagonal of the unit square has length $\sqrt{2}$, but its slope is 1.
Beth:	Well, that doesn't matter. We can just turn the geoboard so that the diagonal is horizontal, and then we can see squares with side length $\sqrt{2}$.
Caitlin:	Sure, but the square roots of two would just cancel. I think they always would, so you can't get $\sqrt{2}$ as a slope.
Dan:	That's not right, because we can make one length of $\sqrt{2}$ and another length of 1 and use them as the rise and the run.

Which of the student statements gives the best insight into Edward's question?

A) Andy's statement

B) Beth's statement

C) Caitlin's statement

D) Dan's statement

21. Ms. Collingwood is teaching a unit on graphing. Some of the students in her class speak a language other than English at home and when they work in small groups. As she prepares for the unit, she makes a note of vocabulary words that she believes will be challenging for her students, especially words that have different meanings in different contexts. Of the following vocabulary words that she identified, which will require the least clarification regarding differences in meaning?

    A) Plane

    B) Origin

    C) Intercept

    D) Parabola

22. Mr. Baas' students were solving inequalities. Cheryl wrote the following solution on the board.

$$\frac{x-2}{x} < 1$$
$$x - 2 < x$$
$$0 < 2$$

She concluded that because this is always true, every x would work. After some discussion, students decided that Cheryl's solution was not correct, but they were unsure why. Of the following explanations, which best identifies what is problematic about Cheryl's work on the problem?

A) It's true that 2 is always greater than 0, but because you have eliminated all x, we cannot say what x is.

B) We can see that the numerator always is two less than the denominator, so the fraction will always be less than 1 for all x. However, we have to require the denominator $x \neq 0$.

C) You don't know what x is, so when you multiply with x like this, you must assume $x > 0$.

D) She should have simplified the left-hand side of the inequality to $1 - \frac{2}{x}$ and then subtracted 1 from both sides and added $\frac{2}{x}$ to both sides. This would yield $0 < \frac{2}{x}$, which is true for $x > 0$.

APPENDIX B

COACTIV – 1 MODIFIED ASSESSMENT

Max-Planck-Institut für Bildungsforschung
Max Planck Institute for Human Development

Befragung im Rahmen von

COACTIV-I

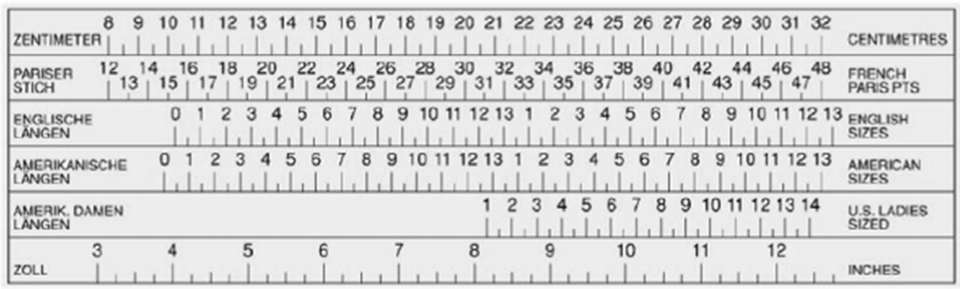
ASSESSMENT BOOKLET 1

COACTIV-I

ID

2. How does the surface area of a square change when the side length is tripled? Please write down as many different ways of solving (and reasoning thought) this problem as possible.

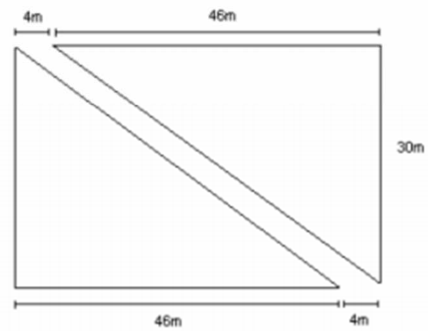
The table below is printed on a lot of shoe boxes in Germany



In Germany, shoe sizes are measured in “Paris Points”. How long (in cm) would be the feet of a person who has a shoe size of 63 in Paris Points?
Show your working.

3. Please write down as many different ways of solving this problem as possible.

A path runs through a rectangular field (see diagram). What is the total area of the field excluding the path?



4. Please write down as many different ways of solving this problem as possible.

5. Luke says "The square of a number is always one less than the product of the numbers on either side of it."

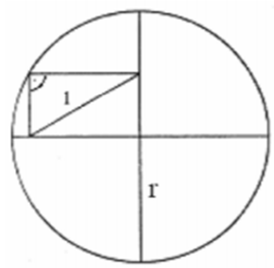
Is Luke right? Please write down as many different ways as possible of solving this problem.

6. A student says "I don't understand why (-1) x (-1) = 1." Please outline as many different ways as possible of explaining this concept to the student

A student calculates $1\frac{3}{4}$ divided by $\frac{1}{2}$ correctly, but says that it "doesn't mean a thing" to her.

7. Please outline one way that this concept could be explained to her.

What is the length of the hypotenuse l of a right-angled triangle in a circle with radius r (see diagram)?

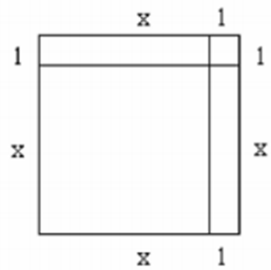


8. Describe the relationship between l and r in the clearest way possible.

9. Which of the following versions of Bayes Problem do you think would be easiest to solve? Please give a reason for your choice.

Version A 10 of a group of 1000 people with no symptoms have TB. A test used to identify those people who have the disease is successful in 8 of the 10 genuine cases. However, it also gives a "false positive" result for 99 of the 990 healthy people. How many of the people with a positive test result actually have TB?	**Version B** 1% of all people with no symptoms have TB. A test used to identify people who have the disease is successful in 80% of cases. However, it also gives a "false positive" result for 10% of healthy people. What percentage of people with no symptoms, but a positive test result, have TB?
Version C The probability of a person with no symptoms having TB is 1%. The probability of people with the disease being identified by a test is 80%. The probability of the test giving a "false positive" result is 10%. What is the probability that a person with no symptoms, but a positive test result, has TB?	**Version D** 10 of 1000 people with no symptoms have TB. A test used to identify the people who have the disease is successful in 800 of 1000 genuine cases. However, it also gives a "false positive" result for 10 in 1000 healthy people. How many of the people with a positive test result actually have TB?

When walking past a colleague's classroom, you see the diagram below on the board:



10. What was likely to have been the purpose of this sketch?

11. An elementary student works out that

$$\begin{array}{r} 542 \\ -\ 242 \\ \hline 310 \end{array} \quad \text{and} \quad \begin{array}{r} 676 \\ -\ 835 \\ \hline 241 \end{array}$$

What is likely to be their answer to 814 - 333 = ___?

A student using a calculator worked out that

$$\sqrt[2]{9^4} = 3$$

$$\sqrt[3]{4^6} = 2$$

What is likely to be their answer to the following problem?

$$\sqrt[5]{8^{10}} = \underline{\hspace{3cm}}?$$

12.

A student works out that

$$\sin 30^\circ + \sin 60^\circ - 1 = 0$$

$$\frac{\sin 45^\circ}{2} = \frac{1}{4}$$

What is likely to be their answer to the following problem?

13. $\sin 10^\circ =$ _________ ?

14. Students frequently have difficulty accepting the definition a° = 1. What might be a reason for this?
Please list as many reasons as possible.

15. Please briefly outline as many ways as possible to make this definition more accessible to students .

16. A student calculates the solutions for the equation (x - 3) (x - 4) = 2 to be x = 5 or x = 6.

How did the student probably come up with this answer?

17. A group of students is given the following problem:

There are S students and P professors at a university. There are 6 students to a professor. Write an equation to show the relationship between S and P.

What might be a common student error?

$$(b_1 + b_2) \cdot \frac{h}{2} \qquad \frac{b_1 \cdot h}{2} + \frac{b_2 \cdot h}{2} \qquad \frac{(b_1 + b_2) \cdot h}{2} \qquad \frac{(b_1 + b_2)}{2} \cdot h$$

18. Here are four formulas that can be used to find the area of a trapezoid.

What might be the pedagogical value of considering each of these formulas?

Please estimate the following without using a calculator

$(0.9)^{50} \approx ?$ (Tip: $0.9^7 \approx \frac{1}{2}$)

19. What is your estimate?

20. How did you come up with your estimate?

21. Please give a definition for the derivative of a function, and use that definition to show that the derivative of the function $f(x) = x^2$ is the function $f'(x) = 2x$.

22. How would you prove that the base angles of an isosceles triangle are congruent?

23. Let C be the field of complex numbers and $\dot{z} = a - bi$ the conjugate of the complex number $z = a + bi$. Why is there no z □ C when $z \cdot \dot{z} = 23$ and a, b □ Z

Let n be a natural number and and let the numbers a and b be defined as follows:

$a = 2n + 1$ $b = ½ (a^2 - 1)$

24. Is b an odd or an even number?

25. Is $a^2 + b^2$ always a square number?

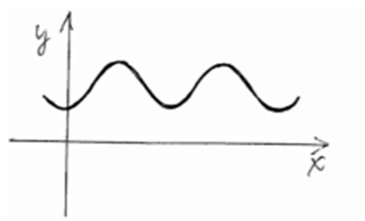
(Hint: Test the statement for some n, make a guess on the basis of your observation, and then try and prove this guess.)

26. How would you prove that $\sqrt{2}$ is irrational?

Consider the function

$$y = a \sin (bx + c) + d$$

The independent variable is x and a, b, c, d are parameters. When these parameters have certain values, the following curve is produced (see sketch):



Please describe in a few words what happens to the graph when

27. parameter a is doubled.

28. parameter b is doubled.

29. parameter c is doubled.

30. parameter d is doubled.

31. Is it true that 0.9999999... = 1 ?   Please explain your answer

32. The coefficients of Pascals triangle come up

a) in combinatorics as "the number of permutations of k objects selected from n objects."

b) in algebra as the coefficients of the development of $(a + b)^{N}$.

Please briefly describe the relationship between these two meanings.

A student calculates the solutions to the equation (x - 3) (x - 4) = 2 to be x = 5 or x = 6. When asked, the student gave the following explanation:

I've learned that it follows from	(x - 3) (x - 4) = 0	that Therefore	x - 3 = 0 or x - 4 = 0 x = 3 or x = 4
So it follows from	(x - 3) (x - 4) = 2	that Therefore	x - 3 = 2 or x - 4 = 2 x = 5 or x = 6

Later another student makes the following calculation.

$$\sqrt{5+7} = \sqrt{5} + \sqrt{7}$$

33. Please describe the difference between the errors these two students are making.

34. A large number of students have been given the problem:

There are S students and P professors at a university. There are 6 students to a professor. Write an equation to show the relationship between S and P.

The most frequently observed student error is the equation $P = 6S$.

Please give possible reasons for this error being made - what might the students have been thinking?

35. Please briefly describe possible teaching strategies that would target this error.

APPENDIX C

COACTIV – 1 RATER CODEBOOK

Codebook

Test of Mathematics Teachers' Professional Knowledge (PCK – CK)

Pedagogical Content Knowledge
&
Content Knowledge

- Rater Version -

Version of 10.10.2008

PART 1 - PCK

General notes on coding:

1. In questions that require multiple responses (i.e., problems that can be solved/explained in various ways), each response is given a separate code. For example, a question with responses numbered 1), 2a), 2b), and 3), would be given 4 codes and a total score.
2. If the respondent misunderstands a question, it should be assumed that the entire question has been misunderstood – in other words, each response given is coded as "misunderstood".
3. If answers refer back to previous responses (e.g., "see point 2"), the same code is given – unless it is obvious that another solution is meant.
4. In questions that require respondents to give only one response, but that can be answered in various ways, the first *correct* response given is coded. If none of the responses given are correct, the first is coded.
5. In case of doubt caused by "two solutions in one," the first *correct* response given is coded. If none of the responses given are correct, the first is coded.
6. Responses given in brackets are not given a code unless they are completely correct and score an additional point.

SQUARE

How does the surface area of a square change when the side length is tripled? Show your reasoning.

Please note down as many different ways of solving this problem (with reasonings) as possible.

CODING SCHEME

Responses	**Response Code** bqua1-bqua5	**Score** bquasc
Missing (no response, dash, or "?")	999	0
Non-classifiable (e.g., "not in the mood," "can't concentrate," "material not covered at the level I teach," etc.)	98	0
Incorrect/unintelligible/incomplete or irrelevant response; question misunderstood (respondent misunderstands the question; e.g., writes down the errors his or her students would make)	0	0
Incorrect: numerical example(s) only (WITHOUT generalization): Solves the question using arbitrary examples (without a system), giving the result: "9 times larger". Example: Let the surface area of the original square be 1 cm^2. The surface area of the "new" square is then – because $(3{\cdot}1cm)^2 = 9cm^2$ – 9 times larger. Includes the response "numerical example(s)".	1	0
Correct: Paradigmatic (example AND generalization): Solves the question using arbitrary examples (without a system), giving the result: "9 times larger". Example: Let the surface area of the original square be 1 cm^2. The surface area of the "new" square is then – because $(3{\cdot}1cm)^2 = 9cm^2$ – 9 times larger. **This response includes a generalization, such as "That is evidently always the case!" The generalization may also be algebraic.**	2	The total number of responses of different structural types scoring Code 2-8 is taken as the Score.*
Correct: Algebraic: Surface area of the original square: a^2. Surface area of the "new" square: $(3a)^2= (a+a+a)(a+a+a)= 3^2a^2$ or $=9a^2$, or 9 times larger. May be accompanied by a sketch. $(3x)^2= 9x^2$ or similar (including responses in which the algebraic solution is accompanied by a sketch) Likewise "$x \dashrightarrow x^2$ and $3x \dashrightarrow 9x^2$" or "$3^2a^2 = 9a^2$" Likewise $(a+2a)^2 = a^2+4a^2+4a^2 = 9a^2$ (may be accompanied by a sketch).	3	
Correct: Geometric I: 9 times the size of the original square! **"Sloppy" drawings** are to be given **Code 4** if it is clear what is meant. (Simply giving the response "graphic solution" **without providing a drawing** is not sufficient = **Code 0!**) *(It is up to raters to decide if drawings are substandard!)* **A drawing must be provided!** **Exception:** The respondent gives a detailed description of how a drawing should be done and what it can/should show.	4	
Correct: Covariative/functional: The surface area is the square of the side length, so you can have to square the lengthening factor to determine the change in the surface area. Therefore, the new square will be 3^2, i.e., 9 times larger. Responses given this code explain the relationship between side length and surface area in written form. The response "Detailed text!" It is not sufficient. Likewise: The side length squared is the area, therefore 3x3 = 9, so 9 times the size. Also: Reasoning draws on 2-dimensionality to explain why each side times 3 => 9x the size.	5	

Correct: Geometric II Central Dilatation: Response refers to the quadratic scale factor (e.g., k^2) or a picture illustrates that the square is stretched/tripled "in two directions" (e.g., upwards and to the right).	6	
Correct: Reference to quadratic function E.g., the function $f(x)=x^2$, if you triple the x value, the y value is 9 times larger, e.g. $f(1)=1$ and $f(3)=9$.	7	
Any other correct solution (of a different structural type from 2-7)	8	

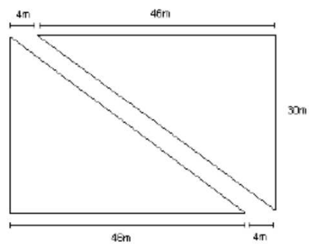
***Note on scoring:**

If the respondent gives two or three responses with slightly different reasonings but the same response code, no additional points are awarded. A maximum of one point per Code type can be awarded.

Example: *The respondent gives (and numbers) three solutions, the first two of which are given Code 3, and the third of which is given Code 2. The total Score given is 2 points (the first two responses are of the same structural type, so only one point is awarded; the third solution is also correct and of a different structural type, so another point is awarded, giving a total Score of 2 points).*

PATH

A path runs through a rectangular field (see diagram). What is the total area of the field excluding the path?



Please note down as many different ways as possible of solving this problem.

General note (based on responses given in 2004): If the respondent does the right calculation, but using the wrong numbers (e.g., "(46-4)x30" instead of "(50-4)x30"), the response is coded as if it were correct.

CODING SCHEME

Responses	**Response Code** bpfad1 - 6	**Score** bpfasc
Missing	999	0
Non-classifiable (e.g., "not in the mood," "can't concentrate," "material not covered at the level I teach," etc.)	98	0
Incorrect/unintelligible/incomplete or irrelevant response, question misunderstood (respondent misunderstands the question; e.g., writes down the errors his or her students would make)	0	0
Calculations partially correct, but final result incorrect. If the respondent takes a correct approach, but the final result is incorrect, Response Code 1 (Score 0) is awarded.	1	0
Correct: Place the triangles together to make a rectangle of area 46 x 30 or to make a parallelogram! Including responses such as "Cut the triangles out and join or glue them together."	2	The total number of responses of different structural types scoring Code 2-5 is taken as the Score.*
Correct: Total area minus path. It must be clear that the area of the path is to be deducted from the total area of the rectangle.	3	
Correct: Sum of the two small triangles	4	
Any other correct response (of a different structural type) Other responses include 1. count the number of unit squares 2. glide reflection: Place the triangles together to make a large triangle (of g=2x46 and h=30) 3. use a computer to solve the problem	5	

Square of a Natural Number

Imagine that your students have been working on the task below:

> Luke says: "The square of a natural number is always 1 more than the product of the numbers on either side of it."
>
> Is Luke right?

Please note down as many different ways as possible of solving this problem.

Correct responses (see next page for codes and scoring):

1. Algebraic:

a) Let n be any given natural number.

$(n - 1) \cdot (n + 1) = n^2 - 1$, which is 1 smaller than n^2

b) $a, (a+1), (a+2) \Rightarrow (a+1)^2 = a^2 + 2a + 1$

$a^* (a+2) = a^2 + 2a$

(If a proof [e.g., complete induction] is given as response, it should also be conducted – at least in part. It is not sufficient simply to state that a proof could be used!)

2. Paradigmatic:

a) Let us take a number such as 18. The numbers on each side are 17 and 19.

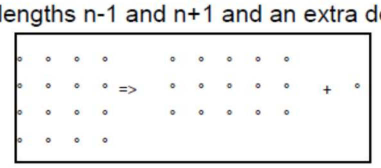
$(18 - 1) \cdot (18 + 1) = 18^2 - 1$

It is evidently always the case!

b) Tests the result using arbitrary examples. Then gives the general formula (as in 1.)

3. Geometric:

Graphic representation of a square of dots with a side length of n dots. The number of dots is thus n^2. One row is taken away and moved, e.g., to the "right" of the square. This gives a rectangular pattern of dots with the side lengths n-1 and n+1 and an extra dot. For example:



Incorrect responses – numerical examples only:

1. A single numerical example:

For example, $17 \cdot 19 = 323$ and $18^2 = 324$, so it's correct.

2. A series of numerical examples:

Examples:

$4 \cdot 4 = 16$, $3 \cdot 5 = 15$: which is 1 less

$5 \cdot 5 = 25$, $4 \cdot 6 = 24$: which is 1 less

$6 \cdot 6 = 36$, $5 \cdot 7 = 35$: which is 1 less. Luke is right!

Tables are also given this code:

n	n²	n-1	n+1	(n-1)(n+1)	Difference
2	..				1
3	…				1
4	…				1

CODING SCHEME

Responses	**Response Code** bnach1 - bnach6	**Score** bnacsc
Missing	999	0
Non-classifiable (e.g., "not in the mood," "can't concentrate," "not material covered at the level I teach," etc.)	98	0
Incorrect response 1: A single numerical example For example, 17 · 19 = 323 and 18² = 324, so it's correct. Includes the responses: "Example" (in the singular) or "try it out with numbers " (where it's not clear whether one or a series of examples are meant)	0	0
Incorrect response 2: A series of numerical examples Examples: 4 · 4 = 16, 3 · 5 = 15 : which is 1 less 5 · 5 = 25, 4 · 6 = 24 : which is 1 less 6 · 6 = 36, 5 · 7 = 35 : which is 1 less. Luke is right! c) Tables are also given this code. Includes "examples" (in the plural)	1	0
Other incorrect solution: Including "$x^2 + 1 = (x+1)\cdot(x-1)$" or "$n^2 < (n-1) \cdot (n+1) = n^2 - 1$", Also insufficient: "$a^2$-(a+1)(a-1)." The response must refer to the question. It must be clearly stated: "1 more" or "Luke is right." Also: "Use a computer to solve the problem" (with no further explanation)	2	0
Correct solution 1: Algebraic a) Let n be a natural number. $(n - 1)\cdot(n + 1) = n^2 - 1$, which is 1 smaller than n^2 b) a, (a+1), (a+2) => $(a+1)^2 = a^2 + 2a + 1$ $a \cdot (a+2) = a^2 + 2a$ (If a proof [e.g., complete induction] is given as response, it should also be conducted – at least in part. It is not sufficient simply to state that a proof could be used!) Complete induction (at least in part!) is thus given Code 3! **Also includes** (based on responses given in 2004): **n^2 <--> (n-1)·(n+1) or** **n, (n+1), (n+2) or** **"third binomial formula" (if it is clear that this statement is valid.)**	3	The total number of responses of different structural types scoring Code 3-6 is taken as the Score.*
Correct solution 2: Paradigmatic **Paradigmatic:** a) Let us take a number such as 18. The numbers on each side are 17 and 19. $(18 - 1)\cdot(18 + 1) = 18^2 - 1$ **It is evidently always the case!** b) Tests the result using arbitrary examples. Then gives the general formula (as in 1.) **Includes the response: "Trying it out and generalizing."** **What is important here is that it's clear that the examples are an integral part of the response and not just extraneous detail.**	4	
Correct response 3: Geometric Graphic representation of a square of dots with a side length of n dots. The number of dots is thus n^2. One row is taken away and moved, e.g., to the "right" of the square. This gives a rectangular pattern of dots with the side lengths n-1 and n+1 and an extra dot. Includes: "Compare areas n^2 and $(n-1)\cdot(n+1)$" or "compare the area 5x5 with the rectangle 4x6 (i.e., example based on areas)"	5	
Any other correct response (of a different structural type) e.g., Consider functions: $f(x)=x^2$ and $g(x)=(x-1)\cdot(x+1)=x^2-1$, showing that if one shifts the graph of g by 1 on the y-axis, one obtains the graph f. => Luke is right.	6	

DIVIDING BY FRACTIONS

A student calculates $1\frac{3}{4}$ divided by $\frac{1}{2}$ correctly, but says that it "doesn't mean a thing" to her.

How might this mathematical concept be explained to the student?

Correct response:

What is decisive here is that the response refers to the "goes into" aspect of division. We are also interested in whether the responses are entirely text based (e.g., I'd ask: "How may half cakes are there in 1¾ cakes?") or use graphical representations. An example of a graphical representation would be:
Imagine 1 whole cake and a ¾ cake:

How many half cakes would you get from these 1¾ cakes?

CODING SCHEME

Caution: Many of the responses given entail dividing by 2, rather than ½ (→Code 0!)

Responses	**Response Code** bbru	**Score** bbrusc
Missing	999	0
Non-classifiable (e.g., "not in the mood," "can't concentrate," "material not covered at the level I teach," etc.)	98	0
"Incorrect" (implausible or "woolly" explanation) Application of formula (reciprocal), with/through examples and insufficient/incomplete explanations (e.g., merely mentions the division aspect or gives a bad explanation of the division aspect), etc. • Examples of dividing by **"half a person,"** etc., are wrong! • Formulating the reciprocal task: $x \cdot ½ = 1¾$, what is x? • "Halving the 7 ¼ pieces of cake gives twice as many pieces."	0	0
Correct: "goes into" explanation (text only) What is decisive here is that the response refers to the "goes into" aspect of division. We are also interested in whether the responses are entirely text based (e.g., I'd ask: "How may half cakes are there in 1¾ cakes?") or use graphical representations. An example of a graphical representation would be: Imagine 1 whole cake and a ¾ cake: How many half cakes would you get from these 1¾ cakes?	1	1
Correct: "goes into" explanation (with graphic representation) (e.g., cakes or real line or line segment)	2	1
Other didactically suitable explanation Explanations that do not refer to the "goes into" aspect, but to the division aspect: "*Imagine refilling 1 ¾ liters of liquid into ½-liter bottles. How many ½-liter bottles can you fill with 1 ¾ liters of liquid?"* = "filling bottles" Also acceptable: Principle of permanence. Can be formulated in verbal terms only.	3	1

BAYES

Which of the following four versions of the same problem on Bayes' theorem will it be easiest for students to solve (even if they are not familiar with Bayes' theorem)? Please give reasons for your choice.

Version A 10 of a group of 1000 people with no symptoms have TB. A test used to identify those people who have the disease is successful in 8 of the 10 genuine cases. However, it also gives a "false positive" result for 99 of the 990 healthy people. How many of the people with a positive test result actually have TB?	**Version B** 1% of all people with no symptoms have TB. A test used to identify people who have the disease is successful in 80% of cases. However, it also gives a "false positive" result for 10% of healthy people. What percentage of people with no symptoms, but a positive test result, have TB?
Version C The probability of a person with no symptoms having TB is 1%. The probability of people with the disease being identified by a test is 80%. The probability of the test giving a "false positive" result is 10%. What is the probability that a person with no symptoms, but a positive test result, has TB?	**Version D** 10 of 1000 people with no symptoms have TB. A test used to identify the people who have the disease is successful in 800 of 1000 genuine cases. However, it also gives a "false positive" result for 100 in 1000 healthy people. How many of the people with a positive test result actually have TB?

Correct response:
Empirical studies identify **A** as the version that most people are able to solve.
There are several reasons for this that respondents may give in their answers.
To be awarded Score 1, the respondent should give at least one of the reasons listed below:

a)	There are fewer steps to carry out in Version A (just 8 / 8+99) Includes reasons such as A is "easier" or "quicker" to solve.
b)	The way the information is presented in Version A means that some of the calculation has already been done (= another perspective on Argument a)
c)	Version A doesn't involve the concept of conditional probability.
d)	There's no need to apply Bayes' formula in Version A (only a simplified version of it, see Argument a)
e)	In all other versions, the information is standardized (B: 100% of people, C: 100% probability, D: 1000 people), with "100" meaning something different in each case.
f)	All the numbers in Version A are adjusted to a sample. All the information given relates to subsets of this sample of size 1000 (= another perspective on Argument d).
g)	The information presented in Version A can be entered in a simple tree diagram or table; it is then easy to read the answer off that diagram/table.
h)	It's easier to work with integers than with percentages. Integers are "more natural," because (unlike percentages and probabilities) we come across them all the time in daily life. Arguments such as "no percentages and probabilities" are common.

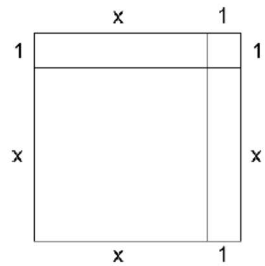
Respondents choosing Version A and giving one of the reasons a)-h) above are awarded Code 4 and Score 1 (even if one or more of the reasons given are wrong – what's important is that at least one correct reason is given).

Coding Scheme

Responses	Response Code bbay	Score bbaysc
Missing	999	0
Non-classifiable (e.g., "not in the mood," "can't concentrate," "material not covered at the level I teach," etc.) Including respondents who state that they "can't decide" on an answer.	98	0
Respondent chooses version **B**	0	0
Respondent chooses version **C**	1	0
Respondent chooses version **D**	2	0
Respondent chooses version **A,** but the **reason** is **incorrect or not given**	3	0
Respondent chooses version **A and gives at least one correct reason**	4	1

Blackboard Sketch

When walking past a colleague’s classroom, you see the diagram below on the board:



What was likely to have been the purpose of this sketch?

Correct:

The sketch illustrates (represents!) an algebraic equation geometrically: If you know the areas of the three different components (x^2, 1x and 1by1), you can use the diagram to work out the area of, for example, $(x + 1)^2$ by adding up the individual parts.
Or the respondent refers to the potential for generalization: “He or she probably wanted to illustrate $(x+k)^2$ in graphic form, starting with k = 1.”

Incorrect / Trivial Responses:

Responses such as “They are working out the area of a plot of land with side length x + 1“ are of course not incorrect in the normative sense, but they barely go beyond everyday knowledge and are thus not an indicator for a teacher’s expertise. Explanations that refer to areas, e.g., increasing the size of a tennis court, etc., (estimating increase in area) are also classed as "trivial responses" and given Code 1.

Coding Scheme

Responses	**Response Code** btaf	**Score** btafsc
Missing	999	0
Non-classifiable (e.g., "not in the mood," "can't concentrate," "material not covered at the level I teach,” etc.)	98	0
Incorrect response	0	0
Trivial response, e.g., calculation of an area	1	0
Correct response as defined above	2	1
Other correct response	3	1

To the Power of Zero a)

Students frequently have difficulty accepting the definition $a^0 = 1$.

a) What might be the reasons for this difficulty? Please list as many reasons as possible (and number each).

Correct responses:

1) "a to the power of something" is otherwise always larger than a, for example in a^2. (C*oncrete numerical perspective.)*

2) It's impossible to imagine zero times the factor a. (C*oncrete operation-based perspective.)* If there's a zero there, then the answer must be "zero." *(Dominance of "zero" as the supposedly only identity element.)*

3) Students have to develop a "relational" perspective. (R*efers to a general cognitive difficulty.)*
"*Students can no longer look at every number individually, as they may have been used to doing. Now they have to think in terms of "relations between the numbers," or take a "relational" approach. And that's more difficult."*
What is important here is that words such as **"relation/s" or "relational"** are used. This Code is given if, and only if, this specific reasoning is given. If the response is more general, but understandable, it is more likely to be awarded Code 4. (MN)

4) Students have to free themselves of a purely numbers-related perspective, and that is a "difficult" cognitive process. (*refers to individual difficulties.)*
Students cannot imagine/find it difficult to imagine that the exponent "0" always gives the same result – namely "1" – even when the bases differ. (MB/MN)

a) Students cannot imagine/find it difficult to imagine this situation because there are no practical, real-life examples of it. It is impossible/very difficult to illustrate the point. (MN)

b) Students cannot imagine this situation!

These two responses seem identical on the surface, but teachers very often list them separately. Both are reported here to emphasize that – beyond a lack of understanding of the principle of permanence – both the lack of opportunity to relate the definition to real-life examples and the simple statement that "students can't imagine this situation" (without specification of any reasons) (MB)

5) The definition is based on the principle of permanence, which permits a logical procedure to be extended meaningfully and consistently beyond the realm of experience. This is the first context in which this kind of thinking is required in school.
Code 5 is awarded for the response "principle of permanence" or "continuation of a series of exponents" or "consistency with the laws of exponents," or similar reasoning. Here again, if the response is more general, Code 4 should be awarded.

Coding Scheme (If the respondent gives two or three responses with slightly different reasonings but the same response code, no additional points are awarded. A maximum of one point per Code type can be awarded.)

Responses	**Response Code** bnula1-6	**Score** bnuasc
Missing	999	0
Non-classifiable (e.g., "not in the mood," "can't concentrate," "material not covered at the level I teach," etc.)	98	0
"Incorrect" (or didactically unsatisfactory) response or responses such as "the student can't imagine the situation" **without provision of any further explanation** (equivalent of "I don't know"). (MN)I	0	0
1) concrete numerical perspective *(rarely given)*	1	The total number of responses **of different structural types** given Code 1-6 is taken as the Score.
2) concrete operation-based perspective	2	
3) general–cognitive	3	
4) individual–cognitive	4	
5) principle of permanence	5	
6) other didactically plausible explanation.	6	

To the Power of Zero b)

b) Please briefly outline as many ways as possible to make this definition accessible to students (and number each method)!

Correct responses:

1) Principle of permanence, numerical: $2^3 = 8$, ... $2^2 = 4$, ... $2^1 = 2$, ... The result of the previous step is always "divided by 2." Therefore, 2 "to the power of 0" must be 2:2 = 1.

2) Principle of permanence with variable: a^3 , ... a^2 , ... $a^1 = a$, ... The result of the previous step is always "divided by a." Therefore, a "to the power of 0" must be a:a = 1.

3) Laws of exponents must remain in force; **permanence of formal computing laws.** It must remain the case that $a^{x+y} = a^x a^y$.

4) Compute $a^0 = \frac{a^n}{a^n}$. Also by reference to an example (a = 2, n = 4)

Incorrect responses:

1) Have students memorize the rule, in the same way as they have to learn other rules, e.g., the response: "It's a definition!" *(methodological procedure of learning facts or reference to definitions.)*

Coding Scheme

(If the respondent gives two or three responses with slightly different reasonings but the same response code, no additional points are awarded. A maximum of one point per Code type can be awarded.)

Responses	**Response Code** bnulb1-5	**Score** bnubsc
Missing	999	0
Non-classifiable (e.g., "not in the mood," "can't concentrate," "material not covered at the level I teach," etc.)	98	0
Insufficient or incorrect suggestion	0	0
Incorrect response **1)** Learn the rule by heart	1	0
Correct response **1)** **NB: Code 2**	2	The total number of responses **of different structural types** given Code 2-6 is taken as the Score.
Correct response **2)**	3	
Correct response **3)**	4	
Correct response **4)**	5	
Other "promising" approaches a) Examples from physics (e.g., exponential growth) b) Other accessible examples – what is important is that the example really is accessible (and can therefore be considered promising). It is not sufficient to use a single word or short sentence to outline an approach or refer to a subdomain of mathematics. In case of doubt: It is vital that the response identifies **another promising** approach!	6	

Professors a)

Please imagine that your students have been set the following problem:

There are **S** students and **P** professors at a university. For each professor there are 6 students. Write an equation to show the relationship between **S** and **P**.

a) What might be a common student error?

Coding Scheme

Responses	Response Code cb1v	Score cb1sc
Missing	999	0
Non-classifiable (e.g., "no idea," "material not covered at the level I teach," "need more time," etc.)	98	0
All other responses	0	0
Correct response (6P = 1S) is assumed to be a student error.	1	0
P = 6S, i.e., 6 on the wrong side of the equation	2	1

PROFESSORS B)

Continuation of Professors a)…

There are **S** students and **P** professors at a university. For each professor there are 6 students. Write an equation to show the relationship between **S** and **P**.

Several studies on this problem have shown that the most frequently observed student error across all grade levels is the following: **P = 6S**

b) Please give possible reasons for this error being made – what might the students have been thinking? Please note down as many reasons as possible *(and number each reason)*.

CODING SCHEME

Responses	**Response Code** cb2g1-5	**Score** cb2sc
Missing	999	0
Non-classifiable (e.g., "no idea," "material not covered at the level I teach," "need more time," etc.)	98	0
Respondent writes: The student didn't read the question properly, made a careless mistake, or other general responses that don't really identify a cognitive cause for the error, e.g., "doesn't understand equations," "proportional function," or "students don't insert numbers." None of these responses identify **errors in the students' thinking** that explain WHY the error was made; all are given code 0.	0	0
Word-for-word translation: For each professor there are six students. ↓ ↓ ↓ ↓ P = 6 S <u>Note</u>: Although empirical studies have shown that the error is made irrespective of the word order, respondents should be evaluated for their flexibility in generating hypotheses here (plausible!). (Variables tend to be written down in the order in which they occur.)	1	The **total number of responses given Code 1-4** is taken as the Score here.
Unclear understanding of variables (it would be correct if P were a unit of measurement!): Students don't distinguish between numbers / variables / unit of measurement ("juggle with them ") Students don't distinguish between numbers / variables / unit of measurement and multiply on the basis of their content knowledge ("There are more students so I'll multiply S by 6")	2	
Proportional relationship incorrect. Students have problems translating proportions into equations. Improper use of the equals sign to reflect that "for each x there are 6y", e.g., "The students thought that they could express this relationship using "=""	3	
Other didactically meaningful and plausible **cognitive** sources of error (in student thinking).	4	

PROFESSORS C)

Continuation of Professors a) and b)…

c) Please briefly describe possible didactic interventions targeting this error. Please note down as many interventions as possible *(and number each approach)*.

CODING SCHEME C

Responses	**Response Code** cb3g1-5	**Score** cb3sc
Missing	999	0
Non-classifiable (e.g., "no idea," "material not covered at the level I teach," "need more time," etc.)	98	0
No / unsuitable intervention Not a didactic intervention, e.g., "Be more careful!"	0	0
Content-related intervention: There can't be more professors than students at a university.	1	The **total number of responses given Code 1-7** is taken as the Score here.
Understanding of variables: S and P stand for *numbers* (not just for students and professors). E.g., suggestion: it might help for students to use X and Y rather than S and P.	2	
Insert numbers: Test the equation formulated to detect any problems.	3	
Graphic illustration (e.g., graph with 6, 12, 18 students on one axis)	4	
Understanding of the equation (e.g., derive the equation from a table of values or by looking at proportions such as S : P = 6 : 1)	5	
Calculate an **analogous example** (e.g., cows – legs)	6	
Reformulate or reword the text	7	

TRAPEZIUM A)

The following formulas all give the surface area of a trapezium.

$(g_1 + g_2) \cdot \frac{h}{2}$	$\frac{g_1 \cdot h}{2} + \frac{g_2 \cdot h}{2}$	$\frac{(g_1 + g_2) \cdot h}{2}$	$\frac{(g_1 + g_2)}{2} \cdot h$

a) What might be the didactic value of considering all of these formulas in the classroom?

Please give reasons for your answer.

CODING SCHEME

Responses	Response Code ctr1g	Score ctr1sc
Missing	999	0
Non-classifiable (e.g., "I need more time and a fresh sheet of paper," "no idea," etc.)	98	0
Respondent offers only weak, implausible reasons or no good reasons at all: - It's pointless / confusing / too much trouble to look at all the formulas: they all express the same thing. - There's no point; it's only worth looking at the "simplest" formula, which students are actually going to use.	0	0
Plausible arguments, but which show that the respondent isn't aware that the formulas represent four different ways of breaking down the surface area of a trapezium. - Transformation of terms (it's a good way of practicing algebra.) - The students have to learn how to deal with different formulas, because formulas can be used to express/describe situations.	1	1
Arguments that show that the respondent is aware that the formulas represent four different ways of breaking down the surface area. - Each formula expresses a different "approach," a different break-down. And exposure to these different approaches contributes to the students' mathematical education. - Each formula represents another way of determining the area.	2	2

Estimate

Please estimate the following without using a calculator (show your working):
$(0.9)^{50} \approx$? (Tip: $0.9^7 \approx \frac{1}{2}$)

Correct responses:

1a) $0.9^{50} \approx 0.9^{49} = \left(0.9^7\right)^7 \approx \left(\frac{1}{2}\right)^7 = \frac{1}{128} \approx 0.008$

(possibly also $\approx \frac{1}{100} = 0.01$)

$\frac{1}{2^7}$ is not sufficient.

1b) As above, with additional step:

$0.9^{50} = 0.9^{49} \cdot 0.9 \approx 0.008 \cdot 0.9 \approx 0.007$

or: $\frac{1}{2^7} \cdot \frac{9}{10} = \frac{9}{1280}\left(\approx \frac{9}{1260} = \frac{1}{140}\right)$

2a)

$0.9^2 = 0.81 \approx 0.8$

$0.9^4 \approx 0.8^2 \approx 0.64$

$0.9^8 \approx 0.64^2 \approx 0.4$

$0.9^{16} \approx 0.4^2 \approx 0.16 \approx 0.2$

$0.9^{50} \approx 0.9^{48} \approx \left(0.9^{16}\right)^3 \approx 0.2^3 \approx 0.006$

(similar approaches based on the principle of permanence also possible)

2b) As above, with additional step

$0.9^{50} \approx 0.9^{48} \cdot 0.9^2 \approx 0.008 \cdot 0.8 \approx 0.006$

Coding Scheme

Responses	Response Code erech	Score erecsc
Missing	999	0
Non-classifiable (e.g., "not in the mood," "can't concentrate," "material not covered at the level I teach," etc.)	98	0
Incorrect response: wrong approach, overly rough estimate, or errors in applying the laws of exponents	0	0
Correct approach, but response is incomplete or includes computational error.	1	0
Code 2 has been cut – has been collapsed with Code 0!		
1a	3	1
1b	4	1
2a	5	1
2b	6	1
Other correct response	7	1

Complex Number

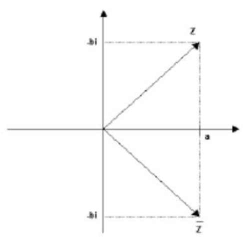
Let **C** be the field of complex numbers and $\bar{z}$ = a – ib the conjugate of the complex number z = a + ib. Why is there no z$\in$ **C** with $z \cdot \bar{z} = 23$ and a, b $\in$**Z**?

Correct response:

$$z \cdot \bar{z} = ... = a^2 + b^2$$

(And conclusion)
"The sum of 2 natural square numbers (0/1/4/9/.....) is evidently never 23"

All other correct responses are extensions on the above and are awarded Code 1, for example:…



$$|z| = |\bar{z}| = \sqrt{a^2 + b^2}$$

Since $z \cdot \bar{z} = 23 \geq 0$,

therefore $z \cdot \bar{z} = |z \cdot \bar{z}| = |z| \cdot |\bar{z}| = (\sqrt{a^2 + b^2})^2 = a^2 + b^2$

Coding Scheme

Responses	**Response Code** ekzahl	**Score** ekzasc
Missing	999	0
Non-classifiable (e.g., "not in the mood," "can't concentrate," "material not covered at the level I teach," etc.)	98	0
Incorrect/incomplete response or incorrect reasoning	0	0
Correct response	1	1

THREE NATURAL NUMBERS A), B)

PART A):

Let n be a natural number. Let the numbers a and b be defined as follows:
$a = 2n + 1$, $b = 1/2\,(a^2 - 1)$.
a) Is b an odd or an even number?

Correct response:

1. Insert definition in the equation::

$b = ½ [(2n + 1)^2 – 1] = ½ [4n^2 + 4n +1 – 1] = ½\, 4n [n + 1] = 2n [n + 1]$

Because of factor 2: b is an even number

CODING SCHEME

Responses	**Code** enza1	**Score** enza1s
Missing	999	0
Non-classifiable (e.g., "not in the mood," "can't concentrate," "material not covered at the level I teach," etc.)	98	0
Incorrect response	0	0
Correct response, but incorrect/missing/incomplete reasoning	1	0
1. Insert definition in the equation	2	1
Other correct response	3	1

PART B):

b) Let a, b be as defined above. Is $a^2 + b^2$ always a square number? (Tip: Test the statement for some n, make a guess on the basis of your observations, and then prove this guess)

Correct solution:

1. Try out numbers as suggested in tip (e.g., in a table):

n	a	b	$a^2 + b^2$
1	3	4	25
2	5	12	169
3	7	24	625
4	9	40	1681

On this basis, generate the idea that: $a^2 + b^2 = (b + 1)^2$
(does not have to be stated explicitly, but must at least be apparent from the numbers: $25 = 5^2$, $169 = 13^2$,...)
Subsequently: Prove the idea by inserting the definitions and transforming the terms.
The generation of the idea does not have to be apparent from the numerical examples; this code is awarded for any response presenting numerical examples.

2. Calculate $a^2 + b^2$ as a function of a, b or n **without using the tip**.

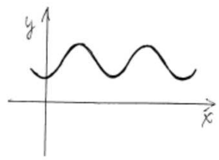
CODING SCHEME

Responses	**Code** enza2	**Score** enza2s
Missing	999	0
Non-classifiable (e.g., "not in the mood," "can't concentrate," "material not covered at the level I teach," etc.)	98	0
Incorrect/incomplete solution	0	0
1.	1	1
2.	2	1
Other correct responses	3	1

SINE A)-D)

Consider the function $y = a \sin(bx + c) + d$

The independent variable is x and a, b, c, d are parameters. When these parameters have certain values, the following curve is produced (see sketch):



Please describe <u>in a few words</u> what happens to the graph when...

***a)** parameter a is doubled:*

<u>Correct response</u>:
a describes the “amplitude.“ The difference between the maximum and the minimum value doubles. When a is doubled, “the wave moves further up and down” – The maximum values of the function with the increased a are higher (to be precise: d + 2a)

***b)** parameter b is doubled*

<u>Correct response</u>:
b influences the “frequency“ and the “period“ of the wave. When b is doubled, the frequency doubles and the period is halved – so “there are twice as many waves in one interval“ – “there are more waves of the same height” – “there are more peaks/valleys“…

***c)** parameter c is doubled*

<u>Correct response:</u>
c describes the “phase displacement.” When c is doubled, the curve cuts the y-axis later – the curve is displaced along the x-axis (it makes no difference if the respondent says that the curve is displaced “to the left” or “to the right.”)

***d)** parameter c is doubled*

<u>Correct response:</u>
d describes the “baseline.” When d is doubled, “the curve is higher up” – the curve is displaced along the y-axis.

CODING SCHEME: PARTS A), B), C) AND D) ARE CODED SEPARATELY

Responses	**Response Code** esin1-esin4	**Score** esi1sc-esi4sc
Missing	999	0
Non-classifiable (e.g., "not in the mood," "can't concentrate," "material not covered at the level I teach” etc.)	98	0
Incorrect solution	0	0
Correct solution	1	1

Each subquestion is to be given a separate response code and score

RECURRING DECIMAL

Is $0.\overline{9} = 1$ *? Please give reasons for your answer.*

Correct solutions:

1) Let $0.\overline{9} = a$. Therefore $10a = 9.\overline{9}$

It follows that 10a – a = $9.\overline{9} - 0.\overline{9}$, therefore 9a = 9, therefore a=1

2) $\frac{1}{3} = 0.\overline{3}$ therefore $0.\overline{9} = 3 \cdot 0.\overline{3} = 3 \cdot \frac{1}{3} = 1$

(or an analogous solution using $\frac{1}{9} = 0.\overline{1}$ or $0.\overline{9} = 9 \cdot \frac{1}{9} = 1$)

3) "Principle of permanence":

$1 \div 9 = 0.\overline{1}$

$2 \div 9 = 0.\overline{2}$

$3 \div 9 = 0.\overline{3}$

....

$1 = 9 \div 9 = 0.\overline{9}$

4) Calculate $1 - 0.\underbrace{99.........9}_{n\ times 9} = 0.\underbrace{00.........0}_{n-1\ times 0}1$

(possibly written out as:
$$\begin{array}{r} 1{,}0 \\ -\,0{,}9.............9 \\ \hline 0{,}......\quad ....01 \end{array}$$

For $n \to \infty$ the final –1 is shifted any number of positions to the right, therefore $1 - 0,\overline{9} = 0,\overline{0} = 0$

5) "Geometric Series":

$$0,\overline{9} = 9 \cdot 0,\overline{1} = 9 \cdot \sum_{i=1}^{\infty} \left(\frac{1}{10}\right)^i = 9 \cdot \left(\frac{1}{1 - \frac{1}{10}} - 1\right) = 9 \cdot \left(\frac{10}{9} - 1\right) = 1$$

But also the following (another interpretation of 4):

6) 0.999........<1, if the 9s ever come to an end (there's no saying that they do!)

Further:

7)
Let $0,\overline{9} = a$. Assuming a < 1, then the arithmetical mean $\frac{1}{2}(a+1)$is precisely in the middle of a and 1.

a) $(a+1) : 2 = 1,\overline{9} : 2 = 0,\overline{9} = a$ Contradiction!

b) The same idea from a constructive–direct perspective, rather than an indirect one:

$\frac{a+1}{2} = a$, therefore ... $a = 1$

Incorrect responses:

a) $0,\overline{9} < 1$, because even at the ∞th position there is always a gap between the number and 1; the difference will never be zero, no matter how many nines there are.

b) $0,\overline{9} < 1$, because the series 0,9; 0,99;......... can never reach the value 1.
(keyword: approximation)

c) Every number that starts with 0. ... is smaller than 1.

d) $0,\overline{9} = 1$ only because of the laws of rounding numbers

e) other responses $0,\overline{9} < 1$

f) other responses $0,\overline{9} = 1$ with incorrect reasoning

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CODING SCHEME

Responses	Response Code edezi	Score edezsc
Missing	999	0
Non-classifiable (e.g., "not in the mood," "can't concentrate," "material not covered at the level I teach," etc.)	98	0
Incorrect solution a)	0	0
Incorrect solution b)	1	0
Incorrect solution c)	2	0
Incorrect solution d)	3	0
Incorrect solution e)	4	0
Incorrect solution f)	5	0
Correct solution 1)	6	1
Correct solution 2)	7	1
Correct solution 3)	8	1
Correct solution 4)	9	1
Correct solution 5)	10	1
Correct solution 6)	11	1
Correct solution 7a)	12	1
Correct solution 7b)	13	1
Other correct solution	14	1

APPENDIX D

MKT SCORES

years	N	Number Correct	Std. Deviation	Median	Formula Scoring	Std. Deviation	Median
-3.00	5	13.20	5.541	11.0	7.60	17.799	7.0
-2.00	1	20.00	.	20.0	32.00	.	32.0
-1.00	4	11.25	12.176	9.5	13.50	15.089	8.0
.00	14	16.07	8.931	17.5	22.71	14.584	25.0
1.00	4	5.00	1.414	5.5	5.75	4.787	7.0
2.00	3	12.67	9.452	16.0	25.33	16.862	33.0
3.00	1	23.00	.	23.0	26.00	.	26.0
4.00	2	3.00	2.828	3.0	4.00	2.828	4.0
5.00	1	24.00	.	24.0	29.00	.	29.0
6.00	1	19.00	.	19.0	16.00	.	16.0
8.00	3	23.67	5.859	26.0	37.67	10.066	39.0
9.00	1	2.00	.	2.0	6.00	.	6.0
11.00	2	18.00	5.657	18.0	11.50	19.092	11.5
12.00	3	20.67	5.774	24.0	23.00	17.436	31.0
13.00	2	17.00	12.728	17.0	25.00	22.627	25.0
14.00	1	17.00	.	17.0	24.00	.	24.0
15.00	5	15.00	7.141	14.0	18.00	13.565	19.0
16.00	1	26.00	.	26.0	37.00	.	37.0
17.00	1	10.00	.	10.0	14.00	.	14.0
18.00	1	.00	.	.0	-1.00	.	-1.0
19.00	2	3.50	.707	3.5	.00	5.657	.0
21.00	2	11.50	7.778	11.5	18.00	11.314	18.0
22.00	3	18.33	16.073	25.0	31.67	29.687	39.0
23.00	2	8.00	5.657	8.0	6.50	3.536	6.5
24.00	3	6.67	5.859	9.0	5.33	6.429	8.0
25.00	4	13.00	10.296	13.5	7.75	19.923	.5
26.00	1	2.00	.	2.0	.00	.	.0
29.00	1	17.00	.	17.0	17.00	.	17.0
31.00	1	27.00	.	27.0	40.00	.	40.0
32.00	2	23.50	2.121	23.5	31.00	8.485	31.0
45.00	1	4.00	.	4.0	-3.00	.	-3.0
Total	78	13.96	9.078	14.5	17.33	16.035	13.0