

An Examination of the Effect of a Secondary Teacher's Image of Instructional
Constraints on His Enacted Subject Matter Knowledge

by

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ABSTRACT

Teachers must recognize the knowledge they possess as appropriate to employ in the process of achieving their goals and objectives in the context of practice. Such recognition is subject to a host of cognitive and affective processes that have thus far not been a central focus of research on teacher knowledge in mathematics education. To address this need, this dissertation study examined the role of a secondary mathematics teacher's image of instructional constraints on his enacted subject matter knowledge.

I collected data in three phases. First, I conducted a series of task-based clinical interviews that allowed me to construct a model of David's mathematical knowledge of sine and cosine functions. Second, I conducted pre-lesson interviews, collected journal entries, and examined David's instruction to characterize the mathematical knowledge he utilized in the context of designing and implementing lessons. Third, I conducted a series of semi-structured clinical interviews to identify the circumstances David appraised as constraints on his practice and to ascertain the role of these constraints on the quality of David's enacted subject matter knowledge.

My analysis revealed that although David possessed many productive ways of understanding that allowed him to engage students in meaningful learning experiences, I observed discrepancies between and within David's mathematical knowledge and his enacted mathematical knowledge. These discrepancies were not occasioned by David's active compensation for the circumstances and events he appraised as instructional constraints, but instead resulted from David possessing multiple schemes for particular ideas related to trigonometric functions, as well as from his unawareness of the mental actions and operations that comprised these often powerful but uncoordinated cognitive

schemes. This lack of conscious awareness made David ill-equipped to define his instructional goals in terms of the mental activity in which he intended his students to engage, which further conditioned the circumstances and events he appraised as constraints on his practice. David's image of instructional constraints therefore did not affect his enacted subject matter knowledge. Rather, characteristics of David's subject matter knowledge, namely his uncoordinated cognitive schemes and his unawareness of the mental actions and operations that comprise them, affected his image of instructional constraints.

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CHAPTER 1

INTRODUCTION AND STATEMENT OF THE PROBLEM

[C]omplaining about public education has long been a popular American indoor sport.

(Berliner & Biddle, 1995, p. 9)

While Americans once considered their education system a source of national pride and a global exemplar, it is now commonly perceived as dysfunctional and a threat to the long-term welfare of our nation (Ravitch, 2013). This drastic shift in public opinion can be traced back to two pivotal events: the successful launch of the Soviet satellite, Sputnik, in early October of 1957, and the publication of *A Nation at Risk: The Imperative for Educational Reform* by the National Commission of Excellence in Education in 1983.

Sputnik was the world's first artificial satellite. Although harmless in itself, Sputnik's launch accentuated among the American public an existing state of anxiety that originated from Cold War tensions. Sputnik, in addition to suggesting the Soviet Union's capacity to deliver nuclear weapons, instilled the impression among Americans that the United States was behind its international competitors in science and technology and, more fundamentally, education. While reforms in U.S. education had been underway in the years preceding the launch of the Soviet satellite, the overwhelming and unanticipated public reaction to Sputnik forced education into the national limelight. Less than a year after the launch Congress passed the National Defense Education Act (1958), which poured billions of dollars into the U.S. education system and set a precedent for stressing high quality instruction in mathematics, science, and technology.

American education experienced a proliferation of radical reform initiatives in the anxious decades following the Sputnik launch. These initiatives sought to reestablish the United States as a powerhouse of technology and innovation and as a global standard for public education. To this end, many of the impulsive reforms of the 1960s and 1970s sought to liberate students from the humdrum conventionality of institutionalized education by rejecting graduation requirements, grades, assessments, and a prescribed curriculum in an effort to foster creativity and innovation (Ravitch, 2010, p. 23). Progressively declining SAT scores from 1963 to 1980 suggested the ineffectiveness of the imaginative reforms that characterized U.S. education during this era.¹ Then, in 1983, at the request of President Ronald Regan, the National Commission on Excellence in Education, under the direction of Secretary of Education Terrel Bell, issued the seminal report *A Nation at Risk: The Imperative for Educational Reform*² in response to the ineffective education reforms of the late 1960s and early 1970s. This report was a portrayal of the state of education in the United States with provocative rhetoric and alarming conclusions. Its accusation:

We report to the American people that while we can take justifiable pride in what our schools and colleges have historically accomplished and contributed to the United States and the well-being of its people, the educational foundations of our society are presently being eroded by a rising tide of mediocrity that threatens our very future as a Nation and a people. What was unimaginable a generation ago has begun to occur—others are matching and surpassing our educational attainments. . . . We have, in effect, been committing an act of unthinking, unilateral educational disarmament. Our society and its educational institutions seem to have lost sight of the basic purposes of schooling, and of the high

¹ Berliner and Biddle (1995, p. 14-23) claim that as a result of poor data analysis techniques, the decline in SAT scores from 1963 to 1980 contributed to the illusion of an education system in peril—an illusion used by policymakers to advance a particular political agenda. This objection notwithstanding, the American public's lack of confidence in the education reforms of the 1960s and 1970s was not illusory.

² A Nation at Risk for short.

expectations and disciplined effort needed to attain them (National Commission on Excellence in Education, 1983, p. 9).

Just as Sputnik had accomplished a quarter of a century earlier, *A Nation at Risk* succeeded in ushering in a feeling of discontent with the American education system. It suggested that the state of U.S. education was even more perilous than during the Sputnik era, claiming that “achievement of high school students on most standardized tests is now lower than 26 years ago when Sputnik was launched” (National Commission on Excellence in Education, 1983, p. 8). The report conveyed to policymakers, educational researchers, and the general public that the United States was again experiencing a crisis in education. *A Nation at Risk* renewed concern among citizens that the U.S. was falling behind its international competitors in education, and criticized the freewheeling reforms that flourished in the two decades immediately following the Sputnik launch for diluting the content of the secondary school curriculum and diminishing academic expectations in the interest of cultivating students’ creativity and individuality.

Since the publication of *A Nation at Risk*, hypothetical causes for the perceived dire state of education in the United States have littered the public discourse. Educational researchers, politicians, union leaders, billionaire entrepreneurs, corporate executives, philanthropists, public intellectuals, students, parents, and teachers have all contributed to the conversation, often vehemently, through thousands of books, magazine and newspaper articles, videos, blogs, and podcasts. Despite diverse participation in the education debate over the past three decades, opinions regarding the source of the crisis are as polarized as ever. Two perspectives dominate the conversation concerning what is wrong with American schools.

Two Perspectives for the Condition of U.S. Education

The first perspective (Perspective I), supported largely by corporate leaders and politicians, assumes that deficits in teacher aptitude and motivation lie at the heart of modest student achievement. The second perspective (Perspective II), often endorsed by teachers and their unions, contends that teachers operate under progressively crippling circumstances that make it nearly impossible for them to effectively do their work. The latter criticize the former for framing education too simplistically by failing to appreciate the complex effects that social circumstances and ill-conceived educational policies have on teachers and their students. The former condemn the latter for underestimating the significance of the teacher and taking as unproblematic the absence of a quality control mechanism in the education system. One group looks at U.S. education and sees a perfect storm of insufferable constraints while the other group sees a crippling deficit of human capital.

These two perspectives stand in direct opposition to one another. Adherents to Perspective II appraise the reform initiatives that have been supported by those who champion Perspective I as the very constraints that make meaningful teaching and learning essentially unattainable. Proponents of Perspective I consider teachers' and their unions bemoaning these "constraints" as suggestive of a workforce that does not possess the knowledge and skill required to navigate the challenges that are an inevitable byproduct of much needed reform.

While both perspectives have merit, if what each takes as the source of the state of education in the United States were resolved, it would not be likely that our education

system would substantially improve. Understanding why requires a closer look at the policies that derive from these two perspectives.

Perspective I: The Deficit of Human Capital Perspective

If one considers complacent, unknowledgeable, and apathetic teachers the reason for our nation's educational mediocrity, then a natural intervention to this problem involves recruiting and incentivizing capable teachers while holding all teachers accountable for their performance. Initiatives like the infusion of market-based principles into the education sector, high-stakes standardized tests, merit pay, Teach for America, the charter movement, vouchers, parent-trigger laws, the No Child Left Behind Act of 2001, and Race to the Top³ all take teacher aptitude as the primary cause of our nation's underachievement in education. If each of these initiatives were successful in cultivating human capital in education by recruiting and incentivizing highly qualified teachers while removing ineffective ones, what is to ensure teachers will uncompromisingly bring their capabilities to bear in the act of teaching? That is, how are we to know that teachers' aptitude is not significantly compromised by their image of the social and affective pressures of their work environment?

Individuals who design policies in response to their view that the state of education in the United States is attributable to deficiencies in teacher quality take the instantiation of teacher knowledge as unproblematic. If the evocation of a teacher's knowledge is conditioned by his or her appraisal of the environmental context within which it is employed, as I'll argue in Chapter 3, then designing reforms that seek to ensure every student is taught by a highly qualified teacher, while important, may not be

³ The Race to the Top fund is a part of the Obama administration's American Recovery and Reinvestment Act of 2009.

enough. It may also be essential to discern whether teachers' image of the environmental context in which they teach influences the knowledge they utilize while teaching, and if so, how?

In essence, those who endorse Perspective I appear to make the implicit assumption that a highly qualified teacher *will* produce high quality instruction. However, to say a teacher is "highly qualified" is to characterize his or her *potential*, whereas to say a teacher produces high quality instruction is to characterize his or her *behavior*. My criticism, then, of Perspective I is that it is insensitive to the cognitive processes that mediate teachers' potential and behavior.

Perspective II: A Perfect Storm of Constraints Perspective

Let us now consider the alternative perspective, the contention that the present-day teacher is deluged with various pressures that are not conducive to engaging students in experiences whereby they are able to learn meaningful content. One often hears it said that engrained cultural practices, schizophrenic educational policies, an obsession for high-stakes testing and accountability, politicians' aspirations to privatize education, decaying teacher-student relationships, the de-professionalization of teaching, unsupportive administrators and colleagues, insufficient emphasis on the importance of early childhood education, increasing proportions of students living in poverty, and the escalation of unsupportive parents all impose obstacles that severely limit what many teachers can achieve in the classroom. In response to this cocktail of perceived constraints, many have advocated for less punitive accountability and more trust and responsibility afforded to teachers (Sahlberg, 2011); the use of standardized tests for diagnostic and instructional purposes instead of as a metric to justify punishing teachers

and schools (Nichols & Berliner, 2007); better resources for teachers and their students (Ravitch, 2013); high quality curricula in all subjects, not just mathematics and science (Ravitch, 2010); higher compensation for teachers that is not tied to student performance (Figlio, 1997); less competition among schools (Sahlberg, 2004, 2006); more teacher autonomy and fewer bureaucratic mandates (Hargreaves & Shirley, 2009); the expansion of early childhood education programs (Barnet, 1995); implementing social programs that seek to reduce poverty and racial segregation (Ravitch, 2013, p. 8); more responsibility delegated to students for their own learning (O'Neill & McMahon, 2005); higher disciplinary standards for students (Owens, 2013, p. 222); and the cultivation of a professional culture in teaching (Stigler & Hiebert, 1999).

Like the proposals advanced by proponents of Perspective I, the above measures, if implemented, would not necessarily guarantee significant improvement in U.S. education. The aforementioned suggestions focus on manufacturing an environment wherein teachers feel respected, safe, efficacious, and professional by removing those constraints that impede teachers' capacity to provide students with meaningful learning experiences. These proposals, however, assume teachers' are generally equipped to afford students the opportunity to construct powerful understandings of the content in the absence of these constraints. Many advocates of Perspective II recognize that while there may be isolated instances in which teachers are ill-prepared for the demands of their work, on the whole they view teachers as willing and able to teach effectively when not operating under oppressive conditions that stifle their aptitude. That is, the policy initiatives proffered by advocates of Perspective II at best underestimate the significance

of teacher knowledge and at worst assume the quality of teachers' knowledge is adequate for providing students with high quality learning experiences.

Problem Statement

In the final analysis, educational policies that originate from Perspective I seek to promote human capital in education by incentivizing teachers and holding them accountable for student performance. Policies that derive from Perspective II aspire to engineer an environment wherein teachers are encouraged to exploit their knowledge and skill by minimizing the constraints that deter teachers from realizing their potential. The former targets teacher knowledge and the latter focus on teachers' working conditions. Neither of these two common perspectives, nor the policy initiatives that descend from them, attend to the *interaction* between teachers' image of their work environment and the knowledge teachers bring to bear in the act of teaching. How does teachers' knowledge inform their appraisal of what they view as constraints on their teaching? How do the constraints that teachers believe to be operating under condition the knowledge they use in the process of designing learning opportunities for students? The polarized perspectives discussed above leave such important questions unanswered. Proponents of Perspective I justify their position by claiming, "Teachers cannot teach what they don't know!" without emphasizing that teachers may not teach all of what they *do* know. Alternatively, advocates of Perspective II exclaim, "Teachers cannot utilize the full extent of their professional knowledge in the repressive conditions of contemporary education!" without appreciating the degree to which teachers' lacking knowledge base imposes limits on the quality of their instruction in the absence of instructional constraints.

Understanding the interaction between teacher knowledge and teachers' image⁴ of instructional constraints is imperative for achieving an accurate diagnosis for the condition of education in the United States and, more importantly, for postulating effective interventions that seek to reestablish the United States as an international model for high quality education.

The focus of this dissertation is on understanding one direction of the interaction between teachers' knowledge and their image of those aspects of their work environment they view as impositions to effective teaching and learning. I specifically focused on characterizing how the constraints a secondary mathematics teacher believed to be operating under conditioned the subject matter knowledge he utilized in the context of teaching. Accordingly, this dissertation explored the following research questions:

RQ1: Are there incongruities between a teacher's subject matter knowledge and the subject matter knowledge he invokes while teaching?⁵

RQ2a: If so, in what ways does the teacher's image of instructional constraints condition the subject matter knowledge he utilizes while teaching?

⁴ The "image of" qualifier suggests my constructivist approach to defining instructional constraints; I take the position that environmental circumstances per se in the absence of a teacher's construal of them cannot constrain one's practice, but a teacher's appraisal of environmental circumstances can and often does. I therefore define instructional constraints as an individual teacher's subjective construction of the circumstances that impede the teacher's capacity to achieve his or her instructional goals and objectives. Accordingly, I locate instructional constraints in the mind of individuals, not the environment. I define instructional constraints more thoroughly in Chapter 3.

⁵ I note that the identification of incongruities between the teacher's subject matter knowledge and the subject matter knowledge he invokes while teaching is from my perspective. Similarly, characterizing the effect of a teacher's image of instructional constraints on his enacted mathematical knowledge is also a characterization from my perspective.

RQ2b: If not, how is the teacher appraising and/or managing what he perceives as instructional constraints so that these constraints do not condition the mathematical knowledge he enacts while teaching?

It is essential in the contemporary climate of U.S. education to apprehend whether teachers do indeed teach what they know. If policymakers and educational researchers assume teachers use their knowledge in an uncompromised way, they design professional development programs that focus on supporting teachers in constructing more advanced knowledge structures—knowledge that ultimately may not inform teachers’ instructional practices and thus the content that students may learn. Policymakers and educational researchers are ultimately interested in teacher knowledge only to the extent that it may affect the quality of students’ learning. Therefore, understanding the factors that condition teacher knowledge is necessary for designing effective educational policies, professional development opportunities, and teacher preparation programs that are less influenced by the simplistic and polarized perspectives on the condition of U.S. education discussed above.

Research that aspires to understand how teachers’ image of the constraints under which they work conditions the knowledge they employ in the context of practice while not respecting content area idiosyncrasies will not suffice. Mathematics teachers experience not only different constraints in their work, but the same constraints differently than do their colleagues in history, English, music, and physical education (Grossman & Stodolsky, 1994). Mathematics teachers’ appraisal of such constraints, then, are likely to differ, as will the influence that these appraisals have on their enacted knowledge. Without achieving clarity relative to the ways in which mathematics teachers’

image of the constraints under which they operate conditions the knowledge they invoke to support students' learning, the quality of U.S. mathematics education is likely to remain unsatisfactory. The present dissertation seeks to contribute to this end.

CHAPTER 2

LITERATURE REVIEW

Critical features of teaching, such as the subject matter being taught, the classroom context, the physical and psychological characteristics of the students, or the accomplishment of the purposes not readily assessed on standardized tests, are typically ignored in the quest for general principles of effective teaching.

(Shulman, 1987, p. 6)

In response to the rhetoric of Perspective I and the ensuing emphasis on teacher knowledge in the education reform discourse, scholars have sought to characterize the knowledge that teachers must possess to provide students with opportunities to construct deep understandings of the content. Researchers have also investigated how such knowledge develops and have attempted to design assessments to measure it. The scholarship on teacher knowledge has a long and varied history. I begin this chapter with a rather summary account of this history and proceed to appraise various research programs in the area of teacher knowledge in mathematics education. In doing so I argue that while the existing literature advances our collective understanding of teacher knowledge on several fronts, this literature is limited in its inattention to the processes that mediate mathematics teachers' knowledge—as it resides in the minds of teachers—and the knowledge they employ in the context of teaching.

Brief History of Scholarship in Teacher Knowledge in Mathematics

The conception of the role of the teacher during the two decades following the Sputnik launch was that of being a manager and facilitator of an expert-made, teacher-proof curriculum (Even & Tirosh, 1995, p. 2). Accordingly, during this period educational research on teaching quality predominantly assumed a *process-product* orientation (Dunkin & Biddle, 1974) in which researchers attributed desired learning

outcomes to observable teaching behaviors (Sherin, Sherin, & Madanes, 2000). Within this research paradigm, scholars observed teachers employing a particular instructional practice (often some classroom management strategy), assessed student performance on tests of achievement or attitude, and, via correlational analyses, quantified the strength of the causal link between the teaching behavior and student performance. This research paradigm therefore did not seriously attend to the subject matter being taught (Shulman & Quinlan, 1996). Research of this type sought to reveal those teaching behaviors—such as wait time, questioning techniques, practices of reinforcement, and lesson structure—that would predict student achievement. “This research shared several characteristics with the foundational field of educational psychology,” Shulman and Quinlan (1996) explain, “such as essentially ignoring the role of subject matter as a central feature of teaching and learning” (p. 409). Shulman and Quinlan go on to argue that what was notably absent from process-product research on teaching during the 1960s and 1970s “was appropriate attention to the centrality of subject matter, not as just another context variable, like class size or student social class, but as a central and pivotal construct in any studies of classroom teaching” (1996, p. 409).

Around the time *A Nation at Risk* was published in 1983, educators were increasingly examining cognitive, social, and affective phenomena related to the practice of teaching instead of taking teachers’ behavior as the primary analytical unit—a trend that dominated educational research for the first half of the 20th century. An integral part of this shift involved researchers in characterizing the knowledge that a teacher must possess to effectively create opportunities for students to construct the understandings that the teacher intends to promote. In the process, researchers began to take seriously the

subject matter being taught. Consequently, propelled by the distressing conclusions of *A Nation at Risk* and sustained by the rising prominence of qualitative research methods ushered in during the cognitive revolution in educational psychology, mathematics education researchers have increasingly devoted attention to understanding what teachers must know as well as examining the extent to which teachers' mathematical and pedagogical knowledge informs their instructional actions.

To this end, Shulman (1986) proposed a theoretical framework for teacher knowledge in response to characterizations of relevant knowledge for teaching that were polarized on a continuum ranging from strict mathematical content knowledge to knowledge of pedagogy independent of any particular content domain. Shulman's influential work brought to the fore the notion that the knowledge instantiated in the practice of teaching is indeed diverse and complex. In response to Shulman (1986), mathematics education researchers have sought to characterize the knowledge that mathematics teachers bring to bear in the practice of teaching. This mathematical knowledge base has come to be called *mathematical knowledge for teaching* (MKT) (Thompson & Thompson, 1996).

Some researchers have taken an empirical approach to identifying the "types" of mathematical knowledge employed in the act of teaching (Ball, 1990; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008; Hill & Ball, 2004; Hill, Ball, & Schilling, 2008; Hill, Schilling, & Ball, 2004) while others have theorized about its character and development (Harel, 2008; Harel & Lim, 2004; Silverman & Thompson, 2008) or demonstrated that teachers do not possess the mathematical knowledge required to provide students with meaningful learning experiences (Bryan, 2002; Carlson, Oehrtman,

& Engelke, 2010; Cooney, Shealy, & Arvold, 1998; Even & Tirosh, 1995; Stigler & Hiebert, 1999). Still others strived to demonstrate the causal link between teachers' MKT and student performance (Hill et al., 2008; Hill, Rowan, & Ball, 2005). As a consequence of differing research objectives and epistemological stances, a variety of MKT characterizations and developmental frameworks currently exist in the literature.

In this chapter, I discuss four prominent treatments of teacher knowledge in the mathematics education literature. To convey the need for the present study, I devote particular attention to explicating the degree to which these four treatments attend to describing the factors that condition the instantiation of teacher knowledge. I conclude with an explanation of why it is necessary to do so if one is to ensure that students have the opportunity to construct meaningful ways of understanding mathematical ideas.

Shulman's Theoretical Framework for Teacher Knowledge

Shulman's (1986) theoretical framework for teacher knowledge was motivated by what he terms, "the missing paradigm problem," which refers to "the absence of focus on subject matter among the various research paradigms for the study of teaching" (p. 6). Shulman does not advocate for the neglect of pedagogical knowledge in favor of content knowledge, but recognizes that "to blend properly the two aspects of a teacher's capacities requires that we pay as much attention to the content aspects of teaching as we have recently devoted to the elements of teaching process" (1986, p. 8). As a consequence of this recognition, and motivated by the need to describe the relationship between content knowledge and general pedagogical knowledge, Shulman devised a theoretical framework that provides a model for how content-related knowledge is organized in the minds of teachers. I illustrate the constructs within this framework and

their relations in *Figure 1*. For my purposes, it suffices to discuss only the three categories of content knowledge within this framework: (a) *subject matter content knowledge*, (b) *pedagogical content knowledge*, and (c) *curricular knowledge*. Indeed these three constructs have been the most influential in the development of more recent conceptualizations of MKT.

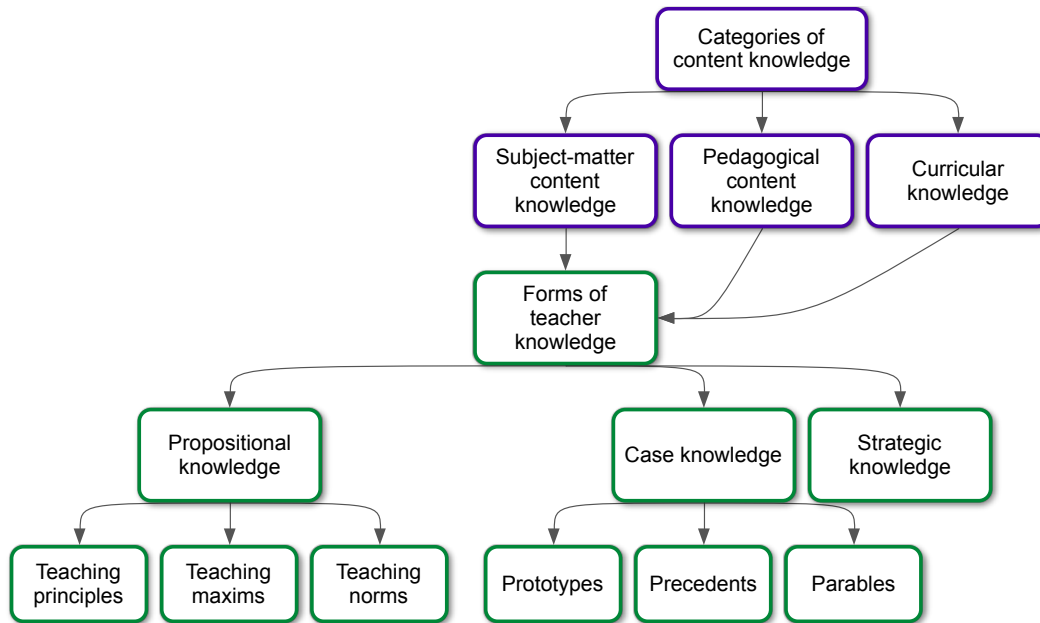


Figure 1. Shulman's theoretical framework for teacher knowledge.

Subject matter content knowledge “refers to the amount and organization of knowledge per se in the mind of the teacher” (Shulman, 1986, p. 9) and requires an understanding of both *substantive* and *syntactic structures* of a discipline (Schwab, 1978). Shulman (1986) explains,

The substantive structures are the variety of ways in which the basic concepts and principles of the discipline are organized to incorporate its facts. The syntactic structure of a discipline is the set of ways in which truth or falsehood, validity or invalidity, are established (p. 9).

Accordingly, knowledge of the substantive structures of mathematics includes an understanding of the truths and techniques of mathematics, whereas knowledge of the syntactic structures of mathematics includes an understanding of the means by which truths are established (e.g., proof) and why such truths are worth knowing.

Pedagogical content knowledge (PCK) refers to the character of content knowledge needed for the practice of teaching. Shulman defines PCK as the knowledge of content that informs “the ways of representing and formulating the subject that make it comprehensible to others” (1986, p. 6). Possessing the knowledge that allows a teacher to represent and formulate concepts in a way that supports students’ learning presupposes an understanding of what makes learning such concepts challenging as well as having a model of the mathematical meanings students bring with them to the instructional context (Shulman, 1986, p. 9).

Pedagogical content knowledge, therefore, specifies what is intuitive—that mere knowledge of mathematics content or exclusive knowledge of effective pedagogical techniques is insufficient for the conceptual teaching of mathematics—but what is in actuality rather elusive. PCK is based on the notion that effective pedagogy cannot exist independently of content knowledge, but is instead fashioned by the way one knows the content of his or her discipline. In this way, PCK is a *form* of content knowledge that *facilitates* effective pedagogy instead of being merely an amalgam of pedagogical and subject matter content knowledge.

The third and final component of content knowledge in Shulman’s theoretical framework is *curricular knowledge*. Curricular knowledge includes

understandings about the curricular alternatives available for instruction ... familiar[ity] with the curriculum materials under study by his or her students in other subjects they are studying at the same time ... [and] familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school, and the materials that embody them (Shulman, 1986, p. 10).

To Shulman, teachers are seen as the medium through which the content of curricula get conveyed to students in the context of instruction. Therefore, in Shulman's framework teachers' knowledge of curriculum is essential for students to experience the content a curriculum designer intends. Moreover, teachers' possessing strong knowledge of curriculum is a necessary condition for lending coherence to instruction, both within and among courses.

Shulman describes the necessity of his framework by citing the absence of theoretical tools available to support disciplined inquiry into the complexities of teacher understanding and transmission⁶ of content knowledge (1986, p. 9). Moreover, Shulman claims that questions like, "What are the domains and categories of content knowledge in the minds of teachers? How, for example, are content knowledge and general pedagogical knowledge related? ... What are promising ways of enhancing acquisition and development of such knowledge?" (1986, p. 9) are not accessible using existing theoretical frameworks for teacher knowledge.

The primary contribution of Shulman's theoretical framework for teacher knowledge lies in his conceptualization of PCK. Shulman was the first to popularize the notion that pedagogical knowledge is fashioned by one's knowledge of the subject

⁶ Shulman uses the phrase "transmission of content knowledge" in the following context: "As we have begun to probe the complexities of teacher understanding and transmission of content knowledge, the need for a more coherent theoretical framework has become rapidly apparent" (1986, p. 9). I use the term here to reflect the epistemological underpinnings of Shulman's framework.

matter, exposing the long-held assumption that effective pedagogical practices transcend disciplinary boundaries and establishing subject matter knowledge as a fundamental component of teachers' knowledge base. Shulman's work revealed that effective pedagogy in biology differs from effective pedagogy in physical education, which differs from effective pedagogy in mathematics, and that all are fashioned by the content of these respective disciplines. Since teachers implement pedagogical practices to create a space in which students may construct intended understandings, the pedagogical knowledge from which these practices arise is necessarily fashioned by the specific meanings a teacher intends to convey.

Shulman's theoretical framework defines various categories and forms of teacher content knowledge. While Shulman outlines other non-content specific forms of teacher knowledge—such as general pedagogical knowledge, knowledge of learners and their characteristics, knowledge of educational contexts, and knowledge of educational ends, purposes, and values (Shulman, 1987, p. 8)—he does not describe how these various categories of teacher knowledge interact in the process of teachers constructing learning opportunities for students. Specifically, Shulman does not address whether teachers' pedagogical content knowledge changes as their knowledge of learners and their characteristics evolves. Nor does he examine how teachers in the same work environment come to have different knowledge of educational context and how this knowledge might influence teachers' enacted subject matter knowledge or pedagogical content knowledge. Shulman also does not attend to how a teacher's knowledge of educational ends, purposes, and values is fashioned by his or her subject matter knowledge. Nor does Shulman's framework address how content knowledge facilitates or impedes a teacher's

capacity to develop knowledge of learners and their characteristics. Concerns such as these were indeed beyond the scope of Shulman's efforts, but are important nonetheless if one seeks to understand the complex cognitive processes from which teachers' instructional actions derive.

In addition to defining categories and forms of teacher knowledge, Shulman (1987) postulates phases by which teachers develop the capacity to transform their own understandings into powerful learning opportunities for students, or what Shulman refers to as *pedagogical reasoning and action*. “[P]edagogical reasoning and action involve a cycle through the activities of comprehension, transformation, instruction, evaluation, and reflection” (1987, p. 14). *Comprehension* comprises achieving a mature understanding of a set of ideas to be taught—which involves knowing ideas in multiple ways and understanding how ideas relate to other concepts within and across subjects—and deciphering the educational utility of students' knowing the ideas being taught.

Transformations, Shulman explains,

require some combination or ordering of the following processes, each of which employs a kind of repertoire: (1) preparation (of the given text materials) including the process of critical interpretation, (2) representation of the ideas in the form of new analogies, metaphors, and so forth, (3) instructional selections from among an array of teaching methods and models, and (4) adaptation of these representations to the general characteristics of the children to be taught, as well as (5) tailoring the adaptations to the specific youngsters in the classroom” (1987, p. 16).

Instruction includes teachers' observable teaching actions including classroom management practices, providing instructions and explanations, and interacting with students. *Evaluation* involves formative and summative, formal and informal assessment.

Finally, *reflection* is the process of retrospectively recalling, reenacting, and reviewing one's experience of an event, including actions, emotions, and outcomes.

Like Shulman's categories of teacher knowledge, his pedagogical reasoning and action phases do not attend to the possibility that the transformation of teacher's content knowledge into students' learning experiences is influenced by other domains of teacher knowledge, such as knowledge of educational context and of learners and their characteristics. For instance, the five processes within the transformation phase (*preparation, representation, selection, and adaptation*) do not address the possibility that the meanings derived during the comprehension phase are not readily available for transformation as a result of teachers' image of students and their understanding of their educational context. In short, while Shulman recognizes that non-content specific varieties of knowledge are a fundamental component of teachers' knowledge base, he does not address the way in which these non-content specific knowledge domains affect pedagogical reasoning and action. It is not likely that these non-content specific forms of teacher knowledge remain inactive while teachers' subject matter knowledge, pedagogical content knowledge, and curricular knowledge work in unison to create optimal learning experiences for students.

MKT in the Framework of Ball, Hill, and Colleagues

Ball, Hill, and colleagues have undoubtedly contributed the most literature in the area of mathematical knowledge for teaching. The research program of this group of scholars is multifaceted as they have aspired to develop a theory—or “portrait” in their usage—of MKT by examining what is entailed in the work of teaching mathematics (Ball, 1990, 2000; Ball & Bass, 2000, 2003; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps,

2008; Hill & Ball, 2004; Hill, Ball, & Schilling, 2008; Hill, Schilling, & Ball, 2004). Ball, Hill, and colleagues have also worked to develop an instrument that seeks to measure teachers' MKT (Ball, et al., 2005; Hill & Ball, 2004; Hill, Ball, et al., 2008; Hill, et al., 2004), and have investigated the effects of teachers' MKT on student achievement (Hill, Rowan, & Ball, 2005) and the quality of their instruction (Hill et al., 2008). *Figure 2* displays the phases of Ball, Hill, and colleagues' research program.

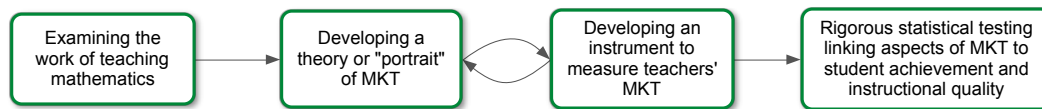


Figure 2. Phases of Ball, Hill, and colleagues' research program.

The focus of this section is on chronicling the evolution of Ball, Hill, and colleagues' MKT research program. I begin by discussing their approach to developing, what they refer to as, a practice-based theory for MKT. I then describe the variety of theoretical constructs related to teacher knowledge that arose throughout their work, emphasizing those that comprise their culminating portrait of MKT. Thereafter, I review Ball, Hill, and colleagues' work to develop an instrument that aspires to measure teachers' MKT and discuss their efforts to demonstrate the causal relation between teachers' MKT and student achievement, and the quality of teachers' mathematics instruction. I conclude by discussing the affordances and limitations of Ball, Hill, and colleagues' MKT research program.

Developing a Practice-Based Theory for MKT

Ball, Hill, and colleagues' have approached the problem of characterizing MKT by describing the knowledge teachers' employ in the practice of teaching mathematics. Ball and Bass (2003) explain,

instead of investigating what teachers need to know by looking at what they need to teach, or by examining the curricula they use, we decided to focus on their work. . . . We began to try to unearth the ways in which mathematics is entailed by its regular day-to-day, moment-to-moment demands. These analyses help to support the development of a *practice-based theory of mathematical knowledge for teaching*” (p. 5, italics in original).

Accordingly, the analytical unit of Ball, Hill, and colleagues’ research is the manifestation of a teacher’s mathematical knowledge as he or she is situated in the context of mathematics teaching.

Ball, Hill, and Bass (2005) construe the problems encountered in the practice of teaching as problems of a mathematical quality in that a type of mathematical knowledge is needed to navigate the complexities of classroom teaching. This is to say the problems of teaching require mathematical reasoning as much as pedagogical thinking. Therefore, in an effort to reveal the nature of mathematical knowledge as it is used in the practice of teaching, Ball, Hill, and colleagues examined “what teachers *do* in teaching mathematics, and in what ways does what they do demand mathematical reasoning, insight, understanding, and skill” (Ball, Hill, and Bass, 2005, p. 17, italics in original).

Following Shulman (1986; 1987), Ball, Hill, and Bass criticize the traditional assumption that subject matter knowledge and pedagogical knowledge are disjoint entities that coalesce in the practice of teaching. Instead, they advance the notion that content knowledge must be organized in a way that is informed by one’s pedagogical knowledge and specialized for the demands of teaching. For instance, Ball and Bass (2000) contend, “the prevalent conceptualization and organization of teachers’ learning tends to splinter practice, and leave to individual teachers the challenge of integrating subject matter knowledge and pedagogy in the contexts of their work” (p. 86). As a

proposal for closing the gap between teachers' subject matter knowledge and pedagogy, Ball and Bass (2000) advance three problems that demand research attention: (1) "we would need to reexamine what content knowledge matters for good teaching" (p. 95); (2) we would need to assess "*how* subject matter must be understood in order to be usable in teaching. We need to probe not just *what* teachers need to know, but to learn how that knowledge needs to be held and used in the course of teaching" (p. 97, italics in original); and (3) we would need to understand "how to create opportunities for learning subject matter that would enable teachers not only to know, but to learn to use what they know in the varied contexts of practice" (p. 99). As Ball, Thames, and Phelps (2008) suggest, the majority of Ball, Hill, and colleagues' research program focuses on the first of these three problems—"we see our work as developing in more detail the fundamentals of *subject matter knowledge for teaching* by establishing a practice-based conceptualization of it, by elaborating subdomains, and by measuring and validating knowledge of those domains" (p. 402).

Ball, Hill, and Colleagues' "Portrait" of MKT

Consistent with the focus on developing the subdomains of subject matter knowledge for teaching, Ball (1990) examines the subject matter knowledge of 252 pre-service teacher candidates during the beginning of their formal teacher education program. Ball found the pre-service teachers had narrow understandings of division, considered mathematical proficiency to be principally characterized by procedural fluency, and generally did not maintain mathematical understandings that promote the conceptual teaching of mathematics. As a consequence of these findings, Ball identified two categories of subject matter knowledge for teaching: (1) *substantive knowledge of*

mathematics and (2) *knowledge about mathematics*. Ball characterizes substantive knowledge of mathematics as satisfying three criteria: (1) “teachers’ knowledge of concepts and procedures should be correct,” (2) teachers “must understand the underlying principles and meanings,” and (3) “teachers must appreciate and understand the connections among mathematical ideas” (Ball 1990, p. 458). Knowledge about mathematics, alternatively, “includes understandings about the nature of mathematical knowledge and of mathematics as a field. What counts as an ‘answer’ in mathematics? What establishes the validity of an answer? What is involved in doing mathematics?” (Ball, 1990, p. 458). These two domains of subject matter knowledge for teaching parallel knowledge of substantive and syntactic structures of a subject that Shulman (1986) discusses. Ball’s notion of knowledge about mathematics, however, differs from knowledge of syntactic structures of a discipline in that Ball’s construct includes more subjective elements, such as beliefs about the nature of mathematics, whereas Shulman’s conceptualization of knowledge of syntactic structures appears limited to knowledge about how the truths of a discipline are established and how disparate truths may be conceptually organized to achieve a coherent image of a subject.

Ball, Hill, and colleagues’ grounded approach⁷ to characterizing the knowledge brought to bear in the practice of teaching has given rise to a variety of other knowledge “types” that are encapsulated in their general notion of MKT. Among them include *specialized knowledge of content* and *common content knowledge*. Specialized knowledge of content is mathematical knowledge specific to the practice of teaching and extends

⁷ Ball, Hill, and colleagues’ approach is grounded in the sense that their conceptualization of MKT derives from their characterization of the various types of mathematical knowledge demanded by the context of classroom teaching.

beyond generic content knowledge. For instance, teachers need to be able to explain why a problem-solving procedure is valid, “appraise student methods for solving computational problems, and when students use novel methods, be able to determine whether such methods would be generalizable to other problems” (Hill & Ball, 2004, p. 332). Alternatively, common content knowledge is knowledge that is necessary to operationalize the content for oneself in generic circumstances. Hill, Ball, and Schilling (2008) explain that common content knowledge is “roughly described as knowledge that is used in the work of teaching in ways *in common with* how it is used in many other professions or occupations that also use mathematics” (p. 377). Ball, Hill, and colleagues originally considered the union of specialized knowledge of content and common content knowledge as comprising the knowledge base needed to satisfy the demands of mathematics teaching. Thus, according to Ball, Hill, and colleagues’ earlier work, not only does MKT entail both specialized and common content knowledge, it is entirely encompassed by these two knowledge domains. Hill and Ball (2004) explain, “By teachers’ *mathematical knowledge for teaching*, we mean not only their common content knowledge but also their specialized knowledge for teaching mathematics” (p. 335, italics in original).

Guided by their research focus to determine if there is one construct that can be called “mathematical knowledge for teaching” or several distinct mathematical competencies held by practicing mathematics teachers (Hill, Ball, & Schilling, 2004), Ball, Hill, and colleagues’ proceeded to identify and define empirically distinguished subdomains of subject matter knowledge and pedagogical content knowledge. For instance, Hill, Ball, & Schilling (2008) recognize *knowledge of content and students* as a

type of “content knowledge intertwined with knowledge of how students think about, know, or learn this particular content” (p. 375). Ball, Thames, and Phelps (2008) introduce *knowledge of content and teaching* and *knowledge at the mathematical horizon* as further subdivisions of pedagogical content knowledge. Knowledge of content and teaching “combines knowing about teaching and knowing about mathematics ... It is an amalgam, involving a particular mathematical idea or procedure and familiarity with pedagogical principles for teaching that particular content” (Ball, Thames, & Phelps, 2008, p. 401-02). Knowledge at the mathematical horizon involves “an awareness of how mathematical topics are related over the span of mathematics included in the curriculum” (Ball, Thames, & Phelps, 2008, p. 403). *Figure 3* illustrates a domain map for MKT that contains the variety of knowledge types that collectively comprise Ball, Hill, and colleagues’ empirically-derived framework for mathematical knowledge for teaching.

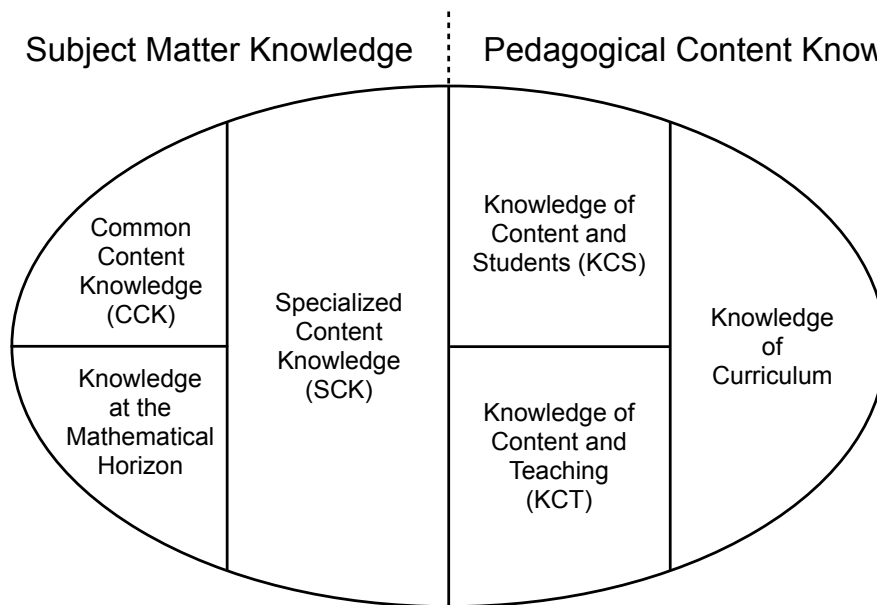


Figure 3. Domain map for mathematical knowledge for teaching (Hill, Ball, & Schilling, 2008, p. 377).

Developing an Instrument to Measure Teachers' MKT

Two foundational questions guiding Ball, Hill, and colleagues' research program include: "Is there a body of mathematical knowledge for teaching that is specialized for the work that teachers do?" (Ball, Hill, & Bass, 2005, p. 22) and "Given the structure of teachers' mathematical knowledge for teaching, can we construct scales that measure such knowledge reliably?" (Hill, Schilling, & Ball, 2004, p. 12). To address these questions, Ball, Hill and colleagues designed a multiple-choice instrument to measure teachers' mathematical knowledge for teaching. These scholars criticized existing measures of teachers' mathematical knowledge on account that they "do not examine teachers' ability to unpack mathematical ideas, explain procedures, choose and use representations, or appraise unfamiliar mathematical claims and solutions" (Hill & Ball, 2004, p. 335). Accordingly, the multiple-choice instrument developed by Ball, Hill, and colleagues' emphasized two key elements of subject matter knowledge: (1) common knowledge of content and (2) specialized content knowledge, and were grounded in the day-to-day and moment-to-moment tasks of mathematics instruction they identified from their grounded approach to characterizing MKT. Hill and Ball (2004) explain, "We use data drawn from instruments that present teachers with particular situations and mathematical problems that arise in those situations, then infer facility in teaching from their ability to solve those mathematical problems in the context of practice" (p. 347).

Hill, Schilling, and Ball (2004) administered three pilot forms of their multiple-choice instrument to 1,552 participants in California's Mathematics Professional Development Institutes (MPDI) to determine the organization of teachers' mathematical

knowledge and to infer whether MKT can be measured reliably using a multiple-choice instrument. Factor analyses found evidence that

teachers' content knowledge for teaching is at least somewhat domain specific, and that scholars who have hypothesized about the categories around which teacher knowledge might organize are at least partially correct. Subject-matter content does play a role; so do the different ways mathematical knowledge is used in classrooms (Hill, Schilling, & Ball, 2004, p. 24).

Accordingly, using their multiple-choice instrument, Hill, Schilling, and Ball (2004) obtained empirical evidence to justify their parsing of MKT into distinct knowledge categories.

For instance, Hill, Ball, and Schilling (2008) garnered empirical support for the existence of *knowledge of content and students* (KCS). The research team designed and administered multiple-choice items to participants of MPDI in an effort to ascertain whether KCS constitutes an identifiable knowledge domain of MKT. Hill, Ball, and Schilling's results indicate that teachers do indeed have an identifiable knowledge of content and students. The results also suggest that the multiple-choice items the research team designed to measure teachers' KCS did not entirely capture KCS but also captured teachers' mathematical content knowledge and test-taking skills.

Demonstrating the Link Between MKT and Student Achievement

In addition to measuring teachers' MKT and providing empirical support for their partition of MKT into distinct knowledge domains, Ball, Hill, and colleagues' multiple-choice instrument also seeks to demonstrate the relationship between teachers' MKT and student achievement. Ball, Hill, and Bass (2005) argue,

only developing grounded theory about the elements and definition of mathematical knowledge for teaching is not enough. If we argue for professional knowledge for teaching mathematics, the burden is on us to demonstrate that

improving this knowledge also enhances student achievement (p. 22, italics in original).

Hill, Rowan, & Ball (2005) demonstrate the contribution of elementary teachers' MKT to student achievement. Their work was motivated by a deficit of studies that provide empirical evidence for this rather intuitive causal relation. As one would expect, their results confirm "teachers' mathematical knowledge for teaching positively predicted student gains in mathematics achievement during the first and third grades" (Hill, Rowan, and Ball, 2005, p. 399). In a similar vein, Hill et al. (2008) conducted a correlational study in which they demonstrate "strong links between teachers' knowledge and the mathematical quality of their classroom practice" (p. 496).

Contributions and Limitations of MKT in the Framework of Ball, Hill, and Colleagues

Ball, Hill, and colleagues' research has focused on developing "the fundamentals of *subject matter knowledge for teaching* by establishing a practice-based conceptualization of it, by elaborating subdomains, and by measuring and validating knowledge of those subdomains" (Ball, Thames, & Phelps, 2008, p. 402, italics in original). Their work has succeeded in raising consciousness among the mathematics education community regarding the various types of knowledge that elementary mathematics teachers must possess in order to effectively navigate the complexities of classroom practice. These categories of MKT have informed teacher preparation and professional development programs in productive ways. Moreover, the assessment these researchers developed to gain insight into teachers' mathematical knowledge is a vast improvement over the previously used proxy variables for mathematical knowledge (e.g.,

degrees earned, mathematics courses taken) that were unable to tease out the nuances in teachers' mathematical understandings and were insensitive to capturing the specific ways in which teachers must know their subject. Accordingly, Ball, Hill, and colleagues' MKT instrument is better suited for assessing the effectiveness of teacher preparation and professional development programs than any instrument that preceded it.⁸

As mentioned above, Ball, Hill, and colleagues' constructed their "portrait" of MKT by postulating the function that subject matter knowledge has on mathematics teachers' instructional actions. Ball (2000) explains, "our understanding of the content knowledge needed in teaching must start with practice. We must understand better the work that teachers do and analyze the role played by content knowledge in that work" (p. 244). I respectfully draw attention to two limitations of this approach. First, deducing categories of knowledge required for effective mathematics teaching by attending to teachers' observable actions in the context of practice does not allow a researcher to gain insight into *how* such knowledge needs to be organized in the mind of the teacher in order to facilitate effective teaching. For instance, claiming teachers must possess specialized knowledge of fractions (content), since their work demands their examining the validity of students' solution methods while solving problems involving fractions, does not specify the ways in which teachers must understand the concept of fraction in order to effectively provide students with opportunities to construct rich meanings of the concept. That is, Ball, Hill, and colleagues do not attend to clarifying underlying mathematical ideas, nor do they propose a means by which teachers may construct such meanings. This

⁸ Although, at the time of writing, Ball, Hill, and colleagues have not released their assessment.

is perhaps a result of Ball, Hill, and colleagues not casting “content knowledge” in terms of teachers’ ways of knowing and understanding particular mathematical concepts.

Second, Ball, Hill, and colleagues’ approach to defining the various domains of MKT by identifying categories of knowledge teachers employ in the practice of teaching does not allow these researchers to ascertain how teacher’s knowledge is compromised to meet the demands of classroom practice. That is, Ball, Hill, and colleagues’ methodology is insensitive to identifying the factors that condition the evocation of teachers’ subject matter knowledge or pedagogical content knowledge in the context of teaching. In addition to not describing what it means to possess the various categories of MKT in their framework, Ball, Hill, and colleagues do not focus on understanding the mechanisms by which teachers “choose,” consciously or not, to employ particular aspects of their knowledge in instructional contexts. For instance, upon observing a teacher performing observable instructional action X , Ball, Hill, and colleagues ask, “In what way does content knowledge play a role in the teacher’s execution of action X ?” They do not additionally ask, “How is this teacher appraising the various aspects of his or her environmental context so as to recognize the need for utilizing the content knowledge needed to perform action X ?” As a result, there are potential aspects of teachers’ knowledge that Ball, Hill, and colleagues cannot deduce from observing teachers’ instructional actions alone.

Silverman and Thompson’s (2008) Developmental Framework for MKT

Silverman and Thompson’s (2008) developmental framework for MKT was motivated by their recognition that “Teachers teach *what they know*” where “to know” is broadly construed “to include the images of mathematical activity and beliefs about the

enterprise of learning and teaching mathematics” (Thompson, 1994, p. 3, italics in original). The assertion that teachers teach what they know is no cause for concern if what teachers know is appropriate for students to learn. However, Thompson (2013b) argues that American teachers’ mathematical knowledge often lacks coherence and is devoid of conceptual meaning. A number of scholars in mathematics education have strongly echoed this allegation (e.g., Bryan, 2002; Carlson, Oehrtman, & Engelke, 2010; Cooney, Shealy, & Arvola, 1998; Even & Tirosh, 1995). Accordingly, Silverman and Thompson’s developmental framework seeks to outline phases by which teachers may construct powerful mathematical ways of understanding “that carry through an instructional sequence, that are foundational for learning other ideas, and that play into a network of ideas that does significant work in students’ reasoning” (Thompson, 2008, p. 45).

While Silverman and Thompson subscribe to the adage that teachers teach what they know, they do not trivialize the process whereby teachers transform personal mathematical understandings into understandings that have pedagogical utility. In their developmental framework, Silverman and Thompson “see a person’s MKT as being grounded in personally powerful understandings of particular mathematical concepts and as being created through the transformation of those concepts from an understanding having pedagogical *potential* to an understanding that does have pedagogical power” (2008, p. 502). For this statement to be meaningful, we must examine what Silverman and Thompson mean by “to understand.”

In Silverman and Thompson’s usage, “to understand” means to achieve a cognitive state that results from a subject having assimilated an experience to a scheme

(Thompson, 2013b). Silverman and Thompson's notion of understanding is distinct from their notion of meaning.

A person's *understanding* of a word, object, sentence, utterance, mathematical inscription, or situation is the result of assimilating it to a scheme of actions, operations, and implications. A person's *meaning* for a word, object, sentence, utterance, mathematical inscription, or situation is the scheme to which it is assimilated (Thompson, Carlson, & Wilson, 2012).

Hence, to have meaning or “to know” is to possess a scheme to which an experience may be assimilated. Such schemes need not be desirable or ideal. Thus, for Silverman and Thompson mathematical knowledge for teaching consists of the schemes a teacher possesses, defined independently of value judgment.

Comprehending Thompson, Carlson, and Wilson's definition of understanding and meaning is, of course, dependent upon what they mean by “assimilation to a scheme.” The process of assimilation “comes about when a cognizing organism fits an experience into a conceptual structure it already has ... [and] reduces new experiences to already existing sensorimotor or conceptual structures” (von Glasersfeld, 1995, p. 62-63). Piaget and Inhelder (1969) explain that a scheme is “the structure or organization of actions as they are transferred or generalized by repetition in similar or analogous circumstances” (p. 4). Thus, “to assimilate to a scheme” is to fit an experience into an existing conceptual structure.

Silverman and Thompson (2008) propose a developmental framework for MKT that is based on the mathematical understandings that resonate throughout an instructional sequence and are foundational for lending coherence to related ideas. They adopt a variation of Gess-Newsome's (1999) transformative model for PCK, which is characterized as “the result of a fundamental transformation of knowledge and the

creation of new knowledge that, though possibly similar to existing mathematical or pedagogical understandings, possesses distinct characteristics that were not present in their original form” (Silverman & Thompson, 2008, p. 501). In other words, Gess-Newsome’s (1999) transformative model asserts that PCK emerges from purposefully integrated experiences that culminate in the *integration* of one’s mathematical knowledge and pedagogical understandings. Silverman and Thompson see utility in the transformative model of PCK in that it “necessitates purposefully integrated experiences that provide teachers with opportunities to extend and connect their mathematical and pedagogical understandings to create a ‘new’ knowledge” (2008, p. 502). Assuming the transformative model for PCK, Silverman and Thompson contend that MKT is grounded in a teacher’s possession of a *key developmental understanding* (Simon, 2002, 2006) as well as the capacity to transform that understanding into a pedagogically useful form via the process of *decentering* (Piaget & Inhelder, 1969). Silverman and Thompson describe the development of MKT as the entailment of five processes or phases that operationalize the process of transforming a KDU into an understanding that is pedagogically powerful—what Silverman and Thompson call a *key pedagogical understanding* (KPU).

The first phase of Silverman and Thompson’s (2008) developmental framework involves a teacher constructing a *key developmental understanding* (KDU). That is, Silverman and Thompson’s framework proposes that teachers should develop a way of understanding a fundamental mathematical concept that aids in their construction of conceptually related mathematical ideas. “Individuals who possess a KDU tend to find different, yet conceptually related ideas and problems understandable, solvable, and

sometimes even trivial” (Silverman & Thompson, 2008, p. 502). It is for this reason that KDUs are *developmental*—they support one’s understanding of related concepts.

In the first phase of their developmental framework, Silverman and Thompson propose the Piagetian construct *reflective abstraction* as a means by which one develops a KDU. Piaget viewed reflective abstraction as the functional mechanism involved in the development of operative thought—thought that allows one “to make propitious decisions about what to do next, and allows them to see what they might do next in relation to what has already taken place” (Thompson, 1985, p. 194). Reflective abstraction is the process whereby an individual constructs more advanced knowledge structures by abstracting properties of action coordinations (Silverman & Thompson, 2008, p. 506). Silverman and Thompson postulate that the construction of a KDU occurs as a consequence of previous knowledge being abstracted via a first reflecting abstraction. Transforming this abstracted mathematical understanding into a form that has pedagogical utility requires a second reflecting abstraction where the teacher abstracts the KDU.

The second phase of Silverman and Thompson’s developmental framework has the teacher construct models of students’ various ways of understanding through the process of *decentering*—the act of adopting the point of view of another. Decentering involves constructing models of potential ways in which students may understand particular mathematical ideas. It is noteworthy that teachers have a tendency to progress through phases of decentering. A teacher’s initial model of students’ understanding is likely to resemble that of themselves as a result of projecting his or her own cognitive behavior onto students. Slightly more sophisticated decentering involves a teacher in

being aware that students' understandings are diverse and differ from his or her own. More sophisticated still are those teachers who are able to use their model of different *epistemic ways of understanding*⁹ to anticipate how students will interpret their instruction.

The third phase of Silverman and Thompson's framework emerges from the first two by requiring the teacher to construct an image of how someone else might come to understand a mathematical concept in the way he or she intends. The fourth phase involves the teacher in becoming aware of the type of specific pedagogical actions that maintain the potential to enable someone else to achieve the way of understanding the teacher envisions. Thompson (2002) proffers *didactic objects* and *didactic models* as a means by which a teacher may create a space in which students have the opportunity to construct the understandings the teacher intends. "The phrase *didactic object*" Thompson explains, "refers to a 'thing to talk about' that is designed to support reflective mathematical discourse involving specific mathematical ideas or ways of thinking" (2002, p. 211). A *didactic model* is "a scheme of meanings, actions, and interpretations that constitute the instructor's or instructional designer's image of all that needs to be understood for someone to make sense of the didactic object in the way he or she intends" (Thompson, 2002, p. 212). The conversations a teacher facilitates surrounding didactic objects are in the service of promoting abstraction of mental operations and operative mathematical structures by creating a context wherein students may participate in mathematical reasoning (Thompson, 2002, p. 211).

⁹ A teacher's model of epistemic ways of understanding is not a model of any *individual* student's way of understanding a particular mathematical idea, but rather an understanding held by a generalized or *epistemic* subject that encapsulates aspects of individual student's understandings that appear, from the teacher's perspective, prevalent among a population of students.

The fifth and final phase of Silverman and Thompson's developmental framework for MKT requires teachers to understand how students' newly developed ways of understanding empower them to learn other related mathematical concepts. At this point, teachers are said to have developed a *key pedagogical understanding* (KPU), as they now recognize how the KDU they constructed in the first phase enables them to support students' learning of related concepts (Silverman, 2005).

Contributions and Limitations of Silverman and Thompson's Developmental Framework for MKT

Silverman and Thompson's framework establishes a foundation for systematic improvement in teacher quality by outlining a developmental trajectory grounded in constructivist epistemology whereby teachers are able to transform personally powerful mathematical understandings into a form that is pedagogically efficacious (Silverman & Thompson, 2008, p. 501). Additionally, their developmental framework posits cognitive mechanisms (i.e., reflective abstraction) that drive such transformations.

Although Silverman and Thompson's developmental framework for MKT is grounded in the idea that teachers teach what they know, they by no means imply that teachers *always* teach what they know. Silverman and Thompson (2005) report on a study in which they investigated the influence of pre-service teachers' mathematical understandings on their instructional practices. Their conclusion:

Despite the fact that pre-service teachers did develop a more robust, coherent understanding of functions as covariation of quantities—an understanding of function that supported their ability to speak conceptually about functional relationships—their instruction remained grounded in a variant of the more traditional correspondence conception of function (ibid., p. 1).

Generally speaking, the following summarizes Silverman and Thompson's position on the instantiation of teacher knowledge:

If a teacher's conceptual structures comprise disconnected facts and procedures, their instruction is likely to focus on disconnected facts and procedures. In contrast, if a teacher's conceptual structures comprise a web of mathematical ideas and compatible ways of thinking, *it will at least be possible* that she attempts to develop these same conceptual structures in her students (Thompson, Carlson, & Silverman, 2007, p. 416-17, my emphasis).

While Silverman and Thompson recognize that teachers may not teach the mathematics they know, their framework does not focus on characterizing the factors that mediate teacher knowledge (as it exists in the mind of the teacher) and the meanings teachers employ to create a space in which students may construct desired understandings. Recall that after developing a KDU, the second, third, and fourth phases of Silverman and Thompson's framework explain for a teacher to develop knowledge that supports the conceptual teaching of a particular mathematical idea, he or she

(2) has constructed models of the variety of ways students *may* understand the content (decentering); (3) has an image of how someone else might come to think of the mathematical idea in a similar way; (4) has an image of the kinds of activities and conversations about those activities that might support another person's development of a similar understanding of the mathematical idea (2008, p. 508, italics in original).

Regarding the second phase, while it was beyond the scope of Silverman and Thompson's efforts, one is left to wonder how inclined a teacher would be to construct models of students' ways of understanding a particular mathematical idea via the act of decentering if the teacher views students as disinterested, apathetic, and otherwise intellectually incapable of constructing the way of understanding she values. Similarly, I consider it probable that the image a teacher constructs of how someone else might come to think about a mathematical idea (Phase 3) is fashioned both by the teacher's

understanding of the mathematical idea *and* her image of the “someone else.” A teacher may possess a mature understanding of a mathematical idea but have an image of how a student might come to think about the idea that reflects a much more impoverished way of understanding as a consequence of her belief that the learner will not engage in instruction in a way that allows him to construct the type of understanding the teacher possesses. Regarding the fourth phase of Silverman and Thompson’s framework, it seems reasonable to expect that a teacher would be disinclined to facilitate rich mathematical discourse around well-conceived didactic objects if she believes that her primary responsibility as a mathematics teacher is to prepare students to demonstrate proficiency on, what she considers, a rote and frivolous standardized assessment.

A teacher would likely be equipped to provide students with opportunities to construct powerful mathematical understandings were he or she to progress through the phases that Silverman and Thompson outline in their developmental framework. However, Silverman and Thompson do not focus on attending to those factors that influence the operationalization of the knowledge that teachers develop upon progressing through their five phases. In addition to many affordances Silverman and Thompson’s developmental framework has contributed, it may be of interest to ascertain how teachers’ appraisal of instructional constraints (e.g., student engagement, educational policies, relationships with colleagues, administrative support) obstructs or promotes their recognizing the need to propitiously decenter and construct didactic objects that allow students to construct powerful mathematical understandings. More generally, it may be of consequence to apprehend the circumstances under which teachers believe the only

consideration that imposes an obstacle on the quality of their instruction is their own knowledge and not the instructional constraints they believe to be working against.

Teachers' Knowledge Base

Teachers' knowledge base (TKB) is Harel's (2008b) analogue to mathematical knowledge for teaching and is situated within an elaborate conceptual framework called *DNR-based instruction in mathematics* (DNR for short)¹⁰. I begin this section with a general overview of DNR followed by a summary of the DNR premises relevant to TKB. I then describe Harel's (1998, 2007, 2008a, 2008b, 2008c, 2010) definition of mathematics as well as his notion of TKB. I conclude this subsection with a discussion of the contributions and limitations of Harel's conception of TKB.

DNR Premises

DNR-based instruction in mathematics “stipulates conditions for achieving critical goals such as provoking students’ intellectual need to learn mathematics, helping them acquire mathematical ideas and practices, and assuring that they internalize, organize, and retain the mathematics they learn” (Harel, 2008c, p. 267). The DNR theoretical framework consists of three categories of constructs: *premises*, *concepts*, and *claims*. DNR premises are explicit assumptions upon which the DNR concepts and claims are based. DNR concepts consist of constructs, consequent to the DNR premises, concerning the cognitive phenomena of teaching and learning mathematics. DNR claims are instructional principles that derive from DNR premises and concepts. Although TKB is situated within the concepts domain of the DNR framework, understanding this

¹⁰ I explain what the abbreviation “DNR” stands for later in this section.

construct of interest requires a review of the DNR premises that informed Harel’s conceptualization of it.

Harel presents the eight underlying premises of the DNR theoretical framework in four categories: Mathematics, Learning, Teaching, and Ontology. *Figure 4* illustrates the categories within which the eight DNR premises reside. The Mathematics, Knowing, Knowing-Knowledge Linkage, Context Dependency and Teaching premises are of particular significance to Harel’s notion of TKB.

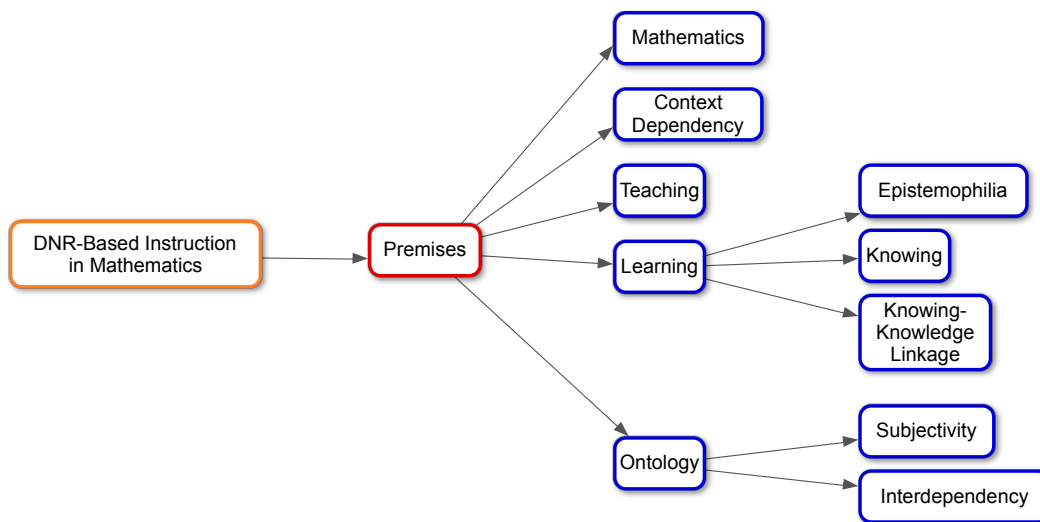


Figure 4. Organization of DNR premises.

The Knowing Premise asserts, “Knowing is a developmental process that proceeds through a continual tension between assimilation and accommodation, directed toward a (temporary) equilibrium” (Harel, 2008b, p. 894). Relatedly, the Knowing-Knowledge Linkage Premise states, “Any piece of knowledge humans know is an outcome of their resolution of a problematic situation” (Harel, 2008b, p. 894). The Knowing and Knowing-Knowledge premises ensue from Piaget’s (1971) genetic epistemology and von Glasersfeld’s (1995) radical constructivism. The Context

Dependency Premise—inferable from the theory of situated cognition—claims that learning is context dependent. The Teaching Premise holds that “Learning mathematics is not spontaneous. There will always be a difference between what one can do under expert guidance or in collaboration with more capable peers and what he or she can do without guidance” (Harel, 2008b, p. 894). The Teaching Premise derives from Vygotsky’s (1978) theory of the zone of proximal development. Finally, the mathematics premise asserts that “Knowledge of mathematics consists of all *ways of understanding* and *ways of thinking* that have been institutionalized throughout history” (Harel, 2008b, p. 894). Of particular significance for conceptualizing TKB is the Mathematics Premise, which claims that mathematical knowledge is the entailment of historically institutionalized mathematical *ways of understanding* and mathematical *ways of thinking*.

The Triad Mental Act, Way of Understanding, and Way of Thinking

The triad, *mental act*, *way of understanding*, and *way of thinking* is central to the concepts domain of the DNR framework. Mental acts are the basic applications of cognition to our experiential world; they encompass behaviors such as interpreting, proving, conjecturing, inferring, justifying, explaining, generalizing, and predicting. It is noteworthy that the observable byproducts of mental acts serve as a medium through which one makes inferences about another’s cognition. Harel (2008a) defines the cognitive constructs *way of understanding* and *way of thinking* and describes their respective relations to mental acts:

A person’s statements and actions may signify cognitive products of a mental act carried out by the person. Such a product is the person’s *way of understanding* associated with that mental act. Repeated observations of one’s ways of understanding may reveal that they share a common cognitive characteristic. Such

a characteristic is referred to as a *way of thinking* associated with that mental act (p. 490).¹¹

Harel (2008a) further describes ways of thinking as involving *proof schemes*, *problem-solving approaches*, and *beliefs about mathematics*, the last of which encompasses “the characteristics of one’s interpretation of (1) what mathematics is or is not, (2) how it is created, and (3) its intellectual or practical benefits” (p. 493). I illustrate the relationship between mental act, way of thinking, and way of understanding in *Figure 5*.

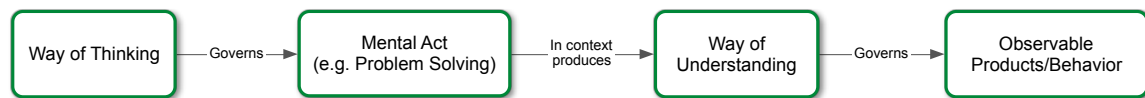


Figure 5. Relation between mental act, way of thinking, and way of understanding.

It is important to note that Harel uses the preface “way of” to suggest that there is no objectively correct way of understanding specific mathematical concepts and no universally efficacious mathematical way of thinking (G. Harel, personal communication, April 19, 2012). Hence, the constructs *way of understanding* and *way of thinking* are defined independently of value judgment. Harel (2007) explains,

the ultimate goal is for students to develop ways of understanding and ways of thinking compatible with those that have been institutionalized in the discipline of mathematics, those that the mathematics community at large accepts as correct and useful in solving mathematical and scientific problems (p. 7).

The “appropriateness” of one’s ways of understanding and ways of thinking is therefore determined by the extent to which they deviate from those that have been

institutionalized in the discipline of mathematics.

¹¹ Although Harel explains that one may observe that another’s ways of understanding reveal a common cognitive characteristic (i.e., a way of thinking), ways of understanding themselves are not observable but instead must be inferred from the observable behaviors they govern. I believe Harel to be making this point when he says, “A person’s statements and actions may signify cognitive products of a mental act carried out by the person” (Harel, 2008a, p. 490).

Harel (2008b) summarizes the reciprocity between one’s way of understanding and his or her way of thinking in the *duality principle*, which states, “Students [or teachers] develop ways of thinking through the production of ways of understanding, and, conversely, the ways of understanding they produce are impacted by the ways of thinking they possess” (p. 899). I illustrate the triad mental act, way of thinking, and way of understanding, with the incorporation of the duality principle, in *Figure 6*.

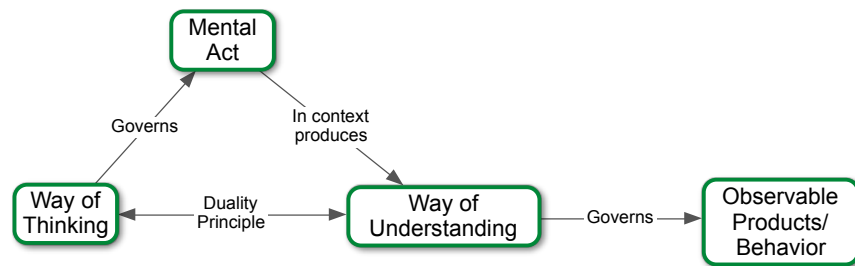


Figure 6. Triad of mental act, way of thinking, and way of understanding.

Teachers’ Knowledge Base

Having recounted the relevant DNR premises and constructs, I am equipped to discuss Harel’s (2008b) notion of *teacher’s knowledge base* (TKB). Harel (2008b) defines teacher’s knowledge base as the compilation of three components: (a) *knowledge of mathematics*, (b) *knowledge of student learning*, and (c) *knowledge of pedagogy*.

In accordance with the Mathematics Premise, teachers’ knowledge of mathematics is composed of their mathematical ways of understanding and ways of thinking. Knowledge of student learning “refers to the teacher’s understanding of fundamental psychological principles of learning, such as how students learn and the impact of their previous and existing knowledge on the acquisition of new knowledge” (Harel, 2008c, p. 276). “Learning” in this definition is consistent with the operational definition of learning proposed by Harel and Koichu (2010), which asserts that learning is

“a continuum of disequilibrium-equilibrium phases manifested by (a) *intellectual needs* and *psychological needs* that instigate or result from these phases and (b) *ways of understanding* or *ways of thinking* that are utilized and newly constructed during these phases” (p. 116, italics in original). Accordingly, knowledge of student learning involves teachers’ awareness of the cognitive and epistemological obstacles that initiate disequilibrium—including students’ intellectual and psychological needs—in the context of students coming to acquire institutionalized ways of understanding and ways of thinking.

Knowledge of pedagogy refers to teachers’ understanding of how to teach in accordance with the psychological principles of learning and is composed of teachers’ *teaching practices* and *instructional principles*. Teaching practices are further composed of *teaching actions*: culturally accepted classroom practices, and *teaching behaviors*: characteristics of teaching actions. Instructional principles derive from one’s anticipation of how specific teaching actions will influence student learning. The concept map in *Figure 7* illustrates the relationship between the DNR constructs mentioned thus far.

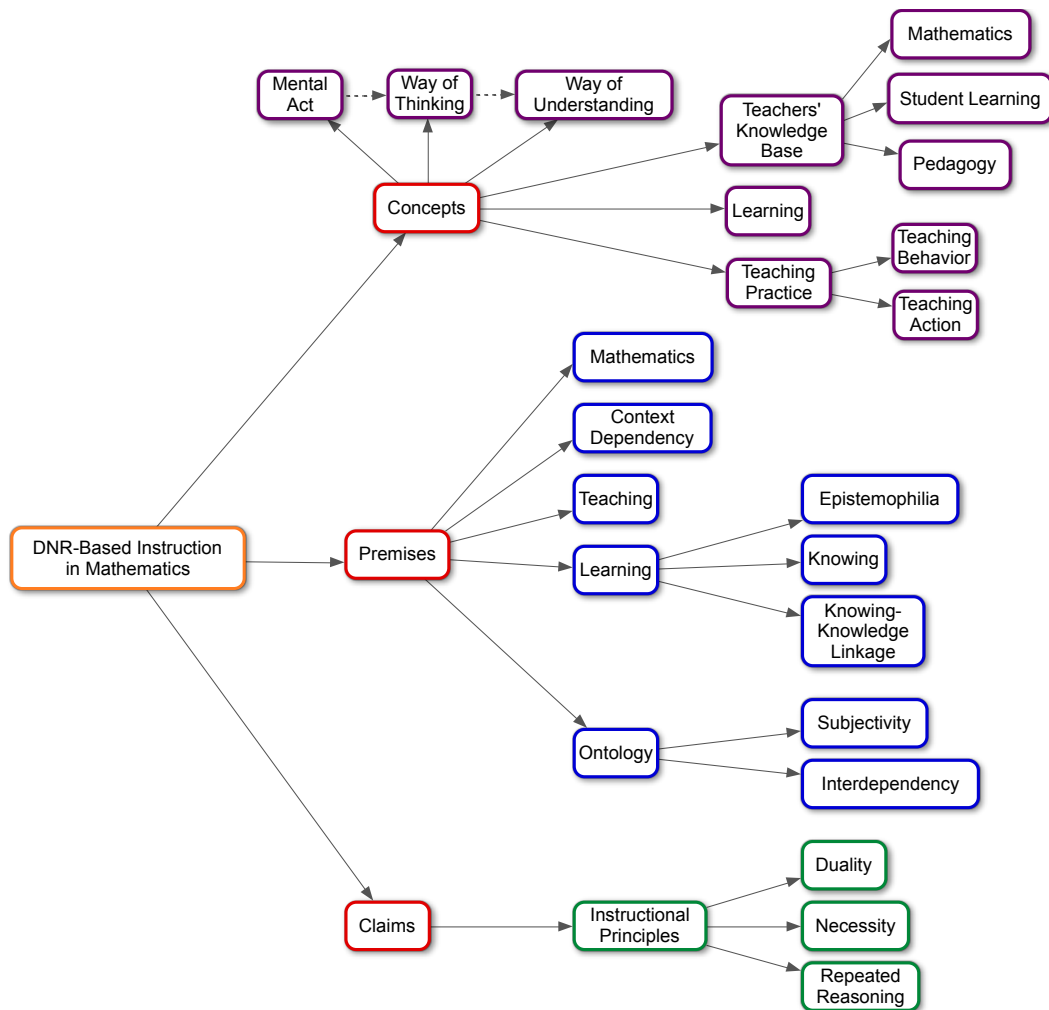


Figure 7. DNR-based instruction in mathematics.

It is noteworthy that in DNR, a teacher’s pedagogical knowledge and knowledge of student learning are heavily informed by the teacher’s knowledge of mathematics. That is, a teacher’s knowledge of pedagogy and of student learning do not exist independently of the teacher’s mathematical knowledge, but instead are fashioned by the teacher’s mathematical ways of understanding and ways of thinking. Accordingly, in DNR the institutionalization of teachers’ ways of understanding and ways of thinking is necessary to sustain the co-development of teachers’ knowledge of pedagogy and knowledge of student learning.

Contributions and Limitations of Harel's *Teachers' Knowledge Base*

Harel's conceptual framework, DNR-based instruction in mathematics, and consequently his notion of TKB, is established on explicit theoretical premises. In addition to the Mathematics, Knowing, Knowing-Knowledge Linkage, Context Dependency, and Teaching premises, defined above, the following additional premises underlie Harel's conceptual framework: Epistemophilia, Subjectivity, and Interdependency. The Epistemophilia Premise asserts,

Humans—all humans—possess the capacity to develop a desire to be puzzled and to learn to carry out mental acts to solve the puzzles they create. Individual differences in this capacity, though present, do not reflect innate capacities that cannot be modified through adequate experience (Harel, 2008b, p. 894).

The Subjectivity and Interdependency premises—collectively comprising the Ontology category of the DNR premises—respectively maintain, “Any observations humans claim to have made is due to what their mental structure attributes to their environment” and “Humans’ actions are induced and governed by their views of the world, and, conversely, their views of the world are formed by their actions” (Harel, 2008b, p. 894). Together, Harel's DNR premises are assertions that derive from a variety of theoretical orientations including Aristotelian epistemology, Piagetian genetic epistemology, situated cognition, Vygotskian social constructivism (namely the theory of the zone of proximal development), and von Glasersfeld's radical constructivism.

A second contribution of Harel's conceptualization of TKB is his definition of mathematical knowledge as the entailment of mathematical ways of understanding and mathematical ways of thinking. Many schoolteachers, college instructors, and educational researchers perceive mathematical knowledge as the aggregation of discrete ways of

understanding particular mathematics concepts while mathematical ways of thinking remain on the periphery of common characterizations of teachers' mathematical knowledge, if included at all. Harel (2008a) cautions such inattention to ways of thinking: "I claim that with the perception of mathematics as a discipline consisting of ways of understanding, it is difficult to form a pedagogically convincing position on the question: What mathematics do mathematics teachers need to know to be effective?" (p. 493). Thus, by including *ways of thinking* into his conceptualization of TKB, Harel acknowledges that the habitual forms of reasoning that govern teachers' concept-specific understandings affect the quality of their instruction, and therefore constitute an essential component of teachers' knowledge.

Harel's notion of TKB is based on an explicit and coherent epistemological foundation and respects the dualistic nature of mathematical knowledge. While Harel's conceptualization of TKB is an articulate, comprehensive, and epistemologically intelligible description of three fundamental categories of mathematics teachers' knowledge—mathematical knowledge, knowledge of student learning, and knowledge of pedagogy—the DNR framework does not address how teachers' appraisal of their environmental context affects the instantiation of these three knowledge domains.

Conclusion

The four perspectives on teacher knowledge that I have reviewed in this chapter make substantial contributions on several fronts. They are collectively limited, however, in not addressing some important factors that condition the instantiation of the knowledge they characterize or discuss.

Educational researchers care about teacher knowledge for the obvious reason that it influences what students may learn. As Thompson (2008) explains,

students' mathematical learning is the reason our profession exists. Everything we do as mathematics educators is, directly or indirectly, to improve the learning attained by anyone who studies mathematics. Our efforts to improve curricula and instruction, our efforts to improve teacher education, or efforts to improve in-service professional development are all done with the aim that students learn a mathematics worth knowing, learn it well, and experience value in what they learn. So, in the final analysis, the value of our contributions derives from how they feed into a system for improving and sustaining students' high quality mathematical learning (p. 45).

Identifying various categories of mathematics teachers' knowledge, characterizing the specific meanings that allow teachers to create rich learning opportunities for students, understanding the experiences by which teachers may construct these meanings, and developing instruments to measure teachers' knowledge, while essential to the enterprise of improving students' mathematics learning, does not ensure teachers will utilize the full extent of their knowledge in the act of teaching. Teachers must recognize the knowledge they possess as appropriate to employ in the process of achieving their goals and objectives in the context of practice. This recognition is subject to a host of cognitive and affective processes that have thus far not been a central focus of research on teacher knowledge in mathematics education (Day & Qing, 2009, p. 17; Hargreaves & Shirley, 2009, p. 94; Meyer, 2009, p. 89; Nias, 1996, p. 293; Schutz et al., 2009, p. 207).

Identifying the factors that condition the knowledge teachers utilize in the context of teaching, and ascertaining the effect of such factors on the quality of teachers' enacted knowledge, is imperative for improving students' mathematics learning. In other words, for the current research on teacher knowledge to realize its intended effect of ensuring teachers present students with opportunities to construct rich mathematical ways of

understanding and develop productive mathematical ways of thinking, it is crucial to apprehend the effect of those factors that condition the knowledge teachers *do* possess in addition to characterizing the knowledge that teachers *should* possess. Ascertaining the factors that mediate the knowledge that resides in teachers' minds and the knowledge they bring to bear while teaching is indispensable to evolving the education reform discourse in the United States beyond the polarized arguments for the state of U.S. education discussed in Chapter 1, and for fashioning well-informed teacher preparation and professional development programs and educational policies that take seriously the effect of both teacher knowledge and those factors that compromise it.

Finally, and by way of transition, the extent to which the four perspectives on teacher knowledge explicate what they mean by "knowledge" varies substantially. Harel's (2008b) notion of TKB and Silverman and Thompson's (2008) developmental framework for MKT are based on explicit epistemological foundations whereas Ball, Hill, and colleagues' and Shulman's work lacks an intelligible theoretical basis. Research on teacher knowledge cannot realize its intended effect if researchers do not explain precisely what they mean by "knowledge." As Thompson (2013b) explains,

The area of mathematics education research that is most wanting today regarding attention to meaning is research on teachers' mathematical knowledge for teaching (MKT). First, the verb 'to know' is used in this research as a primitive, undefined term. The question of what 'to know' means in regard to knowing mathematics is unaddressed (p. 79).

Accordingly, I now turn to describing the theoretical foundation guiding the present study. I devote particular attention to describing my perspective of what it means "to know," as well as clarifying the process by which knowledge develops and is instantiated.

CHAPTER 3

THEORETICAL PERSPECTIVE

[F]acts are only facts within some theoretical framework.

(Guba & Lincoln, 1998, p. 199)

The focus of this chapter is on explicating the theoretical foundation on which I based this dissertation research. I begin with a brief discussion of the role of theory in the present study and in mathematics education research more generally. I then outline the central principles of von Glasersfeld's (1995) radical constructivism and Piaget's (1971, 1977) genetic epistemology. Thereafter, I expound upon Harel's definition of mathematical knowledge as the union of complementary subsets mathematical ways of understanding and mathematical ways of thinking. Then I discuss two foundational ways of thinking that informed the design and analysis of the present study: *quantitative reasoning* (Smith & Thompson, 2007; Thompson, 1990, 2011) and *covariational reasoning* (Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson, 1994b). I conclude this chapter with a discussion of the factors that inform the operationalization of one's knowledge structures. Throughout the chapter, I explain how various aspects of my theoretical perspective informed the design and analysis of the present study.

Role of Theory in Mathematics Education Research

Cobb (2007) characterizes the enterprise of mathematics education as a design science, "the collective mission of which involves developing, testing, and revising conjectured designs for supporting envisioned learning processes" (p. 3). Mathematics educators are therefore responsible for contributing to the development, revision, and validation of designs (pedagogical, curricular, mathematical, institutional, social, political,

and otherwise) that aim to improve the quality of students' mathematics learning. This enterprise is based on researchers' investigation of complex systems such as students' and teachers' cognition, classroom social practices, institutional structures, and the like. These systems are complex in the sense that there are a large number of interacting independent variables within the system, far too many for a researcher to attend to in a disciplined and systematic way. Accordingly, the role of theory in mathematics education research at large, and in the present study in particular, is to orient and constrain the researcher's attention to those variables the researcher assumes to be most fundamental to explaining a particular phenomenon, thereby making the complex system of interest accessible to empirical investigation. The theoretical perspective a researcher assumes serves as a lens through which he or she is able to "control" specific aspects of the complex system under investigation so as to permit the construction of a viable characterization of the system in a particular state, or of the system undergoing transformation. In other words, researchers adopt theoretical perspectives in an effort to isolate what their theoretical perspective prescribes as variables within a complex system that maintain the most explanatory or descriptive utility. Adopting a theoretical perspective is therefore useful because it brings into conscious awareness the lens through which one interprets the phenomena under investigation, which affords the researcher a higher degree of agency over his or her interpretations and conclusions.

For instance, theoretical constructs such as *equilibration* in Piaget's genetic epistemology (Piaget, 1985), *parallel distributed processing* (Rumelhart & McClelland, 1986) in cognitive information processing theory, and *legitimate peripheral participation* (Lave & Wenger, 1991) in situated cognition all explicitly limit the scope of a

researcher's interpretation of the complex system under investigation since researchers who subscribe to these respective theories interpret the complex systems they study through the lens of such theoretical constructs. It is important to note that imposing such restrictions is necessary, and indeed unavoidable, since the complex systems researchers examine are often far too elusive to effectively characterize without the conceptual lenses that accompany the adoption of a theoretical perspective.¹² What a researcher is able to perceive relative to the analytical unit of a complex system is fashioned by the assumptions the researcher adopts regarding this analytical unit; that is, a well-defined theoretical orientation necessarily desensitizes one to particular aspects of the complex system under investigation while simultaneously increasing awareness to others. For this reason, the theoretical perspectives we adopt impose a set of assumptions and expectations about the complex systems we study that serve "to constrain the types of explanations we give, to frame our conceptions of what needs explaining, and to filter what may be taken as a legitimate problem" (Thompson, 2002, p. 192).

For the present study, I adopted a theoretical perspective based largely on von Glasersfeld's (1995) radical constructivism and Piaget's (1971, 1977) genetic epistemology. I intend my discussion of these theories of knowing to reveal the way in which they framed my perception of my research subject's cognition by directing my attention to particular variables within this complex system.

¹² I do not intend this statement to imply that one consciously dismisses relevant causal variables of a changing system based on the assumptions of his or her theoretical orientation, but rather that one's theoretical orientation imposes conceptual "blindness" that limit the causal variables one is able to perceive.

Radical Constructivism

von Glasersfeld (1995) developed the psychological learning theory of radical constructivism as an elaboration of Piaget's genetic epistemology (1971, 1977). The "radical" qualifier emphasizes von Glasersfeld's position that cognitive processing is the foundation of the only reality an organism may come to know. Two fundamental propositions of radical constructivism are:

- (1) Knowledge is not passively received either through the senses or by way of communication; knowledge is actively built up by the cognizing subject.
- (2) The function of cognition is adaptive, in the biological sense of the term, tending towards fit or viability; cognition serves the subject's organization of the experiential world, not the discovery of an objective ontological reality (von Glasersfeld, 1995, p. 51).

These propositions suggest that radical constructivism is grounded in a strong skeptical disposition regarding truth and reality (Thompson, 2013a). Radical constructivism

starts from the assumption that knowledge, no matter how it be defined, is in the heads of persons, and that the thinking subject has no alternative but to construct what he or she knows on the basis of his or her own experience. What we make of experience constitutes the only world we consciously live in (von Glasersfeld, 1995, p. 1).

It is important to note that radical constructivism does not deny the existence of an intersubjective ontological reality. Rather, provided the aforementioned epistemological assumptions, radical constructivism asserts that the only reality an individual may come to know is the reality he or she constructs from the basis of experience, or interaction with the environment.

Reality is a subjective construction because one's experience of external reality is necessarily a manifestation of the cortical processing that is unique to each individual and, consequently, cannot be transcended. As Damasio (1994) explains,

our very organism rather than some absolute external reality is used as the ground reference for the constructions we make of the world around us and for the construction of the ever-present sense of subjectivity that is part and parcel of our experiences (p. xx).

von Glasersfeld thus distinguishes objective reality from *experiential reality*, which "is the domain of the relatively durable perceptual and conceptual structures which we manage to establish, use, and maintain in the flow of our actual experience" (1995, p. 118). Because experiential reality is a subjective construction that cannot be said to resemble a reality that exists independently of the knower, the notion of objective truth is untenable. To claim knowledge of objective truth is to profess access to an external reality that is not fashioned by one's subjective ways of perceiving and conceiving the world, or to suggest that one's ways of experiencing the world permits access to the world as it *really is*. This is not to say that one's knowledge *does not* resemble objective truth, as such a statement is contradictory from a radical constructivist perspective since one would need to possess knowledge of objective truth in order to make the comparison. Accordingly, radical constructivism does not claim that objective truths are unknowable in the sense that one can be certain that their knowledge does not resemble truth, but instead asserts that there is no means by which one can make such a determination.

Since it is impossible to verify whether one's knowledge corresponds to external reality, radical constructivism "replaces the notion of 'truth' (as a true representation of an independent reality) with the notion of 'viability' within the subjects' experiential

world” (von Glasersfeld, 1995, p. 22). In other words, an individual’s knowledge of the world is “correct” not because it resembles external reality but because the individual has no reason to believe that it does not.

Piaget’s Genetic Epistemology

In this section I outline the primary epistemological stance on which the design and analysis of the present study is based. Piaget’s genetic (developmental) epistemology (Piaget 1971, 1977) is a theory dealing with the formation and meaning of knowledge. Piaget explains, “Genetic epistemology attempts to explain knowledge and in particular scientific knowledge, on the basis of its history, its sociogenesis, and especially the psychological origins of the notions and operations upon which it is based” (1971, p. 1). Piaget’s theory arose out of his concern to develop a viable model of how individuals are able to construct a relatively stable image of reality from the flow of their subjective experience (von Glasersfeld, 1995, p. 57).

The focus of Piaget’s psychological work varied greatly throughout his nearly six decades of research. These foci include, but are certainly not limited to, the child’s conception of language, judgment, morality, causality, number, quantity, movement, speed, time, space, geometry, chance, and logic. However disparate these foci may seem, the entirety of Piaget’s psychological work was in the service of elaborating a theory of knowledge development, or genetic epistemology. I thus begin this section with a summary of the goals of Piaget’s psychological research program, including the theoretical premises on which it is based. This summary seeks to provide a context in which to interpret various theoretical constructs within Piaget’s genetic epistemology.

Goals of Piaget's Psychological Research Program

The attention of Piaget's nearly six decades of psychological research was on elaborating a theory of cognitive development—that is, a theory of how individuals come to know their world and achieve increasingly viable and coherent representations of reality. While Piaget's approach was indeed multidisciplinary, the core of his theory originated from the parallelisms he drew between biological and psychological processes (Gallagher & Reid, 2002). In contrast to the empiricist notion that knowledge results from a simple registering of experience, and the innatist or nativist position that knowledge derives from nervous system maturation, Piaget's biological perspective of cognitive development emphasized the *interaction* between the knowing subject and his or her environment.¹³ Cognition, according to Piaget, is an instrument of adaptation instead of the producer of representations of an ontological reality, and evolved to enable individuals to establish a fit between their conceptual model of reality and the reality they experience (von Glasersfeld, 1995, p. 59). Accordingly, central to Piaget's theory, and consistent with its biological underpinnings, is the *action* of the subject, where Piaget broadly defines action to encompass all movement, thought, and emotion that responds to a need (Piaget, 1967, p. 6).

To simply say that knowledge derives from a subject's interaction with the environment does not explain how the subject constructs knowledge. For this reason, Piaget elaborated the concept of *equilibration*, the mechanisms of which are *assimilation* and *accommodation*. Briefly defined, equilibration is the self-regulatory process by which

¹³ Because Piaget did not consider cognitive development a process that is deterministically influenced by environmental pressures nor exclusively a process of maturation, his theory is often referred to as a middle-ground position, third possibility, or *tertium quid* (Gallagher & Reid, 2002, p. 23).

the individual actively compensates to external disturbances (Piaget & Inhelder, 1969); assimilation is the process whereby a subject incorporates experiences into existing cognitive structures, and thus entails the meanings the subject holds; and accommodation is the process by which individuals modify their cognitive structures so as to permit assimilation. In what follows, I describe in greater detail the meaning of assimilation, accommodation, equilibration, and related theoretical constructs within Piaget's genetic epistemology.

Scheme and Assimilation

Central to Piaget's genetic epistemology is the notion of a *scheme* and *assimilation* to a scheme. Piaget generally defines a scheme as "whatever is repeatable and generalizable in an action" (1971, p. 42). More specifically, Piaget described a scheme as "the structure or organization of actions as they are transferred or generalized by repetition in similar or analogous circumstances" (Piaget & Inhelder, 1969, p. 4). It is important to note that schemes are cognitive entities that individuals construct as a consequence of their natural disposition to make sense of their experiences. Schemes serve to organize the individual's reality and impose order on an ever-changing world by equipping the individual with the conceptual tools to systematically act on his or her environment and expect particular outcomes.

Following Piaget, von Glasersfeld (1995) identified three parts of what he called an *action scheme*: (1) perceiving a particular situation, (2) employing a specific activity associated with the perceived situation, and (3) expecting the result of the activity to be the same as a certain previously experienced result (p. 65). Thus, an action scheme involves an individual's learned inclination to act in specific ways when encountered

with a particular situation and his or her anticipation of a specific outcome from responding to the perceived situation. Therefore, in von Glasersfeld's usage, to "assimilate to a scheme" means to fit an experience into an existing conceptual structure by recognizing a particular situation as analogous to previously encountered ones, to be inclined to employ a specific action, to anticipate the outcome of that action, and to experience the outcome of the action align with the expectation. Simply put, assimilation involves "treating new material *as an instance of something known*" (von Glasersfeld, 1995, p. 62, italics in original). Thus, what an individual is able to assimilate is wholly dependent upon his or her current cognitive schemes. "In short, assimilation always reduces new experiences to already existing sensorimotor or conceptual structures" (von Glasersfeld, 1995, p. 63). This suggests the very important point that what one already knows plays a significant role in what he or she may come to know.

von Glasersfeld's definition of an action scheme differs slightly from my understanding of what Piaget generally meant by scheme in that the perception, action, and expectation aspects of von Glasersfeld's definition, in my view, do not constitute one's scheme but are instead processes governed by it. Simply stated, I consider a scheme to be an internalized organization of actions and operations to which one assimilates experiences of reality. My discussion of *reflecting abstraction* below will clarify this definition.

Accommodation and Equilibration

Often in the flow of experience, one's perception of a particular situation induces him or her to employ an action that produces an effect that violates the actor's expectation. In this case the scheme governing the perception, action, and expectation is

not viable with experiential reality. Recognizing such discrepancy induces in the individual a state of cognitive conflict, perturbation, or *disequilibrium*, and the individual thus has a reason to amend his or her existing scheme or construct a new scheme, a process Piaget called *accommodation* because one modifies or constructs cognitive schemes to accommodate for the unanticipated experience.

As a consequence of his constructivist epistemology, Piaget “relinquished the notion of cognition as the producer of representations of an ontological reality and replaced it with cognition as an instrument of adaptation the purpose of which is the construction of viable conceptual structures” (von Glasersfeld, 1995, p. 59). So long as an individual has no reason to believe that her model of the world is not accurate, she will remain in a state of cognitive equilibrium, a state that suggests her conceptual structures are viable with experiential reality. Such a state is preserved so long as the individual is able to assimilate new experiences to existing schemes. When an individual does not have an appropriate scheme to which she may assimilate an experience, she experiences a state of cognitive perturbation, or *disequilibrium*. This state induces a need for the individual to accommodate the experience by modifying an existing scheme or creating a new scheme so that her conceptual structures remain viable with her experiential reality.

von Glasersfeld (1995) puts it nicely,

The ‘recognition’ in part 1 [of the action scheme] is always the result of assimilation. ... The activity, part 2, then produces a result that the organism will attempt to assimilate to its expectation in part 3. If the organism is unable to do this, there will be a perturbation (Piaget, 1974, p. 264). The perturbation, which may be either disappointment or surprise, may lead to all sorts of random reactions, but one among them seems particularly likely: if the initial situation 1 is still retrievable, it may now be reviewed, not as a compound triggering situation, but as a collection of sensory elements. This review may reveal characteristics that were disregarded by assimilation. If the unexpected outcome of the activity

was disappointing, one or more of the newly noticed characteristics may effect a change in the recognition pattern and thus in the conditions that will trigger the activity in the future. Alternatively, if the unexpected outcome was pleasant or interesting, a new recognition pattern may be formed to include the new characteristic, and this will constitute a new scheme, in both cases there would be an act of learning and we would speak of an ‘accommodation’. The same possibilities are opened, if the review reveals a difference in the performance of the activity, and this again could result in an accommodation (p. 65-66).

In summary, cognitive change, or learning, occurs when one’s existing schemes are insufficient to assimilate an experience, causing perturbation which leads to an accommodation that reestablishes equilibrium (von Glasersfeld, 1995, p. 68). Cognitive development, then, is a process of expanding equilibration—a process of *cognitive reorganization* that occurs as a consequence of an individual experiencing perturbation upon recognizing that his or her conceptual structures are not viable with experiential reality, achieving an accommodation, and re-establishing cognitive equilibrium by assimilating the experiential situation to a new or modified scheme.

Knowledge and Understanding

Now that I have described a few of the central constructs within Piaget’s genetic epistemology, I am able to define “knowledge” and “understanding” from a Piagetian constructivist perspective—notions central to the present study.

To Piaget, meaning and understanding were synonymous and both meant “to assimilate to a scheme” (Thompson, 2013b). Thus, one’s meaning of a particular object or idea is the scheme to which it was assimilated. Moreover, “to know” something is to have a scheme to which the something may be assimilated. Piaget (1967) explains, “to know an object implies its incorporation in action schemes, and this is true on the most elementary sensorimotor level and all the way up to the highest logical-mathematical

operations” (p. 17). To speak of one’s knowledge of a particular concept or idea, then, is to speak of her scheme of meaning, or perhaps a network of related schemes of meaning, pertaining to the concept or idea. Similarly, to generally speak of one’s knowledge is to reference the aggregate collection of her schemes of meanings.

With this Piagetian definition of knowledge, one can interpret the research questions I presented in Chapter 1 as follows:

RQ1: Are there incongruities between a teacher’s mathematical schemes of meanings (as they reside in the mind of the teacher) and the mathematical schemes of meanings he invokes while teaching?

RQ2a: If so, in what ways does the teacher’s image of instructional constraints condition the mathematical schemes of meanings he utilizes while teaching?

RQ2b: If not, how is the teacher appraising and/or managing what he perceives as instructional constraints so that these constraints do not condition the mathematical schemes of meanings he utilizes while teaching?

Piagetian Abstraction

Assimilation and accommodation, and thus equilibration, rely heavily on the notion of abstraction, of which Piaget distinguished five varieties: *empirical*, *pseudo-empirical*, *reflecting*, *reflected*, and *meta-reflection*. Piaget explained that higher forms of knowledge derive from abstractions of the subject’s actions and the results of such actions. As Piaget’s career progressed, he increasingly recognized the utility of equilibration and abstraction for providing a functional explanation for the development of knowledge—so much so that by the end of his career, these two constructs were at the

nucleus of his theory. I therefore turn to defining the four of five categories of Piagetian abstraction that informed the design and analysis of the present study (empirical, pseudo-empirical, reflecting, and reflected). I also illustrate these four forms of abstraction by discussing a hypothetical student engaged in a sequence of four tasks that respectively seek to engender each.

Empirical abstraction. Empirical abstraction consists of extracting information from any source the subject considers exogenous, and thus results in *exogenous knowledge*. In other words, in empirical abstraction, the subject extracts properties from objects that are external (from the subject's perspective). It is important to note that these objects need not be physical. Empirical abstraction thus entails a subject engaged in perceptive action whereby he or she draws information from objects (see *Figure 8*). It is also important to note that the characteristics the subject extracts are not *from* the object, in a literal sense, but from the subject's *perception* of the object. This is to emphasize the notion that a passive subject cannot extract properties of objects and that the properties the subject is able to extract are fashioned by the cognitive structures the subject brings to the perceptive act. Since the information one is able to draw from objects is conditioned by her own subjective ways of perceiving and conceiving, empirical abstractions rely upon previous reflecting abstractions¹⁴.

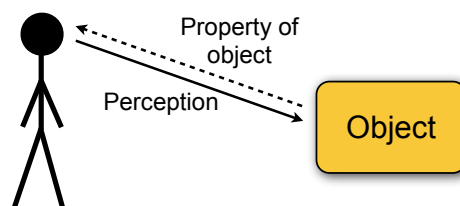


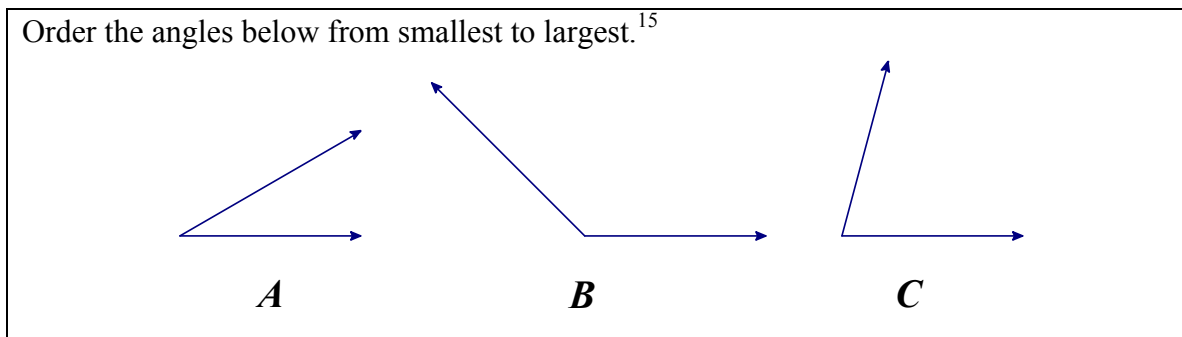
Figure 8. Empirical abstraction.

¹⁴ See my discussion of reflecting abstraction below.

To illustrate empirical abstraction in a mathematical context, consider a student with no formal experience with angle measure attempting to solve the task in Table 1.

Table 1

Task that Supports Empirical Abstraction



Although the student has limited knowledge of what it means to measure an angle, the student would likely be able respond sensibly, if not correctly, to this task. The student might examine the three geometric objects—which she recognizes as existing independently of her thought and action—and abstract the property of “openness” between the rays, where the student vaguely conceives “openness” as the space between two rays. Upon doing so, the student might recognize that Angle *A* has the smallest amount of openness while Angle *B* has the largest amount of openness. It is noteworthy that, via empirical abstraction, this task is accessible even without the student having an understanding of what it means to measure “openness.” This is an example of empirical abstraction because the student abstracts a property from geometric objects that she recognizes as existing independently of her own thought and action.

Pseudo-empirical abstraction. Pseudo-empirical abstraction involves a subject abstracting properties from objects that have been modified by or created through the

¹⁵ This task is intentionally vague and imprecise so as to allow an individual with limited knowledge of angle measure to assimilate it.

subject's actions and enriched by the properties drawn from the coordination of these actions (Piaget, 1977). In pseudo-empirical abstraction, the subject does not disassociate the action from the object nor the object from the action. What the subject abstracts, then, is not simply a property of the object, as in empirical abstraction, nor the action that produces or modifies the object, as in reflecting abstraction (see below), but a property of the object that is produced by and *represents* the action of the subject (see *Figure 9*). That is, in pseudo-empirical abstraction the subject sees the properties of an object as the result of having acted upon the object and as encapsulating these actions so that the properties the subject abstracts capture both the action and the result of acting. In this way, pseudo-empirical abstraction relies on what Piaget called the *semiotic function*, or the means by which a subject represents her world. The individual, when engaged in pseudo-empirical abstraction, conceptualizes the object, or a momentary property of the object, as a symbol that captures both the object and action that produced or modified it so that the individual may then operate on these symbols without having to engage in the action that produced or modified the object.

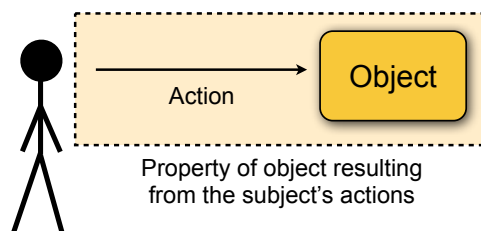


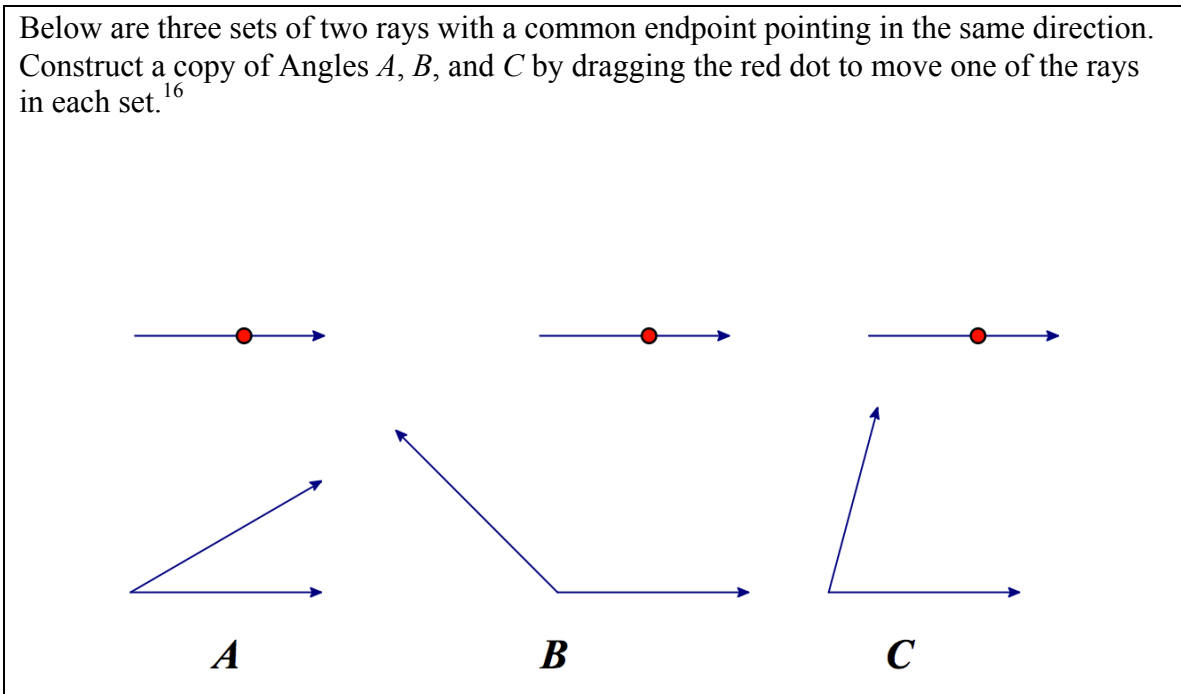
Figure 9. Pseudo-empirical abstraction.

To illustrate pseudo-empirical abstraction, consider our hypothetical student engaged in the task provided in Table 2, which is an extension of the task discussed in Table 1 to illustrate empirical abstraction.

Table 2

Task that Supports Pseudo-Empirical Abstraction

Below are three sets of two rays with a common endpoint pointing in the same direction. Construct a copy of Angles *A*, *B*, and *C* by dragging the red dot to move one of the rays in each set.¹⁶



After completing the task, the student may have produced something like the following:

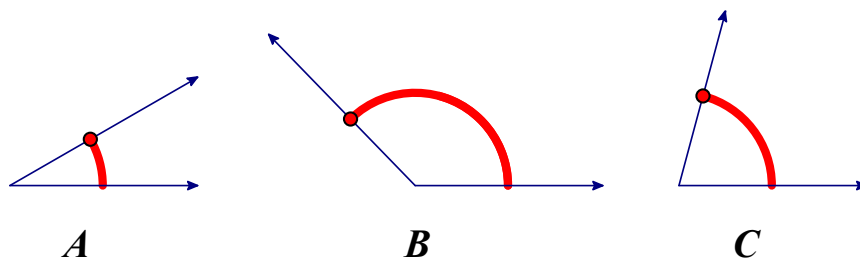


Figure 10. Example of pseudo-empirical abstraction.

After having engaged in the action of rotating the terminal ray of each angle by dragging the red dot, the student is prepared to abstract the property that the red dot traces

¹⁶Note that each red dot is equidistant from the vertex of the respective angle. The student is expected to move the red dot by using Geometer's Sketchpad, a geometric visualization program.

out an arc. The student may also abstract the property that a larger angle subtends an arc that is longer than the arc that a smaller angle subtends, so long as each red dot is the same distance from the vertex of the respective angle. Crucially, the student recognizes that this property resulted from having engaged in the action of rotating the red dot, and therefore *represents* this action. In other words, the student does not see the property as existing independently of the action she engaged in to create it. This inability to dissociate the property of the object that the student abstracted from the action she engaged in to create it is the distinguishing feature of pseudo-empirical abstraction.

Reflecting abstraction. In contrast to empirical and pseudo-empirical abstraction, reflecting abstraction involves the reconstruction on a higher cognitive level of the coordination of actions from a lower level (Chapman, 1988). Reflecting abstraction is thus an abstraction of actions and occurs in three steps: (1) the *differentiation* of an action from the effect of the action, (2) the *projection* of the action from the level of material action to the level of representation, and (3) the *reorganization* that occurs on the level of representation of the action projected from the level of material action (see *Figure 11*).

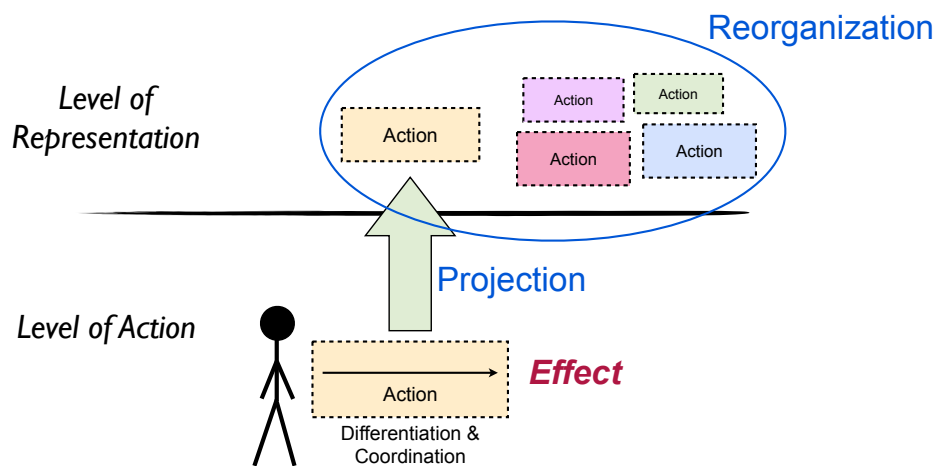


Figure 11. Reflecting abstraction.

A subject must differentiate actions from their effects before she can construct an internalized representation of these actions, what Piaget called projecting actions to the level of representation. Additionally, the subject must coordinate the actions that produced the effect before she can project and represent them on a higher cognitive level. Once a subject differentiates actions from their effect and coordinates these actions, the subject is prepared to project these coordinated actions to the level of representation where they may be organized into schemes.

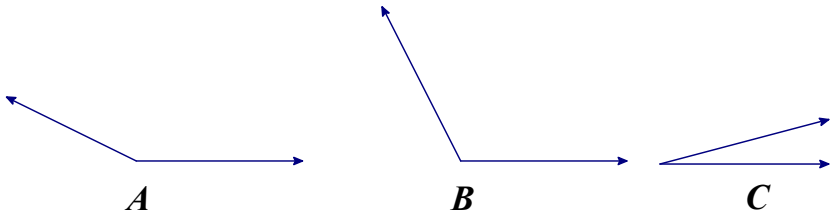
It is important to note that reflecting abstraction constitutes an accommodation to structures at the level of representation and, like all forms of accommodation, occurs to resolve a cognitive perturbation (disequilibrium). Reflecting abstraction resolves such disequilibrium by constructing actions at the level of representation and organizing these actions into coherent schemes at this higher cognitive level. Reflecting abstraction is thus the mechanism of constructing cognitive schemes.

To illustrate reflecting abstraction, consider our hypothetical student engaged in the task in Table 3.

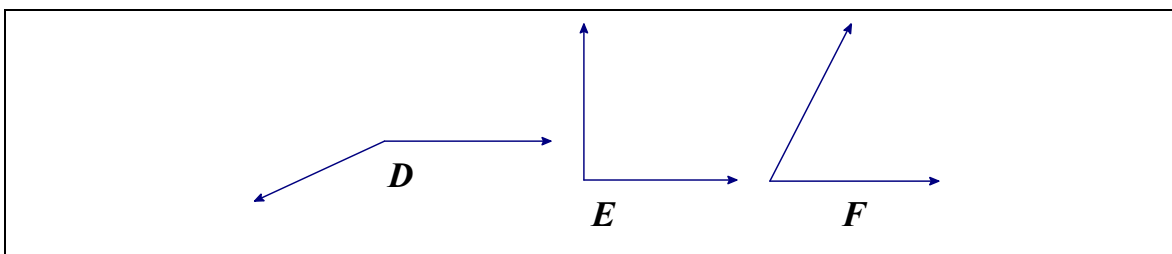
Table 3

Task that Supports Reflecting Abstraction

Order the angles below from smallest to largest and justify your ordering.



The diagram shows three angles labeled A, B, and C. Angle A is an obtuse angle formed by a horizontal ray pointing right and another ray pointing up and to the left. Angle B is a larger obtuse angle formed by a horizontal ray pointing right and another ray pointing up and to the left, with a larger opening than angle A. Angle C is an acute angle formed by a horizontal ray pointing right and another ray pointing up and to the right.



This task is similar to the one I presented to illustrate a context in which the student engaged in empirical abstraction. However, after having pseudo-empirically abstracted the property that moving the terminal ray of an angle by rotating a red dot on the terminal ray results in this red dot tracing out a subtended arc (*Figure 10*), the task in Table 3 engenders a reflecting abstraction by providing an occasion for the student to differentiate the action of rotating the red dot from its effect of producing an angle with a subtended arc length. By asking the student to justify her ordering, the task encourages the student to reflect on the physical actions she engaged in while solving the task in Table 2, namely moving the red dot to create angles with a particular amount of “openness.”

The student may approach the task by examining each of the angles in Table 3 and *imagining* having moved a red dot on the terminal ray to create each angle. The length of the arc that the angle subtends therefore determines the “size” of the angle.¹⁷ The student’s recognition that she could have created these angles by moving a point on the terminal ray demonstrates that the student is capable of thinking about this action independent of its effect (an angle with a particular amount of openness that subtends an arc that was generated by tracing the red dot as it moved) and thus constitutes evidence

¹⁷ For this to be an appropriate method for comparing the “size” of the angles, the student must recognize that the red dot must be the same distance away from the vertex of each angle—a fact that the student had the opportunity to pseudo-empirically abstract while completing the task in Table 2.

that the student has differentiated the action of moving a red dot on the terminal ray from the outcome of the action (having an angle that subtends an arc that was generated by tracing the red dot as it moved).

That the student is able to *imagine* moving the red dot (i.e., that the student is able to perform mentally what was in the previous task (see Table 2) a physical action of moving the red dot to create an angle with a particular amount of “openness”) suggests that the student has projected this action to the level of representation. In other words, the student has constructed a mental representation of the physical action of moving a red dot on the terminal ray of an angle to create an angle with a specific amount of “openness.” This mental representation of coordinated physical actions makes this an example of reflecting abstraction.

Reflected abstraction. Reflected abstraction involves operating on the actions that result from prior reflecting abstractions at the level of representation, which results in a coherence of actions and operations accompanied by conscious awareness of those actions and operations.¹⁸ To consciously operate on actions at the level of representation suggests that one has symbolized the action at this higher level. Reflected abstraction, like pseudo-empirical abstraction, thus relies on the semiotic function. The subject symbolizes coordinated actions at the level of representation so as to reify the material actions the symbol represents into a form she can use as an object of thought at the level of representation. On this higher level, the individual can consciously manipulate these symbols independently of reproducing the coordinated actions they represent. The semiotic function is thus the mechanism by which reflecting abstraction turns into

¹⁸ It is the act of deliberately operating on the actions and operations that result from prior reflecting abstractions that brings these actions and operations into conscious awareness.

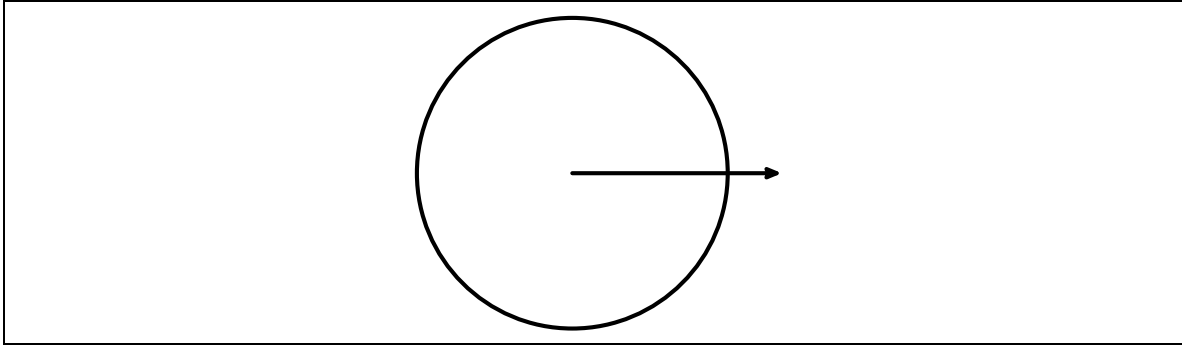
reflected abstraction. As a result of the conscious awareness of actions at the level of representation that accompanies reflected abstraction, the subject's ability to assimilate new experiences to the reflected level (i.e., the level of representation) provides evidence that the subject has engaged in reflected abstraction. Additionally, performing operations on the symbols an individual constructs to represent coordinated actions at the level of representation results in increasingly organized and refined cognitive schemes. Reflected abstraction is therefore the mechanism of injecting coherence into systems of organized actions at the level of representation.

To illustrate reflected abstraction, suppose that our hypothetical student, after having engaged in a number of empirical, pseudo-empirical, reflecting, and even reflected abstractions, has constructed the following as a measurable attribute of an angle in standard position: *the horizontal distance that the terminus of a class subtended arcs is to the right of the vertical line passing through the vertex of the angle, measured in units of the radius of each respective arc*. Further, suppose the student has constructed “ $\cos(x)$ ” as a symbolic representation for this attribute of an angle in standard position that subtends a class of arcs, each of which are x times as large as their respective radius. To engender reflected abstraction, one might ask the student the question in Table 4.

Table 4

Task that Supports Reflected Abstraction

Draw the terminal ray of the angle in the figure below to illustrate the equality $\cos(x) = x$.



This task provides an occasion for the student to operate on a symbol she has constructed to represent the coordinated actions of determining a value for the measurable attribute of an angle in standard position defined above. This symbol may take the form of a mental image like that provided in *Figure 12*. By constructing a mental image of this type, the student has symbolized, at the level of representation, the coordination of actions involved in determining a value for the attribute of an angle defined above, and is therefore prepared to operate on these coordinated actions without having to reproduce them.

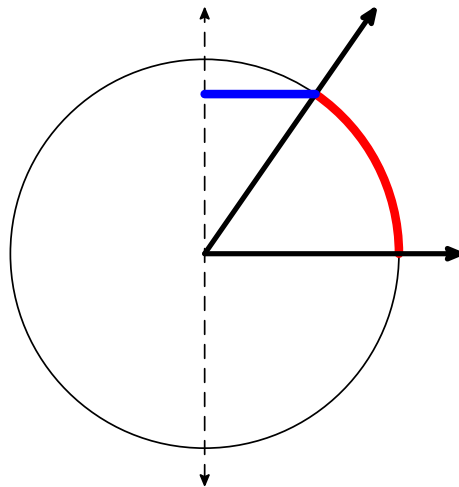


Figure 12. Symbol representing a coordination of actions at the level of representation.

With an image like that in *Figure 12* in mind, the student may interpret the task in the following way: “Move the terminal ray of the angle so that the length of the blue line, which represents $\cos(x)$, is equal to the length of the red arc, which represents x .” Doing so constitutes performing an operation on the symbol (i.e., mental image) that represents a coordination of actions at the level of representation. Operating on this symbol at the reflected level allows the student to organize the coordinated actions the symbol represents into a coherent scheme of actions and operations accompanied by conscious awareness.

Implications of Radical Constructivism and Genetic Epistemology for the Present Study

von Glasersfeld’s radical constructivism and Piaget’s genetic epistemology informed the present study in three important ways. First, recall radical constructivism’s assertion that cognition does not afford one access to an objective ontological reality but instead serves an adaptive function whereby an individual constructs a reality that is viable with his or her experience. This premise of radical constructivism suggests that the results of the present study are not results about the participating teacher but are instead results about my *construction* of the participating teacher’s observable actions. Second, this same premise of radical constructivism implies that instructional constraints are subjective constructions that reside in the minds of teachers and thus do not maintain an ontological designation as instructional constraints. Third, being a proposal for how knowledge structures develop, I made use of Piaget’s notion of abstraction as a principle of designing tasks as well as an explanatory construct that allowed me to construct a

viable model of the participant's mathematical ways of understanding and ways of thinking. I discuss each of these influences in more detail below.

Models as the Researcher's Construction

Constructing a model of a teacher's mathematics is necessary to determine if there are incongruities between the teacher's subject matter knowledge and the mathematical ways of understanding and ways of thinking he employs in the context of teaching. In accordance with the second tenant of radical constructivism, the model I construct of the teacher's mathematics is useful only to the extent that it is viable with my interpretation of the teacher's language and actions. The claims I make about the teacher's mathematics are not claims about the teacher's mathematics as it exists in his mind, but are instead claims about my construction of plausible conceptual operations that explain my interpretation of the teacher's observable behaviors. von Glasersfeld (2000) nicely conveys this implication of radical constructivism in the following recommendation:

Especially in discussing education, we tend to focus on the child or the student as we see them, and we may not stress often enough that what we are talking about is but *our* construction of the child, and that this construction is made on the basis of our own experience and colored by our goals and expectations. ... [W]hen we describe our constructivist orientation, we should take even more care to stress and repeat that we are constructing a model that should be tested in practice, not another metaphysical system to explain what the ontological world might be like (p. 8).

As a researcher, I cannot escape the subjectivity and limitations of my own cortical research equipment. The model I construct of the teacher's mathematics is fashioned by my ways of perceiving and conceiving the teacher's observable actions. Mason (2002) eloquently summarizes this point: "what we learn from an observation is something about the researcher, as well as, perhaps, something about the phenomenon"

(p. 181). It was therefore imperative to document and explicate the decisions and interpretations I made throughout the development of my model of the teacher's mathematics, and to chronicle the evolution of this model throughout the research process. Doing so allowed me to afford the reader a window, however narrow, into the conceptual origins of my results and conclusions.

Instructional Constraints as a Cognitive Entity

The "image of" qualifier in the title of this dissertation suggests my constructivist approach to defining instructional constraints. I take the position that environmental circumstances per se in the absence of a teacher's construal of them cannot constrain his or her practice, but the teacher's construction and appraisal of environmental circumstances can and often does.¹⁹ For this reason, I contend that particular circumstances cannot maintain an ontological designation as instructional constraints, however consensual are teachers' construction and appraisal of such circumstances. Therefore, in consonance with radical constructivism's skeptical position on reality, I define instructional constraints as an individual teacher's subjective construction of the circumstances that impede the teacher's capacity to achieve his or her instructional goals and objectives. Such subjective constructions are the only "constraints" that maintain the potential to influence teachers' instructional actions. Accordingly, I locate instructional constraints in the mind of individuals, not the environment. This conceptualization stands

¹⁹ Defining instructional constraints as an individual teacher's subjective construction suggests that my use of the phrase, "teachers' image of instructional constraints" implies some form of metacognition. By "teachers' image of instructional constraints," I do not mean "teachers' image of their subjective construction of environmental circumstances." This is because individuals tend not to appraise their image of environmental circumstances as subjective constructions but rather as depictions of an ontological reality. Therefore, when I say, "teachers' image of instructional constraints," I really do mean teachers' image of circumstances existing independent of their construction of such circumstances, because this is how teachers tend to perceive them.

in stark contrast to the common perception of instructional constraints as external pressures that exert influence on the quality of teachers' instruction. According to this view, the pressure comes from without instead of from within. My interest in understanding how a secondary teacher's image of instructional constraints conditions the mathematical ways of understanding and ways of thinking he utilizes in the context of teaching necessitated my constructing a model of the teacher's construction of those circumstances he appraised as constraints on his practice.

As a result of my view that instructional constraints are subjective constructions that reside in the minds of teachers, I consider anything that a teacher appraises as an imposition to achieving his or her instructional goals and objectives to be an instructional constraint. The appraisal need not even be of an external circumstance. A teacher may appraise internal characteristics such as his or her mathematical self-efficacy, social endowments, creativity, tolerance, attitude, perseverance, temperament, empathy, confidence, etc., as imposing limits on the quality of his or her instruction. Since a teacher's appraisal of such intrinsic characteristics is a subjective construction in the same way that a teacher's appraisal of external circumstances is, both types of appraisals have the capacity to influence teachers' practice in the same way.²⁰

Piagetian Abstraction as a Design Principle

Finally, Piaget's notion of empirical, pseudo-empirical, reflecting, and reflected abstraction, as well as equilibration, heavily informed the design and analysis of the present study. I must, however, outline my methodology before I am able to explain how

²⁰ While a teacher's appraisal of external and internal constraints influences their teaching in the same way, the teacher will likely manage these constraints differently since the teacher is likely to recognize that the latter, being intrinsic characteristics, are more within the teacher's locus of control.

I made use of Piagetian abstraction as a design principle and as an explanatory construct. I thus withhold discussion of the way in which Piagetian abstraction was instrumental in the design and analysis of this study until Chapter 5.

Mathematical Knowledge as The Union of Mathematical Ways of Understanding and Mathematical Ways of Thinking

As I mentioned above, addressing my first research question—that of understanding whether there are incongruities between a teacher’s subject matter knowledge (i.e., schemes of meaning) and the subject matter knowledge he invoked while teaching—required that I construct a model of the teacher’s mathematics outside the context of classroom practice. I follow Harel (2008) in defining a teacher’s mathematics as the union of mathematical *ways of understanding* and mathematical *ways of thinking*, and thus attended to characterizing both of these aspects of the teacher’s mathematical knowledge. Recall Harel’s (2008b) definitions of these constructs from Chapter 2:

A person’s statements and [observable] actions may signify cognitive products of a mental act carried out by the person. Such a product is the person’s *way of understanding* associated with that mental act. Repeated observations of one’s ways of understanding may reveal that they share a common cognitive characteristic. Such a characteristic is referred to as a *way of thinking* associated with that mental act (p. 490).

To illustrate the distinction between *ways of understanding* and *ways of thinking*, consider the following problem:

$$\text{Let } f(x) = \frac{3\cos(\pi x^2)}{\ln(x)}. \text{ Interpret the meaning of the point } \left(x, \frac{df}{dx}\right).$$

When presented with this problem, one may interpret the mathematical symbolism given in the function rule of f as something to be acted upon using the rules of

differentiation. This inclination is a specific cognitive product of the mental act of interpreting the problem statement and thus reveals a specific way of understanding. One who maintains an orientation across a variety of mathematical domains to manipulate symbols in a way that is unsupported by interpreting the meaning of those symbols possesses a cognitive characteristic of the mental act of interpreting that can be summarized as *non-referential symbolic* (Harel, 2007, p. 4; Harel, Fuller, & Rabin, 2008). This general disposition associated with the mental act of interpreting symbolic mathematics is, by definition, a way of thinking. It is in this way that one's way of thinking associated with a particular mental act governs the mental act itself, whereas the *product* of a specific mental act in a particular context constitutes a way of understanding associated with the mental act in that context.

Ways of understanding may therefore reveal the schemes to which external stimuli are assimilated since schemes themselves produce a cognitive product of a mental act. In particular, schemes produce cognitive products of mental acts because they are established inclinations of behavior that are initiated by the individual's recognition of *specific* external stimuli (von Glasersfeld, 1995). For example, the task presented above is the external stimulus that a student may encounter, and problem solving is the mental act in which the student engages.²¹ An individual may assimilate the $\frac{df}{dx}$ in the problem statement as a cue to differentiate f and upon doing so will expect to have solved the problem. The observable product of such an individual's actions may be something like the following:

²¹ Problem solving is not a solitary mental act but rather an amalgam of interpretation, inference, prediction, etc. (Carlson & Bloom, 2005).

$$\frac{d}{dx} f(x) = \frac{-6\pi x \sin(\pi x^2) \ln(x) - \frac{1}{x} [3 \cos(\pi x^2)]}{[\ln(x)]^2}.$$

This outcome is the observable product of the mental act of solving this *particular* problem. The association between one’s action schemes and his or her ways of understanding lie in the particularity of situations in which they are employed. That is, the recognition of a *certain* situation (Part 1 of von Glasersfeld’s an action scheme) and the action it initiates can only produce a *single* observable cognitive product.²² Hence, ways of understanding are limited to *particular* schemes. Since “way of understanding” is a cognitive construct that we model through observation, it is not entirely analogous to one’s action scheme but rather is the observable product of the action that the individual employs in the action scheme. In this way, it allows an observer to construct a viable model of the scheme itself.²³

The emphasis on particularity in the association between one’s ways of understanding and her action schemes suggests the difference between ways of understanding and ways of thinking. Ways of thinking, being cognitive characteristics of mental acts, are evidenced by one’s recurrent employment of an activity that is initiated by a variety of perceived situations. In Thompson, Carlson, and Wilson’s (2012) words, “Harel’s phrase *way of thinking* is, in our terms, a person’s habitual employment in reasoning of a particular set of meanings”—it thus involves bringing one’s schemes to

²² One may have trouble with “observable cognitive product.” The observable product of a mental act is cognitive only in the sense that it allows an observer to construct a model of one’s cognitive activity (i.e., his or her scheme).

²³ Since one’s way of understanding is an observable product of an action scheme that allows an observer to construct a model of the scheme itself, hereafter I use “way of understanding” and “action scheme”—or just “scheme”—interchangeably since the access one has to another’s action scheme is limited to the observable products produced by the scheme.

bear on encountered situations.

Foundational Ways of Thinking: Quantitative and Covariational Reasoning

I leveraged explicit formalizations of two foundational ways of thinking in the design of the present study and my analysis of its data: *quantitative reasoning* (Smith & Thompson, 2007; Thompson, 1990, 2011) and *covariational reasoning* (Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson, 1994b). I present the situation in Table 5 to provide a context in which to discuss quantitative reasoning.

Table 5

Quantitative Situation

Two brothers, Everett and Harrison, run a 400-meter race against each other. Since Everett is faster, he gives Harrison a 50-meter head start. Everett runs at a constant speed of 6.7 meters per second and Harrison runs at a constant speed of 5.4 meters per second. Both brothers run up to their respective starting lines so that they run the entire race at their respective constant speeds.

Quantitative Reasoning

Quantitative reasoning is a characterization of the mental actions involved in conceptualizing situations in terms of *quantities* and *quantitative relationships*. A *quantity* is an attribute, or quality, of an object that admits a measurement process (Thompson, 1990). One has conceptualized a quantity when she has identified a particular quality of an object and has in mind a process by which she may assign a numerical value to this quality in an appropriate unit (Thompson, 1994b). There are many quantities that one may conceive while thinking about the situation in Table 5 including: *Everett's race distance*, *the time elapsed since Everett passed his starting line*, *Everett's running speed* (all attributes of the race Everett runs), *distance of Harrison's head start*, *the time elapsed since Harrison passed his starting line*, and *Harrison's running speed*

(all attributes of the race Harrison runs). It is important to note that quantities do not reside in objects or situations, but are instead constructed in the mind of an individual perceiving and interpreting an object or situation. Quantities are therefore conceptual entities (Thompson, 2011).

Conceptualizing a quantity does not require that one assign a numerical value to a particular attribute of an object. Instead, it is sufficient to simply have a measurement process in mind and to have conceived, either implicitly or explicitly, an appropriate unit. *Quantification* is the process by which one assigns numerical values to some quality of an object (Thompson, 1990). Note that one need not engage in a quantification process in order to have conceived a quantity, but must have in mind a quantification process whereby she may assign numerical values to the quantity (Thompson, 1994b).

The quantities that one may construct upon analyzing a situation are not limited to those whose numerical values are provided, or attainable from direct measurements. For instance, considering the situation presented in Table 5, one may recognize as a quantity *the total amount of time it takes for Everett to complete the race*. As stated above, doing so involves conceptualizing a means of quantification (i.e., imagining a way to assign a numerical value to this attribute of Everett's race). Defining a process by which one may assign numerical values to this quantity involves an operation on two previously defined quantities, *Everett's race distance* and *Everett's running speed*. Since Everett runs at a constant speed for a fixed distance (i.e., the quantities *Everett's race distance* and *Everett's running speed* do not vary), then the total amount of time it takes for Everett to complete the race is given by the ratio of Everett's race distance to his running speed. Furthermore, because conceptualizing a way to quantify *the total amount of time it takes*

for Everett to complete the race involves an operation on these two other quantities, we say that this new quantity results from a *quantitative operation*—its conception involved an operation on two other quantities. Quantitative operations result in a conception of a single quantity while also defining the relationship among the quantity produced and the quantities operated upon to produce it (Thompson, 1990, p. 12)²⁴. It is for this reason that quantitative operations assist in one’s comprehension of a situation (Thompson, 1994b).

It is important to note the distinction between a quantitative operation and a numerical or arithmetic operation. Arithmetic operations are used to calculate a quantity’s value whereas quantitative operations define the relationship between a new quantity and the quantities operated upon to conceive it (Thompson, 1990). In the example above, the process by which one may quantify *the total amount of time it takes for Everett to complete the race* involves an operation on the measures of two other quantities (*Everett’s race distance* and *Everett’s running speed*). Therefore, imagining a way to quantify this new attribute of Everett’s race simultaneously defines a relationship between the new quantity and the two quantities operated upon to measure it. Alternatively, claiming that the total amount of time it takes Everett to complete the race is $400/6.7$ seconds is a numerical operation, not necessarily a quantitative one. While this numerical operation may proceed from one’s construction of *the total amount of time it takes for Everett to complete the race* as a quantitative operation, it is not necessarily the case. Should one not have in mind the quantities whose respective values are 400 and 6.7, then the statement, “The total amount of time it takes Everett to complete the race is $400/6.7$ seconds” does not define a relationship between quantities but is rather a

²⁴ The page number cited here, as well as in all subsequent references to Thompson (1990), refers to the non-publication draft available at <http://www.patthompson.net/Publications.html>.

statement of fact. Therefore, numerical operations do not guarantee that one has constructed a *quantitative relationship*—“the conception of three quantities, two of which determine the third by a quantitative operation” (Thompson, 1990, p. 13). As Thompson (2011) notes, “Quantitative and numerical operations are certainly related developmentally, but in any particular moment, they are not the same” (p. 42). Other quantitative operations that may be deduced from the task in Table 5 include the following:

- *Harrison’s race distance* (determined by the difference of *Everett’s race distance* and *the distance of Harrison’s head start*),
- *Harrison’s total race time* (determined by the ratio of *Harrison’s race distance* and *Harrison’s running speed*),
- *Everett’s distance run after some number of seconds since passing his start line* (determined by the product of *Everett’s running speed* and *the number of seconds elapsed since Everett passed his start line*),
- *Harrison’s distance run after some number of seconds since passing his start line* (determined by the product of *Harrison’s running speed* and *the number of seconds elapsed since Harrison passed his start line*),
- *the distance between Everett and Harrison after any number of seconds since the brothers passed their respective starting lines* (determined by the absolute value of the difference of *Everett’s distance run after some number of seconds since passing his start line* and *Harrison’s distance run after the same number of seconds since passing his start line*), and

- *the win margin as a distance (determined by the difference of Everett's race distance and the distance that Harrison has run after the number of seconds it takes for Harrison to finish the race)²⁵ or as a time (determined by the absolute value of the difference of the total amount of time it takes for Everett to complete the race and the total amount of time it takes for Harrison to complete the race).*

One may clearly deduce several quantitative operations in the simple situation presented in Table 5. Each of these quantitative operations implies a different quantitative relationship. Achieving a complete understanding of the situation involves coordinating these quantitative relationships into a coherent network, or *quantitative structure*. The diagram in *Figure 13* illustrates such a structure that one may construct after analyzing the situation in Table 5. The process of constructing a quantitative structure is called *quantitative reasoning*—the analysis of a situation into a network of quantitative relationships (Thompson, 1990)—and results in achieving a quantitative understanding of the situation. Thompson (1994) explains, “A person comprehends a situation quantitatively by conceiving of it in terms of quantities and quantitative operations. Each quantitative operation creates a relationship: The quantities operated upon with the quantitative operation in relation to the result of operating” (p. 14).

²⁵ This assumes one is aware of the fact that Everett wins the race.

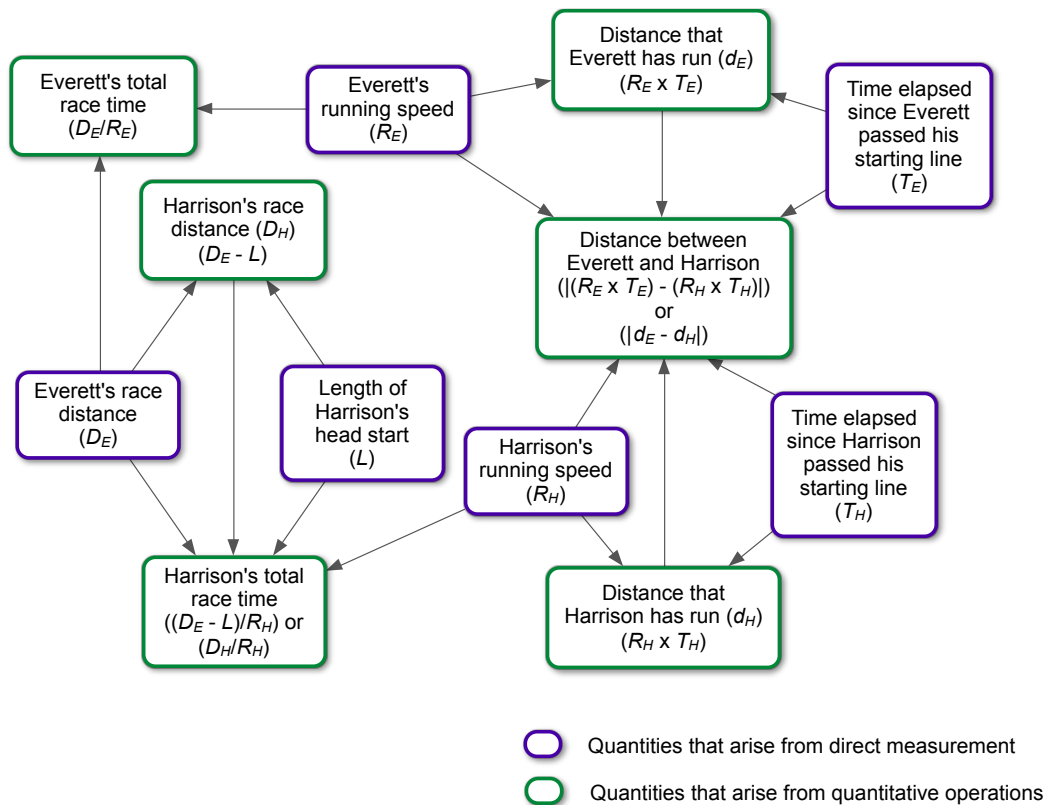


Figure 13. Quantitative structure.

A growing body of research (e.g., Castillo-Garsow, 2010; Confrey & Smith, 1995, Ellis, 2007, Moore, 2012, 2014; Moore & Carlson, 2012; Oehrtman, Carlson, & Thompson, 2008; Thompson 1994b, 2011) has identified quantitative reasoning as a particularly advantageous way of thinking for supporting students' learning of a wide variety of pre- and post-secondary mathematics concepts. Additionally, this body of research has demonstrated the diagnostic and explanatory utility of quantitative reasoning as a theory for how one may conceptualize quantitative situations.

Covariational Reasoning

Covariational reasoning refers to the mental actions involved in coordinating the values of two varying quantities while attending to how these values change in relation to each other (Carlson et al., 2002).

A study by Saldanha and Thompson (1998) gained insight into the mental operations involved in students conceptualizing and reasoning about the continuous covariation of quantities. The researchers conducted a teaching experiment with one 8th grade student to test their hypothesis that students' engagement with tasks requiring the coordination of two sources of information simultaneously is favorable for conceiving of a graph as composed points that record the simultaneous state of two covarying quantities. According to Saldanha and Thompson, covariation entails coupling two quantities so that one may form a *multiplicative object* of the two quantities (p. 1-2). When forming a multiplicative object of two quantities, one develops the immediate and persistent realization that for every possible value that a given quantity can assume, the other quantity also has a value (Saldanha & Thompson, 1998, p. 2).

To Saldanha and Thompson (1998), images of covariation are developmental. An early developmental stage involves one's non-simultaneous coordination of the values of two quantities (i.e., one attends to the value of a quantity, then the value of other, then the value of the first, and so on). In a slightly more sophisticated form of covariational reasoning, one understands time as a continuous quantity, which supports the realization that two quantities' values persist. More sophisticated still is the ability to imagine both quantities being tracked for some duration and recognize the correspondence between the two quantities as an emergent property of the image. Saldanha and Thompson describe

continuous covariation as the understanding that if a quantity assumes different values at different moments in time, the quantity assumed all intermediary values during this interval of time.

Carlson, et al. (2002) propose a framework for characterizing students' mental actions while engaged in tasks involving dynamic function events. Their theoretical framework consists of a hierarchy of five mental actions of covariational reasoning along with five corresponding covariational reasoning levels. The authors define the first mental action (MA 1, *coordination of quantities*) as an individual's recognition that a change in the value of one quantity corresponds to a change in the value of another. The second mental action (MA 2, *coordination of direction of change*) describes not only a recognition that the values of two quantities vary in tandem, but requires one to coordinate the direction of change in the value of one quantity with changes in the value of another. The third mental action (MA 3, *coordination of amounts of change*) involves one in attending to the amount of change in the value of one quantity with respect to the amount of change in the value of another. The fourth mental action (MA 4, *average rate of change*) focuses on one's capacity to attend to the average rate of change of the value of one quantity with respect to the value of another. Finally, the fifth mental action (MA 5, *instantaneous rate of change*) describes one's ability to attend to the instantaneous rate of change of the value of one quantity with respect to the value of another. Carlson et al. explain that the purpose of their proposed framework is to aid in the evaluation of covariational thinking to a greater extent than had been done previously.

Thompson (2011) provides an additional account of the mental operations involved in conceptualizing and coordinating the simultaneous variation of the values of

two quantities. He explains that to imagine variation in a quantity's value is to expect the value of the quantity to differ at two different moments in (conceptual) time and to realize that the quantity's measure assumed all values between the measure of the quantity at the beginning of the interval of time and the measure of the quantity at the end of the interval of time. Thinking about continuous variation therefore amounts to first anticipating an interval of time over which variation in a quantity's value occurs, which allows one to expect that the quantity's value will vary by a specific amount over this interval of time. One then imagines the quantity's value varying in microscopic bits, each of which occurs over a very small interval of time. To imagine a quantity's value varying continuously, one must realize that variation occurred within these very small intervals of time (i.e., one has to imagine that for every value between the initial and final values that the quantity assumed over this small interval of variation, there was a time within the small interval of time over which the variation occurred that the quantity assumed this value).

Implications of Quantitative and Covariational Reasoning for the Present Study

The theories of quantitative and covariational reasoning maintained analytical utility in the present study in that they constituted dimensions along which I situated my model of the participant's ways of understanding specific mathematical ideas. These respective theories also contain useful explanatory constructs that I used to characterize my participant's mathematical ways of understanding and ways of thinking. In Chapter 4 I explain the specific ways in which I made use of these respective theories to analyze my data.

Operationalization of Knowledge Structures

Following Piaget and von Glasersfeld, I have previously outlined my perspective on what it means “to know.” However, the focus of the present study was principally that of characterizing the factors that conditioned the *evocation* of a teacher’s subject matter knowledge (i.e., mathematical schemes of meaning). This chapter would be incomplete, therefore, without proposing a way to think about how knowledge is instantiated in the flow of everyday experience.

To Piaget and von Glasersfeld, cognition serves an adaptive function tending toward viability. Damasio (1994) explains, “the overall function of the brain is to be well informed about ... the environment surrounding the organism, so that suitable, survivable accommodations can be achieved between organism and environment” (p. 90). This position suggests the following two important points: (1) the knowledge an organism enacts is in the service of preserving the homeostasis (both biological²⁶ and psychological²⁷) of the organism, and (2) the knowledge an organism brings to bear in the flow of experience is fashioned by the organism’s interaction with the environment. I will discuss these two points in turn.

Knowledge Serves to Maintain Homeostasis

Being the product of biological evolution, the principal function of the brain, like all aspects of our physiology, is to ensure survival and preserve homeostasis. The central nervous system did not evolve to become an instrument of reason and rationality, as there was minimal survival advantage to reasoned and rational thought throughout the

²⁶ Biological homeostasis is the state of an organism in which physiological stability is maintained.

²⁷ Psychological homeostasis is the state of an organism in which psychological equilibrium is obtained by eliminating or reducing cognitive perturbations.

evolutionary history of our species. Ornstein and Dewan (1991) argue, “The primary billet of the mental system is not self-understanding, self-analysis, or reason, but adaptation to the world, to get nourishment and safety, to reproduce and so pass on descendants” (p. 5).

A cognitive apparatus that guides the organism to achieve a state of well-being and positivity was naturally selected as a result of the survival and reproductive advantages such a cognitive apparatus affords. The decisions an organism makes in the ebb and flow of daily life, and thus the knowledge that informs these decisions, are fashioned by the organism’s ever-present desire to achieve a state of psychological and physiological positivity and avoid a state of negativity. This is arguably the evolutionary role of drives, instincts, emotions, and the somatic sensations that accompany them.

Damasio (1994) explains,

In general, drives and instincts operate either by generating a particular behavior directly or by inducing physiological states that lead individuals to behave in a particular way, mindlessly or not. Virtually all the behaviors ensuing from drives and instincts contribute to survival either directly, by performing a life-saving action, or indirectly, by propitiating conditions advantageous to survival or reducing the influence of potentially harmful conditions. Emotions and feelings ... are a powerful manifestation of drives and instincts, part and parcel of their workings (p. 115).

[T]he brain structures involved in basic biological regulation are also part of the regulation of behavior and are indispensable to the acquisition and normal function of cognitive processes. The hypothalamus, the brain stem, and the limbic system intervene in body regulation *and* in all neural processes on which mind phenomena are based, for example, perception, learning, recall, emotion and feeling, and ... reasoning and creativity. Body regulation, survival, and mind are intimately interwoven (p. 123).

Accordingly, an organism’s cognitive faculties are brought to bear in the service of achieving psychological and biological homeostasis and of preserving a state of fitness,

positivity, and well-being. As Damasio describes, this motivation underlies all cognitive activity.

Enacted Knowledge is Fashioned by Interaction With the Environment

In contrast to the accepted view that knowledge is a static entity that resides in one's mind, waiting to be dispensed whenever the individual who holds it wishes to do so, I consider the evocation of knowledge to be contingent upon the knower encountering stimuli that serve as cues to activate specific cognitive schemes. In particular, specific knowledge is not made manifest until an individual interprets a certain stimulus in such a way that his or her construction of the stimulus serves as a cue for the enactment of a particular cognitive scheme or network of related schemes.²⁸ Therefore, while many believe knowledge is invariantly accessible across time and space, I consider knowledge to be the set of one's schemes of meanings that may become operational in the space of stimuli in which the individual is situated. Note that such stimuli are not limited to external stimuli acquired through sensory input but also include internal stimuli such as the activation of related schemes, emotions, somatic states, self-identity, expectations, goals, and perceived responsibilities, all of which are mediated by the individual's construction of his or her environmental context.

Accordingly, an individual's way of perceiving his or her environmental context

²⁸ I do not intend this statement to take agency away from the individual. That is, I do not mean to convey that external stimuli cause the evocation of particular knowledge structures in an individual. On the contrary, the knowledge one enacts is fashioned by his or her assimilation of the environmental stimulus. The only stimulus that has the potential to affect the cognition of the individual is the stimulus the individual constructs through assimilation, not a stimulus that resides "out there" in an external reality independent of the individual's perception and assimilation. Therefore, the assimilatory schemes of the individual play a critical role in the knowledge that the individual evokes. It is in this way that my view regarding the evocation of knowledge structures is consistent with the enactivist theory of knowing (Proulx, 2013).

constitutes a space of stimuli that maintain the potential to make a subset of his or her schemes of meaning (knowledge) operational. An individual's knowledge, then, can be thought of as the set of cognitive schemes that *may* become enacted as a consequence of the individual's interpretation of the stimuli inherent to a given environmental context. Notice that this definition describes knowledge as the set of cognitive schemes that *may* become enacted as opposed to defining knowledge irrespective of the way it is capable of informing one's actions and behaviors. It is for this reason that we may consider enacted knowledge as situated in the individual's perception of his or her environmental context, which constitutes the space of stimuli that may activate particular cognitive schemes. It is therefore appropriate to say that enacted knowledge is conditioned by the individual's interpretation of his or her environmental context. This is not to suggest that knowledge resides external to the knower since individuals interpret their environmental context—an interpretation affects the knowledge one employs. To speak of employing knowledge, then, is to speak of the particular cognitive schemes that become activated upon an individual's appraisal of stimuli (internal and external) in a particular environmental context.

Since the knowledge one is able to employ is fashioned by his or her perception and interpretation of the stimuli within a specific environmental context, it is not productive to speak of knowledge as an invariant set of schemes irrespective of the factors that condition the potential for these schemes to inform an individual's actions and behaviors. Accordingly, my use of "knowledge" always refers to the *a priori* set of schemes that *may* become operational within a *particular* environmental context and not the set of schemes that may be activated in *any* context. In this way, I consider

knowledge to be a situated, and thus transient, entity.

My view that knowledge is situated relative to the individual's perception of her environment differs from the epistemology of situated cognition in one important way. Many situated cognitive theorists (e.g., Brown, Collins, & Dugid, 1989; Greeno, 1991; Lave & Wenger, 1991; Wenger, 1998) situate knowledge in a domain external to the knower, such as social groups, communal practices, reasoning artifacts, and the like. To say that knowledge is situated relative to the individual's *perception* of his or her environmental context is not to say that knowledge is situated in a social, physical, or environmental domain external to the knower, as such a claim would presume individuals have access to an objective reality independent of their construction. As I have discussed, from a radical constructivist perspective an individual's perception of reality constitutes the only reality he or she experiences. Reality is therefore an individual construction fashioned by one's current cognitive schemes. It is only appropriate, then, to speak of knowledge as being situated relative to an individual's *perception* and *interpretation* of his or her environmental context and not some environmental context independent of such perception and interpretation, thus situating knowledge where it belongs²⁹, in the mind of the individual.

The present study focused on understanding the ways in which a teacher's image of instructional constraints conditioned the mathematical ways of understanding and ways of thinking he brought to bear in the act of teaching. Given my view that knowledge is conditioned by the individual's interpretation of his environmental context, this focus

²⁹ I say, "where it belongs" because I take the view that a theory should, to the greatest extent possible, respect the ontology of the phenomena the theory seeks to explain. Since knowledge is a manifestation of physiological events of the central nervous system, it is not simply a theoretical supposition but a biological fact to say that knowledge and cognition occur "between the ears."

required that I identify the stimuli the participating teacher assimilated and to which he reacted by constructing a model of how the teacher perceived and interpreted his environmental context. Moreover, my position that knowledge is enacted through interaction with the individual's environment to preserve the psychological homeostasis of the individual necessitated my discerning those aspects of the teacher's reality that evoked his knowledge in addition to understanding how the teacher's enacted knowledge served to establish or preserve his psychological and emotional well-being.

Conclusion

In this chapter, I have outlined the theoretical foundation on which the conceptualization, design, and analysis of the present study was based. In particular, I have detailed the central propositions of von Glasersfeld's (1995) radical constructivism and Piaget's (1971, 1977) genetic epistemology, as well as discussed my view concerning the conditions under which knowledge is instantiated in the flow of everyday experience. While I have emphasized how the theoretical propositions discussed are relevant to the present study, two points to this end are worth reiterating. First, the notion that individuals construct their own subjective realities is of paramount importance to the present study. It is not very productive to bemoan the way in which initiatives such as high-stakes standardized tests, merit pay, Teach for America, the charter movement, vouchers, parent-trigger laws, No Child Left Behind, and Race to the Top, degrade instructional quality while sidestepping the mediating role that teachers' appraisal of these circumstances have on the quality of their instruction. The aforementioned circumstances are constraints only to the extent that individual teachers appraise them as constraints and thus do not maintain some objective existence as constraints independent

of one's construction of them as such.

Second, my attention to factors that condition the instantiation of knowledge had implications for how I analyzed the participating teacher's actions in the context of classroom practice. As I have argued throughout these first few chapters, it is of little consequence to simply characterize the knowledge that teachers *should* possess without understanding the factors that condition the knowledge that teachers *do* possess, which constitutes the only knowledge that may affect students' learning. Claiming that the instantiation of knowledge is fashioned by (1) the individual's desire to achieve a state of homeostasis and well-being, and (2) the individual's construction of the environmental context, directed my analysis at discerning the ways in which the knowledge the participating teacher utilized while making instructional decisions was in the service of maintaining his psychological homeostasis, and necessitated my constructing a model of how the teacher perceived his environmental context.

CHAPTER 4

METHODOLOGY

What a frightening thing is the human, a mass of gauges and dials and registers, and we can only read a few and those perhaps not accurately.

John Steinbeck, *The Winter of Our Discontent*

Recall that the focus of the present study was to determine if there were incongruities between a teacher's mathematical knowledge and the mathematical knowledge he leveraged in the context of teaching, and if so, to ascertain how the teacher's image of instructional constraints conditioned his enacted subject matter knowledge. The purpose of this chapter is to describe the experimental and analytical methodology I employed to accomplish this focus. I begin by discussing the context of the study and proceed to outline the three phases of my experimental methodology: (1) series of task-based clinical interviews; (2) classroom observations, pre-lesson interviews, and a teacher journal; and (3) series of semi-structured clinical interviews. I then describe the three phases of my analytical methodology: (1) preliminary analysis, (2) ongoing analysis, and (3) post analysis.

Experimental Methodology

Determining whether there were incongruities between the participating teacher's subject matter knowledge and the subject matter knowledge he utilized while teaching (the first research question) required that I construct a model of the teacher's mathematical knowledge in a context that is independent of the stimuli that may condition the evocation of the teacher's mathematical knowledge in the context of

teaching.³⁰ Moreover, understanding the way in which the teacher's image of instructional constraints conditioned the subject matter knowledge he employed while teaching (the second research question) required that I: (1) construct a model of how the teacher perceived his environmental context while discerning what constituted constraints for the teacher and why, (2) construct a model of the subject matter knowledge the teacher utilized in the context of teaching, and (3) apprehend the way and extent to which the teacher's image of instructional constraints influenced the subject matter knowledge he employed while teaching. The first phase of the experimental methodology I outline in this chapter allowed me to obtain data that facilitated my construction of a model of the teacher's mathematical knowledge independent of its instantiation in the context of classroom practice. In the second phase, I collected data that enabled me to construct a model of the mathematical knowledge the teacher employed while teaching. Finally, the data I generated from the second and third phases allowed me to construct a model of the teacher's perception of the constraints he recognized within the context of teaching, and to ascertain the effect of these constraints on the quality of the teacher's enacted mathematical knowledge.

In the first phase of data collection, I conducted a series of task-based clinical interviews (Clement, 2000; Goldin, 1997; Hunting, 1997) that allowed me to construct a model of the participating teacher's mathematics. In the second data collection phase, I used data from pre-lesson interviews and classroom observations to construct a model of

³⁰ Determining the role of the teacher's image of instructional constraints on his enacted subject matter knowledge required that I construct a model of the teacher's mathematics in a context that, to the greatest extent possible, was free of the stimuli that the teacher may appraise as constraints on his practice. I collected the data that I used to construct a model of the teacher's mathematics in a setting in which the teacher's mathematical knowledge was not likely to be conditioned by his image of instructional constraints. I do not mean to convey that I was able to completely divorce the teacher's mathematical thinking from issues of classroom practice, but this was indeed what I aspired to achieve.

the mathematical knowledge the teacher utilized in the context of classroom practice. Finally, I used teacher journals and semi-structured clinical interviews as my primary data sources for constructing a model of the teacher’s perception of instructional constraints. Table 6 outlines the three models that I constructed and the means of data collection that facilitated my construction of these models. Prior to outlining the details of my three phases of data collection, I briefly describe the context of the study, including the course and content in which I collected data, and the criteria by which I selected the research participant.

Table 6

Data Corresponding to the Construction of Three Models

<i>Model constructed</i>	<i>Phase I</i>	<i>Phase II</i>	<i>Phase III</i>
Teacher’s mathematical ways of understanding and ways of thinking	Task-based clinical interviews		
Teacher’s enacted subject matter knowledge		Pre-lesson interviews and classroom observations	
Teacher’s image of instructional constraints		Pre-lesson interviews and teacher journal	Semi-structured clinical interviews

Context of the Study

Course and content. The sole participant for this study was a secondary mathematics teacher, David³¹, teaching *Pathways Algebra II* (Carlson, O’Bryan, & Joyner, 2013). The series of task-based clinical interviews I conducted in Phase I of the study was based on a subset of the content in Module 8 (Trigonometric Functions) of the

³¹ A pseudonym.

Pathways Algebra II curriculum. Accordingly, the videos of David's classroom teaching I obtained in Phase II of the study were from those class sessions in which David taught a portion of this module. I conducted the semi-structured clinical interviews in Phase III after I completed the series of task-based clinical interviews in Phase I and after David completed his instruction of Module 8.

Pathways Algebra II is a student curriculum and teacher support system that presents content in ways that are informed by the latest research on how students learn foundational concepts in algebra, precalculus, and calculus, and how instruction can support the development of mathematical thinking and problem solving skills. The curriculum was designed to support students in developing confidence in their ability to make sense of ideas and develop meaningful formulas to represent patterns and relationships.

Since the arguments of trigonometric functions can be thought of as angle measures, Module 8 of *Pathways Algebra II* begins by having students investigate approaches for measuring the "openness" of two rays with a common endpoint. The curriculum then introduces students to the idea of periodic function by prompting them to model circular motion. After introducing a context involving circular motion, the curriculum supports students in identifying two quantities to model such circular motion, with discussions typically leading to students' deciding to covary the values of an angle measure with the values of particular linear measurements as an object rotates. The curriculum then introduces students to the ideas of sine and cosine functions as a way to represent specific relationships between the covarying quantities of angle measure and specific linear measurements. Students then explore ideas of domain, range,

transformations, composition, and inverse function in the context of trigonometric functions. The module concludes with an investigation of the Pythagorean identity. While the content of Module 8 spans many topics, I focused only on those pertaining to sine and cosine functions. In the following chapter I provide a detailed conceptual analysis of my interpretation of the mathematical understandings that the designers of Module 8 of the *Pathways Algebra II* curriculum intend to promote.

Recruitment criteria. I recruited my research subject, David, according to three criteria: (1) he demonstrated the potential for having strong mathematical understandings, (2) he did not have extensive experience with the *Pathways Algebra II* curriculum, and (3) he exhibited the capacity to clearly communicate his reasoning while engaged in mathematics tasks. The following is a discussion of my rationale for these recruitment criteria.

Recall that in my opening chapter I identified the shortcomings of educational policy initiatives that derive from two polarized perspectives for the condition of education in the United States. Adherents to Perspective I assume that deficits in teacher aptitude lie at the heart of modest student achievement while proponents of Perspective II contend that teachers operate under progressively crippling circumstances that stifle their aptitude and make it nearly impossible for them to effectively do their work. I claimed that if what each perspective takes as problematic (teacher aptitude and environmental constraints respectively) were resolved, it is unlikely that teaching quality in the United States would drastically improve since the policies that originate from these two perspectives do not attend to the *interaction* between teachers' image of the constraints under which they work and the knowledge teachers bring to bear in the context of

practice. The present study attends to this interaction by investigating how David's image of the constraints under which he believed to be operating conditioned the subject matter knowledge he utilized while teaching.

Understanding how teachers' image of instructional constraints affects the instantiation their content knowledge in the context of teaching is a necessary prerequisite to addressing questions like, "How can teacher preparation and professional development programs be designed to support teachers in appraising what they commonly consider constraints on the quality of their teaching so as to minimize the effect of these perceived constraints on the quality of students' learning experiences?" and "How can teachers' work environments be engineered so as to minimize the potential that they will believe to be operating under constraints that have proven to condition the evocation of their subject matter knowledge in undesirable ways?" Answers to questions of this variety maintain the potential to have an effect on that which mathematics educators really care about: *the mathematics that students have the opportunity to learn.*

The present study cannot contribute to clarifying the answers to such questions should the participating teacher maintain impoverished mathematical ways of understanding and unproductive mathematical ways of thinking. It is, after all, of little consequence to examine how a teacher's weak mathematical knowledge is conditioned by his image of instructional constraints since students are not likely to substantially benefit from the teacher's uncompromised application of such knowledge in the absence of these constraints. Alternatively, it is far more valuable to the enterprise of mathematics education to understand how a teacher's image of instructional constraints conditions the subject matter knowledge he brings to bear in the context of practice when the knowledge

being conditioned would otherwise provide students with opportunities to construct powerful ways of understanding mathematical ideas. In such cases, the teacher's image of instructional constraints conditions mathematical understandings and ways of thinking that would otherwise likely have a profound effect on the mathematics students may learn. In sum, a teacher's appraisal of the constraints under which he works imposes an obstacle on the quality of his instruction if and only if the teacher's subject matter knowledge is sophisticated enough to be conditioned in undesirable ways by his image of such constraints. Accordingly, I attempted to recruit a participating teacher who demonstrated strong underlying meanings for the mathematics he teaches. I gauged the extent of David's content knowledge through closely interacting with him in the context of curriculum workshops that I co-conducted, with Alan O'Bryan, in the summer of 2013.

I also recruited David because he had only taught *Pathways Algebra II* once before, and therefore did not have extensive experience with the course. I insisted on this recruitment criterion because I did not want the participating teacher to have lost intimacy with his rationale for instructional choices. David's ability to articulate his justification for instructional decisions was instrumental to my constructing a viable model of the constraints under which David believed to be operating, and to understanding how these constraints influenced the knowledge that he brought to bear in the context of teaching. Since David had only one year of experience with the *Pathways Algebra II* curriculum, he was able to communicate with greater clarity the rationale for his instructional decisions and pedagogical actions.

Finally, I recruited David because he demonstrated the capacity to clearly communicate his reasoning while engaged in mathematics tasks. This criterion was

important because the model I constructed of David's mathematical ways of understanding and ways of thinking was entirely dependent on the observable products of his reasoning David provided. Because David was adept at communicating his thinking, and because he was not easily threatened by being questioned, the model I constructed of his mathematics was viable with his language and actions and maintained some predictive capacity over his subsequent behavior. I relied upon my interaction with David in the curriculum workshop that I co-conducted in the summer of 2013 to gauge the extent to which he was comfortable and practiced at clearly communicating his reasoning.

Phase I: Task-Based Clinical Interviews

In the first phase of data collection, I employed a series of task-based clinical interviews (TBCIs) to construct a model of David's mathematical knowledge relative to angle measure and sine and cosine functions. In this subsection, I discuss what went into the design and administration of these TBCIs.

The goal of the series of TBCIs was to facilitate my construction of a model of David's ways of understanding and ways of thinking relative to angle measure, the outputs and graphical representation of sine and cosine, and the period of sine and cosine. Constructing a model of an individual's cognition by projecting or imputing one's cognitive schemes to the individual constitutes developing a *first-order model* (Steffe & Thompson, 2000). This is in contrast to developing a *second-order model*, in which the researcher attempts to make sense of the individual's actions by interpreting them through the lens of his or her model of the individual, not through his or her own cognitive schemes (ibid.). It is important to note that the goal of the series of TBCIs was to construct a second-order model of David's mathematical knowledge. Although I

attempted to construct a second-order model of David's mathematics, the model I constructed (which I present in Chapter 6) does not constitute a direct representation of David's knowledge, but rather a viable characterization of plausible mental activity from which his language and observable actions may have derived. Constructing such a model involved my generating prior to, within, and among TBCIs, tentative hypotheses of David's ways of understanding that explained my interpretation of the observable products of his reasoning. I developed these provisional hypotheses by attending to David's language and actions and abductively postulating the meanings that may lie behind them. I designed and modified tasks for subsequent TBCIs to test, extend, articulate, and refine my tentative hypotheses of David's mathematical knowledge.

Prior to administering the first TBCI, I conducted an initial clinical interview for the purpose of constructing preliminary hypotheses relative to David's ways of understanding angle measure, the outputs and graphical representation of sine and cosine, and the period of sine and cosine. In this interview, I asked David to respond to a number of predetermined questions (see Table 7) and probed his responses to these questions until I felt confident that I had generated enough data to develop a detailed preliminary model of David's mathematical knowledge. It is noteworthy that the preliminary model of David's ways of understanding angle measure and sine and cosine functions was neither extremely robust nor substantiated by a litany of evidence. My purpose in the initial clinical interview was simply to generate initial hypotheses of how David conceptualized ideas related to angle measure and sine and cosine functions so that I could modify the protocol for subsequent TBCIs to test these initial hypotheses. The initial clinical interview was video recorded for analytic purposes. Additionally, I

collected and scanned all written work David produced during the initial clinical interview.

Table 7

Initial Clinical Interview Questions

1. What is an angle?
2. When one measures an angle, what is he/she measuring?
3. What does it mean to say that an angle has a measure of one degree? 47 degrees?
4. Describe how you might create a protractor to measure the openness of an angle in degrees?
5. What does it mean to say that an angle has a measure of one radian? 2.1 radians?
6. How does one measure an angle?
7. What is the sine function?
8. What quantities does the sine function relate?
9. How does the output of the sine function vary as the input of the sine function increases from zero?
10. What is the cosine function?
11. What quantities does the cosine function relate?
12. How does the output of the cosine function vary as the input of the cosine function increases from zero?
13. What does the graph of the sine function look like and why?
14. What does the graph of the cosine function look like and why?
15. What does *period* mean?
16. What is the period of the sine function?
17. What is the period of the cosine function?

I conducted eight TBCIs throughout David's instruction of those concepts of Module 8 relevant to angle measure and sine and cosine functions. Initially, I intended to conduct the series of TBCIs at least two weeks prior to David beginning Module 8. However, as a result of his department's decision to incorporate a brief unit on probability at the end of the semester to prepare students for a district standardized assessment, David started teaching Module 8 two weeks earlier than he anticipated. Although I conducted the TBCIs concurrently with David's teaching of Module 8, I administered each TBCI prior to David's teaching the concepts that were the focus of

each respective interview. Table 8 outlines the schedule, duration, and content of the series of TBCIs.

All interviews took place in David’s classroom after school on the days that best suited his schedule. I attempted to schedule the interviews so that there was at least one day between TBCIs to accommodate for ongoing analysis, and accomplished this with the exception of the last two TBCIs. In each interview, I obtained video recordings that captured David’s writing, expressions, and gestures. I also created videos of my computer screen via *QuickTime Player* to capture the didactic objects David and I discussed as well as any work David completed on the computer. Additionally, I collected and scanned all written work that David produced during the interviews.

Table 8

Schedule, Duration, and Content of TBCIs

Interview	Date	Duration	Content
Initial Clinical Interview	03/26/2014	47:53	Angle measure; Outputs, graphical representation, and period of sine and cosine
TBCI 1	03/28/2014	1:19:49	Angle measure
TBCI 2	03/31/2014	1:01:25	Angle measure
TBCI 3	04/02/2014	51:08	Angle measure
TBCI 4	04/07/2014	1:00:47	Outputs of sine and cosine
TBCI 5	04/09/2014	58:19	Outputs of sine and cosine
TBCI 6	04/14/2014	58:41	Graphical representation of sine and cosine
TBCI 7	04/17/2014	20:44	Graphical representation of sine and cosine
TBCI 8	04/18/2014	1:01:51	Period of sine and cosine

Between TBCIs, I conducted a surface-level analysis of the videos of the previous interview for the purpose of elaborating and refining my tentative hypotheses regarding David’s ways of understanding and ways of thinking. Moreover, conducting analysis between TBCIs gave me the opportunity to modify the tasks I planned on using in subsequent TBCIs, or design additional tasks, so that I could test my provisional

hypotheses concerning David's mathematical knowledge. Additionally, I created memos between each TBCI that attended to the following:

- (1) Explicating hypothetical ways of understanding that explained my interpretation of David's language and actions during the interview;
- (2) Identifying David's specific utterances and actions that contributed to my construction of these hypothetical ways of understanding;
- (3) Explaining how I interpreted these utterances and actions so as to make their contribution to my model of David's ways of understanding explicit;
- (4) Justifying my instructional actions during the interview and describing my interpretation of David's response to these instructional actions;
- (5) Describing my rationale for the tasks I planed on using in the subsequent interview(s).

My assimilatory schemes with which I made sense of David's language and actions likely changed throughout the series of TBCIs. Accordingly, I did not consider the decisions, interpretations, hypotheses, and conclusions that I made throughout the series of TBCIs absolute, but instead treated them as provisional and subject to reexamination. I intend the memos I generated between TBCIs to facilitate such retrospective analysis. The purpose of these memos was therefore to create artifacts of my thinking to which I could refer during post-analysis to critically examine the rationale for my decisions and ongoing interpretations of David's observable actions that led to my emerging model of his mathematical knowledge. I outline specific details regarding the analytical procedures I employed between TBCIs in the following section.

There were, of course, limitations to critically examining my actions and interpretations during the TBCIs. For this reason, another researcher, Kristin Frank, viewed all videos and met with me between each interview. Kristin's role was fourfold: (1) to offer alternative interpretations of David's language and actions, (2) to analyze the communication between David and myself, (3) to assess the effectiveness of my tasks, and (4) to participate in the planning of subsequent TBCIs. I met with Kristin after each interview to discuss her interpretation of David's actions and her evaluation of the communication between David and myself. I also presented my current hypotheses regarding David's ways of understanding and ways of thinking and discussed the tasks I planned on using in the subsequent interview. Kristin then had the opportunity to propose modifications to either my tentative model of David's mathematics or the tasks for the upcoming interview.

The tasks I used in the TBCIs mirror the trajectory of concepts as they are developed in Module 8 of the *Pathways Algebra II* curriculum.³² I designed these tasks to engage David in experiences that, to the greatest extent possible, elicited observable products of his reasoning. I then constructed provisional hypotheses regarding David's ways of understanding and ways of thinking and tested these provisional hypotheses with subsequent questions and tasks. A viable model of David's mathematical knowledge emerged after having performed several iterations of generating, testing, and revising my provisional hypotheses of David's thinking.

The tasks I employed in the series of TBCIs maintained the additional objective of providing opportunities for David to advance his understandings and construct more

³² I outline the tasks I used in the series of TBCIs, along with the conceptual analysis that informed their construction, in the next chapter.

coherent schemes of meaning. To accomplish this, I designed tasks to engender particular forms of abstraction (empirical, pseudo-empirical, reflecting, and reflected). I attempted to advance David's understanding through the design of my tasks because I wanted to support him in constructing coherent schemes of meaning that he *could* leverage in his instruction. As I stated above, it is of little consequence to apprehend how weak mathematical understandings and impoverished ways of thinking are conditioned by a teacher's image of his or her environmental context since students are unlikely to benefit from the teacher's uncompromised conveyance of such deficient mathematical knowledge. It is for this reason that it was important that I had the opportunity to provide David with experiences that facilitated his construction of desirable understandings and ways of thinking so that he actually possessed the type mathematical knowledge that merits investigation of how it is conditioned by his image of instructional constraints. It is important to note that the only means by which I attempted to facilitate David's learning was through the design of the TBCI tasks. I did not ask David leading questions or offer any guidance during the interviews. Moreover, and crucially, I did not insist that David learn the mathematics that I know. Doing so would have necessarily focused my attention on my own mathematics instead of attending to David's ways and means of operating. I present the task sequence, and explain how I designed the tasks to provide occasions for David to advance/reorganize his understandings, in Chapter 5.

Phase II: Pre-Lesson Interviews, Classroom Observations, and Teacher Journal

The primary purpose of the second phase of my experimental methodology was to collect data that allowed me to construct a model of the mathematical knowledge David employed in the context of practice so that I could determine if there were incongruities

between David's mathematics and the mathematical understandings and ways of thinking he brought to bear in the process of teaching. Accordingly, the pre-lesson interviews and classroom observations sought to provide insight into those aspects of David's professional life that demand the evocation of mathematical knowledge. The purpose of the teacher journal was to generate data that I used, in conjunction with the data I obtained during Phase III, to construct a model of how David's image of instructional constraints influenced his enacted subject matter knowledge.

Classroom observations. David taught two sections of Honors Algebra II every weekday (4th hour and 6th hour). I video recorded both classroom sessions over a seven-and-a-half-week period, which resulted in 37 videos of classroom teaching (see Table 9 for a schedule of the classroom sessions that I video recorded). The only days I did not intend to collect videos of David's classroom teaching were those days students were testing or the days David was teaching content unrelated to the angle measure, sine, or cosine. In addition to the video recordings of David's classroom teaching, I generated observational memos and collected artifacts of classroom practice (e.g., quizzes, tests, homework assignments). While the classroom observations did not demand the type of ongoing analysis that was part and parcel of the series of TBCIs, I identified, in the form of memos, moments in which David made crucial instructional decisions so that I could analyze these instances in further detail during post analysis. Such moments included, but were not limited to, instances in which David responded to students' questions or comments, occasions where David acted spontaneously based on his image of students' understanding, and cases in which David was required to resolve some sort of mathematical conflict. The post analysis of these key instructional moments, in addition

to the data I obtained in Phase III, contributed to my construction of a model of David's image of his environmental context and the instructional constraints he recognized therein.

Table 9

*Schedule of Classroom Video Recordings*³³

	4th Hour Honors Algebra II	6th Hour Honors Algebra II	
March	31-2014	31-2014	
	01-2014	01-2014	
	02-2014	02-2014	
	03-2014	03-2014	
	09-2014	09-2014	
	10-2014	10-2014	
	11-2014	11-2014	
	14-2014	14-2014	
	April	15-2014	15-2014
		16-2014	16-2014
17-2014		17-2014	
21-2014		21-2014	
--		25-2014	
28-2014		28-2014	
29-2014		29-2014	
30-2014		30-2014	
May		13-2014	13-2014
		14-2014	14-2014
	15-2014	--	
	16-2014	--	

The memos I generated during classroom observations also, and more importantly, documented the mathematical understandings and ways of thinking David afforded his students the opportunity to construct. I must emphasize that I characterized the ways of understanding and ways of thinking David *allowed* his students to construct, and not the

³³ I did not obtain a video recording of David's 4th hour class on April 25th because of a traffic accident. I did not obtain video recordings of David's 6th hour class on May 15th or 16th because of prior engagements.

understandings and ways of thinking his students *actually* constructed.³⁴ Accordingly, my characterization was likely an overestimate of the mathematics students learned since the lens through which I interpreted David's instruction was not fashioned by the unproductive understandings and deficits in prior knowledge common to many secondary students. In essence, I documented the understandings that I would be able to construct, and the ways of thinking that I would be able to develop, were I an engaged student in the class with sufficient background knowledge, uninhibited by unproductive understandings or disadvantageous ways of thinking.

I outline the specific means by which I analyzed the videos of classroom teaching and the material artifacts of classroom practice in the Analytical Methodology section later in this chapter.

Pre-lesson interviews. A teacher's mathematical knowledge is not exclusively brought to bear in the classroom but equally so in the lesson planning process. It is during lesson planning that a teacher articulates the mathematical understandings he or she wants students to develop, and devises and/or selects instructional materials to facilitate students' construction of these understandings. I expected that a teacher's image of the instructional constraints under which he or she operates would figure largely in this mathematically intensive aspect of the teacher's work. Accordingly, I intended to conduct

³⁴ My rationale for doing so was twofold. First, as a consequence of the lack of interaction between David and myself during classroom observations, any effort to determine the meanings David utilized in the context of classroom practice entail significant limitations. In my view, the most effective way to determine the mathematical understandings and ways of thinking that David utilized while teaching was to approximate the meanings David deemed appropriate to convey in the context of classroom practice by characterizing the mathematics that David's students had the opportunity to learn. Second, obtaining insight into the mathematical meanings David employed in the context of teaching by examining what his students learned is, in my view, unproductive. The meanings that David's students actually constructed provide, at best, a convoluted image of the meanings David utilized or aspired to convey since a variety of intervening variables mediate the connection between the understandings and ways of thinking a teacher wants his or her students to develop and the understandings and ways of thinking students actually develop.

pre-lesson interviews with David prior to each day's lessons to obtain data that would allow me to construct a model of his enacted subject matter knowledge. When I disclosed to David my intention of conducting daily pre-lesson interviews, he explained that he does not plan lessons before every class session but rather for each investigation, which often span more than one class session.³⁵ Therefore, David and I agreed that I would conduct pre-lesson interviews prior to his teaching of each investigation. I audio recorded each pre-lesson interview for analytic purposes.

The primary purpose of the pre-lesson interviews was to make explicit David's instructional intentions and to reveal potential constraints that compromised the quality of the mathematical meanings David intended to support in his instruction (see Table 10 for the list of questions I asked David in our first pre-lesson interview). By apprehending the specific meanings David sought to convey, I was able to recognize moments during each lesson in which the mathematics David allowed students to learn differed from the meanings he intended to support. Identifying these "moments of instructional deviation" were essential to the content of the clinical interviews I conducted in Phase III of data collection, as I explain below. The audio recording of the pre-lesson interviews, together with the Phase III semi-structured clinical interviews, contributed to revealing the role of David's image of instructional constraints on the mathematical knowledge he employed to design lessons.

Table 10

³⁵ The *Pathways Algebra II* curriculum is organized into Modules, which are comprised of Investigations. Often, an investigation would span two or more days of instruction. Rather than planning each classroom session, David planned each investigation. Therefore, per David's suggestion, I conducted pre-lesson interviews prior to David's teaching of each investigation rather than each individual lesson.

Pre-Lesson Interview Questions

1. Describe the process you go through to plan investigations.
2. Have you planned for your next investigation?
3. What is the mathematical topic of the next investigation?
4. Walk me through your plan for the upcoming investigation.
5. How do you want students to understand the central mathematical concept of the upcoming investigation?
6. Why, in your view, is this the best way to understand the central mathematical concept of the upcoming investigation?
7. How will your teaching of the upcoming investigation allow students to develop the understanding that you would like them to have?
8. How confident are you that students will develop the understanding that you would like them to have?
9. Complete the following sentence, “My teaching of the next investigation would be a success if ...”
10. While planning the next investigation, did you feel that you needed to make any compromises for any reason? If so, what compromises did you have to make and why did you have to make them?
11. What are the strengths of your lessons for the upcoming investigation? What are the weaknesses?
12. While planning your lessons for the upcoming investigation, did you feel like there were any constraints that you had to keep in mind? If so, what were they?
13. Is there anything about your lesson plans for the upcoming investigation that you recognize as not ideal? If so, what are these things and why are you planning on doing them?
14. What are some things that you anticipate might not allow you to accomplish your learning objective for the upcoming investigation?
15. What are some issues that you anticipate students’ will encounter in the upcoming investigation?
16. Why do you think students will have these difficulties?
17. How will you know if students develop the understanding that you would like them to have?
18. What are some key problems or activities that the students will work on in the upcoming investigation?
19. Why are these problems important?
20. What would a perfect solution to these problems look like?
21. What kind of solution to this problem would you be satisfied with?
22. Describe the level at which you anticipate your students will engage or participate in the upcoming investigation. Is the level of their engagement sufficient for learning the key ideas in the way you intend?

Teacher journal. David completed journal entries after his students completed

each investigation.³⁶ The purpose of these journal entries was to provide insight into the ways in which the mathematical knowledge he brought to bear in the context of teaching was fashioned by his image of instructional constraints. The teacher journal was, first and foremost, an outlet for David to express his thoughts about the lessons that spanned each investigation. Accordingly, the teacher journal was not rigidly structured. However, I encouraged David to consider general prompts while writing his journal entries. Refer to Table 11 for these general prompts.

Table 11

General Teacher Journal Entry Prompts

1. What is your assessment of how the investigation went?
2. Did you accomplish what you wanted to while teaching the investigation? Explain.
3. Did you deviate at all from your lesson plan? If so, how and why?
4. Would you change anything if you had to teach this investigation again?
5. Is there anything that would have made your teaching of the investigation more effective?
6. Was there anything problematic or unexpected about your teaching of the investigation?
7. To what degree do you think your students understood the main ideas of the investigation? Explain.
8. Did anything inhibit your ability to accomplish your lesson objective? If so, what? Explain.
9. When talking with teachers after they teach, they sometimes tell me, “My teaching of this investigation would have been better if only ...” If you feel this way, how would you complete the sentence?
10. If you had the power to change anything that could have improved the quality of how you taught this investigation, what would you change and why?

I also asked David to respond to specific prompts related to the content of his lessons pertaining to a specific investigation. For instance, I often asked David to justify a particular instructional decision or solicited his reaction to a specific event from his

³⁶ My initial intention was to have David complete journal entries daily. David expressed concern about the time commitment that daily journal entries would impose so I decided to administer journal entries for each investigation instead.

teaching. The types of occasions to which I asked David to respond were the same as those I selected for discussion during the Phase III semi-structured clinical interviews: *moments of instructional deviation*, *moments of mathematical concession*, and *moments of instructional incoherence*. I define these three categories of classroom events in the following subsection. David responded to the prompts for each journal entry electronically in a word document that I placed in a shared *Dropbox* folder. The teacher journal constituted an important source of data since, as opposed to the semi-structured clinical interviews I conducted in Phase III, they provided as close to an immediate sample of David's reaction to classroom events, and of his justification for instructional choices, as was feasible.

Phase III: Semi-Structured Clinical Interviews

The objective of the third phase of my experimental methodology was to obtain data that allowed me to construct a model of David's image of those aspects of his environmental context that he appraised as constraints on the quality of his instruction, and to determine the way in which this image conditioned the mathematical knowledge he employed in the practice of teaching. Constructing such a model and determining the effect that David's image of instructional constraints had on his enacted subject matter knowledge involved my conducting a series of three semi-structured clinical interviews after David completed his instruction of Module 8. I conducted these interviews on April 28th, April 30th, and May 4th of 2015. The first and third interviews lasted two hours in duration and the second interview lasted just over one hour.

The content of these semi-structured clinical interviews was heavily informed by my analysis of the data I obtained from Phases I and II of my experimental methodology.

Based on my analysis of the TBCIs, pre-lesson interviews, videos of classroom practice, and David's journal entries, I selected video clips and written artifacts to discuss with David during the clinical interview sessions in an effort to discern the role of David's image of instructional constraints on the quality of his enacted mathematical knowledge. While my analysis of all these forms of data influenced the content of the Phase III clinical interviews, segments of videos from David's classroom teaching were the primary objects around which the conversation in the clinical interviews was based. In particular, I identified and selected from the videos of classroom practice three types of occasions for David and I to discuss in the clinical interviews: (1) *moments of instructional deviation*, (2) *moments of mathematical concession*, and (3) *moments of instructional incoherence*. Discerning David's rationale for these instructional moments allowed me to gain insight into how his image of instructional constraints informed his enacted subject matter knowledge.

Understanding the effect that David's image of instructional constraints had on the quality of mathematical knowledge he allowed students to construct required that I ascertain David's rationale for instances in which he did not allow his students to construct the types of mathematical understandings and develop the ways of thinking that he described in the pre-lesson interviews. Therefore, in the Phase III clinical interviews I used as discursive objects those instances of classroom teaching in which there appeared to be a discrepancy (from my perspective) between the mathematics David intended his instruction to promote and the mathematics he actually taught. Identifying and analyzing these *moments of instructional deviation* allowed me to apprehend the effect that David's image of his immediate social context had on his enacted subject matter knowledge.

In addition to discerning David's rationale for instructional moments in which he conveyed mathematical meanings, and instances in which David deviated from the intended curriculum, I devoted particular attention in the Phase III clinical interviews to ascertaining David's rationale for those instructional actions in which the mathematics he allowed students to construct differed from the mathematical ways of understanding and ways of thinking he demonstrated during the series of TBCIs. Discussing these *moments of mathematical concession* were fundamental to revealing aspects of David's image of instructional constraints that conditioned the mathematical knowledge he brought to bear in the process of designing and administering learning experiences for students.

During the Phase III clinical interviews, I also presented David with moments from his instruction in which he conveyed discrepant or contradictory meanings. My intention for presenting these *moments of instructional incoherence* was to provide occasions for David to identify circumstances or events that compelled him to support inconsistent ways of understanding. I expected these circumstances or events might constitute instructional constraints to which David spontaneously responded while teaching.

It is essential to point out that I did not assume David recognized the discrepancies I noticed in the videos I selected as examples of *moments of instructional deviation*, *moments of mathematical concession*, and *moments of instructional incoherence*. Therefore, after having presented pairs of videos to David that I believed demonstrated him conveying/supporting discrepant meanings, I asked him to compare the ways of understanding he communicated in both videos. My rationale for doing so was to determine if David recognized the same inconsistencies that I noticed in the ways of

understanding he demonstrated/conveyed.

Analytical Methodology

Since the aim of the present study was to build theory that provides a functional explanation for the way in which a secondary teacher's image of instructional constraints conditioned the mathematical knowledge he brought to bear in the practice of teaching, the procedures I used to analyze the data I obtained derive from Strauss and Corbin's (1990) and Corbin and Strauss's (2008) grounded theory approach. In grounded theory, data analysis proceeds in a cyclic fashion whereby the researcher continually generates and refines hypotheses until a stable and viable inductively-derived theory emerges. Refining provisional hypotheses requires purposeful data collection informed by ongoing analysis. Thus, a hallmark of grounded theory is the reciprocal relationship between data collection and analysis; that is, the data a researcher collects influences the hypotheses he or she constructs, and the hypotheses a researcher constructs informs subsequent data collection. Therefore, while I have attempted to separate these aspects of my methodology in writing, the reader should note that, in actuality, the boundary between data collection and analysis is often blurred when one employs grounded theory analytical procedures. This point is well exemplified by the several remarks pertaining to analysis that I could not avoid making in the previous section detailing my experimental methods.

As the preceding section describing my experimental methodology suggests, the data I collected for the present study originated from a variety of sources for a variety of purposes. Thus, for simplicity I present my analytical methodology in the following subsections: (1) preliminary analysis, (2) ongoing analysis, and (3) post analysis. In my

discussion of these phases of analysis, I explain the specific analytic procedures I employed with the particular data within each phase. (See Table 12 for an indication of the data sources that comprise each analytic phase. Also, see *Figure 14* for a chronology of my analytical methods.) Additionally, after discussing the procedures by which I analyzed data during the ongoing analysis phase, I outline the grounded theory analytical framework (Strauss & Corbin, 1990) that I used to analyze data during the post analysis phase.

Table 12

Data Corresponding to the Four Analytic Phases

Preliminary	Ongoing	Post I	Post II
Initial clinical interview video			
	TBCI videos	TBCI videos	
	Pre-lesson interview audio recordings	Pre-lesson interview audio recordings	Pre-lesson interview audio recordings
	Classroom videos	Classroom videos	
	Artifacts of classroom practice	Artifacts of classroom practice	
	Teacher journal	Teacher journal	Teacher journal
			Phase III semi-structured clinical interviews

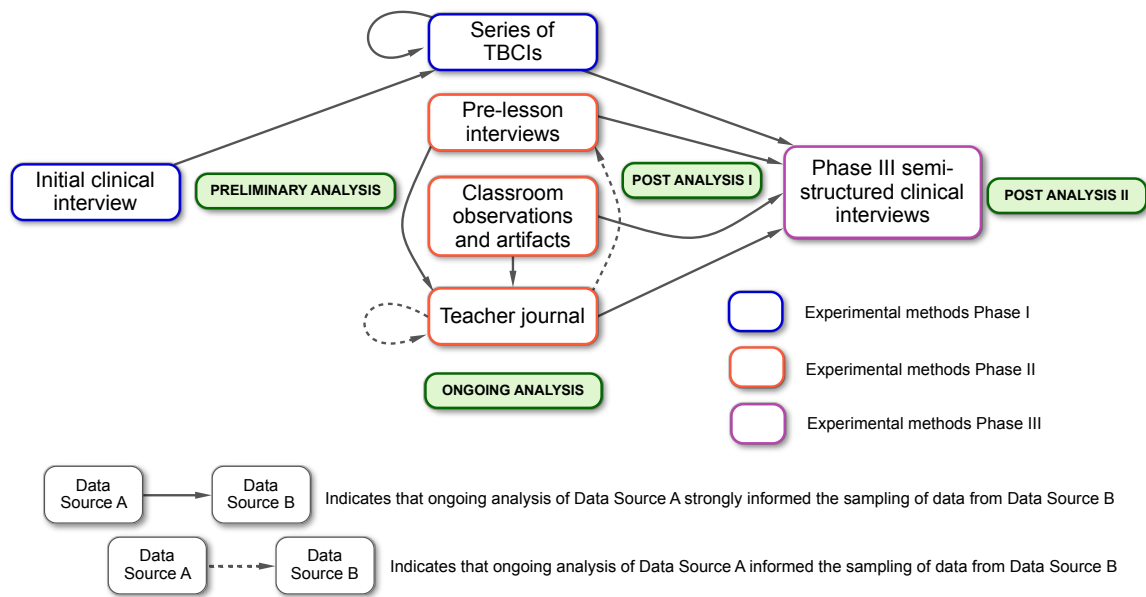


Figure 14. Chronology of analytical methods.

Preliminary Analysis

The initial clinical interview was the exclusive means by which I obtained data for preliminary analysis. As mentioned above, the purpose of this interview was to develop provisional hypotheses relative to David’s ways of understanding ideas related to sine and cosine functions. Since these hypotheses were provisional, and because I tested them in subsequent TBCIs, my analysis of the initial clinical interview was not exhaustive.

My analysis of the initial clinical interview was consistent with the *generative* approach for analyzing clinical interviews (Clement, 2000), which involves “the open interpretation of large episodes by an individual analyst” and focused on “constructing new observation concepts and theoretical models” (p. 342).³⁷ The generative approach fosters the construction of insightful explanatory models of hidden mental processes via

³⁷ My analysis of the Phase III semi-structured clinical interviews followed these same principles.

the process of *abduction*, as opposed to induction. Abduction is the process of generating a viable explanatory model for an observed phenomenon, and differs from induction in that abduction aims to develop a hypothetical explanation for some observation whereas induction seeks to develop conclusions based on trends in observation.

In particular, upon analyzing the video of the initial clinical interview, I constructed a hypothetical way of understanding sine and cosine functions that parsimoniously accounted for my interpretation of David's actions and utterances. That is, while examining each of David's responses, I asked myself, "How might David be understanding sine and cosine functions so as to produce this observable product of his reasoning?" I refined and articulated my model of David's understanding to make it viable with his subsequent language and actions. From this preliminary analysis I emerged with a number of provisional hypotheses to test throughout the series of TBCIs.

Ongoing Analysis

I have emphasized throughout this chapter that the grounded theory methodology relies upon interweaving data collection and analysis. Therefore, a substantial portion of my analysis occurred throughout the data collection process. This type of analysis is commonly referred to as *ongoing analysis*. In this subsection, I discuss how I analyzed the data I obtained from the following sources: the series of TBCIs, pre-lesson interviews and teacher journal, classroom videos and artifacts of classroom practice, and Phase III semi-structured clinical interviews.

Task-based clinical interviews. As I explained in the section outlining my experimental methodology, between each task-based clinical interview I watched the video of the previous interview and constructed memos of my initial thoughts and

impressions regarding what transpired in the interview. Recall that these memos focused on the following:

- (1) Explicating hypothetical ways of understanding that may explain my interpretation of David's language and actions during the interview;
- (2) Identifying David's specific utterances and actions that contributed to my construction of these hypothetical ways of understanding;
- (3) Explaining how I interpreted these utterances and actions so as to make their contribution to my model of David's ways of understanding explicit;
- (4) Justifying my instructional actions during the interview and describing my interpretation of David's response to these instructional actions;
- (5) Describing my rationale for the tasks I planned on using in the subsequent interview(s).³⁸

The memos I generated during ongoing analysis of the series of TBCIs consisted of a 188-page document entitled, "Ongoing Analysis of Task-Based Clinical Interviews."

I organized my memos for each interview using the following section headings:

"Tasks/Didactic Objects," "Description of Events," "Tentative Model of David's Mathematics," and "Possible Adjustments to Subsequent TBCI(s)." I outlined the tasks I used in each interview in the respective "Tasks/Didactic Objects" section. I provided a descriptive narrative, in broad strokes, of what occurred in each interview in the "Description of Events" section corresponding to that interview. The "Tentative Model of David's Mathematics" section of the memos for each interview attended to foci (1)-(4)

³⁸ Designing/modifying tasks for subsequent interviews relied not only upon what I learned from the previous interview, but also from what I learned from all TBCIs prior to it. Therefore, I consulted all the preceding memos I created between TBCIs prior to designing/modifying the tasks I used in the upcoming interview.

above. Finally, I documented in the “Possible Adjustments to Subsequent TBCI(s)” section any modifications/additions to the task sequence for subsequent TBCIs and provided my rationale for these modifications/additions. In addition to informing subsequent data collection and providing artifacts of my thinking throughout the series of TBCIs, the memos I generated during ongoing analysis outlined provisional hypotheses of David’s mathematical thinking that I examined more closely during post analysis.

Pre-lesson interviews. I administered the pre-lesson interviews for the following purposes: (1) to discern the role of David’s image of instructional constraints on the mathematical meanings he designed his lessons to promote, and (2) to identify the meanings that David wanted his students to construct. The second purpose facilitated my contrasting the mathematical ways of understanding David planned on conveying in his lessons with those he actually conveyed. Since both of these purposes had implications for subsequent data collection, my ongoing analysis of the pre-lesson interviews attended to both.

Between each pre-lesson interview I listened to the audio recording of the previous interview and identified instances in which David revealed aspects of his image of instructional constraints and the mathematics he intended to provide students the opportunity to construct. This analysis informed subsequent data sampling through both the teacher journal and Phase III semi-structured clinical interviews.

For instance, suppose that in a pre-lesson interview David indicated that he planned on teaching (from his perspective) an understanding of some mathematical concept that is less sophisticated than he is capable of teaching because of his anticipation of how students would engage in instructional activities that seek to engender a more

sophisticated understanding. Identifying such deliberate instructional compromise would allow me to use the teacher journal and Phase III clinical interviews as outlets to examine the way in which David's image of student engagement conditioned the mathematical meanings he enacted in the context of teaching. For example, in the journal entry following the pre-lesson interview I would ask,

In the most recent pre-lesson interview you said [David's statement], which gave me the impression that under different circumstances the understandings you would allow students to develop relative to [the key concept of the day's lesson] might be different. If my interpretation of your statement is correct, please describe what these different circumstances would be and elaborate on how the understandings you wish to impart would differ. If my interpretation of your statement is incorrect, please clarify what you meant by [David's statement].

The implication of identifying in each pre-lesson interview the mathematical understandings and ways of thinking David intended his lesson to promote is equally apparent. As I explained above, a primary purpose of the teacher journal and the Phase III clinical interviews was to solicit the David's justification for the following occasions of classroom teaching: *moments of instructional deviation*, *moments of instructional incoherence*, and *moments of mathematical concession*. Identifying moments of instructional deviation, in particular, relied upon my identification of the meanings that David intended his students to construct. It is for this reason that after each pre-lesson interview I listened to the audio recording to identify instances that revealed the mathematics David intended to teach.

Teacher journal. Recall that the purpose of the teacher journal was to provide insight into the way in which the mathematical knowledge David brought to bear in the context of teaching was fashioned by his image of instructional constraints. As with the ongoing analysis of the pre-lesson interview audio recordings, I identified in each of

David's journal entries instances in which he revealed aspects of his image of instructional constraints, paying particular attention to the way in which the mathematical knowledge David employed in the context of teaching was conditioned by his image of these instructional constraints. This ongoing analysis informed subsequent data collection by providing content to discuss, probe, refine, and elaborate in the pre-lesson interviews and journal entries that followed, as well as in the Phase III semi-structured clinical interviews.

Classroom videos and artifacts of classroom practice. Recall that my analysis of the videos of classroom teaching and artifacts of classroom practice facilitated my developing a model of David's image of his environmental context including the instructional constraints he recognized therein, as well as the mathematical understandings and ways of thinking David afforded his students the opportunity to construct. I generated memos during classroom observations that served as a first-level analysis to this end. Specifically, I focused on documenting occasions that revealed David's image of instructional constraints as well as the mathematics he allowed students to learn. As with my analysis of the pre-lesson interviews, my observational memos of David's classroom teaching informed the prompts of the succeeding teacher journal entry³⁹ and the content of the Phase III semi-structured clinical interviews.

During ongoing analysis of the Phase II data, I recorded my initial impressions of the pre-lesson interviews, the memos I generated during classroom observations, and the questions I asked in each teacher journal entry in a single 153-page word document

³⁹Note that all of the specific journal prompts refer to David's teaching of a specific investigation and thus were informed by a surface-level analysis of the videos of the lessons in which David taught a particular investigation.

entitled, “Phase II Ongoing Analysis.” Generally speaking, these notes documented my interpretations, reactions, inferences, rationalizations, reflections, questions, concerns, confusions, tensions, recommendations, and conclusions relative to my ongoing analysis of the Phase II data. The objective of these notes was twofold: (1) to assist in the organization of my thinking, and (2) to create artifacts of my ongoing analysis to which I could refer during post analysis.

Analytical Framework for Post Analysis

While the details of the analytic techniques I employed were unique to the specific data being analyzed, my analysis of the data within the ongoing and post analysis phases of my analytical methods largely followed Strauss and Corbin’s (1990) grounded theory methodology. Accordingly, the focus of this subsection is on outlining the salient features of this methodology, with particular emphasis on open, axial, and selective coding. I explain the nuances of how I applied this analytical framework to analyze specific categories of my data corpus in the proceeding subsections.

Grounded theory is a qualitative research methodology that consists of a set of analytic procedures and techniques that allow a researcher to construct an inductively-derived theory about some phenomenon (Strauss & Corbin, 1990, p. 24). Grounded theory, like all theory, is built from concepts and relationships among them. Hence, the analytic procedures that comprise grounded theory principally aspire to assist the researcher in defining, examining, articulating, characterizing, conceptualizing, categorizing, and relating concepts through the process of *coding*. Strauss and Corbin (1990) distinguish among three types of coding procedures, which are at the nucleus of their methodology: (1) *open coding*, (2) *axial coding*, and (3) *selective coding*. It is

important to note that the analyst does not progress through these three types of coding sequentially, but rather moves flexibly between them.

Open coding. Open coding is the process whereby a researcher breaks down, examines, compares, conceptualizes, and categorizes data (Strauss & Corbin, 1990, p. 61). The first step in open coding is to closely examine the data to identify *concepts*—the conceptual labels an analyst uses to represent incidents, ideas, or events. The second step is to categorize concepts by grouping those that appear to pertain to a particular phenomenon, and to place a more abstract conceptual label to each category of concepts. The third step is to develop each category in terms of its *properties* and *dimensions*. Properties “are the characteristics or attributes of a category” and dimensions “represent locations of a property along a continuum” (ibid., p. 69). Since every category has several properties, each of which vary along a dimensional continuum, every occurrence of a category gives rise to a separate *dimensional profile* (ibid., p. 70). The final step in open coding is to identify patterns in data by grouping dimensional profiles of occurrences of a category.

Axial coding. Through the process of open coding, the analyst breaks data down into categories of concepts, and develops each category in terms of their properties and dimensions. The purpose of axial coding is to begin to put data back together by refining categories so that the analyst is equipped to make connections between categories in selective coding.

Strauss and Corbin (1990) describe the *paradigm model* as the analytical framework of axial coding. The paradigm model seeks to allow the analyst to give

precision to categories⁴⁰ by specifying the *causal conditions*⁴¹ that give rise to each category, the *context*⁴² within which each category exists, the *action/interactional strategies*⁴³ involved in each category—including the *intervening conditions*⁴⁴ bearing upon these action/interactional strategies, and the *consequences*⁴⁵ of the action/interactional strategies (see *Figure 15* for a diagram of Strauss and Corbin’s paradigm model). Elaborating each category by means of the paradigm model allows the analyst to develop each category and prepares him or her to establish relationships between categories (e.g., a consequence of Category *A* is a causal condition for Category *B*), thereby providing the foundation for constructing an analytical story line via the process of selective coding.

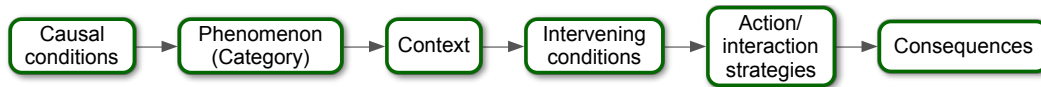


Figure 15. Paradigm model (Strauss & Corbin, 1990).

Selective coding. Developing grounded theory involves the analyst in integrating categories to form a coherent whole with both explanatory and predictive power. The methods and procedures that comprise selective coding assist the analyst in achieving such integration. The purpose of selective coding is similar to that of axial coding: to

⁴⁰ Recall that a category is a set of concepts that pertain to a particular phenomenon.

⁴¹ Causal conditions are the incidents or circumstances that lead to the manifestation of a phenomenon.

⁴² Context consists of the set of properties belonging to a particular phenomenon.

⁴³ Action/interactional strategies are those strategies “directed at managing, handling, carrying out, [and/or] responding to a phenomenon as it exists in context or under a specific set of perceived conditions” (Strauss & Corbin, 1990, p. 104).

⁴⁴ Intervening conditions are those conditions pertinent to action/interactional strategies that pertain to a particular phenomenon.

⁴⁵ Outcomes of action/interactional strategies as they pertain to a particular phenomenon.

define the relationships among categories, validate those relationships, and, in the process, further refine and develop categories (Strauss & Corbin, 1990, p. 116).

The selective coding process begins by constructing a *story line*, or *core category*. The story line consists of “the conceptualization of a descriptive story about the central phenomenon of the study” (ibid., p. 119), which becomes the core category. It is important to note that the core category, like all categories, must be developed in terms of its properties and dimensions. It is also noteworthy that, initially, the story line is *descriptive* in nature. The procedures that comprise selective coding seek to allow the researcher to construct an analytic version of the story, thereby providing the foundation of a grounded theory.

The next step in the selective coding process is to define the relationship between subsidiary categories and the core category by means of the paradigm model (see *Figure 15*). This process is similar to that engaged in during axial coding whereby the analyst, by means of the paradigm, defines the relationship between concepts and the category within which they are organized. In selective coding, however, the analyst identifies *categories* of concepts as constituting the conditions, context, strategies, and consequences of the core category.

The third step in the selective coding process involves the analyst in transforming the *descriptive* story line into an *analytic* story line by using the relationships he or she defined between subsidiary categories and the core category using the paradigm model as a guide. Finally, the researcher must validate the analytic story line against the data and elaborate/articulate categories in need of further development or refinement.

I used these grounded theory procedures to various extents during post analysis. In the following section, I explain how I applied these abstract analytical procedures to my specific data corpus.

Post Analysis

The primary objective of my ongoing analysis was to facilitate purposeful data collection. After I completed data collection, I conducted the fine-grained analysis that was necessary to address my research questions. Since the analytical practices I employed during post analysis was not in the service of informing (immediate) subsequent data collection, these analytical practices were far more elaborate and extensive.

As I emphasized at the beginning of this chapter, addressing the central research question of the present study involved:

- (1) Constructing a model of David's mathematical ways of understanding and ways of thinking relative to sine and cosine functions;
- (2) Constructing a model of David's enacted subject matter knowledge in the context of classroom practice by characterizing the ways of understanding and he afforded students the opportunity to construct as well as the ways of thinking he supported students in developing;
- (3) Constructing a model of David's image of constraints on his instruction and ascertaining the role of his image of these constraints on his enacted subject matter knowledge.

Recall that I used the data I obtained from the series of TBCIs to construct the first model. To construct the second model, I used the data I acquired from the pre-lesson

interviews and the videos of classroom practice. Finally, I used the data generated from the pre-lesson interviews, the teacher journal, and the Phase III semi-structured clinical interviews⁴⁶ to develop the third model. The procedures I utilized to construct these three models were based on the analytic methods of grounded theory discussed above (Strauss & Corbin, 1990).

Phase I clinical interviews: Initial Clinical Interview and series of TBCIs. I

used the qualitative data analysis program *Studiocode* to facilitate my coding of the initial clinical interview and the series of TBCIs. For each interview, I merged the video I took of my computer screen via *QuickTime Player* with the video I obtained from the video camera, and then imported this merged video into *Studiocode* for analysis. Coding in *Studiocode* involves constructing a code window that consists of an organization of code buttons that the analyst clicks on while the video is playing to tag segments of video with particular conceptual labels (see *Figure 16* for an image of the code window I used while coding the video of the initial clinical interview and the videos from the series of TBCIs). These coded segments of video are recorded in a “Timeline” (see *Figure 17*).

⁴⁶ I analyzed the Phase III semi-structured clinical interviews according to the generative approach for analyzing clinical interviews described by Clement (2000), as discussed in the “Preliminary Analysis” subsection of this chapter.

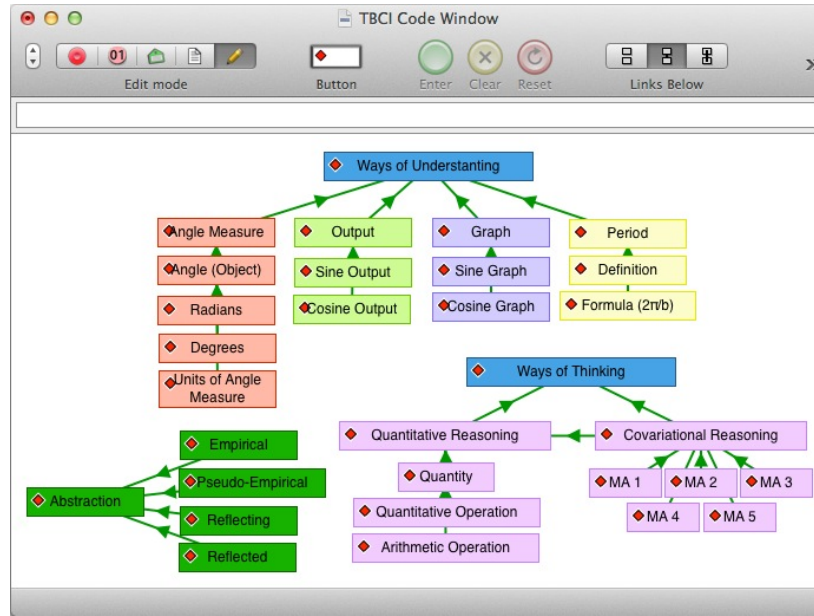


Figure 16. Code window for the series of TBCIs.

Code	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	
1 Sine Output																										
2 Output																										
3 Cosine Output																										
4 Radians																										
5 Angle Measure																										

Figure 17. Studiocode video and timeline.

My coding of the initial clinical interview and the series of TBCIs differed from the procedure for open coding outlined by Strauss and Corbin (1990) in one important respect: I specified almost all of my codes prior to analysis instead of having these codes

emerge from my analysis. As is apparent from my code window in *Figure 16*, many of my codes represent constructs within various domains of my theoretical perspective—particularly Piaget’s genetic epistemology and the theories of quantitative and covariational reasoning. Other codes within my code window represent the mathematical topics that David and I discussed in the initial clinical interview and the series of TBCIs (e.g., “Angle Measure”, “Sine Output”, “Sine Graph”, “Period”, “Formula ($2\pi/b$)”). Since the codes I used to analyze the initial clinical interview and the series TBCIs originated from my theoretical perspective and from the mathematical content of these interviews, these codes were necessarily organized into predetermined categories.

I began my analysis of each clinical interview by performing a pass of coding in which I labeled segments of the video with codes that represent mathematical ideas (e.g., “Radians”). The segments of video I labeled with these codes were those in which David provided observable products of his way(s) of understanding the mathematical idea represented by the respective code. I then performed a second pass of coding in which I identified instances from the video that revealed characteristics of David’s quantitative and covariational reasoning and/or demonstrated his engaging in particular types of abstraction. I made a final pass of each video to edit and refine my coding.

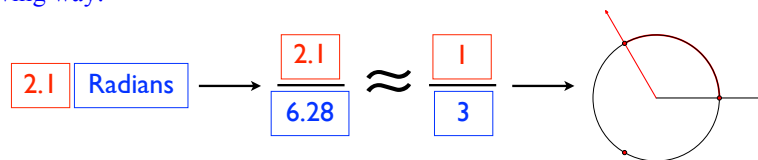
After I coded the video of each Phase I clinical interview, I transcribed all coded segments of the video. This generally amounted to transcribing almost the entire video, excluding extraneous conversations and interruptions by David’s students and colleagues. I imported the transcription of each video into a single word document. I then performed a line-by-line conceptual analysis of the transcript in which I documented in blue text underneath each of David’s statements the mental actions and operations that explain my

interpretation of David’s language and actions. Additionally, I described the extent to which David engaged in the abstractions that I designed my tasks to engender and discussed what David’s language and actions reveal about his mathematical ways of thinking. Finally, I provided justifications for my actions as an interviewer on those occasions in which I asked David spontaneous questions or probed his responses. (See Table 13 for a sample analysis from a line of the transcript from the initial clinical interview.)

Table 13

Sample Analysis of the Initial Clinical Interview Transcript

Michael: What does it mean to say that an angle has a measure of 2.1 radians?
 David: So to say that it has a measure of 2.1 radians would mean that about, now about a third of the way around a complete circle if I’m just approximating so that I can kind of understand where it would be in my head, um, without doing any real math, uh, involved. Um if I, you know, wanted to I could actually then, you know, take that and convert using that as a proportion of the circumference and figure out an actual length; the circumference being 2π .
 David’s response here is consistent with the way of understanding angle measure as “so many out of so many” that he demonstrated previously. David does not appear to be attending to a unit of measure (in this case the radius of the circle). It seems to be the case that David understands that an angle subtends a particular fraction of a circle’s circumference. The “quantity” David seems to have in mind is therefore the portion of the circle’s circumference that the angle subtends. My hypothesis is that David assimilated the statement “2.1 radians” in the following way:



After I performed a line-by-line conceptual analysis of the transcript for each video, I carefully read through my notes and identified themes in my conceptual analysis. I constructed a list of themes and identified specific lines from the transcript that provided evidence of each theme. Identifying themes in my conceptual analysis constituted an iteration of open coding; the themes I identified were themselves the

conceptual labels I extracted from the data to represent my conceptual analysis of segments of the transcript. After this iteration of open coding, I carefully reviewed the lines of the transcript that exemplified each theme and wrote a detailed summary that elaborated each theme. I recorded each of these summaries beneath the transcript of the respective interview. These summaries constituted my emerging model of David's mathematics. I then compared my summaries of the themes of David's thinking with the memos I generated during my ongoing analysis of the series of TBCIs. My purpose in doing so was to ensure that I did not miss anything important and to compare my provisional model of David's mathematical knowledge that I constructed during ongoing analysis with the emerging model that I was constructing during post analysis. Where there were discrepancies in these models, I returned to the relevant data and modified my emerging model of David's mathematics accordingly.

I entitled the 244-page document that contained the transcription of the Phase I clinical interviews (with my conceptual analysis recorded within) and my summaries of the themes of David's thinking, "Post Analysis of Phase I TBCIs." After completing this artifact of post analysis, I compared the themes I identified from examining my conceptual analysis of each clinical interview against all relevant data. Where my model of David's mathematics was inconsistent or contradictory, I went back to the transcript to either modify my model or to explain these inconsistencies (on occasion, these inconsistencies resulted from David having learned throughout the series of TBCIs). This process allowed me to construct the comprehensive and viable model of David's mathematical knowledge that I present in Chapter 6.

Phase II data: Videos of classroom practice and audio recordings of the pre-lesson interviews. As I did with the Phase I data, I used *Studiocode* to facilitate my coding of the videos of David’s classroom teaching as well as the audio recordings of the pre-lesson interviews. I began my analysis of the videos of David’s teaching by making an initial coding pass in which I identified instances in which David conveyed some way of understanding. I coded these occasions for the specific category of understanding David conveyed (e.g., “Units of Angle Measure”, “Cosine Output”, “Period”). I then made a second coding pass in which I verified and refined my initial codes, as well as identified instances in which David conveyed understandings that deviated from those he professed the intention to support during the pre-lesson interviews (i.e., *moments of instructional deviation*).⁴⁷ I also identified during this second iteration of coding occasions in which the meanings David supported in his teaching differed from those he demonstrated in the series of TBCIs (i.e., *moments of mathematical concession*). I used the code window displayed in *Figure 18* to code the videos of David’s teaching. I defined all but one of the codes within this code window prior to my analysis. I added the “Instructional Goals” code button early in my analysis to capture moments of David’s teaching in which he made explicit to students the understandings or skills he intended them to construct/acquire.

⁴⁷ I more accurately identified *moments of instructional deviation* after having carefully analyzed the audio recordings of the pre-lesson interviews. My coding for *moments of instructional deviation* during this second pass of coding was rather tentative, and served the primary purpose of identifying segments of video to more carefully examine later.

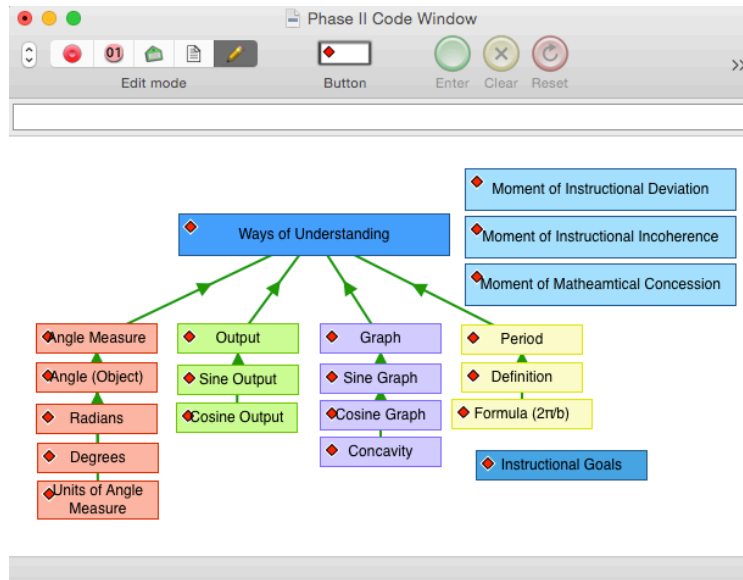


Figure 18. Code window for videos of David’s classroom teaching.

My analysis of the audio recordings of the pre-lesson interviews followed a similar progression as my analysis of the videos of David’s classroom teaching. I made an initial pass of coding in which I identified categories of meanings David expressed the intention to support in his instruction. These categories were the same as those contained in the code window I used to analyze the videos of David’s instruction (see *Figure 18*), and were therefore defined prior to my analysis. The other codes that comprise the code window I used to analyze the pre-lesson interview audio recordings emerged from my analysis of these interviews (see *Figure 19* for an image of the code window I used to code the pre-lesson interview audio recordings). I conducted a second pass of coding that involved my attending to these codes while validating and refining the codes from my initial pass. I created the “Instructional Constraint” code to identify David’s remarks that revealed his image of instructional constraints. I included the “Lesson Planning” code to label segments of the recording in which David disclosed his general approach to designing lessons, and I included the “Lesson Plan” code to identify occasions in which

David communicated his plan for the upcoming lesson. I added the “Compromises/Concessions” code to capture instances in which David made conscious concessions to the quality of the mathematical ways of understanding he intended his instruction to support. I created the “Behavior vs. Mental Process” code to label segments of the audio recording in which David defined his instructional objective in terms of behavioral outcomes as opposed to the mental activity that characterizes particular ways of understanding. I included the “Lack of Reflection” code to capture David’s statements that revealed the extent to which he reflected on his own ways of understanding while defining his instructional goals and objectives. I used the “Question” code to simply identify David’s responses to specific questions in the interview protocol.

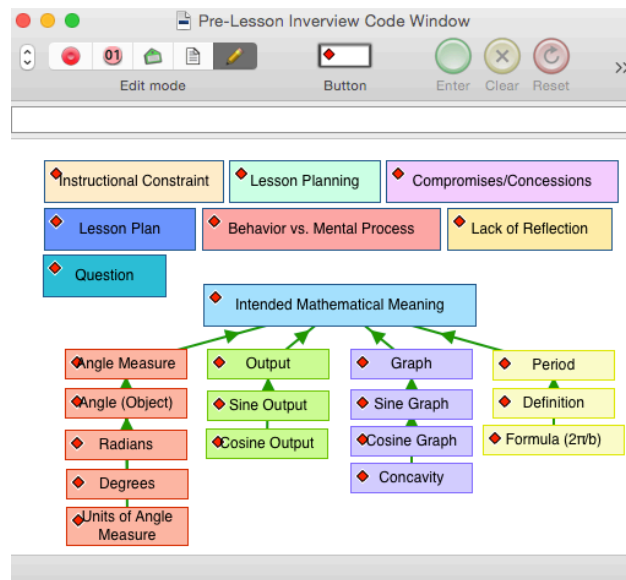


Figure 19. Code window for pre-lesson interviews.

After having coded the 37 videos of David’s classroom teaching, I produced a 57-page document entitled, “Phase II Post-Analysis Memos” in which I summarized each coded instance of the videos and performed selective transcriptions of, what appeared at the time to be, particularly revealing moments of David’s instruction (see Table 14 for a

sample summary). I carefully read through these memos and organized into themes the coded segments of video that corresponded to the following categories of codes: (1) angle measure, (2) output quantities and graphical representations of sine and cosine functions, and (3) period of sine and cosine functions. For example, the following four themes emerged from my analysis of the summaries of the coded segments of video within the angle measure category:

- (1) To measure or not to measure ... a length?
- (2) Radians measure length. Degrees, quips, and marks measure “space.”
- (3) Radians are advantageous because radius lengths are “based on the circle.”
- (4) The size of the circle is immaterial to the measure of the angle.

After having organized the coded segments of video into themes, I examined the data within each theme and elaborated my model of David’s enacted knowledge relative to each theme. I present this model in Chapter 7.

Table 14

Sample Summary of a Coded Segment of Video from Lesson 15

Output #1/Cosine Output #1:

A student asked what the output of cosine means in the context of Task 3 on Investigation 7. David explained,

Cosine usually takes the cosine or sine of an angle. We take that angle and we look it up on the circle and that’s how we get the output values. So an angle measure is what part of the circle we have. So when we’re saying we have an angle with a measure of one degree, we have $1/360^{\text{th}}$ of an entire circle. ... So instead of using the word circle, replace it with cycle. So I am $1/360^{\text{th}}$ of the way through the cycle of going all the way around. In a problem like this what we’re really talking about is cycles. So we’re saying when the cosine is this (*points to the argument of the function*) far through its cycle, ... then the output that it gives us ... is an output ... that will cycle back and fourth between positive one and negative one because that’s what cosine always gives me.

David's explanation reveals his understanding of the argument of sine or cosine as representing "a location in the cycle" and the output of these respective functions as representing "a percentage of an output value."

Once I coded the ten pre-lesson interviews using the code window in *Figure 19*, I summarized coded segments of the pre-lesson interview audio recordings and documented these summaries in a 15-page document entitled, "Pre-Lesson Interview Post-Analysis Memos." I then organized into themes the coded segments of the audio recordings that corresponded to the three aforementioned categories of codes (angle measure, output quantities and graphical representations of sine and cosine functions, and period of sine and cosine functions). With a few exceptions, these themes aligned with those that emerged from my analysis of the videos of David's classroom teaching.

After having summarized the coded segments of the pre-lesson interview audio recordings, I used the "Pre-Lesson Interview Post-Analysis Memos" and "Phase II Post-Analysis Memos" documents to identify *moments of instructional deviation* by comparing coded instances of the videos of David's classroom teaching with coded instances from the pre-lesson interview audio recordings within each theme. I then returned to the themes of David's mathematical knowledge that resulted from my analysis of the series of TBCIs and compared and contrasted relevant instances from David's classroom teaching and the pre-lesson interviews with coded segments of the TBCI data to identify *moments of mathematical concession*. To compare these instances, I grouped coded segments from the TBCI videos with those from the videos of David's classroom teaching and the pre-lesson interview audio recordings in which David demonstrated/conveyed similar ways of understanding. To contrast coded instances of David's classroom teaching and the pre-lesson interviews with coded segments of the

TBCI data, I grouped coded segments from the TBCI videos with those from the videos of David’s classroom teaching and the pre-lesson interview audio recordings in which David demonstrated/conveyed discrepant, inconsistent, or incompatible ways of understanding. I recorded these groupings, and relevant notes associated with them, in a 22-page document entitled, “Comparison and Contrast of David’s Mathematical Knowledge with His Enacted Mathematical Knowledge.” I discuss the results of this comparison and contrast in Chapter 7.

Post analysis phase II: Pre-lesson interview audio recordings, teacher journal, and Phase III semi-structured clinical interviews. As previously mentioned, I asked David questions in the pre-lesson interviews that sought to reveal his image of constraints on his practice and to ascertain how David’s enacted mathematical knowledge was conditioned by his response to such constraints. I coded the audio recordings of the pre-lesson interviews for statements that revealed the instructional constraints David recognized in the context of classroom practice or lesson planning, as well as for the compromises and concessions David consciously made in the context of teaching (see *Figure 19* for an image of the code window I used). While I applied the “Instructional Constraint” and “Compromises/Concessions” codes to any relevant segment of the pre-lesson interview audio recordings, I most often applied these codes for the segments in which David responded to the following questions from the pre-lesson interviews:⁴⁸

- (1) While planning your lesson, did you feel that you needed to make any compromises for any reason? If so, what compromises did you have to make and why did you have to make them?

⁴⁸I asked these questions in slightly different forms in different interviews—the wording was not always precisely the same but I attempted to maintain the spirit of the questions.

- (2) While planning your lesson, did you feel like there were any constraints that you had to keep in mind? What were they?
- (3) Is there anything about your lesson plan that you recognize as not ideal? If so, what are these things and why are you planning on doing them?
- (4) What are some things that you anticipate might not allow you to accomplish your lesson objective?

After having coded all pre-lesson interview audio recordings, I created memos for each coded segment of audio that summarized the content of the coded segment. These memos resulted in a seven-page document entitled, “Pre-Lesson Interview Instructional Constraint and Compromises/Concessions Codes.” For those segments of audio labeled with the “Instructional Constraint” code, I inferred the instructional goal that appeared to be obstructed. Similarly, for the segments of audio labeled with the “Compromises/Concessions” code, I identified the instructional goal that David’s actions seemed to be in the service of maintaining or achieving. I then examined each of David’s responses to the four questions above and identified trends and patterns in these responses. I present my analysis of these responses in Chapter 8.

Recall that the purpose of the teacher journal entries was to provide insight into the ways in which the mathematical knowledge David enacted in the context of teaching was fashioned by his image of instructional constraints. Each journal entry form consisted of a number of prompts common to all journal entry submissions, as well as a number of specific prompts related to the content of his lessons pertaining to a specific investigation. I designed the prompts common to all journal entry forms to expose David’s image of instructional constraints and to provide an occasion for him to reveal

how he compromised the quality of his enacted mathematical knowledge in response to such constraints. My analysis of the teacher journal entries consisted of examining David's responses to the questions that were common to all eleven entries and to identifying trends in these responses. I paid particular attention to inferring the instructional constraints David's responses revealed as well as to ascertaining the instructional goals these constraints obstructed David from achieving. I recorded these inferences and observations in a 21-page document entitled, "Teacher Journal Entry Analysis Notes." I present my analysis of David's responses to the journal entry prompts in Chapter 8.

As I previously discussed, the purpose of the Phase III semi-structured clinical interviews was to reveal David's image of instructional constraints and to discern the role of these constraints on the mathematical knowledge David recruited to support his students' learning. During the Phase III clinical interviews, I presented David with pairs of audio and/or video excerpts from the series of TBCIs, pre-lesson interviews, and his classroom instruction that illustrated *moments of instructional deviation*, *moments of instructional incoherence*, and *moments of mathematical concession*. While there were several of each of these three types of instructional occasions from which to choose, I presented David with only three pairs of excerpts exemplifying each of these three instructional occasions. My reason for doing so was to limit the number of Phase III clinical interviews. The main criterion I used to select the pairs of audio and/or video excerpts was that David clearly demonstrated/conveyed a particular way of understanding in each excerpt.

I coded each of the Phase III clinical interviews using the code window displayed in *Figure 20*. I used the “Instructional Goals/Objectives” code button to label segments of video in which David’s statements revealed his instructional goals and/or objectives. I used the “Instructional Constraint” code button to identify the portions of video in which David identified particular circumstances or events as constraints on the quality of his teaching. I labeled the segments of video in which David revealed his perspective of what learning is and how it occurs with the “Theory of Learning” code. Finally, I identified with the “Awareness of Mental Actions” code those portions of video in which David revealed the extent to which he was consciously aware of the mental activity that comprise particular ways of understanding mathematical ideas. Both the “Theory of Learning” code and the “Awareness of Mental Actions” codes emerged from my analysis of the Phase II and Phase III data. My discussion of results in Chapters 6 and 7 reveal the necessity of these codes. After having made one pass of coding for each of the three Phase III clinical interviews, I made a second pass during which I refined and validated my initial codes. I then summarized the content of each coded instance in a 32-page document entitled, “Phase III Clinical Interview Notes.” I then made several passes through this document, adding selective transcriptions and analytical notes each time.

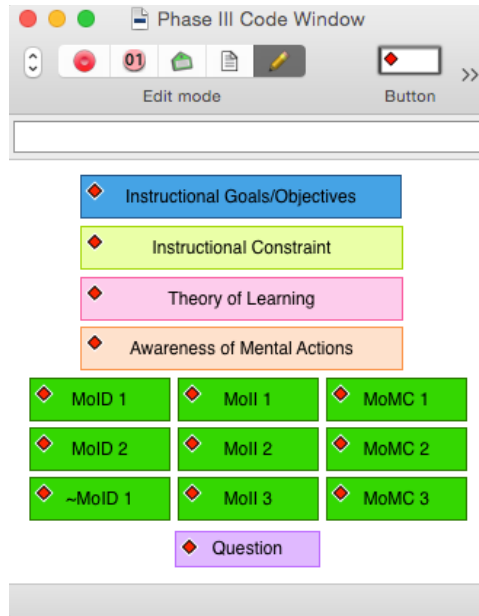


Figure 20. Code window for Phase III semi-structured clinical interviews.

CHAPTER 5

CONCEPTUAL ANALYSIS

Content gradually becomes subordinated to form with the growth of knowledge.

(Wason, 1977, p. 119)

Chapter 3 focused on outlining the theoretical foundation on which the design and analysis of the present study was based. I continue this emphasis in the present chapter, wherein I describe a system of ideas relative to angle measure and sine and cosine functions. These ways of knowing constituted an aspect of the interpretative lens through which I perceived David's actions and utterances throughout the series of TBCIs.

Constructing a model of David's mathematical ways of understanding and ways of thinking relative to sine and cosine functions required that I present him with tasks⁴⁹ designed to reveal, to the greatest extent possible, these ways of understanding and ways of thinking. My design of such tasks relied heavily upon my interpretation of Piaget's notion of abstraction, which I discussed in Chapter 3.

Piagetian abstraction has clear implications as a principle of curriculum design. If a curriculum designer follows Piaget in considering action the catalyst for knowledge development, then he or she seeks to design mathematics tasks to engage learners in the types of actions that promote their construction of desired understandings, as well as place them in situations that foster abstractions from these actions. Determining the actions most propitious for achieving desired understandings results from a process called *conceptual analysis* (Thompson, 2008). A conceptual analysis engages the curriculum designer in describing the ways of knowing an idea that are most advantageous for

⁴⁹ Following Thompson (1985), I consider curriculum to entail "a collection of activities from which students may construct the mathematical knowledge we want them to have" (p. 191). Therefore, I hereafter use "activities," "tasks," and "curriculum" interchangeably.

individuals' subsequent learning, discerning the actions and abstractions from these actions the curriculum designer expects will allow the learner to construct such desired understandings, and proposing activities, didactic objects, and/or conceptualizing conversations that are likely to engender such actions and abstractions.

In the pages that follow, I outline a conceptual analysis of sine and cosine functions. I present my conceptual analysis in four sections corresponding to the four topics that comprise the content of Module 8 of the *Pathways Algebra II* curriculum relevant to sine and cosine functions: (1) angle measure, (2) outputs of the sine and cosine function, (3) graphical representations of sine and cosine functions, and (4) period. In particular, I expound upon the following for each of these four topics:

- What are the meanings that constitute a mature understanding of the concept?
- What are the mental actions in which I suspect a learner must engage, and what are the abstractions upon these actions that I hypothesize a learner needs to make, to construct the meanings that I intend to promote?
- What experiences (i.e., participation with activities and didactic objects) are likely to engender such abstractions?

By addressing these three points for each of the four topics of Module 8 pertaining to sine and cosine functions, I hope to satisfy Thompson's (1985) recommendation for curriculum design: "We must make explicit the nature of the knowledge that we hope is constructed and make a case that the chosen activities will promote its construction" (p. 192).

Angle Measure

An angle is a geometric object (two rays extending from a common endpoint) with one particular measurable attribute of interest: its “openness.” One may understand quantifying the openness of an angle as measuring the length of the arc of a circle, centered at the angle’s vertex, that the angle subtends (see *Figure 21*). For the measure of the angle to be independent of the size of the circle, this arc length must be measured in units that covary with the subtended arc length so that the ratio of *subtended arc length* to *unit length* is always constant for an angle with a fixed amount of openness. This is to say that the unit of measure must be proportional to the subtended arc length and, by extension, the circumference of the circle (Thompson, Carlson, & Silverman, 2007).

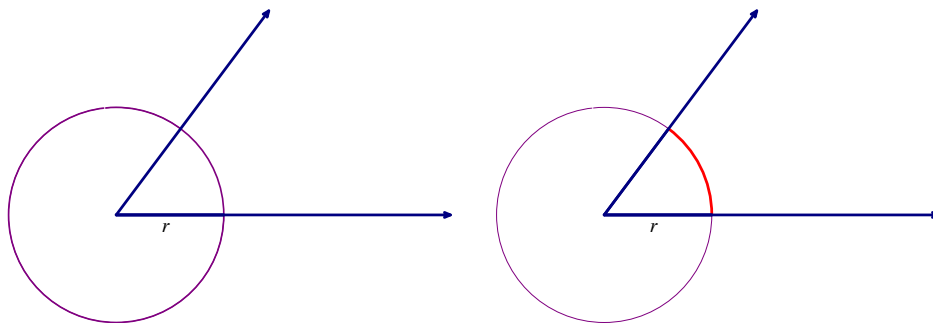


Figure 21. Angle measure as subtended arc length.

Understanding the need for the unit of measure to be proportional to the circumference of the circle that contains the subtended arc involves realizing that a fixed angle always subtends the same fraction of the circumference of *any* circle centered at the vertex of the angle (Moore, 2010). Suppose a fixed angle subtends a/b^{th} of the circumference of a particular circle centered at the vertex of the angle where a is the length of the subtended arc measured in some unit of length m (i.e., $S/m = a$ where S is the length of the subtended arc and S and m are measured in the same linear unit) and b is

the circumference of the circle measured in the same unit of length m (i.e., $C/m = b$ where C is the circumference of the circle and C and m are measured in the same linear unit). For the measure of the angle, a , not to depend on the size of the circle, the circumference of all circles centered at the vertex of the angle must be b times as long as the unit of length m . This implies that the circumference needs to be measured in a unit that keeps its “length” measured in this unit invariant. This is to say that the unit of length m must be proportional to the circumference of the circle. For example, an angle with a measure of one *radian* subtends an arc that is $1/2\pi^{\text{th}}$ times as long as the circumference of *any* circle centered at the vertex of the angle. Similarly, an angle with a measure of one *degree* subtends an arc that is $1/360^{\text{th}}$ times as long as the circumference of *any* circle centered at the vertex of the angle.

A common unit of measure for angles, and the one I’ll use almost exclusively in the pages that follow, is *radians*. Conceptualizing angle measure in units of *radians*, as opposed to radii or radius lengths, entails understanding that the measure of an angle in radians is an equivalence class of arc lengths, each measured in units of the length of the radius of the arc that the angle subtends; that is, angles measured in radians are an equivalence class of the multiplicative comparison of a subtended arc length and a radius length (Thompson, 2008; Moore, 2014). Similarly, conceptualizing angle measure in units of degrees involves understanding that the measure of an angle in degrees is an equivalence class of arc lengths, each measured in units that are $1/360^{\text{th}}$ of the circumference of the circle centered at the vertex of the angle that subtends the arc length.

In summary, my vision of a mature conceptualization of angle measure involves the following:

- (1) Understanding that one can quantify the “openness” of an angle by measuring the length of the arc of a circle centered at the vertex of the angle that the angle subtends;
- (2) Understanding that any particular angle subtends the same fraction of the circumference of all circles centered at the vertex of the angle;
- (3) Understanding that the unit with which to measure this subtended arc length must be proportional to the circumference of the circle centered at the vertex of the angle so as to make the size of the circle inconsequential; and
- (4) Understanding angles measured in radians and degrees as equivalence classes of arc lengths, measured in units of the length of the radius of the circle centered at the vertex of the circle and measured in units of $1/360^{\text{th}}$ of the circumference of the circle centered at the vertex of the angle respectively.

I conjecture that the process by which one constructs these four understandings involves a number of empirical, pseudo-empirical, reflecting, and reflected abstractions. I now turn to describing the abstractions, and the mental actions that constitute them, that I anticipate might allow someone to construct the aforesaid understandings.

Conjectured Role of Abstraction on Conceptualizing Angle Measure

I hypothesize that coming to understand that one may quantify the openness of an angle by measuring the length of the arc the angle subtends involves a pseudo-empirical abstraction in which the object the subject acts upon is an angle and the action the subject engages in is varying the openness of the angle.⁵⁰ From this action, the subject may pseudo-empirically abstract the property that any point on the varying ray of the angle

⁵⁰ Acting upon an angle of course requires some sort of manipulative or geometric visualization software.

traces out an arc, or a portion of the circumference of a circle centered at the vertex of the angle, that he⁵¹ may then recognize as providing a means by which to quantify the openness of the angle. Achieving this pseudo-empirical abstraction involves the following mental actions: (1) visualizing one ray of an angle—which initially extends from its vertex in the same direction as the initial ray so that the angle initially has no openness—move in the counterclockwise direction; (2) focusing on a specific point on the varying ray of the angle; (3) noticing that when the varying ray of the angle moves “all the way around,” this specific point traces out a circle whose center is at the vertex of the angle; (4) recognizing that when the varying ray of the angle moves “part of the way around,” the specific point on the varying ray of the angle traces out a portion of the circle’s circumference; and (5) noticing that the openness of the angle covaries with the portion of the circle’s circumference traced out by the point on the varying ray of the angle. Constructing this covariation in (5) allows one to see that the openness of the angle is in direct correspondence with the portion of the circle’s circumference that the angle subtends, and thus provides a means by which one may measure, or quantify, the openness of the angle. That is, through the process of varying the size of an angle, the subject has the opportunity to transform an elusive quantity like “openness” into a concrete quantity that he can imagine measuring in some linear unit.

As I explained above, to make the measure of an angle irrespective of the size of the circle centered at its vertex, one must measure the arc length subtended by the angle in units that are proportional to the circumference of the circle. There are many ways one might construct this understanding. In what follows, I describe the abstractions, as well as

⁵¹ Since this conceptual analysis informed my design of tasks that I used in a series of task-based clinical interviews with my research participant, David, I use the male pronoun throughout this chapter.

the mental actions that constitute them, that lead to what is, in my view, a particularly efficacious way of understanding the need for the subtended arc length to be measured in units that are proportional to the circumference of the circle centered at the vertex of the angle.

I anticipate that coming to understand the need for the unit of measure to be proportional to the circumference of the circle involves first that the subject engage in a pseudo-empirical abstraction (P-EA 1) whereby he varies the size of the circle centered at the vertex of a fixed angle and abstracts the property that the subtended arc length appears to be the same fraction of the circumference of the circle. In particular, I expect that this pseudo-empirical abstraction relies upon the subject's recognition that, as he varies the size of the circle centered at the vertex of the angle, the circumference always appears to be the same number of times as large as the length of the subtended arc. This recognition involves simultaneously coordinating the lengths of the circumference and subtended arc and comparing these lengths multiplicatively.

Additionally, I expect that the subject will need to engage in a second pseudo-empirical abstraction (P-EA 2) whereby he notices that while the size of the circle centered at the vertex of the angle varies—causing the circumference and subtended arc length to vary in tandem—the “openness” of the angle does not change. However, if the openness of the angle changes, the fraction of the circle's circumference that the angle subtends changes so that the openness of the angle and the ratio of subtended arc length to circumference are in direct correspondence. Achieving this understanding involves the subject in attending to how the openness of the angle and the ratio of subtended arc length to circumference covary. In particular, the subject must coordinate the amounts of

change of these quantities (MA 3) and pseudo-empirically abstract the property that the subtended arc length changes by equal amounts every time the openness of the angle changes by the same amount.⁵²

These two pseudo-empirical abstractions (P-EA 1 and P-EA 2) allow the subject to make the following complementary deductions, each of which rely on the subject having made a reflecting abstraction⁵³:

- (D1) Since, for an angle with a fixed openness, the ratio of subtended arc length to circumference (i.e., the length of the subtended arc measured in units of the circumference) is constant (P-EA 1), and since the openness of the angle varies directly with the ratio of subtended arc length to circumference (P-EA 2), one may quantify the openness of the angle by measuring the length of the subtended arc in units of the circumference of the circle centered at the vertex of the angle.
- (D2) Since the length of the arc that a fixed angle subtends covaries with the circumference of the circle centered at the vertex of the angle in such a way that the subtended arc length (measured in some standard linear unit) is always the same fraction of the circumference of this circle (measured in the same standard linear unit), one may quantify the openness of the angle by measuring the subtended arc length in any unit that covaries with the circumference so that the subtended arc length measured in this unit is always constant.

⁵² In this context, “amount” is conceptualized qualitatively and not numerically.

⁵³ These deductions involve reflecting abstractions because they require the subject to construct a mental representation of the actions from which the subject made P-EA 1 and P-EA 2.

One way to justify the condition that a unit of angle measure must correspond to a length that is proportionally related to the circumference of the circle centered at the vertex of the angle is to construct symbolic representations for the relationships between subtended arc length, circumference, and openness of an angle constructed through P-EA 1, D1, and D2. Constructing symbolic representations for these relationships allows one to operate on these symbolic representations to define covariational relationship between circumference and a unit of angle measure.

Since a fixed angle subtends an arc that is always the same fraction of the circle's circumference (P-EA 1), we have $S/C = k$ where k is a constant, S represents the length of the subtended arc in some standard linear unit, and C represents the length of the circumference of the circle in the same standard linear unit. Similarly, since, via the second deduction (D2) from P-EA 1 and P-EA 2, the unit of measure for the subtended arc length must covary with the length of the subtended arc so that the subtended arc length measured in this unit is always constant, we have $S/m = p$ where p is a constant and m is the length of the unit of measure. Using these two equations, one can express the relationship between m and C as follows:

$$\frac{S}{C} = k \Rightarrow S = Ck$$

$$\frac{S}{m} = p \Rightarrow m = \frac{S}{p}$$

$$\text{Thus, } m = \frac{Ck}{p} = \left(\frac{k}{p}\right)C \text{ by } *.$$

This computation shows that the unit of length m used to measure the subtended arc length S must covary with the circumference C in such a way that the unit of length m is

proportional to the circumference of the circle C centered at the vertex of the angle (with constant of proportionality k/p).

Symbolic manipulation is, of course, not the only, or perhaps even most desirable, way that one may understand that a suitable unit with which to measure the openness of an angle must correspond to a length that is proportional to the circumference of the circle centered at the vertex of the angle. The subject may imagine the arc length subtended by a fixed angle being measured in a unit of some length by multiplicatively comparing the magnitude of the subtended arc length with the magnitude of the unit. When the subject varies the size of the circle centered at the vertex of a fixed angle he may reflectingly abstract the property that, since the measure of the subtended arc must remain the same (P-EA 2), the unit of measure must be the same fraction of the subtended arc.⁵⁴ This further implies that since the length of the subtended arc is always the same fraction of the circumference of the circle (P-EA 1), the unit of measure is the same fraction of the circumference of the circle that includes the subtended arc (see *Figure 22*). It is noteworthy that this process of concluding that the unit of measure for the subtended arc must be proportional to the circumference of the circle centered at the vertex of the angle, as with the symbolic method discussed above, relies upon the subject having made P-EA 1 and P-EA 2.

⁵⁴ This constitutes a reflecting abstraction for the following reason: Constructing this understanding relies upon the subject imagining the length of the subtended arc being measured in some linear unit of length m and visualizing this subtended arc length vary as the size of the circle centered at the vertex of the angle varies. As the length of the subtended arc varies, the subject imagines the unit of measure varying in tandem so as to make the subtended arc length measured in this unit invariant (i.e., the subtended arc length remains the same number of times as large as the length of the unit). What is noteworthy is that one constructs this image in thought. Therefore, one must differentiate the actions on which P-EA 1 and P-EA 2 are based, namely varying the size of the circle centered at the vertex of the angle and attending to how the subtended arc length varies, from their effect and project these actions to the level of representation.

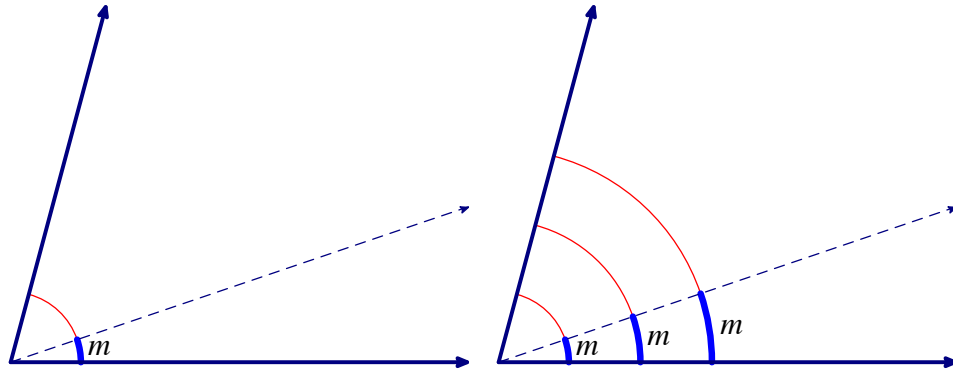


Figure 22. Unit of measure proportional to circumference.

I have now outlined two ways in which I envision one coming to understand that a unit of angle measure must be proportional to the circumference of the circle centered at the vertex of the angle. It is worth noting that I have intentionally not opted for the equivalent, and more accessible, condition that the unit of measure must be proportional to the subtended arc length. I insist on supporting my subject in constructing the understanding that any unit of angle measure must be proportional to the *circumference*, and not just the subtended arc length, because the most obvious quantity that is proportional to the circumference of every circle is its radius ($C = 2\pi r$). This makes *radians* as a unit of measure very natural. This is why I go the extra step to establish the proportional relationship between the unit of measure and the circumference of the circle, not just the proportionality of the unit of measure and the subtended arc length.

Understanding radians and degrees as units of measure for the openness of an angle involves one in recognizing that these units of measure correspond to lengths that are proportional to the circumference of the circle centered at the vertex of the angle. The portion of any circle's circumference that a particular angle subtends will always be the same number of times as large as the radius of the circle (in the case of radians) or $1/360^{\text{th}}$

of the circumference of the circle (in the case of degrees). I anticipate that understanding degrees as an appropriate unit of measure for the openness of angles involves assimilation, assuming the subject has a sufficiently robust proportionality scheme. A degree, as corresponding to $1/360^{\text{th}}$ of the circumference of a circle, is proportional to the circumference of the circle by definition, but only to those individuals who understand that if quantity A is always the same fraction of quantity B , then quantities A and B are proportionally related. Similarly, I propose that understanding radians as an appropriate unit of angle measure involves assimilation if one knows that the circumference of a circle is 2π times as large as the radius (i.e., $C = 2\pi r$) and has an understanding for what it means for two quantities to be proportionally related. If one does not have a proportionality scheme that allows him to assimilate the proportional relationship between the radius and the circumference of a circle, then I hypothesize that this understanding may emerge from a pseudo-empirical abstraction whereby one places radius lengths end to end along the circumference of circles of various sizes and notices that the circumference is always 2π , or approximately six-and-a-third, times as large as the radius, regardless of the size of the circle.

If one has constructed the understanding that a unit of measure for the openness of an angle must be proportionally related to the circumference of the circle centered at the vertex of the angle, and recognizes that radians and degrees satisfy this condition, then I propose that he has constructed angles measured in radians and degrees as equivalence classes of arc lengths. To say that a unit of length m with which to measure the openness of an angle is proportionally related to the circumference of the circle C centered at the

vertex of the angle is to acknowledge that as C varies⁵⁵, m varies in such a way so as to make the ratio C/m constant. Therefore, since the subtended arc length S is always the same fraction of the circumference of the circle of which it is a part (P-EA 1), the measure of the angle, calculated by S/m , is necessarily constant.^{56, 57} Thus, the measure of any angle in radians or degrees is an equivalence class of arc lengths measured in units of the radius of the circle centered at the vertex of the angle or $1/360^{\text{th}}$ of the circumference of this circle respectively.

Smooth Variation of Angle Measure

Understanding the outputs of sine and cosine functions involves one in constructing a representation of the process whereby he obtains output values of these respective functions from a *continuum* of input values without having to go through the process of obtaining output values in action. Accordingly, I designed and selected tasks to support David in conceptualizing the measure of an angle varying continuously by

⁵⁵ Imagining the circumference of the circle centered at the vertex of the angle varying is crucial. Claiming that two quantities whose values do not vary are proportionally related is trivially true (k/p is constant for constants k and p).

⁵⁶ A brief proof: Let S represent the length of the subtended arc and let C represent the circumference of the circle. Also, let m be the length of the unit of measure for the subtended arc length. $S = kC$ for some constant k via P-EA 1. Also, if m is the length of the unit of measure for the subtended arc length, m and C are proportionally related. That is, $m = qC$ where q is a constant. Therefore, we have

$$\frac{S}{m} = \frac{kC}{qC} = \frac{k}{q}.$$

Thus, the measure of the angle in units of length S/m is constant. \square

⁵⁷ We can see that this construction, in the way that I have presented it, relies upon reversible mental operations, in the Piagetian sense. While constructing the condition that a unit of measure for the openness of an angle must satisfy, I hypothesized that one needs to deduce, from P-EA 1 and P-EA 2, that the subtended arc length S measured in units of length m must remain invariant. From representing this result, along with P-EA 1, symbolically, I argued that one could derive the condition that m must be proportional to the circumference of the circle centered at the vertex of the angle. In constructing angle measure in radians and degrees as equivalence classes of arc lengths in the way that I have presented, one assumes the proportionality between m and C and deduces that S measured in units of m is constant. It is important to note that one need not represent this construction symbolically.

providing opportunities for him to see the openness of an angle as a representation of the process by which he measures the angle, and for him to imagine this openness, and thus the measure of the angle, varying *smoothly* (Castillo-Garsow, Johnson, & Moore, 2013). Imagining the smooth variation of the openness of an angle measured in radians involves understanding that, for an angle whose openness is varying, the arc length subtended by the angle contains some number of full radius lengths, and a continuously varying portion of a radius length. In general, conceptualizing smooth variation in the value of any quantity involves imagining the value of the quantity varying by infinitesimal amounts over very small intervals of conceptual time while understanding that variation occurs within each interval of conceptual time as well (Thompson, 2011).

Conjectured Role of Abstraction on Conceptualizing Smooth Variation of Angle Measure

I anticipate that visualizing the measure of an angle varying smoothly presupposes a reflecting abstraction on the act of measuring an angle in radians. Once the subject has repeatedly engaged in the act of measuring angles in radians by multiplicatively comparing the length of an arc that an angle subtends with the radius of the circle centered at the vertex of the angle of which the subtended arc is a part (while understanding that this multiplicative comparison remains invariant for a multiplicity of subtended arcs), he is positioned to differentiate this action from the result it produces (obtaining a value for the openness of an angle). It is important to note that the act of measuring an angle in radians is itself a sequence of actions that the subject must coordinate before he can project these actions to the level of representation. This sequence of actions includes: (1) imagining a portion of a circle of some radius centered

at the vertex of the angle being subtended by the angle, and (2) multiplicatively comparing this subtended arc length and the radius of the circle. When the subject coordinates the actions that constitute measuring an angle and projects them to a higher cognitive level, he can imagine an angle whose openness is continually varying as always having a measure, which, by necessity, also varies. Additionally, since the subject has constructed the action at the level of representation, he is able to do this without having to pause the variation to consider a specific angle.

I now turn to outlining the instructional sequence for angle measure. I designed the tasks in all four instructional sequences to accomplish two objectives: (1) to reveal David's ways of understanding and ways of thinking relative to the topic of each instructional sequence, and (2) to support David in engaging in the abstractions I hypothesize are involved in coming to understand the ideas in the way I describe in my conceptual analysis. Recall that, because of my intention to understand how a teacher's *powerful* mathematical ways of understanding and ways of thinking are conditioned by the teacher's image of instructional constraints, I designed my tasks to advance David's thinking, in addition to eliciting observable products of his reasoning. As I mentioned in the previous chapter, it is of little consequence to understand how a teacher's weak mathematical meanings are compromised by his image of instructional constraints since students are unlikely to benefit in significant ways from the learning experiences that derive from a teacher's weak mathematical knowledge.

Before proceeding, I must point out that I designed all of the tasks I present in this chapter *prior* to conducting the series of TBCIs in the first phase of data collection. I added tasks throughout the series of TBCIs based on my emerging model of David's

mathematical ways of understanding and ways of thinking. I present these additional tasks, as well as my rationale for including them, in the next chapter.

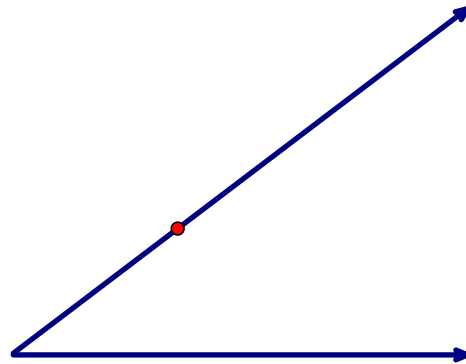
Instructional Sequence for Angle Measure

I designed the following instructional sequence to support David in constructing the ways of understanding angle measure discussed above. In particular, the tasks in this instructional sequence sought to engage David in the types of actions, and engender abstractions upon these actions, that might allow him to construct the aforementioned meanings of angle measure. I present each task of the instructional sequence individually and describe the abstractions I designed each task to promote.

David completed all tasks in this first instructional sequence in *Geometer's Sketchpad* (Jackiw, 2011). Some tasks prompted David to act upon geometric objects using *Geometer's Sketchpad* and other tasks involved the use this program to present didactic objects designed to facilitate conversation. I therefore spent some time prior to this first instructional sequence getting David acquainted with *Geometer's Sketchpad* and its features. In what follows, I provide each task as well as what David saw on the computer screen when I presented the task.

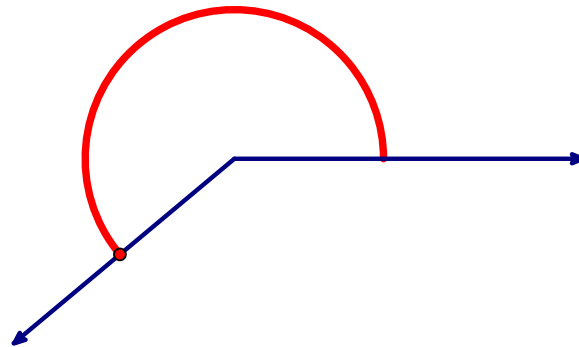
Table 15

Task 1 of Instructional Sequence for Angle Measure



- (a) What do you see when you look at this picture?
- (b) Moving this red point changes the size of the angle. What are some things that you notice while moving this point?

Animate




- (c) [Press the “Animate” button] What do you see this doing?
- (d) How does the “openness” of the angle change with the portion of the circle subtended or cut off by the angle?
- (e) How might you measure the amount of openness between the two rays?


I designed this first task to engender the pseudo-empirical abstractions I anticipate are necessary for one to conceptualize angle measure as an arc length. In this task, David was able to change the openness of the angle by dragging one point on one ray of the angle or by pressing the “Animate” button. I designed Part (a) to reveal how David conceptualizes the geometric object of an angle. The purpose of Parts (b)-(d) of this task

was to support David in noticing that the red point traces out a portion of a circle while simultaneously changing the openness of the angle. In other words, I intended this task to enable David to recognize that the arc generated by moving the red point and the openness of the angle change together so that the openness of the angle is in direct correspondence with the arc length that the angle subtends. This pseudo-empirical abstraction, I expect, would allow David to answer Part (e) by recognizing that he may quantify the openness of an angle by measuring the subtended arc length.

Table 16

Task 2 of Instructional Sequence for Angle Measure

Subtended Arc 

Circumference 

(a) By clicking on the “Vary Radius” button or dragging the red dot, the size of the circle changes. The solid orange line and the dashed green line represent the subtended arc and the circumference of the circle centered at the vertex of the angle respectively. As the radius of the circle varies, compare the length of the subtended arc and the length of the circumference. What do you notice? Represent your observation mathematically. Does the relationship between the subtended arc length and the circumference change as the openness of the angle changes? Explain.

(b) Press the “Vary Radius” button. What quantities are varying in this situation? What quantities are not varying? How might you measure this angle?

(c) What condition must a unit of measure for the openness of an angle satisfy? Explain.

Vary Radius

Hide Radius

Hide Measurements

Circumference = 24.27 cm
Subtended Arc Length = 8.09 cm

Radius ———

Subtended Arc ———

Circumference - - - - -

(d) Press the “Show Measurements” button. We see that the length of the subtended arc is 8.09 cm and the length of the circumference is 24.47 cm. What is the length of the subtended arc if the circumference were changed to 16.8 cm?

(e) If 16 quips rotate a circle, what is the measure of an angle (in quips) that has a circle with a circumference of 19.4 cm centered at its vertex and subtends an arc length of 7.6 cm?

(f) Let m be the length of a unit of measure for the openness of an angle. Suppose the circumference of the circle C centered at the vertex of the angle is 6 times as large as m . Also, suppose that the subtended arc length S is 0.4 times as large as C . What is the measure of the angle in units of length m ? Explain.

My intention with Part (a) of Task 2 was to engage David in the action of determining the relative size of the arc length subtended by the angle with respect to the circumference of the circle centered at the vertex of the angle, and to pseudo-empirically abstract a property of the relative size of these two lengths—namely that the subtended arc length is always the same fraction, or percentage, of the circumference (P-EA 1). Additionally, Part (a) provided an occasion for David to pseudo-empirically abstract the property that when the openness of the angle changes, the relationship between the subtended arc length and the circumference change accordingly. The purpose of the first

two questions in Part (b) was to support David in pseudo-empirically abstracting the property that the subtended arc length and circumference of the circle vary while the openness of the angle remains fixed (P-EA 2). Before answering the third question in Part (b), “How might you measure this angle?” David had the opportunity to realize the following: (1) The subtended arc length S and the circumference C are varying such that the subtended arc length is always the same fraction of the circumference; (2) The openness of the angle is not varying; (3) If the openness of the angle changes, the ratio S/C changes accordingly. I designed the question “How might you measure this angle?” in Part (b) to support David in reflectingly abstracting something like the following (see D1 on page 155): “Since the openness of the angle and the ratio S/C are invariant, and since the openness of the angle covaries with the ratio S/C , perhaps I can quantify the openness of the angle by measuring the length of the subtended arc in units of the circumference of the circle.”

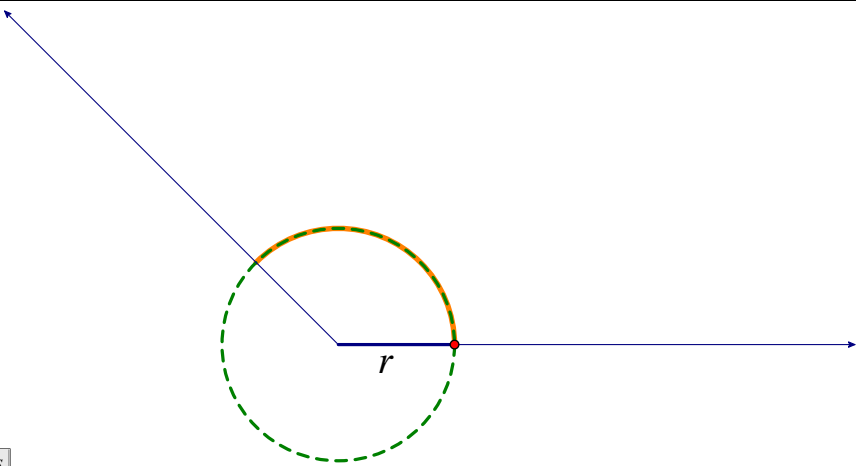
Part (c) sought to engender an additional deduction (see D2 on page 155) from P-EA 1 and P-EA 2 whereby David may conclude that measuring the angle involves measuring the arc length that the angle subtends in units that covary with the circumference so that the subtended arc measured in this unit is always constant.⁵⁸ Upon probing David’s response to Part (c), I hoped to support him in engaging in a reflecting abstraction whereby he may construct the understanding that the unit of measure for the openness of the angle must be proportional to the circumference of the circle centered at the vertex of the angle. Parts (d) and (e) provided contexts in which David may utilize the understanding that I designed Part (a) to engender: that the arc length an angle subtends is

⁵⁸ I do not mean to convey that the unit of measure is constant but rather that the unit of measure and the subtended arc length vary together so that the measure of the subtended arc length in this unit is constant.

always the same fraction of the circle's circumference. Finally, I intended Part (f) to promote a reflected abstraction since David must consciously symbolize and operate on the actions at the level of representation that I designed Part (c) to support. I expected that providing an occasion whereby David may symbolize the relationship between the relative size of the circumference of the circle with respect to the subtended arc length would allow him to reify this action into a form that he could then use as the object of thought at this higher cognitive level without having to reengage in the coordinated actions the symbols represent.

Table 17

Task 3 of Instructional Sequence for Angle Measure



The diagram shows a circle with a center point. A blue radius line extends from the center to the right, labeled with the variable r . A blue angle is formed by this radius and another blue radius extending towards the top-left. A solid orange arc is drawn along the top-right part of the circle, subtending the angle. A dashed green line follows the entire circumference of the circle. Below the diagram is a legend with three items: 'Vary Radius' in a button, 'Radius' with a blue line segment, 'Subtended Arc' with an orange line segment, and 'Circumference' with a dashed green line segment.

Radius —————

Subtended Arc —————

Circumference - - - - -

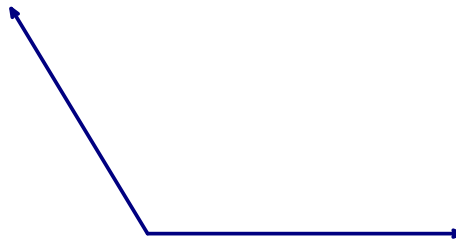
(a) By clicking on the “Vary Radius” button, the size of the circle changes. The solid orange line and the dashed green line represent the subtended arc and the circumference of the circle centered at the vertex of the angle respectively. The solid blue line represents the length of the radius. Is the radius of the circle an appropriate unit for measuring the angle? Explain.

- (b) Suppose you are discussing this applet with a colleague. As you increase the radius of the circle she observes, “The arc length is getting longer so the angle is getting bigger.” Respond to your colleague’s observation.
- (c) Suppose you are discussing this applet with another colleague. As you vary the radius of the circle, he remarks, “It stays the same!” What are the different things he could mean by *it*? Name as many as you can.
- (d) Suppose your colleagues are looking at a picture like this one on their computers except their angles are different sizes and the circles centered at the vertex of their angles are different sizes. What could you tell them to do so that their angle is approximately the same measure as yours?

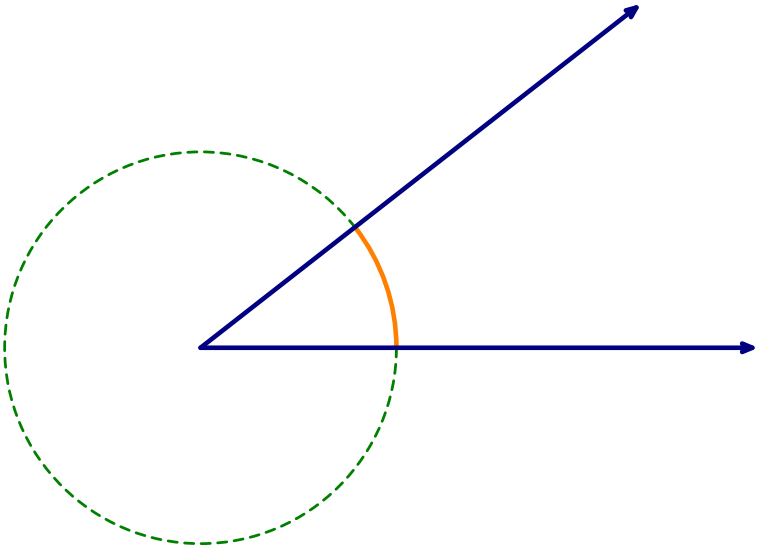
Table 18

Task 4 of Instructional Sequence for Angle Measure

- (a) What does it mean to have an angle of 3.92 radians?
(b) What does it mean to have an angle of 51.7 degrees?
Using the features of *Geometer’s Sketchpad*, determine the following:



- (c) Measure this arbitrary angle (pictured above) in radians.
(d) Measure this arbitrary angle (pictured above) in degrees.
(e) Construct an angle with a measure of 1.6 radians.
(f) Construct an angle with a measure of 34 degrees.



Circumference = 47.21 cm

(g) The measure of the angle above is 38 degrees and the circumference of the circle centered at the vertex of the angle is 47.21 cm. What is the length of the subtended arc?

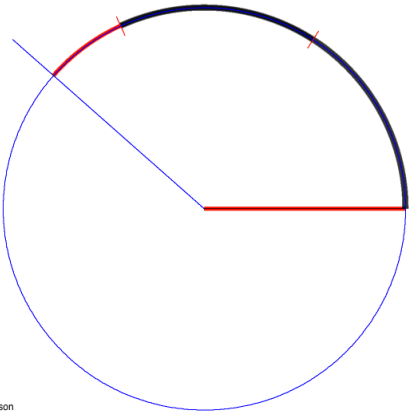
As I mentioned above, understanding angles measured in radians and degrees as equivalence classes of arc lengths involves one in assimilating these units of measure as satisfying the condition that a unit of measure for the openness of an angle must be proportional to the circumference of the circle centered at the vertex of the angle.

Accordingly, the purpose of Task 3(a) was to assess whether David recognized that the radius of the circle centered at the vertex of the angle satisfies the conditions he proposed as a solution to Task 2(c). I designed Tasks 3(b) and 3(c) to reveal the extent to which David understood the idea that the unit of measure for the openness of an angle must covary with the circumference of the circle centered at the angle's vertex in such a way so as to make the length of the subtended arc constant when measured in this unit. The purpose of Task 3(d) was to assess whether David had constructed an understanding for angle measure that does not depend on the size of the circle centered at the vertex of the

angle. Finally, I designed Task 4 is to reveal whether David had constructed meanings for radians and degrees as units of measure for angles in a sufficiently robust way so as to allow him to use the features of *Geometer's Sketchpad* to measure an arbitrary angle in these units and construct angles of specific measures. Together, I intended Tasks 3 and 4 to reveal the extent to which David had constructed a scheme to which he may assimilate radians and degrees as units of angle measure.

Table 19

Task 5 of Instructional Sequence for Angle Measure



(2011) Pal Thompson

What are you seeing this doing? What are the black segments showing? What are the red segments showing? Now draw another circle with a different radius and different center. Construct the same angle on the new circle.

To support David in conceptualizing angle measure varying continuously, I presented him with an animation of an angle measure varying from a measure of zero radians to a measure of 2π radians (Thompson, 2011). The angle traced out an arc length of some number of whole radius lengths, represented by black segments, and some fraction of a radius length, represented by a red segment. The purpose of Task 5 was to determine if David visualized angle measure varying continuously or in discrete intervals.

This task provided a context in which David could engage in reflecting abstraction on the action of measuring angles from Task 4.⁵⁹

Output of the Sine and Cosine Functions

Before one is able to imagine the association of input and output quantities of the sine and cosine functions, he or she must construct input and output quantities to associate. If one has constructed the aforementioned meanings of angle measure, conceptualizing the output of the sine (cosine) function involves recognizing that, for any angle measure in standard position, there exists a vertical (horizontal) distance that the terminus of the subtended arc length is above (to the right of) the horizontal (vertical) diameter of the circle centered at the vertex of the angle of which the subtended arc is a part (see *Figure 23*).^{60, 61} Moreover, understanding the output of the sine (cosine) function involves recognizing that one can obtain this vertical (horizontal) distance only if an angle measure has been specified. In other words, conceptualizing the outputs of the sine and cosine functions relies upon one's recognition that obtaining values for these outputs is the result of a *process* that begins with knowing an angle measure. Accordingly, I designed my instructional sequence to engage David in the actions involved in determining the vertical and horizontal distances that represent the outputs of the sine and cosine functions respectively when given the measure of an angle in radians. Additionally,

⁵⁹ I discuss what is involved in this reflecting abstraction the section “Conjectured Role of Abstraction on Conceptualizing Smooth Variation of Angle Measure” above.

⁶⁰ Note that one ray of the angle must lie on the horizontal diameter as indicated in Figure 23. Hereafter my use of “angle” assumes this property.

⁶¹ Hereafter I use “vertical distance above the horizontal diameter” to refer to the output quantity of sine, and “horizontal distance to the right of the vertical diameter” to refer to the output quantity of cosine.

my instructional sequence presents tasks designed to focus David's attention on the relationship between the inputs and outputs of the sine and cosine functions.

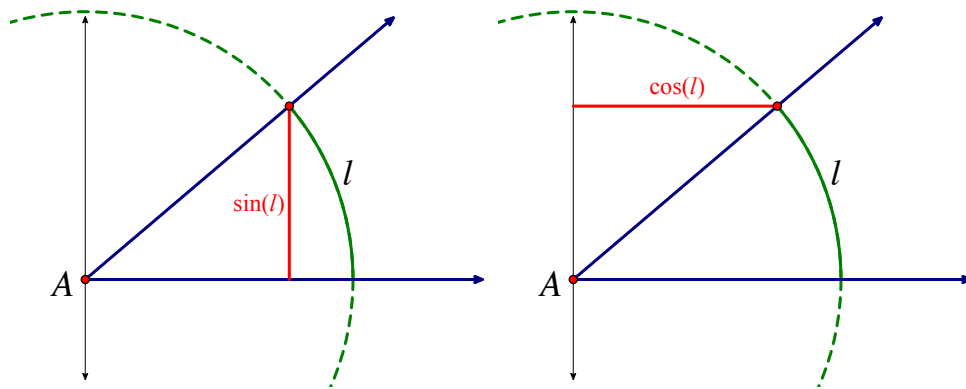


Figure 23. Output quantities $\sin(l)$ and $\cos(l)$ where l is a subtended arc length measured in units of the radius.

Conceptualizing the *quantities* that the outputs of the sine and cosine functions represent (a vertical and horizontal displacement respectively) involves not just identifying a measurable attribute of a geometric object but anticipating a means by which one may *quantify*, or obtain a value for, this attribute (Thompson, 1990, 1994, 2007). Constructing a quantity therefore requires that one conceptualize an appropriate unit of measure. Accordingly, conceiving the quantities that the outputs of the sine and cosine function represent involves conceptualizing the radius of the subtended arc as a unit of measure for the vertical and horizontal lengths that respectively correspond to the output quantities of the sine and cosine functions. In particular, one must see that measuring these vertical and horizontal lengths in units of the radius makes the value of these quantities independent of the size of the circle of which the subtended arc is a part. When one conceptualizes the outputs of sine and cosine in this way, I argue that he or she implicitly understands that the output of the sine (cosine) function constitutes an

equivalence class of the multiplicative comparison of the vertical (horizontal) length that the terminus of the subtended arc is above (to the right of) the horizontal (vertical) diameter of the circle centered at the vertex of the angle, and a radius length.

Conjectured Role of Abstraction on Conceptualizing the Output of the Sine and Cosine Function

In what follows, I outline my perspective regarding the role of abstraction on one's construction of the output quantity of the sine function. I focus exclusively on sine because the abstractions I expect being involved in one's construction of the output of the cosine function are analogous.

I hypothesize that one must engage in pseudo-empirical abstraction to understand that for any angle measure there exists a vertical distance that the terminus of the subtended arc is above the horizontal diameter of the circle centered at the vertex of the angle. Achieving this pseudo-empirical abstraction involves one in varying the openness of an angle, attending to distance the terminus of the subtended arc is above the horizontal diameter of the circle, and abstracting the property that this vertical distance varies with the angle measure, and thus exists for an angle with any amount of openness.

I further anticipate reflecting abstraction being involved in understanding that obtaining a value for the distance that the terminus of the subtended arc is above the horizontal diameter of the circle is the result of a process, or sequence of actions, in which one starts with an angle and ends with this vertical distance measured in units of radius lengths. Understanding the measure of this vertical distance as being the result of a process that begins with knowing an angle measure is important if one is to conceptualize this vertical distance as the output of the sine function (i.e., the result of a process). The

sequence of mental actions that characterize this process are as follows: (1) imagining a fixed angle with a particular amount of openness; (2) imagining a circle, or a portion of a circle that the angle subtends, with a particular circumference centered at the vertex of the angle; (3) visualizing the distance that the terminus of the subtended arc is above the horizontal diameter of the circle; and (4) measuring this vertical distance in units of the radius of the circle by multiplicatively comparing the radius and this vertical distance. After the subject has repeatedly engaged in this sequence of actions—whereby he obtains a value for the vertical distance that the terminus of the subtended arc is above the horizontal diameter of the circle centered at the vertex of the angle—he is positioned to differentiate the process of obtaining this value from the specific length he measured and the value he obtained. By coordinating the sequence of actions involved in measuring this vertical distance, I conjecture that the subject will be able to project this collection of coordinated actions to a higher level of thought where they are represented independently of the effects they produced at the level of material action. After having projected the sequence of coordinated actions to the level of representation, the subject is equipped to understand that for all angles there exists a vertical distance that the terminus of the subtended arc is above the horizontal diameter of the circle centered at the vertex of the angle, and that this vertical distance has a measure, in radius lengths, without having to perform the measurement in action.

Finally, I conjecture that the subject's ability to assimilate the notation $\sin(\theta)$ to a scheme of actions and operations at the reflected level, and to perform operations on this notation, suggests that he has engaged in reflected abstraction. In particular, the subject's ability to assimilate and operate on the symbolic notation $\sin(\theta)$ suggests that he has

constructed a mental image at the reflected level that symbolizes the coordinated actions involved in determining a value for the output quantity of the sine function. This mental image serves as an object upon which the subject is able to act and operate at the level of representation without having to access or engage in the actions that this mental image represents. The subject's ability to act on this representation of coordinated actions suggests that he has achieved a level of coherence at the level of representation that results from reflecting on the products of prior reflecting abstractions.

Instructional Sequence for the Outputs of Sine and Cosine Functions

I designed tasks in this instructional sequence to engage David in the sequence of actions involved in determining the output value of the sine and cosine functions, and to support him in engaging in reflecting abstraction upon this sequence of actions so that he may be able to envision obtaining output values in thought without having to do it in action. Additionally, I designed tasks in this instructional sequence to encourage reflected abstraction by providing occasions for David to assimilate the symbolic notation $\sin(x)$ and $\cos(x)$ in novel contexts. In what follows, I present the task sequence and indicate the abstractions I designed each task to engender.

As with the instructional sequence on angle measure, David used *Geometer's Sketchpad* as an aid to answering most of the tasks in this instructional sequence. Therefore, for the tasks that involved the use of *Geometer's Sketchpad*, I include an image of what David saw on the computer screen while engaged in the task. I presented the tasks that require the use of *Geometer's Sketchpad* in the context provided in Table 20.

Table 20

Context of Tasks in the Instructional Sequence for the Outputs of Sine and Cosine

Functions

Suppose Joe is riding his bike on Euclid Parkway, a perfectly circular road that defines the city limits of Flatville. Ordinate Avenue is a road running vertically (north and south) through the center of Flatville and Abscissa Boulevard is a road running horizontally (east and west) through the center of Flatville [note that Ordinate Avenue and Abscissa Boulevard are perpendicular]. Assume Joe begins riding his bike at the east intersection of Euclid Parkway and Abscissa Boulevard in the counterclockwise direction.

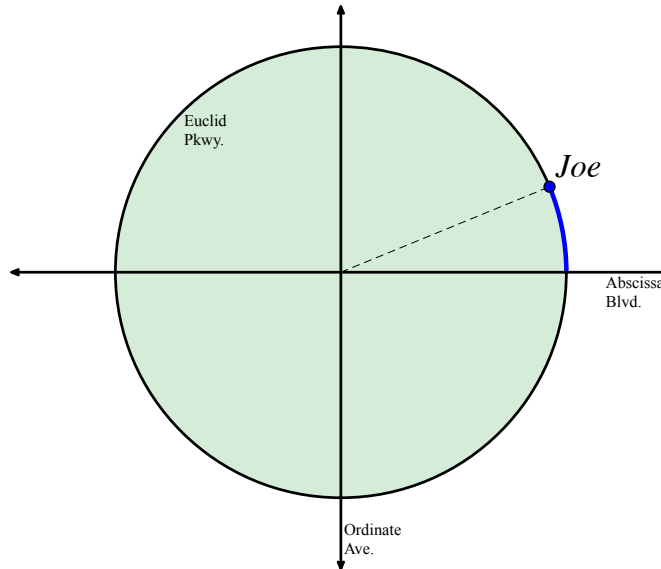


Table 21

Task 1 of Instructional Sequence for the Outputs of Sine and Cosine Functions

- (a) If the radius of Flatville is 10.24 kilometers and if Joe's path has swept out an angle of 1 radian (from the intersection of Ordinate Avenue and Abscissa Boulevard), approximately how many radius lengths is Joe north of Abscissa Boulevard?
- (b) If the radius of Flatville is 7.59 kilometers and if Joe's path has swept out an angle of 1 radian, approximately how many radius lengths is Joe north of Abscissa Boulevard?
- (c) If the radius of Flatville is 9.23 kilometers and if Joe's path has swept out an angle of 1.5 radians, approximately how many radius lengths is Joe east of Ordinate Avenue?
- (d) If the radius of Flatville is 11.47 kilometers and if Joe's path has swept out an angle of 1.5 radians, approximately how many radius lengths is Joe east of Ordinate Avenue?
- (e) If Joe has ridden four times the distance of the radius of Flatville, how many radius lengths is Joe north of Abscissa Boulevard?

- (f) As the angle traced out by Joe's path increases from 0 to $\pi/2$ radians, how does the number of radius lengths that Joe is north of Abscissa Boulevard change? Be as specific as possible.
- (g) As the angle traced out by Joe's path increases from 0 to $3\pi/4$ radians, how does the number of radius lengths that Joe is east of Ordinate Avenue change? Be as specific as possible.

I designed the first task of this instructional sequence to engage David in the sequence of actions involved in determining the output values of the sine and cosine functions⁶² in a novel context so that in subsequent tasks David may be positioned to coordinate these actions, dissociate them from their effect, and then project these coordinated actions to a higher cognitive level where they may be represented in thought independent of their origin in material action. In other words, I designed this task to establish the foundation for future reflecting and reflected abstractions.

Parts (a) and (b), and Parts (c) and (d), of this first task had the additional objective of assessing whether David had constructed the understanding that the output of the sine function (in the case of Parts (a) and (b)) and the cosine function (in the case of Parts (c) and (d)), measured in units of radius lengths, constitute equivalence classes. For instance, should David recognize the radius of Flatville as inconsequential to the solution of these tasks, thereby making Part (b) the same question as Part (a) and Part (d) the same question as Part (c), then it seems possible that he has an intuitive sense that the measure of these quantities constitutes equivalence classes of linear distances measured in units of Flatville's radius. Finally, the purpose of Parts (f) and (g) was to strengthen David's

⁶² It is important to note that while completing this task, David did not initially know that he was determining output values of the sine and cosine functions. Therefore, when I say that I designed this task to engage David in the sequence of actions involved in determining sine and cosine values, I am characterizing the action I designed the task to promote from my perspective, not his.

understanding of how the input and output quantities of the sine and cosine functions are related by prompting him to consider how these quantities change together.

Table 22

Task 2 of Instructional Sequence for the Outputs of Sine and Cosine Functions

- (a) Where is Joe on Euclid Parkway when the angle traced out by his path (in radians) is twice as large as the number of radius lengths that he is east of Ordinate Avenue?
- (b) Where is Joe on Euclid Parkway when the number of radius lengths that he is north of Abscissa Boulevard is $\frac{1}{2}$ times as large as the angle swept out by his path (in radians)?
- (c) Approximately what is the angle swept out by Joe's path when the number of radius lengths that he is north of Abscissa Boulevard is half of the number of radius lengths that he has ridden?
- (d) If Joe is -1 radius length to the east of Ordinate Avenue, approximately what is the angle swept out by Joe's path?

I designed Task 2 to focus David's attention on the simultaneous variation of the input and output quantities of the sine and cosine functions. Solving each part of this task involves comparing the magnitude of the input and output quantities of the sine function (in the case of Parts (b) and (c)) and cosine function (in the case of Parts (a) and (d)) as one varies the value of the input until the values of the input and output quantities satisfy the conditions specified in each task. Attending to the simultaneous variation of the input and output quantities of the sine and cosine functions constitutes an action that, when reflectively abstracted upon, may result in David's recognition that as the value of the input quantity varies continuously, the value of the output quantity varies in tandem.

Table 23

Task 3 of Instructional Sequence for the Outputs of Sine and Cosine Functions

- (a) If Joe's path has swept out an angle of x radians, explain how you would use this program to estimate the number of radius lengths Joe is north of Abscissa Boulevard and east of Ordinate Avenue.

- (b) If Joe is k radius lengths north of Abscissa Boulevard, explain how you would use this program to estimate the angle swept out by Joe's path.
- (c) If Joe is q radius lengths east of Ordinate Avenue, explain how you would use this program to estimate the angle swept out by Joe's path.

The purpose of Task 3 was to support David in differentiating the action of determining the output value of the sine or cosine function from the result of the action (i.e., obtaining a sine or cosine value). By removing the constraint of being given a specific value for either the input or output quantity of the sine or cosine function, as was the case in the previous task, David was obliged to make explicit to himself, and thereby represent, the sequence of actions I designed the previous two tasks to engender (i.e., the sequence of actions involved in determining either the input or output value of the sine or cosine function when a given specific value for the output or input respectively).

Accordingly, I attempted to engender reflecting abstraction by providing a context that encouraged David to construct a mental representation of the actions I designed Tasks 1 and 2 to support.

Table 24

Task 4 of Instructional Sequence for the Outputs of Sine and Cosine Functions

If we know the angle swept out by Joe's path (in radians), there are two buttons on our calculator that tell us how many radius lengths Joe is north of Abscissa Boulevard and east of Ordinate Avenue respectively. What are they?

The purpose of Task 4 was simply to introduce notation for the output of the sine and cosine functions. In an effort to facilitate reflected abstraction, the solution to many of the remaining tasks in this instructional sequence encourage David to assimilate this symbolic notation to a scheme of coordinated actions and operations at the level of representation. This provides an occasion for the actions resulting from David's prior

reflecting abstractions to become increasingly organized and coherent at the reflected level.

Table 25

Task 5 of Instructional Sequence for the Outputs of Sine and Cosine Functions

By dragging the blue dot to change Joe's position, estimate the following:

- (a) $\sin(0.5)$
- (b) $\cos(\frac{3}{4})$
- (c) $\sin(2)$
- (d) $\cos(4)$
- (e) In general, how could we use this program to determine what $\sin(k)$ is for some k ?
How about $\cos(j)$ for some j ?

I designed Parts (a)-(d) of Task 5 to assess whether David was able to assimilate the symbolic notation for the output of the sine and cosine functions as a representation for the *result* of the action of determining the output values of these respective functions for specific values of the input. For instance, I expected the solution to Part (a) to involve David in interpreting $\sin(0.5)$ as the number of radius lengths Joe is north of Abscissa Boulevard after he has ridden 0.5 radius lengths on Euclid Parkway. I hypothesized that were David to interpret $\sin(0.5)$ in this way, he would be able to engage in the action of moving Joe so that he has traveled 0.5 radius lengths on Euclid Parkway, and then compare the relative size of the distance that Joe is north of Abscissa Boulevard to the length of the Flatville's radius. Alternatively, the generality of Part (e) prompted David to assimilate the symbolic notation as a representation of the *action* of determining the output of the sine and cosine functions, differentiated from the result of this action.

Table 26

Task 6 of Instructional Sequence for the Outputs of Sine and Cosine Functions

- (a) Place Joe in a position where the angle swept out by his path x is such that $\sin(x) = 0.6$.
- (b) Place Joe in a position where the angle swept out by his path x is such that $\cos(x) = -0.25$.
- (c) In general, how could we use this program to determine the value of x for which $\sin(x) = q$ for $0 \leq q \leq 1$? How about $\cos(x) = w$ for $0 \leq w \leq 1$?
- (d) Write an equation that represents the following statement: The angle swept out by Joe's path (in radians) is twice as large as the number of radius lengths that he is east of Ordinate Avenue.
- (e) Write an equation that represents the following statement: The number of radius lengths that Joe is north of Abscissa Boulevard is $\frac{1}{2}$ times as large as the angle swept out by his path (in radians).

Table 27

Task 7 of Instructional Sequence for the Outputs of Sine and Cosine Functions

- (a) Suppose I typed the sine of some number into my calculator and got 0.75. Use this program to estimate the number I may have put in.
- (b) Suppose I typed the cosine of some number into my calculator and got 0.4. Use this program to estimate the number I may have put in.

I designed Parts (a) and (b) of Task 6 for the same purpose as Parts (a)-(d) of Task 5 with the minor distinction that Parts (a)-(d) of Task 5 specify an input value and asked David to determine the output value whereas Parts (a) and (b) of Task 6 provide an output value and asked David to determine the corresponding input value. I intended this emphasis on reversibility to support David in constructing the sequence of actions involved in determining the output values of sine and cosine as mental operations. Similarly, the purpose of Task 7 is the same as that of Parts (a) and (b) of Task 6 and the objective of Part (c) of Task 6 is the same as that of Part (e) of Task 5. I designed problems from different tasks for the same purpose to allow David to engage in repeated reasoning and to increase the amount of data from which I was able to construct a model of David's ways of understanding these mathematical ideas.

My intention for Parts (d) and (e) of Task 6 was to reveal the extent to which David had constructed an organization of actions at the reflected level to which he might assimilate the contextual statements in these respective tasks, and to determine whether David was able to represent these actions with symbolic notation. These tasks stand to have educative value for David by occasioning and prompting his construction of the targeted organization of actions.

Table 28

Task 8 of Instructional Sequence for the Outputs of Sine and Cosine Functions

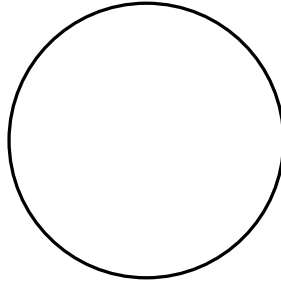
Place Joe in a position on Euclid Parkway to illustrate the following equalities:
(a) $\sin(x) = x/2$
(b) $\cos(x) = 0.7x$
(c) Compare Part (a) to Task 2(b).

The purpose of Task 8 is the same as that of Task 2 Parts (a)-(c) with the additional objective of providing insight into whether David is able to assimilate the symbolic notation in the task statement to an organized scheme of actions at the reflected level. To resolve Part (a) of Task 8, I anticipated that David would need to understand $\sin(x)$ as a representation of the output of a process that begins with angle measure and produces a vertical distance. Thus, the solution to the equation $\sin(x) = x/2$ represents the measure of the angle (in radians) swept out by Joe's path when the number of radius lengths that is north of Abscissa Boulevard is half as large as this angle measure. David's ability to answer this question, as well as Task 8(b), provides evidence that he has constructed an organization of actions at the reflected level and has represented these actions symbolically.

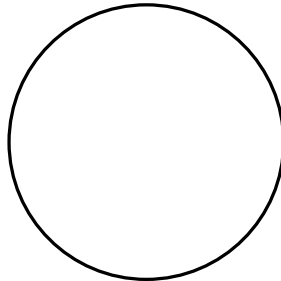
Table 29

Task 9 of Instructional Sequence for the Outputs of Sine and Cosine Functions

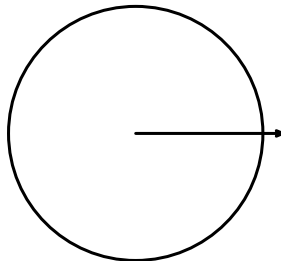
- (a) Use the circle below to illustrate what the values 2.5 and 0.60 represent in the equality $\sin(2.5) = 0.60$.



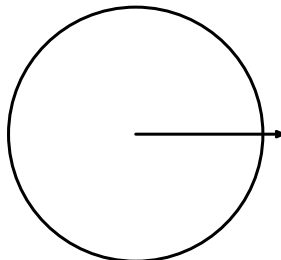
- (b) Use the circle below to illustrate what the values 3.77 and -0.809 represent in the equality $\cos(3.77) = -0.809$.



- (c) Use the circle below to draw the second ray of the angle θ (measured in radians) that has the property $\cos(\theta) = -\frac{1}{4}$.



- (d) Use the circle below to draw the second ray of the angle θ (measured in radians) that has the property $\sin(\theta) = \frac{1}{3}$.



- (e) How would your responses to Parts (a)-(d) change if the size of each circle was twice as large? Half as large? 4.7 times as large? Explain.
- (f) As the measure of an angle in radians, θ , changes from 0 to $\pi/4$, how does $\sin(\theta)$ change? Draw an image that displays how you are conceptualizing this change.

(g) As the measure of an angle in radians, θ , changes from 0 to $3\pi/2$, how does $\cos(\theta)$ change? Draw an image that displays how you are conceptualizing this change.

I designed Parts (a)-(d) of Task 9 for the same reason as Task 8: to provide an occasion for David to assimilate symbolic notation to a scheme of actions at the reflected level and to attend to the simultaneous variation of the input and output quantities of the sine and cosine functions. Parts (a)-(d) of Task 9, however, do not refer to a contextual situation, which allowed me to assess David's ability to apply the meaning for the symbolic notation of the output of the sine and cosine functions he demonstrated in previous tasks to this less concrete setting. Part (e) of Task 9 provided an occasion for me to examine whether David's understanding of the input and output quantities of the sine and cosine function depends on the size of the circle centered at the vertex of the angle. Finally, as with Parts (f) and (g) of Task 1, I designed Task 9(f) and 9(g) to allow David to strengthen his understanding of how the input and output quantities of the sine and cosine functions are related by prompting him to consider how these quantities change together.

Graphical Representations of the Sine and Cosine Functions

The sine function defines the covariational relationship between the value of the input quantity *openness of an angle* and the value of the output quantity *vertical distance that the terminus of the arc subtended by the angle is above the horizontal diameter of the circle centered at the vertex of the angle of which the subtended arc is a part*. Similarly, the cosine function defines the covariational relationship between the value of the input quantity *openness of an angle* and the value of the output quantity *horizontal distance that the terminus of the arc subtended by the angle is to the right of the vertical diameter*

of the circle centered at the vertex of the angle of which the subtended arc is a part.

Graphically representing these respective covariational relationships relies upon the understanding that a function defines the relationship between the values of two quantities that change together.⁶³

The focus of the first instructional sequence was to advance and assess David's understanding of what it means to measure the openness of an angle. The emphasis of the previous instructional sequence was to support David in conceptualizing the output quantity of sine and cosine functions and to reveal his ways of understanding these output quantities. While there are several productive ways of understanding the graphical representations of the relationship between the input and output quantities of the sine and cosine functions, it seems to me that a particularly useful one involves attending to how the values of these input and output quantities change individually as a function of time. One may then coordinate the changes in the values of these input and output quantities relative to each other, thereby making the construction of the sine and cosine functions essentially parametric. In the case of sine, one may conceptualize this parametric function as follows:⁶⁴

$x(t)$ = class of subtended arc lengths of a circle centered at the vertex of an angle,
each measured in units of their respective radius length t seconds after the

⁶³ I recognize that a function need not define the covariational relationship between the values of two quantities. For the purpose of understanding sine and cosine functions, however, I argue that a conception of function based on a covariational way of thinking is advantageous.

⁶⁴ Constructing the cosine function is conceptually analogous with a different output quantity. There is, however, one potentially important difference worth noting. The magnitude orientation for the output of the cosine function is horizontal instead of vertical. Therefore, representing the covariation of the input and output quantities of the cosine function on the conventional Cartesian coordinate plane involves one in making a transformation where horizontal magnitudes from the circle are rotated to vertical magnitudes on the Cartesian coordinate plane.

openness of the angle began increasing at a constant rate⁶⁵ from a measure of zero radians,

$y(t)$ = the number of radius lengths the terminus of the arc subtended by the angle is above the horizontal diameter of the circle centered at the vertex of the angle t seconds after the openness of the angle began increasing at a constant rate from a measure of zero radians.

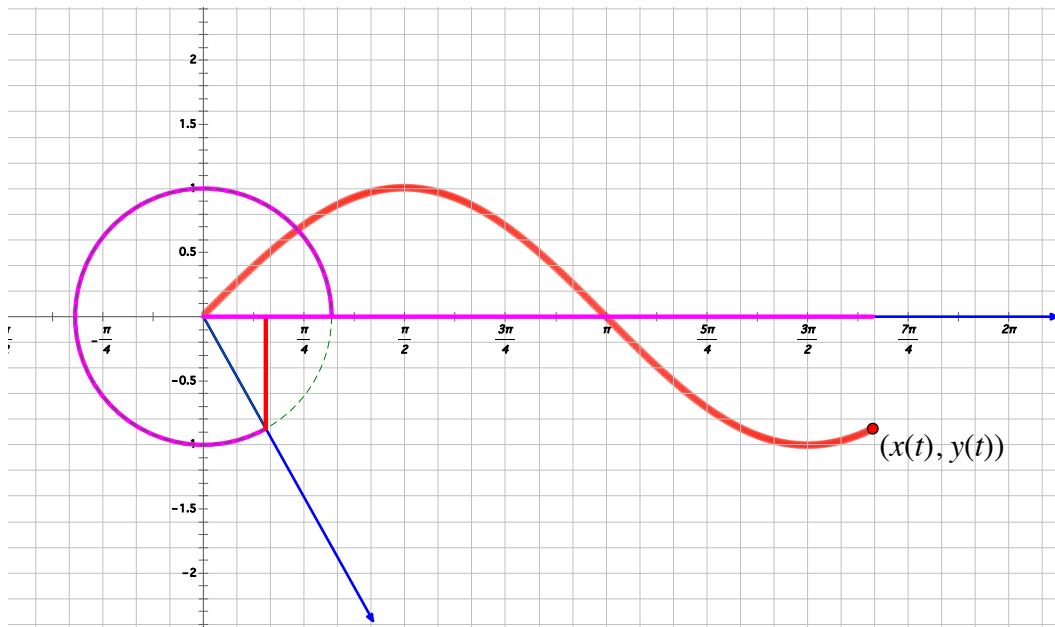


Figure 24. Parametric construction of the sine function.

One cannot divorce conceptual time from his or her image of quantitative change. In other words, to speak of change in a quantity's value is to acknowledge that the value of the quantity differs at two different moments in *time*. Constructing graphical representations of the relationship between the input and output quantities of the sine and cosine functions involves one in discerning how the value of the output quantity changes

⁶⁵ It is not necessary that the openness of the angle increase at a constant rate. However, this is not a minimalist parametric definition but rather a description of what is, in my view, an advantageous way of conceptualizing the graphical representation of the sine function.

as the value of the input quantity varies. Characterizing the variation of the input quantity necessarily implies treating the value of that quantity as a function of time. One may then attend to how the value of the output quantity changes as time varies and then coordinate the simultaneous variation of the value of these two quantities, both as functions of time.

One's ability to describe how the values of the input and output quantities of the sine and cosine function vary with time, and then coordinate the changes in the values of these quantities, does not imply that one is able to represent this covariation on a coordinate plane. Doing so involves the additional understanding that on the Cartesian coordinate plane the value of the input quantity is represented as a horizontal displacement from the vertical axis and the value of the output quantity is represented as a vertical displacement from the horizontal axis. Constructing graphs of the sine and cosine functions, then, involves coordinating changes in horizontal and vertical displacements that respectively correspond to changes in the values of the input and output quantities. Achieving such coordination allows one to construct a locus of points in the plane, each of which represent a state of the simultaneous variation of the values of the input and output quantities.

Constructing a graph of the sine and cosine functions by treating the value of their input and output quantities as functions of time, coordinating the variation of the values of these quantities, and representing this simultaneous variation on a coordinate plane does not necessarily give one an appreciation for why the graphs of the sine and cosine functions are curved the way they are. Understanding the concavity of the sine and cosine functions involves comparing successive changes in the value of the output quantity for equal changes in the value of the input quantity. For instance, the sine function is concave

down for values of the input between zero and π because successive changes in the value of the output quantity decrease as the value of the input quantity change by equal amounts between zero and π radians (see *Figure 25*). Similarly, the cosine function is concave up for values of the input between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ since successive changes in the values of the output quantity increase as the value of the input quantity changes by equal amounts between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

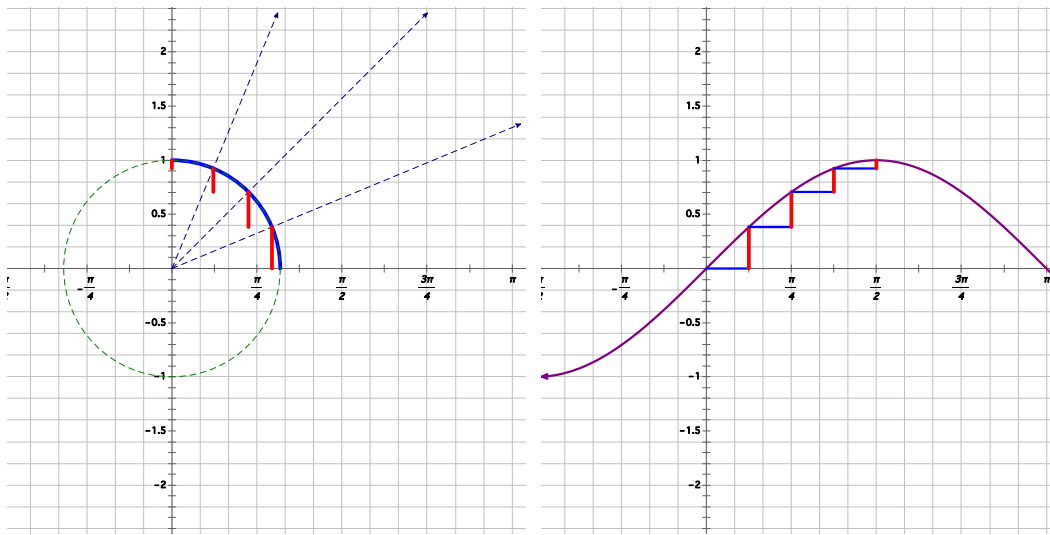


Figure 25. Concavity of the sine function.

Conjectured Role of Abstraction on Conceptualizing the Graph of the Sine and Cosine Functions

Coordinating the simultaneous variation of the values of the input and output quantities of the sine and cosine functions relies upon one's ability to represent how each of these quantities' values change with respect to time. Representing how the value of these input and output quantities vary with time presupposes one's ability to envision the

value of these quantities changing continuously, or smoothly.⁶⁶ Additionally, attending to how the values of the input and output quantities of the sine and cosine functions change together relies upon the understanding that for every value of the input quantity, there simultaneously exists a single value of the output quantity. I designed the first and second instructional sequences to support these understandings.

I conjecture that one can support a subject in constructing a representation of the variation of the input and output quantities of the sine and cosine functions by presenting an angle whose openness is continuously varying and asking the subject to describe how the value of each of these quantities change with time. Such a description involves a pseudo-empirical abstraction in that the subject must abstract properties from a dynamic geometric object (the properties being how the input and output quantities of the sine and cosine functions change with time). It is noteworthy that the properties David abstracts from such a dynamic geometric object will likely result from the reflecting abstractions he achieved/demonstrated during the first two instructional sequences.

Constructing an internalized representation of how the values of the input and output quantities of the sine and cosine functions vary together involves, by definition, a reflecting abstraction. I designed to support David in making this reflecting abstraction by engaging him in a sequence of actions whereby he represented the variation of the input and output quantities of the sine and cosine functions individually and then simultaneously. I then posed problems that required David to construct an internalized

⁶⁶ As I mentioned above, conceptualizing smooth variation in the value of any quantity involves imagining the value of the quantity varying by infinitesimal amounts over very small intervals of conceptual time while understanding that variation occurs within each interval of conceptual time as well (Thompson, 2011).

representation of these actions. Of course, becoming aware of and acting/operating on this internalized representation involves a reflected abstraction.

One cannot construct a meaningful understanding of the concavity of sine and cosine functions if he or she does not understand the graphs of these functions as a representation of the relationship between the values of two quantities that vary simultaneously. Thinking about a graph in this way allows one to reason about the curvature of the graph quantitatively and not imagistically or pictorially. I hypothesize that a reflecting abstraction is involved in the process of coming to view a graph representationally, where the coordinated actions one projects to the level of representation are the actions involved in constructing a graph by attending to how the values of the input and output quantities covary, first as functions of time and then with respect to each other. Until one constructs graphs as a representation of the relationship between the values of two quantities that vary simultaneously, questions about the meaning of concavity are largely inaccessible. Additionally, reasoning about concavity constitutes an action on the result of a previous reflecting abstraction (understanding a graph as a symbol that represents the covariational relationship between two quantities' values) and thus involves a reflected abstraction; the subject must assimilate questions about concavity to a sequence of coordinated actions at the level of representation.

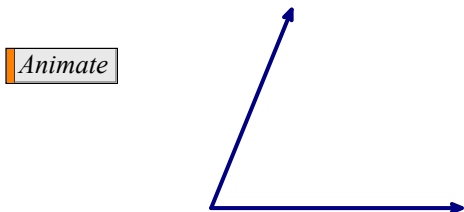
Instructional Sequence for the Graphical Representations of Sine and Cosine Functions

I designed this third instructional sequence to advance David's conception of the graphs of the sine and cosine function as a representation that emerges from tracking the simultaneous variation of input and output quantities by providing occasions for David to

engage in the abstractions discussed above. Each task in this instructional sequence references a didactic object that I created in *Geometer's Sketchpad*. Accordingly, as with the previous instructional sequences, I present the task statement as well as what David saw on the computer screen when I presented the task.

Table 30

Task 1 of Instructional Sequence for the Graphical Representations of Sine and Cosine Functions



(a) Describe how the input quantity of the sine function varies when you press the “Animate” button.⁶⁷ Explain.

(b) Describe how the output quantity of the sine function varies when you press the “Animate” button. Explain.

(c) Describe how the input quantity of the cosine function varies when you press the “Animate” button. Explain.

(d) Describe how the output quantity of the cosine function varies when you press the “Animate” button. Explain.

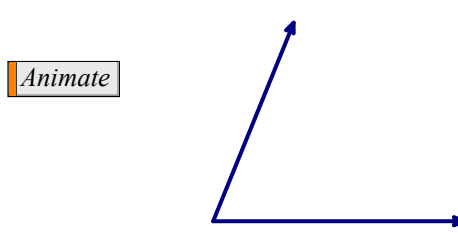
The purpose of this first task was to direct David’s attention to the properties of a dynamic geometric object, which I expected him to coordinate in subsequent tasks to construct a representation of how the input and output quantities of the sine and cosine function covary. In other words, I designed Task 1 to promote a pseudo-empirical abstraction. After having pseudo-empirically abstracted how the input and output quantities change with respect to time for an angle whose measure is increasing from zero at a constant rate, I conjectured that David would be prepared to coordinate the

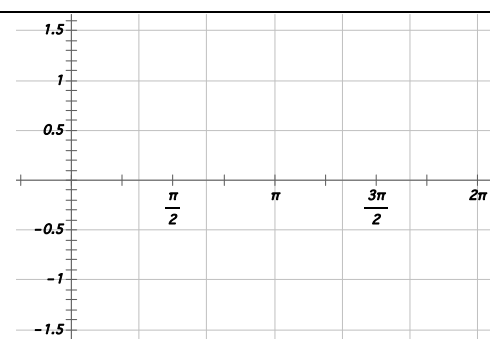
⁶⁷ Note that prior to pressing the “Animate” button, the angle has a measure of zero. This is true for Tasks 2 and 3 as well.

simultaneous variation of the input and output quantities of the sine and cosine functions respectively in Tasks 2, 3 and 4.

Table 31

Task 2 of Instructional Sequence for the Graphical Representations of Sine and Cosine Functions (Courtney, 2010; Thompson, 2002)





(a) When I press the “Animate” button, track the angle measure along the horizontal axis with your right index finger.

(b) When I press the “Animate” button, track the output of the sine function along the vertical axis using your left index finger.

(c) When I press the “Animate” button, do Parts (a) and (b) together. Keep your angle measure finger along the horizontal axis and your sine output finger along the vertical axis.

(d) When I press the “Animate” button, do Parts (a) and (b) together. Keep your angle measure finger along the horizontal axis and your sine output finger directly above your angle measure finger.

(e) On the coordinate system provided, sketch a graph of the relationship between the input and output of the sine function. Explain why you drew your graph the way you did and justify the curvature of your graph.

(f) I’ve identified three points on the graph of the sine function below. Draw a picture that displays what the coordinates of each of these three points represent.

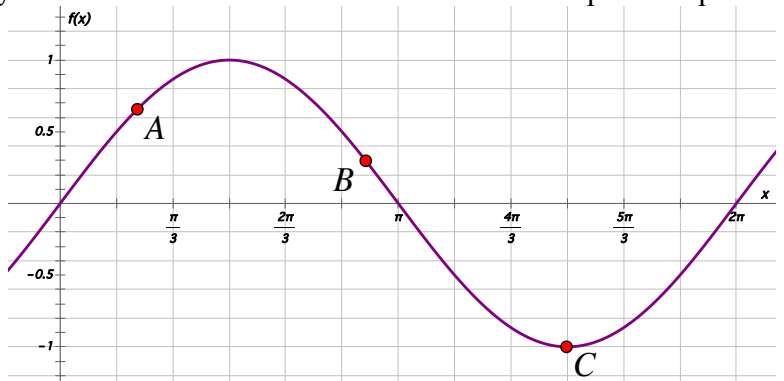
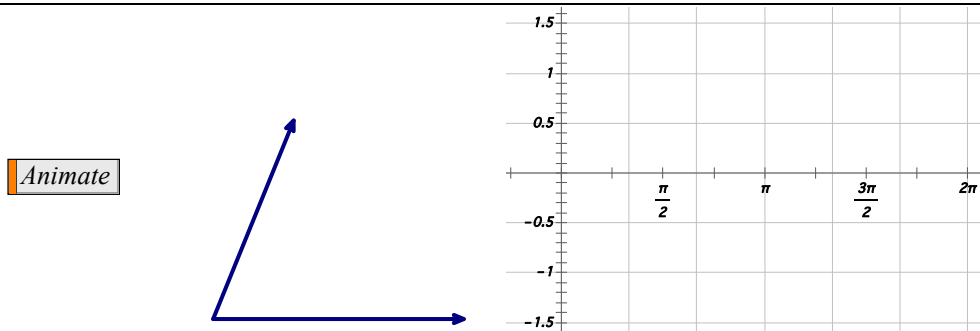


Table 32

Task 3 of Instructional Sequence for the Graphical Representations of Sine and Cosine

Functions (Courtney, 2010; Thompson, 2002)



(a) When I press the “Animate” button, track the angle measure along the horizontal axis with your right index finger.

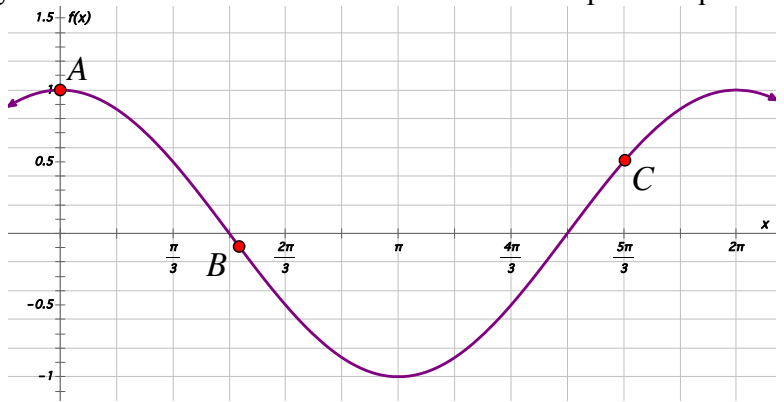
(b) When I press the “Animate” button, track the output of the cosine function along the vertical axis using your left index finger.

(c) When I press the “Animate” button, do Parts (a) and (b) together. Keep your angle measure finger along the horizontal axis and your cosine output finger along the vertical axis.

(d) When I press the “Animate” button, do Parts (a) and (b) together. Keep your angle measure finger along the horizontal axis and your cosine output finger directly above your angle measure finger.

(e) On the coordinate system provided, sketch a graph of the relationship between the input and output of the cosine function. Explain why you drew your graph the way you did and justify the curvature of your graph.

(f) I’ve identified three points on the graph of the cosine function below. Draw a picture that displays what the coordinates of each of these three points represent.

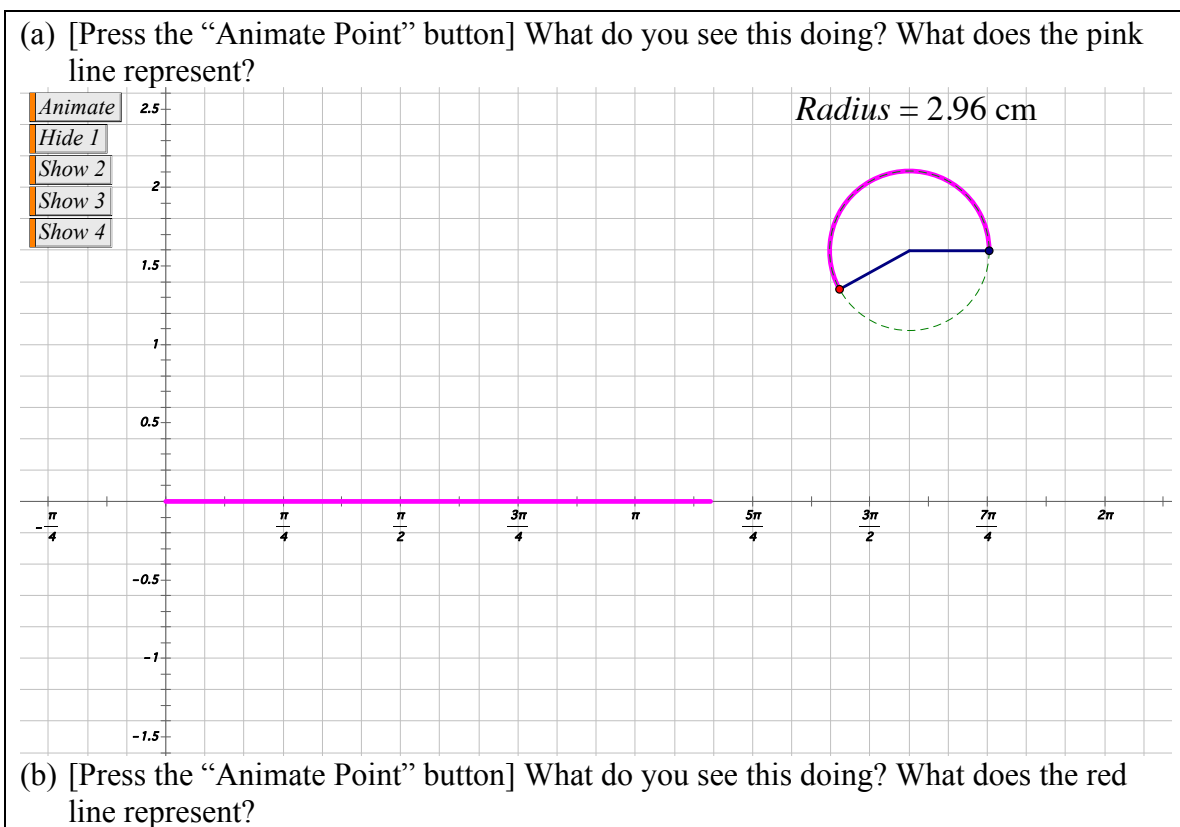


Parts (a)-(d) of Tasks 2 and 3 engaged David in a sequence of kinesthetic actions whereby he represented the variation of the input and output quantities of the sine and

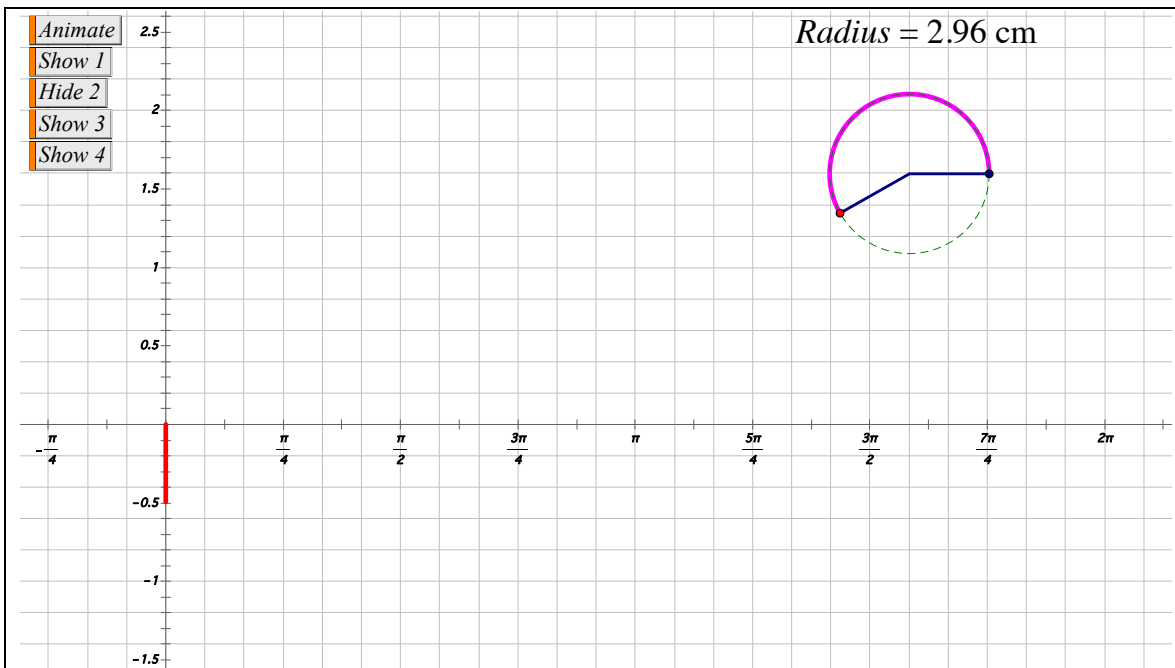
cosine function respectively. These actions form the foundation for viewing graphs of the sine and cosine functions as a representation of the covariational relationship between the input and output quantities of these respective functions. Parts (e) and (f) of Tasks 2 and 3 seek to promote reflecting abstraction on the actions I intended David to engage in during Parts (a)-(d) by prompting him to view the graphs of the sine and cosine functions representationally as opposed to imagistically.

Table 33

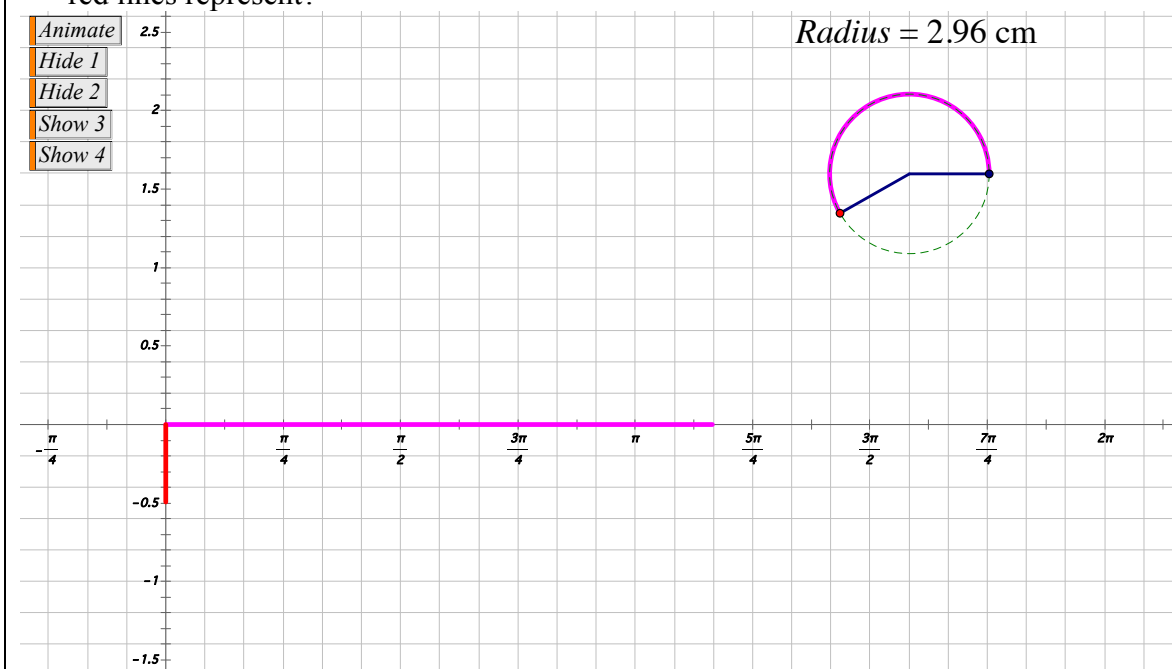
*Task 4 of Instructional Sequence for the Graphical Representations of Sine and Cosine Functions*⁶⁸



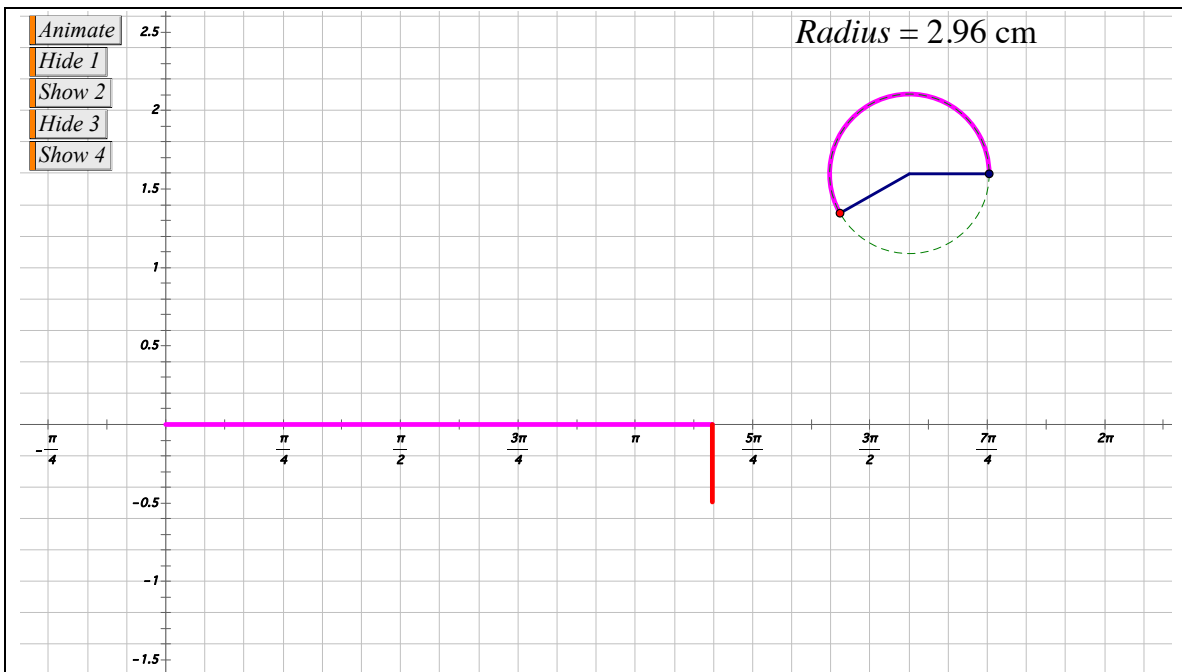
⁶⁸ Each of the images in this table is a snapshot of a dynamic *Geometer’s Sketchpad* simulation. Note that prior to the interviewer pressing the “Animate Point” button, the angle centered at the origin of the coordinate plane has a measure of zero radians.



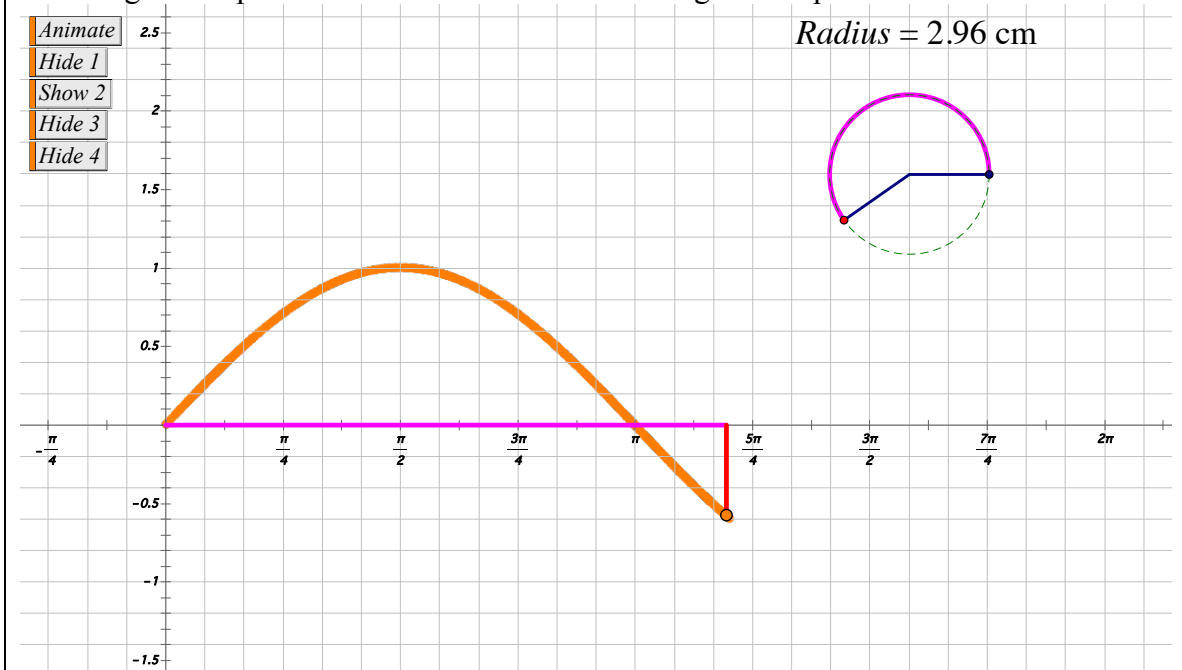
(c) [Press the “Animate Point” button] What do you see this doing? What do the pink and red lines represent?



(d) [Press the “Animate Point” button] What do you see this doing? What do the pink and red lines represent?



(e) [Press the “Animate Point” button] What do you see this doing? What does the orange dot represent? What does the curved orange line represent?



- (f) Label the axes of this graph.
- (g) If I change the radius of the circle to 4.37cm, how would the trace of the orange line change? Explain.
- (h) How do you think I programed *Sketchpad* to create this graph? Let’s work together to create a similar sketch for cosine.

The purpose of Task 4 was to assess whether David had constructed an understanding of the graph of the sine function as a representation of the covariational relationship of the values of input and output quantities by being able to assimilate each didactic object to a scheme of actions and operations at the reflected level. I intended Part (f) to provide an additional occasion for David to reveal his way of understanding the inputs and outputs of the sine function. I included Part (g) to assess whether David viewed the construction of the graph as being dependent upon the size of the circle centered at the vertex of the angle. Additionally, I designed Part (h) to give David the opportunity to communicate his way of understanding the graphical representation of the cosine function. In particular, David's approach to using the features of *Geometer's Sketchpad* to construct a similar simulation for the cosine function revealed his way of understanding the graphical representation of cosine, as well as the extent to which he conceptualized the inputs and outputs of the cosine function as representing the values of quantities.

Period of Sine and Cosine Functions

The period of a sine or cosine function is the length of the interval of input values over which the output quantity varies through one complete cycle of values (Carlson et al., 2013, p. 277).⁶⁹ For sine and cosine functions whose input quantity is the openness of an angle, the output values of these functions complete one full cycle when the input

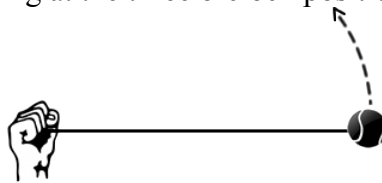
⁶⁹The period of trigonometric functions is often defined as the interval of input values over which the output values vary through one complete cycle. However, to my knowledge one rarely, if ever, presents period as an *interval* but rather as the *length* of the interval of input values over which the output values vary through one full cycle. In other words, no textbook that I have ever encountered presents the period of $f(x) = \cos(\pi x)$ as $[0, 2]$ but rather as just two. Additionally, if one defines period as the interval of input values over which the output values vary through one full cycle without specifying where the interval must begin, then this interval is certainly not unique (e.g., as the input varies through each interval in the infinite set $\{[k, k+2] | k \in \mathbb{R}\}$ the output values of $f(x) = \cos(\pi x)$ complete one full cycle).

values vary by 2π radians. However, the input quantity of sine and cosine functions need not be angle measure. Consider, for example, the task provided in Table 34.

Table 34

Sample Task (Carlson, O'Bryan, & Joyner, 2013, p. 485)

José extends his arm straight out, holding a 2.3-foot string with a ball on the end. He twirls the ball around in a circle with his hand at the center, so that the plane in which it is twirling is perpendicular to the ground. Answer the following questions assuming the ball twirls counter-clockwise starting at the three o'clock position.



Suppose the ball travels two radians per second. Define a function h that relates the ball's *vertical distance above* José's hand (in feet) as a function of the number of seconds elapsed since the ball passed the three o'clock position.

If we let t represent the number of seconds since the ball passed the three o'clock position and if we let $h(t)$ represent the ball's vertical distance above José's hand (in feet), then $h(t) = 2.3\sin(2t)$. We notice that the function h is still sinusoidal—in the sense that the graph of h is a smooth oscillatory curve—but the input quantity represents an amount of time, not an angle's measure. Instead, the *argument* of the function h represents the angle measure, in radians, traced out by the ball's path.

To accommodate for situations in which the input quantity of the sine or cosine function is not the openness of an angle, it is useful to conceptualize the argument of the function as such since all sine and cosine functions are 2π -periodic with respect to their argument. Thinking about the argument of sine and cosine functions as representing angle measure allows one to reason quantitatively about the covariational relationship between the argument, and thus input quantity, and the output quantity of sine and cosine

functions.⁷⁰ This further allows one to determine the period of a sine or cosine function by attending to the amount the input quantity's value must vary to make the argument (i.e., openness of an angle) vary by 2π radians.

For example, consider the function $f(t) = \cos(2t - 1)$. If one conceptualizes the argument of f as representing angle measure, θ , in radians, then $f(t) = \cos(g(t))$ where $\theta = g(t) = 2t - 1$. Additionally, if one understands that the output values of cosine complete one full cycle whenever an angle measure, θ , varies by 2π radians, then the period of f is the length of the interval that the input variable needs to vary by to make the argument vary by 2π radians. Since $g(t) = 2t - 1$ is a linear function with a constant rate of change of two, whenever t varies by π , $g(t)$ will vary by 2π . Therefore, the period of f is π because this is the length of the interval that the input variable t must vary by to make the argument $2t - 1$ vary by 2π . In sum, conceptualizing the argument of sine and cosine functions as representing angle measure allows one to understand the period of these functions by attending to the relationship between the argument and the input.

A common understanding of period (or, more appropriately, how to compute the period) of sine and cosine functions of the form $f(x) = \sin(bx + c)$ and $g(x) = \cos(bx + c)$, for real numbers b and c , is to divide 2π by b , the coefficient of the independent variable. Understanding the argument of sine and cosine functions as representing angle measure allows one to construct a more powerful understanding of why this procedure produces

⁷⁰ Unlike functions whose input and output are related by a sequence of arithmetic operations (e.g., $f(x) = 3x - 4$), the symbolic representations $f(x) = \sin(x)$ and $g(x) = \cos(x)$ provide no insight into how one obtains values of f and g for particular values of x , beyond pressing a button on the calculator. Therefore, without a means by which to understand the argument and output of the sine and cosine functions as quantities, I hypothesize that students cannot construct a meaningful understanding how the input and output quantities of the sine and cosine function covary, and thus cannot attain the understanding period discussed above.

the length of the interval that the input variable must vary by to cause the output values to complete one full cycle. The period of the functions $f(x) = \sin(bx + c)$ and $g(x) = \cos(bx + c)$ is the length of the interval that x must vary by to cause $bx + c$, the argument of these respective functions, to vary by 2π . For the functions f and g , the change in the argument is always b times as large as the change in the input. Therefore, for the argument to vary by 2π , the input must vary by $\frac{2\pi}{b}$.

Conjectured Role of Abstraction on Conceptualizing the Period of the Sine and Cosine Functions

As emphasized above, determining the period of sine and cosine functions relies upon the understanding that all sine and cosine functions are 2π -periodic with respect to their arguments. Achieving this understanding involves one in conceptualizing the argument of sine and cosine functions as representing the measure of an angle in radians and noticing that the output values complete one full cycle when the angle measure varies by 2π radians. One can make this observation by pseudo-empirically abstracting this property from an angle whose terminal ray is rotating in the counter-clockwise direction, thereby varying the openness of the angle. It is noteworthy that this pseudo-empirical abstraction relies upon the subject having constructed angle measure in radians as a multiplicative comparison of a class of subtended arc lengths and their corresponding radius lengths. This pseudo-empirical abstraction further requires the subject to have constructed the outputs of sine and cosine functions as quantities (vertical and horizontal displacements of the terminus of the subtended arc respectively). These understandings allow the subject to assimilate the dynamic geometric object of an angle whose terminal ray is varying to a scheme of meanings that allow him to abstract the property that the

output values of the sine and cosine functions complete one full cycle when the argument of these respective functions varies by 2π . In other words, achieving this pseudo-empirical abstraction relies upon the subject having made the abstractions that I designed the first two instructional sequences to support.

Understanding that all sine and cosine functions are 2π -periodic with respect to their arguments allows one to compute the period of sine and cosine functions by determining the amount the input quantity's value must vary by to make the argument vary by 2π radians. My hypothesis is that this understanding emerges from a pseudo-empirical abstraction on the result of determining the relationship between the value of the input quantity and the value of the function's argument—a relationship that results from comparing the values of these two quantities for a specific sine or cosine function. Constructing a method for computing the period of sine and cosine functions involves representing the sequence of actions in which the subject engaged from the pseudo-empirical abstraction that produced the relationship between values of the input and argument for a specific sine or cosine function—thus constituting a reflecting abstraction—and organizing these actions into a coherent scheme at the reflected level to which one may assimilate more advanced tasks.

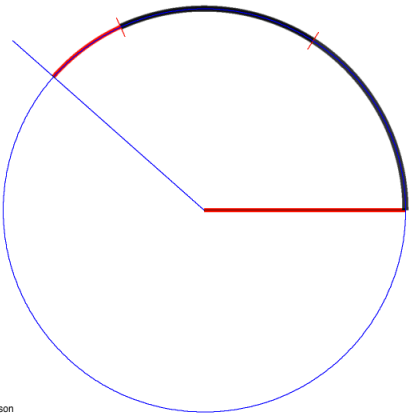
Instructional Sequence for the Period of Sine and Cosine Functions

I designed this final instructional sequence to engage David in tasks that foster the abstractions I hypothesized were necessary for him to achieve the understandings of the period of sine and cosine functions discussed above. In particular, these tasks sought to allow David to understand period of sine and cosine functions as the length of the interval

of input values needed to make the argument of these respective functions vary by 2π radians.

Table 35

Task 1 of Instructional Sequence for the Period of Sine and Cosine Functions



(2011) Pat Thompson

Recall this animation from the first instructional sequence.

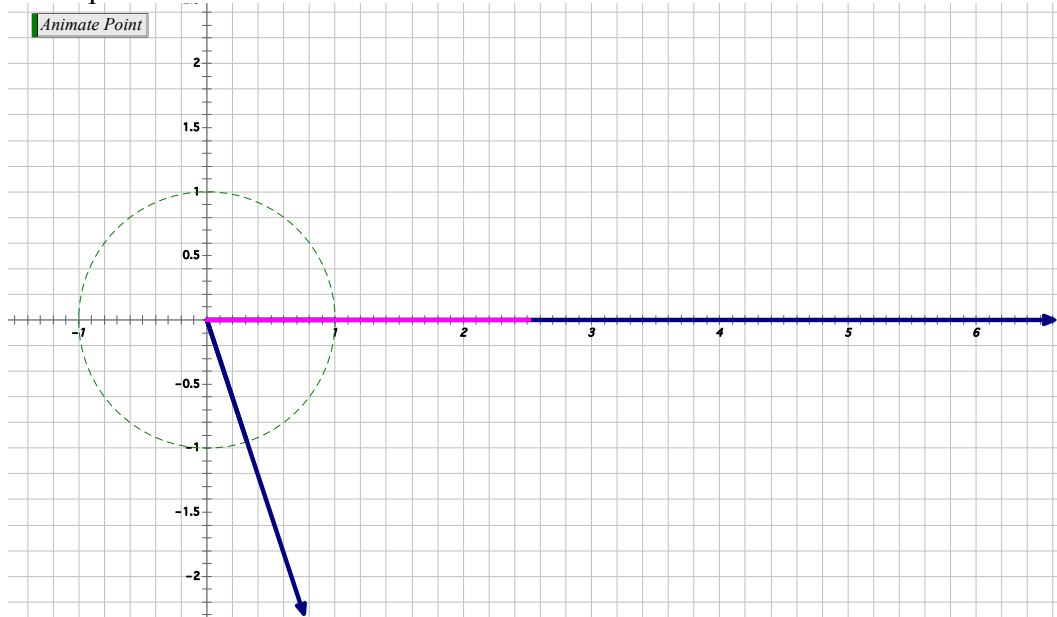
- (a) Consider the function $f(x) = \sin(x)$. What does x represent? What does $f(x)$ represent?
- (b) How much does x have to vary by for $f(x)$ to vary through one full cycle of output values? Explain.
- (c) Consider the function $g(x) = \cos(x)$. What does x represent? What does $g(x)$ represent?
- (d) How much does x have to vary by for $g(x)$ to vary through one full cycle of output values? Explain.

I began this instructional sequence by presenting the animation of an angle measure varying from a measure of zero radians to a measure of 2π radians (Thompson, 2011) from the first instructional sequence. Recall that this animation displays an angle tracing out an arc length of some number of whole radius lengths, represented by black segments, and some fraction of a radius length, represented by a red segment. The purpose of this task was to provide an opportunity for David to engage in pseudo-empirical abstraction whereby he recognizes that as the measure of the angle varies by 2π radians, the output values of the sine and cosine function vary through one complete cycle.

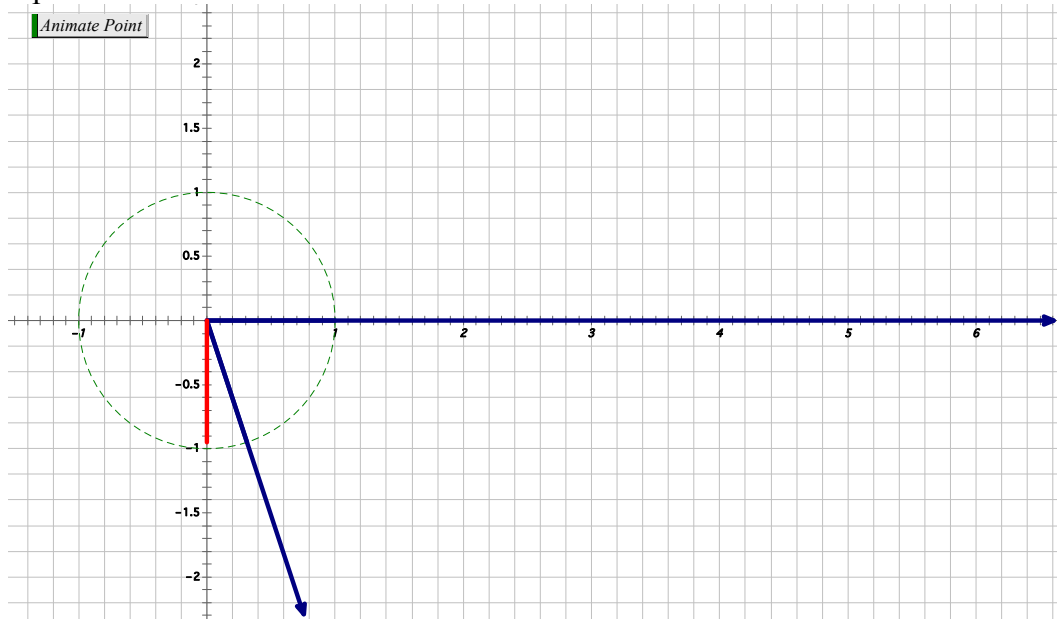
Table 36

Task 2 of Instructional Sequence for the Period of Sine and Cosine Functions⁷¹

(a) [Press the “Animate Point” button] What do you see this doing? What does the pink line represent?

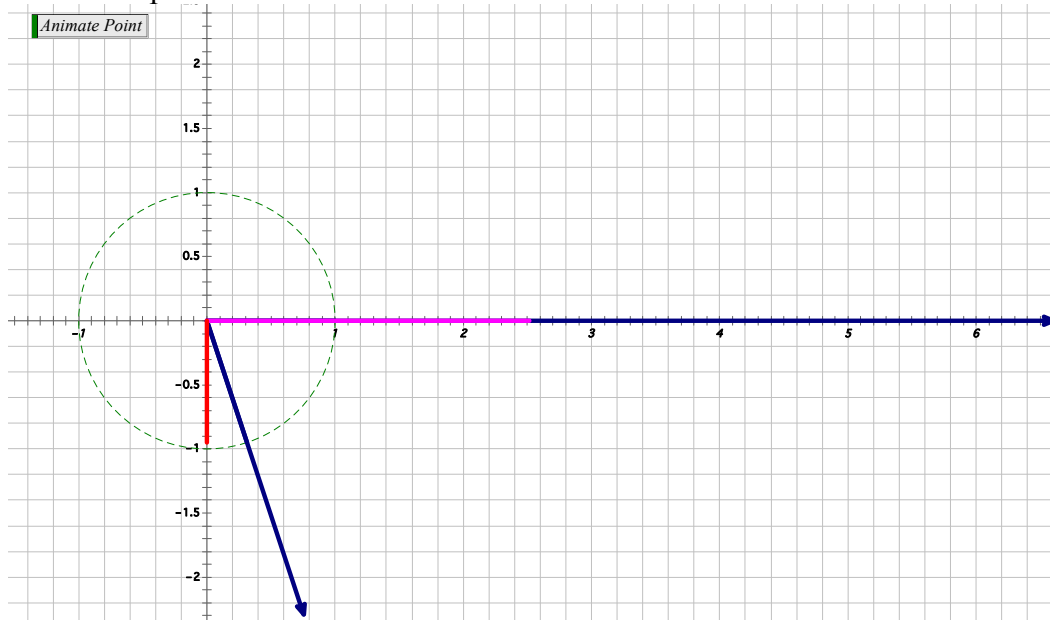


(b) [Press “Animate Point” button] What do you see this doing? What does the red line represent?

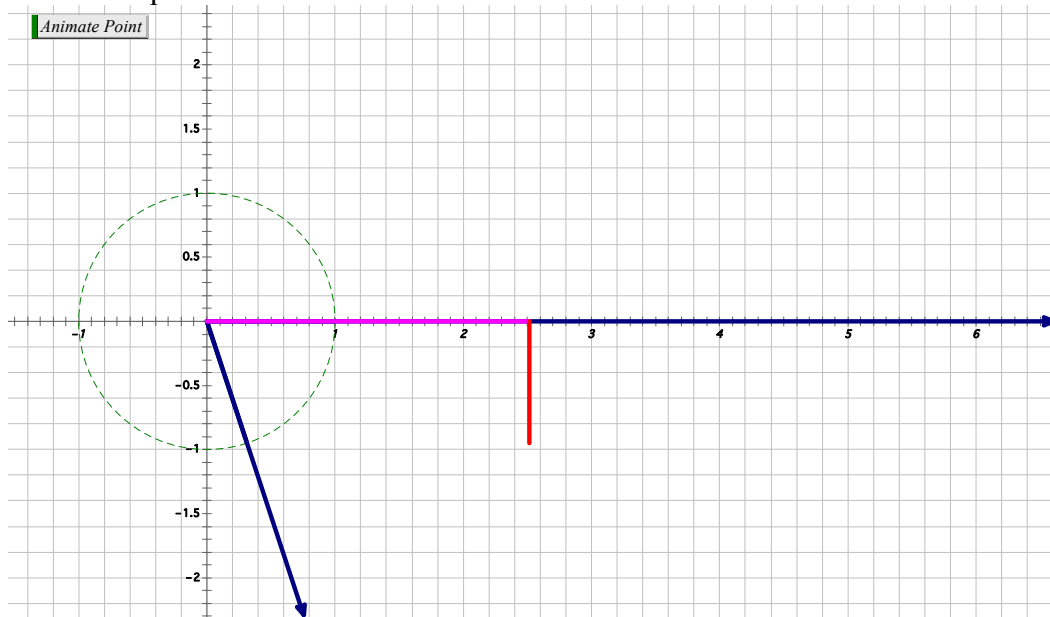


⁷¹ Each of the images in this table is a snapshot of a dynamic *Geometer's Sketchpad* simulation. Note that prior to the interviewer pressing the “Animate Point” button, the angle centered at the origin of the coordinate plane had a measure of zero radians.

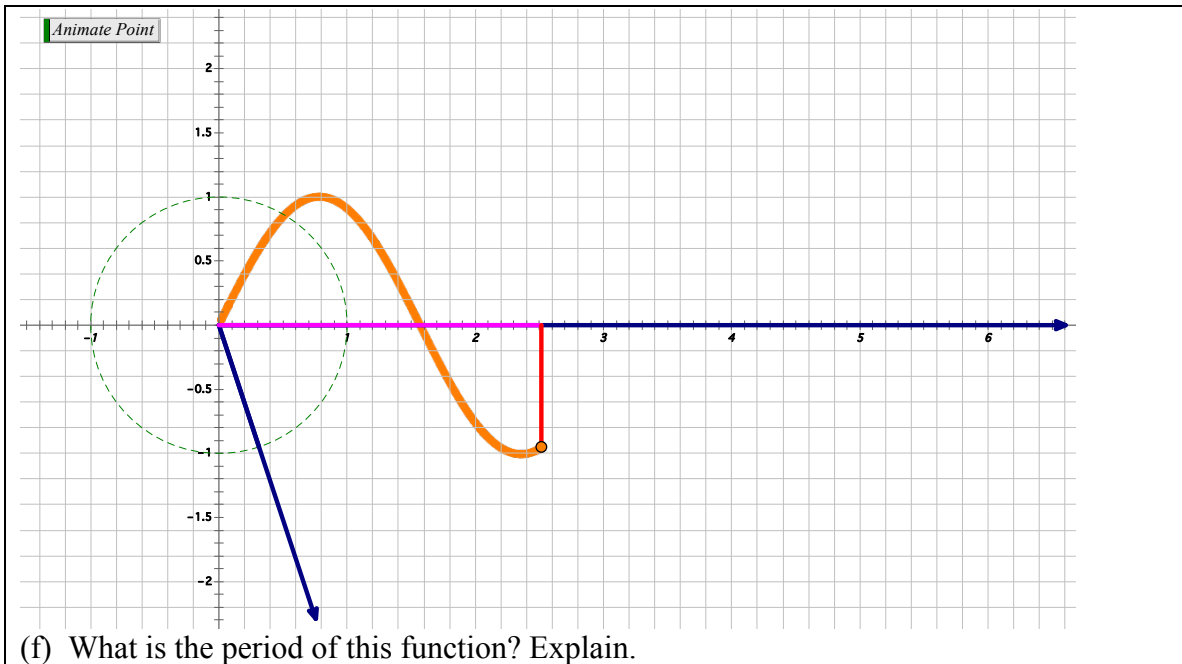
(c) [Press the “Animate Point” button] What do you see this doing? What do the pink and red lines represent?



(d) [Press the “Animate Point” button] What do you see this doing? What do the pink and red lines represent?



(e) [Press the “Animate Point” button] What do you see this doing? What is the input quantity to this function? What is the output quantity to this function? What is the relationship between the measure of the varying angle (in radians) and the measure of the input quantity? Write a rule for this function and explain why your function rule expresses the same relationship between the input and output quantities as shown in the graph.



The purpose of Part (a) of Task 2 was to engender a pseudo-empirical abstraction whereby David may recognize that the pink line, and thus input quantity, represents time since the angle measure began varying from zero radians. If David constructed the meanings that I designed Task 4 of the previous instructional sequence to promote, Parts (b)-(d) involve assimilation only. The purpose of these tasks was for David to attend to the quantities being covaried in the construction of the graph displayed in Part (e). I designed Part (e) to provide an occasion for David to conjecture a relationship between the value of the input quantity (i.e., number of seconds since the angle measure began varying from zero radians) and the measure of the angle (in radians). Writing the rule for the function whose graph is displayed in Part (e) involves representing this relationship symbolically. Since the relationship between the value of the input quantity and the angle measure emerges from comparing these two quantities, representing this relationship symbolically constitutes a pseudo-empirical abstraction. I anticipated that David would

easily answer Part (f), assuming that he would assimilate “period” as the interval that the value of the input quantity must vary by to make the value of the output quantity vary through one full cycle of values. However, I expected that after David had constructed a definition for the function whose graph is displayed in Part (e), he would be prepared to compare the period of the function, as represented graphically, and the function definition. In other words, assuming David would produce the correct function definition, I expected that David would examine the function rule $f(t) = \sin(2t)$ and the period of π seconds, and begin to attend to the relationship between the argument of a sine or cosine function and the period.

It is noteworthy that I designed four didactic objects that follow the same progression as Task 2 with different periods. This provided me with the opportunity to engage David in the same line of questioning as in Task 2 with these other didactic objects if David’s reasoning appeared fragile or underdeveloped.

Table 37

Task 3 of Instructional Sequence for the Period of Sine and Cosine Functions

- (a) What is the period of the function $g(x) = \cos(3.7x)$? Explain.
- (b) What is the period of the function $h(t) = \sin(at + b)$ for real numbers a and b ? Explain.
- (b) Suppose a colleague suggests to you, “To find the period of a sine or cosine function, all you have to do is divide 2π by the coefficient of the input variable” but is unsure why this works. How would you explain to your colleague why her observation is true?

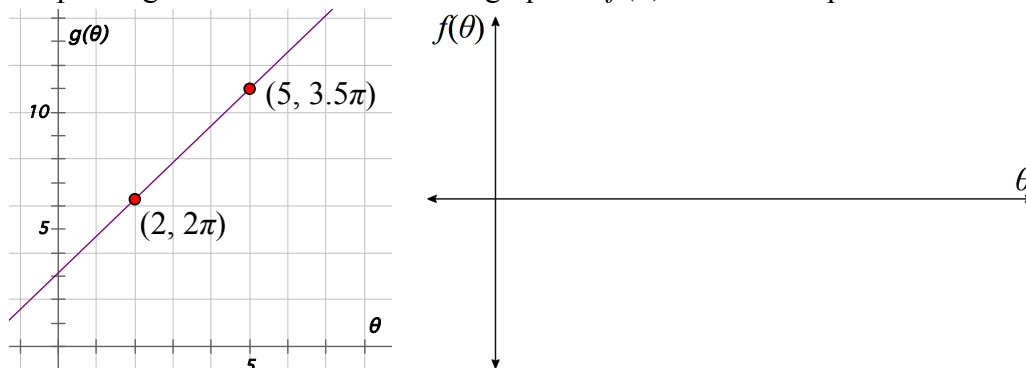
I designed Tasks 3 to provide an opportunity for David to see the argument of sine and cosine functions as a representation of the relationship between the measure of an angle and the value of the input quantity. In addition, I intended Task 3 to assess the

extent to which David understood period as the length of the interval that the value of the input quantity must vary by to make the argument vary by 2π .

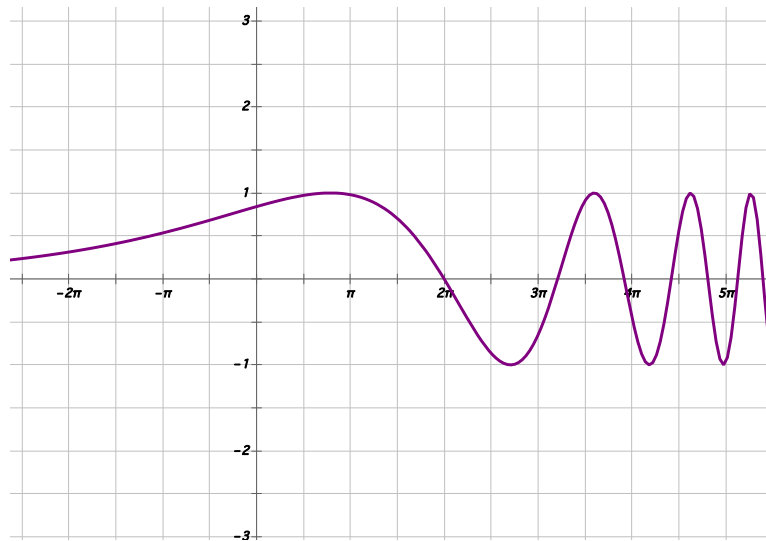
Table 38

Task 4 of Instructional Sequence for the Period of Sine and Cosine Functions

- (a) Suppose f is a linear function with a constant rate of change of π . What is the period of $g(x) = \sin(f(x))$?
- (b) Let $f(\theta) = \cos(g(\theta))$. The graph of $g(\theta)$ is given below with the coordinates of two points given. Sketch an **accurate** graph of $f(\theta)$ on the axes provided.



- (c) Suppose $f(x) = \sin(x)$. The following is a graph of $f(g(x))$. Sketch a possible graph of $g(x)$.



I designed Task 4 to reveal the extent to which David has constructed a scheme of meanings for period to which these novel problems may be assimilated. I hypothesized that for David to successfully solve these problems, he must have constructed a

representation for the action of comparing the values of the argument and input quantity at the reflected level.

CHAPTER 6

DAVID'S MATHEMATICAL KNOWLEDGE

The end may justify the means as long as there is something that justifies the end.

Leon Trotsky, *Their Morals and Ours*

This chapter presents my model of David's ways of understanding the four main ideas that were the focus of the series of task-based clinical interviews (TBCIs) I conducted in Phase I of the present study (angle measure, outputs of sine and cosine, graphical representations of sine and cosine, and period).

In addition to presenting my model of David's mathematics, I provide insight into the conceptual origins of this model by chronicling its evolution throughout the series of TBCIs and by making explicit the factors that led to its emergence. In particular, in this chapter I focus on the following:

- (1) Explicating hypothetical conceptual operations that seek to explain my interpretation of David's language and actions during the series of TBCIs.
- (2) Identifying David's specific utterances and actions that contributed to my construction of these hypothetical conceptual operations.
- (3) Explaining how I interpreted the observable products of David's reasoning so as to make their contribution to my model of David's ways of understanding and ways of thinking explicit.
- (4) Justifying my actions during the series of TBCIs and describing my interpretation of David's responses to these actions.
- (5) Justifying modifications or additions to the TBCI protocol that were informed by my ongoing analysis.

I present my model of David's mathematics in three main sections that correspond to the central topics of the series of TBCIs: angle measure, output and graphical representations of sine and cosine⁷², and period of sine and cosine. I begin each of these main sections with a brief summary of the clinical interviews that focused on the respective topic. I then present my analysis of each of these clinical interviews. This analysis begins with a general description of what occurred in the interview, followed by an outline of the major themes that emerged from my analysis of the interview. I then present data to illustrate and support each theme. This presentation and discussion of the empirical evidence is followed by a summary of my emerging model of David's mathematics and an explanation of, and justification for, any additions or modifications to the tasks for the following clinical interview.

Angle Measure

As with each of the four main mathematical topics of the series of TBCIs, I constructed a preliminary model of David's way of understanding angle measure through my analysis of the initial clinical interview. While ideas of angle measure were present throughout the series of TBCIs, the meaning of assigning a numerical value to the "openness" of an angle was the central focus of the first three TBCIs. David completed Tasks 1-5 (see Chapter 5) during the first TBCI. Broadly speaking, I intended these tasks to support David in conceptualizing a process by which one might quantify the openness of an angle. In the second TBCI, David completed tasks I designed during ongoing analysis for the purpose of refining my emerging model of his way of understanding

⁷² Since David tended to reveal his way of understanding the outputs of sine and cosine while discussing their graphical representations, and vice versa, I present my model of David's way of understanding the output of sine and cosine and the graphical representations of these functions in one section.

angle measure. Since I suspected that David possessed more advanced ways of understanding than he demonstrated in the first TBCI, I designed tasks for the second TBCI that were less accessible to memorized computational techniques. The third TBCI addressed the condition that a unit of angle measure must satisfy and provided David with opportunities to organize into a coherent scheme two uncoordinated ways of understanding he had previously demonstrated for what it means to quantify the openness of an angle. As with the second TBCI, I designed the tasks for TBCI 3 during ongoing analysis to refine my model of David's thinking and to provide opportunities for him to advance his understanding of what it means to quantify the openness of an angle.

Initial Clinical Interview

When discussing ideas of angle measure, David often referred to multiplicatively comparing the length of the arc that an angle subtends and the circumference of the circle centered at the vertex of the angle that contains the subtended arc. This observation led to my identification of the first theme of David's mathematics: *Angle measure as a comparison of subtended arc length and circumference.*

Consistent with this way of understanding angle measure, David explained that a unit of angle measure must be a particular "amount of circle." By this David meant the number of units contained in the whole circle must be constant and known. This understanding suggested a second theme of David's mathematics: *Unit of angle measure as a fraction of a circle's circumference.*

Theme 1: Angle measure as a comparison of subtended arc length and circumference. In contrast to conceptualizing the process of quantifying the openness of an angle as measuring the length of the arc an angle subtends in a unit that is a fractional

part of the circle's circumference, David repeatedly demonstrated a way of understanding angle measure as a multiplicative comparison of the length of the subtended arc and the length of the circumference. Consider, for instance, David's responses to my asking him what it means to say an angle has a measure of one and 2.1 radians respectively.

Excerpt 1

- 1 Michael: What does it mean to say that an angle has a measure of one radian?
- 2 David: Then, um (*pause*) (*sigh*). So one radian (*pause*), sorry, means that the complete circumference of a circle is made up of 2π , so 6.28. So one radian is approximately a sixth way around the circle if we're going to do approximates, um, it would be (*pause*), it would be one over 2π if we wanted to associate a number value kind of going with the length of any unit of measure. ...
- 3 Michael: So then a similar question: What does it mean to say that an angle has a measure of 2.1 radians?
- 4 David: So to say that it has a measure of 2.1 radians would mean that about, now about a third of the way around a complete circle if I'm just approximating so that I can kind of understand where it would be in my head, um, without doing any real math, uh, involved. Um if I, you know, wanted to I could actually then, you know, take that and convert using that as a proportion of the circumference and figure out an actual length; the circumference being 2π .

While David conceptualized the portion of the circle's circumference subtended by the angle as an attribute to which "one radian" and "2.1 radians" may be assigned as measures, nothing in David's response suggests that he was attending to a unit with which to measure this attribute. In Line 2 David claimed, "the complete circumference of a circle is made up of 2π , so 6.28." It does not appear that, to David, 2π or 6.28 radians referred to 2π or 6.28 *of anything*; there are 2π , or approximately 6.28, radians in a circle just because there are. David went on to explain that because there are 2π radians in a circle, one radian "is approximately a sixth way around the circle" and 2.1 radians is about "a third of the way around a complete circle." David did not seem to have conceptualized one radian and 2.1 radians as referring to an attribute of a geometric object that is respectively one and 2.1 of some unit of measure. Instead, it appears that David took for granted the fact that there are 2π radians in a circle and then used this fact to approximate the fraction of a circle's circumference that an angle with a measure of one and 2.1 radians respectively subtend.

Not only is there no evidence in Excerpt 1 to indicate that David had in mind a unit to which one radian and 2.1 radians refer, David's response in Line 4 suggests that he was likely not attending to a unit of measure. David explained that he could determine the length of the arc that an angle with a measure of 2.1 radians subtends by taking $1/3^{\text{rd}}$, the approximate fraction of the circle's circumference that the angle subtends, and "convert using that [$1/3^{\text{rd}}$] as a proportion of the circumference and figure out an actual length; the circumference being 2π ." Throughout the series of TBCIs David used the word "proportion" synonymously with "fraction" and "ratio." David's method for determining the length of the subtended arc, while assuming the circumference has a

length of 2π , was therefore to multiply $1/3$ and 2π to obtain approximately 2.1. Since I did not ask David to pursue his suggestion, I do not know what meaning David would have attributed to this value; that is, I do not know whether David would have interpreted the product of $1/3$ and 2π as representing a subtended arc length that is 2.1 radius lengths. Nevertheless, David's suggestion reveals his anticipation that determining the length of the arc that an angle with a measure of 2.1 radians subtends results from a computation rather than being self-evident in the phrase "2.1 radians" itself. In other words, David's circuitous proposal to multiply $1/3^{\text{rd}}$ by 2π to determine the length of the subtended arc indicates that he likely did not interpret an angle with a measure of 2.1 radians as subtending an arc that is 2.1 times as large as a unit of measure, namely a radius length.

Following the exchange in Excerpt 1, I asked David how he might measure an angle without a protractor. David's response, and the ensuing conversation, provides evidence in support of my hypothesis that David understood angle measure as a comparison of subtended arc length and circumference.

Excerpt 2

- 1 Michael: If you're looking at an angle and you don't have a protractor, how could you measure it?
- 2 David: *(Pause)*
- 3 Michael: Maybe if you had a ruler and a string or something.
- 4 David: Oh if you had a ruler and a string?
- 5 Michael: Right.
- 6 David: Um. So to do a, if you had a ruler and a string and you had, say you have some arbitrary angle here (*draws an angle*). So I would

measure a length from any given point on one of the sides so I would measure this length (*the length from the point on one of the rays to the vertex of the angle*) and say that's two inches.

7 Michael: Does it matter? So why did you pick two?

8 David: Because I'm going to end up doing a ratio here in a minute so the units won't matter anyways.

9 Michael: Okay.

10 David: So then from using the piece of string I could put my finger on one piece of the string and have the other attached to the thing and create an entire circle.

11 Michael: Okay.

12 David: So I'd end up with a full circle eventually. Okay? Then I can take the piece of string and measure the total and then I could then do a proportion if I wanted to do it based solely on measurements of things that I could measure because then I'm relating the angle measurement as a proportion of the total circumference so it would be an arc length (*inaudible*).

13 Michael: So suppose you measure the circumference of the circle and you get some number. We'll just call that number C . So here you have, um, the distance from the vertex of the, um, of the angle and where the side of angle intersects the circle is two inches.

14 David: Um-hum (*yes*) (David writes " $C = 4\pi$ " on his paper).

15 Michael: Okay. So the circumference is 4π ?

16 David: Um-hum. Yeah. ... So let's just say that when I measured this length here (*pointing to the subtended arc*) that this is, um, let's just say that this is one inch. Then if I did that as a proportion it is one-inch out of 4π inches (*David writes "1/4 π "*). The inches would cancel so that ratio would be my radian.

It is noteworthy that David used inches to measure the radius, circumference, and subtended arc length. David justified this decision in Line 8 by explaining his intention to compute a ratio of lengths, "so the units won't matter." David's elusive remark in Line 12 reveals that he was thinking of the circumference as one of these lengths. In an effort to prompt David to represent symbolically what he had in mind, thereby making his reasoning from Line 12 more intelligible, I asked him in Line 13 to denote the length of the circumference as C and reminded him that the radius of his circle is two inches. David recognized that because the radius of the circle is two inches, the circumference C is 4π inches. David then proposed a hypothetical length of one inch for the subtended arc and claimed that the measure of the angle in radians is $1/(4\pi)$ (see David's written work in *Figure 26*).

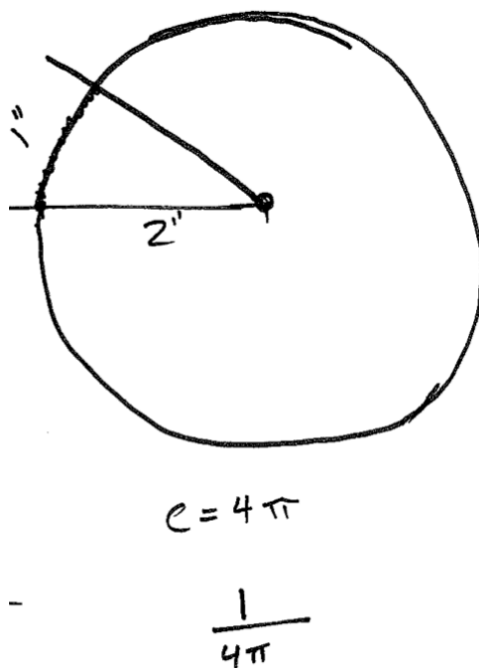


Figure 26. David's written work (Excerpt 2).

David's responses in Excerpt 2 suggest his way of understanding angle measure in radians as a multiplicative comparison of subtended arc length and circumference. To David, since the angle subtends an arc length of one inch, and since there are 4π inches in the circumference of his circle, the measure of the angle *in radians* is $1/(4\pi)$. Of course, $1/(4\pi)$ is not the measure of the angle in radians; it is the fraction of the circle's circumference subtended by the angle. The angle measure in radians, in this case, is 0.5 since the angle subtends half of a radius length. David's responses in Excerpt 2 suggest he was not conceptualizing angle measure in radians as a measure of the subtended arc length in units of radius lengths but instead as the ratio of subtended arc length to circumference.

Curious about David's reliance on measuring the circumference and subtended arc length in inches, and intrigued by his claim that the ratio of these lengths constitutes

the measure of an angle *in radians*, I asked him if it would be appropriate to measure an angle in inches. My intention was not only to establish if David considered it possible to measure an angle in inches, but if he did, to determine whether he expected the process of measuring an angle in inches to differ from the process he explained in Excerpt 2.

Excerpt 3

- 1 Michael: We usually use radians and degrees but if somebody wanted to could they use inches to measure an angle?
- 2 David: Yes you could. Um, if (*pause*) because while radians are a– (*pause*). Radians are the ratio of distances of the arc compared to that of the whole, um, circle. So I could see measuring in inches as being a measure and you could call them inches because it would still be, it's a ratio of the circle. Technically the units would end up canceling out but you could call them inches if you, if you wanted to and just kind of ignore the fact that the units of the arc and the units of the, uh, circle as a whole would cancel, um, but I see no reason why you couldn't call them inches.
- 3 Michael: So could you say a bit more about when you said, um, 'a ratio of the circle?' I think that's what you said.
- 4 David: Um, a ratio of the, um, the arc length, um, of the part of the circle that the angle, um, cuts off compared to the, um, whole, um, the circumference of the whole (*inaudible*).

David's responses in Lines 2 and 4 further demonstrate his way of understanding angle measure as a ratio of subtended arc length to circumference. David explained,

“Radians are the ratio of distances of the arc compared to that of the whole, um, circle.”

Again it appears that David did not conceptualize angle measure in radians as the length of the subtended arc measured in units of the arc’s radius. David’s statement in Line 2, “I could see measuring in inches as being a measure and you could call them inches” reveals something important about how he understood the idea of radian measure. For David, there were 2π , or approximately 6.28, radians in a circle. So if an angle cuts off two of these 6.28 segments of a circle, then the measure of the angle, in radians, is $2/6.28$, or approximately $1/3$. David’s reasoning with inches as a unit of angle measure appears analogous. David claimed that one could measure an angle in inches if both the subtended arc and the circumference of the circle were measured in inches. If one divides the subtended arc length by the circumference (with both lengths measured in inches) then, according to David’s thinking, it is perfectly acceptable to say that one has measured the angle *in inches*. In this way, “inches” and “radians” are little more than labels David put on the end of the ratio of subtended arc length to circumference whenever he measured these quantities in these respective units. David did not appear to conceptualize this ratio as representing the fraction of the circle’s circumference that the angle subtends irrespective of the units he used to measure the subtended arc length and the circumference. Nor did David seem to conceptualize the circumference as a unit of measure for the length of the subtended arc when he computed the ratio of subtended arc length to circumference. It is noteworthy that David did not appear perturbed by the fact that, according to his way of understanding, the same angle measured in inches and radians has the same numerical value.

Theme 2: Unit of angle measure as a fraction of a circle's circumference. My question in Line 1 of Excerpt 3 served an additional, albeit subordinate, purpose. I wanted to assess the extent to which David recognized the need for a unit of angle measure to covary with the size of the circle centered at the vertex of the angle. David's response to my question did not adequately reveal his understanding of the condition that a unit of angle must satisfy. I therefore asked David the question in Line 1 of Excerpt 4.

Excerpt 4

- 1 Michael: So if people thought about measuring an angle in inches in the way that we're talking about and I asked, um, somebody in one room to draw me an angle of three inches and somebody in the other room to draw me an angle of three inches (*David interrupts*)
- 2 David: There would be a problem because of the way that they, um, (*pause*) the way that we— because my issue would be are we saying, you know, that it's, that the three inches are a result of— because again the units of the arc length of the circumference really should have, if we're actually measuring distance in inches and measuring arc length in inches, then those units should have canceled and so that doesn't leave us a space for that unit. Um, which can lead to the confusion that we're having here so, um, the issue would be were you talking about when you did the three inches was that the three inches could be the circumference and it could have been the, or it could be the arc length like I had mentioned before. Uh, so the problem is you'd have to have some kind of rule on how you were

measuring a, um, you know, measuring a circle so that it worked out in inches. Um, but yeah it doesn't make me happy to use inches but I imagine that you could do it but you would have to define inches differently than the way that we typically define them as the distance or length.

David's vague and imprecise response in Line 2 reveals weaknesses in his conception of angle measure. David did, however, seem to have had a revelation when he said, "the problem is you'd have to have some kind of rule on how you were measuring ... a circle so that it worked out in inches." David appears to have recognized that the two people in different rooms would have to agree on how many "inches" are contained in the circumference of a circle. This recognition suggests David's understanding that if the measure of the angle is not to depend on the size of the circle centered at the angle's vertex, then one cannot commit to saying that an inch is an invariant length, but instead must acknowledge that a length of one "inch" should covary with the circumference of the circle. Perhaps this is what David meant when he said, "you would have to define inches differently than the way that we typically define them as the distance or length." To get a sense of the difference David envisioned, I asked him the question in Line 1 of Excerpt 5.

Excerpt 5

- 1 Michael: How would the definition be different?
- 2 David: (*Sigh*) Well because you would have to define it all in the same fashion that you would define like, you know, degrees, you know, so a degree is a proportion of the whole circle. One radian is a

proportion or, sorry, is a ratio of the whole circle. So you would have to say one inch will count for so much of a whole circle and so that's the problem is that, um, you've kind of lost what that means.

- 3 Michael: How much of the whole circle?
- 4 David: Right. That's the, that is the issue is, you know, one degree is three hundred and, or one, sorry, one three-hundred-and-sixtieth of the circle and that's because I understand what the whole circumference or the entire circle has been given a unit of. Um, the, you know, radians it's, I know that it's 2π is, you know, the radians measure for an entire circle so I can understand what one radian would be, or 2π radians. The issue with doing it with as far as inches is that you run into the, 'Well it's out of what? One inch out of how many inches?' and so, um, that would need to be somehow defined what that was.
- 5 Michael: So the two people in the different rooms, what would they have to agree on?
- 6 David: They would have to agree on how many inches are needed to make up a circle.
- 7 Michael: Okay. And if you change the size of the circle, does that change the number of inches?
- 8 David: Uh, well it would depend on how you're, again because now we're talking about inches as the amount of circle. So, um, if you, you

know, if one person thought that the whole circle would be 14 inches and one person said it was seven inches, well they're really not talking about the same proportions, you know. One of the seventh would be one-seventh of the circle and one of the fourteenth would be one-fourteenth of the other circle. That's not the same angle so that's why it would need to be some kind of, they would have to agree on a circle always has this length so that we can always use inches so that we can use that distance measure.

David's remark in Line 2, "one inch will count for so much of a whole circle" suggests that he recognized that for "inches" to be an appropriate unit of angle measure, one inch must constitute a particular fraction of the circumference of all circles. In Line 4 David explained that he knows how many radians and degrees are contained in "an entire circle" and claimed that such information would need to be made explicit for one to measure an angle in "inches." It does not appear that in this excerpt David was thinking of inches as the invariant lengths on a ruler but instead as a length that is a specified fraction of the circumference of any circle. In Line 8 David justified his proposal that one would have to know how many inches are contained in the circumference of any circle by explaining that if this were not the case—if one person claimed there are seven inches contained in the circumference of a circle while another claimed there are 14 inches—then the openness of angles that subtend an arc length of one "inch" will differ, thereby negating the most essential criterion that a unit of measure must satisfy: one's ability to apply the unit to definitively assign numerical values to an attribute of an object.

The condition David specified for “inches” to be considered an appropriate unit of angle measure may have derived from his need to calculate the measure of an angle, which for David entailed dividing the length of the subtended arc measured in some unit by the length of the circumference measured in the same unit. This multiplicative comparison of subtended arc length and circumference relied upon his knowing the measure of these respective lengths in a particular unit.

Summary of Initial Clinical Interview. David’s language and actions during the initial clinical interview suggests he understood angle measure as a multiplicative comparison of subtended arc length and circumference of the circle centered at the vertex of the angle that contains the subtended arc. In particular, David appeared to conceptualize angle measure as a the ratio of subtended arc length to circumference and not as the length of the subtended arc measured in a unit that is a particular fraction of the circle’s circumference. To elaborate, I hypothesize that David did not see an angle measure of two radians as subtending an arc that is two $1/(2\pi)^{\text{th}}$ of the circle’s circumference, or two radius lengths, but rather an angle that subtends $2/(2\pi)^{\text{th}}$ of the circle’s circumference. This is to say David did not see angles as subtending a number of fractional portions of a circle’s circumference (each of which is a unit), but instead as subtending a fraction of the circle’s circumference.

When presented with an angle and the task of measuring it in radians, I conjecture David constructed a sequence of mental images that resemble those illustrated in *Figure 27*.

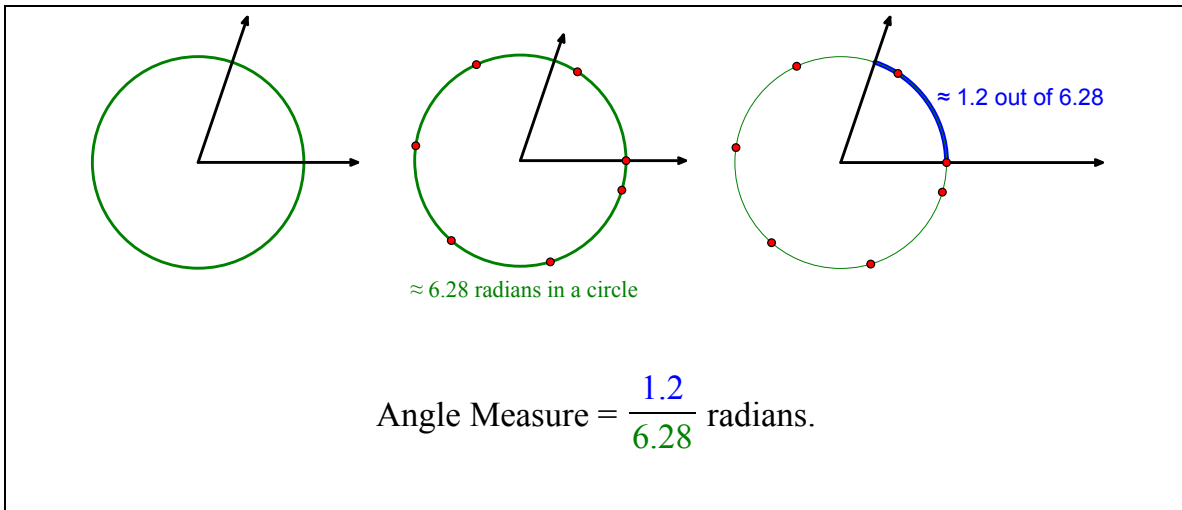


Figure 27. David's image of an angle with a measure of 1.2 radians.

David seemed to understand that there are 2π , or approximately 6.28, radians in a circle⁷³. There was no evidence from the initial clinical interview that, for David, these were 6.28 of anything. David also appeared to recognize that every angle subtends a portion of the circle that contains a certain number of these 6.28 pieces. To David, the measure of the angle in radians was therefore the number of these 6.28 pieces that the angle subtends divided by 6.28, the total number of pieces in the circle. David claimed *radians* as the unit of this ratio because the subtended arc length and the circumference were both measured in this unit. In David's usage, the unit "radians" appeared to be little more than a label to put on the end of the ratio of subtended arc length to circumference.

David also appeared to understand that a unit of angle measure must be a particular "amount of circle." David may have said this out of his need to calculate the measure of an angle, which for him involved dividing the length of the subtended arc by

⁷³ It is of course nonsense to claim that there are a number of radians "in a circle." My use of this phrase in this context and hereafter reflects my interpretation of David's reasoning, not a desirable or intended way of understanding radians as a unit of angle measure.

the circumference. According to this way of understanding, one cannot calculate the measure of an angle without knowing what to divide the length of the subtended arc by.

In summary, when presented with an angle and the task of determining its measure, David appeared to imagine: (1) a circle centered at the vertex of the angle, (2) the subtended arc measured in some unit (i.e., the subtended arc split up into some number of this unit), (3) the need to determine how many of this unit are contained in the circumference of the circle centered at the vertex of the angle, and (4) the measure of the angle as being how many units the angle subtends divided by how many units are contained in the circumference of the circle.

Task-Based Clinical Interview 1

During the first TBCI, David continued to demonstrate his understanding of angle measure as a multiplicative comparison of subtended arc length and circumference. This way of understanding, however, did not monopolize David's reasoning. On several occasions in the first TBCI, David exhibited a way of understanding angle measure as *a comparison of subtended arc length and a unit of measure*, thus constituting a new theme of David's mathematics. While David's language and actions revealed two different ways of understanding what it means to quantify the openness of an angle, TBCI 1 was perhaps most notable for David's persistent, almost slavish, reliance on setting up the equality of two ratios to solve problems involving angle measure—a calculational orientation (A. G. Thompson et al., 1994) that I summarize with the phrase *pseudo-proportional reasoning*. Finally, during the first TBCI David revealed with greater clarity his understanding of the condition that a unit of angle measure must satisfy, namely that one needs to know how many of the unit “make up an entire circle.”

Theme 1: Angle measure as a comparison of subtended arc length and circumference. In the initial clinical interview David demonstrated his way of understanding angle measure as the multiplicative comparison of subtended arc length and circumference. In particular, David claimed that quantifying the openness of an angle involves computing the ratio of these two quantities. David exemplified a similar way of understanding throughout TBCI 1, including in his responses to Tasks 3(b) and 3(d) provided in Table 39.

Table 39

Tasks 3(b) and 3(d)

Vary Radius

Radius —————

Subtended Arc —————

Circumference - - - - -

(b) Suppose you are discussing this applet with a colleague. As you increase the radius of the circle she observes, “The arc length is getting longer so the angle is getting bigger.” Respond to your colleague’s observation.

(d) Suppose your colleagues are looking at a picture like this one on their computers except their angles are different sizes and the circles centered at the vertex of their angles are different sizes. What could you tell them to do so that their angle is approximately the same measure as yours?

While engaged in this task David was able to vary the length of the radius manually by dragging the red dot on the initial ray of the angle, or automatically by pressing the “Vary Radius” button. Excerpt 6 contains David’s response to Task 3(b).

Excerpt 6

- 1 Michael: Suppose you’re discussing this, um, this image with a colleague and you’re changing the size of the radius and your colleague observes, she says, ‘The arc length, as you increase the radius, the arc length is getting longer. So the angle is getting bigger.’ What would you say to your colleague?
- 2 David: Uh, well the, um, it’s the ratio of the arc length to circumference that really shows the, uh, measure of the angle because it’s the part of the circle being cut off by the angle. So the ratio is remaining constant and we can see that by just looking. If we actually measured it or we could say, you know, this one (*pointing to the subtended arc*) is about three times the length of the circumference (*David misspeaks. I think he meant to say that three of the subtended arcs equal one circumference*). ...
- 3 Michael: Okay. Um, suppose you’re discussing this with a colleague and you hit the ‘Vary Radius’ button (*clicks on the ‘Vary Radius’ button*) and as you vary the radius of the circle, your colleague says, ‘It stays the same.’ What are the different things he could mean by ‘it’?

- 4 David: Uh, it could mean the, um, the ratio, okay, of um, the radius to circumference; the ratio of the arc to circumference; the ratio of the, uh, arc to radius. He could also mean the openness, for lack of a better term, of the angle. Um, or the angle measure itself.

David explained in Line 2 that the measure of the angle is determined by the ratio of the subtended arc length to circumference. It is noteworthy that David responded to this question while the radius of the circle was varying. David's remark, "the ratio is remaining constant" suggests that he pseudo-empirically abstracted the property that the subtended arc always appears to be the same fraction of the circle's circumference.⁷⁴ Moreover, David's response in Line 4 indicates he engaged in an additional pseudo-empirical abstraction whereby he recognized that, while the radius of the circle varied, the openness of the angle remains fixed.⁷⁵ It therefore appears likely that David considered the ratio of subtended arc length and circumference an appropriate means by which to quantify the openness of the angle because, as the radius of the circle varied, the openness of the angle and the ratio of subtended arc length to circumference remained invariant.⁷⁶ Accordingly, David appears to have engaged in P-EA 1, P-EA 2, and D1 (see Chapter 5).

David's response to Task 3(d) in Table 39 further illustrates David's reliance on multiplicatively comparing the subtended arc length and the circumference when discussing angle measure.

⁷⁴ I call this pseudo-empirical abstraction P-EA 1 in my conceptual analysis in Chapter 5.

⁷⁵ I call this pseudo-empirical abstraction P-EA 2 in my conceptual analysis in Chapter 5.

⁷⁶ I call this deduction D1 in my conceptual analysis in Chapter 5.

Excerpt 7

- 1 Michael: So suppose your colleagues are looking at a picture like this on their own computers except for their angles are a different size than this one. ... So here's what yours looks like. Their angles are a different size and the circles centered at the vertex of their angles are different sizes. What could you tell them to do so that their angle is approximately the same measure as yours? ...
- 2 David: So if we were just doing the angle then I would say that my subtended, uh, arc is about a third of the length of my circumference so they need to move their terminal side so that the subtended arc of their circle is a third of the size of their circumference.
- 3 Michael: Okay. And so then, um, if they asked you, 'What's approximately the measure of your angle in radians?' what would you say?
- 4 David: Uh, two, um, two-pi-thirds.
- 5 Michael: Why is that?
- 6 David: Uh, because it's about a third of the, uh, circle and so, uh, the entire circle in radians would be 2π so it's a third of 2π , which is $\frac{2}{3}\pi$.

David could have said a number of things in response to Task 3(d). For instance, David could have advised his colleagues to place their terminal ray in such a place so as to make the length of their subtended arcs twice as large as their respective radii. Instead David advised his colleagues to move their terminal rays so that the arc lengths their

angles subtend are the same fraction of the circumference of their respective circles. Therefore, David's response in Line 2 further exemplifies his understanding that quantifying the openness of an angle involves computing the ratio of subtended arc length to circumference.

In Line 3, I asked David to approximate the measure of his angle in radians. Recall that during the initial clinical interview David claimed that the measure of an angle in radians was the ratio of subtended arc length to circumference. David placed the word "radians" at the end of this ratio as if it were a label instead of denoting a unit with which to measure an attribute of a geometric object. David's responses in Lines 4 and 6 suggest a more sophisticated way of understanding than he demonstrated in the initial clinical interview. In particular, David appeared to understand that since the angle subtends a third of the circumference of the circle, the measure of the angle in radians must be a third of the number of radians in an entire circle. David's response in Line 6, however, does not indicate that David had conceptualized the radius as a unit of measure for the length of the subtended arc. According to David's thinking, the measure of the angle is $(2/3)\pi$ because there are 2π radians in a circle and the angle subtends $1/3$ of them. Although David did not claim that the measure of the angle was simply the ratio of subtended arc length to circumference, his solution relied upon a multiplicative comparison of these quantities.

Theme 2: Angle measure as a comparison of subtended arc length and a unit of measure. In my analysis of the initial clinical interview, I explained that David repeatedly demonstrated his understanding of angle measure as a multiplicative comparison of subtended arc length and circumference as opposed to a comparison of

subtended arc length and a unit of measure that is a fractional part of the circle's circumference. After analyzing TBCI 1, I could no longer say, "as opposed to." There were occasions during the first TBCI in which David exhibited an understanding of angle measure as the length of the subtended arc measured in a particular unit.

David's language and actions during the initial clinical interview and the first TBCI up until and including his response in Line 6 of Excerpt 7 suggested that he understood angle measure in one and only one way: as a multiplicative comparison of the arc length an angle subtends and the circumference of the circle centered at the vertex of the angle containing the subtended arc. My interest in discerning whether this meaning characterized the entirety of David's understanding of angle measure compelled me to remove the dotted green line representing the length of the circumference from the bottom of the screen. I then asked David to approximate the measure of the angle displayed on the screen in radians (see *Figure 28*).

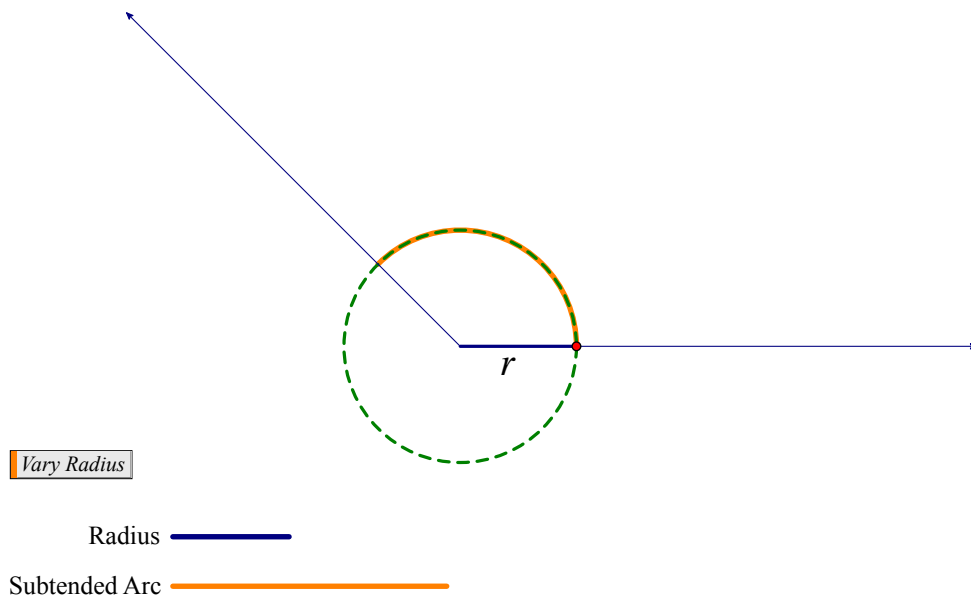


Figure 28. Approximate the measure of the angle (again).

Excerpt 8

- 1 Michael: Okay. If you had the same question, um, if they ask you, ‘About what’s the measure of your angle in radians?’ but this is what you saw (*hides the length of the circumference at the bottom of the screen*), what would you say?
- 2 David: (*Long pause*) I would say it’s about two.
- 3 Michael: Why is that?
- 4 David: Because there are about two, um, radius lengths in the, um, arc and I know that, (*pause*) yeah.

David explained that the measure of the angle in radians is approximately two because there are two radius lengths contained in the subtended arc. David’s response demonstrated his capacity to reason about angle measure in radians as a comparison of the subtended arc length and a unit of measure, namely the radius length. It is worth noting that I simply removed a piece of information David used to estimate the measure of the angle in Excerpt 7. Without this information, David was able to assimilate the task and produce an approximation for the measure of the angle in a completely different way than he had previously.

Immediately following David’s remark that the measure of the angle in *Figure 28* is approximately two radians, I asked him to explain what it means to say that an angle has a measure of 3.92 radians.

Excerpt 9

- 1 Michael: What does it mean to say that an angle has a measure of 3.92 radians?

2 David: Um, (*pause*) that if I was to, um, (*pause*) pick a point on one of the, um, rays and measure the length from the vertex to that point, that would give me the radius of a circle that it subtends and then that length, um, if I went ahead and made it so that, um, ... 3.92 times the length of the radius, whatever that is, um, ... is the length of the arc. Sorry so, okay so let's, uh, the, I would take the length of the radius, I would multiply the length of the radius by 3.92 and then I would use that to create the length of arc desired, um, of a circle with the radius that I measured.

David proposed to construct an angle with a subtended arc that is 3.92 times as large as arc's radius. The way of understanding David's response in Line 2 reveals differs substantially from that which he demonstrated in the initial clinical interview and in Excerpt 6 and Excerpt 7 from the first TBCI. David's remark in Line 2 demonstrates his understanding that an angle with a measure of n radians subtends an arc length that is n times as large as the radius of the subtended arc. David therefore appeared to have conceptualized the radius as a unit of measure for the length of the arc an angle subtends.

David continued to demonstrate his understanding of angle measure in radians as a multiplicative comparison of subtended arc length and radius length. For instance, consider David's response to my asking him to explain how to construct an angle with a measure of 1.6 radians using the features of *Geometer's Sketchpad*.

Excerpt 10

1 David: So I would, um, start by constructing, uh, two rays (*pause*), or actually, I'd start by constructing one ray ... and put a point on it

(Michael constructs a ray in standard position with a point on it).

And then I want a circle with a center at the end of the ray and the radius length to some other point on the ray (*Michael draws a circle as David advised*). That's fine. So now I can measure a radius (*Michael measures the radius of the circle*). ... So it would be, I want the arc length to be 1.6 radians times the length of the radius. So I would need to measure the radius ... and I would need to multiply that by, uh, 1.6 (*Michael performs this calculation in Sketchpad*) and that would give me the, uh, length of the, um, sector, or the arc length that I want. Okay, so then I would— (*Michael places a point on the circumference of the circle in preparation for drawing the terminal ray and constructs an arc on the circumference of the circle*).

- 2 Michael: Okay so there's an arc and you're saying that this (*the arc length*) has to be as long as (*David interrupts*)
- 3 David: I want that length to be the same as the radius times 1.6.
- 4 Michael: (*Moves the terminal point of the arc so that the arc length is $1.6r$ and then constructs the terminal ray of the angle*) So then this angle here would be the angle?
- 5 David: Correct.

David completed the task by advising me to do the following: (1) construct an initial ray; (2) construct a circle centered at the endpoint of this initial ray; (3) measure the radius of the circle; (4) multiply the radius by 1.6; and (5) draw the terminal ray so

that the length of the subtended arc is the same as the radius of the circle times 1.6. Instead of instructing me to create an angle that subtends an arc that is $1.6/(2\pi)$ of the circle's circumference, David suggested that I construct an angle that subtends an arc that is 1.6 times as large as the length of the radius. In sum, David's remarks in Excerpt 10 further demonstrate his understanding of angle measure as the length of the subtended arc measured in units of the radius.

Theme 3: Pseudo-proportional reasoning: "Part-whole" and "new-old" proportion formula scheme. David's language and actions up to and including the exchange in Excerpt 10 suggested that he possessed two complementary but conceptually distinct ways of understanding angle measure. The first involved a multiplicative comparison of subtended arc length and circumference while the second entailed a multiplicative comparison of subtended arc length and some unit of measure, often the radius. These two ways of understanding angle measure relied upon David's identification of quantities (e.g., subtended arc length, circumference, and radius length) and his construction of quantitative relationships (e.g., the ratio of subtended arc length to radius length as a measure of the length of the subtended arc in units of radius lengths). While David demonstrated his ability to reason quantitatively about angle measure, TBCI 1 is most notable for his evasion of quantitative reasoning by way of a memorized "proportion formula." David often assimilated novel problems to what I refer to as a "part-whole" or "new-old" proportion formula scheme, which allowed him to avoid attending to quantities and relationships between them.

To support David in pseudo-empirically abstracting the property that the openness of an angle covaries with the arc length the angle subtends (P-EA 1), I asked him to

explain how the length of a subtended arc changes while varying the openness of an angle.⁷⁷ After asking this question I began to increase the openness of the angle by dragging a red dot on the terminal ray of the angle that traced out a subtended arc (see *Figure 29*).

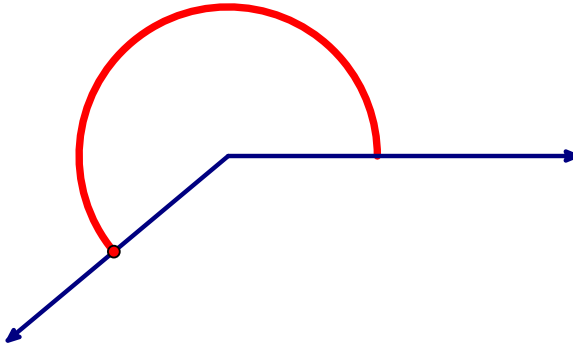


Figure 29. Context for Excerpt 11.

Excerpt 11

- 1 Michael: As I increase the size of the, the openness of the angle, or the size of the angle, how does the length of this arc change as I change the size of the angle? As I increase it?
- 2 David: As the (*pause*), as the angle, um, increases, measuring the angle in degrees, then it would be angle measure is to 360 the same way that, uh, arc length, uh, so I should put x , is to circumference (*writes* " $A/360 = x/C$ ").
- 3 Michael: So angle measure is to 360 as the arc length, x being the length of this red line (*David interrupts*)
- 4 David: x being the arc length.

⁷⁷ See Task 4(d) in the Angle Measure section of Chapter 5.

- 5 Michael: Okay, is to the circumference. Um, so what is the left-hand side of that equation? Um, the ... what does the angle measure divided by 360, what does that ratio represent?
- 6 David: Uh (*pause*), that represents the part of the circle created by the, um, sector as measured by degrees (*pause*). So the interior angles, how they relate, this is measuring, this is showing how the lengths of the outside edge of the circle, so the circumference and arc length are changing together.
- 7 Michael: And why do those two ratios have to be equal?
- 8 David: (*Long pause*) Because they are the (*pause*), if we think of this as part-whole, whole circumference (*pointing to "C" in his equation*), whole degree measure of an entire circle (*pointing to "360" in his equation*) the central angle that we're talking about creating the sector or the arc cut off by the sector, you know, by the angle. So it's the part-whole idea.

It is noteworthy that in Line 2 David did not describe how the openness of the angle and the subtended arc length covary, but instead verbalized a formula that he appeared to have memorized (" $A/360 = x/C$," where A represents the angle measure in degrees, x represents the length of the subtended arc, and C represents the length of the circumference). I asked the question in Line 5 to determine if David understood that the left-hand side of this equation represents the fraction of the circle's circumference subtended by the angle; that is, I wanted to assess whether David understood " $A/360$ " as a quantitative operation. David's inarticulate response in Line 6 suggests that he did not

interpret " $A/360$ " as the fraction of the circle's circumference subtended by the angle. In an attempt to ascertain how David understood the equation he proposed, I asked him to explain why " $A/360$ " and " x/C " are equivalent expressions (Line 7). Again David's reply indicates that this equality did not derive from a quantitative understanding of the situation. My interpretation of David's response in Line 8 is essentially, "The parts are on top of both fractions and the wholes are on bottom. When you do these proportions, the parts always have to go on top and the wholes always have to go on bottom." David did not appear to interpret " $A/360$ " and " x/C " as representing the fraction of the circle's circumference that the angle subtends. David seemed to be reasoning from a memorized formula without relying on a quantitative conceptualization of the relationship between angle measure and subtended arc length the formula conveys.

While David's response in Line 2 of Excerpt 11 was the first time he mentioned a "proportion formula," David's use of such formulas continued throughout the remainder of the first TBCI. Consider, for example, David's response in Excerpt 12 to Task 2(d), restated in Table 40.

Table 40

Angle Measure Task 2(d)

Vary Radius
Hide Radius
Hide Measurements

Circumference = 24.27 cm
Subtended Arc Length = 8.09 cm

Radius —————
Subtended Arc —————
Circumference - - - - -

(d) Press the “Show Measurements” button. We see that the length of the subtended arc is 8.09 cm and the length of the circumference is 24.47 cm. What is the length of the subtended arc if the circumference were changed to 16.8 cm?

Excerpt 12

- 1 Michael: So what is the length, without moving the radius now, what’s the length of the, or what would the length of the subtended arc be if the circumference were changed to 16.8? ...
- 2 David: Uh, the ratio of, um, 24.27 to 16 (*David unintentionally omits the “0.8”*). Um, I would take that and multiply it by the, or sorry 16(.8) divided by 24.27, and multiply it by 8.09 and that would give me the length.
- 3 Michael: Okay. Why is that? First, could you write that down because I’m going to forget that.
- 4 David: Yes I can. Okay so we said the new length was 16?

- 5 Michael: Um, so the new, we're changing the circumference to be (*David interrupts*)
- 6 David: So the new circumference.
- 7 Michael: The new circumference is 16.8.
- 8 David: Okay (*writes "new cir = 16.8"*).
- 9 Michael: And we want to know what the subtended arc length is going to be.
- 10 David: Okay. So I would do (*writes while talking*) 16.8 is to 24.27 the same way that 8.09 is to the new length. So the new value of S (*writes "16.8/24.27 = S/8.09"*). And then again we've got to get rid of the division so 8.09 times the ratio of 16.8, the new circumference, to the old circumference.
- 11 Michael: Okay. So here you set up, um, the, what does the 16.8 divided by 24.27 represent? That ratio?
- 12 David: It is the ratio of, uh, new circumference to old circumference and then this is new arc length to old arc length (*pointing to the right-hand side of his equation*).

David responded to the task with the vague and imprecise procedural description in Line 2. Therefore, in an effort to elicit more concrete observable products of David's reasoning, I asked him to write down what he had in mind (see David's written work in *Figure 30*). Recall the purpose of Task 2(d) was to provide a context in which David may utilize the understanding that I designed Task 2(a) to engender: that the arc length an angle subtends is always the same fraction of the circle's circumference (P-EA 1). Although David demonstrated this understanding on several occasions prior (e.g., see

Excerpt 6) he did not appear to employ it in his solution to Task 2(d). Instead, David seemed to identify from the given information pairs of “new” and “old” values, from which he made fractions—taking care to ensure that the “new” values were in the numerator and the “old” values were in the denominator—and then set these fractions equal to each other. In particular, David recognized that the 16.8 and the 24.27 “go together” because both are values of circumference. David also understood that he was being asked to determine the new value of the subtended arc length, which he called S . Moreover, David claimed that S and 8.09 “go together” because both are values of subtended arc length. In general, David appeared to assimilate the given information to a “proportion formula scheme” like the following:

$$\frac{\text{New Value (of type } X)}{\text{Old Value (of type } X)} = \frac{\text{New Value (of type } Y)}{\text{Old Value (of type } Y)}$$

new cir . 16.8

$$\frac{16.8}{24.27} = \frac{S}{8.09}$$

$$8.09 \cdot \frac{16.8}{24.27} = S$$

Figure 30. David’s written work to Task 2(d).

In David’s language, “[New value of type X] is to [Old value of type X] the same way that [New value of type Y] is to [Old value of type Y].” David solved his equation for S , the new value of the subtended arc length. David’s solution to this problem suggests that his understanding of the invariant multiplicative relationship between the subtended

arc length and the circumference was perhaps tentative, or that he did not make the effort to attend to meaning since he had a memorized formula on hand. To assess whether David's actions while solving Task 2(d) were informed by an understanding of quantities and quantitative relationships, I asked him the question in Line 11. David's response suggests that he likely did not see both sides of his equation as representing the fractional part of the circle's circumference subtended by the angle, but rather as pairs of "new" and "old" values that are a particular instantiation of his proportion formula.

I designed Task 2(d) to provide an occasion for David to employ his understanding that the arc length an angle subtends is always the same fraction of the circle's circumference (P-EA 1). After seeing David's solution, however, I realized that the information provided in the task (two known values of circumference, one known value of subtended arc length, and one unknown value of subtended arc length) almost begged to be set up in a "proportion formula" and algebraically manipulated. I was certain that a more complicated⁷⁸ problem, like Task 2(f), would encourage, perhaps even require, David to attend to quantities and quantitative relationships.

Excerpt 13

- 1 Michael: So now if ... let's just say m is the length of a unit of measure for the openness of an angle. It's some unit. Um, suppose the circumference of the circle, uh, C we're calling it, centered at the vertex of the angle is six times as large as the unit of measure (*David writes "m unit of measure / C circum. / C = 6m"*). ... Also suppose that the subtended arc length S is 0.4 times as large as the

⁷⁸ I say "more complicated" only because it is quite difficult to assimilate this task to a "proportion formula scheme." In a way Task 2(f) is not at all complicated, assuming one is able to reason proportionally.

circumference. ... (*David writes* " $S = 0.4C$ "). ... What's the measure of the angle in units of length m ?

2 David: (*Long pause*) Okay. ... (*Long pause. David speaks quietly to himself*) So you would do (*long pause*). (*David writes* " $S/0.4 = C$ "). (*Long pause*) ... (*David draws a circle with an angle in standard position and writes* " $m = (1/6)C$ "). (*Sigh*) S is to C as 0.4 (*writes* " $S/C = 0.4$ "). S is to C the way that my angle is to $6m$ (*writes* " $S/C = \theta/(6m)$ "). (*Long pause*) So $6m$ times 0.4 is equal to θ (*writes* " $6m(0.4) = \theta$ "). Okay so $2.4m$ is the units so $2.4ms$ (*writes* " $2.4m = \theta$ ").

3 Michael: Okay so you're going to have to walk me through this.

4 David: Okay. So you said that m was the unit of measure. There are six ms in a complete circle. ... There are six ms in a circle so I know that the, um, that it's 0.4 of the circumference is S so that means the ratio of S to C is 0.4. So if we do set up the ratio like we have been doing, the, um, arc divided by circumference is the same relationship as the angle that is desired divided by the equivalent angle measure for the entire circle, which was $6m$. Okay, so, uh, substituting, I knew this ratio (*points to* S/C) was 0.4 and I multiply by $6m$ and so then 0.4 times six gives me 2.4 with a unit of m .

To correctly solve Task 2(f), one may reason as follows: Since the angle subtends 0.4 of the circumference, the angle subtends an arc that is 0.4 times as large as the

circumference measured in any unit. Therefore the subtended arc length measured in units of length m is $0.4(6)$, or 2.4 .

It is worth noting that David's several long pauses in Line 2 indicate that Task 2(f) perturbed him. That said, David's ability to solve this problem by assimilating it to his proportion formula scheme—which is to say by reasoning pseudo-proportionally and calculationally instead of quantitatively—is remarkable. In particular, David's remark in Line 4, “So if we do set up the ratio like we have been doing, the, um, arc divided by circumference is the same relationship as the angle that is desired divided by the equivalent angle measure for the entire circle” suggests that he had assimilated the information in the task to his part-whole proportion formula scheme:

$$\frac{\text{Part (of type } X\text{)}}{\text{Whole (of type } X\text{)}} = \frac{\text{Part (of type } Y\text{)}}{\text{Whole (of type } Y\text{)}}.$$

Essentially, David set up the “proportion” $S/C = \theta/(6m)$ and substituted 0.4 for the ratio S/C . David then solved the equation $0.4 = \theta/(6m)$ for θ to obtain the correct answer (see David's written work in *Figure 31*). In general, David solved the problem by writing down algebraic expressions from the given information and then reasoning from these algebraic expressions without attending to the quantities or quantitative operations the symbols represent.

M unit of measure

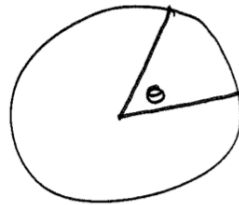
C circum

$$C = 6m \rightarrow m = \frac{1}{6}C$$

$$S = 0.4C$$

$$\frac{S}{0.4} = C$$

$$\frac{S}{C} = 0.4$$



$$\frac{S}{C} = \frac{\theta}{6m}$$

$$6m(0.4) = \theta$$

$$2.4m = \theta$$

Figure 31. David's written work on Task 2(f).

David continued to not employ understandings he had previously demonstrated and instead reasoned based on memorized formulas and procedures. I asked David to use the features of *Geometer's Sketchpad*—which amount to the technological equivalent of a ruler and a piece of string—to determine the measure of an arbitrary angle in radians (Task 4(c)). The purpose of Task 4(c) was to assess whether David had constructed a way of understanding angle measure in radians as a multiplicative comparison between subtended arc length and the length of the radius.

Excerpt 14

- 1 Michael: Um, so we're now going to use the features of Sketchpad to measure this angle. ... So how would we measure this angle here in radians?

- 2 David: Okay. So I would use the circle tool. I would put the, um, vertex, or the center at the vertex ... (*student enters the room and interrupts*)
- 3 Michael: (*Draws a circle centered at the vertex of the angle*) So there's the circle. First, does it matter what size I make it?
- 4 David: Uh, no.
- 5 Michael: Okay.
- 6 David: Okay. So then I would measure the length of the radius (*Michael measures the length of the radius using the features of Geometer's Sketchpad*) ... and then I would measure the length of the arc, uh, that subtends (*Michael measures the subtended arc length*). Okay.
- 7 Michael: Okay. So we have the radius and we have the, um, length of the subtended arc.
- 8 David: Okay. So then (*pause*) it would be (*picks up pen and situates paper*), I know that, um, (*long pause*). (*Speaks softly to himself*) Let me think how I want to do this. I know that the circumference is equal to 2π times the radius. So circumference is 2π times the radius (*writes " $C = 2\pi r$ "*). I know that the arc length, um, for this particular one, I know it's length so I can do S and I can do C (*writes " S/C "*) and then, what units are we trying to measure the angle?
- 9 Michael: Um, radians.

10 David: In radians. So I would compare that to 2π , and angle measure (*writes* “ $\theta/(2\pi)$ ”). So it would be S is to C (*talks inaudibly to himself and writes* “ $2\pi \frac{S}{2\pi r} = \theta$ ”). The 2π would cancel so it’s the ratio of arc length to radius (*writes* “ $S/r = \theta$ ”) would be the angle measure in radians.

11 Michael: Okay. So the ratio of arc length to radius (computes this ratio using the Calculate feature of Geometer’s Sketchpad and finds this ratio to be 1.31). Does that seem reasonable?

12 David: Yes.

David measured the arbitrary angle in radians by instructing me to: (1) construct a circle centered at the vertex of the angle, (2) measure the radius of the circle, and (3) measure the length of the subtended arc. David then set up the “proportion” $S/C = \theta/(2\pi)$ and solved this equation for θ . After expressing “ C ” as “ $2\pi r$ ” David simplified his expression to obtain “ $\theta = S/r$.” David then divided the length of the subtended arc by the length of the radius to find the measure of the angle in radians (see David’s written work in *Figure 32*). It is noteworthy that after David and I determined values for the length of the subtended arc and the length of the radius in Line 6, all he needed to do to compute the measure of the angle in radians was divide the length of the subtended arc by the radius length. Doing so would have suggested that David relied on his understanding of angle measure in radians as a multiplicative comparison of subtended arc length and radius length—an understanding that David had previously demonstrated on several occasions (see Excerpt 8 and Excerpt 10). Instead, David assimilated the measurements

we obtained into his “proportion formula scheme.” David recognized the subtended arc length as a part of the whole circumference of the circle. Similarly, David understood that the measure of the angle, θ , in radians is a part of 2π , the total number of radians in a circle. David therefore had enough information to plug into his part-whole proportion formula and solve for θ , the measure of the angle in radians.

$$\frac{s}{c} = \frac{\theta}{2\pi}$$

$$\cancel{2\pi} \frac{s}{\cancel{2\pi} \cdot r} = \theta$$

$$\frac{s}{r} = \theta$$

Figure 32. David’s written work to Task 4(c).

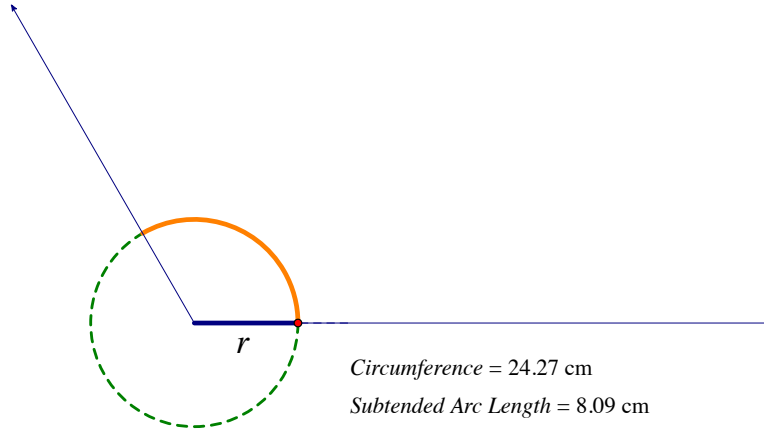
David’s responses in Excerpt 14 suggest his inclination to assimilate information to his proportion formula scheme rather than rely on his understanding of what it means to measure an angle in radians. When I confronted David with the task of determining the value of a quantity from given information, his calculational orientation dominated and did not necessitate his reliance on, or perhaps even permit him access to, more meaningful ways of understanding. Excerpt 14 demonstrates that David allowed himself to solve problems in ways that were uninformed by an image of quantities and quantitative relationships. For this reason, David’s mathematics appeared somewhat utilitarian in nature: he did what he had to do to solve problems correctly.

Theme 4: “I need to know how many units make up the whole circle.” During the initial clinical interview, David explained that a unit of angle measure must be a

particular “amount of circle” so that the number of units contained in the circumference of a circle is invariant. David continued to exhibit this way of understanding throughout TBCI 1, and justified his claim that he must know how many of a particular unit is contained in an entire circle by appealing to the information he would need to know to compute the measure of an angle using his part-whole proportion formula.

An objective of the first TBCI was to provide opportunities for David to deduce from P-EA 1 (the subtended arc length is always the same fraction of the circumference) and P-EA 2 (subtended arc length and circumference vary while the openness of the angle remains fixed) that measuring an angle involves measuring the arc length the angle subtends in units that covary with the circumference so that the subtended arc measured in this unit does not depend on the radius of the subtended arc. I call this deduction D2 in my conceptual analysis (see Chapter 5). To provide an occasion for David to make this deduction, I asked him to describe the condition that a unit of angle measure must satisfy (Task 2(c)). Upon probing David’s response to this task, I intended to engender a reflecting abstraction whereby he may construct the understanding that the unit of measure for the openness of an angle must be proportional to the circumference of the circle centered at the vertex of the angle. Excerpt 15 contains my question as well as the ensuing conversation. During this conversation David referred to the image in *Figure 33*.

Vary Radius
Hide Measurements



Subtended Arc 

Circumference 

Figure 33. Context of Excerpt 15 (Task 2(c)).

Excerpt 15

- 1 Michael: So we know that we have degrees to measure angles. We have, um, radians. You know, we may have quips from the other day. What does, what condition does a unit of measure for an angle have to satisfy so that you could use it as a unit of measure for an angle?
- 2 David: Um, I would say that the only thing that it would really have to satisfy is it would have to be, have some kind of, uh, universal acceptance for, so that when we said a quip we knew what a quip was or a radian, we know what a radian is, or a degree, and how it relates to the entire circle. So, you know, we need to know how many radians make up an entire circle, how many degree measures would make up an entire circle, how many quips make up an entire circle.

- 3 Michael: So say you wanted to make up a unit of angle measure, call them [Davids]. So what would you have to tell, if you wanted me to start using [Davids] to measure all of my angles, what would you have to tell me so that I would be able to do that?
- 4 David: I would have to tell you the number of [Davids] in one, um, circumference of circle.
- 5 Michael: Okay. Um, ... let's just make something up.
- 6 David: Okay so, five.
- 7 Michael: Okay. So five [Davids] make up a whole circle. So if I wanted to measure this angle here (*points to the angle on the screen*) in [Davids], what would I, how would I do that?
- 8 David: Then it would be, uh, 8.09 divided by 24.27 so that we get that ratio of arc to circumference, so part to whole, and then I would multiply by the whole, which was five [Davids] to get the part in [Davids].
- 9 Michael: So without specifying how many, um, without specifying how many [Davids] make up a whole, um, circle, why could I not measure this angle in [Davids]?
- 10 David: Um, (*long pause*) because (*pause*) if we think in terms of the, um, part-whole relationship in this, um, proportion that I've set up ... there's just too many unknowns in order to make any useful unit.

David explained in Line 2 that one would have to know how a unit “relates to an entire circle” or, more precisely, how a many of the unit “make up an entire circle.” If

David's response to my question in Line 3 was informed by an understanding that a unit of angle measure must be proportional to the subtended arc length, or, by extension, the circumference of the circle that contains the subtended arc, he might have said something like, "The subtended arc length, or the circumference, always needs to be the same number of times as large as the unit of measure so that when I measure the length of any arc the angle subtends in this unit I always get the same number." Instead, David explained that he would have to tell me the number of *Davids* (a hypothetical unit) contained in the circumference of a circle.

In an effort to get a sense for why David proposed that one would need to know how many of a particular unit make up an entire circle, I prompted him to explain how he would measure an angle in a unit that satisfies this condition. For the sake of definiteness, we settled on there being five *Davids* in the circumference of a circle. I then asked David to measure the angle in *Figure 33* in *Davids* (Line 7). In particular, I wanted to determine if David's condition for a unit of angle measure derived from his understanding of quantities and quantitative relationships or if it resulted from his need to obtain the information required to use his part-whole proportion formula to determine the measure of an angle in a particular unit. An answer to my question that is based on an understanding of quantities and quantitative relationships might be something like, "Since an angle that subtends an entire circle has a measure of five *Davids* , and since the angle in *Figure 33* subtends $8.09/24.27^{\text{ths}}$ of the circle's circumference, then the measure of the angle in *Figure 33* , in *Davids* , is $8.09/24.27$ of five." In contrast to an explanation of this type, David described a sequence of calculations that appeared to be based on his image of plugging values into the formula

$$\frac{\text{Measure of the angle in Davids}}{\text{Number of Davids in a whole circle}} = \frac{\text{Subtended arc length}}{\text{Circumference}}$$

and solving for “Measure of the angle in *Davids*.” David’s remark in Line 8—particularly when he said, “so part to whole, and then I would multiply by the whole”—suggests that he algebraically manipulated his part-whole proportion formula mentally and verbalized the arithmetic operations represented in the result.

I conjectured that David’s condition for a unit of angle measure was based on his need to have enough information to solve his part-whole proportion formula. To assess this conjecture, I asked David to explain why I could not measure the angle in *Figure 33* if I didn’t know how many *Davids* make up a whole circle (Line 9). David’s response in Line 10—namely his statement that there’s just “too many unknowns”—suggests that his reason for needing to know how many *Davids* are contained in a circle was informed by his need to be able to solve the equation

$$\frac{\text{Angle measure (in unit } X)}{\text{Number of unit } X\text{s in a whole circle}} = \frac{\text{Subtended arc length}}{\text{Circumference}}$$

for “Angle measure (in unit *X*).” David understood that without knowing the number of *Xs* in a whole circle, one could not solve this equation for “Angle measure (in unit *X*).” The information that was required for him to use his part-whole proportion formula to determine the measure of an angle in a particular unit appears to have been the primary motivation behind David’s description of the condition that a unit of angle measure must satisfy.

Later in the interview I asked David if the radius of a circle centered at the vertex of an angle satisfies the condition he proposed for a unit of angle measure (Task 3(a)).

Excerpt 16

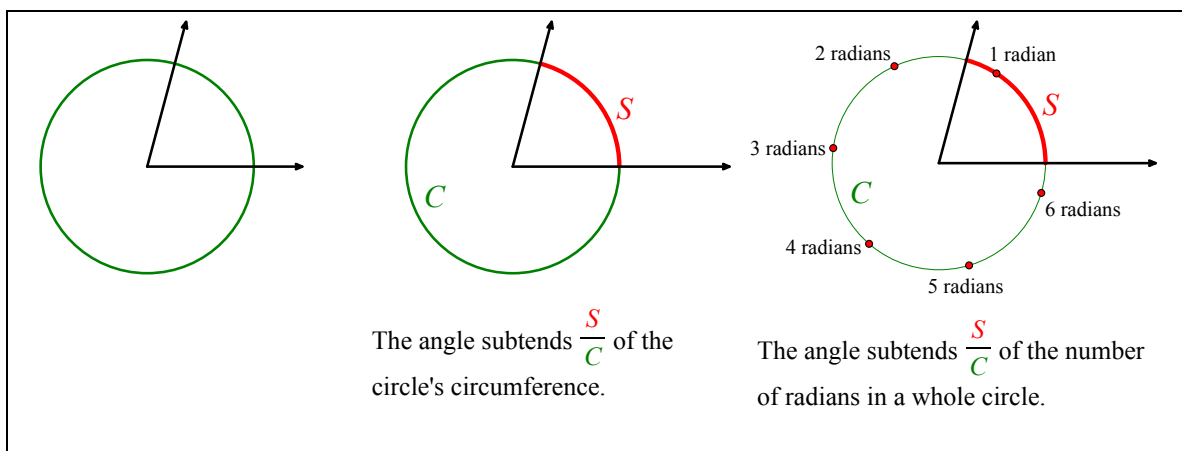
- 1 Michael: Is the radius of the circle, um, this length, an appropriate unit for measuring the angle? ...
- 2 David: Yes.
- 3 Michael: Why is that?
- 4 David: Because the length of the radius is 2π to the circumference. So I know how many radius are in a, um, circumference. So if it has a fixed radius length I could measure that. Um, (*pause*) I am trying to think if I even need it to be fixed because I do know how many radiuses are in a circumference so.
- 5 Michael: Is that the essential condition?
- 6 David: Yes. ... Uh, because before we quantified that as long as I could, I knew how the unit of measure related to the, uh, length of the circumference, or how much of it made up a circle, then it could be an appropriate measure.

David's response in Line 4 is consistent with the condition for a unit of measure he provided in his response to Task 2(c) in Excerpt 15. David explained that the radius is an appropriate unit with which to measure the openness of the angle because he knows how many radiuses are contained in the circumference of the circle. At first, David did not seem to be thinking about the fact that there are 2π radiuses in the circumference of every circle. David's remark in Line 4, "So if it has a fixed radius length I could measure that. Um, (*pause*) I am trying to think if I even need it to be fixed because I do know how many radiuses are in a circumference" suggests he noticed that, if the circumference of

the circle changes, the radius will change accordingly so that there are always 2π radius lengths in the circumference of the circle.

Summary of TBCI 1. David demonstrated a way of understanding angle measure as a comparison of subtended arc length and circumference on several occasions during TBCI 1. However, David did not claim that the measure of an angle in radians is the ratio of subtended arc length to circumference, as he did in the initial clinical interview. Instead, David demonstrated that he understood that the measure of an angle in radians is a fraction of 2π , the number of radians “in a whole circle.” In particular, David explained that the measure of an angle in radians is the fractional part of the circle’s circumference the angle subtends of the number of radians in a whole circle.

My interpretation of David’s language and actions during the first TBCI compelled me to amend by initial conjecture regarding the mental imagery in which David engaged while determining the measure of an angle in radians (see *Figure 27* for an illustration of my initial conjecture). When presented with an angle and the task of measuring it in radians, I conjecture that David visualized something similar to the images displayed in *Figure 34*.



$$\text{Angle Measure} = \left(\frac{S}{C} \right) \cdot 2\pi \text{ radians.}$$

Figure 34. Model of David's conception of angle measure in radians.

While the way of understanding angle measure illustrated in *Figure 34* is not simply a multiplicative comparison of subtended arc length and circumference (as it was in the initial clinical interview), it nonetheless relies upon this multiplicative comparison. While demonstrating this way of understanding, David attended to quantifying the fraction of the circle's circumference an angle subtends. David noticed that a particular angle subtended $1/3^{\text{rd}}$ of the circumference of the circle centered at the angle's vertex and claimed that the measure of the angle was therefore $1/3^{\text{rd}}$ of 2π , the number of radians in a whole circle. Accordingly, David quantified the openness of the angle by determining how many of the 2π , or approximately 6.28, radians in the whole circle the angle subtended. It is important to note that this process of measuring the angle in radians relied upon David's knowledge that there are 2π radians in a whole circle. David was not measuring the length of the subtended arc in a particular unit, but rather determining how many of the 2π pieces of circumference an angle subtended. An important characteristic of David's way of understanding angle measure in radians as a comparison of subtended arc length and circumference is therefore its inattention to the radius as a unit of measure.

During TBCI 1, David also demonstrated an understanding of angle measure in radians as a comparison of subtended arc length and the length of the radius of the circle centered at the vertex of the angle. In particular, David approximated the measure of an angle by estimating the number of radius lengths contained in the subtended arc. David also explained that an angle with a measure of 3.92 radians subtends an arc length that is

3.92 times as large as the length of its radius. Additionally, David instructed me to use the features of *Geometer's Sketchpad* to construct an angle with a measure of 1.6 radians by creating a subtended arc that was 1.6 times as large as its radius. These three occasions, together with several others, demonstrated David's understanding that an angle with a measure of n radians subtends an arc that is n times as long as the arc's radius.

David's understanding of angle measure in radians as a multiplicative comparison of subtended arc length and radius length was based on his image of quantities and his construction of quantitative operations. David concretized the "openness" of an angle—a opaque and ill-defined attribute—as the arc length the angle subtends.⁷⁹ Moreover, David appeared to conceptualize a way to quantify the openness of an angle by constructing the ratio of subtended arc length to radius length as a quantitative operation. One would expect that this quantitative way of understanding angle measure would support David in conceptualizing novel problems involving angle measure. However, during the first TBCI David assimilated many tasks to a calculational scheme that allowed him to avoid reasoning quantitatively.

During TBCI 1, David appeared to be very familiar with what I refer to as a *part-whole proportion formula*. Specifically, David maintained a strong disposition to identify values in a problem statement as representing parts and wholes of certain types, and inserted these values into a formula template like the following:

$$\frac{\text{Part (of type } X)}{\text{Whole (of type } X)} = \frac{\text{Part (of type } Y)}{\text{Whole (of type } Y)}$$

⁷⁹ I say, "*the* arc length the angle subtends" because David's understanding of angle measure in radians did not, at the conclusion of the first teaching episode, appear to account for a multiplicity of subtended arcs.

When David did so, I claim he assimilated a task to his “part-whole proportion formula scheme.” David even had a phrase he would recite upon reading or hearing a problem statement: “[*Part of type X*] is to [*Whole of type X*] the same way that [*Part of type Y*] is to [*Whole of type Y*].” David applied his part-whole proportion formula with remarkable fluency and flexibility throughout TBCI 1. David’s only justification for this equality was that the parts of different types were in their respective numerators and the wholes of different types were in their respective denominators. David did not appear to understand these equivalent expressions as different representations of the fraction of a circle’s circumference subtended by an angle, which indicates David likely did not conceptualize the expressions on each side of this equality as quantitative operations.

David’s confidence in this part-whole proportion formula as a means of allowing him to solve problems related to angle measure was so unwavering that David rarely attempted to reason quantitatively when confronted with a novel task; his part-whole proportion formula was often his first recourse. Not only did David’s part-whole proportion formula negate any necessity to attend to quantities and quantitative relationships, his consistent reliance on this formula seemed to prevent him from utilizing meanings he had previously demonstrated. That David plugged values into his part-whole proportion formula and performed a circuitous computation to determine the measure of an angle in radians instead of simply dividing the subtended arc length by the radius of the circle centered at the angle’s vertex is but one example of how his part-whole proportion formula deterred him from reasoning quantitatively (see Excerpt 14).

David’s part-whole proportion formula was so dominant in his thinking that it was the basis for his justification for the condition that a unit of angle measure must satisfy.

David explained that one would need to know how many of a particular unit “make up an entire circle.” This condition for a unit of angle measure appeared to derive more from David’s need to calculate the measure of an angle using his part-whole proportion formula than it did from an understanding that, for the measure of an angle to be invariant, the unit one uses to measure the angle must covary with the length of the subtended arc so that the subtended arc length measured in this unit is always constant. In other words, when presented with the task of measuring an angle, David seemed to have the following instantiation of his part-whole proportion formula in mind:

$$\frac{\text{Subtended arc length}}{\text{Circumference}} = \frac{\text{Angle measure (in unit } X \text{)}}{\text{Number of } X\text{s in a whole circle}}$$

David appeared to anticipate that he could measure the subtended arc length and circumference (presumably in a standard linear unit like centimeters or inches). To use this formula to determine the measure of the angle in unit X , David recognized that he would need to know the number of X s contained in the circumference of a circle. Accordingly, David claimed that the number of a particular unit contained in a whole circle must be known for the unit to be appropriate for measuring angles.

Adjustments to TBCI 2. After having conducted TBCI 1, and performed a surface-level analysis, I was dissatisfied with the fact that David was able to assimilate so many of my tasks to his part-whole proportion formula scheme. While David’s engagement with these tasks did reveal his calculational orientation to solving problems involving angle measure, they did not engender the abstractions nor elicit the reasoning I designed them to. There were, however, instances during TBCI 1 where David demonstrated more profound ways of understanding angle measure than his pseudo-

proportional reasoning suggested. I was not confident at the conclusion TBCI 1 that I had collected enough data to construct a robust and viable model of David's way(s) of understanding angle measure. I therefore designed tasks between the first and second TBCIs that were less accessible to David's pseudo-proportional reasoning and that sought to provide opportunities for David to coordinate his two ways of understanding what it means to measure an angle in radians. I discuss only those tasks I designed during ongoing analysis that are relevant to my discussion of the reasoning David demonstrated in the second and third TBCIs (see Table 41).

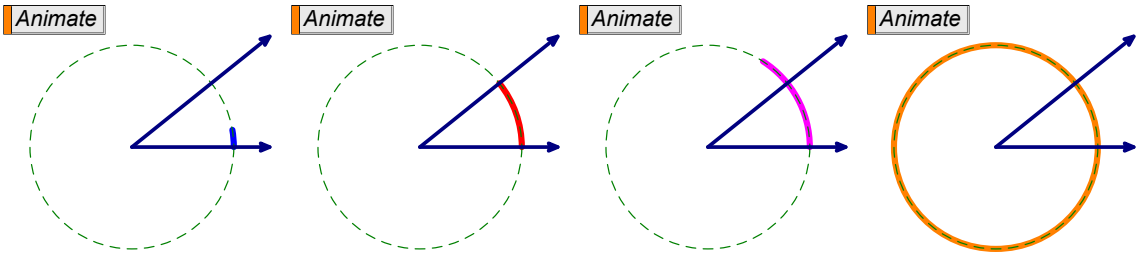
Table 41

Tasks Added During Ongoing Analysis for TBCI 2

Task 1: [Animation in *Geometer's Sketchpad*] Which of these units of measure is appropriate for measuring the openness of an angle? Explain.

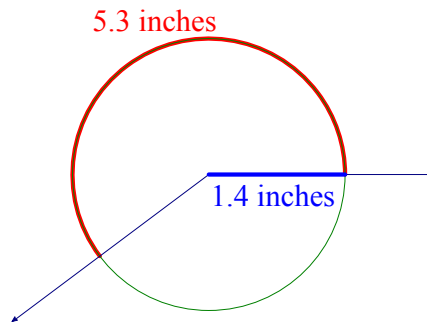
Unit 1
Unit 2
Unit 3
Unit 4
Circumference

Task 2: [Animation in *Geometer's Sketchpad*] Which of these units of angle measure is appropriate for measuring the openness of the angle? Explain.

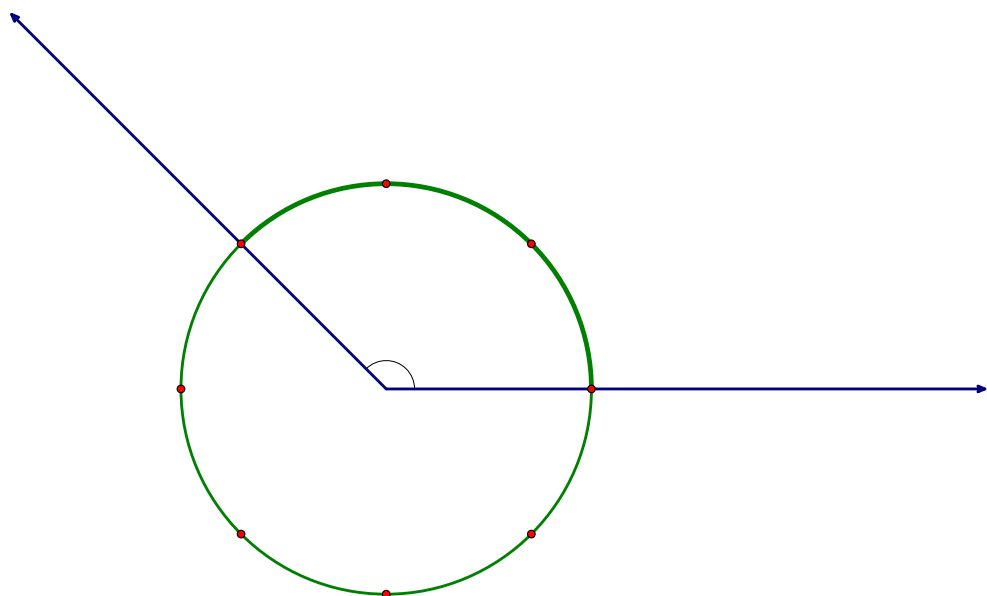


Task 3: Draw an angle with a measure of 2.5 radians.

Task 4: Suppose an angle has a circle of radius 1.4 inches centered at its vertex and suppose the angle subtends an arc length of 5.3 inches. What is the measure of the angle in radians?



Task 5: Nick claims that the measure of the angle shown is $\frac{3}{8}^{\text{th}}$ and Meghan claims that the measure of this angle is three. How is Nick thinking about measuring this angle? How is Meghan thinking about measuring the angle? Are they both correct?



Task 6: Courtney claims that measuring an angle in radians means measuring the arc length that the angle subtends in units of the radius of the circle centered at the vertex of the angle. Rebecca says that to measure an angle in radians, you take the length of the arc that the angle subtends divided by the length of the

circumference of the circle centered at the vertex of the angle and multiply this ratio by 2π . Are Courtney and Rebecca correct? How could they be talking about the same thing?

During TBCI 1, David explained that for one to use a particular unit to measure the openness of an angle, one would need to know how many of those units are contained in the circumference of a circle. David justified this condition by appealing to his need to have enough information to use his part-whole proportion formula to determine the measure of an angle in a particular unit. I designed Tasks 1 and 2 in Table 41 to assess whether David could articulate a rationale for the condition a unit of angle measure must satisfy that is not based on his need to calculate.

Pressing the “Animate” button in Tasks 1 and 2 causes the size of the circle centered at the vertex of the angle to vary. In Task 1, four potential units of angle measure, whose magnitudes are displayed on the bottom of the screen, covary with the circumference in different ways. The circumference is related to the lengths of Units 1 and 4 additively and is related to the lengths of Units 2 and 3 multiplicatively (i.e., the lengths of Units 2 and 3 are proportional to the circumference of the circle while Units 1 and 4 are not). In contrast to Task 1, the four potential units of angle measure in Task 2 are placed along the subtended arc and all are proportional to the circumference of the circle.

I intended Tasks 3 and 4 to elicit observable products of David’s reasoning that might allow me to further refine my model of David’s way of understanding angle measure. In particular, I wanted to determine if David would utilize his “angle measure as a comparison of subtended arc length and radius length” way of understanding or if he

would employ his “angle measure as a comparison of subtended arc length and circumference” way of understanding while engaged in these tasks.

I designed Tasks 5 and 6 to provide occasions for David to reconcile the two ways of understanding angle measure he demonstrated during TBCI 1. I intended these tasks to support David in becoming consciously aware of his two ways of understanding angle measure by allowing him to perform mental operations on them (i.e., I designed these tasks to engender a reflected abstraction).

Task-Based Clinical Interviews 2 and 3

I present my analysis of TBCIs 2 and 3 together because of the similarity in the content of these interviews as well as the consistency in the themes that emerged from my analysis of them.

Tasks 1 and 2 that I designed during ongoing analysis supported David in constructing a more coherent understanding of the condition that a unit of angle measure must satisfy, namely that *a unit of angle measure must be proportional to the circumference of the circle centered at the vertex of the angle* (Theme 1). In particular, the didactic objects to which my questions referred supported David in articulating his understanding in a way that was uninformed by his anticipation of needing to solve his part-whole proportion formula, which was not the case during the previous clinical interview. Additionally, Tasks 3 and 4 in Table 41 supported David in reflecting upon his two ways of understanding angle measure, which ultimately allowed him to coordinate these understandings into a coherent scheme. Accordingly, a second theme that emerged from my analysis of TBCIs 2 and 3 was, *David’s coordination of his two ways of understanding angle measure*.

Theme 1: A unit of angle measure must be proportional to the circumference of the circle. I began TBCI 2 with Task 1 in Table 41. I displayed Unit 1 and clicked on the “Animate” button, which made the size of the circle centered at the vertex of the angle vary. I then asked David if Unit 1 was an appropriate unit of measure for the openness of the angle (recall Unit 1 was related to the circumference of the circle additively and not multiplicatively). David immediately stated that Unit 1 was not an appropriate unit of measure and explained his rationale by stating, “as your radius increases and decreases, the circumference and the unit measure length, um, seem to be changing at the same rate and not, uh, in proportion to each other.” I was uncertain as to what David meant by “in proportion to each other” so I asked him to elaborate why his observation implied that Unit 1 was not an appropriate unit with which to measure the angle. I provide David’s reply in Excerpt 17.

Excerpt 17

- 1 David: When we’re measuring angles we’re talking about the openness or the part of the circle that is being retained so (*pause*) the ratio of the two needs to be constant, um, but since (*Michael interrupts*)
- 2 Michael: Can I ask, the ratio of what two?
- 3 David: Of the unit and the circumference should be constant because then we should be seeing that the angle measure is retained. ... The problem is they’re not growing, uh, proportionally to each other.

David’s remark in Line 1, “When we’re measuring angles we’re talking about the openness or the part of the circle that is being retained” suggests his understanding that quantifying the openness of an angle involves measuring the length of the arc the angle

subtends. David then claimed, as if by way of implication, that the ratio of the unit length and the circumference must be constant. This appears to be what David meant by “in proportion” when he claimed that Unit 1 and the circumference of the circle were not “in proportion to each other.” David explained that if this condition is satisfied, the measure of the angle is constant.

David was clearly able to verbalize the condition that a unit of angle measure must satisfy but did not clearly communicate his rationale while discussing Unit 1. Instead of probing David’s response in Line 3 of Excerpt 17, I decided to introduce Unit 4—another unit that varied additively with the length of the circumference—in hopes that he would articulate his reasoning more clearly while discussing this potential unit of angle measure.

Excerpt 18

- 1 Michael: Is Unit 4 an appropriate unit of measure?
- 2 David: (*David watches the animation*) No.
- 3 Michael: Okay. Why is that?
- 4 David: Um, because this one you can see Unit 4 at one point actually goes all the way down to like zero but the circumference is not zero so when you’re thinking about proportion of a circle it’s, you know, at one point it seems to be a greater proportion than at other points. Again it doesn’t seem that the length, the difference between the two is not changing, is staying, remaining constant. Instead of the ratio remaining constant. ...

- 5 Michael: So what is it about Unit 4? What's wrong with Unit 4 as a potential unit of measure for the angle?
- 6 David: Uh, the (*pause*), instead of the lengths, um, growing at the same, with the same scale factor, or growing or shrinking according to the same scale factor, the length of the circumference and the length of the unit, they're, um, growing by, um, equal changes. So if one changes by a tenth the other one also increases by a tenth and, where it should be if one is increases by ten percent then the other one would increase by ten percent. So it's a, it's an additive, um, you're adding to both lengths instead of scaling both lengths.

David noticed that, in contrast to Unit 1, the length of Unit 4 collapsed to zero during the animation (Line 4). Additionally, David's remark in Line 4, "at one point [in time the length of Unit 4] seems to be a greater proportion than at other points [in time]" suggests that he was attending to the multiplicative comparison of the length of Unit 4 and the circumference at different moments during the animation. David then pseudo-empirically abstracted the property that the circumference is always a constant length larger than the length of Unit 4, which, for David, necessarily meant that the ratio of unit length to circumference was not constant (Line 4).

Although, in Line 4 of Excerpt 18, David explained that for a unit to be appropriate for measuring the openness of angles (i.e., for measuring the length of the subtended arc) the ratio of the length of the unit and the circumference of the circle centered at the vertex of the angle must be constant, he had not satisfactorily explained why this condition needs to be satisfied. Accordingly, I asked David to articulate why

Unit 4 is not an appropriate unit of angle measure (Line 5). David's response in Line 6 was essentially a restatement of the property he pseudo-empirically abstracted in Line 4: that the unit length and the circumference were related additively instead of "growing or shrinking according to the same scale factor." David's response in Line 6 still did not reveal his rationale for why the length of a unit of angle measure must be proportional to circumference. I therefore probed David's response by asking the question in Line 1 of Excerpt 19.

Excerpt 19

- 1 Michael: (*Referring to David's remark in Line 6 of Excerpt 18*) So why is that a problem for a unit of measure, or a potential unit of measure for an angle?
- 2 David: Uh, because we're comparing part-whole (*points to the line segments representing the length of Unit 4 and the length of the circumference on the bottom of the computer screen*) so we're doing proportions and so then the lengths should stay proportional so that the measure is constant, um, so that when the circle is, has a longer radius, the proportion will take care of the length unit of measurements in the proportion will end up canceling and we just are left with the proportion and the proportion stays the same. If I shorten the radius, the proportion of the circle to the, um, unit length should be the same. Uh, it shouldn't have, um, where when we're just adding, you know, to that then that ratio doesn't remain constant.

3 Michael: And the ratio you're talking about is (*David interrupts*)

4 David: The ratio of unit to, um, one unit length to the, um, circumference of the whole circle.

As with David's response in Line 6 of Excerpt 18, his reply in Line 2 of Excerpt 19 does not convey why the measure of an angle does not depend on the size of the circle centered at the angle's vertex when the angle is measured in a unit whose length is proportional to the circumference of the circle. David's response in Line 2 is a more vague and imprecise restatement of the condition a unit of angle measure must satisfy. David began his response to my probing question in Line 1 by claiming, "we're comparing part-whole so we're doing proportions." The part-whole comparison to which David referred was the comparison of the length of Unit 4 and the circumference, represented by separate line segments at the bottom of the computer screen (see Task 1 in Table 41). David proceeded to explain, "so then the lengths should stay proportional so that the measure is constant." This was again a restatement of the essential property of a unit of angle measure David had already proposed. That this conclusion followed by way of implication from David's recognition that "we're comparing part-whole so we're doing proportions" suggests that David may have reasoned as follows: "Because we're comparing part (i.e., unit length) to whole (i.e., circumference) the part has to always be the same part, or fraction, of the whole." David was still unable to articulate why the measure of an angle is constant when measured in a unit that satisfies the condition he had repeatedly stated.

The final noteworthy aspect of David's response in Line 2 is his observation that when the circle centered at the vertex of an angle "has a longer radius, the proportion will

take care of the length unit of measurements in the proportion will end up canceling and we are just left with the proportion and the proportion stays the same.” This remark indicates that David may have imagined a unit of angle measure (call it Unit X) and the circumference being measured in the same standard linear unit, and recognized that, as the size of the circle varies, the ratio of the lengths of these quantities reduce and the units cancel, resulting in a constant ratio, or “proportion” in his usage. In other words, to David, the ratio of the length of Unit X to circumference is a ratio of lengths measured in the same standard linear unit that, when reduced, results in a constant ratio. There is not enough in David’s response to claim with certainty whether he anticipated the ratio of the length of Unit X to circumference being constant prior to imagining these quantities being measured in a standard linear unit. Based on David’s remark above, I find it plausible that he did not initially think of the ratio of unit length to circumference as a quantitative operation that represents the fraction of the circle’s circumference subtended by an angle with a measure of one Unit X . In other words, David did not appear to imagine a unit length and the circumference covarying so that, at all moments during the variation of these quantities, the circumference was the same number of times as long as the unit of measure, or that the length of the unit of measure was always the same fraction of the circumference. Thinking in this way would not have involved imagining measuring these quantities in a standard linear unit, nor would it have necessitated reducing the ratio of these quantities’ values and canceling their units. That the length of Unit X is the same fraction of the circle’s circumference at all moments during the variation of these quantities’ values appeared to result from David’s anticipation that at any moment during the variation of these quantities, the ratio of the length of Unit X to circumference would

reduce and the result would be constant. For David, the constant ratio of the length of Unit X to circumference therefore appeared to emerge from anticipating the result of an arithmetic operation (reducing a fraction) rather than from a quantitative operation. In sum, David was attending to the measure of the length of Unit X and the length of the circumference and *then* appeared to think about the ratio of these quantities' values reducing and being constant. David's statement, "we're comparing part whole so we're doing proportions" suggests that conceptualizing the multiplicative relationship between the length of Unit X and the circumference as a quantitative operation was not David's motivation for conceiving the ratio of these quantities' values.

In the context of discussing Task 2 in Table 41, David did demonstrate the understanding that, while the size of the circumference of the circle centered the vertex of an angle varied, the length of a proper unit of angle measure varied in tandem so that the length of the unit was always the same fraction of the circle's circumference. Consider, for instance, David's remarks in Excerpt 20.

Excerpt 20

- 1 Michael: Let's take a look at another visual here. Um, again same question now. We have three different units; the question is, which of these units of measure is appropriate for measuring the openness of the angle? ... Let's go ahead and take a look at Unit 1 (*clicks on* "Show Unit 1"). So there's, I've put the length of Unit 1 in that orientation versus down here. And so is Unit 1 an appropriate measure for the openness of the angle?
- 2 David: Yes it appears to be.

- 3 Michael: Why is that?
- 4 David: Because one Unit 1 appears to be, um, if I put a, if I traced along the, um, terminal end of that arc that's not part of the actual angle
(Michael interrupts)
- 5 Michael: So the terminal, like where Unit 1 ends?
- 6 David: Right. If I was to put a dot on the end of that and have it trace it out, what I would end up seeing is a line going back into the vertex.
And so that tells me that, um, this unit shows me the same angle for all radius lengths. Um, so Unit 1 is always giving me that same angle measurement no matter what the radius is. Unit 1 counts for that portion of the subtended arc, so that portion of the circle.
- 7 Michael: So why does that, why does the fact that if you were to put a dot on the end of this, um, *(David interrupts)*
- 8 David: Because if we're saying that that is one unit of measure then that means that no matter what the length is if I draw an angle with a measure of one of those units I should get the same angle at all times. And since that arc length (*Unit 1*) does seem to grow and shrink and it does seem to be at this, like, like I said if I traced on the end, so that means I know that the proportion is, uh, being, uh, maintained. The ratio of circumference to unit, um, or sorry unit length to circumference. So Unit 1 seems to be maintaining its relationship to the circle as a whole. ... Could we pause it just for one second? *(Michael pauses animation)* Let's see *(pause)* the

measure of the angle would be maybe, um, four Unit 1s. Let's see one, two, three, yeah somewhere around four Unit 1s. I think I can fit those little, the blue arc lengths that represent Unit 1, I think I could fit four of them in that space.

9 Michael: Okay. And does that (*David interrupts*)

10 David: It seems to hold true as we vary the radius.

11 Michael: (*Michael changes the radius*) So it doesn't matter? I could make it, I could have paused it there?

12 David: Right. The subtended arc would still be four Unit 1s.

David explained in Lines 4 and 6 that Unit 1 is an appropriate unit of angle measure because, as the size of the circle centered at the vertex of the angle varies, an angle that subtends an arc length equal to the length of Unit 1 has a fixed amount of openness. David's statements, "this unit shows me the same angle for all radius lengths" and "Unit 1 is always giving me the same angle measurement no matter what the radius is" support this claim. David pseudo-empirically abstracted the property that, as the radius of the circle centered at the vertex of the angle varied, the terminal point of an arc whose length is equal to that of Unit 1 traced out the terminal ray of an angle with a measure of one Unit 1 (see *Figure 35*).

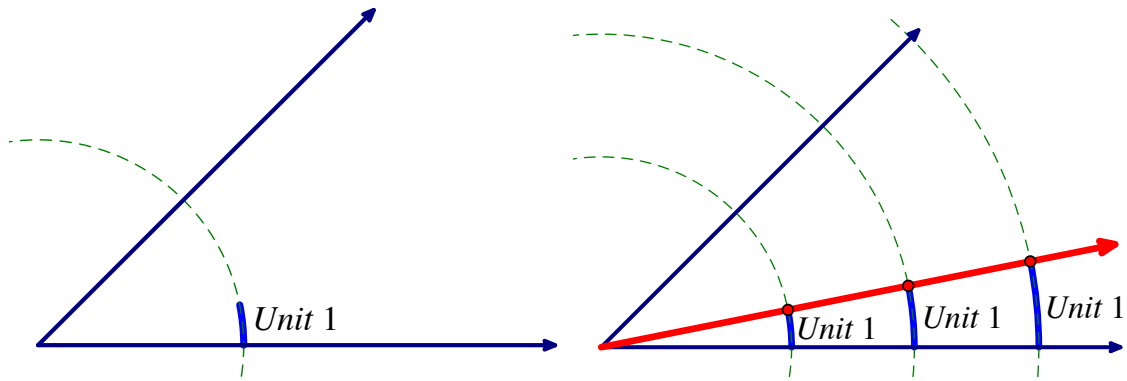


Figure 35. David's pseudo-empirical abstraction about Unit 1 in Task 2.

In Line 7 I began to ask David to justify the proposition, “if an angle with a measure of one Unit 1 has a constant amount of openness, then Unit 1 is an appropriate unit of angle measure.” Before I could finish the question, David restated the hypothesis of this proposition and explained that, as the size of the circle varies, Unit 1 appears to be the same fraction of the circle's circumference. In contrast to his remarks in Excerpt 19, David's responses in Excerpt 20 suggest that he constructed the ratio of the length of Unit 1 to the circumference as a quantitative operation that represents the fraction of the circle's circumference subtended by an angle with a measure of one Unit 1. Additionally, David approximated the measure of the angle shown on the screen by estimating how many Unit 1s fit into the arc length the angle subtends and explained, without hesitation, that the measure of the angle in Unit 1s is constant as the size of the circle centered at the vertex of the angle varies. David's response in Line 12 suggests that he understood that, whatever the size of the radius, the arc length the angle subtends is always four times as long as the length of one Unit 1. David therefore appeared to attend to the fact that an appropriate unit of angle measure is not just a constant fraction of the circle's circumference, but also a constant fraction of the length of the arc that a static angle

subtends. David's remarks in Excerpt 20 are similar to those he provided in the context of discussing Units 2, 3, and 4.

After having discussed Tasks 1 and 2 in Table 41, I again prompted David to explain why a unit whose length is proportional to the circumference of a circle is appropriate for measuring the openness of angles. My previous efforts to this end resulted in David either restating the condition he had already proposed or providing an elusive rationale for this condition. I had hoped that, after having discussed Tasks 1 and 2, David would be positioned to more clearly articulate his reasoning.

Excerpt 21

- 1 Michael: So what has to be true of a unit of measure so that if I were to change the radius of the circle, the measure of this angle (*referring to the angle from Task 2 displayed on the screen*) is always the same in that particular unit?
- 2 David: The unit needs to have a proportional relationship with circumference so that it's either a fixed, um, proportion or percent of the circle's circumference or it, um, (*pause*) yeah actually that's it (*inaudible*).
- 3 Michael: So why does that condition have to be satisfied? ... So units that satisfy that condition, why is it as I move the radius around, the measure of the angle in that particular unit is always the same?
- 4 David: Because when we think about units, um, when we think about units for angle measures we're really saying, um, what part or percent of a circle is the subtended arc, um, representing (*points with his pen*

to the length of the arc the angle on the screen subtends). And so if we don't have that proportional relationship then that means depending on the size of the radius we'll get, um, different angle measurements for the same angle. So we need to make sure that it's, that ratio is maintained.

5 Michael: Why will the measure of the angle change if (*David interrupts*)

6 David: Because then it (*the length of the subtended arc*) won't be the same portion of the whole circle.

In Line 2 David explained that a unit of angle measure “needs to have a proportional relationship with circumference.” David's response in Line 4 to my asking him why this condition needs to be satisfied, like so many of his previous efforts, suggests a multiplicity of potential mental images and conceptual operations. In retrospect, I wish I had asked David to draw an image to represent his reasoning. Being that I did not, the following is my best attempt at postulating the mental activity that informed David's response in Lines 4 and 6. My analysis of these responses is informed by my interpretation of David's language and actions while engaged in Tasks 1 and 2 in Table 41.

David began his response in Line 4 of Excerpt 21 by claiming that measuring the openness of an angle involves determining the fraction of a circle's circumference the angle subtends. This remark is consistent with David's way of understanding angle measure as a multiplicative comparison of subtended arc length to circumference. David then proceeded to explain that if a unit of angle measure is not proportional to the circumference of the circle centered at the angle's vertex, then the measure of an angle

with a fixed amount of openness depends on the radius of this circle. I asked David why this is the case (Line 5) and he stated that, if a unit of angle measure is not proportional to the circumference of the circle, then as the size of the circle changes, an angle with the same measure in this particular unit will not subtend an arc that is the same fraction of the circle's circumference.

On several occasions David explained that an angle subtends a constant fraction or percentage of all circles centered at the vertex of the angle. David's remarks in Excerpt 20 also reveal his understanding that, if Unit X is an appropriate unit of angle measure, then an angle that subtends one Unit X will have a constant amount of openness, irrespective of the size of the circle centered at the vertex of the angle. Given these understandings, I hypothesize David's responses in Lines 4 and 6 of Excerpt 21 were informed by a mental image similar to diagram in *Figure 36*, and derived from a chain of observations and deductions like those provided in Table 42.

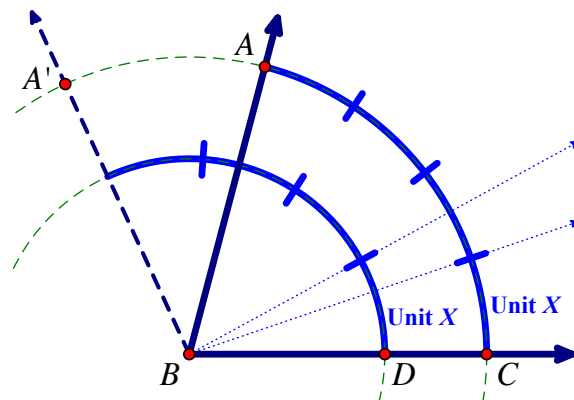


Figure 36. My model of David's mental imagery that informed his remarks in Excerpt 21.

Table 42

My Model of the Reasoning that Informed David's Remarks in Excerpt 21

Statement	Warrant
1. If we consider the circle in <i>Figure 36</i> with radius	David's way of understanding

\overline{BC} , then $\angle ABC$ has a measure of three Unit X s.	angle measure as a comparison of subtended arc length and a unit of measure.
2. We notice that Unit X does not vary proportionally with the circumference of the circle centered at B because the openness of an angle that subtends one Unit X is not constant.	David's remarks in Excerpt 20.
3. Because Unit X does not vary proportionally with the circumference of the circle, an angle with a circle of radius \overline{BD} centered at its vertex that subtends an arc of three Unit X s does not have the same amount of openness as the angle with a circle of radius \overline{BC} centered at its vertex that subtends an arc of three Unit X s (i.e., $m\angle ABC \neq m\angle A'BC$).	David's remarks in Excerpt 20 and his response in Line 4 of Excerpt 21.
4. Therefore, Unit X is not an appropriate unit of angle measure because an angle with a measure of three Unit X s subtends a different fraction of a circle's circumference depending on the size of the circle.	David's statement in Line 6 of Excerpt 21.

Theme 2: David's coordination of his two ways of understanding angle measure.

In TBCI 1 David demonstrated two complementary but conceptually distinct ways of understanding angle measure in radians: (1) angle measure as a comparison of subtended arc length and circumference, and (2) angle measure as a comparison of subtended arc length and radius length. For simplicity, I hereafter refer to these two ways of understanding as *WoU 1* and *WoU 2* respectively. These two ways of understanding informed how David interpreted the measure of an angle in radians as well as how he measured angles in radians. I summarize these ways of understanding in Table 43.

Table 43

David's Two Ways of Understanding Angle Measure in Radians

<i>WoU 1</i>	<i>WoU 2</i>
<u>Interpretation:</u> An angle with a measure of n radians	<u>Interpretation:</u> An angle with a measure of n radians

subtends $n/(2\pi)$ of the circumference of the circle centered at the vertex of the angle.

Measurement:

The measure of an angle, θ , in radians is determined by the formula $\theta = (s/c) \cdot 2\pi$ where S represents the length of the subtended arc and C represents the length of the circumference.

subtends an arc that is n radius lengths, or n times as large as the radius of the subtended arc.

Measurement:

The measure of an angle, θ , in radians is determined by the formula $\theta = s/r$ where S represents the length of the subtended arc and r represents the length of the radius of the subtended arc.

During TBCI 2 I asked David to sketch on paper an angle with a measure of 2.5 radians (Task 3 in Table 41). David did this very precisely using a compass and a piece of string. David began by drawing the initial ray in standard position and constructed a circle centered at the endpoint of the initial ray. David then measured the length of the radius with a string and used this string to measure an arc length equal to 2.5 radius lengths. David completed the task by drawing the terminal ray so that the arc length the angle subtended was 2.5 times as large as the radius of the circle. I then asked David to sketch an angle with a measure of 0.7 radians. After having sketched the angle in a very similar fashion to how he sketched the previous one, David explained his process for constructing a subtended arc length that was 0.7 radius lengths: “I first, um, did half of a radian, um, and then I knew if I cut that in half again that would give me point-seven-five so I just did a little under that.” David constructed these angles based on his understanding of angle measure in radians as the length of the arc an angle subtends measured in units of the radius of the circle centered at the angle’s vertex (*WoU 2*). Additionally, while David was engaged in Task 1 (see Table 41) during TBCI 2, I regularly asked him to approximate the measure of an angle displayed on the computer screen in Units 1–4. David often approximated the measure of these angles by measuring

the length of the arc the angle subtended in these respective units, thereby employing *WoU 2*.

I presented David with Task 4 (see Table 41) twenty-five minutes after he relied upon *WoU 2* to carefully sketch two angles with measures of 2.5 and 0.7 radians respectively. Task 4 asked for the measure of an angle in radians provided the angle subtends an arc that is 5.3 inches and the circle centered at the vertex of the angle has a radius of 1.4 inches. The following is an excerpt of our conversation.

Excerpt 22

- 1 Michael: Suppose an angle has a circle of radius 1.4 inches centered at its vertex and suppose the angle subtends an arc of 5.3 inches. What is the measure of the angle in radians?
- 2 David: Okay. So, uh, circumference would be 2π times 1.4 (*writes “ $C = 2\pi(1.4)$ ”*) and the arc length then is 5.3. So 5.3 is to 2π times 1.4 as the ratio of arc length to circumference (*writes “ $\frac{S}{C} = \frac{5.3}{2\pi(1.4)}$ ” and computes $2\pi(1.4)$ in the calculator*). So (*writes “ $\approx 5.3/8.796$ ” next to the right hand side of his equation, performs this division in his calculator, and writes “ $= 0.602$ radians.”*)
- 3 Michael: Okay. So what’s the measure of the angle?
- 4 David: Uh, it’s about 0.6 (*pause*) radians.

To my surprise, David did not employ his way of understanding angle measure in radians as a multiplicative comparison of subtended arc length and radius length, in which case he would have simply divided 5.3 by 1.4. Instead, David determined the

length of the circumference, by computing the product of 2π and 1.4, and divided the subtended arc length by the circumference, a ratio of approximately 0.6. David then claimed that the measure of the angle was 0.6 radians. While solving this problem, David did not appear to be thinking about angle measure as a quantitative operation involving the quantities *subtended arc length* and *radius length*. Rather, David computed the ratio of subtended arc length to circumference and claimed that the value of this ratio was the measure of the angle *in radians*. It is noteworthy that David had not demonstrated this way of understanding angle measure since the initial clinical interview.

After David claimed that the measure of the angle was 0.6 radians, he sat quietly, fixating on his written work for about one minute. The following is an excerpt of the ensuing discussion.

Excerpt 23

- 1 David: *(Long pause) (whispers to himself)* That's not right. *(Long pause)*
- 2 Michael: You seem dissatisfied with something. Can you verbalize what
 (David interrupts)
- 3 David: Yeah I am dissatisfied with it. Oh I know why, cause I was being
 stupid, *(pause)* because I divided out the 2π . *(Long pause)* Yeah
 that's the reason why. Okay I know why. 'Cause I was not thinking
 about what I was actually doing *(crosses out his work)*. Uh, my
 dissatisfaction is it has a circumference of, um, 8.796 inches
 approximately and yet I said the arc length was 5.3 so that would
 be at about two-thirds of the complete distance around the circle,
 um, so it should not be, um, *(pause)* uh, so that doesn't make sense

‘cause half way around the circle would be three so there’s no way that that measure actually made any sense to me once I actually, uh, finished my computations ‘cause that doesn’t (*Michael interrupts*)

4 Michael: So you know that the angle measure has to be, for the reason you just said, bigger than three.

5 David: Right. Uh, okay. So let me think about what I did. Oh, that’s why. ‘Cause really all I need to do is break it up into 5.3 divided by, because I converted it into percentage of and then I would have to multiply it back by 2π to figure out what it, yeah, okay. I know what I did wrong. (*Long pause*) Okay. Instead of just doing (*writes “5.3/1.4”*) 5.3 divided by 1.4 (*speaks while typing the fraction into the calculator*) gives me approximately 3.785 radians. That is much better. Okay. So (*sigh*) um, looking at this (*points to where he wrote “5.3/1.4 = 3.785”*) the (*long pause*), I know that the radius is going to be, um, (*long pause*). (*Now talking to himself*) Okay. The circumference, sorry, is 8.79. Okay. And that’s now measured in inches. And so this would be the arc length that’s left (*points to where he wrote “S = 5.3”*), um, (*inaudible*) ... Yeah. That makes, I’m much happier with that.

David’s cognitive conflict revealed in Lines 1 and 3 arose from his recognition that the measure of the angle must be at least π radians because the angle subtends an arc length that is more than half of the circumference of the circle. Early in David’s response in Line 5 he explained, “ ‘Cause all I need to do is break it up into 5.3 divided by,

because I converted it into a percentage of and then I would have to multiply it back by 2π to figure out what it, yeah, okay.” This remark suggests that David understood that the ratio $5.3/(2\pi \cdot 1.4)$ represented the percentage of the circle’s circumference the angle subtended. David’s statement “I would have to multiply it back by 2π ” indicates that in this context David conceptualized angle measure in radians as a fraction of the number of radians in a circle (*WoU 1*). This is noteworthy because when David wrote “5.3/1.4” in Line 5, he *still* did not appear to think about angle measure in radians as the length of the subtended arc measured in units of the radius. David mentally simplified the product $(5.3/(2\pi \cdot 1.4)) \cdot 2\pi$ to $5.3/1.4$ but did not see this ratio as representing the number of times the subtended arc length is longer than the radius of the subtended arc. In other words, David did not seem to interpret the ratio $5.3/1.4$ as the result of measuring the length of the subtended arc of 5.3 inches in units of the radius (1.4 inches). *Figure 37* displays David’s written work.

The image shows handwritten mathematical work. At the top left, it says "radius 1.4\" and "s = 5.3\". To the right, it says "C = 2\pi(1.4)". Below these, a calculation is crossed out with a large diagonal line: $\frac{s}{C} = \frac{5.3}{2\pi(1.4)} \approx \frac{5.3}{8.796} = 0.602 \text{ radians}$. Below the crossed-out work, a new calculation is written: $\frac{5.3}{1.4} = 3.785 \text{ rad}$.

Figure 37. David’s written work to Task 4.

It is noteworthy that David did not assimilate Task 4 to a cognitive scheme that contained his understanding of angle measure in radians as a multiplicative comparison of subtended arc length and radius length. David’s difficulty with this task suggests that

his two ways of understanding what it means to measure an angle in radians were not coordinated into a coherent scheme. Nor did David appear to be consciously aware of his two ways of understanding angle measure in radians. Consequently, David was not equipped to strategically apply his understandings of angle measure in novel problem-solving situations.

I attempted to support David in becoming consciously aware of his two ways of understanding angle measure in radians by providing opportunities for him to perform mental operations on both of these ways of understanding. This is to say that I attempted to engender reflected abstraction⁸⁰. At the beginning of TBCI 3, I presented David with Task 5, shown in Table 44. I provide our discussion of this task in Excerpt 24.

Table 44

Task 5 from TBCI 3

Nick claims that the measure of the angle shown is $\frac{3}{8}$ ths and Meghan claims that the measure of this angle is three. How is Nick thinking about measuring this angle? How is Meghan thinking about measuring the angle? Are they both correct?

⁸⁰ See the description of reflected abstraction in Chapter 3.

Excerpt 24

- 1 Michael: Taking a look at this picture, what do you see in this picture here?
- 2 David: Uh, I see an, a central angle inscribed inside a circle. Uh, we have an arc, um, that seems to be in a bolder line that, um, subtends the circle and that would be our angle measure.
- 3 Michael: Okay. What about the red dots?
- 4 David: Uh, the red dots seem to be evenly spaced around the, um, around the circle. So it appears that we have basically divided the circle into, um, eight pieces. ...
- 5 Michael: Okay. So let's assume that that's the case. Let's assume that these red dots are evenly spaced around the circumference of the circle so that we're splitting up the circumference into eight equal pieces. And let's suppose that Nick claims that the measure of this angle is $3/8^{\text{ths}}$ and let's say that Meghan claims that the measure of this angle is three. Um, how might Nick, who claimed that the measure of the angle is $3/8^{\text{ths}}$, how might he be thinking about measuring this angle?
- 6 David: (*Pause*) Well if he's thinking of it as $3/8^{\text{ths}}$ then he is thinking that the arc is $3/8^{\text{ths}}$ of the whole. So his unit of measure is not the red dots, his unit of measure is the full circumference. Uh, so, um, where Meghan is saying it's three so her unit of measure is the, um, individual, um, arc lengths, uh, between the dots. And so she's

saying that the arc is three units of measure where eight units of measure, where eight make up the whole. And he is saying it's $3/8^{\text{th}}$ s of the whole. And so his unit of measure is the whole not the parts.

7 Michael: So are they both correct in their, in saying that the measure of this angle is, in Nick's case $3/8^{\text{th}}$ s and in Meghan's case three?

8 David: Yes. It's just that they're using different units.

I designed Task 5 during ongoing analysis to confront David with the apparent distinction between measuring an angle by determining the length of the subtended arc in units that are a fractional part of the circle's circumference, and measuring an angle by determining the fraction of the circle's circumference an angle subtends.⁸¹ These two ways of understanding what it means to measure an angle parallel David's way of understanding angle measure as a comparison of subtended arc length and a unit of measure and his way of understanding angle measure as a comparison of subtended arc length and circumference respectively. On some occasions during the first three TBCIs, David conceptualized the process of quantifying the openness of an angle as measuring the arc length the angle subtends in a unit that is a fractional part of the circumference of the circle containing the subtended arc. On other occasions David imagined the process of measuring angles as determining the fraction of the circumference of the circle centered at the vertex of the angle that the angle subtends. Note that in the latter case,

⁸¹ I say "apparent" because this distinction is not a distinction if one has conceptualized the circumference as a unit of measure for the length of the subtended arc. Until I presented David with Task 5, he had not conceptualized the circumference as a unit with which to measure the length of the subtended arc.

David did not seem to conceptualize the circumference as the unit of measure. David often said things like, “There are no units” or “The units cancel.”

David’s response in Line 6 of Excerpt 24 suggests that he had begun to coordinate his two ways of understanding angle measure. In particular, David recognized that Nick measured the subtended arc length in units of circumference and Meghan measured the subtended arc length in units of $1/8^{\text{th}}$ of the circumference. David therefore assimilated Nick and Meghan’s claims as two different instantiations of the same process: *measuring the subtended arc length in a particular unit*. David’s response in Line 6 was the first occasion in the series of TBCIs in which his two ways of understanding angle measure did not appear disjoint.

Immediately following David’s work on Task 5, I presented him with Task 6, restated in Table 45 below.

Table 45

Task 6 from TBCI 3

Courtney claims that measuring an angle in radians means measuring the arc length that the angle subtends in units of the radius of the circle centered at the vertex of the angle. Rebecca says that to measure an angle in radians, you take the length of the arc that the angle subtends divided by the length of the circumference of the circle centered at the vertex of the angle and then multiply this ratio by 2π . Are they both correct?

David was immediately convinced that Courtney’s claim (which represents *WoU 2*) is correct but was much more skeptical regarding the accuracy of Rebecca’s claim (which represents *WoU 1*). David translated Rebecca’s statement into symbolic notation by writing “ $(\frac{s}{c})2\pi$ ” and then claimed, “I’m not convinced that the second one works.”

After a few minutes thinking quietly to himself, David justified the accuracy of

Rebecca's claim by testing the formula $\theta = \left(\frac{s}{c}\right)2\pi$ for specific values of the ratio S/C to ensure the formula produces the angle measures in radians he expected it to. For instance, David let the ratio S/C equal $\frac{1}{2}$ and verified that the measure of the angle in radians (θ) was π , half of the number of radians in a whole circle. David then asserted with confidence that both Courtney and Rebecca's claims are correct. Following this, David made the observation in Excerpt 25.

Excerpt 25

- 1 David: And actually if we look at what I did on the last question (*Task 4 in Table 41*), that is basically what happened. When we do this

(*points to where he wrote* “ $\frac{S}{C} = \frac{5.3}{2\pi(1.4)} \approx \frac{5.3}{8.796} = 0.602$ radians”

in Figure 37), this is the second method that was being described, the ratio of arc length to circumference, and I realized that this is just the percentage of the circle and if I multiplied it by 2π I would be back at the answer. ... So this is Rebecca's method (*points to the crossed out work in Figure 37*) except I didn't multiply it by 2π Where this (*pointing to where he wrote* “5.3/1.4”) is taking the arc length and dividing it by the radius measurements to get the answer in, um, radians. ... This is the first girl's (*Courtney's*) method. What we're doing here (*points to where he wrote* “5.3/1.4”), we're saying we have the arc and we're saying how many of the radiuses go into it. And then here (*points to the work that he crossed out in Figure 37*) ... we figure out the percentage

of the circle that we have and then by multiplying it by 2π , then we figure out the percentage of radians, or we figure out the number of radians that we have where 2π is basically that there should be 2π radians in a complete circle and this is the percentage of the circle (*points to the ratio "5.3/8.796"*).

David's inability to employ his understanding of angle measure in radians as a multiplicative comparison of subtended arc length and radius length while completing Task 4 suggests that he was not consciously aware of his two different ways of understanding what it means to measure an angle in radians. Asking David to reflect on the accuracy of Courtney and Rebecca's statements appeared to engender a reflected abstraction by providing an occasion for him to operate on his two ways of understanding angle measure in radians. After engaging in this reflected abstraction, David could not only describe the validity of Courtney and Rebecca's claims, but could also identify how their meanings for angle measure were represented in his solution to Task 4, a problem he previously struggled to solve and explain. Moreover, upon reflecting on his solution to Task 4, David made the observation in Excerpt 26.

Excerpt 26

- 1 Michael: You ended up dividing 5.3, which was the length of the subtended arc, and 1.4, which was the length of the radius. So why did you do the division there?
- 2 David: Well because again here I said that really all I, I realized I needed to take this and multiply it by 2π . Um, so if I just remove that from the problem then we get it. And when we're measuring in radians

what we're doing is we're saying that it's one radius length so really I should be thinking about how one radius goes into 5.3 instead of how 5.3 goes into the circumference. That was the difference in my thinking so. Up here in this one (*pointing to the work that he had crossed out*) I was thinking more of what percentage it was of the circle instead of just making it, doing it the easy way and saying how many radians are going to actually be in here where a radian is that length. So how many times does 1.4, uh, inches go into 5.3? Which is what I should have done.

The ease with which David recognized both *WoU 1* and *WoU 2* in his solution to Task 4, and the fluency with which David diagnosed his faulty reasoning while completing Task 4, suggests that David had reorganized his two ways of understanding what it means to measure an angle in radians into a coherent scheme and became consciously aware of possessing these understandings.

Summary of TBCIs 2 and 3. While discussing Unit 4 of Task 1, David imagined the length of a unit of angle measure and the circumference of the circle centered at the vertex of an angle measured in the same standard linear unit and recognized that, as the size of the circle varies, the ratio of the lengths of these quantities reduce and the units cancel, resulting in a constant ratio. David did not appear to initially conceptualize the ratio of unit length to circumference as a quantitative operation that represents the fraction of the circle's circumference subtended by an angle with a measure of one unit. Rather, this ratio had a quantitative meaning for David after he imagined measuring the unit length and the circumference in the same standard linear unit, and reducing the ratio

of these quantities' values. While discussing Task 2, however, David demonstrated a quantitative way of understanding the condition that a unit of angle measure must satisfy. In particular, he demonstrated the understanding that, while the size of the circumference of the circle centered the vertex of an angle varies, the length of a proper unit of angle measure varies in tandem so that the length of the unit was always the same fraction of the circle's circumference. In contrast to his responses while engaged in Task 1, David's remarks in the context of discussing Task 1 suggest that he constructed the ratio of the length of Unit 1 to the circumference as a quantitative operation that represents the fraction of the circle's circumference subtended by an angle with a measure of one Unit 1.

During TBCIs 2 and 3, David repeatedly explained that a unit of angle measure must covary with the circumference of the circle centered at the vertex of the angle so that the length of the unit of measure is proportional to this circumference. While David correctly stated the condition that a unit of angle measure must satisfy, his responses while discussing Tasks 1 and 2 in Table 41 were replete with conditional statements he struggled to justify (e.g., If the length of a unit of measure is proportional to the circumference of the circle, then the measure of the angle in that unit does not depend on the size of the circle). David's difficulty articulating how the conclusions of these conditional statements follow logically from their hypotheses may have been due to his lack of fluency in explaining complicated ideas, or may suggest the implicitness of his meanings. That being said, in Excerpt 21 David demonstrated the understanding that if a unit of angle measure is not proportional to the circumference of a circle, then the unit will be a different fraction of the circle's circumference for circles of different sizes. So if two angles with circles of radius r_1 and r_2 (with $r_1 \neq r_2$) centered at their respective

vertices each have a measure of n Unit X s, then these angles will subtend different fractions of the circumference of their respective circles and therefore have different amounts of openness. Unit X is therefore not an appropriate unit of angle measure.

Throughout the first three TBCIs, David demonstrated two ways of understanding angle measure in radians. The first relied on a comparison of subtended arc length and circumference (*WoU 1*) and the second involved a comparison of subtended arc length and radius length (*WoU 2*).

Although David had demonstrated *WoU 2* on several occasions during the first three TBCIs, he did not rely on this understanding when I asked him to determine the measure of an angle in radians provided a subtended arc length and a radius length (Task 4). David instead divided the subtended arc length by the circumference and claimed that the value of this ratio was the measure of the angle in radians. After having recognized that this solution was incorrect, David employed *WoU 1* and multiplied the ratio of subtended arc length to circumference by 2π . In essence, David did not assimilate this task to a cognitive scheme that contained his understanding of angle measure in radians as a multiplicative comparison of subtended arc length and radius length. David's failure to deliberately employ an appropriate understanding he had demonstrated on several occasions prior suggests that he did not appear to be consciously aware of his two ways of understanding angle measure in radians.

To support David in becoming consciously aware of the two ways he understood angle measure, I designed two tasks during ongoing analysis that provided opportunities for him to perform mental operations on both of these ways of understanding. The first task (Task 5) confronted David with the distinction between two processes for measuring

angles that paralleled his two ways of conceptualizing angle measure: (1) measuring an angle by determining the length of the subtended arc in units that are a fractional part of the circle's circumference, and (2) measuring an angle by determining the fraction of the circle's circumference an angle subtends. David recognized that these methods differ only in the unit one uses to measure the length of the subtended arc, and thus constitute two different instantiations of the same process. Accordingly, Task 5 appeared to support David in coordinating his two ways of understanding angle measure.

The second task I designed during ongoing analysis (Task 6) supported David in becoming consciously aware of his two ways of understanding angle measure by prompting him to assess the validity of two claims regarding what it means to measure an angle in radians (resembling *WoU 1* and *WoU 2* respectively). David's consideration of these claims engendered a reflected abstraction by providing an opportunity for him to operate on his two ways of understanding angle measure. After having engaged in this reflected abstraction, David explained that while engaged in Task 4 he attempted to utilize *WoU 1* where employing *WoU 2* would have been far more useful and efficient. Additionally, David more strategically and deliberately employed these ways of understanding in subsequent tasks. So, in the final analysis, Tasks 5 and 6 appeared to support David in becoming consciously aware of his meanings for angle measure and in organizing these ways of understanding into a coherent scheme of actions and operations at the reflected level.

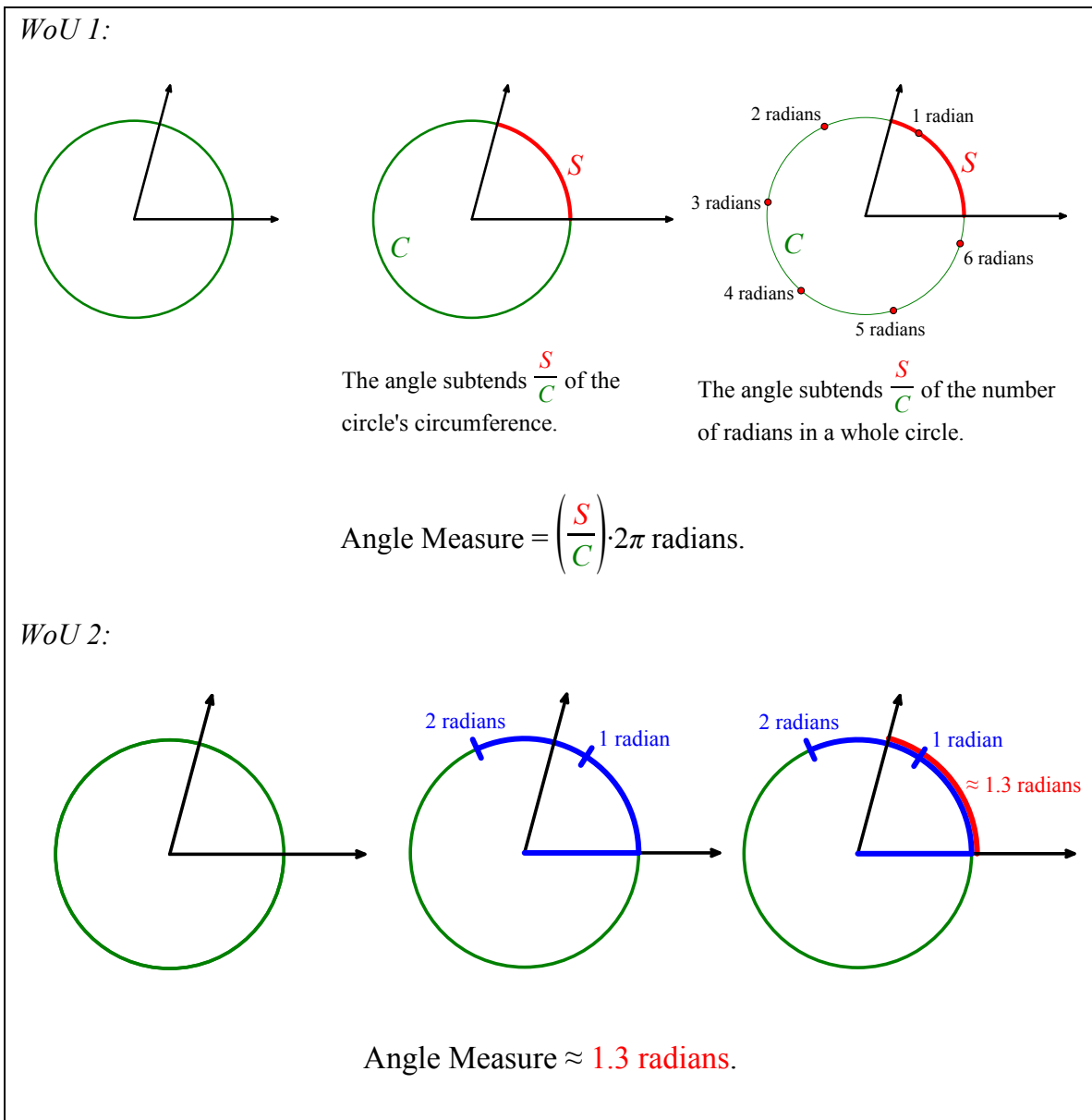
Summary of David's Way of Understanding Angle Measure

David's language and actions during the initial clinical interview and the first three TBCIs revealed his way of understanding angle measure as well as his way of

understanding the condition that a unit of angle measure must satisfy. I summarize these ways of understanding below. I also discuss the way of thinking that appeared to govern the ways of understanding David demonstrated during the initial clinical interview and the first three TBCIs.

Angle measure. During the initial clinical interview, David explained that there are approximately 6.28, radians in a whole circle, although he did not appear to understand 6.28 radians as referring to 6.28 of anything. Also during this interview, David appeared to conceptualize the measure of an angle in radians as the number of these 6.28 pieces that the angle subtends divided by 6.28, the total number of pieces in the circle. Throughout the first three TBCIs, in contrast, David demonstrated two ways of understanding what it means to measure an angle in radians that differed markedly from the (incorrect) understanding he exhibited in the initial clinical interview. These two ways of understanding were evident when David measured angles in radians as well as when he interpreted the measure of an angle in radians (see Table 43 for a summary of these two ways of understanding). David's first way of understanding angle measure in radians was as a comparison of subtended arc length and circumference. David's second way of understanding angle measure in radians was as a comparison of subtended arc length and radius length. According to the first way of understanding David demonstrated (*WoU 1*), measuring an angle in radians involves multiplying the fraction of the circle's circumference the angle subtends by 2π , the number of radians in a whole circle, to obtain the number of radians the angle subtends. Therefore, to say an angle has a measure of n radians means the angle subtends an arc that is $n/(2\pi)$ of the circumference of the circle centered at the vertex of the angle. According to David's second way of

understanding (*WoU 2*), an angle with a measure of n radians subtends an arc that is n radius lengths, or n times as large as the radius of the subtended arc, which implies the measure of an angle in radians is determined by the ratio of subtended arc length to radius length. *Figure 38* below illustrates David's two ways of understanding angle measure in radians by displaying my interpretation of what David imagined when presented with an angle and the task of determining its measure in radians.



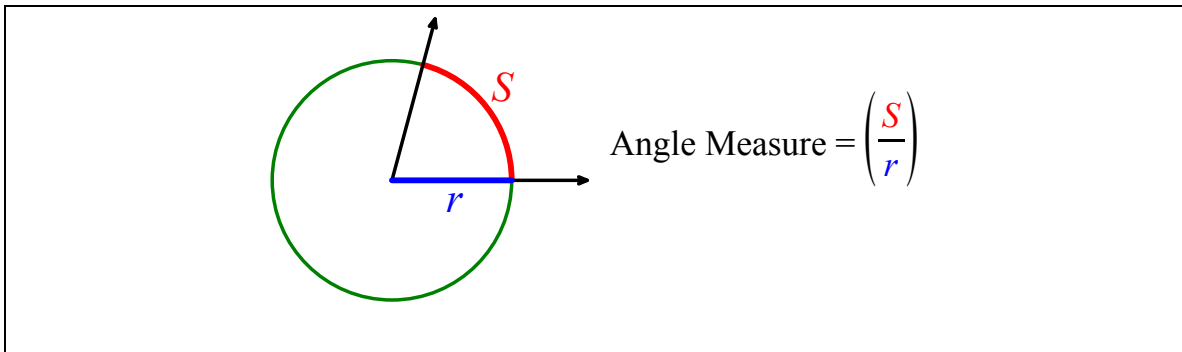


Figure 38. Mental imagery of *WoU 1* and *WoU 2*.

Although David demonstrated *WoU 1* and *WoU 2* on several occasions throughout the first three TBCIs, he did not deliberately employ the understanding most advantageous for solving a particular task. Accordingly, David did not appear to be consciously aware of his two ways of understanding what it means to measure an angle in radians. I designed tasks during ongoing analysis that engendered a reflected abstraction by providing opportunities for David to perform operations at the level of representation on the mental actions and operations that constitute his two ways of understanding angle measure in radians, which resulted in a coherence of these mental actions and operations at the reflected level accompanied by conscious awareness.

Condition for a unit of angle measure. In the initial clinical interview, David explained that for a unit to be appropriate for measuring angles, there must be a certain number of these units in the circumference of a circle. This proposed condition for a unit of angle measure appeared to derive from David's need to calculate the measure of an angle, which during the initial clinical interview involved his dividing the length of the subtended arc by the circumference. While employing this way of understanding angle measure, David could not calculate the measure of an angle without knowing what to

divide the length of the subtended arc by, hence his need to know how many units are contained in the circumference of a circle.

During TBCI 1, David continued to justify the condition that a unit of angle measure must satisfy by appealing to his need to calculate angle measure. David often calculated an angle's measure in a particular unit by applying the following instantiation of his more general part-whole proportion formula:

$$\frac{\text{Subtended arc length}}{\text{Circumference}} = \frac{\text{Angle measure (in unit } X)}{\text{Number of } X\text{s in a whole circle}}$$

David anticipated that he could measure the subtended arc length and circumference in a standard linear unit and would therefore need to know how many of a particular unit is contained in the circumference of a circle before he could solve this formula for the measure of the angle in this particular unit. In essence, at the conclusion of the first TBCI, David's condition for a unit of angle measure was not informed by an understanding of angle measure as an invariant multiplicative relationship between subtended arc length and the length of a unit of measure.

During TBCIs 2 and 3, David demonstrated a more sophisticated way of understanding the condition that a unit of angle measure must satisfy. David repeatedly stated that a unit of angle measure must covary with the circumference of the circle centered at the vertex of the angle so that the length of the unit of measure is proportional to the circumference. David explained that if this condition is not satisfied, then the proposed unit of angle measure will be a different fraction of the circle's circumference for circles of different sizes, implying that two angles could have the same measure but different amounts of openness.

David's mathematical way of thinking. During TBCI 1, David demonstrated a strong calculational disposition by heavily relying on the following part-whole proportion formula to solve problems involving angle measure:

$$\frac{\text{Part (of type } X\text{)}}{\text{Whole (of type } X\text{)}} = \frac{\text{Part (of type } Y\text{)}}{\text{Whole (of type } Y\text{)}}.$$

David struggled to articulate the meaning of the expressions in this formula and to justify their equality. Although David did not appear to understand this formula as the representation of two equivalent quantitative operations, he applied it with remarkable fluency to solve a variety of problems involving angle measure. In fact, so steadfast was David's confidence in this formula that it was often his first recourse; he rarely attempted to interpret novel tasks quantitatively when he thought a variation of this formula might apply.

Additionally, as I mentioned above, David demonstrated two ways of understanding angle measure in radians that he was unable to strategically utilize to solve novel problems. This suggests that David had not previously recognized the need to reflect on what it means to measure an angle in radians. Doing so might have engendered a reflected abstraction whereby David may have become consciously aware of his two ways of understanding angle measure. Additionally, such a reflected abstraction would have resulted David in constructing a more coherent organization of the mental actions and operations that constitute the ways of understanding angle measure he demonstrated throughout the series of TBCIs. That David had not found the need to reflect on his understandings of angle measure suggests that, from his perspective, obtaining solutions to problems matters more than the means by which one obtains them. David was often

content with “hacking away” at a problem until he obtained an answer that he thought was correct. This observation, combined with David’s persistent reliance on a formula for which he had little meaning, reveals the utilitarian nature of David’s mathematical way of thinking; David appeared to value his capacity to obtain correct answers more than the ways of understanding he relied upon to obtain them.

Output Quantities and Graphical Representations of Sine and Cosine Functions

My analysis of the initial clinical interview resulted in my constructing an initial model of David’s way of understanding the output quantities and graphical representations of sine and cosine functions. TBCIs 4 and 5 focused on the output quantities of sine and cosine while TBCIs 6 and 7 focused on the graphical representations of sine and cosine. I began TBCI 4 by presenting David with a task I designed during ongoing analysis to test my initial model of his way of understanding the output quantities of sine and cosine. David worked on Tasks 1-6 (see “Outputs of Sine and Cosine” section of Chapter 5) for the majority of TBCI 4. Generally speaking, I designed Tasks 1 and 2 to engage David in the sequence of actions involved in determining sine and cosine values. The purpose of Task 3 was to support David in differentiating this sequence of actions from their effect of producing values of sine or cosine. I intended Tasks 4-6 to provide occasions for David to coordinate and project this sequence of actions to the level of representation and to perform operations on them so that he may construct a coherent scheme of actions and operations at the reflected level accompanied by conscious awareness. Accordingly, Tasks 1-6 sought to occasion reflecting and reflected abstractions. David completed Tasks 7-9 during TBCI 5. These tasks provided opportunities for David to assimilate symbolic notation to a scheme of

actions and operations at the reflected level, thereby revealing the extent to which David had engaged in the reflecting and reflected abstractions I designed Tasks 1-6 to engender.

I modified Tasks 1-4 in the “Graphical Representations of the Sine and Cosine Functions” section of Chapter 5 for David to complete during TBCI 6. My modification of these tasks was informed by my emerging model of David’s way of understanding the output quantities and graphical representations of the sine and cosine functions. I designed Task 1 to support David in describing how the input and output quantities individually change with respect to time (as the measure of an angle increases from zero at a constant rate with respect to time) so that he may coordinate the variation of these quantities to construct graphical representations of the sine and cosine functions in Task 2. I designed Task 3 to assess the extent to which David could interpret the graphs of sine and cosine functions as a representation of the covariational relationship between the values of quantities, as opposed to a picture or image. The purpose of Task 4 was to support David in constructing an understanding of the graph of the sine function as a representation of the covariational relationship of the values of input and output quantities. During the entirety of TBCI 7, David and I discussed a didactic object I designed during ongoing analysis to further refine my model of his way of understanding the concavity of the sine and cosine functions.

Initial Clinical Interview

My analysis of the initial clinical interview led to my identifying two themes relative to David’s way of understanding the outputs and graphical representations of the sine and cosine functions: (1) *sine and cosine values as ratios of lengths*, and (2) *covariation of the input and output quantities of the sine and cosine functions*.

Theme 1: Sine and cosine values as ratios of lengths. On several occasions during the initial clinical interview, David demonstrated a way of understanding the *outputs of sine and cosine as ratios of lengths*, but did not appear to conceptualize a quantity to which sine and cosine values refer; that is, David struggled to describe an attribute of a geometric object to which sine and cosine values may be assigned as measures. David therefore did not appear to conceptualize as quantitative operations the ratios of lengths he claimed represent the respective outputs of the sine and cosine functions. Consider, for example, the conversation depicted in Excerpt 27.

Excerpt 27

- 1 Michael: Let's talk about sine and cosine. What is the sine function?
- 2 David: (*Sigh*) The sine function is, it depends on how we're willing to think of it. If we're going to think of it as the circular function, the sine function, and not just the operation on right triangles, then the sine function is the ratio of the distance from the vertical, or sorry from the horizontal axis compared to the, um, radius of the circle for a point that is cut off by a central angle, um, (*pause*) how do I want to say that, of, um, some degree. So the input is the angle measure, the output is the ratio of vertical length to radius, um, as the, of an angle that, um, intersects an arc or intersects some circle of some radius.
- 3 Michael: Okay. I think I got most of that. Would you mind drawing me something?

- 4 David: Yeah. Let me try to see if I can clean that up. So (*draws a circle centered at the origin of a coordinate plane and an angle with a measure of approximately $\pi/4$ radians in standard position*). So the input is some given angle (*labels the angle " θ "*) and we have a arbitrary, um, circle, um, drawn out there with some given radius and we have some height to a point of intersection of the terminal, of the terminal side of the angle, angle in standard position, sorry I didn't mention that before. So ... it (*the sine value*) is the ratio of this height (*points to line extending from the terminal point to the horizontal axis*) from the horizontal axis to the circle compared to the radius of the circle. So (*writes " $\sin(\theta) = h/r$ "*). The ratio of h to r .
...
- 5 Michael: So I think you answered this in your response but I'll ask it anyway. What quantities does the sine function relate?
- 6 David: It relates the quantity of angle measure and how it relates to the quantity created by the ratio of the height to the radius of a known circle that is with its central angle.

The first noteworthy aspect of David's response is in Line 2, in which he suggested a distinction between sine as a "circular function" and sine as an operation on right triangles. This statement reveals that David did not initially understand the sine function as defining the relationship between the same input and output quantities in both circle and right triangle contexts. In the former context, David explained that the input to the sine function is an angle's measure. In the latter context, David appeared to

understand sine as an operation on right triangles, suggesting that the geometric object of a right triangle is somehow the input to the sine function.

David also explained in Line 2 that the output quantity to sine as a “circular function” is the ratio of the vertical distance that the terminus of the subtended arc is from the horizontal diameter of the circle centered at the vertex of the angle to the radius of this circle. David therefore appeared to conceptualize the output of sine as a ratio of lengths and not as a length measured in a particular unit (i.e., the radius of the circle centered at the vertex of the angle). The elusiveness of David’s response in Line 2 gave me the impression that he was organizing his thoughts while speaking. Accordingly, I asked him in Line 3 to draw a picture to represent what he had in mind. David drew the picture in *Figure 39*.

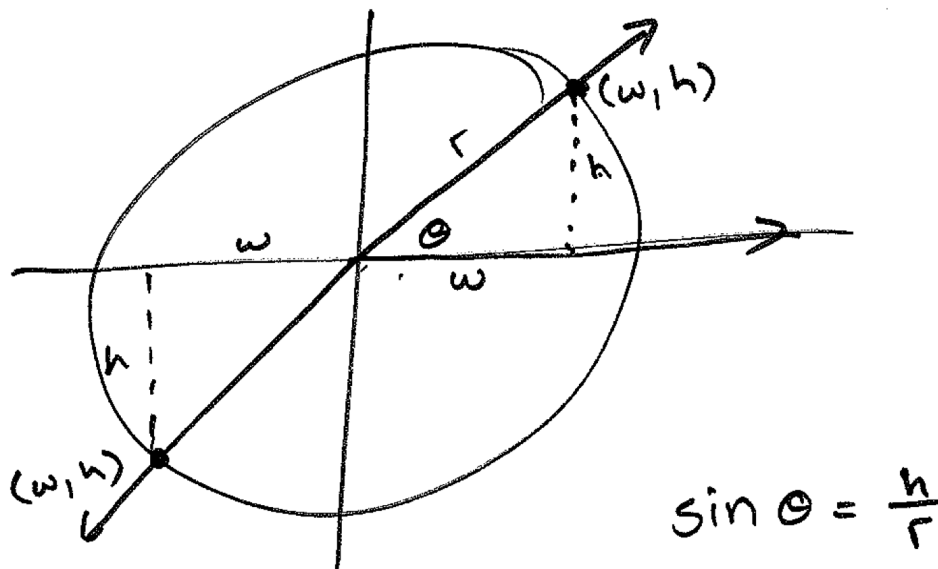


Figure 39. David’s representation for the output of sine.

David’s drawing supports my conjecture that he conceptualized the output of sine as a ratio of lengths. According to this way of understanding, one obtains sine values by

performing an arithmetic operation on the values of two quantities (a vertical distance and a radius length). Since David did not speak of or identify in his drawing an attribute to which values of sine may be applied as a measure, I consider it likely that David understood the ratio he proposed as an arithmetic operation as opposed to a quantitative operation. David reiterated in Line 6 that the sine function defines the relationship between angle measure and the ratio:

$$\frac{\left(\begin{array}{l} \text{Vertical distance that the terminal point is away from the horizontal} \\ \text{diameter of the circle centered at the vertex of an angle in standard position} \end{array} \right)}{\text{Radius of the circle}}.$$

My phrasing for the numerator of this ratio is intentional. David’s language in Excerpt 27 and his drawing in *Figure 39* did not indicate that he understood this vertical distance as directional. David appeared to attend only to the distance that the terminus of the subtended arc is *away* from the horizontal diameter instead of the distance that this terminal point is *above* the horizontal diameter. David did not seem to account for the possibility that this ratio could be negative until I asked him to describe how the outputs of the sine function vary as its input increases from zero. David then realized, “I have an issue with my definition because I defined it as distance instead of as the value because distances would be positive.” David amended his initial definition by imagining a coordinate plane with its origin at the vertex of the angle and defined the numerator of the above ratio as the *y*-value of the terminus of the subtended arc.

After having discussed the output of sine, I asked David to explain the meaning of the cosine function. David’s response, “The cosine function would ... be the horizontal position and its ratio to the radius length of the circle” indicates that he conceptualized the output of cosine analogously to how he understood the output of sine: as a ratio of

lengths. To assess whether David had conceptualized the output of cosine as a quantity (i.e., as an attribute of a geometric object), I asked him to identify the output of cosine on his drawing in *Figure 39*. David did not identify an attribute to which cosine values may be applied as a measure but instead proposed an arithmetic operation on the lengths represented by w and r . In particular, David claimed that cosine values are the ratio of the x -coordinate of the terminal point (w) to the length of the radius (r). Referring to his drawing in *Figure 39*, David expressed the output of cosine symbolically as, “ $\cos(\theta) = \frac{w}{r}$.” Additionally, David explained that the cosine function defines the relationship between “angle measure for an angle in standard position and the ratio of the length of the radius of a circle and the horizontal distance from the vertical axis to that (*the terminal*) point.”

Theme 2: Covariation of the input and output quantities of the sine and cosine functions. David described the covariation of the input and output quantities of the sine and cosine functions by attending to the direction of change in the outputs of these respective functions as the input varied (MA 2). Consider, for example, David’s response to my asking him how the output of cosine covaries with the input quantity.

Excerpt 28

- 1 Michael: How does the output of the cosine function vary as the input of the cosine function increases from zero?
- 2 David: So initially it would decrease because it would start, um, at its, um, highest value of one because at that point the radius is the horizontal value if you had an angle of zero. And then as we increased the angle the horizontal distance to that point would get closer and closer to the vertical axis so it would decrease, um, and

then it would become negative and keep going and decreasing until it became a ratio of negative one because it was the opposite of the radius and its measure and then it would increase back to its maximum and then oscillate between the two.

David began his response in Line 2 by suggesting that if the angle measure is zero, then the horizontal distance that the terminal point is away from the vertical axis is the same as the radius length, thereby making the ratio of these quantities equal to one. David then appeared to imagine the horizontal distance that the terminus of the subtended arc is away from the vertical axis decreasing as the measure of the angle increased from zero. David realized that the value of cosine would decrease until it assumed its minimum value of negative one when the terminal ray is on the horizontal axis, pointing in the opposite direction as the initial ray. David then claimed that the value of cosine would oscillate between one and negative one as the measure of the angle continued to increase. David characterized the covariation of the input and output quantities of the cosine function by describing the direction of change in the output quantity as the input quantity varied, which suggests he engaged in Mental Action 2 (MA 2) in Carlson et al.'s (2002) covariation framework.

David also coordinated the amount of change in the output quantity of the sine function as the input varied by equal amounts (MA 3). During the initial clinical interview, I spontaneously asked David consider what he would say to a student who proposed that a graph of the sine function consisted of two semicircles, one positioned above the horizontal axis and the other adjacently positioned below the horizontal axis. I provide David's reply in Excerpt 29.

Excerpt 29

- 1 David: Initially if you're talking about let's just say the, um, sine function, so we're comparing, we're measuring the ratio of the height to the point on the terminal side versus the radius of that angle. If we were to move along (*motions rotation of terminal ray with hand*) and have it go, you know, one degree, two degree, three degree, and watch it in time frame and have it just move at a nice steady even pace, what we would notice is that at the beginning, um, one degree would have a very big jump of vertical distance, uh, because there is that really big jump in vertical distance it wouldn't make it a very, um, the ratio, um, would change a lot more. But as you swing around and get near the top, when you get to the top of the circle, the vertical distance, um, does not change as quickly. The change in the change, or I'm sorry the change in the vertical for every increment of one degree wouldn't be that noticeable and so you would see it kind of level off a little bit more and it doesn't end up giving you a true circle if you were to kind of copy that path.

David coordinated changes in the output quantity for equal changes in the input quantity of one degree. In particular, David compared the changes in the ratio of vertical distance that the terminal point is above the horizontal axis to the length of the radius for successive one-degree changes in angle measure. David appeared to recognize that, since the vertical distance that the terminal point is above the horizontal axis changes by

smaller amounts for consecutive changes in angle measure of one degree, then the ratio of this vertical distance to radius length (i.e., the sine value) will increase but will change by smaller amounts for successive one-degree changes in angle measure. David therefore described the variation of the input and output quantities of the sine function by engaging in Mental Action 3 (MA 3) of Carlson et al.'s (2002) covariation framework.

Summary of initial clinical interview. While discussing the meaning of the output quantities of the sine and cosine functions, David appeared to visualize an angle in standard position with its vertex at the origin of a coordinate plane. David then described the output of the sine (cosine) function as the ratio of the y -coordinate (x -coordinate) of the terminus of the subtended arc to the radius of the circle centered at the vertex of the angle (see *Figure 40*). David did not appear to interpret these ratios as respectively representing vertical and horizontal distances measured in units of the radius of the circle centered at the vertex of the angle. In other words, David's language and actions during the initial clinical interview suggested that he conceptualized the outputs of the sine and cosine function as ratios of lengths, not as a lengths measured in a particular unit (i.e., the radius). Accordingly, David did not appear to have constructed the *quantities* that sine and cosine values represent. For this reason, David understood the outputs of sine and cosine as arithmetic operations as opposed to quantitative operations.

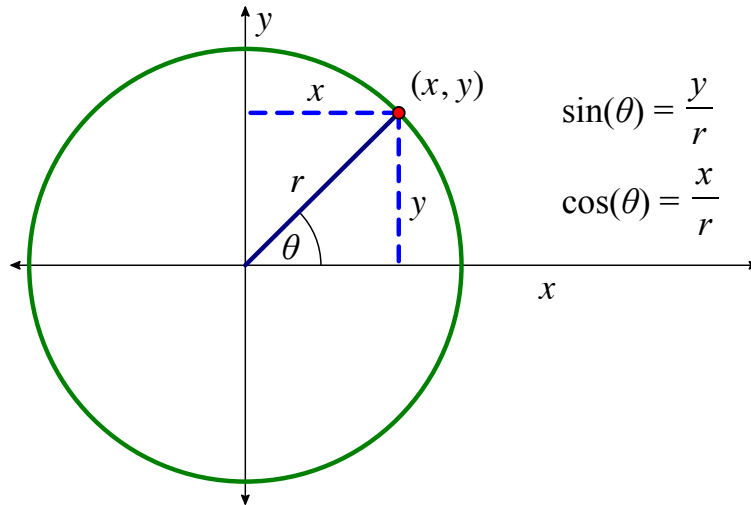


Figure 40. David's initial way of understanding the outputs of sine and cosine.⁸²

During the initial clinical interview, David described the covariation of the input and output values of the cosine function by coordinating the direction of change of the output value while imagining variation in the angle measure (MA 2). Additionally, while justifying the concavity of the sine function, David coordinated the amount of change in the output value for equal changes of the input value (MA 3). Although David did not describe attributes of a geometric object to which the output values of sine and cosine may be applied as measures, while describing the covariation of the input and outputs of the sine and cosine functions, David appeared to visualize how the vertical (horizontal in the case of cosine) distance that terminus of the subtended arc was above (to the right) of the horizontal (vertical) diameter of the circle varied as the angle measure increased. In particular, David seemed to understand that since the radius of the circle remained constant as the measure of the angle varied, the quantity that affected the output value of

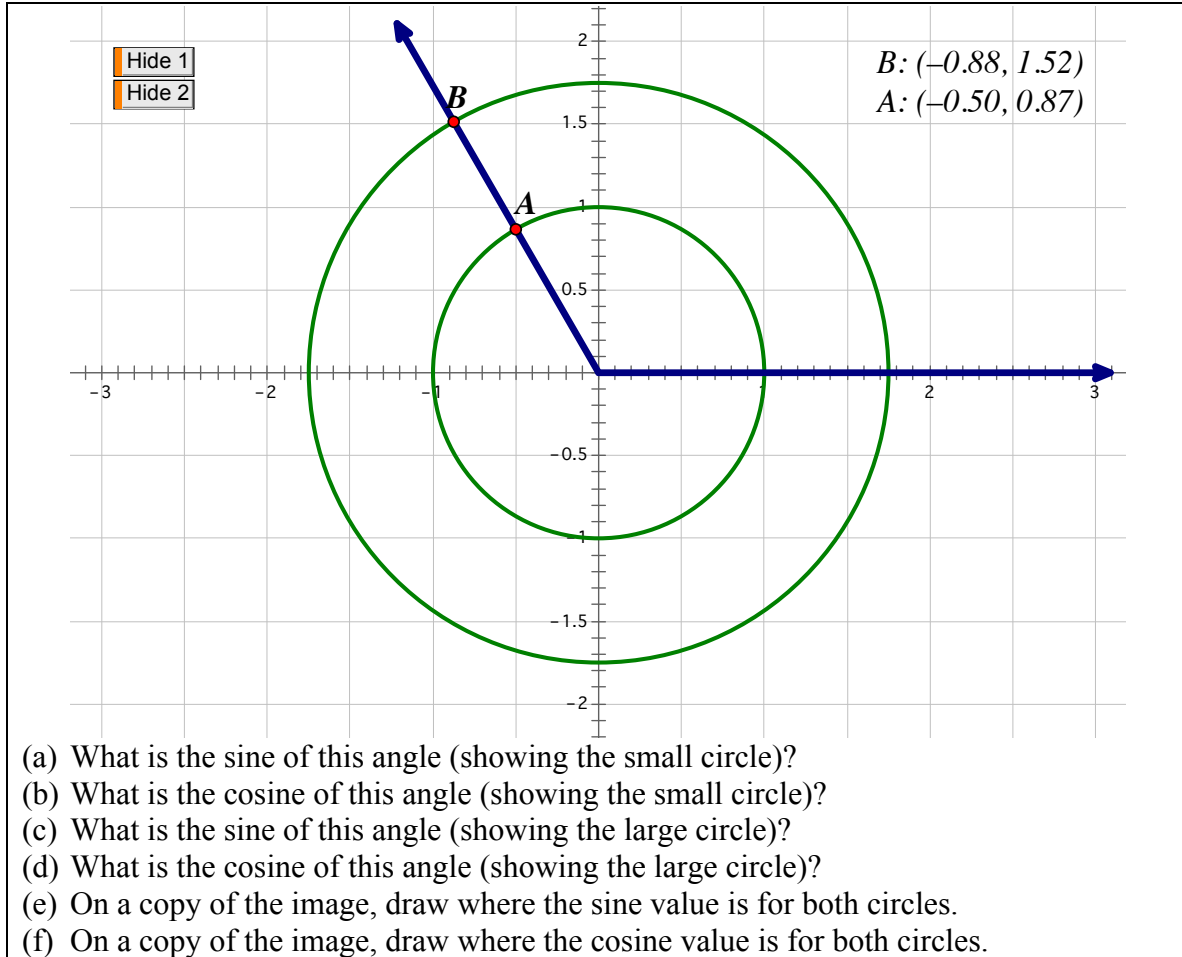
⁸² I intentionally placed the label “ θ ” near the vertex of the angle. Not only did David label his angle in Figure 39 in the same way, his language and actions during the initial clinical interview suggested that, while discussing the outputs of the sine and cosine functions, David thought of the angle measure θ as simply specifying an amount of openness and not as a measure of the subtended arc length in a particular unit.

sine (cosine) was the vertical (horizontal) distance, expressed in the numerator of the ratio he constructed to represent the output of sine (cosine).

Adjustments to TBCI 4. David’s language and actions during the initial clinical interview led me to hypothesize that he understood the output of the sine function as the ratio of the y -value of the terminus of the subtended arc to the radius of the circle. Similarly, I conjectured that David thought about the output of the cosine function as the ratio of the x -value of the terminus of the subtended arc to the radius of the circle. David did not appear to conceptualize quantities to which sine and cosine values respectively refer. To assess the validity of this conjecture, I designed the task in Table 46 after having conducted a surface-level analysis of the initial clinical interview. Pressing the button “Hide 1” hides the circle containing Point A , the coordinates of Point A , and Point A itself. Similarly, pressing the button “Hide 2” hides the circle containing Point B , the coordinates of Point B , and Point B itself. I specifically designed this task to provide an opportunity for David to determine the value of sine and cosine of the measure of the angle shown in at least two different ways: by computing a ratio of lengths or by respectively estimating vertical and horizontal lengths in units of the radius. I also designed this task to reveal whether David recognized that sine and cosine values of the measure of the angle do not depend on the size of the circle centered at the vertex of the angle. Finally, I intended Parts (e) and (f) of this task to provide an opportunity for David to identify the quantities to which sine and cosine values may be applied as measures.

Table 46

Task Added During Ongoing Analysis for TBCI 4



Task-Based Clinical Interviews 4 and 5

David continued to demonstrate his understanding of the output of sine and cosine as ratios of lengths at the beginning of the fourth TBCI. Throughout TBCIs 4 and 5, David engaged in a sequence of tasks that I designed to support him in constructing the outputs of sine and cosine as quantitative operations. I also intended these tasks to allow David to construct internal representations for the sequence of actions involved in determining output values of sine and cosine.

Theme 1: Sine and cosine values as ratios of lengths. I began TBCI 4 by presenting David with the task I designed during ongoing analysis to assess the extent to which he possessed a quantitative understanding of the outputs of sine and cosine (see Table 46). David's response to this task supported my initial conjecture that he conceptualized the outputs of sine and cosine as ratios of lengths, not as lengths measured in units of the radius of the circle centered at the vertex of the angle.

Excerpt 30

- 1 Michael: So we're going to start with this task here (*see task in Table 46*).
... So if I center a circle at the origin, say we have this one here (*displays the coordinates of Point A circle containing Point A*), and there's, I put this point on the intersection of the circle and the terminal ray. And, you know, I can find the coordinates of that point which is this here (*motions pointer to the coordinates of Point A displayed on the screen*). So what's the sine of this angle?
- 2 David: (*Pause*) Uh, the sine of that angle would be 0.87.
- 3 Michael: Okay. Why is that?
- 4 David: Uh, because the radius of, if I'm reading that right, looks like the radius is one so sine is the relationship of the y -coordinate of a point on the terminal side to the radius of the circle. So it'd be 0.87 divided by one.
- 5 Michael: Okay. What's the cosine of this angle?
- 6 David: Uh, negative point-five.
- 7 Michael: Why is that?

- 8 David: Uh, because cosine is the ratio of the x -coordinate to the radius of the circle and it appears it has a radius of one. So it is whatever the x -coordinate is.
- 9 Michael: Okay. So if I take that circle and I have a different circle centered at the origin (*hides the small circle containing Point A and displays the large circle containing Point B*). So now I have this other point, B , uh, that's at the intersection of the circle and the terminal ray and the coordinates of B are here (*points to the coordinates of Point B displayed on the screen*). So what's the sine of this angle?
- 10 David: The sine would be 1.52 divided by whatever the radius is, which it looks like it's maybe 1.75. So it would be the ratio of 1.52 divided by 1.75.
- 11 Michael: And if you actually computed that ratio, what would you get?
- 12 David: I would get the same answer that I got in the last one, which was, um, ... 0.87.
- 13 Michael: Okay. What about the cosine?
- 14 David: Um, point, uh, negative point-five. And I would do the same thing. I would take the x -coordinate and divide it by 1.75 because that appears to be the radius.

Consistent with the way of understanding the outputs of sine and cosine that David demonstrated in the initial clinical interview, his responses affirmed that he viewed the output of the sine function for a particular angle measure as the ratio of the y -

coordinate of the terminal point to the radius of the respective circle centered at the vertex of the angle (Lines 4 and 10). Similarly, David explained that the cosine value of the angle is the ratio of the x -coordinate of the terminal point to the radius of the circle containing that terminal point (Lines 8 and 14). Additionally, while considering the coordinates of Point B , David claimed that the sine and cosine values of the angle would be 0.87 and -0.50 respectively (the y - and x -coordinates of Point A) without having to compute the respective ratios he proposed in Lines 10 and 14. This suggests that David recognized the size of the circle centered at the vertex of the angle as inconsequential to the values of sine and cosine.

To determine whether David's proposal in Excerpt 30 to obtain sine values by computing the ratio of the y -coordinate of the terminal point to the radius, and to obtain cosine values by dividing the x -coordinate of the terminal point by the radius, was motivated by an image of quantities and quantitative relationships, I asked David to represent, on a copy of the image displayed in Table 46, the sine and cosine values of the angle (see *Figure 41* for David's drawing). I specifically wanted to know if David could identify an attribute of the image of which the values of sine and cosine are measures.

Excerpt 31

- 1 Michael: So if you had to draw on this image where the sine of this angle is.
- 2 David: Where the sine is?
- 3 Michael: Yeah. So where's this number that we're getting for sine? This 0.87?
- 4 David: So I would give (*pause*) this would be the sine of my angle (*mumbles to himself inaudibly while drawing a vertical line from*

Point A, the terminus of the subtended arc of the small circle, to the x-axis).

- 5 Michael: Okay.
- 6 David: Do you want me to do cosine?
- 7 Michael: Sure.
- 8 David: Okay. *(David represents the value of cosine on the diagram)*
- 9 Michael: Now if we're just, so if we're just looking at the larger circle now, ... where's the sine and the cosine if we just look at the big circle?
- 10 David: Um, *(long pause)* the sine would still be, would have to do with this length here *(draws a vertical line from Point B, the terminus of the subtended arc of the large circle, to the x-axis)* but it's not the full, um, it's not the full length. It is, um, that length divided by, um, 1.75. So it is, um, *(pause)* it's a portion of this *(points to vertical line from Point B to the x-axis)* so it would still be this *(points to the vertical line from Point A to the x-axis)*, this same portion. But this is the portion of the, um, y-coordinate, um, that correlates to sine, um, because it's a ratio of the whole y-coordinate to the whole radius. ... So it would still be this vertical length but it wouldn't be the entire length. It would be a fraction of that so it's hard to show. So it would really, you'd kind of need to know, if you knew that one *(the vertical length that he drew in the small circle)* that's the easiest way to show it.
- 11 Michael: Okay. ... And same thing for cosine?

12 David: Yeah. What it would be is just a portion of that so, just this one here (*draws on diagram*). So it's harder to show it on here because it's not, uh, an ending part because it's a ratio.

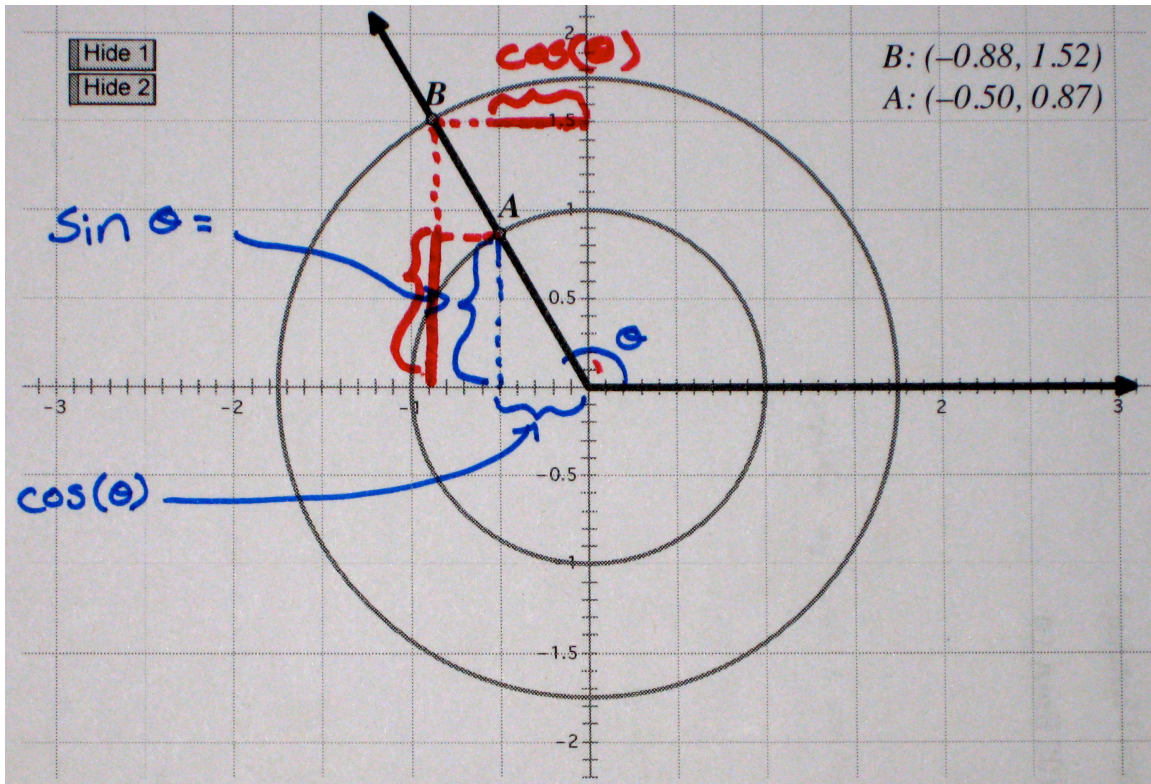


Figure 41. David's identification of sine and cosine values.

David initially represented the sine value of the angle as the vertical distance from the horizontal axis to the terminal point (Point *A*) and the cosine value of the angle as the horizontal distance from the vertical axis to Point *A*. Since the radius of the circle containing Point *A* has a length of one unit, David recognized that the ratio of the *y*-coordinate of Point *A* to the radius of the circle (i.e., the sine value) and the ratio of the *x*-coordinate of Point *A* to the radius of the circle (i.e., the cosine value) are simply the *y*- and *x*-coordinates of Point *A* respectively. When I asked David to represent the sine value of the angle considering the circle centered at the angle's vertex containing Point *B*, he

explained that the sine value would not be the full vertical length from Point *B* to the horizontal axes respectively, but would instead be a portion of this length equal to the length from Point *A* to the horizontal axis (Line 10). David illustrated the cosine value similarly (Line 12). That David did not represent the sine and cosine values of the angle as vertical and horizontal lengths from Point *B* to the horizontal and vertical axes respectively suggests that he had not conceptualized the radius as a unit of measure for these vertical and horizontal lengths. In other words, David did not appear to conceptualize the ratios $(y\text{-coordinate of Point } B)/(\text{radius length})$ and $(x\text{-coordinate of Point } B)/(\text{radius length})$ as quantitative operations; that is, David interpreted these ratios as operations on the values of two quantities without understanding the result as representing the value of a quantity itself.

In an effort to assess the viability of my hypothesis regarding David's way of understanding the output of sine and cosine as ratios of lengths, I spontaneously presented him with a hypothetical student response to the task of representing the sine value of the angle displayed in *Figure 41*. I specifically intended to confront David with a way of understanding the output of sine as a quantitative operation so that I could assess whether he possessed a more sophisticated scheme to which he might assimilate the students' response.

Excerpt 32

- 1 Michael: Okay. So if, looking back at our picture here, if a student had a question that said, 'What's the sine of this angle?' and she was given this particular circle centered at the origin (*the larger circle*) and the student said that this length (*draws the vertical line from*

Point B to the horizontal axis), that length right there is the sine of this angle. Is she correct or incorrect?

- 2 David: She is incorrect. It is not the sine. But it is one of the lengths that we would need to use to calculate the sine.
- 3 Michael: What's the other length?
- 4 David: We would need to know the length of the radius.

David's confident evaluation in Line 2 of the student's representation for the output of sine further supports my claim that he had not conceptualized the radius length as a unit of measure for the vertical distance that the terminal point is above the horizontal diameter of the circle. Additionally, David's remark that the vertical distance from Point *B* to the horizontal axis "is one of the lengths that we would need to use to calculate the sine" further demonstrates his understanding of the output of sine as an arithmetic (as opposed to quantitative) operation.

Theme 2: Constructing the outputs of sine and cosine as quantitative operations.

David's responses to my questions during the initial clinical interview, and while engaged in completing the task in Table 46 during TBCI 4, demonstrated that he had not conceptualized the outputs of sine and cosine as quantities; that is, as a length in which the radius of the circle centered at the vertex of an angle is the unit of measure. After David had completed the task in Table 46, I presented him with a sequence of activities that I designed to support him in conceptualizing the outputs of the sine and cosine functions as quantities. It was my goal to support David in constructing a scheme of mental actions and operations at the reflected level to which he may assimilate novel

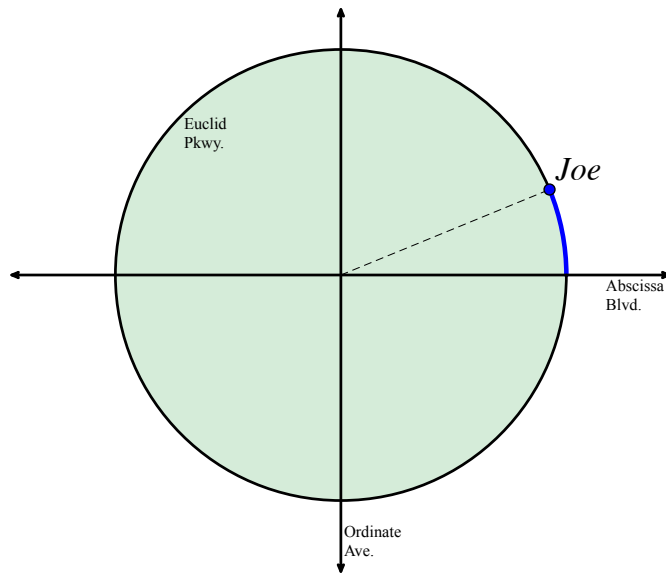
problems involving the symbolic notation “ $\sin(\theta)$ ” and “ $\cos(\theta)$.” I presented this sequence of tasks in the context of the situation provided in Table 47.

Table 47

Context of Tasks in the Instructional Sequence for the Outputs of Sine and Cosine

*Functions*⁸³

Suppose Joe is riding his bike on Euclid Parkway, a perfectly circular road that defines the city limits of Flatville. Ordinate Avenue is a road running vertically (north and south) through the center of Flatville and Abscissa Boulevard is a road running horizontally (east and west) through the center of Flatville. Assume Joe begins riding his bike at the east intersection of Euclid Parkway and Abscissa Boulevard in the counterclockwise direction.



To support David in conceptualizing the outputs of sine and cosine as quantities, I began my instructional sequence with a series of tasks I designed to engage David in the actions involved in determining the output values of sine and cosine (see Table 21 in Chapter 5) so that, in subsequent tasks, he would be positioned to differentiate these actions from their effect and project them to the level of representation. In other words, I

⁸³ Note that I designed the image displayed in Table 47 in *Geometer's Sketchpad* so that, while engaged in the sequence of tasks stated in the context of this situation, David could move Joe's position on Euclid Parkway.

began my instructional sequence by laying the foundation for future reflecting abstractions.

Excerpt 33

- 1 Michael: If the radius of Flatville is, say, 10.4 kilometers, and if Joe's path has swept out an angle of one radian, how many radius lengths is Joe north of Abscissa Boulevard?
- 2 David: *(David moves Joe and places him so that his path has swept out approximately one radius length)* Okay so somewhere like that would be about one radius length, *(pause)* and so he is *(pause)*, um, about 0.8 above Abscissa.
- 3 Michael: Okay. So how did you know that Joe's about, you said 0.8 radius lengths north of Abscissa Boulevard?
- 4 David: Well if he had gotten all the way up to, um, Ordinate Avenue then he would have been one radius length north of Abscissa Boulevard. So I know he's less than one. Um, and it looks like he has gone, you know, more than halfway so, because he's somewhere around this high *(pointing to the place on Ordinate Avenue that matches the vertical distance that Joe is north of Abscissa Boulevard)*.

David estimated that Joe was 0.8 radius lengths above Abscissa Boulevard when his path had swept out an angle of one radian. To make this estimate, David first moved Joe so that he had traversed an arc equal in length to the radius of Flatville. David then appeared to project the vertical distance from Joe to Abscissa Boulevard onto Ordinate Avenue, as evidenced by his pointing to the place on Ordinate Avenue that matched Joe's

vertical position. David's projection of Joe's vertical distance north of Abscissa Boulevard onto Ordinate Avenue allowed him to multiplicatively compare this vertical distance with the radius of Flatville. This led to him approximating Joe's distance north of Abscissa Boulevard in units of radius lengths.

David's responses in Excerpt 33 demonstrate that he engaged in the sequence of actions involved in determining a sine value when given an angle measure, although David did not appear to notice that this was in fact what he was doing. Immediately following David's response in Line 4 of Excerpt 33, I asked him to estimate the number of radius lengths that Joe is north of Abscissa Boulevard when his path had swept out an angle of one radian, supposing the radius of Flatville was 7.59 kilometers instead of 10.4 kilometers. David immediately responded that the number of radius lengths that Joe is north of Abscissa Boulevard "would change in length of kilometers but it wouldn't change in, uh, length of radius lengths." David further explained, "The radius didn't even come into play when I did the first one." David's responses suggest that he recognized the radius of Flatville as inconsequential to measuring Joe's vertical distance above Abscissa Boulevard, so long as he measured this distance in units of radius lengths.

While approximating the number of radius lengths that Joe is to the east of Ordinate Avenue when his path had swept out an angle of two radians, David engaged in an analogous sequence of actions to those his responses in Excerpt 33 reveal. Specifically, David moved Joe so he had traveled a distance approximately twice as large as the radius of Flatville, and then projected the horizontal distance that Joe was to the east of Ordinate Avenue onto Abscissa Boulevard so that he could multiplicatively compare this distance with the radius of Flatville. David accurately approximated that Joe was -0.4 radius

lengths to the east of Ordinate Avenue when his path has swept out an angle of two radians.

The task to which David responded in Excerpt 33, and others like it, required that he first move Joe's position to approximate the measure of an angle and *then* estimate the sine or cosine value associated with that angle measure. In addition to providing opportunities for David to engage in the sequence of actions involved in determining sine and cosine values provided an angle measure, I designed tasks that involved him in simultaneously coordinating the input and output quantities of the sine and cosine functions as he varied the input. I intended these tasks to further direct David's attention to the input and output quantities of the sine and cosine functions and to support him in recognizing that as the value of the input quantity varies continuously, the value of the output quantities of sine and cosine vary in tandem.

Excerpt 34

- 1 Michael: So where is Joe on Euclid Parkway when the angle traced out by his path, in radians, is twice as large as the number of radius lengths that he is to the east of Ordinate Avenue? ...
- 2 David: Radius lengths from the east. So (*long pause*) I would put him somewhere here. This length (*David points to the point on Abscissa Boulevard that is the same distance away from Ordinate Avenue that Joe is*) looks like it's about half of the, um, complete radius all the way across. So (*long pause*) and then I'm wondering if this would be a full radius (*points to the arc length that Joe has ridden*).

- 3 Michael: So can you tell me, what's, in this situation, what is half of what?
- 4 David: Okay. So (*sigh*) if I was to draw, uh, a, um, vertical line from this point (*Joe's position*) down, I would see that I'd have, it would intersect about the midpoint between the center and the outer edge of the circle. So that would make it about half of a radius length. I want to be able to double that to get the angle, and when I look at that I see that when I double it I get about one full radius length here (*points to the arc length that Joe's path has swept out*). And again it's about a third of the way across half the circle so that makes sense.

David's responses in Lines 2 and 4 suggest that he moved Joe's position and then compared the number of radius lengths that Joe was east of Ordinate Avenue with the number of radius lengths that Joe had traveled along Euclid Parkway. While David attended to the input and output quantities of the cosine function as I had intended, he compared the *values* of these input and output quantities while Joe's position was static instead of coordinating the quantities themselves while varying Joe's position. In other words, David moved Joe so that he was half of a radius length to the east of Ordinate Avenue and then determined if the arc swept out by Joe's path was twice as long, or one full radius length. David did not appear to multiplicatively compare the length of the arc swept out by Joe's path with Joe's distance to the east of Ordinate Avenue *while* varying Joe's position. Instead, he placed Joe somewhere on Euclid Parkway and *then* assessed whether the number of radius lengths that Joe had ridden was twice as large as the number of radius lengths that Joe was to the east of Ordinate Avenue.

In contrast, David appeared to simultaneously coordinate the input and output quantities of the sine function when prompted to place Joe in a position on Euclid Parkway so that Joe's position north of Abscissa Boulevard was half as many radius lengths as the angle swept out by his path.⁸⁴

Excerpt 35

- 1 Michael: So where is Joe on Euclid Parkway when the number of radius lengths that he is to the north of Abscissa Boulevard is half times as large as the angle swept out by his path?
- 2 David: So radians north is half of the angle ... (*David thinks quietly for about two minutes and begins moving Joe slowly*).
- 3 Michael: So what are you imagining when you're moving Joe?
- 4 David: Um, as I'm looking at this I'm thinking about, I'd have to take the angle, cut it in half, and then I would want the, um, vertical distance that, if I was to draw a horizontal line over from that point (*Joe's position*) I'd want vertical distance here to be half of the arc length that I see over there. So (*begins moving Joe slowly along Euclid Parkway while speaking inaudibly to himself*). (*Places Joe slightly west of Ordinate Avenue and to the north of Abscissa Boulevard*) Yeah I like that one there. Okay. Uh, so (*moves Joe slightly*). Okay. So what I am basically imagining is that this length here (*the vertical distance that Joe is above Abscissa Blvd*; *David*

⁸⁴ While completing this task, David did not appear to know that he was indeed coordinating the input and output quantities of the sine function. It was therefore from *my perspective* that David coordinated the input and output quantities of sine.

motions this length along Ordinate Avenue), which is the number of radian lengths north, so this percent, so this would be point-nine, or something like that. Okay. And then I'm looking at this angle here (*points to the arc length that Joe has ridden*) and I'm saying that I should get about 1.8 radian lengths around. So there was one (*motioning to approximately one radius length of arc along Euclid Parkway*). So it would be somewhere in there. Okay. I'm happy with there.

David's first sentence in Line 4 suggests that he was coordinating the input and output quantities of the sine function while varying the input quantity. While slowly moving Joe's position, David attempted to determine where to place Joe so that his distance north of Abscissa Boulevard was half as long as the length traced out by his path. After David placed Joe in such a position, he quantified the number of radius lengths that Joe was north of Abscissa Boulevard in the same way he had in Excerpt 33—by projecting onto Ordinate Avenue Joe's vertical distance above Abscissa Boulevard and multiplicatively comparing this vertical distance with the radius of Flatville. David also quantified the angle measure swept out by Joe's path by estimating the length of the arc Joe's path had traced out in units of the radius of Flatville. David then verified that the number of radius lengths that Joe was north of Abscissa Boulevard (approximately 0.9) was half as large as the angle swept out by Joe's path in radians (approximately 1.8). David's responses in Excerpt 34, and his reply in Line 4 of Excerpt 35, demonstrate the fluency with which he conceptualized the input and output quantities of the sine and cosine functions. However, as I mentioned above, David did not appear to recognize

Joe's distance above Abscissa Boulevard and his distance east of Ordinate Avenue (both measured in units of the radius of Flatville) as quantities corresponding to the outputs of the sine and cosine functions respectively.

Recall that my instructional sequence for the output quantities of sine and cosine sought to engender a reflecting abstraction whereby David would construct an organization of actions and operations at the reflected level to which he may assimilate novel problems involving the symbolic notation " $\sin(\theta)$ " and " $\cos(\theta)$." During TBCI 4, up until his response in Line 4 of Excerpt 35, David had engaged in the sequence of actions involved in determining output values of the sine and cosine functions provided an angle measure. To engender the reflecting abstraction I had in mind, I presented David with a number of tasks that supported him in differentiating this sequence of actions from their effect, as well as in projecting this sequence of actions to the level of representation.

Excerpt 36

- 1 Michael: So if Joe's path has swept out an angle of x radius lengths, explain how you would use this program to estimate the number of radius lengths that Joe is, first of all, north of Abscissa Boulevard. So if I just tell you Joe traveled x radius lengths.
- 2 David: Okay so then I would, you know, I would figure out where I'm supposed to be on the outside of the circle depending on what we did, you know. So I said like that was one, two three, four, five, six, and a quarter (*moves Joe so that his path traces out one, two, three, four, five, six and then 6.28 radius lengths approximately*). Alright. So we'd figure out where we were and then if I was looking for

how far he was north of Abscissa Boulevard, then I would compare the vertical, um, height that this point is above, um, or below the horizontal line and, of the Abscissa Boulevard. And if we were talking about, um, the east was distances from Ordinate, then I would look at the, um, horizontal distance that this point is from the axis, or Ordinate Avenue. So this side would be positive and over here we would get negative horizontal distances (*moves Joe to the east, then the west, of Ordinate Avenue*). Okay.

3 Michael: So how do you get numbers for these distances? ...

4 David: Then it's, it would be an approximation of the radius. That's the best that I could do for something like that. ... I would just look at what percent it is of the, of the one length from the center to the, um, outside edge of the circle along the axis.

In Line 2 David discussed the sequence of actions involved in determining Joe's distance north of Abscissa Boulevard and his distance east of Ordinate Avenue (both measured in radius lengths) provided the angle measure in radians swept out by Joe's path. David explained that he would first move Joe's position to the appropriate place on Euclid Parkway, which he appeared to determine by knowing the location of the terminal point of an angle with a measure of one, two, three, etc. radians without having to imagine laying down radius lengths along the circumference of the circle. David's response in Line 2 was underspecified in terms of how he would quantify Joe's distance north of Abscissa Boulevard and east of Ordinate Avenue, so I asked him in Line 3 to explain how he would obtain values for these quantities. David's reply, "I would just look

at what percent it is of the, of the one length from the center to the, um, outside edge of the circle along the axis” suggests that he imagined multiplicatively comparing these respective horizontal and vertical distances to the radius of Flatville. In essence, Excerpt 36 demonstrates that David made explicit to himself the sequence of actions involved in determining output values of sine and cosine, although he seemed to be unaware that this was in fact what he was doing. That David was able to discuss this sequence of actions without being given a particular angle measure and without reference to a particular distance that Joe is north of Abscissa Boulevard or east of Ordinate Avenue suggests that he had differentiated this sequence of actions from the result of employing them.

Before David could assimilate novel tasks involving the symbolic notation for the outputs of sine and cosine to an organization of actions and operations at the reflected level, he had to construct “ $\sin(\theta)$ ” and “ $\cos(\theta)$ ” as symbolic representations for the coordinated actions he projected via reflecting abstraction to the level of representation. The following excerpt demonstrates that David had indeed made this construction.

Excerpt 37

- 1 Michael: If we know the angle swept out by Joe’s path, in radians, there are two buttons on our calculator that tell us how many radius lengths Joe is to the north of Abscissa Boulevard and to the east of Ordinate Avenue. What are these buttons?
- 2 David: (*Long pause*) Um, the sine and cosine buttons.
- 3 Michael: Let’s say Joe travels, uh, 0.7 radius lengths.
- 4 David: So then that means that it’s 0.7 radians. So then we would do the sine of 0.7 and then that would give me, um, a vertical distance in

radius lengths from the center of the circle. And if we did cosine it would give me horizontal distance in radius lengths from the center.

Without any other prompting, David recognized that, if given the measure of the angle swept out by Joe's path in radians, the sine button on his calculator produces the number of radius lengths Joe is north of Abscissa Boulevard and the cosine button yields Joe's distance east of Ordinate Avenue in units of the radius. It is noteworthy that David's long pause before proposing that one may use the sine and cosine buttons on a calculator to determine the value of these respective quantities provided an angle measure in radians suggests that he was unaware that he had been approximating values of sine and cosine while solving the previous tasks. David's explanation that when Joe's path had swept out an angle of 0.7 radius lengths, $\sin(0.7)$ represents "vertical distance in radius lengths from the center of the circle" and $\cos(0.7)$ represents "horizontal distance in radius lengths from the center" indicates that he had constructed sine and cosine as symbols that represent the sequence of actions involved in determining the number of radius lengths that Joe is north of Abscissa Boulevard and east of Ordinate Avenue respectively (Line 4). Additionally, David appeared to understand $\sin(0.7)$ and $\cos(0.7)$ as representing values of quantities.

I designed succeeding tasks in the instructional sequence to assess whether David had engaged in the reflecting abstraction I intended to engender. In particular, I designed subsequent tasks to reveal whether David had projected the sequence of actions involved in determining sine and cosine values to the level of representation, and to ascertain if he had constructed an organization of actions and operations at the reflected level to which he may assimilate novel tasks involving the symbolic notation for the outputs of sine and

cosine. The following excerpts demonstrate that David had indeed constructed such an organization of internalized actions and operations.

Excerpt 38

- 1 Michael: So, um, by dragging the blue dot to change Joe's position, can you estimate sine of point-five?
- 2 David: So we said that this was about one radian (*places Joe so that his path has traced out an angle of one radian*). So this would be about half of a radian, so half of a radius length (*moves Joe so that his path has traced out half of a radius length*). So we could draw across (*motions mouse from Joe's position straight to Ordinate Avenue*) and I'd get about 0.4, um, is the vertical distance north of Abscissa Boulevard. So that would be my sine.
- 3 Michael: Okay. Um, so how about cosine of three-fourths?
- 4 David: Okay. So cosine of three-fourths, I'd go, okay so that's about one radius length (*places Joe so that his path has traced out an angle of one radian*) so three-fourths of that would be something like this (*moves Joe so that his path has swept out approximately 3/4^{ths} of a radius length*). ... So for the cosine it would be this length over here (*motions along Abscissa Boulevard*), which would also be about, we'll actually I guess about three-fourths. Something like that.

David's response in Line 2 suggests that he interpreted the task of approximating the value of $\sin(0.5)$ as, "Estimate how many radius lengths is Joe north of Abscissa

Boulevard when the angle traced out by his path is 0.5 radians.” In particular, David interpreted the 0.5 as representing the number of radius lengths that Joe had traveled along Euclid Parkway and $\sin(0.5)$ as representing Joe’s distance north of Abscissa Boulevard in units of radius lengths. That David interpreted the task in this way suggests that he had coordinated and projected to the level of representation the actions involved in determining output values of sine and had organized these coordinated actions into a scheme to which he assimilated the task of estimating $\sin(0.5)$. As Lines 3 and 4 reveal, David had also coordinated and projected to the level of representation the actions involved in determining output values of cosine and had organized these coordinated actions into a scheme to which he assimilated the task of estimating $\cos(\frac{3}{4})$. David subsequently estimated the value of $\sin(2)$ and $\cos(4)$ in an analogous manner. David’s responses to these tasks demonstrate that he had constructed the outputs of sine and cosine as quantities.

To further assess the organization of internalized actions involved in determining sine and cosine values that David constructed at the reflected level, and to further reveal his way of understanding the outputs of sine and cosine as quantities, I asked David to represent symbolically a number of statements phrased in the context of Joe riding his bike on Euclid Parkway.

Excerpt 39

- 1 Michael: Write an equation that represents the following statement: “The angle swept out by Joe’s path, in radians, is twice as large as the number of radius lengths that he is to the east of Ordinate Avenue.”

- 2 David: Okay. (*David sat quietly for several minutes then wrote “ $2r_e = \theta$ ” followed by “ $2\cos(\theta) = \theta.$ ”*)
- 3 Michael: Okay. So you have “ $2\cos(\theta) = \theta.$ ” So, um, how did you come to that?
- 4 David: Um, the cosine of the angle will give me the, um, number of radius lengths, um, east of Ordinate Avenue, and then I want to double that length to actually figure out the angle. ...
- 5 Michael: Write an equation that represents the following statement: “The number of radius lengths that Joe is north of Abscissa Boulevard is half times as large as the angle swept out by his path in radians.”
- 6 David: Okay. (*Speaks to himself inaudibly and “ $r_N = \frac{1}{2} \theta / \sin(\theta) = \frac{1}{2} \theta.$ ”*)
- 7 Michael: Okay. So how did you know to write that?
- 8 David: ... So (*sigh*) the number of radius lengths north is half as large as the angle, so it’s half of the angle (*pointing to his equation “ $r_N = \frac{1}{2} \theta$ ”*), and then radius length north comes from the sine of the angle (*David replaced “ r_N ” in the first line with “ $\sin(\theta)$ ” in the second*).

That David replaced r_e (which represented Joe’s distance east of Ordinate Avenue measured in radius lengths) in the equation $2r_e = \theta$ with $\cos(\theta)$ suggests that he had constructed $\cos(\theta)$ as a symbolic representation for Joe’s distance east of Ordinate Avenue (in radius lengths) when Joe’s path had traced out an angle of θ radians. Similarly, David’s substituting r_N (which represented Joe’s distance north of Abscissa Boulevard measured in radius lengths) with $\sin(\theta)$ in the equation $r_N = \frac{1}{2} \theta$ reveals that he had constructed $\sin(\theta)$ as a symbolic representation for Joe’s distance north of Abscissa

Boulevard measured in radius lengths when his path had traced out an angle of θ radians. In general, David's responses in the previous excerpt further supports the claim that he had constructed an organization of actions at the reflected level to which he assimilated the symbolic notation " $\sin(\theta)$ " and " $\cos(\theta)$." Additionally, the previous excerpt further demonstrates that David had conceptualized the outputs of sine and cosine quantitatively.

After I became confident that David had constructed an internalized representation for the actions involved in determining output values of sine and cosine, I asked him to move Joe's position on Euclid Parkway to illustrate the solution to the equations $\sin(x) = 0.6$ and $\cos(x) = -0.25$. David's ability to move Joe's position to illustrate these equalities suggests that his construction afforded him simultaneous access to this sequence of actions in two directions—determining a sine or cosine value given an angle measure and determining an angle measure given a sine or cosine value.

Accordingly, David's response to these tasks revealed whether the actions he had constructed at the level of representation via reflecting abstraction were indeed mental operations (i.e., an internalized organization of actions that are reversible).

Excerpt 40

- 1 Michael: Okay. Um, so can you place Joe in a position where the angle swept out by his path, x , is such that $\sin(x)$ equals 0.6?
- 2 David: *(Sigh)* sine of x is 0.6 *(moves Joe)*. Somewhere around there. Okay so that looks like about 0.6. That would be one *(places Joe so that he is about 0.6 radius lengths north of Abscissa Boulevard, but to the west of Ordinate Avenue)* and then I can do another one over

here (*places Joe the same distance to the north of Abscissa Avenue but to the east of Ordinate Avenue*).

3 Michael: Okay. And so how do you know that that's, um, that gives you, uh, 0.6?

4 David: Uh, vertically it's (*Joe's position*) about, I'd say about 60 percent of the radius (*motions pointer to the place on Ordinate Avenue that is 0.6 radius lengths north of Abscissa Boulevard*). So I just went around so that I was about 60 percent. So I knew I would be somewhere along this line (*motions pointer horizontally from the place he identified on Ordinate Avenue as being 0.6 radius lengths north of Abscissa Boulevard*) so I put one on this side here (*places Joe on the intersection of his imaginary horizontal line and Euclid Parkway*), that's 60 percent, and I put one over here on this side (*places Joe on the other intersection of his imaginary horizontal line and Euclid Parkway*) that's about 60 percent of this entire length (*points to the length of Ordinate Avenue above Abscissa Boulevard*).

5 Michael: So how would you know what the actual angle measure is?

6 David: So then it would just come down, after that I would come down to estimating. If I'm, if we're using this program then I would use estimation, uh, in order to, uh, to get it. So I would say, how many, uh, how does it relate to my (*pause*) one radian, two radian, three radian trip around (*motions pointer to the places on Euclid*

Parkway that correspond to the location of the terminal point of an angle with a measure of one, two, and three radians)? So I could estimate it that way.

As Excerpt 38 demonstrated, when David used the applet to approximate a sine value given the measure of an angle in radians, he moved Joe's position so that his path had swept out the number of radius lengths equal to the value of the angle measure. David then visualized Joe's vertical distance north of Abscissa Boulevard, projected this distance onto Ordinate Avenue, and multiplicatively compared this vertical distance to the radius of Flatville, which allowed him to estimate the sine value of the given angle measure. As Lines 2 and 4 of Excerpt 40 reveal, David enacted this sequence of actions in reverse while discussing his approach to approximating the solution to the equation $\sin(x) = 0.6$. First, David identified to the spot on Ordinate Avenue that was 60 percent of a radius length north of Abscissa Boulevard and imagined a horizontal line extending through this point. David then identified where this horizontal line intersected Euclid Parkway (the circumference of the circle) and recognized these points of intersection as the two places on Euclid Parkway where Joe's distance above Abscissa Boulevard is 0.6 radius lengths. Finally, David explained that he would approximate the measure of the relative angles swept out by Joe's path when Joe is positioned at each of these two places on Euclid Parkway. David responded similarly when I subsequently presented him with the following tasks:

- (1) Place Joe in a position to illustrate the equality $\cos(x) = -0.25$.
- (2) Suppose I typed the sine of some number into my calculator and I get as an output 0.75. What number could I have put into my calculator?

(3) Suppose I typed the cosine of some number into my calculator and I get as an output 0.4. What number could I have put into my calculator?

That David was able to solve these tasks by employing in reverse the sequence of actions involved in determining output values of sine and cosine suggests that he had constructed these internalized actions as mental operations.

I presented in Excerpt 35 David's response to my asking him to place Joe in a position on Euclid Parkway so that his distance north of Abscissa Boulevard is half as large as the angle swept out by Joe's path. I asked David the same question (from my perspective) in the following clinical interview but in symbolic form. In other words, I asked David to place Joe in a position on Euclid Parkway to illustrate the equality $\sin(x) = x/2$. David positioned Joe in the correct place on Euclid Parkway and justified his response by explaining that Joe's distance north of Abscissa Boulevard was a little less than one radius length while the angle swept out by Joe's path was a little less than two radius lengths; so Joe's vertical distance north of Abscissa Boulevard was approximately half as large as the angle swept out by his path. David appeared to assimilate the symbolic notation " $\sin(x) = x/2$ " to a scheme that allowed him to interpret the symbols quantitatively. In particular, David appeared to understand " $\sin(x)$ " as the vertical distance that Joe is north of Abscissa Boulevard measured in radius lengths, and he appeared to understand " x " as the length of the subtended arc measured in units of radius length. I then asked David to compare this task with the task I stated in Line 1 of Excerpt 35. I provide David's response in Excerpt 41.

Excerpt 41

- 1 Michael: I want to look back real quick at another question that I asked, which was I think (*pause*) 2(b). On 2(b) I asked, um, where is Joe on Euclid Parkway when the number of radius lengths that he is north of Abscissa Boulevard is half times as large as the angle swept out by his path, in radians? So can you compare this task that you just did, putting Joe in a place that satisfies that equality, with Part (b).
- 2 David: Actually it sounds like the same thing. Let me just read it real quick (*read the question to himself*). Yeah. Actually it sounds like the same thing so (*inaudible*) it's asking the same question.
- 3 Michael: So why is that?
- 4 David: Uh, because we want the, um, we want the output of sine, which is the distance north of Abscissa Boulevard, to be, um, half of the angle that we have put.

I asked the question in Line 1 to assess whether David interpreted the statement, “move Joe so that the number of radius lengths that he is north of Abscissa Boulevard is half times as large as the angle swept out by his path, in radians” to mean the same thing as, “move Joe to illustrate the equality $\sin(x) = x/2$ for x .” That David recognized these tasks as essentially the same demonstrates that he had constructed a scheme of actions and operations at the reflected level to which he assimilated both statements. Additionally, this scheme of actions and operations allowed him to interpret “ $\sin(x)$ ” as representing the value of a quantity. David’s ability to assimilate the symbolic notation for the output of sine to a scheme of actions and operations that he had constructed via prior reflecting

abstractions indicates that he had engaged in the reflected abstraction I designed these tasks to engender. Additionally, and as a result of this reflected abstraction, David appeared to be consciously aware of his quantitative meanings for the output of sine.

The previous few excerpts demonstrated David engaging in tasks whereby he assimilated the symbolic notation for the outputs of sine and cosine to a scheme of actions and operations at the reflected level. Through the processes of reflecting and reflected abstraction, David constructed this scheme in a way that allowed him to interpret the outputs of sine and cosine as representing quantities—a vertical and horizontal length respectively. It is worth noting that David completed these tasks in the context of using the didactic object with Joe riding his bike around Euclid Parkway. Throughout TBCI 5, David demonstrated his ability to assimilate the symbolic notation for the outputs of sine and cosine without the aid of this didactic object. That David was able to apply the actions he constructed at the level of representation to a more general class of geometric objects further suggests that he had constructed these internalized actions as mental operations.

Toward the beginning of TBCI 5, I asked David to explain what the values 0.7 and 0.644 represent in the equality $\sin(0.7) = 0.644$. David replied, “The 0.7 would be the angle measure in radians and the 0.644 would be the, um, vertical distance above the horizontal axis to the terminal point, um, on the circle measured in radius lengths.” Similarly, upon my asking him to describe what the values 1.2 and 0.362 represent in the equality $\cos(1.2) = 0.362$, David explained, “The 1.2 would be the angle measure of the central angle in radians of 1.2 and then the 0.362 would be the distance, um, from the vertical axis; so it’s the horizontal distance from the vertical axis that the point on the

terminal side of the angle, uh, with an angle measure of 1.2, and measured in radius lengths.” Although David did not define the value 0.362 as representing the horizontal distance that the terminal point is *to the right* of the vertical axis, he did appear to interpret the values in these equations quantitatively.

Also during TBCI 5, I asked David to draw pictures to represent the meaning of the numerical values in a number of different equalities (see task statement in Table 29). First, I asked David to illustrate what the values 2.5 and 0.6 represent in the equation $\sin(2.5) = 0.6$ (see David’s drawing in *Figure 42*). David responded by sketching an initial ray in standard position, and then drew the terminal ray so that the arc length the angle subtended was equal to 2.5 radius lengths. David labeled the length of the subtended arc “2.5 radius lengths/radians,” indicating that he did not recognize a distinction between the terms “radians” and “radius lengths.” David then sketched a line segment from the terminal point to the horizontal diameter of the circle and claimed that the 0.6 in the equation represented the length of this line measured in units of radius lengths. David responded analogously to my asking him to illustrate what the values 3.77 and -0.809 represent in the equation $\cos(3.77) = -0.809$ (see David’s drawing in *Figure 43*). In essence, David represented the values for the outputs of sine and cosine as respective vertical and horizontal distances that the terminus of the subtended arc is away from specific reference points, without my phrasing the tasks in the context of Joe riding his bike around Euclid Parkway. David therefore maintained a quantitative understanding of the outputs of sine and cosine in a more general context. In other words, David was able to assimilate these non-contextual tasks to the scheme of actions and operations he

constructed at the reflected level. It is noteworthy that David also assimilated to this scheme the following tasks:

- (1) Draw the terminal ray of an angle with a measure of θ radians that satisfies the condition $\cos(\theta) = -1/4$.
- (2) Draw the terminal ray of an angle with a measure of θ radians that satisfies the condition $\sin(\theta) = 1/3$.

David correctly drew the terminal rays that satisfy the respective conditions specified in these tasks, which exemplifies the coherence of David's scheme for the outputs of sine and cosine.

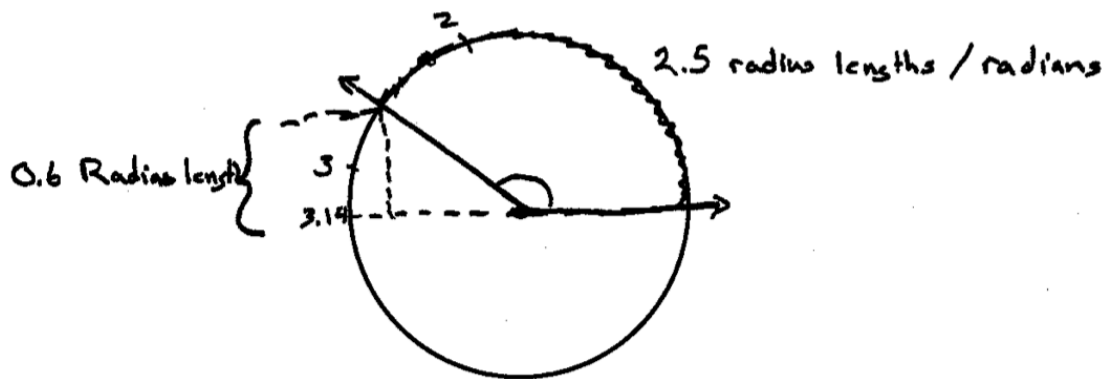


Figure 42. David's illustration of the equality $\sin(2.5) = 0.6$.

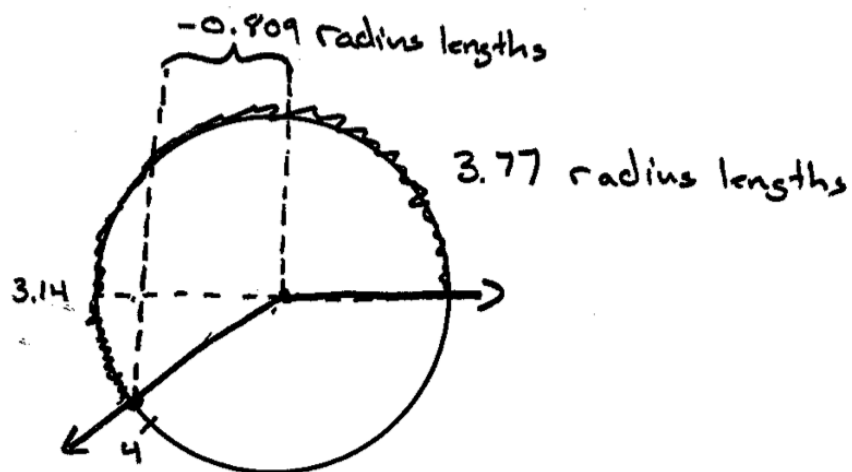


Figure 43. David's illustration of the equality $\cos(3.77) = -0.809$.

After having correctly solved two tasks that prompted David to represent the values in equations of the form $\sin(a) = b$ and $\cos(c) = d$ (for constants a , b , c , and d), and after having correctly drawn the terminal ray of two angles that respectively satisfy a condition of the form $\sin(\theta) = j$ or $\cos(\theta) = k$ (for constants j and k), I asked David if his answers to such tasks would change if the circles provided in the task statement were twice as large. David replied, "well if you made it twice as large, uh, none of my answers for this would change because everything I measured was in radius lengths." David went on to explain, "since I measured according to radius lengths for both my arc length and my vertical height, it would be the same because it's still the same portion of a radius length." David's recognition that the size of the circle centered at the vertex was immaterial to his responses to the previous questions suggests that he was able to apply his scheme for the outputs of sine and cosine to a generalized class of objects without regard to an initial state. David therefore constructed his scheme for the outputs of sine and cosine as an operatory structure (i.e., an organization of mental *operations* at the reflected level).

Theme 1: Sine and cosine values as ratios of lengths (revisited). After having completed the instructional sequence for the outputs of sine and cosine during TBCI 5, I spontaneously decided to revisit the task I designed during ongoing analysis and presented to David at the beginning of TBCI 4 (see Table 46). Recall that in response to this task David described the outputs of sine and cosine as arithmetic as opposed to quantitative operations. In particular, David defined the sine (cosine) value as the ratio of the y -coordinate (x -coordinate) of the terminal point to the radius of the subtended arc.

Additionally, when I prompted David to represent the values of sine and cosine on an image of the angle with the circle centered at its vertex containing Point B , he did not identify quantities to which values of sine and cosine are respective measures. In particular, David did not appear to conceptualize the radius as a unit of measure for the vertical and horizontal distances that correspond to the outputs of sine and cosine respectively.

I expected that the scheme of actions and operations David appeared to have constructed while engaged in the instructional sequence for the outputs of sine and cosine would support him in identifying sine and cosine values as measures of quantities. In other words, I anticipated that David would recognize the value of sine as the vertical distance the terminus of the subtended arc is above the horizontal diameter of the circle centered at the vertex of the angle, and that he would identify the value of cosine as the horizontal distance the terminal point is to the right of the vertical diameter of the circle centered at the vertex of the angle, both measured in units of radius lengths. To assess whether David utilized this understanding—a way of understanding he demonstrated repeatedly throughout the instructional sequence for the output of sine and cosine—I presented him with a hypothetical student response to Part (e) of the task in Table 46.

Excerpt 42

- 1 Michael: Okay. The last question I have is, we looked at this picture the other day (*displays task*) with these two circles centered at the origin and, um, with the coordinates of A and B given, and I asked if a student, you know, if we were just looking at the, um, large circle, and you asked a student to draw the output of the sine

function and if they drew this line (*draws a vertical line from the terminus of the subtended arc to the horizontal axis*), I asked if that would be correct or incorrect. What do you think now?

- 2 David: I think that would be, um, incorrect (*pause*) because they're thinking of the, that that is the vertical distance and they're not thinking of the output of sine as a ratio. They're thinking of it, they're using unit circle reasoning on a non-unit circle.
- 3 Michael: So can you say a little about what you mean by "unit circle reasoning"?
- 4 David: They are, if we go back to that (*finds the papers that have the circle he drew and the circle I drew while discussing the sine curve*), they're using this kind of thought process (*points to where he wrote " $\sin(\theta) = y/r = y/1$ "*) where they've just decided to make the radius one. So it's just the output value, the y -value, is the output. Um, but in this case we can see that the radius is not one, of this circle (*points to screen*) so they're not, they're using, they're treating r like one and it's not. Where they should have treated it more like we did with the red circle (*the circle that Michael drew*) where we treat it as a ratio.

After having completed the instructional sequence on the outputs of sine and cosine—during which David continually demonstrated his understanding of the outputs of sine and cosine as quantities—David claimed it incorrect to think of sine values as representing vertical distances (Line 2). David explained that sine is a ratio, not a distance

or a length. Additionally, David argued that the student who thinks sine values represent vertical distances is employing “unit circle reasoning,” by which he meant the student assumes that the radius of the circle centered at the vertex of the angle is one unit (Line 4). Of course the radius of any circle centered at the vertex of the angle *is* one unit if one has conceptualized the radius as the unit of measure. David’s suggestion that the sine value of the angle is not a vertical distance but the ratio of the y -coordinate of the terminal point to the radius of the circle (measured in units defined by the coordinate system) revealed that he had not constructed the radius as a unit of measure for the vertical and horizontal distances that correspond to the outputs of sine and cosine respectively.

After David had explained that the sine value is a ratio and not a vertical distance, I asked him to use the features of *Geometer’s Sketchpad* to represent the sine value on the image displayed in Table 46 (see *Figure 44* for David’s construction). David explained that the sine value of the measure of the given angle is “the ratio of the blue line to the whole radius.” I then asked David whether or not it would be correct for one to say that the value of sine is the length of the red line in his construction. I provide David’s reply in Excerpt 43.

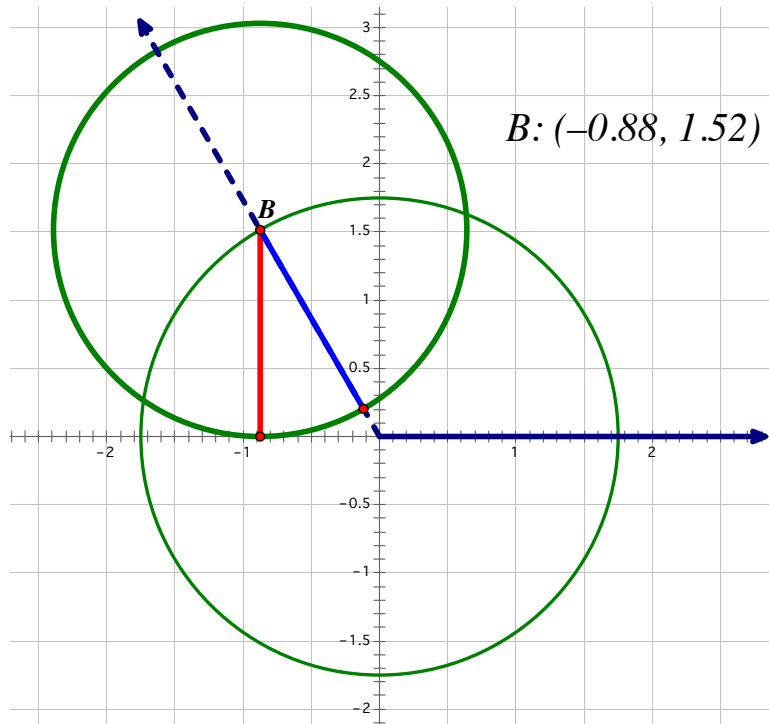


Figure 44. David's representation for the output of sine.

Excerpt 43

- 1 Michael: Okay. So if someone just said, if someone just said that the sine is this distance (*highlights the red line in David's construction*)?
- 2 David: No. But if they said it's, it's how this distance compares to the length of the length of radius, then I'm okay with it. (*Pause*) And I figure that the, since the original prompt was how would they display it and we were talking about visually displaying it, ... somehow we need to physically move this length of radius, or this vertical length next to a radius so that we can compare the lengths. So whether that's doing what we did there where we rotated it around so that it fit on there or if it's like some of the earlier experiments (*Joe tasks*) where you had the radius length and you

had the, you know, the vertical length as lines down here just, you know, off to the side ...

3 Michael: So they have to have some way of comparing this vertical distance to the length of the radius?

4 David: Correct.

David again asserted that the vertical distance that the terminal point is above the horizontal axis is not a correct representation for the output value of sine (Line 2). He explained that to represent the value of sine, “somehow we need to physically move this length of radius, or this vertical length next to a radius so that we can compare the lengths.” David did not appear to interpret this process of multiplicatively comparing the radius length and the vertical distance that the terminal point is above the horizontal axis as a process of *measuring* this vertical distance in units of the radius. That is, David’s remarks in Excerpt 43 further reveal that he had not conceptualized the radius as a unit of measure for the output of sine.

Summary of TBCIs 4 and 5. At the beginning of TBCI 4, while engaged in the task presented in Table 46, David demonstrated a way of understanding the outputs of sine and cosine as ratios of lengths, not as lengths measured in a particular unit. More specifically, David explained that the output of sine is determined by the y -coordinate of the terminal point divided by the radius of the circle centered at the vertex of the angle (assuming the vertex of the angle is positioned at the origin of a coordinate plane). Similarly, David described the output of cosine as the ratio of the x -coordinate of the terminal point to the radius of the circle. David interpreted these ratios as operations on the values of two quantities without understanding the result as representing the value of

a quantity itself. David therefore did not appear to conceptualize the denominator of these respective ratios as a unit of measure for the numerator, which is to say David constructed the outputs of sine and cosine as arithmetic (as opposed to quantitative) operations. Moreover, David was unable to represent quantities to which values of sine and cosine are measures when provided a circle centered at the vertex of the angle that had a radius other than one unit.⁸⁵ David therefore did not appear to conceptualize the outputs of sine and cosine as quantities at the beginning of TBCI 4.

During the majority of the fourth and fifth TBCIs, David demonstrated a different and very consistent way of understanding the outputs of sine and cosine. I presented David with several tasks of the form, “If we have $\sin(m) = n$, what do the constants m and n represent?” David often replied with some variation of the following: “The m represents the angle measure in radians and the n represents the vertical distance that the terminal point is above the horizontal diameter of the circle measured in radius lengths.” Moreover, David was able to move the terminus of the arc the angle subtends to accomplish the following tasks:

- (1) Estimate the value of expressions of the form $\sin(v)$ and $\cos(w)$ (for constants v and w).
- (2) Illustrate what the values a , b , c , and d represent in the equalities $\sin(a) = b$ and $\cos(c) = d$.
- (3) Represent the solutions to equations of the form $\sin(\theta) = j$, $\cos(\theta) = k$, $\sin(\theta) = m \cdot \theta$, and $\cos(\theta) = n \cdot \theta$ (for constants j , k , m , and n).

⁸⁵ Note that David considered a unit to be defined by the coordinate system containing the angle and the circle centered at its vertex. Specifically, David considered the distance between the point $(0, 0)$ and $(0, 1)$ to be the unit of measure for the radius of the circle centered at the vertex of the angle.

David's language and actions while solving tasks of this type convincingly revealed his understanding of the outputs of the sine and cosine as vertical and horizontal distances measured in units of radius lengths.

In the final analysis, David appeared to construct, via reflecting and reflected abstraction, a scheme of actions and operations at the level of representation to which he assimilated novel tasks involving the symbolic notation " $\sin(\theta)$ " and " $\cos(\theta)$." This scheme allowed David to interpret the symbolic notation for the outputs of sine and cosine as representations for the measures of two different quantities. In particular, David assimilated the notation " $\sin(\theta)$ " as, "the vertical distance that the terminal point is above the horizontal diameter of the circle centered at the vertex of the angle measured in units of radius lengths." Similarly, David understood " $\cos(\theta)$ " as, "the number of radius lengths the terminal point is to the right of the vertical diameter of the circle centered at the vertex of the angle." Furthermore, David recognized the size of the circle centered at the vertex of the angle as inconsequential to the output values of sine and cosine since the lengths that correspond to these respective outputs are measured in units of radius lengths. *Figure 45* summarizes my model of David's way of understanding the symbolic notation " $\sin(\theta)$ " and " $\cos(\theta)$," and is based on my interpretation of his language and actions while engaged in the instructional sequence for the outputs of sine and cosine.

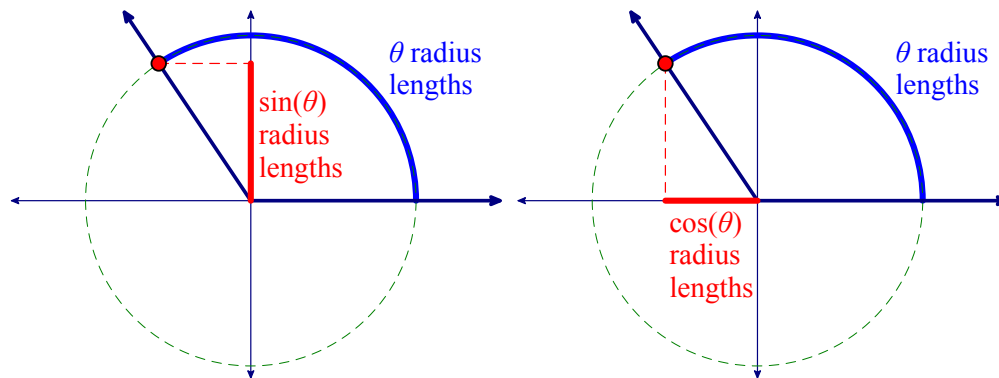


Figure 45. David’s way of understanding “ $\sin(\theta)$ ” and “ $\cos(\theta)$.”

After having demonstrated his understanding that sine and cosine values represent measures of quantities, David reverted back to speaking of the output of sine as the ratio of the y -coordinate of the terminus of the subtended arc to radius length. Moreover, David explained that for an angle with its vertex at the origin of a coordinate system, one may *not* represent the value of sine as the vertical distance that the terminus of the subtended arc is above the horizontal axis if the radius of the circle centered at the vertex of the angle containing the subtended arc is not a unit circle (i.e., if the circle does not pass through the point $(1, 0)$). David’s assertion suggests that he did not conceptualize the radius as a unit of measure for the vertical distance that the terminal point is above the horizontal axis. In other words, David did not interpret the denominator of the fraction,

$$\frac{\text{y-coordinate of terminal point}}{\text{radius of the circle centered at the vertex of the angle}}$$

as a unit of measure for the numerator, which is to say David did not demonstrate a “measurement” way of understanding division. In general, while David did conceptualize the outputs of sine and cosine as quantities, he did not appear to construct the ratios he proposed as representing the outputs of sine and cosine as quantitative operations.

Adjustments to TBCIs 6 and 7. I amended the protocol for the sixth and seventh TBCIs based on my emerging model of David’s way of understanding the outputs of sine and cosine and his way of understanding the graphical representation of sine and cosine functions. In the following paragraphs, I present and explain my rationale for the tasks I designed during ongoing analysis that are relevant to my discussion of TBCIs 6 and 7.

At the beginning of TBCI 4 and at the end of TBCI 5, David defined the outputs of sine and cosine as ratios of lengths. However, throughout the majority of the fourth and fifth TBCIs, David interpreted sine and cosine values as respective vertical and horizontal distances measured in units of radius lengths. David did not appear to recognize the relationship between these two ways of understanding the outputs of sine and cosine because he did not seem to conceptualize a quotient as the numerator measured in units of the denominator, which is to say David did not possess a “measurement” way of understanding division. I designed the task in Table 48 to assess how David conceptualized the outputs of sine and cosine while constructing a graphical representation of these functions. My phrasing of this task is intentionally vague; I intended to provide a context in which David could identify aspects of the image on the screen that his sine and cosine graphs represent without directing his attention to particular quantities in the diagram. Additionally, I designed this task to reveal whether David recognized that the graphical representation of the sine and cosine functions do not depend on the size of the circle centered at the vertex of the angle.

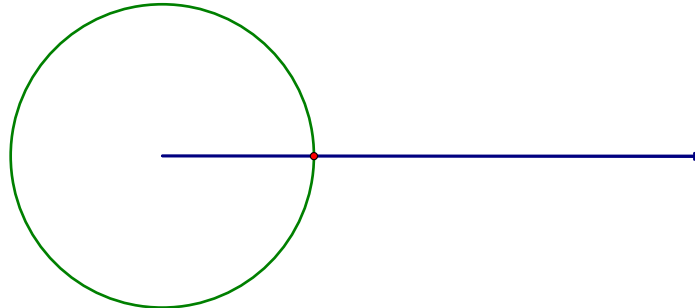
Table 48

Task Added During Ongoing Analysis for TBCI 6⁸⁶

(a) Consider the following simulation:

 *Animate*

Radius = 6.00 cm

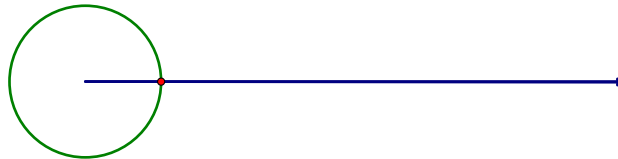


Sketch the sine and cosine curves that correspond to this situation. Explain why you constructed your graphs in the way that you did.

(b) Consider the following simulation:

 *Animate*

Radius = 3.00 cm



Sketch the sine and cosine curves that correspond to this situation. Explain why you constructed your graphs in the way that you did.

I added the task in Table 49 to the protocol for TBCI 7 to assess David's understanding of the concavity of the sine and cosine functions. In particular, I intended this task to reveal the extent to which David's way of understanding the concavity of sine and cosine functions was based on his conceptualization of the input and outputs of the sine and cosine functions as quantities. I designed the didactic objects in Parts (a) and (b) to indicate the change in the respective output quantities of sine and cosine for equal changes of $\pi/6$ radians in the input quantity. The vertical distances between adjacent

⁸⁶ Note that pressing the "Animate" button causes the terminal ray rotates counterclockwise one rotation.

horizontal dashed lines in each didactic object represent displacements of the output quantities of sine and cosine respectively for changes in the input quantity of $\pi/6$ radians. Upon examining these didactic objects, one may recognize how the average rate of change of the sine and cosine functions change over adjacent $\pi/6$ intervals of the input quantity, thereby revealing why the graphs of the sine and cosine functions are curved the way they are.

Table 49

Task Added During Ongoing Analysis for TBCI 7⁸⁷

(a) I designed this animation for one of my classes. What concept do you think I designed this animation to convey? Explain.

(b) I designed this animation for one of my classes. What concept do you think I designed this animation to convey? Explain.

⁸⁷ Note that pressing the “Animate Point” button causes the terminal ray to rotate one full rotation. Additionally, pressing the “Hide Objects” button hides the horizontal dashed lines.

Task-Based Clinical Interviews 6 and 7

Throughout the sixth and seventh TBCIs, David demonstrated a way of understanding the graphs of sine and cosine functions as representations of the covariational relationship between input and output quantities. Moreover, David utilized his quantitative understanding of the inputs and outputs of sine and cosine functions to make sense of the concavity of these functions.

Theme 1: Graphs of sine and cosine as representations of the covariational relationship between input and output quantities. I presented the task in Table 48 to David at the beginning of TBCI 6. In response to Part (a), David sketched the sine curve displayed on the left side of *Figure 46*. I asked David to explain why he drew the sine curve the way he did, to which he replied, “I made my vertical axis go up to one and down to negative one because I know that the, um, output values for sine are a fraction of the radius.” It is noteworthy that David described the output of sine as a fraction of the radius of the circle centered at the vertex of the angle as opposed to the ratio of the y -coordinate of the terminal point to the radius of the circle, a claim David had made several times previously. However, David’s response did not indicate that he was conceptualizing a quantity with a measure that is a fraction of the radius. In an effort to reveal David’s way of understanding the output of sine in this context, I asked him to label what the values on the vertical axis of his graph represented. David labeled the output axis, “Vertical distance from the horizontal axis in radius lengths” and explained that the values on this axis represent the distance that the terminus of the subtended arc is above the horizontal diameter of the circle centered at the angle’s vertex measured in units of radius lengths (see the right side of *Figure 46*). David constructed a graph of the

cosine function similarly. I then presented David with Part (b) of the task in Table 48, which asked him to explain how his graphs would differ if the circle centered at the vertex of the angle had a radius of three centimeters instead of six centimeters. David immediately responded, “The sine and cosine curves would be the same as what I’ve already drawn. ... Since it’s in radius lengths, it doesn’t matter if the radius is six or the radius is three.” David therefore appeared to recognize that the size of the circle centered at the vertex of the angle does not change the graphical representation of the sine and cosine functions because the input and output quantities of these respective functions are measured in units of the radius.

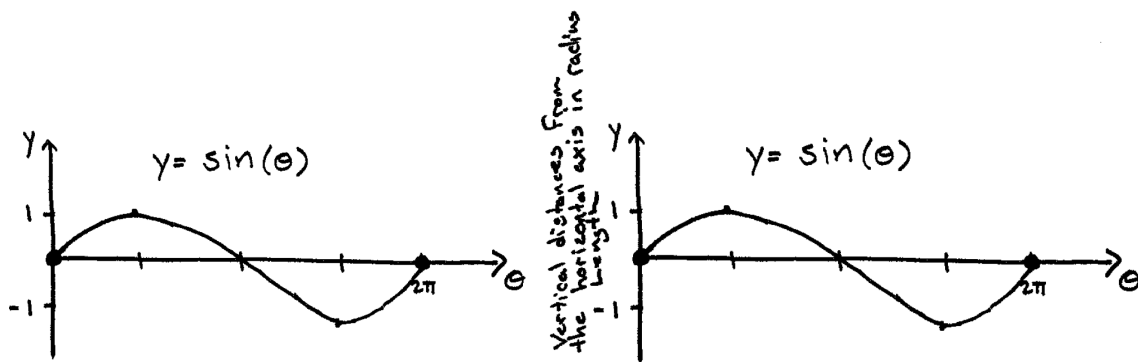


Figure 46. David’s graph of the sine function.

To further reveal the extent to which David understood the graphs of sine and cosine as a representation of the covariational relationship between the values of quantities, I identified three points on the graphs of sine and cosine and asked him to draw pictures to display what the coordinates of these respective points represent. David drew the picture on the right side of Figure 47 to represent the coordinates of Point A. (Note that David’s drawing also illustrates his response to a subsequent task.)

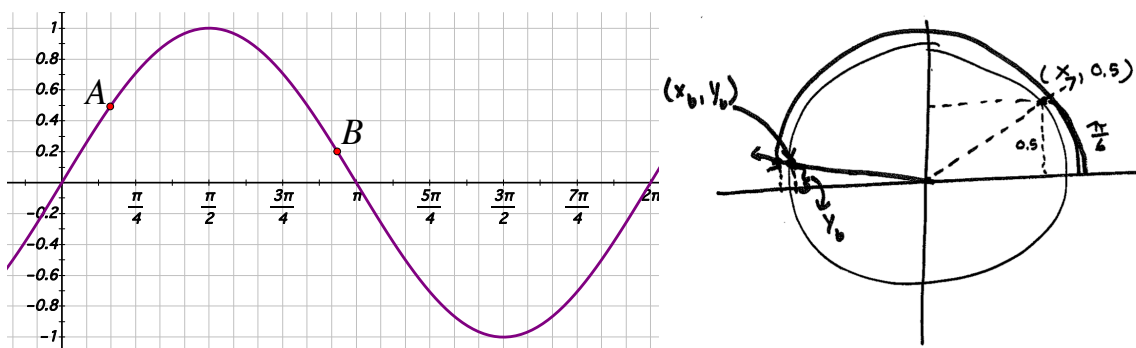


Figure 47. David's representation of the coordinates of Point A.

David first approximated the coordinates of Point A as being $(\pi/6, 0.5)$. He then represented the $\pi/6$ as the length of the arc an angle subtends and the 0.5 as the vertical distance that the terminus of the subtended arc is above the horizontal diameter of the circle centered at the angle's vertex. David explained that the $\pi/6$ and the 0.5 refer to the measure of the respective quantities in units of radius lengths. David similarly constructed a diagram to represent the coordinates of Point B as well as the coordinates of two points on the graph of cosine, except he denoted the y-coordinate of the points on the cosine graph as a horizontal distance to the right of the vertical diameter of the circle. David's ability to represent the coordinates of points on the graphs of sine and cosine as quantities in a geometric diagram demonstrates his understanding that each point on the graph of sine (cosine) represents a particular correspondence between the measure of an angle in radians (i.e., the length of the arc an angle subtends measured in units of radius lengths) and the vertical (horizontal) distance that the terminus of the subtended arc is above (to the right of) the horizontal (vertical) diameter of the circle centered at the vertex of the angle, measured in radius lengths.

To assess the extent to which David understood the graphs of sine and cosine as representations of the covariational relationship between input and output quantities, I

the “Animate” button and asked David to describe what he saw happening on the screen. David recognized that as the measure of the angle varied, the terminal point traced out a subtended arc length, and that the pink line extending along the horizontal axis represented the length of the subtended arc measured in units of radius lengths. I then paused the animation and asked David to describe what would happen to the length of the pink line on the horizontal axis if I changed the radius of the circle. David explained that the length of the horizontal pink line would remain fixed because its length represents the measure of the subtended arc length in units of radius lengths.

I then presented David with Part 2 of the didactic object. I pressed the “Animate” button and asked David to explain what he saw. David replied, “So now it’s showing me the, um, vertical distance from the terminal point of our angle to the, uh, horizontal line going through the diameter of that circle, and it’s measured in radius lengths.” David clearly understood that the length of the red line extending along the vertical axis represented the number of radius lengths that the terminus of the subtended arc was above the horizontal diameter of the circle. As I did while discussing Part 1 of the didactic object, I paused the animation and asked David to anticipate what would happen if I changed the radius of the circle. David explained that the length of the red line extending along the vertical axis would not change “Because we’re still measuring in terms of radius length. So it doesn’t matter what the radius is because the radius is my unit of measure.” David therefore appeared to recognize that as the size of the circle varies, the terminus of the subtended arc is always the same number of radius lengths above the horizontal diameter of the circle centered at the angle’s vertex.

While viewing Part 3 of the didactic object, David explained that the animation “is showing me, um, along the horizontal axis, the input values as they change from zero to 2π and the outputs, um, that correspond with those inputs going vertically.” As he did while viewing Part 2, David remarked that the length of the red line represents the “vertical distance measured in radius lengths.” After having viewed Part 4 of the didactic object, David observed that the length of the red line represented the output value that corresponds with the input value represented by the length of the pink line.

Finally, I presented David with Part 5 of the didactic object and asked him to explain what he saw. David responded, “Now it traced out the sine curve. ... It’s just plotting points there for me so I can see how the inputs and outputs vary together.” David appeared to see the orange dot tracing out a curve that represented the covariational relationship between the input and output quantities of the sine function. I then paused the animation and asked David to explain what the coordinates of the orange dot represent.

Excerpt 44

- 1 Michael: So if I pause this somewhere, just anywhere, say there (*pauses the animation*), what does this orange dot represent?
- 2 David: So that orange dot represents when you have an input, uh, from, uh, represented by the pink line, or the length of the arc, uh, in radius lengths, you will have the corresponding output, which is the red line, which is the vertical distance, um, above the horizontal diameter of our circle, as an output. So it’s just showing how those two quantities are varying together.

In his response in Line 2, David explained that the coordinates of the orange dot represent the input and output quantities of the sine function. This response further reveals David's way of understanding the sine curve as a representation of the relationship between the values of two quantities that change together. In sum, David appeared to understand the graph of the sine function as a representation of the simultaneous variation of the values of two quantities, as opposed to simply an image or picture.

After David explained that the coordinates of the orange dot represent the values of the input and output quantities of the sine function, I reset the animation and changed the size of the circle centered at the angle's vertex. I then asked David to anticipate how the graph of the sine curve would change.

Excerpt 45

- 1 Michael: So what I'm going to do is I am going to change the circle now to make it that big (*increases the radius of the circle*), and I'm going to show exactly what I showed before, um, and I'm going to hit Animate again but what do you expect, where are you expecting this orange dot to go now that I've changed the radius?
- 2 David: I expect it to be in the same place.
- 3 Michael: Why is that?
- 4 David: Because our inputs and outputs will still be measured in radius lengths and so as you change the radius length you, um, it's still one radius length no matter how big you make the radius it's still one radius long.

David's responses in Lines 2 and 4 demonstrate that he understood that the graphical representation of the sine function does not depend on the size of the circle centered at the vertex of the angle. David appeared to recognize that the actual length of the radius is inconsequential because the radius of the circle centered at the vertex of the angle *always* has a measure of one radius length.

Before moving on from our conversation around the didactic object displayed in *Figure 48*, I asked David to label the axes of the graph of the sine curve that we generated using this didactic object. After about a minute of searching for the right words, David decided to label the input axis, "Subtended arc length measured in radius lengths." David then changed his mind and settled on what he considered a more economical but equivalent phrase, "Angle measure in radians." David then labeled the output axis, "Vertical distance above the horizontal diameter of a point on the terminal side of an angle in radius lengths." David's labels for what the values on the input and output axes of the sine function represent demonstrates that he conceptualized the input of the sine function as the length of the arc an angle subtends measured in units of radius lengths, and the output of the sine function as the vertical distance that the terminus of the subtended arc is above the horizontal diameter of the circle measured in units of the radius. It is noteworthy that this way of understanding the output of sine differs markedly from the definition of the output of sine as a ratio of lengths that David proposed in previous tasks/interviews.

After having discussed all five parts of the didactic object displayed in *Figure 48*, I asked David to work with me to create a similar didactic object for the cosine function.

My intention was to provide an opportunity for David to demonstrate his way of understanding what the graph of the cosine function represents.

David began by advising me to leave the angle and the circle centered at its vertex in the top right corner of the screen. He then explained that to construct the horizontal pink line represented by “1” in *Figure 48*, one would need to “create a line segment starting at the coordinate zero-zero with a length equal to the angle measure in radians.” I asked David how we might determine the measure of the angle, to which he replied, “you could measure the length, the arc length, and divide it by the length of the radius.” I performed this computation. Since David was somewhat unfamiliar with the functionality of *Geometer’s Sketchpad*, I carried out his suggestion of constructing the line extending along the horizontal axis whose length represented the measure of the angle in radians by plotting a point with an x -coordinate of the ratio of subtended arc length to radius length and a y -coordinate of zero. I then connected this point to the origin with a bold pink line.

David then advised me as to how to construct the red line extending vertically along the output axis, represented by “2” in *Figure 48*. His first suggestion was to construct a horizontal diameter of the circle in the top right corner of the screen. Then David proposed that I construct a perpendicular to this horizontal diameter that passed through the terminus of the subtended arc. After that, David requested that I measure the distance from the vertex of the angle to the intersection of the vertical line that passed through the terminal point and the horizontal diameter of the circle (see *Figure 49* for an illustration of David’s construction).

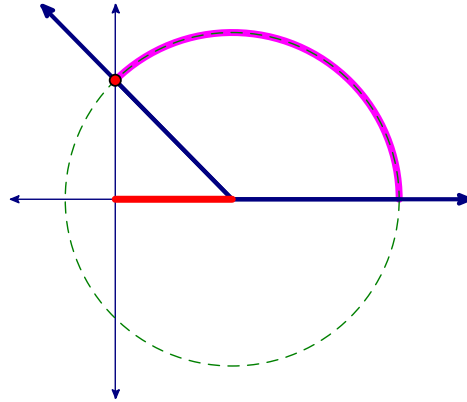


Figure 49. David's construction for the output of cosine.

David then asked me to construct a point with an x -coordinate of zero and a y -coordinate of the ratio of the horizontal distance that the terminal point is away from the vertical diameter of the circle to the radius length. David understood this point on the vertical axis as representing the number of radius lengths that the terminal point was away from the vertical diameter of the circle centered at the angle's vertex. David connected this point to the origin with a bold red line. He then suggested that I plot another point with an x -coordinate of subtended arc length divided by radius length and a y -coordinate of the horizontal distance that the terminal point is away from the vertical diameter of the circle divided by the radius length. David connected this point to the point he constructed on the input axis (see *Figure 50* for an illustration of the three points, and their coordinates, that David instructed me to create).

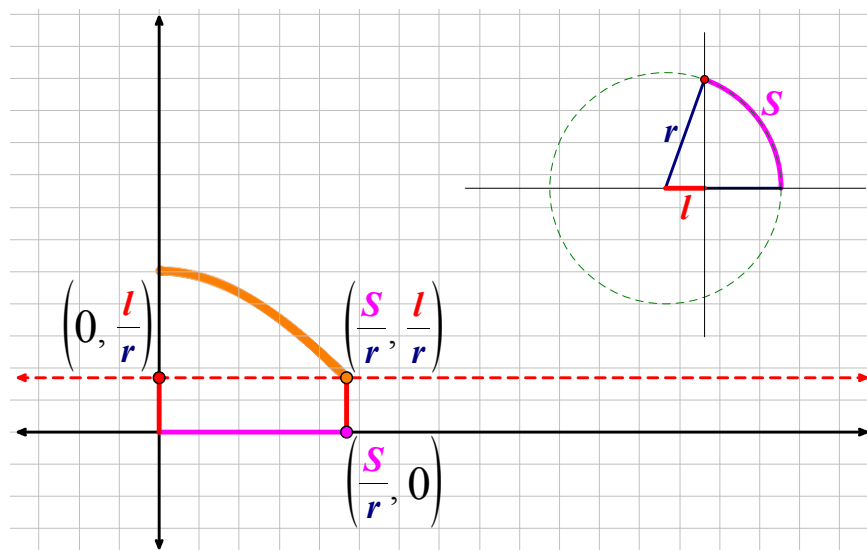


Figure 50. David's construction of the graph of cosine.

David traced the point he advised me to construct in the first quadrant and pressed the “Animate” button. Since David defined the y -coordinate of this point as the horizontal distance that the terminal point is away from the vertical diameter of the circle divided by the radius length, this y -coordinate did not assume negative values. David therefore generated the graph of $f(x) = |\cos(x)|$ (see Figure 51 for an image of what David saw on the screen after he pressed the “Animate” button). Although TBCI 6 ended before David had the opportunity to fix his construction, he explained that he should have defined the y -coordinate of the point he traced to be the number of radius lengths that the terminus of the subtended arc is *to the right* of the vertical diameter of the circle centered at the vertex of the angle.

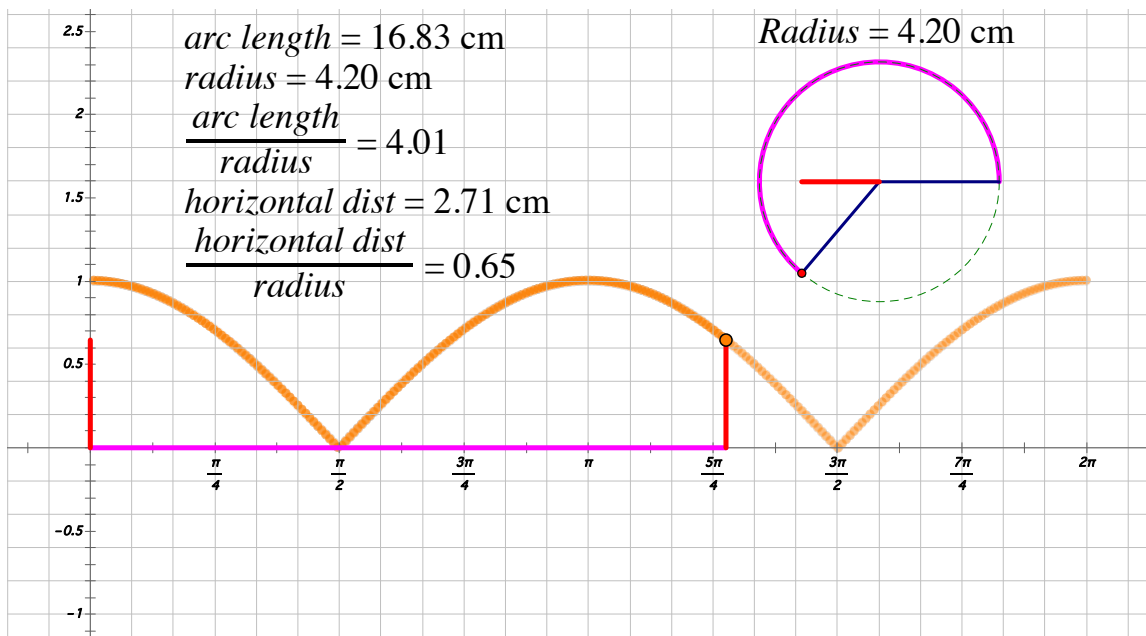


Figure 51. David's graph of $f(x) = |\cos(x)|$.

The process by which David attempted to construct the graph of cosine using the features of *Geometer's Sketchpad*, and his explanation for why his construction produced the graph of $f(x) = |\cos(x)|$ instead of $f(x) = \cos(x)$, demonstrates his understanding of the graph of cosine as a representation of the covariational relationship between angle measure—which he conceptualized as the length of the subtended arc in units of radius lengths—and the horizontal distance that the terminus of the subtended arc is to the right of the vertical diameter of the circle centered at the vertex of the angle.

Theme 2: Concavity of the sine and cosine functions. After David had completed the task in Table 48 during TBCI 6 by constructing graphs of the sine and cosine functions, I asked him to justify the curvature of his graphs (see Figure 46 for David's graph of the sine function). Specifically, I asked David to justify the concavity of his sine and cosine curves on the interval $[0, \pi/2]$. Figure 52 displays an image of the drawings to which David's statements in Excerpt 46 refer.

Excerpt 46

- 1 Michael: So just from zero to $\pi/2$, what about this (*points to the image on the screen*), what you're seeing here, suggests that the graph should be curved in the way that it is as the angle measure varies from zero to $\pi/2$ (*changes the angle measure from zero to $\pi/2$ on the screen*)?
- 2 David: (*David begins drawing on his paper*) So if I was to look at my change in heights (*pause while drawing*) you can see here that the change from here to here is fairly large (*motions vertically from lowest dotted line to middle dotted line on the drawing displayed on the right side of Figure 52*). But when we get up here (*motions vertically from middle dotted line to top dotted line he drew*) the change in y is very small, and these are equally spaced angle measures. ...
- 3 Michael: Okay. Then the cosine?
- 4 David: And so cosine same kind of thinking that was involved except now we're talking about (*begins drawing*) comparing, um, the vertical (*long pause while drawing*). So we can see that we have bigger changes in x (*i.e., horizontal changes*) here when we get closer to the angle of $\pi/2$ than we do here initially when we went from zero to my first little tick mark. (*Pointing to his drawing*) So this is a very small change in the x -values (*horizontal distance*). This is a large change in the x -values. The change in x is negative because

it's going from a longer length to a shorter length. So that's why it's a decreasing rate of change.

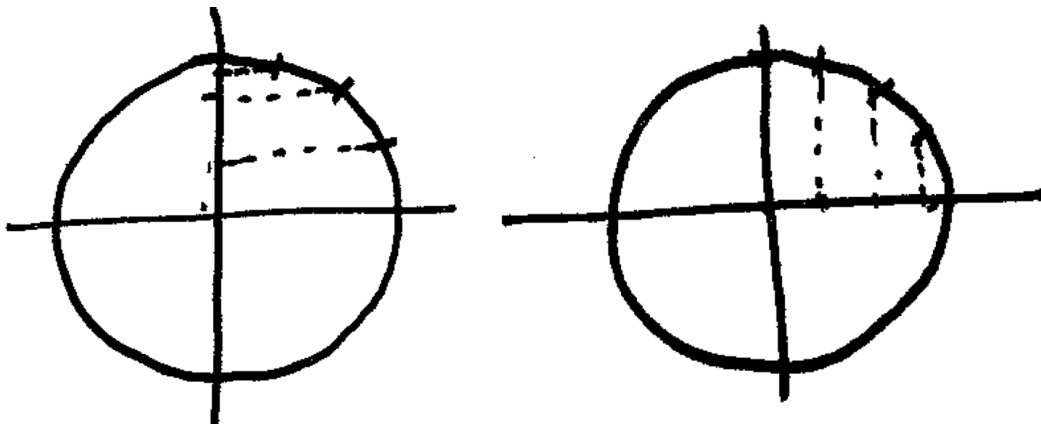


Figure 52. David's justification for the concavity of sine and cosine on $[0, \pi/2]$.

While justifying the concavity of the sine function on the interval $[0, \pi/2]$ in Line 2, David attended to how the output quantity of sine changed for successive uniform changes in the input quantity. In particular, David noticed that as the input quantity changed by equal amounts from zero to $\pi/2$ radians, the output quantity of sine changed by decreasing amounts (see David's drawing on the left side of *Figure 52*). In this context, David conceptualized the input quantity of sine as an arc length and the output quantity as a vertical distance that a terminal point is above the horizontal diameter of the circle. While David represented the output quantity of sine as a vertical distance, he referred to this vertical distance as a y -value, suggesting that he imagined the circle he drew as being centered at the origin of a coordinate plane and having a radius of one unit (i.e., passing through the point $(1, 0)$). David similarly justified the concavity of the graph of cosine on the interval $[0, \pi/2]$ by describing how the horizontal distance of a terminal point to the right of the vertical diameter of the circle centered at the angle's vertex changed for equal changes in the input quantity (Line 4). That David referred to this horizontal distance as

an x -value suggests that he imagined his circle as a unit circle centered at the origin of a coordinate plane. David explained that as the angle measure increased by equal amounts from zero to $\pi/2$ radians, the horizontal distance that the terminal point is to the right of the vertical diameter (or x -coordinate of the terminal point) decreased by larger amounts, resulting in a decreasing rate of change as the angle measure increased from zero to $\pi/2$ radians (see David's drawing on the right side of *Figure 52*).

I began TBCI 7 by presenting David with Part (a) of the task in Table 49. Excerpt 47 recounts the conversation that proceeded.

Excerpt 47

- 1 Michael: (*Plays the animation*) For what purpose do you think I designed this? What purpose could this possibly serve?
- 2 David: Well it shows the, um, by having the horizontal lines there, um, you've got equally spaced, um, angle measures. So you're counting by $\pi/6$. But we can see that the vertical change, um, is greater between some of those points than it is at others. So one of the things that it can be used to show is, you know, basically the rate of change of the vertical position with respect to the angle measure and how it's not constant.
- 3 Michael: Okay. So let me, let me pause this (*pauses the animation*). ... So if we just consider this, the angle measure varying from zero radians to $\pi/2$ radians. ... What are you interpreting here as the angle is varying from zero to $\pi/2$?

- 4 David: So, I'm interpreting that the, as the angle increases at, uh, a constant rate with respect to time, that the height, um, is starting by changing at a, or the sine output, is changing at the beginning by, uh, the difference is large.
- 5 Michael: How can you tell that?
- 6 David: Because you have more space between, um, it has gone up half of a radius length by the time you get to $\pi/6$ and then it's, you know, it hasn't gone up quite half of a radius length by the time you get to $\pi/3$ and then it finishes off that last, you know, part that it has left over by the time you get to $\pi/2$ So the dashed lines are showing me the change in the height because I can look at the differences, um, between, in the space between the, um, horizontal lines, uh, as we get to equally spaced angle measures. ... So it's basically almost stacking vertical changes on top of each other. So this was the change, um, from, when you look at the sine curve the change from the bottom to that first dashed line is the change in the height for the first $\pi/6$ radians. The next one is the change in height for the next $\pi/6$ radians and the last one is the change in the $\pi/6$, um, radians and so we see that that distance gets smaller and smaller, um, but we're still changing the angle by just $\pi/6$. Um, so, you know, you can see that it's, the change is decreasing, um, so you can kind of see how they're varying together.

David explained in Line 2 that the simulation illustrates that the sine function has a non-constant rate of change because the change in output values varied for equal changes of $\pi/6$ in the input values. In particular, in Line 6 David attended to how the vertical distance that the terminus of the subtended arc is above the horizontal diameter of the circle changed as the angle measure varied by successive intervals of $\pi/6$ radians from zero to $\pi/2$ radians. David noticed that the output of sine changed by about half a radius length when the angle measure varied from zero to $\pi/6$ radians, and that the change in the sine value was less than half a radius length as the angle measure changed from $\pi/6$ to $\pi/3$ radians. Additionally, David seemed to understand that the vertical distances between adjacent horizontal lines on the screen represented changes in the output quantity for successive changes of $\pi/6$ radian in the input quantity. David's comment, "So it's basically almost stacking vertical changes on top of each other" supports this claim. David's response in Line 6 also suggests that he pseudo-empirically abstracted the property that the sine values changed by decreasing amounts as the angle measure varied from zero to $\pi/2$ radians in $\pi/6$ radian intervals. David similarly explained that I designed the didactic object from Part (b) (see Table 49) to illustrate that for equal changes in angle measure, the change in the output quantity of cosine is non-constant.

That David justified the concavity of the sine and cosine functions on the interval $[0, \pi/2]$ by attending to how the outputs of these functions change for successive uniform changes in angle measure suggests that he engaged in at least Mental Action 3, but more likely Mental Action 4, in Carlson et al.'s (2002) covariation framework. I hesitate claiming with certainty that David engaged in Mental Action 4 because he did not explicitly verbalize how the average rate of change of the sine and cosine functions vary

for successive uniform changes in angle measure, although David did state in Line 4 of Excerpt 46 that the rate of change of cosine decreased as the input quantity varied by successive uniform amounts from zero to $\pi/2$ radians. I therefore consider it probable that David recognized that when the angle measure changes by equal amounts, one may determine how the average rate of change varies by attending only to the change in the output quantity.

Summary of TBCIs 6 and 7. During the sixth and seventh TBCIs, David revealed his understanding that the x - and y -coordinates of points on the sine and cosine graphs represent values of the input and output quantities of these respective functions. For example, the picture David drew to represent the coordinates of particular points on the graphs of sine and cosine revealed that he understood the x -coordinates of these points as a representation of the measure of an angle in radians—which he conceptualized as the length of the arc an angle subtends measured in units of radius lengths—and that the y -coordinates of the points on the sine (cosine) curve represent the number of radius lengths that the terminus of the subtended arc is above (to the right of) the horizontal (vertical) diameter of the circle centered at the vertex of the angle.

David also convincingly demonstrated his understanding that the graphs of sine and cosine represent the covariational relationship between input and output quantities. For instance, David recognized all five parts of the didactic object displayed in *Figure 48* as representing the variation in the input and/or output quantities of the sine function. Specifically, David claimed that Part 1 displayed the variation of the angle measure in radians; Part 2 showed the variation of the output of the vertical distance that the terminal point is above the horizontal diameter of the circle centered at the angle's vertex; Parts 3

and 4 represented the simultaneous variation of these input and output quantities; and Part 5 displayed the construction of a locus of points that represented all states of the simultaneous variation of the input and output quantities of the sine function (as the input varied from zero to 2π radians). In addition, David walked me through the construction of a similar didactic object for the cosine function, although he recognized that he made an error in how he defined the output of cosine by not allowing this quantity to assume negative values.

In summary, during TBCIs 6 and 7 David was able to: (1) sketch sine and cosine curves to represent the covariation in the input and output quantities of these respective functions, (2) draw pictures to represent the coordinates of points on the graphs of sine and cosine, (3) explain all five parts of the didactic object in *Figure 48* by referring to the (co)variation of the input and output quantities of sine, (4) label the axes of the graph in this didactic object with the names of the quantities that the values on these respective axes represent, and (5) explain how to use the features of *Geometer's Sketchpad* to construct a similar didactic object for cosine. These actions, in addition to the explanations that accompanied them, provide rather strong evidence that David conceptualized the graphs of sine and cosine as representations of the covariational relationship between the input and output quantities of these respective functions.

David also demonstrated his understanding that the size of the circle centered at the vertex of the angle does not have an effect on the graphical representation of the sine and cosine functions since the input and output quantities of these respective functions are measured in units of the radius. For example, after having viewed the terminal ray of an angle that had a circle with a radius of six centimeters centered at its vertex rotate

counterclockwise, David constructed graphs of the sine and cosine functions to represent the relationship between the input and output quantities of these respective functions. David then explained that his graphs would not change simply because I changed the radius of the circle centered at the vertex of the angle from six centimeters to three centimeters since the input and output quantities of the sine and cosine functions are measured in units of the radius. In essence, David recognized that the graphical representations of sine and cosine do not depend on the length of the radius of the circle centered at the vertex of the angle since the process by which he quantified the input and output quantities of the sine and cosine functions does not necessitate the measure of the radius in a standard linear unit being known.

Finally, during TBCIs 6 and 7 David justified the concavity of the sine and cosine functions by attending to how the output quantities of these respective functions change for successive uniform changes in the input quantity. It is noteworthy that David did not justify the concavity of the sine and cosine functions on particular intervals of their domain by explicitly describing how the average rate of change varied for equal changes in the input quantity—in which case he would have unquestionably engaged in Mental Action 4 of Carlson et al.'s (2002) covariation framework. However, David's language and actions suggested that he considered his characterization of how the outputs of sine and cosine changed for successive uniform changes in the input quantity as a proxy for describing how the average rate of change of these respective functions changes as the input quantity varies by successive equal amounts. I therefore consider it likely that David engaged in Mental Action 4 while justifying the concavity of the sine and cosine functions.

Period of Sine and Cosine Functions

As with the other main topics that were the focus of the series of TBCIs, I constructed a preliminary model of David's way of understanding the period of sine and cosine through my analysis of the initial clinical interview. During this interview, David described the period of sine and cosine functions as the interval of input values over which the output values vary through one full cycle. The eighth and final clinical interview focused exclusively on engaging David in tasks that exposed his way of understanding the period of sine and cosine functions.

Initial Clinical Interview

I concluded the initial clinical interview by asking David to explain the meaning of period. I probed David's response to this question in an effort to construct an initial model of David's way of understanding the period of sine and cosine.

Excerpt 48

- 1 Michael: What does period mean?
- 2 David: Um, period is the length (*pause*), um, (*pause*) it is the, um, change in the input that is required for the, um, pattern of the outputs to be repeated. So, um, the, if I change, if we're talking in terms of radians, so every time I change the inputs by either adding 2π or subtracting 2π to the input, I will end up at the corresponding part with the same output. So that way I get this repeating segments.
- 3 Michael: Is that true for both sine and cosine?
- 4 David: Uh, yes.
- 5 Michael: So what's the period of sine?

- 6 David: 2π .
- 7 Michael: And that's because?
- 8 David: That's how long it takes it to repeat its pattern.
- 9 Michael: So the 2π represents?
- 10 David: The 2π would be if we were measuring in, or if we were using radians for the angle measurements, um, and that's because we've said that 2π would be the, um, circumference of the circle, so we're talking about the, um, the angle requirement to go completely around the circle, the arc length to create a full circle, a circle with a radius of one.

David defined the period of sine and cosine in Line 2 as the change in the input required for the outputs of these respective functions to vary through one complete cycle of values. David claimed that the period of sine and cosine is 2π because the circumference of a circle with a radius of one unit is 2π (Line 10). David's remark in Line 10, "we're talking about the, um, the angle requirement to go completely around the circle, the arc length to create a full circle" suggests that he visualized a change in the input of sine and cosine as variation in the length of the arc an angle subtends. For David, the period of sine and cosine was 2π because as the length of the arc an angle subtends varies from zero to 2π units (assuming the radius of the subtended arc is one unit), the angle goes "completely around the circle."

Task-Based Clinical Interview 8

In the eighth and final clinical interview, David demonstrated his way of understanding period of sine and cosine as the length of the interval of input values over

which the argument of these respective functions vary by 2π radians. This way of understanding was based on David having conceptualized the argument of sine and cosine functions as representing the measure of an angle in radians.

Approximately nine minutes into TBCI 8, I generated the graph displayed in *Figure 53* and asked David to explain what he saw. David noticed that the output values complete one full cycle when the input values vary by π , and therefore conjectured that the input quantity was the measure of the angle in units of the diameter of the circle centered at the vertex of the angle. David then noticed the timer at the bottom of the screen and realized that the input quantity of the graph represented the number of seconds elapsed since the terminal ray of the angle began rotating counterclockwise. I asked David to write a function definition to represent the relationship between the input and output quantities displayed in the graph.

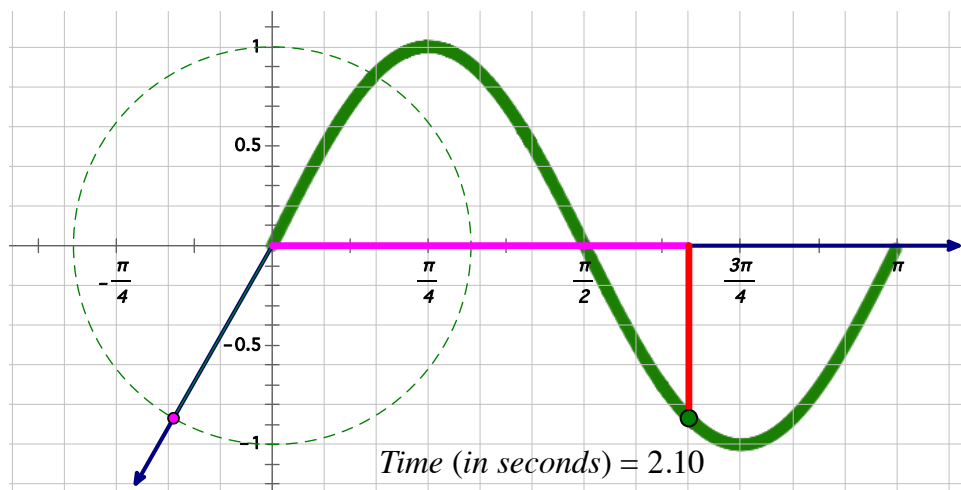


Figure 53. Graph of $f(t) = \sin(2t)$.

Excerpt 49

- 1 Michael: So if we call the input quantity, since it's time in seconds, we just call that t , and we call the output quantity $f(t)$, can you write a

function definition that would represent the same relationship between these quantities that we see in the graph here?

- 2 David: Yeah (*writes* " $f(t) = \sin(2t)$ ").
- 3 Michael: So why, why the $2t$ inside the parentheses for sine?
- 4 David: Um, because I want to make it so that it (*the sine curve*) goes through its complete cycle in half of the time that a sine curve normally does. So that means all of its, um, input values we want to, um, be able to double. So when I put in, when I plug in a value of π for t , so π seconds, then that will actually act, as far as sine is concerned, as 2π . So that would be the end of its cycle. So even though the input is usually half of what its cycle is, by doubling it, um, with the two in there, as far as sine is concerned it acts like 2π .
- 5 Michael: So say a little more about this. As far as sine is concerned it acts like 2π .
- 6 David: So the sine is, the sine is an operation that's acting on an angle measure on the inside. So when I plug in, um, you know, one, the way that sine looks at it is as one times two so it's the input into the sine itself is double what the input to the whole function was.
- 7 Michael: Okay. And so how does that make, I mean, as we know, normal sine curves have a period of 2π . How does multiplying t by two, in other words having $2t$ inside the sine function, get the graph to be (*David interrupts*)
- 8 David: It's compressed the period by half.

9 Michael: Why is that?

10 David: Again because it's doubling all of the input values that I use for t . So because I'm doubling it they (*the outputs of sine*) get through the complete cycle in twice as fast. Since it's twice as fast, it takes half as long to get through the whole cycle.

11 Michael: So what's the period of that function there?

12 David: π it appears.

David began his response in Line 4 by saying, "I want to make it so that it (*the sine curve*) goes through its complete cycle in half of the time that a sine curve normally does." Since the input quantity to the "parent" sine function is angle measure in radians, the sine curve does not go through its complete cycle in a specific amount of *time*, but rather when the angle measure varies by 2π radians. It is difficult to know whether David used "time" in this context in a temporal sense or if he used the term as a euphemism for "interval" or "value on the horizontal axis." In any case, David's response in Line 4 revealed his understanding that the period of the "normal" sine function is 2π , which he appeared to conceptualize in this context as the value of the input at which the sine curve completes one full cycle. Based on this understanding, David justified his function definition as follows: Since the sinusoidal curve completes its cycle in π seconds, the function definition must be $f(t) = \sin(2t)$ because when t is π , the "input" (i.e., argument) to the sine function is 2π , which is what it should be to make the sine curve go through its full cycle.

David's response in Line 4 of Excerpt 49 did not reveal the extent to which he conceptualized the argument of his function definition as representing the value of a

quantity. Accordingly, I asked David to explain what he meant by, “as far as sine is concerned it acts like 2π ” (Line 5). David’s remark in Line 6, “the sine is an operation that’s acting on an angle measure on the inside” suggests that he recognized the argument of sine as representing the measure of an angle. David therefore appeared in principal to make a distinction between the *input* and *argument* (although he did not use this language), claiming that sine “sees” its argument as an angle measure.

David’s responses in Lines 4 and 6 of Excerpt 49 demonstrate his understanding that since the period of the sinusoidal curve displayed in *Figure 53* is π seconds, the argument to sine in the function definition must be $2t$. Moreover, David understood that an input of π makes this argument equal to 2π , which is the angle measure (in radians) at which the outputs of sine complete one full cycle. This way of understanding period does not necessarily entail an image of the input quantity varying. According to the way of understanding David demonstrated in Lines 4 and 6, period is an *input value*, not the length of an *interval of input values*. In contrast to this way of understanding, David’s response in Line 10 reveals that he conceptualized period as the length of the interval of time over which the argument varies by 2π . In particular, David explained that the outputs of the function $f(t) = \sin(2t)$ vary through one full cycle of values “twice as fast” as they do for the function $g(t) = \sin(t)$. Therefore, the amount of time it takes for the outputs of the function f to vary through one complete cycle of values is half the amount of time it takes for the outputs of the function g to do so. Accordingly, David conceptualized the coefficient of t in his function definition as representing the *speed* at which the angle measure varied from zero to 2π radians.

While generating notes during the first phase of post-analysis, I wrote the paragraph in Table 50 in an effort to summarize my interpretation of the reasoning David demonstrated in Excerpt 49. As was standard procedure during this phase of analysis, I attempted to represent David's statements with greater clarity by interpreting them through my model of his way of understanding. For this reason, the paragraph constitutes a second-order model of David's reasoning. I include this summary of David's reasoning because, upon analyzing the excerpt a second time while writing the present section, I considered it a viable characterization of the way of understanding period David demonstrated in Excerpt 49.

Table 50

Second-order Model of David's Way of Understanding Period

Consider $f(x) = \sin(x)$. The input variable x represents angle measure in radians. If we imagine the terminal ray of an angle rotating in the counter-clockwise direction, there is a relationship between the angle measure and the number of seconds elapsed since the angle measure started varying. For simplicity, assume this relationship is $x(t) = t$. We know that the function $f(x) = \sin(x)$ has a period of 2π radians. The function $f(t) = \sin(2t)$ has a period that is half of 2π seconds because the angle measure varies twice as fast, so it only takes half of the time for the angle measure to vary from zero to 2π radians. Since $2t$ represents angle measure, t has to vary only by π seconds to make the angle measure vary by 2π radians.

After David and I finished discussing the function definition he wrote to represent the graph in *Figure 53*, I asked him to define a new function g to represent the relationship between the input and output quantities displayed in the graph in *Figure 54*. Upon viewing the graph being generated, David immediately noticed, "By the time it (*the graph*) got to 2π , which is the normal period of sine, it had only gotten half way through its cycle." David then wrote " $g(t) = \sin(\frac{1}{2} t)$ " and justified his function definition by explaining, "I know that t is the actual input but what's really being worked on to the sine

is being cut basically in half so it's going to take twice as long for sine to go through its whole cycle." As he did in Excerpt 49, David demonstrated a way of understanding period in this context as the interval of the input quantity over which the argument varies by 2π . David appeared to understand that since the input to the function g is multiplied by $\frac{1}{2}$, it will take twice as long for the argument of sine to vary from zero to 2π —the “normal period of sine”—which causes the outputs of sine to vary through one full cycle of values.

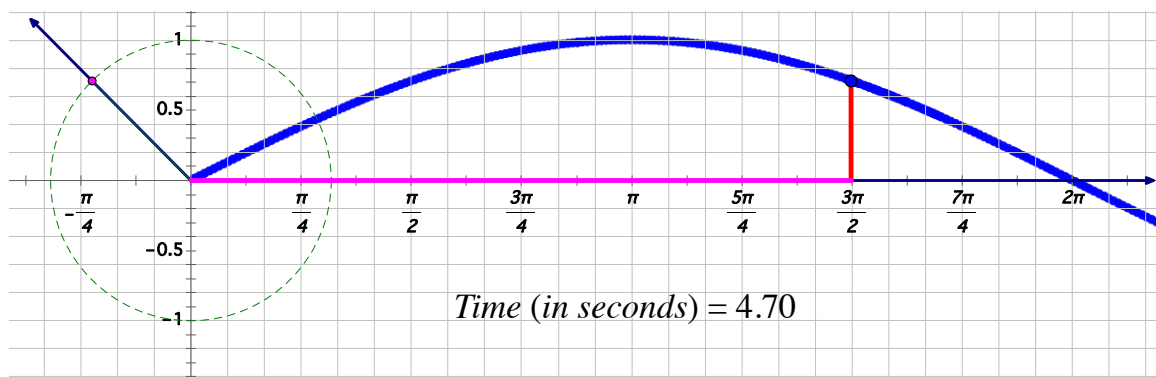


Figure 54. Graph of $g(t) = \sin(\frac{1}{2} t)$.

After David had correctly defined the function rules that respectively correspond to three different sinusoidal graphs, I asked him to determine the period of the function $j(x) = \sin(bx)$, to which he replied, “The reciprocal of b times 2π .” I then prompted David to explain his response.

Excerpt 50

- 1 Michael: Why is it that when we multiply the x in here by b , the period is $1/b$ as much as what it would be if b was just one?
- 2 David: Okay. And again the, the issue really can kind of be, um, also explained if you do a u -substitution. So let u equal b of x (*writes*

“Let $u = bx$ ”). So then, right now, um, if I create another function, um, so let’s do k of u equals the sine of u (*writes “ $k(u) = \sin(u)$ ”*). Okay. So just, we’re talking about the relationship between these. So this is our normal, um, relationship for sine where the input goes directly into the sine function (*points to the function definition, “ $k(u) = \sin(u)$ ”*). Um, what we’re doing up here (*points to the function definition “ $j(x) = \sin(bx)$ ”*) is we’re taking the input and then multiplying it by something. ... So whatever number I plug in for here (*points to the function k*), the input used up here (*points to the function j*) to get that same kind of output will need to be $1/b$ times that u . So, um, so if I, so if I want to have these have the same output, then the input that I use here (*points to the function j*), I’d have to multiply it by $1/b$ to get the input that I want here (*points to the function k*). And so that’s that relationship that we’re talking about. So normally I would want a 2π to end my period. So that means in order to end this one’s period (*points to the function j*) at that same location, to get that same kind of output, then we would have to multiply the 2π times $1/b$ to figure out what my input would need to be there (*points to the argument of the function k*).

While David’s statement Line 2 is somewhat circuitous, he appeared to reason essentially as follows:

- We have $j(x) = \sin(bx)$. Let's do a u -substitution and make $u = bx$. Now consider the function $k(u) = \sin(u)$.
- Let's suppose I plug the value a into both functions. Then I have $j(a) = \sin(ba)$ and $k(a) = \sin(a)$. So for the function j , sine would receive an input that is b times as large as the input that sine receives in the function k .
- Thus, in order to make these two functions produce the same output, the input to the function j will have to be $1/b$ times as large as the input to the function k . We see that $j(1/b \cdot a) = \sin(b(1/b \cdot a)) = \sin(a)$ and $k(a) = \sin(a)$. So $j(1/b \cdot a) = k(a)$.
- The normal period of the sine function is 2π , which means sine will have gone through one full cycle of output values when the argument of sine is 2π .
- Since, to get the same output, the input to j has to be $1/b$ times as large as the input to k , the function j will have gone through one full cycle of output values when the input is $(1/b) \cdot 2\pi$. So the period of $j(x) = \sin(bx)$ is $(1/b) \cdot 2\pi$, or $2\pi/b$.

Ignoring the fact that the u -substitution David proposed served no purpose in his subsequent reasoning (David could have just defined the function k as $k(x) = \sin(x)$), his explanation in Line 2 further demonstrates that he understood period as the value of the input variable that makes the argument of sine equal to 2π .

In Line 10 of Excerpt 49, David demonstrated a way of understanding period as the length of the interval of input values over which the argument varies by 2π . In contrast, David's responses in Lines 4 and 6 of Excerpt 49 and in Line 2 of Excerpt 50 suggest that he understood the period of a function as the value of the input quantity that makes the argument of sine or cosine equal to 2π . In an effort to determine if one of these ways of understanding took precedence in David's thinking, or if his understanding of

period was not as dichotomous as I had interpreted, I asked David to determine the period of the function $g(x) = \cos(3.7x)$ and to explain the meaning of the period in this context. David wrote, " $P = (1/3.7) \cdot 2\pi$ " and explained that this value represents "how much you need to change the input value in order for the outputs to go through an entire cycle." I then pressed David to articulate why the input x has to vary by $(1/3.7) \cdot 2\pi$ for $g(x)$ to vary through one full cycle of values.

Excerpt 51

- 1 Michael: So how do you know that the input has to vary by this number (*points to where David wrote " $P = (1/3.7) \cdot 2\pi$ "*) so that $g(x)$ goes through one full cycle?

- 2 David: ... If this had just been $\cos(x)$, the period would have been 2π . When I multiply, when I put in the 3.7 here, it's $3.7x$. What do I need x to be in order to get to 2π ? So what do I need to take the cosine of? So x would end up being $2\pi/3.7$. So I know when I plug in zero, I get zero here (*points to the argument " $3.7x$ "*). I know when if I plug in this number (*points to where he wrote " $P = (1/3.7) \cdot 2\pi$ "*), then I will get 2π . So cosine will have had a chance to go through its complete cycle. ... So if you think in terms of time, uh, these inputs, because they are being multiplied by 3.7, will happen 3.7 times faster. So it will go through its cycle 3.7 times faster. So since it does that, um, it will take $1/3.7$ times the amount of time to actually complete a normal cycle. And so since a normal cycle for, um, for cosine goes on the interval (*inaudible speech*

while writing the inequality, " $0 \leq 3.7x \leq 2\pi$ "). So this is our normal cycle (from zero to 2π). How do I make this whole thing, the input, go between zero and 2π because I know that's what cosine is going to act on? (David begins solving the inequality he wrote) When I divide, or when I multiply by the reciprocal, I get zero over here and I get $2\pi/3.7$ over here. So then you can see that this is the normal interval (zero to 2π). If I take my input and multiply it by 3.7, how is that going to affect my interval? And then just by solving I know that this is an interval (zero to $2\pi/3.7$) that will give me a complete cycle of this graph because it's acting on these values.

There are several noteworthy aspects of David's response in this excerpt. First, David explained that finding the period of $g(x) = \cos(3.7x)$ amounts to determining the value of x for which the argument $3.7x$ equals 2π . Second, David claimed that the argument of cosine is zero when x is zero and the argument of cosine is 2π when x is $(1/3.7) \cdot 2\pi$. From this observation David concluded, "cosine will have had a chance to go through its complete cycle" when x varies from zero to $(1/3.7) \cdot 2\pi$. This observation suggests that David likely interpreted the process of calculating the value of x for which the argument $3.7x$ is equal to 2π as equivalent to determining the upper bound of the interval over which the outputs of cosine vary through one full cycle of values. In other words, David appeared to think about variation in the input variable even when he demonstrated a way of understanding period as the value of the input that makes the argument of the function equal to 2π . Third, David explained that what the cosine takes

as its input will vary 3.7 times as fast since the input variable is multiplied by 3.7, which implies that it takes $1/3.7$ times as long for cosine to vary through one full cycle of values. Hence, the period of the function g is $1/3.7$ of 2π . This reasoning is consistent with the way David conceptualized the coefficient of the input variable as the *speed* of variation for the input quantity of cosine (or sine) in Line 2 of Excerpt 50. Fourth, David's rhetorical question, "How do I make this whole thing, the input, go between zero and 2π because I know that's what cosine is going to act on?" suggests that he understood that the argument ("input" in David's usage) has to vary from zero to 2π for the output of cosine to vary through a complete cycle of values. Finally, David solved the inequality $0 \leq 3.7x \leq 2\pi$ to determine the interval over which x must vary to make the argument of g vary by 2π . I provide in *Figure 55* the written work David produced while responding to my question in Excerpt 51.

$$g(x) = \cos(\overbrace{3.7x})$$

$$P = \frac{1}{3.7} \cdot 2\pi \quad \begin{array}{l} 3.7x = 2\pi \\ x = \frac{2\pi}{3.7} \end{array}$$

$$0 \leq x \leq 2\pi$$

$$0 \leq \underline{3.7x} \leq 2\pi$$

$$0 \leq x \leq \frac{2\pi}{3.7}$$

Figure 55. David's written work corresponding to Excerpt 51.

David's response in Excerpt 51 reveals a number of different ways of understanding period. In particular, David demonstrated the following meanings for the period of $f(x) = \cos(kx)$:

WoU 1: The period of f is the value of x for which the argument kx equals 2π .

WoU 2: The period of f is the length of the interval of x -values over which $f(x)$ varies through one full cycle of values.

WoU 3: If we think of the input x as representing values of time, the argument kx varies from zero to $2\pi k$ times as fast as does x . Therefore, the period of f is $1/k$ of the period of $g(x) = \cos(x)$, or $1/k$ of 2π .

While these three ways of understanding seem conceptually distinct, David appeared to understand the first and third as being related to the second. In particular, David recognized that because the argument of cosine is zero when x is zero, determining the length of the interval over which x must vary to make the argument *vary by* 2π (*WoU 2*) amounts to determining the value of x for which the argument *equals* 2π (*WoU 1*), thus making *WoU 1* a kind of restatement of *WoU 2*. The third way of understanding expresses a method for determining the period of a function of the form $f(x) = \cos(kx)$ based on conceptualizing the input quantity as time and period as the interval of time over which $f(x)$ varies through one full cycle of values (*WoU 2*). Hence, although not conceptually equivalent, David appeared to invoke *WoU 2* while employing *WoU 3*. Therefore, it appeared that David's primary meaning for period—or at least the one most foundational—was as the length of the interval of input values over which the output values vary through one full cycle (*WoU 2*).

I asked David to explain why one may determine the period of a sine or cosine function by dividing 2π by the coefficient of the input variable (assuming the argument is a linear function). David's response further demonstrated his understanding of period as the length of the interval of the input variable over which the argument of sine or cosine varies by 2π .

Excerpt 52

- 1 Michael: So, um, suppose a colleague suggests to you, ‘To find the period of a sine or cosine function all you have to do is divide 2π by the coefficient of the input variable’ but your colleague is unsure why this is true. How would you explain to your colleague why this is true?
- 2 David: Okay. So, um, let’s go back to our, um, we know that (*pause while writing*) normal sine we can see that one full, um, cycle from zero to 2π , of a normal sine curve (*writes “ $0 \leq \theta \leq 2\pi$ ”*).
- 3 Michael: So what does the theta represent?
- 4 David: And so theta would just be our angle measure, which would be our input into a sine curve. So this whole thing could be theta (*writes “ θ ” above the argument “ $at + b$ ”*). So if I do a substitution then zero is less than or equal to $at + b$ (*pause*) and we do our 2π there (*writes and solves the inequality “ $0 \leq a \cdot t + b \leq 2\pi$ ”*). ... A period is the difference between these two (*the upper and lower values of the solved inequality*) so we can see that one period up here (*points to the inequality “ $0 \leq \theta \leq 2\pi$ ”*) is 2π because it’s 2π minus zero, so that’s how we get 2π . So now we’re saying here’s our start ($-b/a$) and here’s our end ($2\pi/a - b/a$) so it would be, a period would be equal to $2\pi/a$ minus b/a , so that’s my endpoint, minus the opposite of b/a (*writes “ $P = (2\pi/a - b/a) - (-b/a)$ ”*), and

we would get the period is $2\pi/a$ because these would cancel (writes “ $P = 2\pi/a$ ”).

David explained that the argument to the function $h(t) = \sin(a \cdot t + b)$ represents an angle measure, and therefore the period of h is the length of the interval of t -values that satisfy the inequality $0 \leq a \cdot t + b \leq 2\pi$. David solved this inequality to determine the interval of input values over which the argument of h varies from zero to 2π (see David’s written work in *Figure 56*). David then subtracted the lower bound of this interval from the upper bound to derive the formula for the period of a sine (or cosine) function with a linear argument that has a constant rate of change of a with respect to the input variable. David’s solution relied upon his understanding of period as the length of the interval of input values over which the argument varies from zero to 2π , which is the interval over which the argument of sine or cosine must vary in order for the outputs of these respective functions to vary through one full cycle of values.

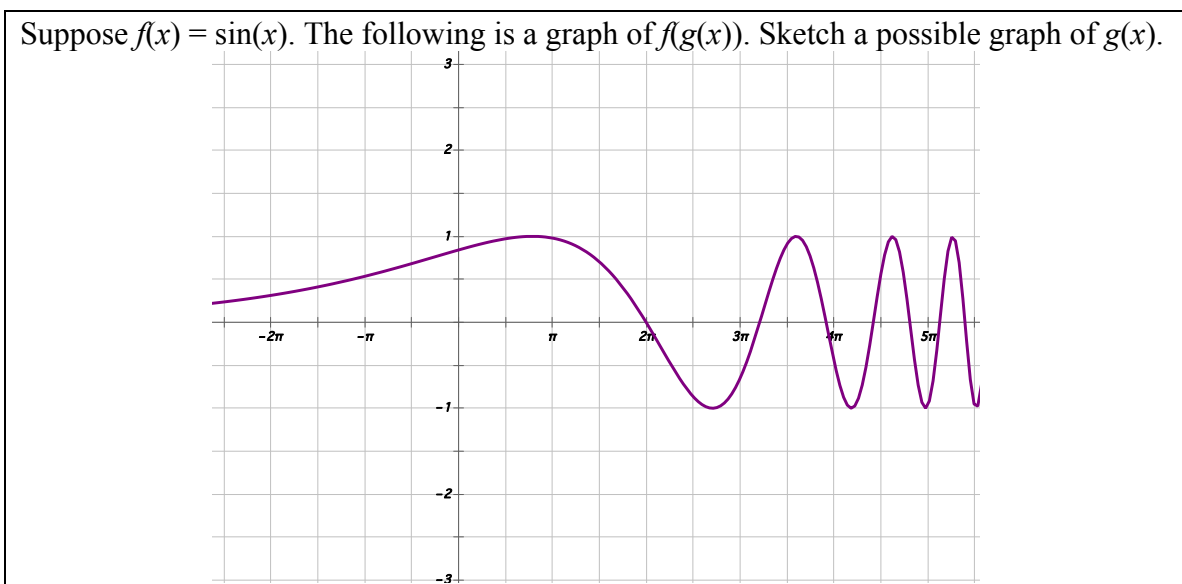
$$\begin{aligned}
 h(t) &= \sin(\overbrace{a \cdot t + b}^{\Theta}) \\
 P &= 2\pi - 0 = 2\pi \\
 \Theta &= 0 \leq \Theta \leq 2\pi \\
 0 &\leq at + b \leq 2\pi \\
 -b &\leq at \leq 2\pi - b \\
 \frac{-b}{a} &\leq t \leq \frac{2\pi - b}{a} \\
 P &= \left(\frac{2\pi}{a} - \frac{b}{a}\right) - \left(-\frac{b}{a}\right) \\
 P &= \frac{2\pi}{a}
 \end{aligned}$$

Figure 56. David's written work corresponding to Excerpt 52.

I concluded the eighth and final clinical interview by presenting David with the task in Table 51. Since the length of the interval of x -values over which $f(g(x))$ varies through one full cycle of values decreases as x increases, the length of the interval over which $g(x)$ varies by 2π must necessarily decrease as x increases, thereby making g an increasing convex function. Therefore, although the composite function $f \circ g$ is aperiodic⁸⁹, I designed this task to reveal the extent to which David had conceptualized period as the length of the interval of input values over which the argument varies by 2π .

Table 51

Final Period Task



I read the task in Table 51 aloud and David immediately began constructing the graph displayed in Figure 57.

⁸⁹ There does not exist a real number P such that $f(g(x)) = f(g(x + P))$ for all x in the domain of $f \circ g$.

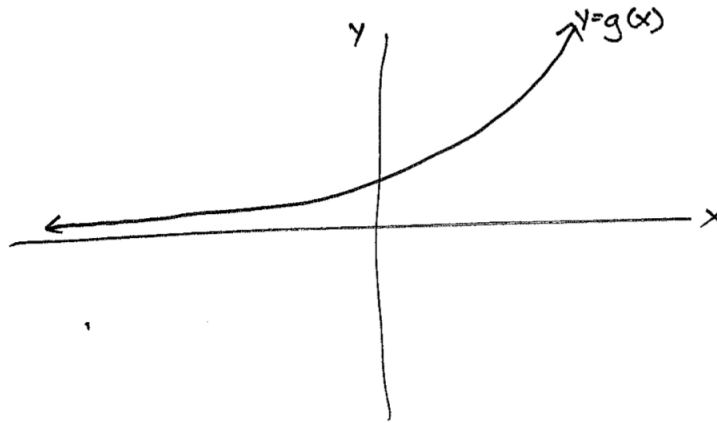


Figure 57. David's solution to the task in Table 51.

Excerpt 53

- 1 David: (*David begins constructing his graph*) $g(x)$ is exponential.
- 2 Michael: Alright so how did you, uh, know to draw $g(x)$ like that?
- 3 David: Well it looks like the, the inputs are increasing at an increasing rate because the period is shrinking. So I know it's not a linear equation, um, because a linear equation, um, just causes the sine wave to be shifted. So I'm pretty sure it's not that. Um, I needed a function that as I went to the right would increase faster and faster and faster, thus shortening the periods, but to the left it looks like it was, um, decreasing and so that's why I went with an exponential equation.
- 4 Michael: How does, how does the fact that your function that you've drawn, as it increases as you go to the right, why does that cause the period here to, uh, to get smaller and smaller?

5 David: Well because it's increasing, the outputs of g are increasing exponentially. ... The outputs for this (*the function g*) are going to become the inputs for sine, uh, so I needed those to grow quicker and quicker and quicker and quicker so that the inputs (*David is using "inputs" here to mean "argument"*) of sine would grow quicker and quicker and quicker, thus shortening its period. On the other side of the graph it appears that it is, um, that growth is slowing. So I needed a function that didn't come back up so that's why I didn't choose like a quadratic, um, was because, you know, a quadratic, um, would make it do that.

David began his response in Line 3 by claiming that the values of $g(x)$ are "increasing at an increasing rate because the period is shrinking." This statement suggests that David recognized that because the period⁹⁰ of $f \circ g$ decreases as x increases, what sine takes as its input (i.e., the argument of sine) cycles through intervals of 2π over shorter intervals of x as x increases. David concluded from this observation, "I needed a function that as I went to the right would increase faster and faster and faster, thus shortening the periods." David recognized that the sine takes as its input the outputs of the function g , and reasoned that since the period of $f \circ g$ decreased as x increased, $g(x)$ must vary through intervals of 2π over shorter intervals of x as the input of $f \circ g$ increased (Line 5). For this reason, David sketched the graph of an exponential growth

⁹⁰ I recognize that an aperiodic function does not have a period to speak of. However, my use of "period" in this context is consistent with my interpretation of David's use of "period" in the context of discussing his solution to the task in Table 51: as the length of a *specific* interval over which the outputs $f(g(x))$ vary through one full cycle of values. Using the term "period" in this way makes it reasonable to speak of the period of $f \circ g$ decreasing as x increases.

function. The reasoning David demonstrated while responding to the task in Table 51 further demonstrates that he understood period as the length of the interval of input values over which the argument of sine varies by 2π .

Summary of TBCI 8. On several occasions throughout TBCI 8, David described the period of sine and cosine functions as the length of the interval of input values over which the outputs vary through one full cycle of values. As a result of having conceptualized the argument of sine and cosine as representing the measure of an angle in radians, David recognized that the outputs of sine and cosine vary through one full cycle of values when the argument of these respective functions vary by 2π . Therefore, David understood period of sine and cosine functions as the length of the interval of input values over which the argument of these respective functions vary by 2π . For example, David reasoned as follows when confronted with the task of determining the period of a generic function of the form $f(x) = \sin(bx)$: Since the input x is being multiplied by b , x only has to vary by $(1/b) \cdot 2\pi$ to make the argument vary by 2π . So the period of f is $(1/b) \cdot 2\pi$, or $2\pi/b$.

David occasionally conceptualized the input variable of sine and cosine as representing time (in seconds) since the terminal ray of an angle in standard position with a measure of zero radians began rotating counterclockwise. On such occasions, and when the argument of sine or cosine was a linear function of the input variable, David conceptualized the coefficient of the input variable as representing the *speed* at which the measure of an angle varies. For instance, David understood that the argument (i.e., angle measure) of the function $f(x) = \sin(bx)$ varies from zero to 2π radians b times as fast as does the argument of the function $g(x) = \sin(x)$ because multiplying the input variable by

b speeds up the variation in the angle measure. Therefore, $f(x)$ will vary through one full cycle of values in $1/b^{\text{th}}$ of the amount of time it takes for $g(x)$ to vary through one full cycle of values.⁹¹ So the period of $f(x) = \sin(bx)$ is $1/b$ of 2π , or $2\pi/b$.

Conclusion

In this chapter, I have presented my model of David's knowledge of to angle measure, the outputs of sine and cosine, the graphical representation of sine and cosine, and the period of sine and cosine. I remind the reader that this chapter is not a characterization of the mathematical knowledge that resides in David's mind, but is instead a description of plausible ways of knowing that explain my interpretation of David's language and actions. The following chapter presents my model of the mathematical knowledge David enacted in the context of classroom practice, and compares and contrasts this model with the one presented in the current chapter.

⁹¹ This statement assumes that David conceptualized the input variable x as representing the number of seconds elapsed since the terminal ray of an angle in standard position with a measure of zero radians began rotating counterclockwise.

CHAPTER 7

DAVID'S ENACTED MATHEMATICAL KNOWLEDGE

If a teacher's conceptual structures comprise disconnected facts and procedures, their instruction is likely to focus on disconnected facts and procedures. In contrast, if a teacher's conceptual structures comprise a web of mathematical ideas and compatible ways of thinking, it will at least be possible that she attempts to develop these same conceptual structures in her students.

(Thompson, Carlson, & Silverman, 2007, p. 416-17)

In the previous chapter, I presented my model of David's knowledge of angle measure, the outputs and graphical representation of sine and cosine, and the period of sine and cosine. The purpose of the current chapter is to present my model of the mathematical knowledge David enacted in the context of lesson planning and classroom teaching, and to compare and contrast this model with the one I presented in Chapter 6. In doing so, I identify consistencies and incongruities between David's mathematical knowledge and the mathematical knowledge he brought to bear in the context of teaching.

I have organized this chapter into three main sections that correspond to the three general topics that were the focus of the series of TBCIs: (1) angle measure, (2) outputs and graphical representation of sine and cosine, and (3) period of sine and cosine. Within each of these main sections, I characterize the mathematics David intended to convey in his teaching—as evidenced by his responses during pre-lesson interviews—and present themes of David's enacted mathematical knowledge that emerged from my analysis of the videos of his classroom teaching. My discussion of these themes constitutes my model of the mathematical knowledge David utilized in the context of classroom

practice.⁹² I then compare and contrast my model of David’s enacted mathematical knowledge with the corresponding characterization of David’s mathematics from Chapter 6.⁹³ I conclude this chapter by advancing an evidence-based hypothesis that seeks to explain the incongruities, inconsistencies, and contradictions I observed between—and within—these two instantiations of David’s mathematical knowledge.

Angle Measure

The following themes emerged from my analysis of David’s teaching of angle measure:

- (1) To measure or not to measure ... a length?
- (2) Radians measure length. Degrees, quips, and marks measure “space.”
- (3) Radians are advantageous because radius lengths are “based on the circle.”
- (4) The size of the circle is immaterial to the measure of the angle.

In what follows, I describe the mathematical meanings David intended to convey in his teaching relative to each theme and provide evidence from David’s instruction that illustrate each theme.

Theme 1: To Measure or Not to Measure ... a Length?

As I discussed in my conceptual analysis in Chapter 5, assigning numerical values to the “openness” of an angle requires that one has identified a quantity to measure and a unit with which to measure it. I explained the necessity of conceptualizing angle measure

⁹² As I explained in Chapter 4, discrepancies that exist between the mathematical meanings David intended to convey and the meanings he actually supported in the context of classroom teaching constitute *moments of instructional deviation*, and were foundational to the content of the Phase III semi-structured clinical interviews, the results of which I present in Chapter 8.

⁹³ As I indicated in Chapter 4, occasions in which David’s enacted mathematical knowledge differed from the mathematics he demonstrated during the series of TBCIs amounted to *moments of mathematical concession*, and were among the content David and I discussed in the Phase III semi-structured clinical interviews.

as the length of an arc an angle subtends in units proportional to the circumference of the circle that contains the subtended arc.

David's instruction of angle measure was often incoherent and occasionally contradictory with regard to the quantity one measures when assigning numerical values to the openness of an angle. On some occasions David supported students in conceptualizing angle measure as the length of an arc the angle subtends, while on other occasions he explained that measuring an angle involves determining the fraction of the circle's circumference subtended by the angle. These meanings of angle measure appeared to be distinct since David never supported students in conceptualizing the circumference of the circle centered at the angle's vertex as a unit of measure for the length of the subtended arc.

Pre-Lesson Interview 1. During the first pre-lesson interview, David conveyed his intention to support students' understanding of angle measure as a "proportion of the circle" or as "part of the whole circle." Consider, for instance, David's response in Excerpt 54.

Excerpt 54

- 1 Michael: What do you want students to understand about angle measure after this lesson?
- 2 David: Um, well today I want them to start to understand that it's hard to measure openness. So the way we can kind of get around that abstract idea is by starting to think in terms of ratios and that the units of measure doesn't really make a difference when we measure arc lengths and circumference and stuff like that because

of the reason that they reduce, they cancel, and that we're just looking at the proportion of the circle.

David's remark suggests his intention to support students in actualizing the process of quantifying the openness of an angle by allowing them to conceptualize angle measure as a "proportion of the circle." David's claim that the unit with which one measures the subtended arc length and the circumference of the circle is inconsequential indicates that he did not plan on supporting students in seeing the circumference as a unit of measure for the length of the subtended arc. Instead, David wanted his students to realize that both the subtended arc length and the circumference are themselves measured in a particular unit, and that when one computes the ratio of these measures the units cancel, resulting in a value that represents the fraction of the circle subtended by the angle.

Throughout the remainder of the first pre-lesson interview, David continued to communicate his intention to support students' understanding of angle measure as the ratio of subtended arc length to circumference. David explained that when students look at an angle, he wants them to see "it really is a part of the whole circle." Additionally, David emphasized that he wanted students to understand angle measure as "a proportion of the total circumference. ... So we're comparing part of the circle to the whole circle."⁹⁴ Moreover, David explained that his lesson would be a success if students developed the understanding of angle measure "as the ratio of the part to the whole" and

⁹⁴ It is worth noting that Module 8 of the *Pathways* Algebra II curriculum begins by supporting students in quantifying the "openness" of an angle by determining the fraction of the circle's circumference the angle subtends. After radians are introduced as a unit of angle measure, the *Pathways* curriculum then emphasizes angle measure as the length of the subtended arc measured in units of the radius of the subtended arc.

if “they start talking about it as a proportion of the circle when they’re talking about angle measures.”

Lesson 1. While David’s instruction during the first lesson of Module 8 reflected his objective of supporting students in understanding angle measure as a fraction of the circle subtended by the angle, he also emphasized angle measure as the length of the arc an angle subtends. For example, David began the first lesson by asking a student to draw two angles on the whiteboard (see *Figure 58*). David then explained and illustrated that the angle on the left is larger than the angle on the right because, if one were to construct two circles of equal radii respectively centered at the vertex of each angle, the angle on the left would subtend an arc that is longer than the arc subtended by the angle on the right (see *Figure 59* for David’s illustration). Immediately following this explanation, David asked the question in Line 1 of Excerpt 55.

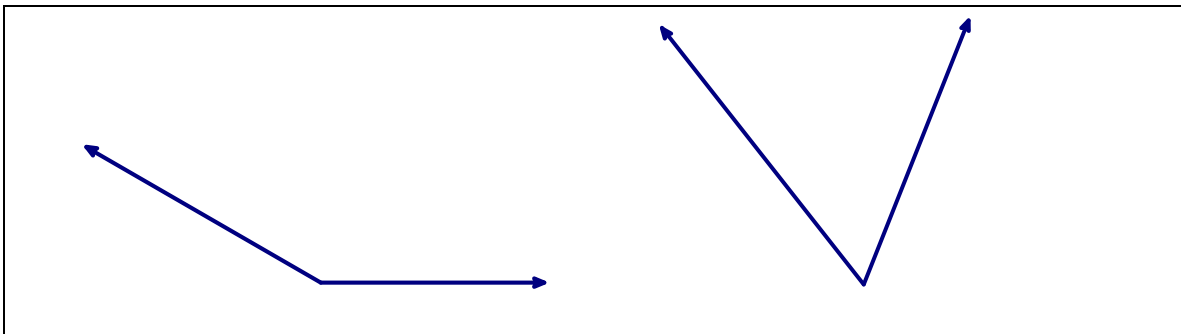


Figure 58. Two angles of unequal “openness.”

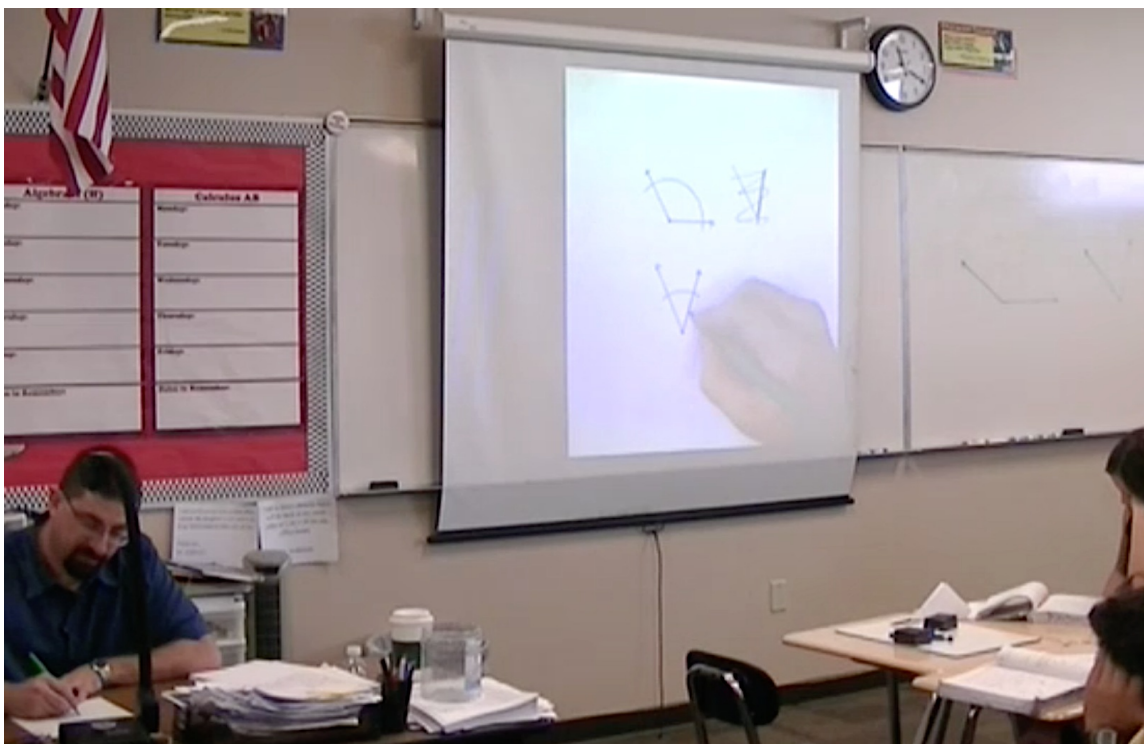


Figure 59. David’s comparison of the “size” of two angles.

Excerpt 55

- 1 David: When we measure an angle what are we really measuring? I mean it’s not like we’re measuring a length, right? How would we describe the thing that I’m measuring when I just look at these two angles? ...
- 2 Student: The openness of the angle.
- 3 David: Yeah. Which is weird. How do you measure openness? ... I’m not measuring length. ... We have to think about what we are actually measuring.

While David previously compared the openness of two angles by attending to the respective arc lengths these angles subtend, he claimed in Lines 1 and 3 of Excerpt 55

that quantifying the openness of an angle does *not* involve measuring a length. Following the dialogue in Excerpt 55, David explained that two angles have the same measure if “the length of the [subtended] arc is the same, as long as I made the circle have the same radius and it was centered at the vertex.” David therefore supported contradictory meanings of angle measure during the first lesson; he pronounced that measuring an angle is not a process of measuring a length, and then proceeded to compare the openness of two angles, as well as define what it means for two angles to have the same measure, by attending to the arc lengths the two angles respectively subtend. In other words, when speaking of angle measure David did not consistently reference the same quantity being measured.⁹⁵

A few minutes after David’s remark in Line 3 of Excerpt 55, he projected an image like the one displayed in *Figure 60* on the board and asked the question in Line 1 of Excerpt 56.

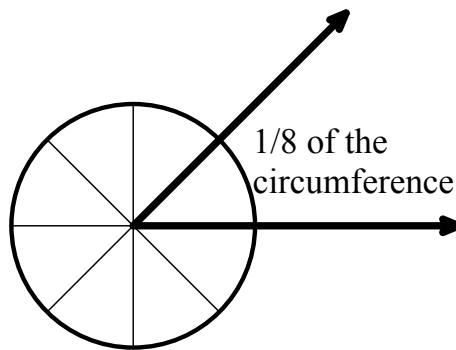


Figure 60. Angle measure as a fraction of the circle’s circumference.

⁹⁵ As I explained in my conceptual analysis in Chapter 5, understanding angle measure quantitatively involves conceptualizing a specific attribute to measure as well as identifying an appropriate unit with which to measure it. When measuring an angle, the attribute one measures is the length of the arc the angle subtends. This subtended arc length must be measured in units that are proportionally related to the circumference of the circle that contains the subtended arc so as to make the size of this circle inconsequential to the measure of the angle. It is important to note that this condition on the unit of measure does *not* change the quantity being measured: *subtended arc length*.

Excerpt 56

- 1 David: So the angle subtends $1/8^{\text{th}}$ of the circumference of the circle.
(Pause) Now do units matter here? ... Why do units not matter here?
- 2 Student: 'Cause you're using a proportion.
- 3 David: Why does that matter? ...
- 4 Student: Because even though you're making the radius larger you're also making the whole circle larger.
- 5 David: So what happens when you do your proportion? Think in science class. *(Long pause)* 'Cause we're comparing it to our circumference, right? We're comparing arc length to circumference? What would happen to the units then? *(Long pause)* So lets just say for the sake of argument $1/8^{\text{th}}$ could be a circumference of, uh, a circumference of 16, that would mean that the arc length would be two, if it's an eighth. So two inches divided by 16 inches is?
- 6 Student: One-eighth.
- 7 David: One-eighth. What are the units now? *(Long pause)* What happens when you put—and again think in terms of science class—what happens when you put two inches divided by 16 inches *(writes "2in/16in")*, your science teacher would say that's $1/8^{\text{th}}$. What are the units?
- 8 Student: It doesn't matter.

- 9 David: It does matter. What are the units?
- 10 Student: Inches.
- 11 David: Inches divided by inches give you inches?
- 12 Student: No.
- 13 David: What does it give you?
- 14 Student: One-eighth.
- 15 David: What are the units?
- 16 Student: It doesn't have units.
- 17 David: It doesn't have units? Why not?
- 18 Student: Because the inches cancel.
- 19 David: 'Cause inches cancel inches! ... 'Cause I'm not just measuring arc length. What am I measuring? I'm measuring arc length and comparing it to what?
- 20 Student: Circumference.
- 21 David: Circumference! How am I comparing them?
- 22 Student: By length.
- 23 David: By length? What operation is going on here? Am I subtracting the circumference? (*Pause*) It's division! We're creating a ratio! Then do the units matter?
- 24 Student: No. ...
- 25 David: What happens when we do the ratio? The units stop mattering, right? Because the units end up canceling. We're interested in the

ratio. We're not interested in the units from the ratio because the units are going to reduce.

After acknowledging in Line 1 that the angle in *Figure 60* “subtends $1/8^{\text{th}}$ of the circumference of the circle,” David exclaimed that the measure of the angle is a value without units. In particular, David explained that if one measured the subtended arc length and circumference in inches, the ratio of these quantities is unit-less because the inches “cancel” as a result of the division. Moreover, David’s statement, “I’m not just measuring arc length. What am I measuring? I’m measuring arc length and comparing it to what? ... Circumference!” did not support students in seeing the ratio of subtended arc length to circumference as the length of the subtended arc measured *in units of the circumference*.⁹⁶ Generally speaking, David’s statements and questions in Excerpt 56 did not provide students with an opportunity to interpret the ratio of subtended arc length to circumference as a quantitative operation, but rather as an arithmetic operation; that is, he did not communicate the division of these quantities as the numerator measured in units of the denominator but simply as the ratio of two lengths. Such an emphasis is necessary if one is to support students in conceptualizing angle measure quantitatively (i.e., as a measure of *something* in some unit). Therefore, while David’s instruction during the first lesson overtly emphasized angle measure as a fraction of the circle’s circumference subtended by the angle—and implicitly conveyed angle measure as the length of the subtended arc—David did not encourage students to see the former meaning as an

⁹⁶ Although the *Pathways* curriculum does not claim that the measure of the angle displayed in Figure 60 is without units, the curriculum does state, “The angle subtends $1/8$ of the circumference of the circle” (Carlson, O’Bryan, & Joyner, 2013, p. 462). Additionally, Investigation 1 of Module 8, which was the context of Excerpt 56, explains that measuring of an angle involves determining “the portion of the circle’s circumference that falls between the angle’s rays” (ibid.). Therefore, like David, Investigation 1 of Module 8 did not support students in conceptualizing the fraction of a circle’s circumference an angle subtends as a measure of the length of the subtended arc in units of the circumference.

application of the latter by not supporting them in conceptualizing the ratio of subtended arc length to circumference as a quantity that represents the length of the subtended arc measured in units of the circumference. In fact, David suggested that these meanings were incompatible by continually asserting that the process of measuring an angle is not one of measuring length.

Pre-Lesson Interview 2. Throughout the second pre-lesson interview, during which David and I discussed his plan for Lessons 2 and 3, he provided conflicting descriptions of how he wanted his students to conceptualize what it means to measure an angle in radians. Consider, for example, David's statements in Excerpt 57.

Excerpt 57

- 1 Michael: The topic of Investigation 2 is, as you mentioned, angle measure in radians. So how do you want students to understand angle measure in radians?
- 2 David: I want them to understand that just like with degrees we're comparing the arc length to the circumference as a whole. ...
- 3 Michael: At the end of Investigation 2, when a student sees some arbitrary angle and you ask them to measure the angle in radians, how do you want them to conceptualize how to do that, or what that means?
- 4 David: I want them to conceptualize how many radius lengths would it take to get that angle? You know, I want them to conceptualize that, um, thinking of the whole circle as just a little bit more than six, um, radians and half of the circle as a little bit more than three,

you know, so they can kind of get those benchmarks of where they are supposed to be with radian measurements.

David's remark in Line 2 revealed his intention to support students' understanding of angle measure in radians as a comparison of subtended arc length and circumference. David's response in Line 4, in contrast, suggests that he wanted his students to conceptualize angle measure as the number of radius lengths contained in the subtended arc. Excerpt 58 similarly reveals the conflicting instructional objectives David communicated during the second pre-lesson interview.

Excerpt 58

- 1 Michael: So at the end of Investigation 2, how will you know if your instruction on the investigation was successful or not?
- 2 David: I'll know if it's successful if I can ask them a question like, 'Can you draw me an angle of one radian or two radians?' and if they have an idea of what that means, and not by going, 'Oh, let's convert it to degrees.' ... [If I give them] three radians and they go, 'Oh well that would just be three radius lengths.' If I said something as far as, you know, π , 2π , 3π , and I actually used the π measurements then hopefully they will be okay saying something like, 'Well 2π is the circumference so a measurement of π radians would be half of the circumference of a unit circle.'

David's response in Line 2 demonstrates that he wanted his students to think about an angle with a measure of three radians as subtending an arc that is "three radius lengths." This way of understanding angle measure makes explicit a quantity being

measured (the length of a subtended arc) and a unit with which to measure it (a radius length). David did not, however, reveal the expectation that students conceptualize an angle with a measure of π radians as subtending an arc whose length is π times as large as the radius of the circle centered at the angle's vertex. Instead, David explained that his lesson would be a success if students develop the understanding that an angle with a measure of π radians subtends an arc that is "half of the circumference of a unit circle."

In contrast to David's statement in Line 2 of Excerpt 58—in which he explained that he would consider his instruction a success if his students conceptualized an angle with a measure of three radians as subtending an arc that contains three radius lengths—David also revealed, in the same pre-lesson interview, his intention to support students' understanding of an angle with a measure of three radians as subtending $3/(2\pi)^{\text{th}}$ of the circumference of the circle centered at the vertex of the angle. In particular, David communicated his desire to support students in conceptualizing an angle measure of three radians as, "Three divided by 2π ... so it's just a little under half." David further explained that if students can correctly draw an angle with a measure of three radians by reasoning in this way, "then they understand that angle measure is a portion of the circle. ... So it's really showing that they know, they understand angle measurement and what it is to measure an angle as a part of the circle." These remarks reveal the inconsistency with which David defined the understanding of angle measure he wanted his students to construct.

In an effort to prompt David to reveal with greater clarity and precision the way of understanding angle measure he aspired to convey, I presented him with two hypothetical student responses to the task, "Construct an angle with a measure of one

radian.” I explained that the first student constructed her angle by drawing a circle with an initial ray extending from its center. She measured the radius of the circle with a string and wrapped the string along the circle with one end extending from the intersection of the circle and the initial ray. Finally, she drew the terminal ray so that it intersected the circle at the other end of the string, which created an angle that subtends an arc equal in length to the radius of the circle. I explained that, as with the first student, the second student drew an initial ray with a circle centered at its endpoint. She then realized that an angle with a measure of one radian subtends $1/6^{\text{th}}$ of the circumference of the circle, since one radian is approximately $1/6^{\text{th}}$ of 2π radians, and proceeded to draw her terminal ray to satisfy this criterion. I asked David to identify which of these two hypothetical responses he preferred and to explain why.

Although David admitted that he would be content with the second student’s response—drawing the terminal ray so that the angle subtends $1/6^{\text{th}}$ of the circumference of the circle—he expressed preference for the first student’s method of constructing the angle so that the subtended arc length is equal to one radius length. David claimed that the first student’s method would likely produce an angle that was closer in measure to one radian than would the second student’s method. In other words, David’s partiality toward the first student’s approach derived from his consideration of accuracy rather than from a justification of the conceptual affordances of the first student’s way of understanding.⁹⁷ This suggests that David might not have been aware of the conceptual utility of the first student’s reasoning, nor of the conceptual limitations of the second

⁹⁷ Such conceptual affordances might include that the first student appeared to conceptualize an attribute of a geometric object (subtended arc length) that has a measure of one in a particular unit (radius lengths). In other words, the first student appeared to conceptualize an angle with a measure of one radian quantitatively.

student's reasoning, which would explain the inconsistency and occasional contradiction in the meanings of angle measure David professed the intention to support in his teaching. I'll return to this point later.

Lessons 2 and 3. On several occasions during the second pre-lesson interview, David explained that he wanted to support students' understanding of angle measure as either the fraction of the circumference of the circle subtended by the angle, or as the length of the arc the angle subtends measured in units of the radius. However, David's instruction during Lessons 2 and 3 (the lessons that were the topic of discussion during the second pre-lesson interview) almost exclusively emphasized the former way of understanding.⁹⁸

Following Investigation 1 of Module 8 in the *Pathways* curriculum, David introduced two non-standard units for measuring angles: *quips* and *marks* (Carlson, O'Bryan, & Joyner, 2013, p. 463-64).⁹⁹ After having introduced these non-standard units of angle measure, David made the remark in Excerpt 59 while contrasting the characteristics of units of angle measure with standard linear units like inches or centimeters.

Excerpt 59

- 1 David: How are degrees, *quips*, or *marks* different from length units like inches and centimeters? (*Long pause*) ... What am I doing when I measure in degrees, measure *quips*, measure *marks*? We're

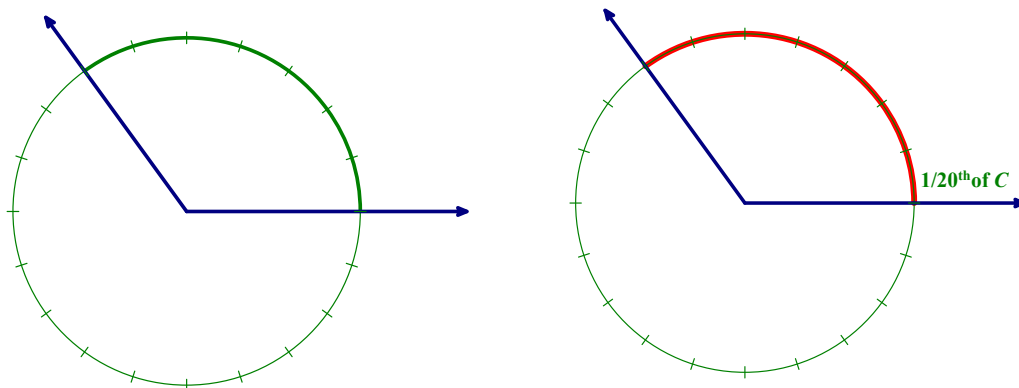
⁹⁸ It is important to acknowledge that on some occasions this emphasis was consistent with meanings promoted in the *Pathways* curriculum, while on other occasions it was not.

⁹⁹ One *quip* represents the measure of an angle that subtends an arc that has a length of $1/8$ measured in units of the circumference of the circle centered at the vertex of the angle. Similarly, an angle with a measure of one *mark* subtends an arc that has a length of $1/20$ measured in units of the circumference of the circle centered at its vertex.

measuring a fraction of the circle, right? (*Long pause*) So we're comparing the part of the circle to the whole circle. So it's a ratio. Where when I say, 'Measure something in inches' what are we doing? We're just saying it's 15 inches and we're just measuring one length. We're not comparing it to anything else. Okay? So degrees are not units for measuring lengths because it's not a fixed size. Instead we're doing the arc and we're comparing it to the entire circle. So degree measurements, *quips* measurements, radian measurements, *marks* are all ways of comparing to the whole circle. We're comparing to something else.

David communicated the process of measuring an angle in degrees, radians, *marks*, and *quips* as one of determining the fraction of the circle subtended by the angle. Immediately following David's remark in Excerpt 59, he explained that an angle with a measure of 10 degrees subtends $10/360^{\text{th}}$ s of the circle centered at the vertex of the angle. David similarly explained that an angle with a measure of 47 degrees subtends $47/360^{\text{th}}$ s of the circumference of the circle centered at the vertex of the angle. These explanations, combined with David's suggestion in Excerpt 59 that the act of measuring an angle is a process of "comparing part of the circle to the whole circle," demonstrates that he supported students in conceptualizing an angle with a measure of n degrees, *quips*, *marks*, or radians as subtending n out of the number of these respective units contained in the circumference of the circle centered at the vertex of the angle. In particular, David's instruction prompted students to first imagine the circumference of the circle being split into a number of equal pieces (360 in the case of degrees, eight in the case of *quips*, 20 in

the case of *marks*, and 2π in the case of radians) and then to attend to the ratio of the number of these pieces the angle subtends to the number of pieces contained in the circumference of the circle (see the left side of *Figure 61*). This meaning for angle measure stands in stark contrast to understanding an angle with a measure of n degrees, *quips*, *marks*, or radians as respectively subtending an arc length that is n times as long as $1/360^{\text{th}}$, $1/8^{\text{th}}$, $1/20^{\text{th}}$, or $1/(2\pi)^{\text{th}}$ of the circumference of the circle centered at the vertex of the angle (see right pane of *Figure 61*). The latter meaning makes explicit a quantity being measured (subtended arc length) and a unit with which to measure it (a fractional part of the circumference of the circle centered at the vertex of the angle).



An angle with a measure of 7 *marks* subtends $7/20^{\text{th}}$ of the circumference of the circle.

An angle with a measure of 7 *marks* subtends an arc that is 7 times as long as $1/20^{\text{th}}$ of the circumference of the circle.

Figure 61. Contrasting meanings of angle measure.

Lessons 4-9. Since Lessons 4-9 focused primarily on the sine and cosine functions, the pre-lesson interviews that preceded these lessons addressed issues of angle measure only occasionally, and often in the context of discussing the input quantity of the these functions. When David and I discussed angle measure in Pre-Lesson Interviews 3-5, he consistently communicated his intention to support students’ understanding of angle measure as “a portion of the circle” (Pre-Lesson Interview 4) or as the “fraction of the

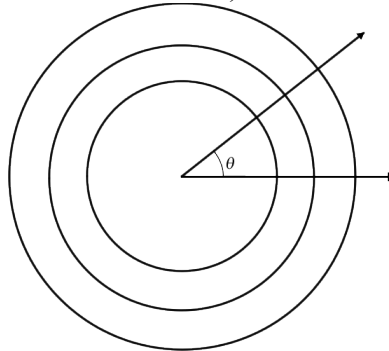
circle's circumference the angle subtends" (Pre-Lesson Interview 6). However, the meanings of angle measure David conveyed in Lessons 4-9 were far less consistent.

Excerpt 60 documents a conversation David facilitated during Lesson 4 while discussing the task shown in Table 52.

Table 52

Investigation 2, Task 10 (Carlson, O'Bryan, & Joyner, 2013, p. 471)

10. Given that an angle measures $\theta = 0.45$ radians, determine the length of each arc subtended by the angle if the circles have radius lengths of 2 inches, 2.4 inches, and 2.9 inches. (*Diagram is not drawn to scale.*)



Excerpt 60

- 1 David: So what do angle measures really tell me? We've been talking about this for three days now. What does an angle measure really tell me? (*Pause*) The portion of the circle right? Okay. So if I said that we had an angle of 10 degrees, that means I have how much of the circle? (*Student responds inaudibly*) Ten out of 360 parts, right? So $1/36^{\text{th}}$ of the circle. So what does it mean when I say 0.45 radians? (*Long pause*) How much of the circle do I have? ... I take the 0.45 and do what?
- 2 Student: You divide 0.45 by 2π .

- 3 David: And that tells me the portion of the circle, right? Okay? So then how would I use that to figure out the arc length?
- 4 Student: You can multiply by— or you find the circumference based on the radius and then you, uh, multiply the circumference by $1/14^{\text{th}}$.
- 5 David: ... Okay so first step for you was to take the 0.45 and divide it by 2π (*writes “0.45/(2 π)” on the board*). This gives me the portion of the circle (*writes “portion of the circle” above the ratio “0.45/(2 π)”*). I’m good with that. ... Okay. So then we are going to multiply by 2π times the radius to get circumference (*writes “(2 πr)” on the board next to “0.45/(2 π)” to indicate multiplication*). Okay? I’m okay with that. So this is how much of the circle you have (*pointing to the expression “0.45/(2 π)”*) and this is the circumference of the whole circle (*pointing to the expression “(2 πr)”*) so when I multiply it, it will tell me how much arc length I have. (*Writes “= Arc Length” next to the expression “0.45/(2 π) · (2 πr)” and calls on a student with her hand up*).
- 6 Student: Couldn’t we just multiply 0.45 times the radius because if we know that it’s only, um, 45 percent of the radius, and a radian is just a length of a radius, then you can just take 45 percent of the radius. ...
- 7 David: So these cancel (*crosses out both 2π s in the expression “0.45/(2 π) · (2 πr)”*) and we can see that’s the reason why. Okay? So we could have just taken the radius measure and multiplied by the radius and

then that would have actually given me the answer. And we can see we end up with the same thing.

In Line 1 David argued that angle measures convey the “portion of the circle” subtended by the angle and supported his claim by explaining that an angle with a measure of 10 degrees subtends $1/36^{\text{th}}$ of the circumference of the circle centered at the vertex of the angle. It is noteworthy that David did not explain that an angle with a measure of 10 degrees subtends an arc that is 10 times as long as $1/360^{\text{th}}$ of the circumference of the circle centered at the vertex of the angle—an explanation that emphasizes angle measure as the length of the subtended arc measured in units that are proportional to the circumference of the circle containing the subtended arc. Furthermore, David prompted students to state that an angle with a measure of 0.45 radians subtends $0.45/(2\pi)^{\text{th}}$ (approximately $1/14^{\text{th}}$) of the circumference of the circle centered at its vertex. He then asked students to think about how they might determine the subtended arc length provided the angle subtends $0.45/(2\pi)^{\text{th}}$ of the circumference of the circle (Line 3). A student suggested multiplying $1/14$, the approximate fraction of the circle subtended by the angle, by $2\pi r$, the circumference of the circle (Line 4). David elaborated the student’s suggestion in Line 5 and wrote the following expression on the board:

$$\frac{\overbrace{0.45}^{\text{Portion of the Circle}}}{2\pi} \cdot \overbrace{(2\pi r)}^{\text{Circumference}} = \text{Arc Length.}$$

David’s immediate inclination to multiplicatively compare the measure of the angle in radians with the number of radians “in a circle” to obtain the fraction of the circle’s circumference subtended by the angle circumvented the curriculum designers’ intention of supporting students’ understanding of radians as the length of the arc an

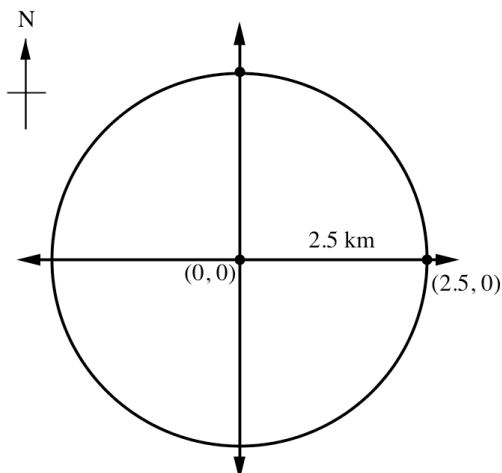
angle subtends measured in units of the radius of the subtended arc. In Line 6 of Excerpt 60, a student suggested multiplying the measure of the angle in radians by the radius of the circle centered at the vertex of the angle to determine the length of the subtended arc. The student justified her suggestion by explaining, “a radian is just a length of radius.” Interestingly, David confirmed the correctness of the student’s suggestion not by appealing to the fact that an angle with a measure of 0.45 radians subtends an arc that is 0.45 times as long as the radius of the subtended arc, but by canceling the $2\pi r$ in the expression above to obtain $0.45r$ as a representation of the length of the subtended arc. David’s validation of the student’s suggestion was therefore exclusively algebraic, as opposed to quantitative. For this reason David still did not support his students in constructing the meaning of angle measure in radians the curriculum developers designed Task 10 to promote (radian measure as the length of the subtended arc measured in units of the radius), even after having to respond to a student who communicated this meaning.

While David’s discussion of Task 10 on Investigation 2 in Excerpt 60 was consistent with the way of understanding angle measure he described in the pre-lesson interview that preceded Lesson 4, his instruction did not consistently support students in constructing this way of understanding. Consider the discussion David facilitated in Excerpt 61 around Task 6(a) from Investigation 4 (see Table 53).

Table 53

Investigation 4, Task 6(a) (Carlson, O’Bryan, & Joyner, 2013, p. 482)

<p><i>For Exercises #6-8, use the following context:</i> An arctic village maintains a circular cross-country ski trail that has a radius of 2.5 kilometers.</p>
--



6. A skier started skiing from the position $(2.5, 0)$ and skied counter-clockwise for 2.75 kilometers before stopping for a rest.
- How many *radius lengths* did the skier travel around the trail? Explain your reasoning.

Excerpt 61

- David: The radius length is 2.5 but he skied for 2.75 kilometers. How many radius lengths did he travel around? (*Long pause*)
- Student 1: I would say 0.175.
- David: How?
- Student 1: Um, there's 6.28 radius lengths around the whole circle so you multiply that by 2.5 and then do how much he skied divided by that number.
- David: Why are we dividing by the whole circumference?
- Student 1: (Responds inaudibly)
- David: Okay. How about you (*Student 2*)?
- Student 2: I divided 2.75, which was how long he skied, by 2.5, which is a radius length.
- David: So you just divided.

In response to Task 6(a), Student 1 divided the length the skier traveled by the circumference of the circular trail ($\frac{2.75}{2.5(6.28)}$). It is not clear whether she thought her task was to determine the number of radius lengths the skier traveled along the circular trail, or the fraction of the circular trail the skier traveled.¹⁰⁰ David nonetheless questioned Student 1's decision to divide the length the skier traveled by the circumference of the circular path (Line 5). This instructional decision is significant considering David unnecessarily computed the portion of the circle subtended by the angle to solve Task 10 on Investigation 2 (see Excerpt 60). I expected David to remediate Student 1's response by supporting her in seeing that she did not multiply the fraction of the circle's circumference subtended by the angle by the circumference of the circle, as he did to solve Task 10 (see Excerpt 60). Instead, David accepted Student 2's suggestion to divide the length the skier traveled by the radius of the circular trail. This endorsement of Student 2's approach is noteworthy since David did not take advantage of the opportunity to communicate the conceptual validity of a student's approach to solving Task 10 on Investigation 2 that emphasized radian measure as the length of the subtended arc measured in units of the radius of the circle centered at the angle's vertex (see Excerpt 60). This was not the only occasion in which David validated a way of understanding that he previously dismissed.

David provided conflicting descriptions of what it means to measure an angle in radians on several other occasions throughout Lessons 4-9. A particularly revealing one occurred during Lesson 6. David asked a student to move the terminal point of the arc an

¹⁰⁰ Interestingly, Student 1's procedure was precisely the one David employed during the series of TBCIs when I asked him to determine the measure of an angle (in radians) with a circle of radius 1.4 inches centered at its vertex and that subtends an arc of 5.3 inches (see Excerpt 22). This makes David's dismissal of the student's approach all the more interesting.

angle subtends along the circumference of a circle centered at the angle's vertex so that the angle has a measure of six radians. *Figure 62* illustrates where the student placed the terminal point A , and Excerpt 62 documents the ensuing conversation.

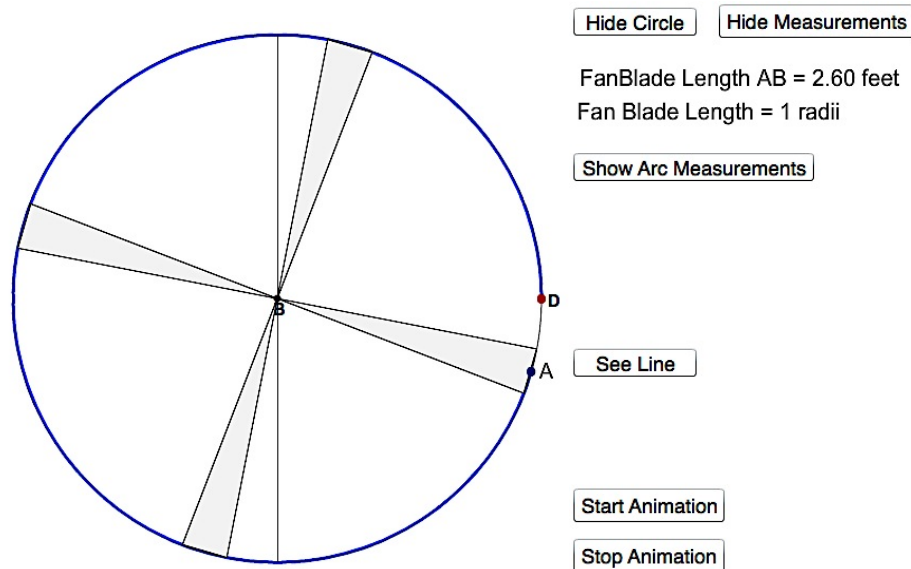


Figure 62. Subtended arc of six radius lengths.

Excerpt 62

- 1 David: So where would I put it? (*Student moves terminal point*) ... Why there?
- 2 Student: Because 6.28 radians are in the entire circle so the left over space is about 0.28.
- 3 David: So this (*points to the compliment of subtended arc*) should be 0.28 of a (*pause*)
- 4 Student: Of 6.28 radians. Of a circle.
- 5 David: What is a radian again?
- 6 Student: A radian is like, um, it's about $1/6^{\text{th}}$ of the circle.
- 7 David: Okay. But where are we getting the radians from?

- 8 Student: Um, isn't it like over 2π ? When you put it over 2π ?
- 9 David: Yeah it's one out of 2π . Okay. But when we're saying there's six of these lengths and then 0.28, what are we talking about? What lengths? ... When we're saying there's a length of six, what lengths of six? (*Long pause*)
- 10 Student: It would be about six radius lengths.
- 11 David: Six radius lengths! So remember that when we're talking about a radian we're saying six radius lengths. Okay so remember that this is a radius length (*points to the radius*) so this part right here is (*points to the compliment of the subtended arc*)
- 12 Student: 0.28 radius lengths.
- 13 David: Okay. So it's about 20 percent of a radius.

Excerpt 62 demonstrates David attempting to direct students' attention to the fact that measuring an angle in radians means measuring the length of the subtended arc in units of radius lengths. The student's suggestion in Line 6 that a radian is "about $1/6^{\text{th}}$ of the circle" is consistent with the way of understanding angle measure David expressed the intention to convey in his instruction during Pre-Lesson Interviews 3-5. Moreover, on several occasions prior to the dialogue in Excerpt 62, David supported students' understanding of angle measure in radians as a fraction of the circle's circumference subtended by the angle (see Excerpt 56, Excerpt 59, and Excerpt 60). Despite David's prior emphasis on this way of understanding, he was dissatisfied with the student's claim in Line 6 that an angle with a measure of one radian subtends $1/6^{\text{th}}$ of the circumference of the circle. Instead, David attempted to support his students in conceptualizing the

radius of the circle centered at the vertex of the angle as a unit of measure for the length of the subtended arc. Accordingly, David emphasized a way of understanding angle measure that differed from that which he previously expressed the intention to support as well as from that which he actually supported in previous lessons.

The inconsistency with which David communicated what it means to measure an angle in radians continued throughout Lesson 6. The conversation in Excerpt 63 transpired during David's 6th hour class on the same day that the episode in Excerpt 62 occurred. Similar to 4th hour, David asked a student to move the terminal point of a subtended arc to illustrate an angle with a measure of one radian. *Figure 63* illustrates where the student placed the terminal point *A*.

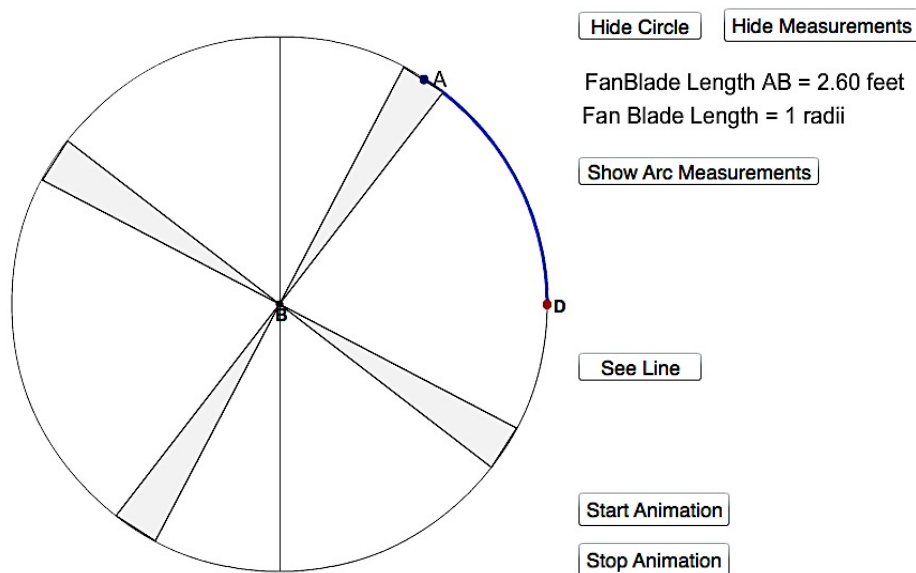


Figure 63. Subtended arc of one radius length.

Excerpt 63

- 1 David: Where would I need to put my Point *A* so that I had a one-radian of angle measure?

- 2 Student: You need to put it at a point that is 2.6 feet around the circumference starting from that point (*Point D*).
- 3 David: Okay. So how am I going to be able to measure that?
- 4 Student: I don't know. (*Pause*) Well actually I do know. So you take the circumference of the circle and divide it by 2.6 and then that's how much of the circle you've moved it. ...
- 5 David: Okay. 5.2π divided by 2.6? (*Performs computation in his calculator*). That gets me 2π .
- 6 Student: Why did I do that?
- 7 David: I don't know. That's what I'm asking. (*Students talk amongst themselves inaudibly*) ... So that's circumference per radian. We need to do it the other way. Okay so (*types "2.6/(5.2 π)" in his calculator*). Okay so what's that telling me?
- 8 Student: That you need to move it 16 percent of the way around the circle.
- 9 David: Okay so how would I do 16 percent of the way around the circle?
...
- 10 Student: Just go like $1/6^{\text{th}}$ of the way around! 'Cause we're estimating.
- 11 David: So if we actually divided our circle into six pieces, would I want to go all the way to the $1/6^{\text{th}}$ mark if I wanted to do one radian?
- 12 Student: It would be a little bit less.
- 13 David: So at least now we have an idea where to go.

In Line 2 the student suggested moving Point *A* so that the length of the subtended arc is 2.6 feet, or one radius length. Indicating that the student's suggestion was somehow

lacking, David asked, “So how am I going to be able to measure that?” (Line 3).

Although the student made a computational error in Line 4 by proposing to divide the circumference by the subtended arc length (instead of the other way around), his recommendation was in essence to determine the fraction of the circle’s circumference subtended by the angle. David endorsed the student’s suggestion to “Just go like $1/6^{\text{th}}$ of the way around!” (Line 10) by saying, “So at least now we have an idea where to go” (Line 13). David’s validation of the student’s proposal to move Point A so that the subtended arc length is approximately $1/6^{\text{th}}$ of the circumference of the circle supported students in conceptualizing an angle with a measure of one radian as subtending $1/6^{\text{th}}$ of the circumference of the circle centered at the vertex of the angle. Moreover, David’s dismissal of the student’s initial suggestion to move Point A so that the angle subtends an arc of one radius length discouraged students from understanding an angle with a measure of one radian as subtending one radius length of arc.

It is noteworthy that through the exchange in Excerpt 62, David guided a student who appeared to understand that an angle with a measure of one radian subtends $1/6^{\text{th}}$ of the circumference of the circle centered at the angle’s vertex to conceptualize an angle with a measure of one radian as subtending an arc that is one radius length. Conversely, in Excerpt 63 David encouraged a student who initially conceptualized an angle measure of one radian as subtending one radius length of arc to understand that an angle with a measure of one radian subtends approximately $1/6^{\text{th}}$ of the circumference of the circle centered at the vertex of the angle. David therefore encouraged meanings that he subsequently did not accept while simultaneously devaluing meanings that he subsequently endorsed. Interestingly, this conflict in the ways of understanding David

communicated occurred on the same day of instruction and with students who were engaged in essentially the same task.

Theme 2: Radians Measure Length. Degrees, *Quips*, and *Marks*, Measure “Space.”

I demonstrated in my discussion of the first theme that on some occasions in his teaching David supported students in conceptualizing angle measure as the length of an arc the angle subtends, while on other occasions he explained that measuring an angle involves determining the fraction of the circle’s circumference subtended by the angle. David did not, however, allow students to recognize the latter meaning as an application of the former because he did not support them in conceptualizing a fraction of the circle’s circumference as a unit with which to measure the length of the subtended arc. In particular, twice during Lesson 3 David suggested that the process of measuring an angle in radians is conceptually different than the process of measuring an angle in degrees, *quips*, or *marks*—the distinction being that radians measure the length of the subtended arc whereas degrees, *quips*, and *marks* are measures of “space” and are used to determine the fraction of the circle subtended by the angle. Consider, for instance, David’s explanation in Excerpt 64.

Excerpt 64

- 1 David: (*Draws an angle of arbitrary measure on the board*) If I give you a random angle and I say, ‘Hey I want you to measure this. No protractor.’ ... If you tried to do it in degrees what would be the problem? ... What would the process take for me to measure it in degrees? ... I would have to take a circle and split it up into 360 equal parts. Do I want to? No! ... What’s involved now if I

measure in terms of radius length? What's the only thing I have to do? ... What's the only tool I need? (*Long pause*) A ruler to measure the [radius] length right? And I'm actually measuring a physical length. ... The problem with measuring in degrees is what? When we're measuring in degrees we're measuring space, right? The openness. How do you measure openness? Do you pull out your ruler? (*Pause*) No! You have to draw a circle and divide it into 360 parts. But if I measure in terms of radius lengths then really actually the only thing I need are the length of the radius and the length of the arc, and those are both things that I measure with a ruler. They're physical measurements that I can physically take, easily.

David contrasted the characteristics of radians and degrees as units of angle measure in his explanation in Excerpt 64. In particular, David suggested that instead of being a "physical length" or a "physical measurement," degrees measure "space" or "openness." David's explanation conveyed that $1/360^{\text{th}}$ of the circumference of the circle centered at the vertex of an angle is not a length, and therefore cannot be a unit with which to measure the length of the arc the angle subtends. David made a similar suggestion during his 6th hour class that same day (Excerpt 65).

Excerpt 65

- 1 David: What's the process for measuring angles in *quips*? What did we do? ... We first had to start by drawing a circle then we had to cut it up into eight parts. When we did *marks* we divided it up into 20

parts. If I do degrees I have to start with a circle then divide it into 360 parts. Until I actually do that circle and divide it all up I can't actually measure that angle. ... When we talk about *quips* or *marks* or degrees, we started with a proportion of the circle. We started with, 'Hey I have to take a circle and divide it up into so many parts and then I can figure out what the angle is.' But when we're talking about [radians] here, we're talking about, I'm comparing physical measurements. I can actually measure a radius length, I can measure an arc length, and I can compare them. ... I'm breaking it up based on an actual physical length as opposed to when I did *quips* I didn't measure a length. ... With a *quip* I was basically doing a percentage. I was saying I want an eighth of the circle. But when I do radians, when I do these radius lengths, I'm saying I want each of these arcs to be four centimeters if my radius is four centimeters so I can just trace the arcs and go about it that way. So there is a little bit of a difference.

As with his statement in Excerpt 64, David's explanation in Excerpt 65 supported students in engaging in the mental action of imposing on an angle a circle centered at its vertex and then imagining the circumference of this circle being partitioned into a certain number of equal-sized arc lengths (360 for degrees, 20 for *marks*, 8 for *quips*). David did not, however, support students in conceptualizing this act of partitioning as one of constructing a unit with which to measure the length of the arc the angle subtends. David explained that *quips* ($1/8^{\text{th}}$ of the circumference of a circle) are not measures of length but

are instead “a percentage” or “an eighth of the circle.” As I emphasized in my conceptual analysis in Chapter 5, understanding angle measure quantitatively involves one in identifying an attribute to measure (i.e., the arc length an angle subtends) and conceptualizing a unit with which to measure it (i.e., a fraction of the circumference of the circle that contains the subtended arc). Crucially, one must use a unit of length to measure a length. By not supporting students in conceptualizing a fraction of the circumference of a circle centered at the angle’s vertex as a unit with which to measure the subtended arc length, David did not promote the understanding that an angle with a measure of n *quips* (degrees) subtends an arc that is n times as long as $1/8^{\text{th}}$ ($1/360^{\text{th}}$) of the circumference of the circle that contains the subtended arc. Instead David conveyed that an angle with a measure of n *quips* (degrees) subtends $n/8^{\text{ths}}$ ($n/360^{\text{ths}}$) of the circle centered at the vertex of the angle. Additionally, David’s suggestion that radians are units of length whereas degrees, *quips*, and *marks* are units of “space” encouraged students to think there is a conceptual difference between measuring an angle in radians and measuring an angle in units that are a particular fraction of the circumference of the circle centered at the vertex of the angle.¹⁰¹ In general, David’s explanations in Excerpt 64 and Excerpt 65 are representative of his lack of attention to supporting students in building a meaning for angle measure independent of a system of measurement.

Theme 3: Radians are Advantageous Because Radius Lengths are “Based on the Circle”

In addition to demonstrating that David’s instruction encouraged students to think that measuring an angle in radians is conceptually different than measuring an angle in

¹⁰¹ Of course a circle’s radius is a particular fraction of the circle’s circumference. At this point in David’s instruction, however, he did not convey this fact.

degrees—the former being a process of measuring length and the latter being a process of measuring “space” or “openness”—Excerpt 64 and Excerpt 65 obscurely reveal David’s preference for radians as a unit of angle measure over degrees. David more explicitly communicated this preference during Pre-Lesson Interview 2 as well as during Lesson 4.

Pre-Lesson Interview 2. During the second pre-lesson interview, while articulating how he wanted his students to understand the key idea of Lessons 3 and 4, David explained that radians are a more convenient unit of angle measure because “the length of the radius has that built-in relationship to circumference through the formula $2\pi r$ We’re relating a physical measurement of the circle to other physical measurements of the circle.” David went on to explain that with degrees, “When you do 360 and you’re breaking the circumference into 360 parts, there’s no physical measurement.” David therefore suggested that $1/360^{\text{th}}$ of the circumference of a circle does not correspond to a “physical measurement of the circle” whereas a radius length obviously does, and that radians are preferable as a unit of angle measure for this reason.

Lesson 4. Twice during Lesson 4 David communicated his rationale for preferring radians as a unit of angle measure to degrees. As he did during Pre-Lesson Interview 2, David emphasized that radians are based on a physical measurement (i.e., radius lengths) whereas degrees are not. David also justified his preference for radians by appealing to a connection between the radius of a circle and its area and circumference. As Excerpt 66 demonstrates, David’s discussed this connection with a certain amount of ambiguity.

Excerpt 66

- 1 David: Because radians have this built-in unit of measure where we’re comparing the radius to the rest of the circle, they naturally are

okay with finding things like areas and circumferences and arc lengths because they are already part of comparing it to the radius. That's built into the problem because when we're talking about a radian we're talking about how a radius length compares to the whole circle. So they have that built-in length component already in them. ... So we talked about why we might want to use radians and not degrees; radians have this natural built-in length component. They want to work with the mathematics of circles. They want to work with area formulas. They want to work with circumference formulas. They like working with circles because we based it on part of the circle itself, it's radius. Where degrees, we decided to cut it into 360 degrees because it just felt fun to do. So I know radians aren't something that you're used to but mathematically they have this connection to area and circumference formulas that other angle measurements just don't have and that's because they're based on radius. Okay? So it's kind of built-in.

David's remark in Excerpt 66 raises a number of immediate questions, namely, "What does it mean to compare the radius to the whole circle?" and "In what way is the fact that the area and circumference formulas are defined in terms of radius a compelling argument for the usefulness of radians as a unit of angle measure?" David's response becomes only slightly more intelligible when one recognizes that by "compares" he meant, "relates to." This clarification notwithstanding, the vagueness and imprecision

with which David communicated the affordances of using radians as a unit of angle measure made it rather challenging for me to characterize the way of understanding David's explanation encouraged. After having carefully examined and reexamined the video and transcript of Excerpt 66, I was unable to discern the specific meaning that David intended his statement to convey. This is perhaps more suggestive of my limitations as an analyst than it is a criticism of David's justification for the utility of radians as a unit of angle measure. Nevertheless, I concluded that David's explanation—and another very similar to it from Lesson 4—encouraged students to think that there is some magical property of radians that makes them more advantageous for measuring angles than degrees, and that this property derives from the fact that radians are “based on the circle.”

Theme 4: The Size of the Circle is Immaterial to the Measure of the Angle

On several occasions in his teaching, and often prompted by specific tasks from the *Pathways* curriculum, David explained that the size of the circle centered at the vertex of the angle is immaterial to the measure of the angle. These explanations, however, did not afford students the opportunity to construct the understanding that the measure of the angle is independent of the size of the circle because units of angle measure are in constant proportion with the circumference of the circle, thereby making a class of subtended arc lengths the same number of times as long as the length of one unit. Instead, David supported students' understanding that the size of the circle centered at the vertex of the angle is immaterial to the measure of the angle because measuring an angle involves the use of ratios.

Lessons 2-5. During Lesson 2, David asked students to determine whether the size of the circle centered at the angle's vertex affects the measure of the angle. Several students confidently asserted that the size of the circle does not matter but struggled to communicate their rationale. After a couple minutes of listening to students' inarticulate justifications, David provided the explanation in Excerpt 67.

Excerpt 67

- 1 David: It's because we're talking about the fraction of a circle. Okay?
Now when we talked about similar shapes, remember when we did similar triangles in geometry, what did we use to figure out similar triangles? *(Pause)* If they were similar, what did we do with similar triangles?
- 2 Student: Proportions.
- 3 David: Yeah we did proportions right! So it didn't matter when we had a triangle like this *(draws a large right triangle and a small right triangle on the board)*. ... We had these two triangles we're similar because they have the same angle measurements and they're not the same size. If this was one and this is two *(labels the base and hypotenuse of the small right triangle "1" and "2" respectively)*, if I say this is three *(labels the base of the large triangle "3")* then this is *(points to the hypotenuse of the large right triangle)*? It's going to be six. It's not because we know the lengths. It's because we're thinking of them about the percent, the ratios. So when we're dealing here with the circles, we're dealing with the ratio of

the circle just like we did here with triangles; we were dealing with the ratio of the parts. So then does it matter which one we have? As long as I have one triangle I know something about I was able to figure out all the other similars. So same kind of idea here. Okay? So it's because we have that idea of a fraction, a ratio, a proportion.

David began his explanation by claiming that the size of the circle centered at the vertex of the angle is immaterial to the angle's measure "because we're talking about the fraction of a circle" (Line 1). It is not clear whether David meant that an angle subtends the same fraction of the circumference of all circles centered at its vertex, or that a unit of angle measure is the same fraction of the circumference of all circles. David proceeded with an attempt to explicate a parallelism between angle measure and similar triangles. David's analogy to similar triangles was inspired by his recognition that ratios, or "proportions" in his usage, are involved in determining unknown side lengths of similar triangles and in computing the measure of an angle. While David's explanation was underspecified, my interpretation of the parallelism he attempted to establish between similar triangles and angle measure is illustrated in *Figure 64*. In particular, David appeared to argue that the ratio of subtended arc length to circumference is constant in the same way that the ratio of corresponding adjacent sides of similar triangles are constant. Based on my interpretation of David's explanation, I suspect his initial remark, "we're talking about the fraction of a circle" was in reference to the fact that an angle subtends the same fraction of the circumference of all circles centered at its vertex, since the analogy to similar triangles David attempted to convey appeared to be based on this

fact. Moreover, David’s statement, “So when we’re dealing here with circles, we’re dealing with the ratio of the circle just like we did here with triangles; we were dealing with the ratio of the parts” supports my claim that David promoted students’ understanding that the size of the circle is immaterial to the measure of the angle by leveraging the fact that an angle subtends the same fraction of the circumference of all circles centered at its vertex. It is worth noting that David’s explanation did not emphasize the condition that a unit of angle measure must satisfy for the measure of the angle to be independent of the size of the circle centered at the vertex of the angle.

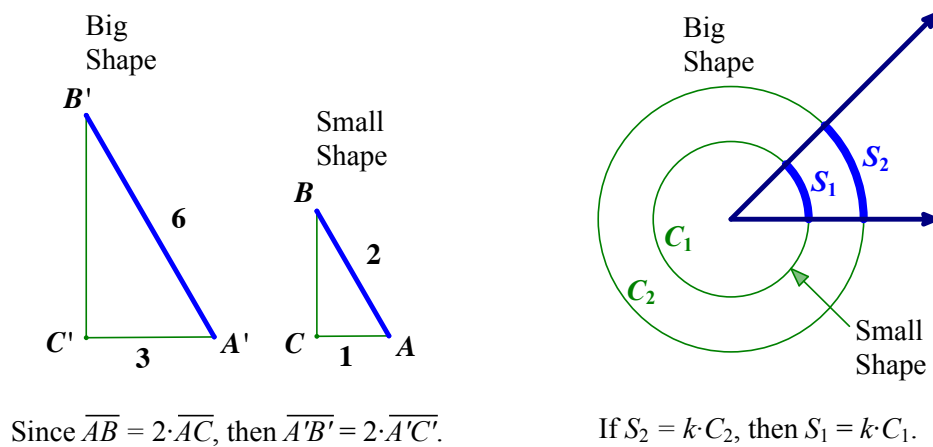


Figure 64. David’s analogy to similar triangles.

Prompted by Task 13 on Investigation 1¹⁰², David provided another explanation for why the size of the circle centered at the vertex of the angle does not affect the measure of the angle (Excerpt 68).

Excerpt 68

¹⁰² This task asked students to describe how degrees (or *quips* or *marks*) differ from length units like inches or centimeters (Carlson, O’Bryan, & Joyner, 2013, p. 466).

- 1 David: How is it different when I'm talking about a degree versus when I'm talking about something measures two inches or five inches or something like that? ...
- 2 Student: It doesn't matter how big a circle is, it will always be 360 degrees but its diameter and radius can all change.
- 3 David: And why is that? (*Long pause*) Why is it that we can use different circles and still get the same answer when we measure angles but yet if I changed the ruler that I was measuring lengths with that would make a difference?
- 4 Student: Degrees are based on kind of like ratios.
- 5 David: Okay. So since angles are based on the ratio, as long as I maintain the ratio we're okay. Okay? Where I wouldn't want to make an inch longer than what it was, but I can make a circle bigger and break it up. So it's because that ratio idea. ... When we're talking about degrees we're talking about proportion of a circle.

Contrary to his explanation in Excerpt 67, David's statement in Line 5 of Excerpt 68 afforded students some insight into the condition that a unit of angle measure must satisfy. However, as with the previous excerpt, David's justification for the fact that the measure of an angle is irrespective of the size of the circle centered at its vertex was rather opaque. His remark, "I wouldn't want to make an inch longer than what it was, but I can make a circle bigger and break it up" suggested that units of angle measure do not correspond to a fixed length. Moreover, David's claim, "When we're talking about degrees, we're talking about proportion of a circle" supported students in understanding

that one degree corresponds to an arc length that is the same fraction of the circumference of all circles. David did not, however, explicitly communicate to students that the length of the arc an angle subtends is independent of the size of the circle that contains it when the subtended arc length is measured in units that are proportionally related to the circumference of the circle.

David did not take advantage of several opportunities in his teaching to support students' understanding of why units of angle measure that are proportionally related to the circumference of the circle centered at the vertex of an angle make the size of the circle inconsequential to the measure of the angle. For example, while discussing a task that asked each student to construct circle of arbitrary radius and to draw an angle whose vertex is at the center of the circle that subtends an arc equal in length to the radius of the circle, David was content with stating that everyone's angle subtends about $1/6^{\text{th}}$ of the circumference of the circle centered at its vertex. David did not support students in realizing that all of their angles have the same "openness" as a result, which is precisely the pseudo-empirical abstraction the task was designed to promote. During the same lesson David claimed, "A radius is the proportion of one over 2π of a circle's circumference" and then proceeded to explain, as if it followed deductively, that one could measure angles using radius lengths. David did not discuss why when one measures the length of the subtended arc using the radius as a unit of measure, the size of the circle centered at the vertex of the angle is inconsequential. Finally, while facilitating a conversation around a task that prompted students to conjecture how the length of a subtended arc changes if the radius of the circle containing the subtended arc doubles, David accepted a student's claim that the subtended arc would be twice as long. David

did not support students in recognizing that the length of the subtended arc measured in units of the radius would remain invariant, nor did he explore the characteristics of the unit of measure that made the length of the subtended arc invariant when measured in this unit.

Summary of David's Enacted Knowledge of Angle Measure

David's instruction of angle measure did not consistently support students in conceptualizing the quantity one measures when assigning numerical values to the openness of an angle. On some occasions David conveyed that the process of measuring an angle is one of determining the length of the arc an angle subtends, while on other occasions he explained that measuring an angle involves determining the fraction of a circle's circumference subtended by the angle. While in certain circumstances one of these ways of understanding may be more natural than the other, David did not support students in making the connection between these ways of understanding. In other words, David did not provide opportunities for students to see these meanings as two instantiations of the same quantification process (measuring the length of the subtended arc) because he never supported students in conceptualizing the circumference of the circle centered at the angle's vertex as a unit of measure for the length of the subtended arc.

During the first pre-lesson interview, David explicitly and consistently communicated his intention to support students' understanding of angle measure as the fraction of the circle's circumference the angle subtends. While David's instruction during the first lesson emphasized this meaning, he also discussed angle measure as the length of the arc an angle subtends, but then suggested that these two meanings are

incompatible by explaining that the process of measuring an angle is not one of measuring a length.

In contrast to the consistency with which David communicated the way of understanding angle measure he intended to support in his teaching during the first pre-lesson interview, David provided conflicting descriptions of how he wanted his students to conceptualize what it means to measure an angle in radians throughout the second pre-lesson interview. On several occasions during this interview, David communicated his intention to support students' understanding of angle measure as either the fraction of the circumference of the circle subtended by the angle, or as the length of the arc the angle subtends measured in units of the radius. However, the vast majority of David's instruction during Lessons 2 and 3 emphasized angle measure as the fraction of the circle's circumference subtended by the angle. Specifically, David's teaching promoted the mental imagery of imagining the circumference of the circle centered at the vertex of the angle being split into a number of equal pieces, and then attending to the ratio of the number of these pieces subtended by the angle to the number of these pieces contained in the circumference of the circle.

The inconsistency between the meanings of angle measure David expressed the intention to convey and those he actually conveyed in his teaching continued. During Pre-Lesson Interviews 3-5, David consistently communicated his intention to support students' understanding of angle measure as a fraction of the circle's circumference subtended by the angle. However, on several occasions during Lessons 4-9, David discussed angle measure as the length of the subtended arc measured in units of radius lengths. Moreover, David's instruction throughout Lessons 4-9 remained contradictory in

that he encouraged meanings that he subsequently did not accept and devalued meanings that he subsequently endorsed.

David's instruction also supported students' understanding that the process of measuring an angle in radians is conceptually different than the process of measuring an angle in degrees, *quips*, or *marks*. In particular, David communicated that radians correspond to the length of the radius of the circle centered at the angle's vertex whereas degrees, *quips*, and *marks* are measures of "space" that do not correspond to physical lengths or physical measurements. Accordingly, David explained that radians measure the length of the arc an angle subtends in units of the radius whereas degrees, *quips*, and *marks* measure the fraction of the circle subtended by the angle. Notably, while David communicated that degrees, *quips*, and *marks* correspond to a particular fraction of the circumference of the circle centered at the vertex of the angle, he did not support students in recognizing the fractions of the circle's circumference that respectively correspond to these units of measure as being units with which one may measure the length of the arc the angle subtends; nor did David explain why the size of the circle centered at the vertex of the angle is inconsequential to the length of the subtended arc when measured in these units.

During both the second pre-lesson interview and the fourth lesson, David communicated his preference for radians as a unit of angle measure over degrees and rationalized this preference by referencing the distinction he drew in the previous lesson between radians and degrees. Specifically, David argued that radians are a more convenient unit of angle measure because they correspond to a physical length that is "based on the circle" whereas degrees are not. David also justified his preference for

radians by appealing to the connection between the radius of a circle and its area and circumference. The vagueness and imprecision with which David communicated the efficacy of this seemingly irrelevant connection encouraged students to think there is some magical property of radians that makes them more advantageous for measuring angles than degrees.

David acknowledged on several occasions in his instruction that the size of the circle centered at the vertex of the angle is immaterial to the measure of the angle. David did not, however, provide students with the opportunity to understand why the measure of the angle is independent of the size of the circle when measured in units that are proportionally related to the circumference of the circle. Alternatively, David supported students in understanding that the size of the circle centered at the vertex of the angle is immaterial to the measure of the angle because measuring an angle involves the use of ratios. David communicated an elusive parallelism between angle measure and similar triangles that was inspired by his recognition that ratios are involved in determining unknown side lengths of similar triangles and in computing the measure of an angle. His explanation promoted students' understanding that the size of the circle is immaterial to the measure of the angle because an angle subtends the same fraction of the circumference of all circles centered at its vertex. David did not, however, emphasize the condition that a unit of angle measure must satisfy for the measure of an angle to be independent of the size of the circle centered at its vertex, nor did he take advantage of several opportunities in his teaching to support students' understanding of why units of angle measure that are proportionally related to the circumference of the circle centered

at the vertex of an angle make the size of the circle inconsequential to the measure of the angle.

Comparison and Contrast of David’s Knowledge of Angle Measure with His Enacted Knowledge of Angle Measure

In this subsection, I compare and contrast the meanings of angle measure David revealed in the series of TBCIs with the meanings he supported in the context of instruction. To compare these instantiations of David’s mathematical knowledge, I identify specific meanings he demonstrated/conveyed and examine instances in which these meanings arose in the context of the series of TBCIs and during his classroom teaching. To contrast David’s knowledge of angle measure with his enacted knowledge, I discuss meanings that arose in either the series of TBCIs or during his instruction. With a few notable exceptions, David engaged his students in experiences that supported them in constructing the ways of understanding angle measure he demonstrated during the series of TBCIs. David therefore did not appear to make many conscious concessions to the quality of his enacted knowledge of angle measure. As a result, and as the present chapter has already revealed, David’s enacted knowledge of angle measure inherited many of the same inconsistencies that he demonstrated throughout the series of TBCIs.

Angle measure as the ratio of subtended arc length to circumference. On several occasions in his instruction and throughout the series of TBCIs, David explained that to measure an angle one must determine the ratio of the arc length an angle subtends to the circumference of the circle centered at the angle’s vertex. David explicitly stated in his teaching that angle measure is a “proportion of the total circumference,” “part of the circle to the whole circle,” “fraction of the circle,” “fraction of the circle’s circumference,”

and “proportion of the circle.” David similarly emphasized in the Initial Clinical Interview that the measure of an angle is “the ratio of arc length to circumference.” During Lesson 4 and in TBCIs 1 and 2, David explained that to measure an angle in radians one must multiply the ratio of subtended arc length to circumference by 2π , the number of radians “in a circle.” Additionally, in the Initial Clinical Interview and during Lesson 6, David conveyed that an angle with a measure of one radian subtends $1/6^{\text{th}}$ of the circumference of the circle centered at its vertex. These statements emphasize angle measure as a multiplicative comparison of the arc length an angle subtends and the circumference of the circle that contains the subtended arc.

Angle measure as the length of the subtended arc. David also demonstrated throughout the series of TBCIs, and conveyed in his teaching, his understanding of angle measure as the length of the arc an angle subtends. On several occasions during the series of TBCIs, David approximated the measure of an angle in radians by determining the length of the subtended arc in units of its radius. Twice in the first TBCI David constructed an angle with a particular measure in radians by constructing a subtended arc that was the angle’s measure times as long as the radius of the subtended arc. Generally speaking, throughout the series of TBCIs David convincingly demonstrated his understanding that an angle with a measure of n radians subtends an arc that is n times as long as the radius of the circle centered at the angle’s vertex.

David’s instruction of angle measure supported students in constructing this way of understanding. For instance, in Lesson 1 David compared the openness of two angles by attending to the lengths of the arcs the angles respectively subtend. During Lesson 6, David prompted a student to approximate the measure of an angle in radians by

determining how many radius lengths are contained in the length of the subtended arc. Also during Lesson 6, David asked a series of questions that guided a student to realize that an angle with a measure of one radian subtends an arc equal in length to its radius. Although David supported students' understanding of angle measure as subtended arc length, he did so far less frequently than he supported their understanding of angle measure as the fraction of the circle's circumference subtended by the angle.

Units of angle measure. During the second and third TBCIs, David demonstrated his understanding that the unit with which one measures the length of the arc an angle subtends must be proportionally related to the circumference of the circle centered at the angle's vertex so as to make the size of the circle immaterial to the measure of the angle. Recall that in the third TBCI David explained that as the size of the circle varies, a proper unit of angle measure must vary in tandem so that an angle that subtends one of these units has an invariant amount of openness. David recognized that this condition is satisfied when the unit of measure is always the same fraction of the circle's circumference. Moreover, in the same TBCI David revealed his understanding that the length of the arc an angle subtends varies as the radius of the subtended arc varies if the subtended arc length is measured in a unit that is not proportionally related to the circumference of the circle that contains the subtended arc.

While David convincingly communicated the necessity for a unit of angle measure to be proportionally related to the circumference of the circle centered at an angle's vertex, his instruction did not support students in constructing this understanding. David was instead content with communicating in Lesson 2 that units of angle measure do not correspond to fixed lengths. David also explained in Lesson 3 that degrees, *quips*,

and *marks* measure “a fraction of the circle.” On no occasion in his teaching did David explicitly support students’ understanding that the length of the arc an angle subtends is independent of the size of the circle that contains it when the subtended arc length is measured in units that are proportionally related to the circumference of the circle.

Circumference as a unit of measure. David’s teaching also did not support students in conceptualizing the process of determining the fraction of a circle’s circumference an angle subtends as one of measuring the length of the subtended arc—an understanding he demonstrated during the third TBCI. In other words, David allowed students to understand angle measure as a fraction of the circumference of the circle centered at the vertex of the angle and as the length of the arc an angle subtends, but he did not support students in understanding the former meaning as an application of the latter. During TBCI 3, David explained that one could conceptualize the measure of the angle shown in *Figure 65* in two ways: (1) as subtending an arc that is $3/8^{\text{th}}$ as long as the circumference of the circle centered at the angle’s vertex, or (2) as subtending an arc that is three times as large as $1/8^{\text{th}}$ of the circumference of the circle centered at the angle’s vertex. Importantly, David recognized these two conceptualizations as being more similar than dissimilar in that they both produce measures of the subtended arc length, but in different units. This recognition was based on David’s understanding that the circumference of the circle centered at the angle’s vertex can itself be a unit of measure for the length of the subtended arc. David’s instruction, however, did not support students in conceptualizing the circumference of the circle as a unit with which to measure the subtended arc length. David’s students therefore did not have the opportunity

to conceptualize the ratio of subtended arc length to circumference as a measure of the subtended arc length in units of the circumference.

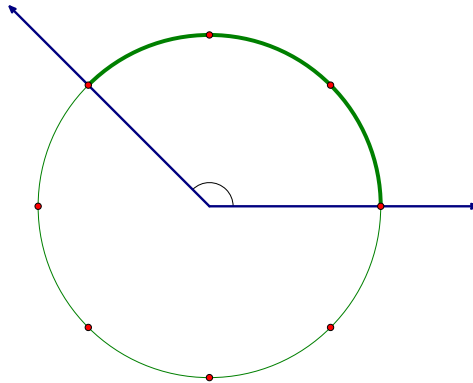


Figure 65. Circumference as a unit of measure for the length of the subtended arc.

Pseudo-proportional reasoning. Another way of understanding David demonstrated throughout the series of TBCIs but did not leverage in his instruction was his approach to determining the measure of an angle using his “part-whole proportion formula.” Recall that on four occasions during the first TBCI, David assimilated information in a problem statement to what I referred to as his “proportion formula scheme.” This scheme assumed the form of a formula template in which David would insert the values of quantities so that he could algebraically determine the value of a quantity of interest. This general proportion formula was of the following form:

$$\frac{\text{New Value (of type } X)}{\text{Old Value (of type } X)} = \frac{\text{New Value (of type } Y)}{\text{Old Value (of type } Y)}$$

It is interesting considering the frequency with which David assimilated information to his proportion formula scheme that at no time in his instruction did he support students in constructing a similar coping mechanism to facilitate their avoidance of reasoning about quantities and the relationships between them.

Conflicting ways of understanding. There were a number of occasions in which David supported meanings in his instruction that not only differed from, but were also inconsistent with, and even contradictory to, the ways of understanding he demonstrated in the series of TBCIs. A couple of examples of the inconsistency and contradiction between David's knowledge of angle measure and his enacted knowledge of angle measure are worth noting. First, in the Initial Clinical Interview David explained that to say an angle has a measure of one radian means the angle subtends $1/6^{\text{th}}$ of the circumference of the circle centered at its vertex (see Excerpt 1). During Lesson 6, David asked a student to explain the meaning of one radian and she replied, "A radian is like, um, it's about $1/6^{\text{th}}$ of the circle." David did not accept this response and engaged the student in a sequence of questions intended to support her in realizing that one radian corresponds to one radius length of arc (see Excerpt 62). Additionally, as I discussed above, in the third TBCI David demonstrated his understanding that a unit of angle measure must be proportionally related to the circumference of the circle centered at the angle's vertex so as to make the length of the subtended arc invariant when measured in this unit. Instead of supporting students in constructing this meaning, David presented students with faint parallelism between angle measure and similar triangles in an effort to justify the fact that the size of the circle centered at the vertex is immaterial to the measure of the angle. Finally, in TBCI 1 David explained that the radius of a circle satisfies the condition that a unit of angle measure must satisfy: its length is proportional to the circle's circumference. During Lesson 3, however, David contrasted the characteristics of degrees, *quips*, and *marks* with radians and suggested that the former units of angle measure correspond to a fraction of the circle's circumference whereas

radians correspond to the length of the radius. This discussion did not support students in understanding that degrees, *quips*, *marks*, and radians correspond to lengths that are necessarily proportionally related to the circumference of the circle centered at the vertex of the angle. These and subsequent instances in which the meanings David supported in his instruction conflicted with the ways of understanding he demonstrated in the series of TBCIs reveal a very significant characteristic of David's mathematical knowledge that compromised the quality of his enacted mathematical knowledge. I discuss this characteristic in detail at the conclusion of the chapter.

Output Quantities and Graphical Representations of Sine and Cosine Functions

The following themes emerged from my analysis of David's enacted knowledge of the output quantities and graphical representations of sine and cosine functions:

- (1) Non-quantitative way of understanding the outputs of sine and cosine.
- (2) Quantitative way of understanding the outputs of sine and cosine.
- (3) Covariation of the input and output quantities of sine and cosine.
- (4) The size of the circle is inconsequential to the output values and graphical representations of sine and cosine.

The organization of this section is the same as the previous one; I describe the mathematical meanings David intended to convey in his teaching relative to each of the aforesaid themes and provide evidence from David's instruction to illustrate each theme.

Theme 1: Non-Quantitative Way of Understanding the Outputs of Sine and Cosine

Recall that David's instruction supported two ways of understanding what it means to measure an angle: (1) the measure of an angle is the ratio of subtended arc length to circumference; and (2) the measure of an angle is the length of the arc the angle

subtends in units that are proportional to the circumference of the circle containing the subtended arc. I argued in the previous section that these meanings were indeed distinct since David did not support students in conceptualizing the ratio of subtended arc length to circumference as the length of the subtended arc measured in units of the circumference. David's instruction similarly supported two distinct ways of understanding the output quantities of sine and cosine: (1) as a ratio of a vertical or horizontal distance to the radius of the circle centered at the vertex of the angle; and (2) as respective vertical and horizontal distances measured in units of the radius of the circle centered at the angle's vertex. As with the meanings of angle measure David emphasized in his teaching, he did not support students in seeing the former meaning as equivalent to the latter by treating the ratios of lengths he used to define the output quantities of sine and cosine as arithmetic, as opposed to quantitative, operations.

Pre-Lesson Interviews 3-5. During the fourth pre-lesson interview, I asked David to describe what would constitute a successful outcome of Lessons 5 and 6. He explained that minimally he would like his students to understand that sine values correspond to vertical distances and cosine values correspond to horizontal distances. David then expressed his preference for understanding sine as the ratio of a vertical distance to the radius of the circle centered at the angle's vertex and cosine as the ratio of a horizontal distance to this radius. Similarly, in Pre-Lesson Interview 3, David explained that initially he would be content with his students understanding the graph of sine as a representation of the relationship between angle measure and a vertical distance, but then admitted that eventually he would like them to conceptualize the output values of points on the sine curve as ratios of lengths. Finally, during the fourth pre-lesson interview David

acknowledged that students tend to have difficulty understanding sine “as an operation” (Excerpt 69).

Excerpt 69

- 1 David: They have problems identifying that sine is actually an operation. ... They haven't really wrapped their mind around the fact that it's an operation.
- 2 Michael: So what do you mean by that?
- 3 David: Um, (*pause*) they'll look at the sine and they're still not sure what the sine is giving them as an output. That takes a while for them to sink in. They don't realize that when I say take the sine of $\pi/4$ what I'm actually asking them to do, that we're asking them to find the ratio. ... They haven't yet clicked their mind over to when they see the word 'sine' that really what it's saying is, 'Do this operation.'
- 4 Michael: And what is the operation that you want them to think about?
- 5 David: So the operation that we want them to think about is to take the, um, angle measure input to find the corresponding vertical distance above the horizontal axis and find its proportion of the length of the radius.

The “operation” David claimed students have difficulty understanding is the division of the vertical distance that the terminus of the subtended arc is above the horizontal diameter of the circle centered at the vertex of an angle in standard position and the radius of this circle. David clearly explained in Line 3 his desire for students to interpret the symbol “sin” as a cue to perform this arithmetic operation. David's remarks

from Pre-Lesson Interviews 3-5 very clearly reveal his intention to support students' understanding of sine and cosine as ratios of lengths, and therefore as arithmetic operations.

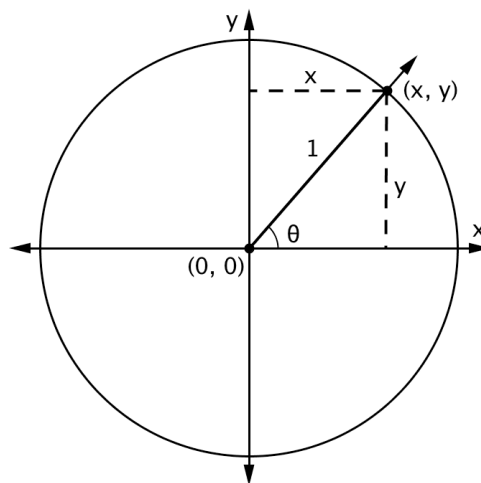
Lessons 4-9. Consistent with his instructional intentions, David engaged students in questions and explanations that supported their understanding of sine and cosine as ratios of lengths. I highlight one instance from David's teaching that is representative of a multiplicity of occasions in which he communicated sine and cosine values as ratios of lengths.

During Lesson 7, while facilitating a discussion about a statement and image that preceded Task 3 on Investigation 8 (see Table 54), David provided the explanation in Excerpt 70.

Table 54

Statement Preceding Investigation 8, Task 3 (Carlson, O'Bryan, & Joyner, 2013, p. 477)

When discussing the sine and cosine functions, it is common to create a circle on the coordinate axes centered at the origin. To simplify the problem, we often use a circle with a radius length of 1 unit, which results in the following sketch. This is referred to as the **unit circle**.



Excerpt 70

1 David: If I make it so that my radius, this length here (*points to the radius in the diagram in Table 54*) is one, that makes it nice. Now we could say a radius is one inch or one foot or one centimeter or, even if it is five inches, we can say that this is one radius length. When I do that then I go, ‘Oh. The y -coordinate is the value of sine. The x -coordinate is the value of cosine.’ If I don’t make it a radius of one, what we need to do is think of it as a proportion. So if the radius is not one, if we have our circle (*draws an angle on the whiteboard with a circle centered at its vertex*) and we have our point here on the terminal side (*labels the terminal point “(x, y)”*) and we have a radius of one, then the sine of theta is equal to y only if r is equal to one (*writes “ $\sin(\theta) = y$ ”*). If r is not one than it’s y over r (*writes “ $\sin(\theta) = y/r$ ”*). Cosine of theta is x only if the radius is one (*writes “ $\cos(\theta) = x$ ”*). ... As long as this radius is one, that’s the answer. But if this is not (*points to the radius of the circle he drew on the whiteboard*), then it’s the ratio of the x -coordinate to r (*writes “ $\cos(\theta) = x/r$ ”*).

David acknowledged that one may conceptualize a radius of any length as a radius of one unit if he or she measures the radius in units of *radius lengths*. David then defined sine and cosine values in reference to a circle centered at the origin of a coordinate system with the initial ray extending along the positive x -axis. He claimed that if the radius of the circle has a measure of one unit, then the sine and cosine values of the angle’s measure are respectively equal to the y - and x -coordinate of the terminus of the

subtended arc. David proceeded to explain that if the radius of the circle centered at the angle's vertex does not have a measure of one unit, then the values of sine and cosine are given by the respective ratios of the y - and x -coordinate of the terminal point to the length of the radius. By defining sine and cosine relative to these two cases, David did not leverage his understanding that one may consider the measure of the radius of any circle to be *one radius length*. Doing so would have negated the necessity for defining sine and cosine by cases since the radius of the circle centered at the angle's vertex *always* has a measure of one radius length. Moreover, David's explanation did not support students in conceptualizing sine and cosine values as the measure of a quantity in a particular unit. In this respect, Excerpt 70 is characteristic of many other instances from David's classroom teaching.

Pre-Lesson Interviews 6-10. In addition to expressing his intention to support students' understanding of sine and cosine values as ratios of lengths, David explained during Pre-Lesson Interview 6 that his instruction seeks to support students in being able to recite sine and cosine values of integral multiples of $\pi/4$ and $\pi/6$. For instance, in response to my asking him to explain how he wants students to understand the key idea of Lesson 10, David said, "I hope that they understand that there are special angles that they can plug into the sine or cosine function and I expect them to give me exact values." He went on to explain, "My expectation becomes when I ask you, 'What is the cosine of $5\pi/6$?' that you can then give me that answer basically from memory." Also during Pre-Lesson Interview 6, David claimed, "Today's lesson will be a success if I can ask a student, 'What is the sine of $5\pi/6$?' and they will be able to give me the exact answer, you know, either by drawing out the situation or from memorization." These remarks plainly

demonstrate that David's instructional intention for Lesson 10 was to support students in being able to recite sine and cosine values of particular angle measures.

In Pre-Lesson Interview 9, during which David and I discussed his plan for Lesson 13, he communicated three instructional goals: (1) to support students' ability to identify and correctly name transformations of periodic functions provided either a function definition or a graphical representation; (2) to graph a transformed periodic function provided a function definition; and (3) to write the function definition of a transformed periodic function provided either a graph or a situation. David expressed concern that there were too few opportunities in the *Pathways* curriculum to facilitate his achievement of the third instructional goal. He therefore decided to supplement Investigation 7 from the *Pathways* curriculum with procedural tasks intended to support students in becoming fluent in writing the function definition of a transformed period functions provided its graph. Excerpt 71 contains a statement of David's rationale for supplementing Investigation 7.

Excerpt 71

- 1 David: Basically what I think that we need in transition into [Investigation] 7 is what most other books do and that is—especially when they're getting into this transformation—is when they go, 'Here's the graph. Give me the equation.' No situation, let's just do transformation and just go, 'Make an equation that matches this graph. Make an equation that matches this graph. Make an equation that matches this graph.' ... So I think that it's almost where usually there is a, before they get to the applications,

they'll do some just, 'Here's the equation. Do the transformation. Here's the equation. Do the transformation. Here's the amplitude. Here's the period that I want. Give me the equation.' And I think that is really kind of what we're missing. ... This would be a basic skill that I would expect them to be able to master.

Twice during Pre-Lesson Interview 10 I attempted to assess the extent to which David recognized the conceptual affordances of the *Pathways* curriculum's focus on supporting students in conceptualizing the inputs and outputs of the sine and cosine functions as quantities. In particular, I was interested in ascertaining whether David recognized the utility of quantitative reasoning on students' ability to construct meaningful function definitions to model periodic phenomena. I asked him to discuss the advantages and limitations of the curriculum's emphasis on supporting students in understanding the input quantity of sine and cosine as representing angle measure and the output quantities as representing vertical and horizontal distances respectively. David explained that students first had to transition from understanding sine and cosine in the context of right triangle trigonometry to understanding these functions in the context of a circle with an angle's vertex at its center. He then argued that the last transition students must make is to apply sine and cosine functions to situations that do not involve circular motion. Regarding this "last transition," David explained, "All we're really worried about is the cyclic pattern. ... Forget about the circle, it's our position in the cycle. ... You leave the circle completely out of it, or if you mention circle it's only to bring in the word cycle." I later asked David to describe how students might leverage a quantitative understanding of the outputs of sine and cosine to model periodic phenomena. David's

immediate response was, “See that I don’t know. I don’t know if there’s an easy way to make that relationship and translation.” After a moment of reflection David added, “Actually I do know how to make the connection. The connection is with transformations of the graph. ... You go straight to, ‘That’s the graph that was created, now everything that we’re talking about is a transformation of that graph.’ ”

There are two noteworthy aspects of David’s remarks. First, David’s statement, “if you mention circle it’s only as to bring in the word cycle” suggests that in the context of thinking about how to support students’ modeling of periodic phenomena, he viewed the circle centered at an angle’s vertex as a symbol for cyclic behavior. Presumably, David understood that this symbol might prompt students to recognize the need to apply sine or cosine functions to model a particular periodic phenomenon. Second, although proffered as a tool to establish a connection between students’ quantitative understanding of the outputs of sine and cosine and their capacity to model periodic phenomena, David’s proposal to leverage transformations of sine and cosine graphs appeared to be a coping mechanism to promote students’ evasion of reasoning quantitatively about the inputs and outputs of the parent sine and cosine functions. David seemed to consider the graphical representations of sine and cosine as reified objects to be manipulated according to specific transformation rules to fit a specific relation between the values of two quantities that covary. In essence, when students are confronted with the task of modeling a particular periodic phenomenon, David’s remarks suggest that he anticipated supporting them in the following way:

- We notice that this situation is cyclic. Circles are cyclic by nature. Remember when we developed sine and cosine we did so in the context of circles. Hence, we can probably apply sine or cosine to model this situation.
- But the parent sine and cosine graphs don't represent the relationship between the values of the quantities in *this* situation, so we'll have to transform them. That is, we'll have to change the amplitude, phase shift, period, and vertical shift to match the variation of the quantities in the situation. When we do, we'll be able to write down the function definition from the transformed graph—a skill we practiced in previous lessons.

In sum, David's remarks during Pre-Lesson Interview 10 reveal that he did not recognize the conceptual affordances of reasoning quantitatively about the inputs and outputs of the sine and cosine functions on one's ability to model periodic phenomena.

Lessons 10-19. David's instruction in Lessons 10 and 11 reflected his intention to support students in being able to recite sine and cosine values of particular angle measures. During Lesson 11, David projected onto the board a unit circle centered at the origin of a coordinate plane like the one displayed in *Figure 66*. He asked his students to refer to this image to recite the sine value of $3\pi/4$. Excerpt 72 contains David's question and the ensuing conversation.

Excerpt 72

- 1 David: What is the sine of $3\pi/4$? (*Long pause*) So what am I asking you for when I ask you to evaluate the sine of $3\pi/4$?
- 2 Student: The y -coordinate. ...
- 3 David: So what's the answer?

- 4 Students: Root two over two ($\sqrt{2}/2$).
- 5 David: Root two over two. All I do is I come over here and go, ‘Oh. There’s $3\pi/4$ (*points to $3\pi/4$ on the unit circle*). There’s $\sqrt{2}/2$ (*points to the y-coordinate of the point that corresponds to $3\pi/4$ on the unit circle*).’ (Pause) As soon as I saw that you were asking me for the sine of something over four, I knew the answer was going to be $\sqrt{2}/2$. The question was is it positive or negative. That was the only thing that ran through my mind. ... $3/4^{\text{th}}$ of π means I haven’t quite gotten to a whole π . That means a whole π would be half way around the circle so I must still be in Quadrant II and I know that in Quadrant II y must be positive.

David’s explanation in Line 5 supported students’ understanding of sine values as y -coordinates on the unit circle (see *Figure 66*). It is clear that David was principally concerned with students’ ability to recite the value of $\sin(3\pi/4)$; his explanation did not support or reinforce students’ understanding of $3\pi/4$ and $\sqrt{2}/2$ as values of quantities. David similarly explained how to determine the value of $\cos(11\pi/6)$, $\sin(-2\pi/3)$ and $\cos(113\pi/6)$. These explanations emphasized sine values as y -coordinates on the unit circle and cosine values as x -coordinates on the unit circle. David concluded his discussion by stating, “I expect if I ask you any of these questions, I expect you to be able to answer them without a calculator and without your unit circle sitting in front of you.” David’s students were noticeably dissatisfied with this expectation. In an effort to reassure them, David said, “Did I memorize the entire circle? No! Really all I had to memorize was three points ... because all the rest of them are just the same point values

except sometimes they're positive and sometimes they're negative." The three point values David claimed to memorize were the coordinates corresponding to $\pi/3$, $\pi/4$, and $\pi/6$ radians. David then summarized for his students the procedure for determining sine and cosine values of integral multiples of $\pi/4$ and $\pi/6$. He explained that they should first associate the input of sine or cosine with $\pi/6$, $\pi/4$, or $\pi/3$ by recognizing the input as an integer multiple of one of these values. He then claimed that they should know the sine and cosine of this value (the respective y - and x -coordinates associated with either $\pi/6$, $\pi/4$, or $\pi/3$) by memory. Finally, David explained that to determine the sign of the sine or cosine value they were asked to evaluate, they would need to establish the quadrant in which the input resides.

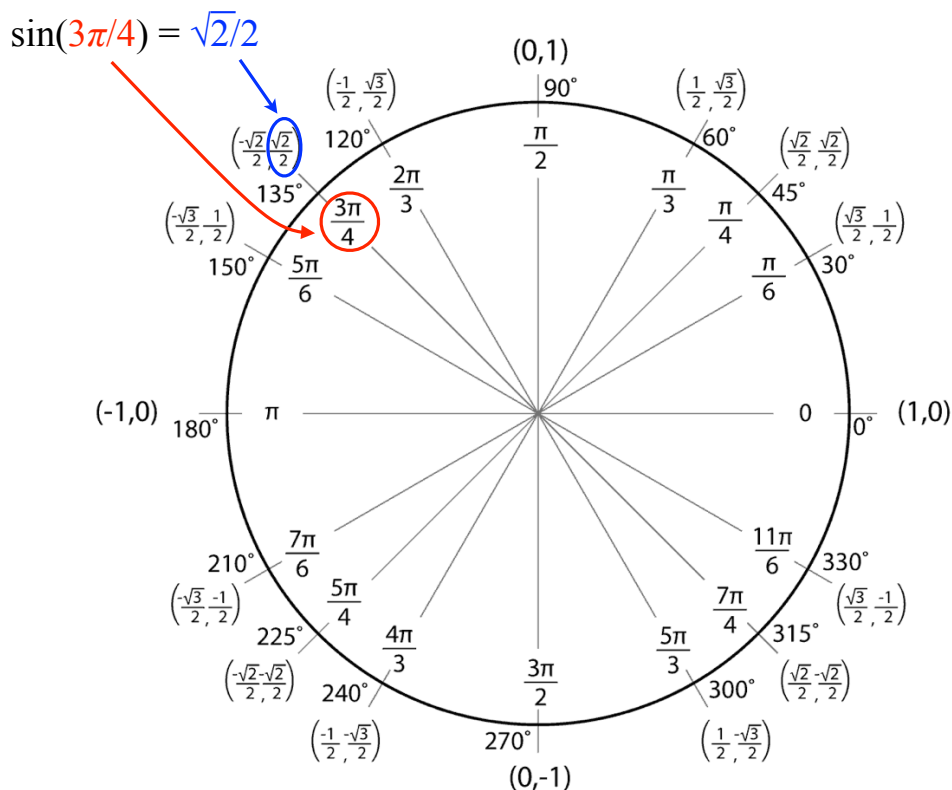


Figure 66. Unit circle and $\sin(3\pi/4)$.

David's instruction also reflected the three instructional goals he revealed during Pre-Lesson Interview 9: (1) to support students' ability to identify and correctly name transformations of periodic functions provided either a function definition or a graphical representation; (2) to graph a transformed periodic function provided a function definition; and (3) to write the function definition of a transformed periodic function provided either a graph or a situation. Regarding the last of these instructional goals, during Lesson 13 David explained, "What we want to be able to do is understand how these transformations here basically apply to us so that we can take the curve and make it do what we want." A few minutes later David communicated his instructional goals more clearly (Excerpt 73). David then proceeded to work a number of examples of constructing function definitions to match graphical representations or to model periodic phenomena by manipulating the values of a , b , c , and d in the generic function definitions $f(x) = a \cdot \sin(b(x + c)) + d$ and $g(x) = a \cdot \cos(b(x + c)) + d$.

Excerpt 73

- 1 David: So basically the gist of what we want to do when we talk about these transformations and we talk about these properties is we want to be able to take these cyclic patterns that has these wavelengths and we want to be able to manipulate a sine or cosine to give us those repeating patterns. So changing the amplitude, vertically shifting it, changing the period, changing the phase shift, so changing where we start, is going to make the difference. Okay so that's basically what we need to make sure that we get out of Investigation 6.

To achieve the first and second of the instructional goals David communicated during Pre-Lesson Interview 9, he assigned students a worksheet to complete in groups during the last 30 minutes of Lesson 15 that required them to construct graphs of transformed periodic functions. David discussed the solutions to this worksheet during Lesson 16. For instance, while explaining how to construct the graph of the function $h(x) = -\sin(3x) + 2.5$, David identified the period, amplitude, phase shift, and vertical shift, and used this information to sketch a graph of h . David similarly demonstrated how to construct the graphs of $h(x) = 0.7\sin(x - \pi/5) - 1$, $f(x) = \cos(x/3 + 5\pi/3) + 2$, and $h(x) = 10\cos(\pi x) + 2$. David's explanations did not emphasize or reiterate the meaning of the input and output quantities of the sine and cosine functions. These were simply activities of graphing transformed sine and cosine functions.

In Lesson 19, during a discussion of inverse trigonometric functions, David asked students to solve the equation $2\sin(x) - 1 = 0$. A student proposed rewriting the equation as $\sin(x) = 1/2$. David then said, "What this is really saying is what is the angle that makes sine have a ratio of $1/2$?" A student claimed $\pi/6$ as a solution and provided the explanation in Line 1 of Excerpt 74.

Excerpt 74

- 1 Student: Well sine is like opposite over hypotenuse so you are saying that the opposite length is one and the hypotenuse is two and sine of the angle that gets you that is $\pi/6$."
- 2 David: Okay. So it's one of our special triangles—our 30-60-90 and our 45-45-90—and this is one of the values that we should have saw on our unit circle, right? Okay. So we get x is equal to $\pi/6$.

The student did not appear to conceptualize the solution to the equation $\sin(x) = \frac{1}{2}$ as the length of the subtended arc (in radius lengths) for which its terminus is $\frac{1}{2}$ radius lengths above the horizontal diameter of the circle centered at the vertex of the angle (in standard position). Instead, the student described the output of sine as “opposite over hypotenuse.” David’s endorsement of this suggestion in Line 2 supported students in conceptualizing the output of sine as an arithmetic operation.

Also during Lesson 19, David asked his students to explain how they might solve the equation $\sin(x) = -x$. Interestingly, David graphed $y = \sin(x)$ and $y = -x$ in his calculator and used the features of his TI-83 to determine the x -coordinate of the intersection. David did not support his students in interpreting the equation $\sin(x) = -x$ in the following way: For what subtended arc length is the vertical distance that the terminus of the subtended arc above the horizontal diameter of the circle centered at the vertex of an angle (in standard position) equal to the length of the subtended arc? Therefore, as with the previous example, David missed an opportunity to support his students in conceptualizing the inputs and outputs of the sine function quantitatively.

Theme 2: Quantitative Way of Understanding the Outputs of Sine and Cosine

On nine occasions during Pre-lesson Interviews 3-5, David communicated his intention to support students’ understanding of the outputs of sine and cosine as respective vertical and horizontal distances measured in units of the radius of the circle centered at an angle’s vertex. David’s instruction reflected this intention; he emphasized this way of understanding the outputs of sine and cosine on 23 occasions throughout Lessons 7-19.

Pre-Lesson Interviews 3-5. During the third pre-lesson interview, I asked David to describe how he intends his students understand the graph of sine at the conclusion of Lesson 4. David explained that he would like his students to conceptualize the output values of sine as, “the vertical distance from the terminal point on a unit circle that is cut off by the angle in standard position on its terminal ray, ... the vertical distance from the axis.” While David’s description of the way of understanding his instruction seeks to promote appears not to allow for negative sine values (he described the output of sine as a vertical distance “*from* the terminal point” and “*from* the axis”), his response did reveal his intention to support students’ understanding of the output of sine as a vertical distance between the terminus of the subtended arc and the horizontal diameter of the circle centered at the angle’s vertex, what David called the “axis.”

In Pre-Lesson Interview 4, I asked David to describe how he wants students to think about the $\sqrt{2}/2$ in the equality $\sin(\pi/4) = \sqrt{2}/2$. David, aware that the decimal approximation of $\sqrt{2}/2$ is 0.707, explained, “When we get that 0.707, I want them to understand that we’re saying that we have 70 percent of a radius length in vertical distance above the horizontal axis to the terminal point for sine.” This response reveals with greater clarity the way of understanding the output of sine David intended his instruction to promote: The vertical distance that the terminus of the subtended arc is above the horizontal diameter of the circle centered at the vertex of an angle in standard position.

In Pre-Lesson Interview 5, during which David and I discussed his instructional intentions for Lessons 7-9, he explained that his primary lesson objective was to support students in seeing how points on the graph of sine and cosine relate “to the circle.”

Specifically, I asked David to describe how he wants students to conceptualize the relationship between an angle with a circle centered at its vertex and the graphical representations of sine and cosine. David explained that the graphs of the sine and cosine functions “are based on the position of a point on the circle, depending on whether it’s a vertical position or a horizontal position measured in radius lengths.” I later prompted David to describe how he wants his students to think about ordered pairs on the graph of the sine function. He explained, “The output value is either the vertical or horizontal distance from the [terminal] point to the axes, depending on whether we’re doing sine or cosine, measured in radius lengths.” David’s responses during Pre-Lesson Interviews 3-5 clearly reveal his intention to support students’ understanding of the outputs of sine and cosine as quantities—respective vertical and horizontal lengths measured in units of radius lengths.

Lessons 7-19. Consistent with his instructional intentions, David supported students’ understanding of the outputs of sine and cosine as quantities on several occasions throughout Lessons 7-19. I summarize a few illustrative examples in Table 55.

Table 55

Summary of Instances in which David Supported Students’ Understanding of the Outputs of Sine and Cosine as Quantities

David explained that the point $(0, 1)$ is on the graph of cosine because the terminus of the subtended arc is one radius length to the right of the vertical diameter of the circle centered at the angle’s vertex when the measure of the angle is zero radians. [Lesson 7]

David explained that the sine of an angle’s measure is the vertical distance that the terminus of the subtended arc is above the horizontal diameter of the circle centered at the vertex of the angle, measured in radius lengths. David then communicated that since this vertical distance is measured in radius lengths, the largest that this distance can be is one and the smallest is negative one. David further explains that measuring

this vertical distance in radius lengths makes the value of sine independent of the size of the circle centered at the angle's vertex. [Lesson 7]

David explained that Pathways defines cosine as $\cos(\theta) = x$ because x is measured in units of radius lengths. He went on to clarify, "Most books define it as $\cos(\theta) = x/r$. This book says, 'Just make sure you measure in terms of radius.' If you measure in terms of radius then this (x) is a fraction of radius. If you don't measure in terms of radius, then you have to make it a fraction of radius (*pointing to the equation $\cos(\theta) = x/r$*)." [Lesson 7]

David supported a student in interpreting the y -value of the coordinate $(2, -0.416)$ on the graph of cosine as the horizontal distance that the terminus of the subtended arc is to the right of the vertical diameter of the circle centered at the vertex of the angle. [Lesson 8]

David asked a group of students to explain the meaning of the coordinates of the point $(2.5, 0.598)$ on the graph of sine. He facilitated a conversation that supported students in understanding that the 0.598 represents the vertical distance that the terminal point is above the horizontal diameter of the circle measured in units of radius lengths. During this conversation David drew on the board the image in *Figure 67*. [Lesson 8]

David asked a student to evaluate $\cos(2)$ and explain the meaning of this value. The student claimed that $\cos(2)$ is equal to 0.416 and explained, "It's the vertical distance from the vertical diameter." David elaborated, "It's the vertical distance from the vertical diameter *measured in radius lengths*." [Lesson 9]

David explained that the ordered pair of the terminus of a subtended arc is given by $(\cos(\theta), \sin(\theta))$. David further described $\cos(\theta)$ as representing a horizontal length measured in units of radius lengths and $\sin(\theta)$ as representing a vertical length measured in units of radius lengths. [Lesson 9]

David explained that the y -coordinates of the points on the graph of sine represent vertical distances from the horizontal diameter of the circle centered at the vertex of an angle to the terminus of the subtended arc, measured in units of radius lengths. Similarly, David explained that the y -coordinates of the points on the graph of cosine represent horizontal distances from the vertical diameter of the circle centered at the vertex of an angle to the terminus of the subtended arc, measured in units of radius lengths. [Lesson 10]

David claimed that $\sin(x) = \cos(x)$ for $x = \pi/4$ and justified his claim by illustrating that when the measure of an angle is $\pi/4$, the vertical and horizontal distances that respectively correspond to the outputs of sine and cosine are equal. [Lesson 10]

David explained that $\sin(\pi/2 + 2k\pi)$ is constant for integer k because "the angle is in the same place so the vertical distance is going to be the same." [Lesson 16]

David discussed the sign of $\tan(x)$ for particular values of x by appealing to whether the vertical and horizontal distances that respectively correspond to $\sin(x)$ and $\cos(x)$ are positive or negative. [Lesson 18]

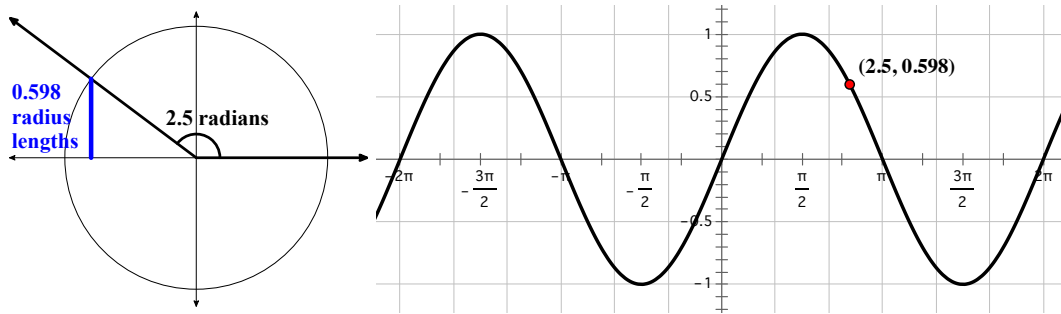


Figure 67. What do the coordinates $(2.5, 0.598)$ on the sine curve represent?

As the examples in Table 55 illustrate, on several occasions David supported students' understanding of the output of sine as the vertical distance that the terminus of the subtended arc is above the horizontal diameter of the circle centered at the vertex of the angle. David's instruction similarly emphasized the output of cosine as the horizontal distance that the terminal point is to the right of the vertical diameter of the circle centered at the angle's vertex. David also supported students in conceptualizing the graphs of sine and cosine as representations of the relationship between an angle's measure in radians, and these respective output quantities.

Theme 3: Covariation of the Input and Output Quantities of Sine and Cosine

During Pre-Lesson Interview 4, David communicated his intention to support students in describing the covariation of the input and output quantities of the sine and cosine functions by first attending to the rate at which the input and output quantities vary with respect to time, and then coordinating these rates of change. On the whole David's instruction during Lessons 6 and 7 reflected this intention. However, in response to a

student's utterance during Lesson 6, David described the covariation of the input and output quantities of the sine function by comparing successive uniform changes in the value of the output quantity for equal changes in the value of the input quantity.

Pre-Lesson Interview 4. During the fourth pre-lesson interview, David explained how he wanted his students to think about the non-linearity of the sine function: “The angle changes at a constant rate but yet the height changes at a varying rate. It goes from increasing to decreasing and sometimes it increases faster than others.” David’s explanation revealed his intention to support students’ understanding of the covariation of the input and output quantities of the sine function by coordinating the rate at which these quantities vary with respect to *time*. Specifically, David appeared to expect students to recognize that as the terminal ray of an angle in standard position varies at a constant rate with respect to time, the vertical distance that the terminal point is above the horizontal diameter of the circle does not vary at a constant rate, but instead “goes from increasing to decreasing.” It is noteworthy that David’s explanation emphasized the *direction of change* (MA 2) in the output quantity as the input quantity varied, as opposed to coordinating the *amount of change* (MA 3) or the *average rate of change* (MA 4) of the output quantity with respect to the input quantity.

Lessons 6 and 7. David’s instruction throughout Lessons 6 and 7 supported students in describing the covariation of the input and output quantities of the sine and cosine functions by coordinating the rate of change of these input and output quantities with respect to time. For example, while facilitating a conversation around Task 4(c) on Investigation 3 (see Table 56) during Lesson 6 using the applet shown in *Figure 68*, David asked his students to identify the varying quantities in an effort to direct their

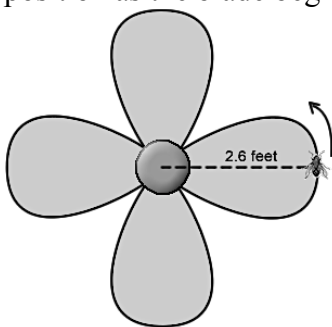
attention to those that respectively correspond to the input and output quantities of the sine function: *the angle swept out by the bug's path* and *the bug's vertical distance above the horizontal diameter of the fan*. After students identified these varying quantities, David explained that as the bug travels from the 3 o'clock position to the 12 o'clock position, "The height is increasing and the angle is increasing. ... So when we're talking about how they vary together, okay, they both are increasing. As the angle increases so does the height." In his explanation, David coordinated the direction of change of the bug's vertical distance above the horizontal diameter of the fan with an increase in the angle measure swept out by the bug's path. In doing so, David supported students in engaging in Mental Action 2 in Carlson et al.'s (2002) covariation framework.

Immediately following David's explanation, he asked the question in Line 1 of Excerpt 75.

Table 56

Investigation 3, Task 4(c) (Carlson, O'Bryan, & Joyner, 2013, p. 473-74)

Imagine a bug sitting on the end of a fan blade as the blade rotates in a counter-clockwise direction. The bug is 2.6 feet from the center of the fan and is located at the 3 o'clock position as the blade begins to turn.



Explain how the bug's vertical distance above the horizontal diameter changes as the bug travels from:

- the 3 o'clock position to the 12 o'clock position
- the 12 o'clock position to the 9 o'clock position
- the 9 o'clock position to the 6 o'clock position
- the 6 o'clock position to the 3 o'clock position

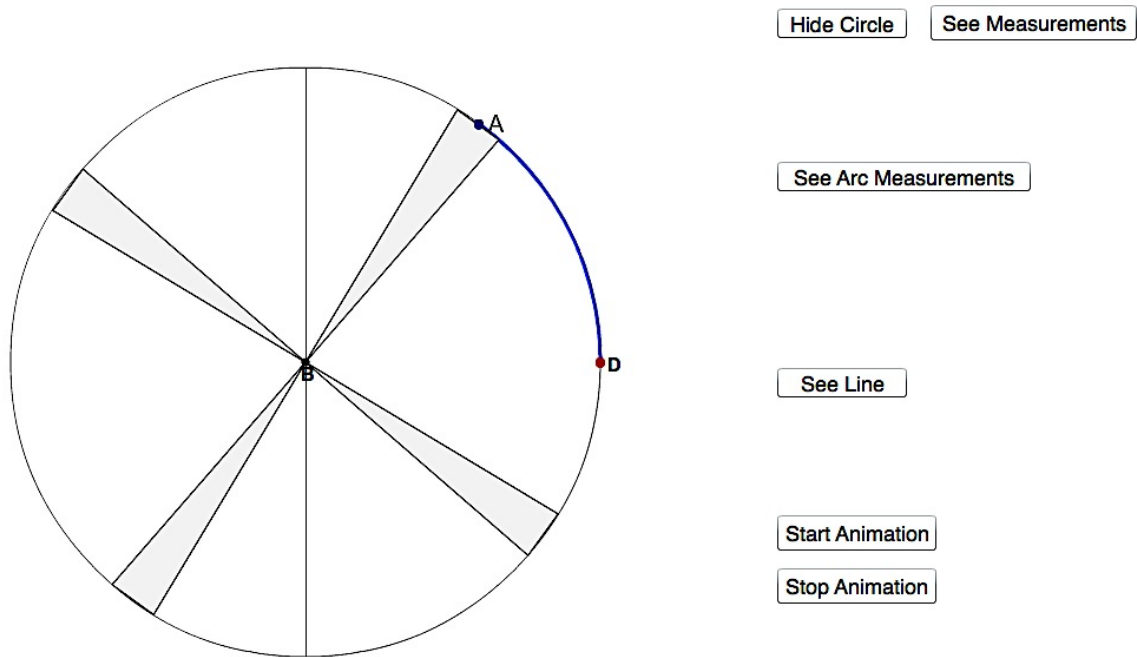


Figure 68. Module 8, Investigation 3 Fan Applet.

Excerpt 75

- 1 David: Is the height changing at a constant rate?
- 2 Student: No.
- 3 David: When is it going faster then?
- 4 Student: Like right at the beginning.
- 5 David: Why?
- 6 Student: Because it's curved. It's not straight.
- 7 David: So how does the curve affect it? Why is it that you said that the height is changing more over here at the beginning (*the 3 o'clock position*) than it is at the end (*the 12 o'clock position*)?
- 8 Student: Because it is more vertical at the beginning and it curves more on top.

9 David: Okay so not only is the height increasing but at the beginning it is increasing really fast but at the end it doesn't, it's still increasing but it is not increasing nearly as fast, right? Does that make sense? Here when the bug first starts off (*pointing to the 3 o'clock position on the circle in Figure 68*), it is going almost completely vertical. So its height is going up really quickly. Then when it gets up here (*pointing to the 12 o'clock position on the circle in Figure 68*), now it's moving more horizontal. So it's not really getting higher. At least it is getting a little bit higher but it's not getting higher really fast, right?

After acknowledging that the bug's vertical distance above the horizontal diameter of the fan increases as the angle measure swept out by the bug's path increases (as the bug travels from the 3 o'clock position to the 12 o'clock position), David prompted students to attend to the rate at which the bug's vertical distance above the horizontal diameter of the fan changed with respect to time (Line 1). A student suggested that the speed at which the bug varies vertically is larger when the bug is near the 3 o'clock position than it is when the bug is near the 12 o'clock position, and justified his claim by appealing to the curvature of the bug's path (Lines 6 and 8). David's elaboration of this justification in Line 9 supported students in comparing the rate at which the bug's vertical distance above the horizontal diameter of the fan changes with respect to time when the bug is near the 3 o'clock position with this rate of change when the bug is near the 12 o'clock position. David's explanation leveraged the fact that the bug's path is "almost completely vertical" when the bug is near the 3 o'clock position and "more

horizontal” when the bug is near the 12 o’clock position. David similarly supported students in describing the variation of the bug’s vertical distance above the horizontal diameter of the fan as the bug travels from the 12 o’clock position to the 9 o’clock position, from the 9 o’clock position to the 6 o’clock position, and from the 6 o’clock position to the 3 o’clock position. In general, David first asked students to identify the direction of change in the bug’s vertical distance above the horizontal diameter of the fan and then led them to describe changes in the speed at which the bug varies vertically.

David’s discussion of Task 4(c) on Investigation 3 during his 6th hour class followed a similar progression: He first prompted students to identify the direction of change in the bug’s vertical distance above the horizontal diameter of the fan as the bug travels from the 3 o’clock position to the 12 o’clock position. David then asked a series of leading questions that guided students to describe how the angle measure swept out by the bug’s path and the bug’s vertical distance above the horizontal diameter of the fan vary with respect to time. Specifically, David led students to observe that the angle measure swept out by the bug’s path varies at a constant rate with respect to time while the bug’s vertical distance above the horizontal diameter of the fan does not. David then asked his students, “When is the height increasing the fastest? ... When would the bug be accelerating vertically?” After hearing a number of faint or evasive responses, David called a student to the board to illustrate his thinking. The student drew an image like the one in *Figure 69*¹⁰³ and explained that the bug’s vertical distance above the horizontal diameter of the fan does not vary at a constant rate (with respect to the angle that

¹⁰³ The student did not use various colors and did not include the phrases “Equal changes in height” and “Unequal changes in angle measure.” The diagram in Figure 69 makes use of different colors for visual convenience and includes these phrases to reflect the student’s statements while explaining his drawing. Figure 69 also includes an image of the student’s drawing.

subtends the arc traversed by the bug) because equal changes in this vertical distance do not correspond to equal changes in the angle measure swept out by the bug's path. David then provided for the rest of the class a viable and coherent précis of the student's thinking.

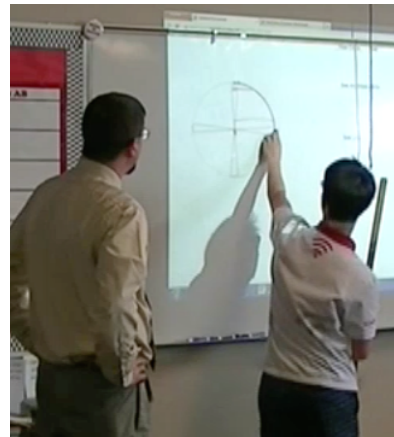
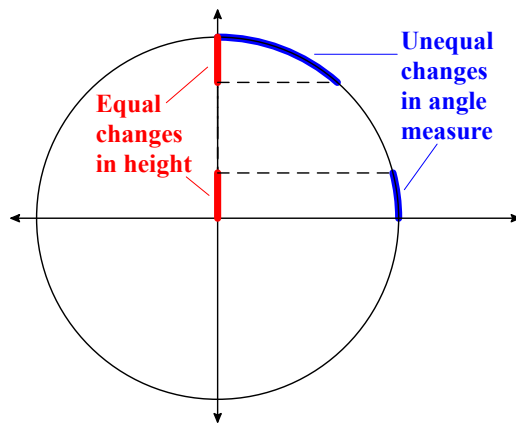


Figure 69. Equal changes in vertical distance do not correspond to equal changes in angle measure.

Immediately following David's summary, another student suggested examining changes in the bug's vertical distance above the horizontal diameter of the fan for equal changes in the angle measure swept out by the bug's path. David pursued this suggestion and drew on the board the diagram illustrated in *Figure 70*¹⁰⁴. Referring to his illustration, David explained that the changes in the bug's vertical distance above the horizontal diameter of the fan decrease for successive uniform changes in the angle measure swept out by the bug's path. Additionally, David leveraged this fact to justify the non-constant rate of change of the bug's vertical distance above the horizontal diameter of the fan with respect to the angle measure swept out by the fan blade.

¹⁰⁴ David did not use various colors. The diagram in Figure 70 makes use of different colors for visual convenience.

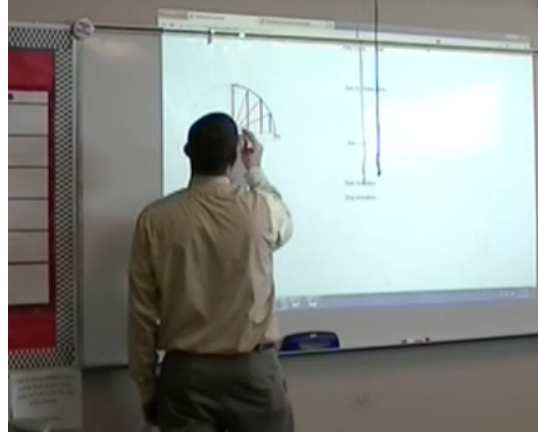
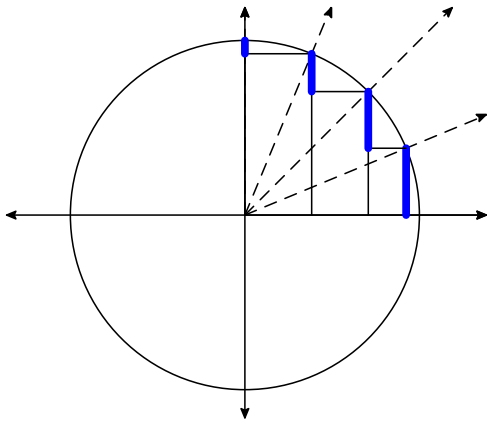


Figure 70. Decreasing changes in the bug's vertical distance.

By providing occasions for his students to create observable products of their reasoning, and by articulating such reasoning, David supported his students in understanding the non-constant rate of change of the bug's vertical distance above the horizontal diameter of the fan with respect to the angle measure swept out by the bug's path. David did so by coordinating changes in the bug's vertical distance above the horizontal diameter of the fan with uniform successive changes in the angle measure swept out by the bug's path—a way of understanding the covariation of the input and output quantities of the sine function that the *Pathways* curriculum seeks to promote. Interestingly, David did not support the same meaning while subsequently discussing how the bug's vertical distance above the horizontal diameter of the fan changes as the bug travels from the 12 o'clock position to the 9 o'clock position, from the 9 o'clock position to the 6 o'clock position, and from the 6 o'clock position to the 3 o'clock position; he reverted to discussing the speed of variation in the bug's vertical distance above the horizontal diameter of the fan with respect to time.

David subsequently engaged students in a number of tasks that the developers of the *Pathways* curriculum designed to more explicitly prompt students to characterize the

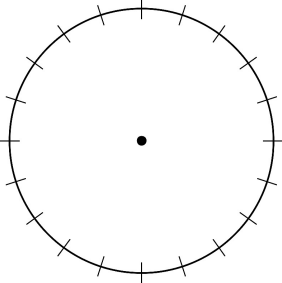
covariation of the angle measure swept out by the bug's path and the bug's vertical distance above the horizontal diameter of the fan by attending to how this vertical distance varies for uniform successive changes in the angle swept out by the fan blade. David's instruction of these tasks, however, did not support students in characterizing the covariation of the aforesaid quantities in this way. Consider David's instruction of Task 4(f) on Investigation 3 (see Table 57). David read this task aloud and immediately said, "We've already answered this. ... Sometimes it's faster. Sometimes it's slower." David then displayed, in succession, the images in *Figure 71* and explained, "There's a big vertical jump here (*pointing to the change in the bug's vertical distance above the horizontal diameter of the fan when the bug is near the 3 o'clock position*) but here there's a little vertical jump (*pointing to the change in the bug's vertical distance above the horizontal diameter of the fan when the bug is near the 12 o'clock position*). ... And then we can see that here (*referring to the graph on the right side of Figure 71*). My vertical change is a lot (*pointing to the change in the output quantity that corresponds to a change in input from zero to $\pi/8$*). My vertical change is not as much (*pointing to the change in the output quantity that corresponds to a change in input from $3\pi/8$ to $\pi/2$*)." David justified the non-constant rate of change of the bug's vertical distance above the horizontal diameter with respect to the angle measure swept out by the fan blade by pointing out that changes in the bug's vertical distance above the horizontal diameter of the fan vary. David's explanation did not support students in *coordinating* changes in the bug's vertical distance above the horizontal diameter of the fan with uniform successive changes in the angle measure swept out by the bug's path, although the developers of the *Pathways* curriculum designed the images in *Figure 71* to promote discussion that might

engender such coordination. Accordingly, David’s instruction did not support students in conceptualizing the concavity of the sine function by attending to changes in successive average rates of change of the output quantity with respect to the input quantity. David similarly missed two opportunities in Lesson 7 to support students’ understanding of the covariation of the input and output quantities of the cosine function by not coordinating changes in output quantity with uniform successive changes in the input quantity.

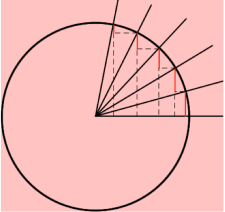
Table 57

Investigation 3, Task 4(f) (Carlson, O’Bryan, & Joyner, p. 474)

Does the bug’s vertical distance above the horizontal diameter change at a constant rate with respect to the angle measure swept out by the fan blade? Explain. (The given diagram, whose circumference is marked off in equal arc lengths, may help you make your case.)



Pathways Algebra II Implementing the Common Core Mathematics Standards I#3

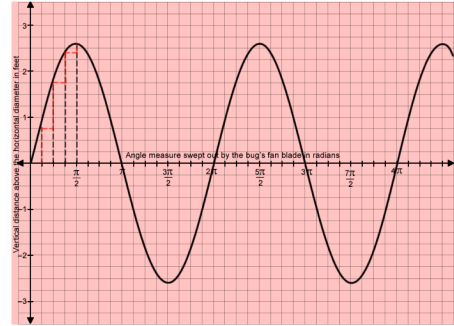


No. Uniform changes in the angle swept out by the fan blade do not correspond with equal changes in the bug’s vertical distance above the horizontal diameter.

For example, as the bug travels from the 3 o’clock position to the 12 o’clock position, for equal changes in the angle measure, the changes in the vertical distance will decrease (but will still be positive). See next slide.

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Figure 71. Images supporting students’ understanding of Task 4(f) on Investigation 3.

Theme 4: The Size of the Circle is Inconsequential to the Output Values and Graphical Representations of Sine and Cosine

During the fourth pre-lesson interview, David stated his intention to support students' understanding that the size of the circle centered at an angle's vertex does not affect the sine or cosine value of the angle measure since the input to these functions remains invariant as the size of the circle varies. David addressed the independence of sine and/or cosine values on the size of the circle centered at the angle's vertex on three occasions in his teaching. The first occurred during Lesson 6 wherein David justified that the size of the circle is immaterial to the value of sine by appealing the invariance of an arithmetic operation (as opposed to a quantitative operation). On the second and third occasions, both occurring during Lesson 7, David simply asserted that the size of the circle centered at the angle's vertex does not affect the value of sine, but did not provide much by way of explanation.

Pre-Lesson Interview 4. During the fourth pre-lesson interview, David communicated his intention to support students' understanding that the sine and cosine values of an angle's measure (in radians) is independent of the size of the circle centered at the angle's vertex. David did not justify this independence by appealing to the fact that the vertical and horizontal lengths that respectively correspond to the output quantities of sine and cosine remain invariant when measured in units of the radius of the circle centered at the vertex of the angle (i.e., David's justification did not rely on a quantitative conceptualization of sine and cosine). Instead, David explained that the input quantity to these functions (i.e., the length of the arc the angle subtends measured in units of radius lengths) does not vary with the size of the circle centered at the angle's vertex (assuming

an angle with a fixed amount of “openness”). Therefore, the sine and cosine values of this invariant input quantity are independent of the size of the circle centered at the vertex of the angle (see *Figure 72*).

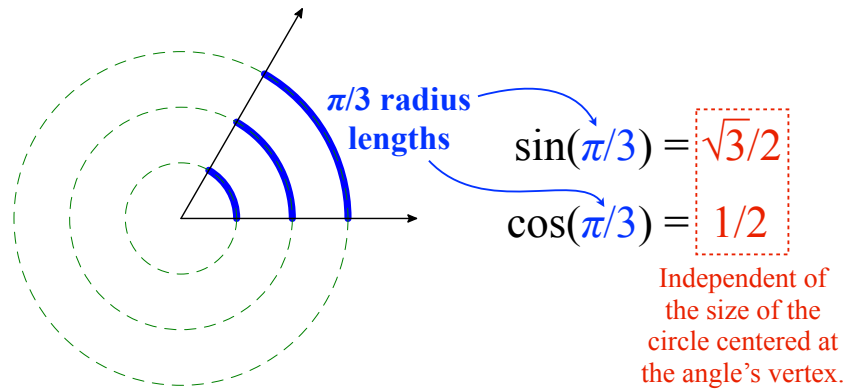


Figure 72. The size of the circle is immaterial to the output values of sine and cosine.

Lessons 6 and 7. During Lesson 6, David explained that the graphical representation of the sine function does not depend on the length of the radius of the circle centered at the angle’s vertex (*Excerpt 76*).

Excerpt 76

- 1 David: Because we’re measuring radius lengths to radius lengths, if I increase the radius, that is just going to increase the height. But when I do my division here I’m going to divide by the new radius so it’s going to end up canceling out so it’s going to give me the same graph. ... If you’re measuring with respect to radius length, if you increase the radius length it is just going to increase the height. It’s then going to increase the circumference and it’ll increase the arc length but when we made our graph we did it as a ratio of the actual height measured in feet divided by the radius length so you

still end up with this ratio of radius length to height. And that ratio doesn't change; it's proportional. So the ratio of the vertical heights compared to the radius lengths wouldn't change so then the graph doesn't change.

David's explanation in Excerpt 76 emphasized the invariance of an arithmetic operation, as opposed to a quantitative operation. That is, he explained that the ratio of the terminal point's vertical distance above the horizontal diameter of the circle centered at the vertex of the angle (measured in some standard linear unit) to the length of the radius of this circle (measured in the same standard linear unit) is constant. David's explanation did not support students' understanding that the value of $\sin(\theta)$ is independent of the size of the circle centered at the vertex of an angle with a measure of θ radians because the terminus of any subtended arcs is the same number of radius lengths above the horizontal diameter of the circle containing the subtended arc.

On two occasions during Lesson 7, David claimed that the size of the circle centered at the vertex of an angle does not affect the sine value of the angle's measure, but offered no explanation. For example, David made the statement in Excerpt 77 while discussing the definition of the sine function.

Excerpt 77

- 1 David: If we change the size of the circle, when we measured in terms of radius lengths did it change our answers?
- 2 Students: No.
- 3 David: No. But when we changed the radius length and we were measuring in terms of feet or centimeters or whatever, then it did

change our answers. So since we want to make it so it works for any circle, if we measure in terms of radius lengths then I know it always works and always gives me that answer. ... We want it to be consistent. I want the sine of π to always be the same thing. I don't want to say, 'Well it depends on the circle.' No. Sine of π should always be the same thing. So make it so that it's always the same thing. That's why we do the radius lengths.

David's question in Line 1 supported students in simply recalling that when they constructed graphs of the relationship between the bug's vertical distance above the horizontal diameter of the fan (in feet) and the angle measure swept out by the bug's path (in radians) in the previous lesson (see Table 56 for context), the radius of the fan affected the graphical representation of this relationship. Conversely, David's remark in Line 3 prompted students to remember that when they constructed graphs of the relationship between the bug's vertical distance above the horizontal diameter of the fan (in radius lengths) and the angle measure swept out by the bug's path (in radians), the radius of the fan was inconsequential to the graphical representation of this relationship. David's comments did not, however, support his students' understanding of *why* the size of the circle is immaterial to the output values, and therefore graphical representation, of the sine function.

It is worth noting that in Line 3 David chose to discuss $\sin(\pi)$ to illustrate the independence of the radius of the circle centered at the angle's vertex on the sine value of the angle's measure. This was an interesting instructional choice considering the terminus of the arc an angle with a measure of π radians subtends is zero inches, centimeters, yards,

light-years, and radius lengths above the horizontal diameter of the circle centered at the angle's vertex; the unit with which one measures the vertical distance that the terminus of the subtended arc is above the horizontal diameter of the circle is inconsequential when the angle has a measure of π radians. Additionally, David's proposal that one should measure the vertical distance that corresponds to the output quantity of sine in units of radius lengths (as opposed to a standard linear unit) so as to make the sine value of a particular angle measure independent of the size of the circle centered at the angle's vertex implied that the unit with which one measures the output of sine is subject to choice. David therefore did not support his students in understanding that if one measures the vertical distance that corresponds to the output quantity of sine in any unit other than radius lengths, the measure of this vertical distance is not a sine value.

Summary of David's Enacted Knowledge of the Output Quantities and Graphical Representations of Sine and Cosine Functions

Through my analysis of the pre-lesson interviews and David's classroom teaching, I identified four themes of David's enacted knowledge of the output quantities and graphical representations of sine and cosine functions. First, David frequently emphasized a non-quantitative way of understanding the outputs of sine and cosine, often defining these outputs as ratios of lengths or as respective y - or x -coordinates of points on a unit circle centered at the origin of a coordinate plane. Second, and conflictingly, David supported his students in conceptualizing the outputs of sine and cosine as respective vertical and horizontal lengths measured in units of the radius of the circle centered at an angle's vertex. Third, David often described the covariation of the input and output quantities of sine and cosine by assuming the input quantity varies at a constant rate with

respect to time, and then attending to the rate at which the output quantity varies with respect to time. Finally, David supported students' understanding that the size of the circle centered at the vertex of an angle is immaterial to the output values and graphical representations of sine and cosine. As with David's instruction of angle measure, the ways of understanding the output quantities and graphical representations of sine and cosine David promoted in his teaching generally resembled the meanings he expressed the intention to convey in the context of the pre-lesson interviews.

David revealed during Pre-Lesson Interviews 3-5 his preference for understanding the outputs of sine and cosine as ratios of lengths, as opposed to respective vertical and horizontal distances measured in units of the radius of the circle centered at the vertex of an angle. David's instruction in Lessons 4-9 was consistent with this intention; he supported students' understanding of the outputs of sine and cosine as arithmetic operations. Moreover, in Pre-Lesson Interview 6, David revealed that he wanted students to be able to recite sine and cosine values of integral multiples of $\pi/4$ and $\pi/6$. David's instruction during Lessons 10 and 11 reflected this aspiration. He first led students through the task of identifying the x - and y -coordinates of the terminus of an arc that an angle with a measure of integral multiples of $\pi/4$ and $\pi/6$ subtends; that is, he led students through the task of "filling out the unit circle" (see *Figure 66*). David then supported his students in memorizing the sine and cosine values of particular angle measures by repetitiously asking them recite y - or x -coordinates of points on the unit circle that correspond to angle measures that are integral multiples of $\pi/4$ and $\pi/6$. Obviously in doing so, David promoted the meaning of sine and cosine values as the y - and x -coordinates of points on the unit circle respectively.

David communicated three instructional goals in Pre-Lesson Interview 9: (1) to support students' ability to identify and correctly name transformations of periodic functions provided either a function definition or a graphical representation; (2) to graph a transformed periodic function provided a function definition; and (3) to write the function definition of a transformed periodic function provided either a graph or a situation. David's teaching attended to all three instructional goals. Specifically, to accomplish the first two instructional goals, David assigned students a worksheet during Lesson 15 that required them to construct graphs of transformed periodic functions. Students were able to successfully complete the worksheet without having to attend to the quantities that the input and outputs of sine and cosine represent. To realize the third instructional goal, David worked a number of examples during Lesson 13 that illustrated how to construct function definitions to match graphical representations or to model periodic phenomena by manipulating the values of a , b , c , and d in the generic function definitions $f(x) = a \cdot \sin(b(x + c)) + d$ and $g(x) = a \cdot \cos(b(x + c)) + d$. David's discussion of these examples emphasized how one must change the values of the constants a , b , c , and d to move the graph of the parent sine or cosine function, but did not emphasize or reiterate the meaning of the input and output quantities of the sine and cosine functions. This lack of emphasis on quantitative reasoning is consistent with David's remarks in Pre-Lesson Interview 10, wherein he revealed that he did not recognize the conceptual affordances of reasoning quantitatively about the inputs and outputs of the sine and cosine functions on one's ability to apply these functions to model periodic phenomena.

Although David frequently supported students' understanding of the outputs of sine and cosine as ratios of lengths or as respective y - or x -coordinates of points on a unit

circle centered at the origin of a coordinate plane, he also supported students in conceptualizing sine and cosine values as measures of vertical and horizontal lengths respectively. On nine occasions during Pre-lesson Interviews 3-5, David communicated his intention to support students' understanding of the outputs of sine and cosine as respective vertical and horizontal distances measured in units of the radius of the circle centered at an angle's vertex. David's instruction throughout Lessons 7-19 reflected this intention on 23 occasions. David also supported students in conceptualizing the graphs of sine and cosine as representations of the relationship between an angle's measure in radians, and the vertical and horizontal distances that respectively correspond to the output quantities of the sine and cosine functions. Although David's instruction emphasized two ways of understanding the outputs of sine and cosine—as ratios of lengths and as respective horizontal and vertical distances measured in units of the radius of the circle centered at the angle's vertex—he did not support students in conceptualizing the former way of understanding as equivalent to the latter because he did not treat the ratios of lengths he claimed defined the outputs of sine and cosine as quantitative operations, but rather as arithmetic operations.

David also provided opportunities for students to describe the covariation of the input and output quantities of the sine and cosine functions. Consistent with the instructional intentions he revealed during Pre-Lesson Interview 4, David's teaching during Lessons 6 and 7 supported students in describing the covariation of the input and output quantities of the sine and cosine functions by first assuming that the input quantity varies at a constant rate with respect to time, and then attending to the rate at which the output quantity varies with respect to time. However, as a result of pursuing a student's

suggestion in Lesson 7, David supported students' understanding of the covariation of the input and output quantities of the sine function by attending to how the output quantity varies for uniform successive changes in the input quantity. David subsequently missed several opportunities to emphasize this way of understanding.

Finally, David supported students in knowing that the size of the circle centered at the vertex of an angle is inconsequential to the output values and graphical representations of sine and cosine. During the fourth pre-lesson interview, David explained that the sine and cosine values do not depend on the size of the circle centered at the angle's vertex because the input to these functions (i.e., the measure of the angle in radians) is independent of the size of this circle. David did not support this way of understanding in his teaching but instead justified that the size of the circle is immaterial to the value of sine by appealing to the invariance of an arithmetic operation (as opposed to a quantitative operation). On two other occasions in his teaching, David simply asserted that the size of the circle centered at the angle's vertex does not affect the value of sine, but did not justify this proposition.

Comparison and Contrast of David's Knowledge of the Output Quantities and Graphical Representations of Sine and Cosine Functions with His Enacted Knowledge

In this subsection, I compare and contrast the ways of understanding the output quantities and graphical representations of sine and cosine functions that David revealed in the series of TBCIs with those he supported in the context of instruction. The format and organization of this subsection parallels that from the concluding subsection of the Angle Measure section of this chapter wherein I compared and contrasted David's

knowledge of angle measure with his enacted knowledge of angle measure. As with angle measure, David generally engaged his students in experiences that supported their constructing the ways of understanding he demonstrated during the series of TBCIs, with a few notable exceptions. David therefore did not appear to intentionally compromise the quality of his enacted knowledge of the outputs and graphical representation of sine and cosine in response to his appraisal of instructional constraints.

Sine and cosine values as ratios of lengths. On a number of occasions throughout the series of TBCIs, David demonstrated his way of understanding sine and cosine values as ratios of lengths (see *Figure 40*). Specifically, David claimed that sine (cosine) values represent the ratio of the vertical (horizontal) distance that the terminus of the subtended arc is above (to the right of) the horizontal (vertical) diameter of the circle centered at the angle's vertex to the radius of this circle. For instance, during the Initial Clinical Interview, David did not appear to conceptualize as quantitative operations the ratios of lengths he claimed represent the respective outputs of the sine and cosine functions. Similarly, during TBCI 4, David defined the sine value of an angle whose vertex is positioned at the origin of a coordinate plane as the ratio of the y -coordinate of the terminal point to the radius of the circle containing that terminal point and the cosine value as the ratio of the x -coordinate of the terminal point to this radius (see Excerpt 30). My subsequent questioning revealed that David interpreted these ratios as operations on the values of two quantities without understanding the result as representing the value of a quantity itself (see Excerpt 31 and Excerpt 32). Even after having completed an extensive instructional sequence on the outputs of sine and cosine—during which he repeatedly demonstrated his understanding of the outputs of sine and cosine as

quantities—in TBCI 5 David claimed it incorrect to think of sine values as representing vertical distances (see Excerpt 42). David reiterated that sine is a ratio, not a distance or a length.

In several pre-lesson interviews David expressed his intention to support—and actually supported in his instruction—students’ understanding of sine and cosine values as ratios of lengths, and therefore as arithmetic operations. In particular, during Pre-Lesson Interviews 3 and 4, David conveyed his preference for students’ understanding of sine and cosine outputs as ratios of lengths. In the fourth pre-lesson interview, David clearly communicated his desire for students to interpret the symbol “sin” as a cue to perform an arithmetic operation (see Excerpt 69). David’s instruction was consistent with these intentions as he repeatedly engaged students in experiences that supported their understanding of sine and cosine as ratios of lengths. David therefore supported students in constructing a way of understanding the outputs of sine and cosine that he continually demonstrated throughout the series of TBCIs.

Outputs of sine and cosine as quantities. In addition to revealing his understanding of sine and cosine outputs as arithmetic operations, on several occasions throughout the series of TBCIs David demonstrated a way of understanding sine and cosine values as measures of vertical and horizontal distances respectively. First, David repeatedly engaged in the sequence of actions involved in determining sine and cosine values provided an angle measure and, through reflecting abstraction, constructed an internalized representation of these actions at the reflected level that he organized into a scheme to which he assimilated novel tasks involving the symbolic notation of sine and cosine. In particular, David understood “ $\sin(\theta)$ ” as a representation of the vertical

distance that the terminus of a subtended arc is above the horizontal diameter of a circle centered at the vertex of an angle in standard position when the subtended arc is θ times as long as the radius of this circle (see Excerpt 38). Similarly, David conceptualized “ $\cos(\theta)$ ” as a representation of the horizontal distance that the terminus of a subtended arc is to the right of the vertical diameter of a circle centered at the vertex of an angle in standard position when the subtended arc is θ times as long as the radius of this circle. David also represented symbolically a number of statements phrased in the context of a situation involving quantities that correspond to the input and outputs of the sine and cosine functions (see Excerpt 39). Furthermore, David drew the terminal ray of two angles that respectively satisfy a condition of the form $\sin(\theta) = j$ or $\cos(\theta) = k$ for $0 \leq j, k \leq 1$ (see Excerpt 40). All of these instances from the series of TBCIs provide rather compelling evidence that David conceptualized sine and cosine values as measures of quantities—respective vertical and horizontal distances measured in units of the radius of the circle centered at the angle’s vertex.

David’s instruction during Lessons 7-19 reflected the nine occasions in Pre-Lesson Interviews 3-5 in which he articulated his intention to support students’ understanding of sine and cosine outputs as quantities (see Table 55 for a summary of the instances from David’s instruction in which he supported a quantitative way of understanding the outputs of sine and cosine). Specifically, David supported students’ understanding of the output of sine (cosine) as the vertical (horizontal) distance that the terminus of the subtended arc is above (to the right of) the horizontal (vertical) diameter of the circle centered at the vertex of the angle, measured in radius lengths. Hence, moments of David’s instruction emphasized a way of understanding the outputs of sine

and cosine that he convincingly exhibited on many occasions throughout the series of TBCIs.

Graphs of sine and cosine as representations of the covariational relationship between input and output quantities. Several of David's actions and utterances throughout the series of TBCIs suggest that he conceptualized the graphs of sine and cosine functions as representations of the covariational relationship between input and output quantities; he did not appear to conceptualize the graphical representations of these functions as simply shapes or images. For example, in TBCI 6 David constructed a graph of the sine function by watching a terminal ray rotating counter-clockwise. He labeled the output axis, "Vertical distance from the horizontal axis in radius lengths" and explained that the values on this axis represent the distance that the terminus of the subtended arc is above the horizontal diameter of the circle centered at the angle's vertex measured in units of the radius of this circle (see *Figure 46*). Also during TBCI 6, I identified three points on the graphs of sine and cosine and asked David to draw pictures to display what the coordinates of these respective points represent. David's ability to correctly do so demonstrated his understanding that each point on the graph of sine (cosine) represents a particular correspondence between the measure of an angle in radians (i.e., the length of the arc an angle subtends measured in units of radius lengths) and the vertical (horizontal) distance that the terminus of the subtended arc is above (to the right of) the horizontal (vertical) diameter of the circle centered at the vertex of the angle, measured in radius lengths.

David enacted his knowledge of the graphs of sine and cosine as representations of the covariational relationship between input and output quantities on a number of

occasions throughout his teaching of Module 8. In Pre-Lesson Interview 5, for example, David disclosed that his primary lesson objective was to support students in seeing how points on the graphs of sine and cosine relate “to the circle.” David explained, “The output value is either the vertical or horizontal distance from the [terminal] point to the axes, depending on whether we’re doing sine or cosine, measured in radius lengths.” David’s instruction reflected this objective. During Lesson 8 he supported students in interpreting the y -value of the coordinate $(2, -0.416)$ on the graph of cosine as the horizontal distance that the terminus of the subtended arc is to the right of the vertical diameter of the circle centered at the vertex of the angle. Also during Lesson 8, David facilitated a conversation that supported students’ understanding of the y -coordinate of the point $(2.5, 0.598)$ on the graph of sine as the vertical distance that the terminus of a subtended arc is above the horizontal diameter of the circle measured in units of radius of this circle. David supported similar meanings in Lessons 7 and 10.

Non-quantitative way of understanding the graphical representations of sine and cosine. In addition to having supported students’ understanding of the graphs of sine and cosine as representations of the relationship between quantities that covary, David’s instruction promoted a non-quantitative way of understanding the graphical representations of sine and cosine. During Pre-Lesson Interview 9, David revealed his intention to support students’ ability to identify and correctly name transformations of periodic functions provided either a function definition or a graphical representation, to graph a transformed periodic function provided a function definition, and to write the function definition of a transformed periodic function provided either a graph or a contextual situation. David’s instruction did indeed support students’ in becoming fluent

at demonstrating these behaviors. In Lesson 13, for instance, David worked a number of examples of constructing function definitions to match graphical representations by manipulating the values of a , b , c , and d in the generic function definitions $f(x) = a \cdot \sin(b(x + c)) + d$ and $g(x) = a \cdot \cos(b(x + c)) + d$. David also assigned students a worksheet during Lesson 15 that engaged them in repetitiously constructing graphs of transformed periodic functions.

During these moments of David's instruction, he treated the graphs of sine and cosine as objects to be manipulated, not as representations of the covariational relationship between input and output quantities. David discussed transformations of sine and cosine functions as a kind of moving, stretching, and bending of shapes that was not justified by attending to the quantities these shapes expressed the relationship between. That David did not demonstrate such ways of understanding in the series of TBCIs may not be surprising since the tasks I designed did not provide an occasion for David to reveal his way of understanding transformations of trigonometric functions, with the exception of period.

Covariation of input and output quantities and concavity of sine and cosine functions. David characterized the covariation of the input and output quantities of the sine and cosine functions, and justified the concavity of the graphs of these functions, on several occasions during the series of TBCIs and throughout his teaching of Module 8. These characterizations and justifications, however, were not always consistent. During the Initial Clinical Interview, David described the covariation of the input and output quantities of the sine and cosine functions by attending to the direction of change in the outputs of these respective functions as the input varied, thereby engaging in Mental

Action 2 from Carlson et al.'s (2002) covariation framework (see Excerpt 28). Also during the Initial Clinical Interview, David characterized the covariation of the input and output quantities of the sine function by coordinating changes in the output quantity for equal changes in the input quantity of one degree, thus engaging in Mental Action 3 (Carlson et al., 2002) (see Excerpt 29). Moreover, in TBCI 6 David justified the concavity of the sine function on the interval $[0, \pi/2]$ by attending to how the output quantity of sine changed for successive uniform changes in the input quantity (MA 3). In particular, David noticed that as the input quantity changed by equal amounts from zero to $\pi/2$ radians, the output quantity of sine changed by decreasing amounts (see Excerpt 46 and *Figure 52*). David similarly justified the concavity of the graph of cosine on the interval $[0, \pi/2]$ by describing how the horizontal distance of a terminal point to the right of the vertical diameter of the circle centered at the angle's vertex changed for equal changes in the input quantity. Also, in the context of discussing the didactic object in Table 49 during TBCI 7, David rationalized the non-constant rate of change of the sine function by attending to how the vertical distance that the terminus of the subtended arc is above the horizontal diameter of the circle changes as the angle measure varies by successive intervals of $\pi/6$ radians from zero to $\pi/2$ radians (see Excerpt 47). David analogously rationalized the non-constant rate of change of the cosine function. That David justified the concavity of the sine and cosine functions on the interval $[0, \pi/2]$ by attending to how the outputs of these functions change for successive uniform changes in angle measure suggests that he engaged in at least Mental Action 3 (Carlson et al., 2002).

While David often engaged in Mental Action 3 while describing the covariation of the input and output quantities of the sine and cosine functions, and while justifying

the concavity of the graphs of these functions, his instruction far less frequently supported students in engaging in such mental actions. Instead, David encouraged students to describe the covariation of the input and output quantities of the sine and cosine functions by attending to the rate at which the output quantity varies with respect to time (assuming the input quantity varies at a constant rate with respect to time). Specifically, David supported students in recognizing that as the terminal ray of an angle in standard position varies at a constant rate with respect to time, the vertical distance that the terminal point is above the horizontal diameter of the circle does not vary at a constant rate (see Excerpt 75). With the exception of responding to a student's suggestion to examine how the output quantity of sine varies for uniform successive changes in the input quantity, David's instruction did not support students in characterizing the covariation of the input and output quantities of the sine and cosine functions by coordinating the *amount of change* (MA 3) or the *average rate of change* (MA 4) of the output quantity with respect to the input quantity.

The size of the circle centered at the angle's vertex is immaterial to the values of sine and cosine. On several occasions throughout the series of TBCIs, David revealed his understanding that the size of the circle centered at the angle's vertex does not affect the sine and cosine values of the angle because the vertical and horizontal distances that respectively correspond to these output values are measured in units of the radius of the circle centered at the vertex of the angle. For instance, in TBCI 5, after having correctly solved two tasks that prompted him to represent the values in equations of the form $\sin(a) = b$ and $\cos(c) = d$ (for constants a , b , c , and d), and after having correctly drawn the terminal ray of two angles that respectively satisfy a condition of the form $\sin(\theta) = j$ or

$\cos(\theta) = k$ (for constants j and k), I asked David if his answers to such tasks would change if the circles provided in the task statement were twice as large. David explained that his answers would not change because the input and output quantities of sine and cosine are measured in radius lengths, and therefore when the size of the circle centered at the vertex of the angle changes, these quantities are “still the same portion of a radius length.” David also explained that the size of the circle centered at an angle’s vertex does not affect the graphical representations of the sine and cosine functions. After having constructed a graph to represent the covariation of an angle’s measure in radians and the vertical distance that the terminus of the subtended arc is above the horizontal diameter of the circle centered at the vertex of the angle in TBCI 6, David argued that if the size of this circle changed, the graph he produced would still represent the covariational relationship of the aforesaid quantities because the subtended arc length and vertical distance that respectively correspond to the input and output quantities of the sine function are measured in units of the radius lengths.

David demonstrated throughout the series of TBCIs his understanding that the actual length of the radius of the circle centered at an angle’s vertex is inconsequential to the output values of sine and cosine because the quantities that correspond to these values are measured in units of the radius of the circle centered at the vertex of the angle, which *always* has a measure of one radius length. The mathematical knowledge David enacted in the context of planning lessons and in the context of teaching, however, did not reflect this way of understanding. During the fourth pre-lesson interview, David discussed his intention to support students’ understanding that the size of the circle centered at an angle’s vertex does not affect the sine or cosine value of the angle measure since the

value of the *input* quantity to these functions remains invariant as the size of the circle varies (assuming an angle with a fixed amount of “openness”). During Lesson 6, David appealed to the invariance of an arithmetic operation to explain that the graphical representation of the sine function does not depend on the length of the radius of the circle centered at the angle’s vertex. In particular, he argued that the ratio of the terminal point’s vertical distance above the horizontal diameter of the circle centered at the vertex of the angle to the length of the radius of this circle (both measured in some standard linear unit) is constant for any circle centered at the angle’s vertex. On two occasions during Lesson 7, David simply asserted, but did not explain, that the size of the circle centered at the angle’s vertex does not affect the value of sine. David’s instruction therefore did not justify the independence of the size of the circle centered at the vertex of an angle on the output values and graphical representations of sine and cosine by appealing to the fact that the vertical and horizontal lengths that respectively correspond to the output quantities of sine and cosine remain invariant when measured in units of the radius of this circle—a way of understanding David demonstrated multiple times throughout the series of TBCIs.

Conflicting ways of understanding. As with angle measure, there were several instances in which David supported meanings in his instruction relative to the outputs and graphical representations of sine and cosine that were incompatible with the ways of understanding he demonstrated in the series of TBCIs. While I have already alluded to a few, two in particular are worth reiterating because they are suggestive of an important characteristic of David’s mathematical knowledge that compromised the quality of the

mathematical knowledge he enacted in the context of classroom practice. I devote the concluding section of this chapter to discussing this characteristic.

As previously mentioned, David convincingly demonstrated in the series of TBCIs his way of understanding values of sine (cosine) as a measure of the vertical (horizontal) distance that the terminus of the subtended arc is above (to the right of) the horizontal (vertical) diameter of the circle centered at the vertex of the angle in units of radius lengths. David also repeatedly supported his students in constructing the outputs of sine and cosine as quantities. However, and contradictorily, during TBCI 5 David asserted that it is incorrect to think of sine values as representing the measure of vertical distances, and reemphasized that the output of sine is a ratio, not a distance or length (see Excerpt 42 and Excerpt 43).

Additionally, in TBCI 4 David placed the terminus of a subtended arc to illustrate the solution to the equation $\sin(x) = 0.6$ by identifying the intersections of the circumference of a circle and the horizontal line that is 0.6 radius lengths above the horizontal diameter of this circle. David employed a similar strategy to approximate the solution to the equation $\sin(\theta) = 1/3$ (TBCI 5) and to estimate a possible input value of sine that produces an output value of 0.75 (TBCI 5). David also placed the terminus of a subtended arc to illustrate the solution to the equality $\sin(x) = x/2$ by interpreting the equation in the following way: Place the terminal point so that the vertical distance of this point above the horizontal diameter of the circle centered at the vertex of the angle is half as long as the length of the subtended arc (TBCI 5). All of these occasions compellingly reveal David's understanding of the output of sine as a quantity.

In contrast, in Lesson 19 David interpreted the task of solving the equation $\sin(x) = \frac{1}{2}$ as follows: What is the angle that makes sine have a *ratio* of $\frac{1}{2}$? A student proposed $\pi/6$ as a solution but did not appear to conceptualize this value as the length of an arc an angle in standard position subtends (measured in radius lengths) for which its terminus is $\frac{1}{2}$ of a radius above the horizontal diameter of the circle centered at the vertex of the angle. Instead, the student described the output of sine as “opposite over hypotenuse.” David’s validation of this justification encouraged students to conceptualize sine as an arithmetic operation. Also during Lesson 19, David asked his students to explain how they might solve the equation $\sin(x) = -x$, but did not support them in interpreting the equation in the following way: For what subtended arc length is the vertical distance that the terminus of the subtended arc above the horizontal diameter of the circle centered at the vertex of an angle (in standard position) equal to the length of the subtended arc? Instead, David graphed $y = \sin(x)$ and $y = -x$ in his calculator and used the features of his TI-83 to determine the x -coordinate of the intersection. As with David’s insistence in TBCI 5 that the output value of sine does *not* represent a vertical distance, but rather a ratio, his approach to solving the equations $\sin(x) = \frac{1}{2}$ and $\sin(x) = -x$ during Lesson 19 not only depart from, but plainly contradict understandings he repeatedly demonstrated in the series of TBCIs.¹⁰⁵

Period of Sine and Cosine Functions

The following themes emerged from my analysis of David’s teaching of the period of sine and cosine functions:

- (1) Non-quantitative way of understanding the period of sine and cosine.

¹⁰⁵ Admittedly, these are contradictions from *my* perspective.

- (2) Quantitative way of understanding the period of sine and cosine.
- (3) Just set the input variable and its coefficient equal to 2π and solve for the input variable.
- (4) Coefficient of the input variable as the speed of variation.

In what follows, I describe the ways of understanding David intended to convey in his teaching relative to each theme and provide evidence from David's instruction that illustrate each theme.

Theme 1: Non-Quantitative Way of Understanding the Period of Sine and Cosine

On several occasions in his teaching, David defined or spoke about the period of sine and cosine in ways that did not reference the input and output quantities of these functions. For instance, David described period as an arithmetic operation, as a duration of time for "it" to complete one full cycle, and as the amount the graph of sine and cosine must "slide over."

Pre-Lesson Interview 8. During the eighth pre-lesson interview, I asked David to describe what he hopes students will understand about period of sine and cosine functions at the conclusion of his instruction of Investigation 5. I provide David's reply in Excerpt 78.

Excerpt 78

- 1 David: I'm hoping that they understand that a normal period is 2π , and that's how long it takes to go through the cycle. When you have a coefficient in front of the input variable inside of a sine or cosine function, then that's going to cause the graph to horizontally stretch or shrink, um, and it's going to be the inverse of whatever

that, um, or sorry the reciprocal of whatever that value is. So, uh, because of that, um, you know they can find the period by taking the normal period and dividing it by whatever the coefficient is, um, you know. So I would like them to have that understanding.

David's description of the understanding he hopes his students will construct is less a characterization of how he intends students to *conceptualize* period and more a statement of fact and an identification of behavior. In other words, David described what he wants his students to *know* (the period of sine and cosine is 2π and the coefficient of the input variable causes the graph to stretch or shrink horizontally) as well as what he wants students to be able to *do* (compute period by dividing 2π by the coefficient of the input variable) instead of characterizing the mental activity involved in constructing the ways of understanding his instruction seeks to support. On several occasions throughout the pre-lesson interviews, David revealed his disposition to defining instructional objectives in terms of statements of fact and descriptions of behaviors. That he did so has significant implications for the main findings of this study, which I discuss at length at the conclusion of this chapter.

Lessons 6-15. David's instruction during Lessons 13 and 15 supported students in attaining the knowledge and enacting the behaviors he specified in Pre-Lesson Interview 8.¹⁰⁶ For instance, in Lesson 13 David described how the value of b in the function definition $f(x) = a \cdot \sin(b(x + c)) + d$ affects the graph of $g(x) = \sin(x)$. Specifically, David asked, "What does multiplying to the inputs do to functions? (*Pause*) Horizontally

¹⁰⁶ My phrasing of this sentence, although incompatible with constructivist epistemology, is intentional. David's instruction reflected the empiricist notion that knowledge is an entity that may be *attained* by a learner through sensory-experience.

stretches or shrinks them, right?” David went on to explain, “If it’s a number between one and zero it’s going to stretch it horizontally. If it’s a number bigger than one, it’s going to condense it horizontally. What’s the scale factor? (*Students respond*) One over b . The reciprocal of b .” David then stated that the value of b affects the period of the function f but did not explain why. David did, however, ask his students to recite the period of $y = \sin(2x)$, $y = \sin(3x)$, and $y = \sin(\pi x)$, and then engendered their pseudo-empirical abstraction by asking, “How do we find the period then?” to which several students replied, “ 2π divided by b .” David was satisfied with this response and neither explained nor ask students to justify the meaning of the division.

Similarly, in Lesson 15 David explained how to determine the period of the function $f(t) = 3.25\cos(12.06t) + 3.25$ (Excerpt 79).

Excerpt 79

- 1 David: This is the number being multiplied in front of the t (*points to the coefficient of the input variable in the function definition*). So that’s going to horizontally condense my graph. So since the period is normally 2π , I’m going to condense it by a factor of one over 12.06. So that’s how we get the 0.52. ... So that’s the period of the function.

In this and the previously mentioned instructional episode, David supported his students in knowing *that* the coefficient of the input variable horizontally stretches or shrinks the graph—thereby affecting the period—and also equipped them with a procedure for computing the period of a function of the form $f(x) = a \cdot \sin(b(x + c)) + d$.

David neither defined period nor justified the procedure for computing it by discussing the input and output quantities of the sine function.

David discussed period at other moments in his teaching that made faint or no reference to the input and output quantities of the sine or cosine functions. For instance, during Lesson 6 David asked, “Why is it that it’s 2π for it to repeat its pattern, in radians?” David answered his own question after a moment of students’ hesitation: “Because 2π is a whole circle! What happens when you get around the circle once? You go around again! What kind of numbers are you going to get? The exact same ones! That’s the reason why.” Moreover, David defined period on several occasions as being “how long it takes it to go through its entire cycle” (Lesson 12, 01:13), “how long it takes it to complete a full cycle” (Lesson 12, 22:11), and “how long it takes to go through its cycle” (Lesson 14, 44:26). These statements, rife with ill-defined pronouns, did not allow students to conceptualize the argument of sine and cosine as a representation for the value of an angle’s measure, and therefore did not support students’ understanding of period as the interval of input values over which the argument of sine or cosine varies by 2π .

David also described period as the amount a graph must “slide over” to lay on top of itself. During Lesson 7, David asked, “How far is it that I had to shift this part of the graph (*points to the graph of sine on the interval $[0, 2\pi]$*) to create this part of the graph (*points to the graph of sine on the interval $[2\pi, 4\pi]$*)? How far to go from this point (*points to a random point $(x, \sin(x))$ where $0 \leq x \leq 2\pi$*) to match it to this point (*points to the point $(x + 2\pi, \sin(x + 2\pi))$*)?” Students replied in concert, “ 2π !” David then explained that a function is periodic if one can slide the graph of the function horizontally to lie on top of itself, and that the period of the function is the amount the graph must slide.

Similarly, in Lesson 13 David read the text on the PowerPoint slide shown in *Figure 73* and asked, “So what does that say? (*Long pause*) ... what happens when we add to the inside like that? What does it do to the graph?” A student replied inaudibly and then David answered his question: “It shifts it to the left P , right? So basically we’re saying, ‘How can we slide it over enough so that it just lays on top of the other one.’ ” These remarks from Lessons 7 and 13 supported students’ understanding of period as the amount a sine or cosine graph must be translated horizontally to look the same as it did prior to the translation. Like David’s description of period as “how long it takes to complete its cycle,” this way of understanding does not involve attending to the input and output quantities of sine or cosine.

The image shows a PowerPoint slide with a blue header and footer. The header contains the text "Pathways Algebra II Implementing the Common Core Mathematics Standards" and "I#6". The main content is enclosed in a black-bordered box. At the top of this box is the title "Period" in bold. Below the title is a paragraph defining the period of a function: "A function has a period p if p is the smallest positive number such that $f(x) = f(x + p)$ for all x . Whenever the input quantity varies by p units, the output quantity completes one full cycles of values." Below this paragraph is a horizontal line, followed by an example: "Ex: The sine function has a period of 2π because for $f(\theta) = \sin(\theta)$ we have $f(\theta) = \sin(\theta + 2\pi)$, and 2π is the smallest positive number that makes this true." The footer of the slide includes a logo for "Rational Reasoning" on the left, the copyright notice "Copyright © 2013 Carlson, O'Bryan & Joyner" in the center, and the number "15" on the right.

Figure 73. Pathways’ definition of period.

Theme 2: Quantitative Way of Understanding the Period of Sine and Cosine

Although David did not reveal in the pre-lesson interviews his intention to support students in constructing a quantitative way of understanding period¹⁰⁷, he did so on one occasion during Lesson 12. David asked his students, “How did we define period?” After a few seconds of silence, a student tentatively responded, “One full revolution around the circle. (*Pause*) So the time it takes to get to the exact same point.” David, noticeably dissatisfied with the student’s response, clarified, “It’s the change in the input required for the output to go through one complete cycle ... It’s the *change* in the input.” David’s statement emphasized that period is the *change* in the input values over which the outputs vary through one full cycle of values. Having previously written three function definitions on the board— $g(t) = 2.3\sin(t)$; $h(t) = 2.3\sin(2t)$; and $j(t) = 2.3\sin(0.5t)$ —David asked his students to determine the period of each function. Students proposed 2π , π , and 4π seconds as the respective periods of the functions g , h , and j (note that the input quantity of these functions represented an amount of time elapsed in seconds). David then asked, “How are we figuring that out?” to which a student replied, “The whole thing equals 2π .” David pushed the student to articulate his reasoning and the student explained that he set the arguments of g , h , and j equal to 2π and solved for the independent variable t . Not satisfied with the student’s emphasis on his enacted procedure, David asked, “Why did you set the inside equal to 2π ?” The student provided a rather unintelligible response, so David persisted: “But again, why are we setting it equal to 2π ?” The student responded, “Because the parentheses still represents theta and 2π would be one period in radians.” David enthusiastically supported the student’s remark, and emphasized for the rest of the

¹⁰⁷ See my conceptual analysis in Chapter 5 for a description of what it means to understand period quantitatively.

class that the arguments of the functions g , h , and j represent the measure of an angle in radians. David then reminded students that the period of a function is the *change* in the input quantity over which the output quantity varies through one full cycle of values. He then explained that when one sets the argument of sine equal to 2π , the 2π represents a *change* in angle measure from zero radians, which is the change over which the outputs of sine vary through one full cycle of values. Thus, solving the equations, David argued, results in determining the change in t (from zero seconds) that results in the angle measure—represented by the argument of the function—changing by 2π .

In the eight minutes of instruction over which this conversation expired, David supported his students' understanding of period as the change in the input quantity over which the output quantity of sine varies through one full cycle of values. Moreover, David allowed his students to see that setting the argument of a sine function equal to 2π and solving for the input variable results in determining the interval over which the input quantity must vary to make the argument, which represents an angle's measure in radians, vary by 2π , which is the interval of angle measure over which the outputs of sine vary through one full cycle of values. In contrast to the previously-discussed moments of David's instruction in which he described period as an arithmetic operation, as a duration of time for "it" to complete one full cycle, and as the amount the graph of sine and cosine must "slide over," the way of understanding period David promoted during these eight minutes of Lesson 12 involved attention to the input and output quantities of the sine function.

Theme 3: Just Set the Input Variable and its Coefficient Equal to 2π and Solve for the Input Variable

While David supported students in constructing a quantitative way of understanding period on one occasion in Lesson 12, his instruction during this and subsequent lessons was more notable for his emphasis on equipping students with procedural knowledge to determine the period of functions of the form $f(x) = a \cdot \sin(bx + c) + d$ and $g(x) = a \cdot \cos(bx + c) + d$. Throughout Lessons 12-15, David supported students in knowing that determining the period of a sine or cosine function involves setting the term bx equal to 2π and solving for the input variable x , which amounts to dividing 2π by b , the coefficient of the input variable. This emphasis on equipping students with procedural knowledge reflected an instructional objective David communicated twice during Pre-Lesson Interview 8.

Pre-Lesson Interview 8. During the eighth pre-lesson interview, I asked David to explain how he wanted his student to think about what it means determine the period of the function $f(x) = 3\sin(1.7x + 2)$.

Excerpt 80

- 1 Michael: Suppose on the next unit test a student sees a question that says, ‘Determine the period of the following function: $f(x) = 3\sin(1.7x + 2)$.’ How do you want them to think about what it means to answer that question?
- 2 David: So I want them to think about, um, well the easy way to think about it is I can ignore the plus two that we had in there and I can ignore the lead coefficient in front of the sine. The only thing that

really made a difference was the coefficient of the input variable.

Um, and then we can go ahead and, um, set that equal to what I

know sine's normal period is, which is 2π .

3 Michael: Set what equal to 2π ?

4 David So we would set the $1.7x$ equal to 2π and then solve for x .

David's response suggests his intention to support students in thinking about what it means to determine the period of $f(x) = 3\sin(1.7x + 2)$ "the easy way," which in David's view was as a solution to the equation $1.7x = 2\pi$. David subsequently explained that he will have accomplished his lesson objective "if [students] are able to give me the answer on what the period is and realize that it's basically the coefficient of the input variable and you divide that by 2π to get the period. Then I'd be happy." Both this remark and David's responses in Excerpt 80 reveal his intention to equip students with a procedure for computing the period of a sine or cosine function with a linear argument. This intention was reflected in David's teaching during Lessons 12-15.

Lessons 12-15. On nine occasions throughout Lessons 12-15, David emphasized that to determine the period of functions of the form $f(x) = a \cdot \sin(bx) + c$ or $g(x) = a \cdot \cos(bx) + c$, one may either set the argument bx equal to 2π and solve for x , or simply divide 2π by b . I discuss two of these occasions that together are representative of the others.

During Lesson 12, David asked a student to share her answer to Task 8(c) on Investigation 5 (see Table 58). The student admitted that she attempted to solve the problem by referring to the method David employed to solve Task 4(c) (see Table 58) but was unsuccessful in doing so. David defined the function $h(t) = 2.3\sin(2t)$ as a solution to

Task 4(b) and then, to answer Task 4(c), set the argument of h equal to 2π and solved for t to determine the interval of time over which the ball completed one full revolution. Prior to asking the student to share her answer to Task 8(c), David wrote “ $k(t) = 2.3\sin(3.8t)$ ” as a solution to Task 8(b). After having noticed the student struggling to articulate a method for solving Task 8(c), David wrote the following on the board as a reminder of his solution to Task 4(c).

$$\begin{aligned}\theta &= 2t \\ 2\pi &= 2t \\ \frac{2\pi}{2} &= \\ \pi &= t\end{aligned}$$

David then said, “So that was back in Question 4. Now the question is how do you do it here (*Task 8(c)*)?” The student advised David to set θ equal to $3.8t$ and then suggested he replace θ with 2π and solve for t as he had done to answer Task 4(c). David wrote the following on the board and said, “Okay, so there is the period. 2π divided by 3.8 .”

$$\begin{aligned}\theta &= 3.8t \\ 2\pi &= 3.8t \\ \frac{2\pi}{3.8} &= t\end{aligned}$$

David concluded his discussion of Task 8(c) by explaining, “How long does it take to go completely around? Well I know that completely around is 2π in θ but I am not sure what it is in time. So we plug in for θ and solve for t .” David subsequently explained determining the period of the function $m(t) = 2.3\sin(\pi/10 t)$ involves setting $\pi/10 t$ equal to 2π and solving for t to obtain $t = 20$.

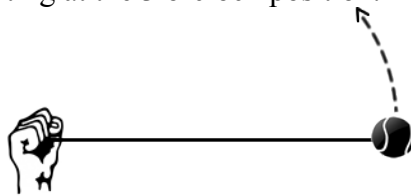
This instructional episode demonstrates David fostering a student’s pseudo-empirical abstraction on the procedure he employed to solve a similar task. As a result,

David conveyed that determining the period of a function amounts to setting the argument equal to 2π and solving for the input variable.

Table 58

Task 4(c) and 8(c) on Investigation 5 (Carlson, O’Bryan, & Joyner, 2013, p. 485-89)

José extends his arm straight out, holding a 2.3-foot string with a ball on the end. He twirls the ball around in a circle with his hand at the center, so that the plane in which it is twirling is perpendicular to the ground. Answer the following questions assuming the ball twirls counter-clockwise starting at the 3 o’clock position.



4. Suppose instead that José twirls the string so that the angle swept out increases by two radians per second.
 - a. How many radians θ does the string sweep out in t seconds?
 - b. Define a function h that relates *the ball’s vertical distance above Jose’s hand* (in feet) as a function of the *number of seconds elapsed*. Define any necessary variables.
 - c. Over what time interval does the ball complete one revolution?
8. Suppose instead that José twirls the string so that the angle swept out increases by 3.8 radians per second.
 - a. How many radians θ does the string sweep out in t seconds?
 - b. Define a function k that relates *the ball’s vertical distance above Jose’s hand* (in feet) as a function of the *number of seconds elapsed*. Define any necessary variables.
 - c. Over what time interval does the ball complete one revolution?

At the conclusion of Lesson 12, David supported students’ generalization by asking, “How can we find out period? ... How can I figure out the period of any sine function?” A student offered an inaudible response that David was clearly not satisfied with. Another student suggested, “Set the theta equal to 2π . So like for Question 8, we had $3.8t$ equals θ .” David responded, “Okay. Good. So in all of these problems we had y equals the sine of something times t So all we did was said, ‘ 2π should equal something times t ’ (writes “ $2\pi = bt$ ” on the board).” David continued, “So when we solve

for t what do we get? (*writes “ $t = 2\pi/b$ ” on the board*) This is one period. It is 2π divided by b .” David then explained that the period of the function $y = \cos(bt)$ is also $2\pi/b$.

David’s explanation equipped students with a general method for determining the period of functions of the form $f(x) = \sin(bx)$ and $g(x) = \cos(bx)$. In particular, David supported his students in knowing that computing the period of a sine or cosine function involves setting the term bx equal to 2π and solving for the input variable x , which amounts to dividing 2π by the coefficient of the input variable.

Theme 4: Coefficient of the Input Variable as the Speed of Variation

On seven occasions throughout Lessons 12-16, David emphasized that the coefficient of the input variable represents the speed at which an angle’s measure varies from zero to 2π radians. David made two remarks during Pre-Lesson Interview 8 that suggested his intention to support this way of understanding.

Pre-Lesson Interview 8. During the eighth pre-lesson interview, David acknowledged that he would like his students to understand that “the coefficient of the input variable has an inverse effect on the graph.” To illustrate this point, David explained that if the coefficient of the input to a sine or cosine function is two, “it takes half the time to actually get through that cycle.” Similarly, David claimed, “If you’re halving the inputs, then it’s going to take twice as long to get through its complete cycle.” David later anticipated that his students might have difficulty achieving this understanding: “They struggle with the idea of, ‘If I’m going twice as fast, it will take me half as long to do it.’ ... Some of them don’t have any idea of what speed is.”

Lessons 12-16. Moments of David’s instruction during Lessons 12-16 were consistent with his intention to support students’ understanding that the period of a sine

or cosine function is reciprocally related to the coefficient of the input variable. I discuss two such moments that together represent the seven occasions throughout Lessons 12-16 in which David conveyed this understanding.

During Lesson 14, after David had written the function definition $j(d) = 23.5\cos(\frac{1}{57.3} d) + 59.5$ on the board, a student asked the question in Line 1 of Excerpt 81.

Excerpt 81

- 1 Student: If the scale factor is less than one, how does it expand the graph?
- 2 David: It's a scale factor for the input. ... The input in this function is d , but cosine thinks of this whole thing as what it's taking the cosine of. ... The reason why it's the reciprocal is because if you multiply the input by two that means you'll get to the input twice as fast. If you get to it twice as fast, it will take half the cycle. Thus it shrinks— it takes it half the time to get through its whole cycle so it shrinks the graph horizontally.

It is unclear what David meant by, “you'll get to the input twice as fast.” Additionally, David's use of pronouns in the sentence, “If you get to it twice as fast, it will take half the cycle” makes the meaning he intended to convey difficult to discern. Based on David's previous remarks, I suspect he meant, “If an angle's measure varies twice as fast—two radians per second as opposed to one—then the output values of cosine will vary through one full cycle of values in half the amount of time it would take otherwise.” Therefore, David's reply in Line 2 appears to have supported the student in conceptualizing the coefficient of the input variable as the speed at which the “input” of cosine (presumably an angle's measure in radians) varies. Generally stated, David's

response seems to have supported the following meaning: The outputs of $f(x) = \cos(bx)$ vary through one full cycle of values b times as fast as do the outputs of $g(x) = \cos(x)$. Therefore, $\cos(bx)$ will vary through one full cycle of values in $1/b^{\text{th}}$ the amount of time it takes for $\cos(x)$ to do so. So since it takes $\cos(x)$ 2π seconds to vary through one full cycle of values, it will take $\cos(bx)$ $1/b(2\pi)$, or $2\pi/b$, seconds to do the same.

In Lesson 16, David computed the period of $h(x) = 10\cos(\pi x) + 2$ by dividing 2π by π as part of his procedure for sketching a graph of the function. After David constructed the graph of h , a student asked, “Can you just explain how you got the 2π over π ?” David’s response in Excerpt 82 supported the same meaning as did his remark in Line 2 of Excerpt 81.

Excerpt 82

- 1 David: I know cosine goes through its normal period in 2π . This (*points to “ π ” in the function definition*) is being multiplied to the inputs. Okay? So it’s making the inputs go through its cycle π times faster. If it goes through its inputs π times faster, then it’s going to take it $1/\pi^{\text{th}}$ less time to go through the cycle. So it shrunk the period by a scale factor of $1/\pi$. So since the normal period is 2π , I’m multiplying the normal period by $1/\pi$, the reciprocal of π .

David’s reply supported students’ understanding that the coefficient of the input variable represents the speed at which the input quantity of cosine (again, presumably an angle’s measure in radians) varies. While ill-defined pronouns again pervade David’s explanation, his statement appeared to convey the following meaning: The output values of cosine ordinarily vary through one full cycle of values when the input varies by 2π .

When we multiply the input quantity by a constant, b , the argument of cosine varies from zero to $2\pi b$ times as fast, which means the output values of cosine will vary through one full cycle of values in $1/b^{\text{th}}$ the amount of time. So the period of $f(x) = \cos(bx)$ is $1/b$ times 2π , the normal period of cosine.

As David's remarks in Excerpt 81 and Excerpt 82 indicate, even when the input variable of a sine or cosine function did not represent values of time, David supported his students in conceptualizing the coefficient of the input variable as representing the speed at which an angle's measure varies.

Summary of David's Enacted Knowledge of the Period of Sine and Cosine

Functions

David's enacted knowledge of the period of sine and cosine was as inconsistent as his enacted knowledge of angle measure and the output quantities and graphical representations of these functions. David's instruction supported both a quantitative and non-quantitative way of understanding period. Additionally, he emphasized the procedure for computing the period of a sine or cosine function with a linear argument, and spoke of the coefficient of the input variable as the speed at which an angle's measure varies. These disparate emphases contributed to an instructional experience that lacked focus and coherence.

In select pre-lesson interviews, and throughout his instruction, David discussed the period of sine and cosine functions in ways that did not reference the input and output quantities of these functions. For instance, in the eighth pre-lesson interview David revealed his intention to support students in being able to compute period by dividing 2π by the coefficient of the input variable. David's instruction during Lessons 13-15

repeatedly reflected this instructional objective. Moreover, David defined period as some variation of “how long it takes it to go through its entire cycle” (Lesson 12), and as the amount a graph must “slide over” to lay on top of itself (Lessons 7 and 13). On many occasions in his teaching, David neither defined period, nor justified the procedure for computing it, by attending to the covariation of the input and output quantities of the sine or cosine functions.

There was one notable moment during Lesson 12, however, in which David did support students’ understanding of period by attending to the covariation of the input and output quantities of sine. David conveyed that the argument of sine and cosine functions represents an angle’s measure in radians, and defined the period of a sine or cosine function as the *change* in the input quantity over which the output quantity varies through one full cycle of values. Additionally, David allowed his students to recognize that setting the argument of a sine function equal to 2π and solving for the input variable results in determining the interval over which the input quantity must vary to make the function’s argument vary by 2π —the interval of angle measure over which the outputs of sine vary through one full cycle of values. David clearly attended to the covariation in the input and output quantities of the sine function while supporting students’ understanding of period in this way.

Though on one occasion during Lesson 12 David supported students in constructing a quantitative way of understanding period, his instruction far more frequently emphasized the procedure for determining the period of sine and cosine functions with linear arguments. For example, nine times throughout Lessons 12-15 David supported his students in knowing that determining the period of a sine or cosine

function involves setting the input variable, with its coefficient, equal to 2π and solving for the input variable, which amounts to dividing 2π by the coefficient of the input variable.

Finally, David frequently discussed the coefficient of the input variable as representing the speed at which an angle's measure varies from zero to 2π radians, even when the input variable of a sine or cosine function did not represent values of time. In particular, David supported his students in constructing the following understanding: The output values of cosine ordinarily vary through one full cycle of values when the input varies by 2π . Since multiplying the input variable by b speeds up the variation of the angle measure, the argument of the function $f(x) = \cos(bx)$ varies from zero to 2π radians b times as fast as does the argument of the function $g(x) = \cos(x)$. Therefore, $f(x)$ will vary through one full cycle of values in $1/b^{\text{th}}$ the amount of time it takes for $g(x)$ to vary through one full cycle of values. So the period of $f(x) = \cos(bx)$ is $1/b$ of 2π , or $2\pi/b$.

Comparison and Contrast of David's Knowledge of the Period of Sine and Cosine Functions with His Enacted Knowledge

In this subsection, I compare and contrast the ways of understanding the period of sine and cosine functions that David revealed in the series of TBCIs with those he supported in the context of instruction. As with angle measure and the outputs and graphical representations of sine and cosine, David generally engaged his students in experiences that supported their constructing the ways of understanding he demonstrated during the series of TBCIs. Accordingly, David did not appear to make any conscious concessions to the quality of his enacted knowledge of the period of sine and cosine.

Recall that in the series of TBCIs, David demonstrated three ways of understanding the period of the functions $f(x) = \sin(bx)$ and $g(x) = \cos(bx)$:

WoU 1: The period of f and g is the value of x for which the argument of these functions, bx , equals 2π . So to compute period, one may just set bx equal to 2π and solve for x .

WoU 2: The period of f and g is the length of the interval of x -values over which the argument of these functions varies from zero to 2π , which is also the interval over which $f(x)$ and $g(x)$ vary through one full cycle of values.

WoU 3: The argument of the functions f and g vary from zero to 2π b times as fast as does x . Thus, $f(x)$ and $g(x)$ will vary through one full cycle of values b times as fast as will $\sin(x)$ and $\cos(x)$. So the period of f and g is $1/b^{\text{th}}$ times as large as the period of $h(x) = \sin(x)$ and $j(x) = \cos(x)$, or $2\pi/b$.

David's instruction supported all three of these ways of understanding. For instance, on nine occasions throughout Lessons 12-15, David emphasized that determining the period of a sine or cosine function involves setting the argument of the function (if the argument is simply a scalar multiple of the input variable) equal to 2π and solving for the input variable, which results in dividing 2π by the coefficient of the input variable (*WoU 1*). The way David supported students in rationalizing this procedure closely resembles the reasoning he demonstrated in Excerpt 50 while comparing the period of the function $j(x) = \sin(bx)$ to that of $k(u) = \sin(u)$. David recognized that the argument for the function j is b times as large as the input to the function k . Therefore, to

make these functions produce the same output, the input to j must be $1/b$ times as large as the input to k . Knowing that the outputs of k will have varied through a complete cycle of values when the input is 2π , David concluded that the outputs of j will have varied through one full cycle of values when the input is $1/b$ times as large as 2π , or $2\pi/b$, since $j(2\pi/b) = k(2\pi)$.

Although in TBCI 8 David demonstrated *WoU 2* more frequently than *WoU 1* and *WoU 3* combined¹⁰⁸, his instruction supported students in constructing *WoU 2* on only one occasion. In Lesson 12, David emphasized that the argument of sine and cosine functions represents an angle's measure in radians, and defined period as the *change* in the input quantity over which the output quantity varies through one full cycle of values. David further conveyed that when one sets the argument of sine or cosine equal to 2π , the 2π represents a *change* in angle measure from zero radians, which is the change over which the outputs of sine and cosine vary through one full cycle of values. Therefore, setting the input variable with its coefficient equal to 2π and solving for the input variable, David argued, results in determining the change in the input values that results in an angle's measure in radians—represented by the argument of the function—changing by 2π . The meaning for period David supported on this one occasion is consistent with the several instances during the Initial Clinical Interview and TBCI 8 in which David defined the period of sine and cosine as the interval of input values over which the argument varies by 2π (see Excerpt 52 and Excerpt 53), or the interval of input values over which the output values vary through one full cycle of values (see Excerpt 48 and Excerpt 51).

¹⁰⁸ David convincingly demonstrated *WoU 2* on nine occasions throughout the series of TBCIs.

David compellingly supported students in constructing *WoU 3* on seven occasions throughout Lessons 12-16. In particular, David conveyed that the outputs of $f(x) = \sin(bx)$ and $g(x) = \cos(bx)$ vary through one full cycle of values b times as fast as do the outputs of $k(x) = \sin(x)$ and $j(x) = \cos(x)$, so $f(x)$ and $g(x)$ will vary through one full cycle of values in $1/b$ the amount of time it takes $k(x)$ and $j(x)$ to do so. Since it takes $k(x)$ and $j(x)$ 2π seconds to vary through a full cycle of values, David argued, it will only take $f(x)$ and $g(x)$ $2\pi/b$ seconds to do the same. This way of understanding period very closely resembles the meanings David revealed during TBCI 8. For example, while discussing the period of the function $g(x) = \cos(3.7x)$, David explained that what the cosine takes as its input will vary 3.7 times as fast since the input variable is multiplied by 3.7, which implies that it takes $1/3.7$ times as long for cosine to vary through one full cycle of values. Hence, the period of the function g is $1/3.7$ of 2π (see Excerpt 51). It is noteworthy that David supported his students in conceptualizing the coefficient of the input variable as representing the speed at which the argument of a sine or cosine function—or an angle's measure in radians—varies, even when the input variable did not represent values of time.

While the three aforementioned ways of understanding period of sine and cosine appear conceptually distinct, David demonstrated his understanding in TBCI 8 that the first and third are related to the second. As I discussed in Chapter 6, David recognized that because the argument of cosine is zero when x is zero, determining the length of the interval over which x must vary to make the argument *vary by* 2π (*WoU 2*) amounts to determining the value of x for which the argument *equals* 2π (*WoU 1*), thus making *WoU 1* a kind of restatement of *WoU 2*. David's instruction supported his students in recognizing this connection. On one occasion during Lesson 12, David encouraged his

students to see that setting the argument of a sine function equal to 2π and solving for the input variable results in determining the *interval* over which the input quantity must vary to make the argument, which represents an angle's measure in radians, vary by 2π —the interval of angle measure over which the outputs of sine vary through one full cycle of values. The third way of understanding, which David both demonstrated in the series of TBCIs and conveyed in his teaching, expresses a method for determining the period of functions f and g based on conceptualizing the input quantity as time and period as the interval of input values over which $f(x)$ varies through one full cycle of values (*WoU 2*). Accordingly, David invoked *WoU 2* while demonstrating *WoU 3* during TBCI 8, and supported *WoU 2* while emphasizing *WoU 3* in his instruction.

There were a few occasions in David's teaching in which he supported ways of understanding the period of sine and cosine that he did not explicitly demonstrate in the Initial Clinical Interview or the series of TBCIs. For example, David stressed that the coefficient of the input variable to a sine or cosine function causes the graphs of these functions to stretch or shrink horizontally. David also defined period in a way that made scant reference to the input or output quantities of the sine or cosine functions. For example, David frequently defined period as “how long it takes to go through its cycle” as well as the amount a sine or cosine graph must “slide over” to lay on top of itself.

Evidence-Based Hypothesis for the Inconsistencies and Incompatibilities Between David's Mathematical Knowledge and His Enacted Mathematical Knowledge

In this concluding section of Chapter 7, I provide an evidence-based hypothesis that seeks to explain the incongruities, inconsistencies, and contradictions I observed between, David's mathematical knowledge and the mathematical knowledge he enacted

in the context of teaching, as well as within these instantiations of David's mathematical knowledge.

Discrepancies Between David's Mathematical Knowledge and His Enacted Mathematical Knowledge

While I have previously discussed a number of occasions in which David's enacted mathematical knowledge was inconsistent with, and sometimes contradictory to, particular ways of understanding he demonstrated in the series of TBCIs, a few of these discrepancies are worth reiterating.

First, recall that throughout the series of TBCIs, David frequently revealed his understanding of angle measure as the length of the arc an angle subtends. While David clearly conceptualized the quantity one measures when assigning numerical values to the openness of an angle, his instruction often supported students' understanding of angle measure as the ratio of subtended arc length to circumference—an arithmetic operation. Although during the third TBCI, David demonstrated that he understood the ratio of subtended arc length to circumference as a quantitative operation—the length of the subtended arc measured in units of circumference— by not emphasizing the circumference as a unit of measure for the length of the subtended arc, his instruction failed to support students' understanding of this ratio as representing the value of a quantity.

Second, in TBCIs 2 and 3, David explained that the unit with which one measures the length of the arc an angle subtends must be proportionally related to the circumference of the circle containing the subtended arc so as to make the size of the circle immaterial to the measure of the angle. However, on no occasion in his teaching

did David explicitly support students in constructing this understanding. He was instead content with stating that units of angle measure do not correspond to fixed lengths and that degrees, *quips*, and *marks* measure “a fraction of the circle.”

Third, several times throughout the series of TBCIs, David appeared to conceptualize sine and cosine values as measures of quantities—respective vertical and horizontal distances measured in units of the radius of the circle centered at the angle’s vertex. However, David’s instruction often supported students’ understanding of sine and cosine values as ratios of lengths without conveying the numerical value of these ratios as measures of an attribute of a geometric object (i.e., a quantity).

Fourth, several of David’s actions and utterances throughout the series of TBCIs suggest that he conceptualized the graphs of sine and cosine functions as representations of the covariational relationship between input and output quantities, as opposed to merely shapes or images. However, moments of David’s instruction promoted a non-quantitative way of understanding the graphical representations of sine and cosine. For example, David discussed transformations of sine and cosine functions as a kind of moving, stretching, and bending of shapes that was not justified by attending to the quantities these shapes expressed the relationship between.

Fifth, during the TBCIs David often engaged in Mental Action 3 (Carlson et al., 2002) while describing the covariation of the input and output quantities of the sine and cosine functions and while justifying the concavity of the graphs of these functions. That is, David compared successive changes in the value of the output quantity for equal changes in the value of the input quantity. David’s instruction, in contrast, frequently encouraged students to describe the covariation of the input and output quantities of the

sine and cosine functions by attending to the rate at which the output quantity varies with respect to time, assuming the input quantity increases at a constant rate with respect to time.

Sixth, throughout the series of TBCIs, David justified the independence of the size of the circle centered at an angle's vertex on the output values and graphical representations of sine and cosine by appealing to the fact that the vertical and horizontal lengths that respectively correspond to the output quantities of sine and cosine remain invariant when measured in units of the radius of this circle. In his teaching, however, David asserted, but did not explain, that the value of the arithmetic operation he proposed as the output of the sine function (the ratio of the terminal point's vertical distance above the horizontal diameter of the circle centered at the vertex of the angle to the length of the radius of this circle) does not depend on the radius of the circle centered at the angle's vertex.

Seventh, and finally, while David revealed three ways of understanding the period of the functions $f(x) = \sin(bx)$ and $g(x) = \cos(bx)$ during the series of TBCIs, he demonstrated the following way of understanding most frequently:

WoU 2: The period of f and g is the length of the interval of x -values over which the argument of these functions varies from zero to 2π , which is also the interval over which $f(x)$ and $g(x)$ vary through one full cycle of values.

David's instruction, in contrast, privileged the following way of understanding:

WoU 3: The argument of the functions f and g vary from zero to $2\pi b$ times as fast as does x . Thus, $f(x)$ and $g(x)$ will vary through one full cycle of

values b times as fast as will $\sin(x)$ and $\cos(x)$. So the period of f and g is $1/b^{\text{th}}$ times as large as the period of $h(x) = \sin(x)$ and $j(x) = \cos(x)$, or $2\pi/b$.

Inconsistencies, Discrepancies, and Contradictions *Within* David's Mathematical Knowledge

The observant reader will notice that some of the aforementioned examples of discrepancies between David's mathematical knowledge and his enacted mathematical knowledge are conveniently selective. Yes, David supported students' understanding of angle measure as the ratio of subtended arc length to circumference, but did he not repeatedly demonstrate this understanding during the series of TBCIs? Were not the occasions in which David supported students in conceptualizing the outputs of sine and cosine as arithmetic operations consistent with the several instances during the series of TBCIs in which he defined these outputs as ratios of lengths? Was the way of understanding period that David promoted in his instruction not reflected in the series of TBCIs? Is it therefore not the case that David's mathematical knowledge and his enacted mathematical knowledge were far less discrepant than these examples suggest? Indeed it is. That one can identify discrepancies between *particular* ways of understanding David demonstrated during the series of TBCIs and *particular* meanings he supported in the context of instruction does not imply the nonexistence of consistencies. Therefore, while there were some true inconsistencies, discrepancies, and contradictions *between* David's mathematical knowledge and his enacted mathematical knowledge, it was more often the

case that David's enacted knowledge reflected the inconsistencies, discrepancies, and contradictions *within* his mathematical knowledge.¹⁰⁹

Were David's mathematical knowledge consistently discrepant with the ways of understanding his instruction supported, we might infer that he willingly compromised the quality of his enacted knowledge in response to circumstances within his environmental context that he appraised as instructional constraints. That this was not the case necessitates alternative explanations. Because David's enacted knowledge reflected the many inconsistencies, discrepancies, and contradictions within his mathematical knowledge, it seems likely that his mathematical knowledge consisted of disorganized and disconnected cognitive schemes. Assimilating external stimuli to such disconnected schemes resulted in David demonstrating observable products of his reasoning that were, from my perspective, incompatible, discrepant, and contradictory.

Recall that in Chapter 3 I defined cognitive schemes as organizations of actions and operations constructed, via reflecting abstraction, at the level of representation. I also explained that reflected abstraction involves operating on the actions that result from prior reflecting abstractions, which results in a coherence of actions and operations accompanied by conscious awareness of those actions and operations.¹¹⁰ Reflected abstraction is therefore the mechanism of injecting coherence into a system of organized actions and operations (i.e., schemes) at the reflected level. Simply stated, it is through the process of reflected abstraction that one consolidates or connects disjoint cognitive schemes, the result being a single coherent scheme or a network of related schemes to

¹⁰⁹ Note that such inconsistencies, discrepancies, and contradictions were from *my* perspective.

¹¹⁰ It is the act of deliberately operating on the actions and operations that result from prior reflecting abstractions that brings these actions and operations into conscious awareness.

which one may assimilate a wider variety of stimuli. That David so often demonstrated incompatible, discrepant, and contradictory ways of understanding throughout the series of TBCIs suggests that he had not sufficiently engaged in reflected abstraction to construct a coherent network of cognitive schemes. Of course, there were occasions in which David appeared to engage in the reflected abstractions I designed select tasks to engender; David's response to the Nick-Meghan Task (see Task 5 in Table 41) and the Courtney-Rebecca Task (see Task 6 in Table 41) are two notable examples. These few occasions, however, often occurred after David demonstrated several discrepant ways of understanding mathematical ideas.

Inconsistencies, Discrepancies, and Contradictions *Within* David's Enacted Mathematical Knowledge

My suggestion that David had not constructed a coherent network of cognitive schemes prior to or during the series of TBCIs does not completely explain the incompatible, discrepant, and contradictory ways of understanding his instruction supported. There is, after all, a significant difference between the context of classroom teaching and the context of a task-based clinical interview. The mathematical knowledge a teacher conveys in the context of instruction is not entirely occasioned by her or his spontaneous responses to verbal or visual stimuli. While there are many external stimuli in the context of classroom practice that elicit observable products of a teacher's enacted mathematical knowledge (e.g., students' comments and questions), such stimuli are not the only, or perhaps even primary, source of these observable products. Prior to instruction, teachers have the opportunity to explicitly define the ways of understanding their lessons seek to promote, and to design or select tasks, activities, didactic objects, or

other curricular tools to support students in constructing these ways of understanding. The teacher's design and selection of such curricular tools is fashioned by the meanings he or she is consciously aware of possessing. Therefore, the mathematical knowledge a teacher demonstrates in the context of classroom practice is not only contingent upon the teacher's immediate assimilation of, and spontaneous responses to, external stimuli, but more so the teacher's conscious awareness of her or his mathematical ways of understanding.

As I have previously explained, one becomes consciously aware of her or his ways of understanding, or cognitive schemes, through the process of reflected abstraction. Recall that reflected abstraction involves operating on the actions that result from prior reflecting abstractions at the level of representation. Performing such deliberate operations necessarily brings the objects on which one operates into conscious awareness. Accordingly, the conscious awareness that results as a byproduct of reflected abstraction is an awareness of the objects on which one operates while engaged in reflected abstraction: the *actions* that comprise a particular cognitive scheme. To be consciously aware of one's way of understanding, therefore, is to be consciously aware of the internalized actions that result from prior reflecting abstractions, which are organized into cognitive schemes at the reflected level.

That David's instruction supported incompatible, discrepant, and contradictory ways of understanding suggests that he was not consciously aware of the mental actions and operations that comprised his cognitive schemes. To illustrate this point, recall that David's instruction of angle measure was often conflicting with regard to the quantity one measures when assigning numerical values to the openness of an angle. On some

occasions David supported students in conceptualizing angle measure as the length of an arc the angle subtends, while on other occasions he explained that measuring an angle involves determining the fraction of the circle's circumference subtended by the angle. During Lesson 6, for instance, David asked a student what a radian is and she replied, "1/6th of the circle." David then engaged the student in a sequence of questions that intended to support her in realizing that one radian corresponds to one radius length of arc (see Excerpt 62). On several occasions prior to this exchange, David supported his students' understanding of angle measure in radians as a fraction of the circle's circumference subtended by the angle (see Excerpt 56, Excerpt 59, and Excerpt 60). Despite David's prior emphasis on this way of understanding, he was dissatisfied with the student's claim that an angle with a measure of one radian subtends 1/6th of the circumference of the circle centered at its vertex. During the same lesson, David asked a student to move the terminal point of a subtended arc to illustrate an angle with a measure of one radian (see Excerpt 63). The student did so and explained that he placed the terminal point so that the subtended arc was equal in length to the radius of the circle centered at the angle's vertex. David was not satisfied with this rationale and eventually accepted another student's observation that the terminal point has gone 1/6th of the way around the circle. David's dismissal of the first student's rationale to move the terminal point so that the angle subtends an arc of one radius length discouraged students from understanding an angle with a measure of one radian as subtending one radius length of arc, a meaning he previously endorsed. This is one example among many in which David's instruction supported conflicting ways of understanding. David's statements in Table 59 further illustrate the discrepant meanings of angle measure his instruction

supported—discrepancies that derived from David’s unawareness of the mental activity involved in the ways of understanding he intended his students to construct.

Table 59

David’s Conflicting Statements of the Meaning of Angle Measure

Lesson 1	“When we measure an angle what are we really measuring? I mean it’s not like we’re measuring a length.”
Lesson 1	“Two angles have the same measure if “the length of the [subtended] arc is the same, as long as I made the circle have the same radius and it was centered at the vertex.”
Lesson 1	“I’m not measuring arc length. I’m measuring arc length and comparing it to what? ... Circumference!”
Lesson 4	“What do angle measures really tell me? The portion of the circle, right?”
Lesson 5	“One radian is one radius length of arc, right?”

We can imagine how David’s instruction might have differed if he was consciously aware of the mental actions that comprise one of his angle measure schemes.¹¹¹ In other words, we can envision how the ways of understanding David supported in his teaching might have been more focused and coherent if, prior to teaching a lesson on angle measure, he answered the question, “When a student looks at an angle and thinks about how to measure it, what should come to mind?” in the following way:

I first want my students to imagine a circle centered at the vertex of the angle.
 Then I want them to recognize that they may quantify the openness of the angle by measuring the length of the arc the angle subtends. I then want my students to

¹¹¹ I say, “one of his angle measure schemes” because David demonstrated two disjoint ways of understanding what it means to measure an angle:

WoU 1: An angle’s measure is the length of the arc the angle subtends in units that are proportionally related to the circumference of the circle centered at the angle’s vertex.

WoU 2: An angle’s measure is the fraction of the circumference of the circle centered at the angle’s vertex that the angle subtends.

imagine the size of the circle containing the subtended arc centered at the angle's vertex varying, and to observe that the length of the subtended arc varies as well. I also want my students to notice that the angle always subtends the same fraction of the circumference of all circles centered at the angle's vertex. As a result, I want my students to recognize the need for measuring the length of the subtended arc in units that not only covary with, but are proportionally related to, the circumference of the circle centered at the angle's vertex. To achieve this recognition, I then want them to imagine partitioning the circle's circumference into a certain number of equal-sized pieces and to multiplicatively compare the length of one of these pieces to the length of the subtended arc. Then I want my students to again imagine the size of the circle varying and to observe that, if the circle centered at the angle's vertex is always partitioned into the same number of pieces, the number of times the subtended arc is longer than the length of one of these pieces remains constant (perhaps they'll have in mind an image like the one in *Figure 74*). As a result, I want them to conclude that the length of the subtended arc measured in units of one of these pieces does not depend on the size of the circle centered at the angle's vertex. Additionally, I want them to generalize their observation and conclude that the unit with which one measures the length of the subtended arc must therefore be the same fractional part of the circumference of *all* circles centered at the angle's vertex so as to make the multiplicative comparison of subtended arc length to unit length independent of the size of the circle centered at the vertex of the angle.

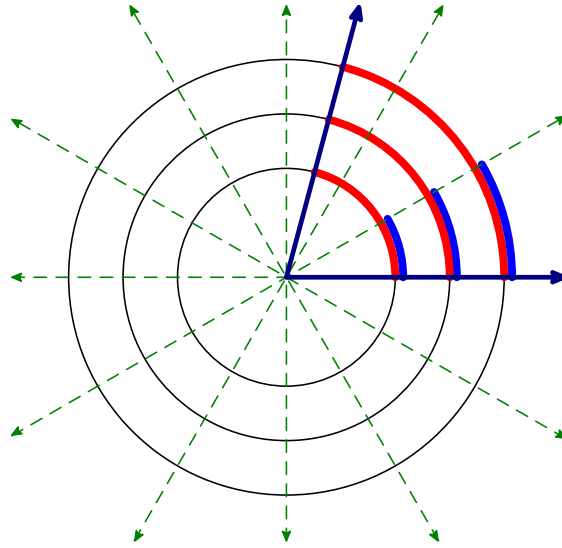


Figure 74. Multiplicative comparison of subtended arc length and a unit of angle measure.

Equipped with a conscious awareness of these mental actions, and the inferences drawn from them, it seems likely (or at least possible) that David would have engaged his students in experiences that promote their engagement in, and internalization of, such mental actions, thereby contributing to an instructional experience lacking the inconsistencies, discrepancies, and contradictions that were so prevalent in David's teaching. I must note that the specifics of David's hypothetical response to the aforesaid question matters less than what the response reveals: the instructional utility of David's awareness of the mental activity that comprise the meaning of angle measure he intends students to construct. David's hypothetical response illustrates that his conscious awareness of the mental activity involved in his own ways of understanding would likely have satisfied a necessary, but probably not sufficient, condition for focused and coherent instruction.

Several of David's remarks during the pre-lesson interviews revealed his unawareness of the mental actions he intended his instruction to promote. For example,

during the second pre-lesson interview David provided conflicting descriptions of how he wanted his students to conceptualize what it means to measure an angle in radians. David explained that an angle's measure in radians is a comparison of "arc length to the circumference as a whole" and then proceeded to profess his intention to support students' understanding of how to measure in angle in radians as "how many radius lengths would it take to get that angle?" (see Excerpt 57). Also during Pre-Lesson Interview 2, David explained that his lesson would be successful if at its conclusion students interpreted an angle with a measure of three radians as "three radius lengths." David later explained that he wanted students to understand an angle with a measure of three radians as "Three divided by 2π ... so it's just a little under half" (see Excerpt 58). That during a single pre-lesson interview David communicated his intention to support two meanings of angle measure, which were from my perspective inconsistent, suggests that he was not consciously aware of the mental activity that comprise the way of understanding angle measure he intended students to construct. Several of David's statements in other pre-lesson interviews revealed his unawareness of the mental activity involved in his own way(s) of understanding the outputs of sine and cosine, the graphical representations and concavity of sine and cosine, and the period of sine and cosine.

As a result of not having been consciously aware of the mental processes that comprised his own cognitive schemes, David was not positioned to define his instructional goals and objectives in terms of mental activity. Instead, David defined the intended outcomes of his instruction in terms of the observable behaviors he wanted students to demonstrate, the skills he hoped they would acquire, or the facts he wanted them to know. For example, I asked David during each pre-lesson interview (with the

exception of Pre-Lesson Interview 7) to explain how he wanted his students to understand the main idea of the lesson (see Table 60). David's responses almost always attended to the facts he expected students to know; he never explicitly discussed the mental images he wanted students to construct, nor did he define the conceptual operations on these mental images that constitute the understandings he intended to support. Moreover, David explained during the pre-lesson interviews that his instruction would be successful if students can "draw me an angle of one radian or two radians," "use radians and be able to tell me what the angle measure is or tell me what the angle looks like," "give me a radian approximation [of an angle]," "match up the angle in standard position on the circle and the coordinate on the unit circle that goes with the corresponding value on the sine or cosine curve," and "give me the answer on what the period is and realize that it's basically the coefficient on the input variable and you divide that by 2π to get the period." As with David's descriptions of how he intended students to understand the main idea of his lessons, he framed his explanations of what would constitute a successful outcome of his lessons in terms of the behaviors he wanted students to demonstrate. David did not characterize the mental activity from which such observable behaviors derive. In my coding of the ten pre-lesson interviews, I identified 33 occasions in which David communicated his instructional goals in terms of demonstrable skills and observable behaviors. I never found the need to create a code button to label segments of video in which David defined his instructional goals in terms of intended mental activity.

Table 60

David's Pre-Lesson Interview Responses to, "How do you want your students to understanding the main idea of the lesson?"

PLI 1	"The unit of measure doesn't make a difference when we measure arc length, and circumference, and stuff like that because of the reason that they reduce, they cancel, and that we're just looking at the proportion of the circle."
PLI 2	"I want them to understand that just like with degrees we're comparing the arc length to the circumference as a whole but since the length of the radius has that built in relationship to circumference through the formula $2\pi r$ that I want them to understand that it makes it a convenient way to measure because you can always measure the radius of the circle and just make that your length."
PLI 3	"I want them to understand that the input values used in a the sine function are the angle measures of an angle in standard position, um, and that the output values are the, for sine, the vertical distance from the terminal point on a unit circle that is cut off by the angle in standard position on its terminal ray."
PLI 4	"The sine and cosine functions are taking an angle and relating the vertical distance to the horizontal axis from the terminal point as a ratio of that distance to the length of the radius, or in terms of radius lengths, that would be okay, so that they understand that we're measuring in radius lengths instead. So I want them to understand that sine is the vertical distances and cosine is the horizontal distances and we're measuring in radius lengths, or a ratio of the length to the radius in other units. ... They either need to do the ratio or they need to think of it as the units of radius."
PLI 5	"Sine and cosine functions are really, the wave patters are based on the position of a point on the circle, depending on whether it's a vertical position or a horizontal position measured in radius lengths. So I want them to get that connection between how they can use a circle, how they can use the unit circle to help them to evaluate and then how we're getting those points and then how that goes then to the graph, how it creates the graph."
PLI 6	"I hope that they understand that there are special values of the, or inputs, special angles, that we can plug into the sine or cosine functions and I expect them to give me exact values. ... My expectation becomes when I ask you what is the cosine of $5\pi/6$ that you can then give me that answer basically from memory, ideally."
PLI 7	<i>No data.</i> ¹¹²

¹¹² I did not ask David in Pre-Lesson Interview 7 to explain how he wanted his students to understand the main idea of the lesson. This was an oversight and was not intentional.

- PLI 8 “I’m hoping they understand that a normal period is 2π , and that’s how long it takes to go through the cycle. When you have a coefficient in front of the input variable inside of a sine or cosine function then that’s going to cause the graph to horizontally stretch or shrink and it’s going to be the inverse, or reciprocal of whatever that value is. So because of that they can find the period by taking the normal period and dividing it by whatever the coefficient is.”
- PLI 9 “The topic of the lesson is how to use transformations to, um, create the sinusoidal function to match whatever situation that you’re in and then to use the trig terminology and how that compares to the function terminology that we had because functions don’t usually have periods so we don’t talk about the period and so it’s important that they understand that when there’s a horizontal expanding or condensing of the function that really that affects period and so with a periodic function we want to make sure we’re specific about that. Again, you know, vertically stretching and condensing a function may or may not have a lot of special meaning to it but when we’re talking about a periodic function that changes its amplitude. Vertically translating it will move the midline. Then horizontally translating it is called a phase shift, even though it is usually just a horizontal translation but in trig they usually refer to it as a phase shift, not a horizontal translation. So it’s kind of taking what they know, adding some new terminology, adding some special case scenarios that only apply really to trig and then how that conversation kind of builds on that.”
- PLI 10 “Sine and cosine, because of the circle, causes them to have these repeating patterns, and that’s in their nature. So anytime we see a repeating pattern in something, then we can relate it to the sine and cosine function, um, even though it might not have a direct correlation to a circle. And that’s because they’re still cyclic in nature and they still have a period that’s involved. So it’s almost, really what we’re doing is kind of unit conversions to use a sine or cosine function to model something that they really weren’t originally probably intended to model. . . . Really we’re kind of tricking sine and cosine to doing something that isn’t almost their intention.”

It is not entirely surprising that David’s inclination to define instructional goals and objectives in easily attainable behavioristic terms that did not reflect how he wanted his students to understand mathematical ideas led to the observed incoherencies in his instruction. When a teacher defines instructional goals in terms of what he or she wants students to be able to *do*, as opposed to how he or she wants students to *understand*, then

it opens up the possibility that he or she, knowingly or not, supports inconsistent, discrepant, and contradictory ways of understanding.

My hypothesis for the discrepancies I observed between David's mathematical knowledge and his enacted mathematical knowledge, as well as for the discrepancies I observed within these instantiations of his mathematical knowledge, is therefore the following: David had not sufficiently engaged in reflected abstraction to construct a coherent network of cognitive schemes and to become consciously aware of the mental actions and operations that comprise these schemes. As a result, David was not positioned to define his instructional goals and objectives in terms of intended mental activity, which contributed to the inconsistencies, discrepancies, and contradictions I observed in David's enacted mathematical knowledge. If David had been consciously aware of the mental actions that comprise the schemes that he wanted his students to construct, and if he had defined his instructional objectives to reflect this aspiration, he would have at least been positioned to employ pedagogical actions in the service of supporting students in engaging in the mental activity that characterize the ways of understanding he intended students to develop, which would likely have contributed to more focused and coherent instruction as a result. It therefore appears that the primary obstacle that conditioned the quality of David's enacted mathematical knowledge was the extent to which he was consciously aware of the mental actions and operations that constitute the mathematical meanings he possessed.

CHAPTER 8

DAVID'S IMAGE OF INSTRUCTIONAL CONSTRAINTS

If you are pained by external things, it is not that they disturb you, but your own judgment of them. And it is in your power to wipe out that judgment now.

Marcus Aurelius, *Meditations*

In this final results chapter, I examine the effect of David's image of instructional constraints on his enacted subject matter knowledge by discussing my analysis of the pre-lesson interviews, teacher journal entries, and the Phase III series of semi-structured clinical interviews. I have organized this chapter into three main sections that respectively correspond to these sources of data collection. I conclude the chapter with a summary of the circumstances David appraised as constraints on his practice, and of the instructional goals and objectives such constraints prevented or inhibited David from achieving. I then advance an explanation for the rather insignificant role that David's image of instructional constraints appeared to have on the quality of his enacted subject matter knowledge.

Pre-Lesson Interviews

In almost¹¹³ all of the pre-lesson interviews, I asked David four questions that I designed to reveal the circumstances he appraised as constraints on his practice (see Table 61). As detailed in Chapter 4, I examined David's responses to the four questions and identified the instructional constraints these responses revealed, as well as the instructional goals or objectives these constraints hindered David from achieving. In this section, I present my analysis of David's responses to each question in Table 61, along

¹¹³ On occasion, I unknowingly skipped one of these questions.

with any other statements he offered during the pre-lesson interviews that revealed his image of instructional constraints.

Table 61

Pre-Lesson Interview Questions Intended to Reveal David's Image of Instructional Constraints

- | |
|--|
| <ol style="list-style-type: none">1. While planning your lesson, did you feel that you needed to make any compromises for any reason? If so, what compromises did you make and why did you have to make them?2. While planning your lesson, did you feel like there were any constraints that you had to keep in mind? What were they?3. Is there anything about your lesson plan that you recognize as not ideal? If so, what are these things and why are you planning on doing them?4. What are some things that you anticipate might not allow you to accomplish your lesson objective? |
|--|

On three of the eight occasions in which David responded to the first question in Table 61, he claimed to have made no compromises or concessions while planning his lesson. This suggests that David was not responding to instructional constraints during lesson planning, or at least was not consciously aware of having done so. On the five occasions in which David claimed to have made compromises while planning his lesson, he cited having insufficient time to cover the curriculum material as the primary source of his compromises. For example, in Pre-Lesson Interview 5 David explained that he had to omit some important tasks from his lesson plan to ensure that his students would finish the investigation in one class period. Similarly, in the eighth pre-lesson interview David disclosed his intention to simply present the solutions to select tasks from the investigation so that he might keep up with departmental pacing recommendations. These responses suggest that David felt responsible for progressing through the content at a

pace that allowed him to cover a particular amount of material in a specific amount of time.

David's response to the second question in Table 61 further revealed that belief that having insufficient instructional time was a constraint on his practice. Five of David's seven responses included remarks suggesting that a deficit of instructional time influenced his lesson plans. In one of the two responses in which David did not mention having insufficient instructional time, he claimed to have not taken any constraints into consideration while planning his lesson. In his other response, David explained that he felt compelled to accommodate for the length and repetitiveness of the investigation, which he anticipated might contribute to deficiencies in students' engagement and motivation.

The eight responses David provided to the third question in Table 61 revealed instructional constraints that were less consistent. In only two of these eight responses did David rationalize the less than ideal aspects of his lesson plan by claiming to have insufficient time to cover the required material. In two other responses, David explained that he intended to supplement the *Pathways* curriculum with tasks that support students in memorizing sine and cosine values of particular angle measures and in becoming fluent with drawing graphs of transformed sine and cosine functions provided a function definition. While David appraised such tasks as a deviation from the material he was required to cover, he justified their inclusion by claiming that such knowledge is necessary for students' success in calculus. These responses reveal that equipping students with the factual and procedural knowledge that will enable them to fluently solve problems in calculus was among David's instructional goals. In the remaining four

responses to the third question in Table 61, David claimed that there is nothing about his lesson plan that he recognizes as not ideal. These four responses suggest that for approximately half of his lessons, David did not consciously make concessions to the quality of his lesson plan in response to the circumstances he appraised as instructional constraints.

Several of David's responses to the fourth question in Table 61 echoed the instructional constraints that his replies to the first three questions revealed. In particular, David explained that having insufficient instructional time (Pre-Lesson Interview 1), feeling responsible for equipping students with the procedural and factual knowledge required for success in calculus (Pre-Lesson Interview 2), and students' insufficient motivation and engagement (Pre-Lesson Interview 3) might inhibit him from achieving his lesson objective. David also claimed in Pre-Lesson Interview 5 that there is no circumstance that he envisions contributing to him not realizing his instructional intentions.

Five of David's nine responses to the fourth question in Table 61, however, revealed instructional constraints that he did not mention when responding to the first three questions. David identified the inconsistency with which students tend to absorb the facts and procedures he communicates as potentially contributing to him not accomplishing his lesson objective.¹¹⁴ For example, in the fourth pre-lesson interview David acknowledged that he would not realize his instructional goals if his students are unable to recite the values of sine and cosine provided specific angle measures. Similarly,

¹¹⁴ I use the word "absorb" in this sentence intentionally to reflect my interpretation of David's instructional disposition to tell students what he expected them to know and to demonstrate what he wanted them to be able to do.

David claimed during Pre-Lesson Interview 8 that he might not achieve his lesson objective if his students do not catch onto the fact that the period “is basically the coefficient of the input variable and you divide it by 2π .” These responses suggest that David also perceived students’ variable capacity to “learn” the facts he tells them as an instructional constraint. David also recognized students’ insufficient prior knowledge as being a probable hindrance to achieving his instructional goals. In Pre-Lesson Interview 6, David anticipated that students’ deficient knowledge of—or at least aversion to—fractions would compromise their ability to understand angle measures in radians “when they have π in them.” In the same vein, David predicted in the ninth pre-lesson interview that he might not accomplish his lesson objective of supporting students in knowing how to graph a transformed sine or cosine function provided its function definition because students tend to not understand function transformations in general.

David made several remarks in the pre-lesson interviews that were not in response to one of the four questions in Table 61, but which nonetheless exposed circumstances he appraised as constraints on his practice. These remarks, in addition to reiterating several of the instructional constraints suggested by David’s responses to the questions in Table 61, revealed a couple of previously unmentioned circumstances that David interpreted as impediments to achieving his instructional objectives. These constraints include (1) students’ inconsistent compliance with directions and instructional expectations and (2) having insufficient time to assess students’ learning and performance. In contrast to some of the instructional constraints David’s responses to the four questions in Table 61 revealed, he mentioned each of these two constraints only once, and almost as a fleeting afterthought.

After having coded all pre-lesson interview audio recordings using the code window in *Figure 19*, I re-examined those instances I identified with either the “Instructional Constraint” or “Compromises/Concessions” codes and specified the instructional constraint David’s remarks suggested. I also inferred the instructional goal or objective each constraint hindered David from achieving. To get a sense for the relative prominence of these instructional constraints, I tabulated the number of occasions in which I identified a segment of audio as revealing a particular instructional constraint. The chart in *Figure 75* illustrates the results of this tabulation.

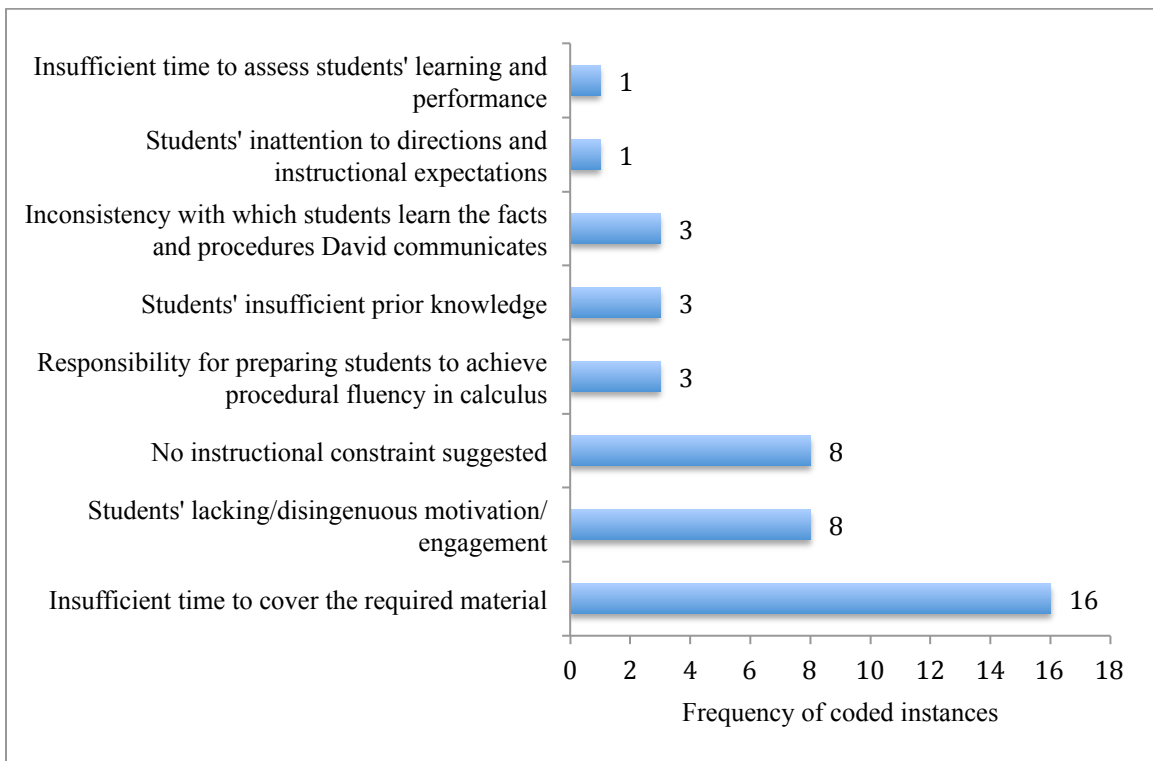


Figure 75. Frequency of David’s references to particular instructional constraints.

Figure 75 shows that the instructional constraint David most frequently cited in the pre-lesson interviews—and thus probably the most imposing circumstance that affected his teaching—was his feeling that he had insufficient instructional time to cover

the required material. David explained that in response to this constraint, he either intended to skip or simply present solutions to tasks that he would otherwise ask students to discuss in groups or think about individually. David also acknowledged that having insufficient instructional time contributed to him feeling pressured to move on to more advanced topics despite having observed some of his students' inability to demonstrate proficiency with basic facts and procedures. Additionally, David conceded that a deficit of instructional time compelled him to work through the investigations as a class, and that doing so allowed him to ensure that students have correct answers to investigation tasks with more economy than would be possible if students worked on the investigations individually or in small groups. These remarks, and others like them, suggest that covering the required amount of curriculum material was among David's primary instructional goals. It is noteworthy that David did not describe the effect of this instructional constraint in terms of the ways in which it compromised his ability to support students in constructing particular ways of understanding. Instead, David's responses suggested that he accommodated for not having sufficient instructional time by occasionally deviating from what he considered to be effective pedagogical practices (e.g., facilitating group discussion and individual thinking) and by failing to modify his instruction in response to his image of students' emerging factual knowledge and procedural competence.

The second most frequent instructional constraint David identified in the pre-lesson interviews was students' lacking or disingenuous motivation and engagement. David recognized that the existence of this instructional constraint necessitated his cultivating students' interest and participation with punitive or superficial strategies,

much to his contempt. David explained that were his students sufficiently motivated to learn and engage in his instruction, the need for nagging and incentivizing them to participate would be significantly reduced, as would unwelcome parental interaction. Interestingly, David did not appeal to the quality of mathematical meanings students' insufficient motivation and engagement prevented them from constructing. It therefore appeared that students' lacking interest and insufficient participation obstructed David from achieving his goal of creating a work environment that is, to the greatest extent possible, free of time-wasting and energy-consuming annoyances and distractions.

The eight occasions throughout the series of pre-lesson interviews in which David suggested that his lesson planning was not conditioned by his responding to instructional constraints illustrates the infrequency with which David believed that his instructional goals were inhibited by encumbering circumstances. It also reveals an important characteristic of the relationship between David's subject matter knowledge and his image of instructional constraints, which I discuss at length at the conclusion of this chapter.

Of the remaining and less frequently identified instructional constraints David's responses in the pre-lesson interviews revealed, he suggested that his lesson planning was affected by only two of them: (1) feeling responsible for preparing students to achieve procedural fluency in calculus and (2) students' insufficient prior knowledge. David appraised the others (the inconsistency with which students learn the facts and procedures he communicates, students' inattention to directions and instructional expectations, and insufficient time to assess students' learning and performance) as not being within his locus of control. These latter constraints were therefore circumstances David lamented

but to which did not actively accommodate. David confessed that he felt obligated to emphasize factual knowledge and procedural skill so that his students advancing to AP Calculus would not be obstructed by their overreliance on degrees as a unit of angle measure or their inability to evaluate sine and cosine functions of particular radian-measure inputs without a calculator. David appealed to this obligation while rationalizing the inclusion of a lesson on “the unit circle,” which emphasized memorization of sine and cosine values of integer multiples of $\pi/6$. David appraised as an instructional constraint his obligation to support students in attaining the factual knowledge he considered necessary for achieving procedural fluency in calculus because it consumed instructional time that otherwise could have been spent progressing through the *Pathways* curriculum. It is noteworthy that David did not appraise this obligation as equipping students with a coping mechanism that facilitated their avoidance of reasoning quantitatively about the inputs and outputs of the sine and cosine functions. Instead, David’s remarks in the pre-lesson interviews exposed this obligation as a constraint only in the sense that it made it even more challenging to adhere to departmental pacing guidelines.

David also explained that he felt the need to review foundational concepts and skills prior to and throughout particular lessons as a result of his belief that students’ prior knowledge was not sufficiently robust to support their learning of the central ideas of these lessons. In particular, David conveyed his intention to support students’ understanding of fractions so that they could more fluently determine the proportion of a circle’s circumference subtended by an angle with a measure that is a “nice fraction of π .” David also communicated the need to remind his students of various function transformation rules that apply to a wide variety of function classes (e.g., adding a

positive value to the input results in a horizontal shift left whereas subtracting a positive value from the input results in a horizontal shift right) so that they can apply these general rules to sketch graphs of transformed sine and cosine functions. Importantly, David's remarks suggest that he appraised students' lacking prior knowledge as imposing a constraint on their ability to solve particular tasks. David did not appear to recognize students' insufficient prior knowledge as imposing an obstacle to the ways of understanding students were capable of constructing.

Teacher Journal Entries

As I explained in Chapter 4, I provided David with a journal entry form at the conclusion of his instruction of each investigation. Recall that the purpose of these journal entries was to provide insight into the ways in which the mathematical knowledge David brought to bear in the context of teaching was fashioned by his image of instructional constraints. Each journal entry form consisted of a number of prompts common to all journal entry submissions, as well as a number of specific prompts related to the content of his lessons pertaining to a specific investigation. I designed the prompts common to all journal entry forms to expose David's image of instructional constraints and to provide an occasion for him to reveal how he compromised the quality of his enacted mathematical knowledge in response to such constraints. Table 62 contains these general prompts. In this section, I present my analysis of David's responses to each question in Table 62. As I did in the previous section, I pay particular attention to identifying the instructional constraints David's responses reveal, and to inferring the instructional goal or objective these constraints impeded David from achieving.

Table 62

General Teacher Journal Entry Prompts

1. What is your assessment of how the investigation went?
2. Did you accomplish what you wanted to while teaching the investigation? Explain.
3. Did you deviate at all from your lesson plan? If so, how and why?
4. Would you change anything if you had to teach this investigation again?
5. Is there anything that would have made your teaching of the investigation more effective?
6. Was there anything problematic or unexpected about your teaching of the investigation?
7. To what degree do you think your students understood the main ideas of the investigation? Explain.
8. Did anything inhibit your ability to accomplish your lesson objective? If so, what? Explain.
9. When talking with teachers after they teach, they sometimes tell me, “My teaching of this investigation would have been better if only ...” If you feel this way, how would you complete the sentence?
10. If you had the power to change anything that could have improved the quality of how you taught this investigation, what would you change and why?

In eight of his eleven replies to the first question in Table 62, David expressed satisfaction with how his lesson went. These positive responses suggest that, whatever David’s instructional goals were, he believed that he was, for the most part, accomplishing them. In one of the three responses in which David expressed dissatisfaction with his lesson, he did not give a reason, claiming only, “I don’t feel that today went as well as I had hoped” (Journal Entry 3). In his other two responses, David explained, “I did not get as far as I had originally thought that I would” (Journal Entry 6) and “I was a little discouraged and surprised by how much my students struggled with matching the point on the graph of the function back to its meaning on the circle” (Journal Entry 7). It is noteworthy that in only one of his eleven responses to this question did David justify his approval of or discontent with the lesson by referencing the knowledge or understandings students could or could not demonstrate. Perhaps more

revealing is that in none of his responses did David assess the outcome of the lesson in terms of ways of understanding his students did or did not appear to construct.

In every one of David's responses to the second question in Table 62, with the exception of the one-word affirmative response he provided in Journal Entry 8, David described whether or not he accomplished his lesson objective in terms of the amount of content he covered (see excerpts of David's responses in Table 63). In the nine responses in which David claimed not to have accomplished his lesson objective, he cited content coverage as the reason. So prevalent was David's reference to curriculum pacing that without knowing that the responses in Table 63 were in fact responses to the question, "Did you accomplish what you wanted to while teaching this investigation?" one would assume that David was responding to, "Did you cover the amount of material that you intended to cover in your lesson?" David's responses to Question 2 in Table 62 transparently reveal the extent to which he appraised not having sufficient instructional time as a constraint on his teaching, thereby demonstrating that covering curricular material at a sufficient pace is among David's primary instructional objectives. As with his responses to the first question in Table 62, David's replies to the second question made no reference to the meanings students appeared to have constructed or the ways of understanding his instruction supported.

Table 63

David's Journal Entry Responses to Question 2

Journal Entry 1	"I would have liked to have gotten a little further into the investigation."
Journal Entry 2	"My goal was to finish Investigation 1 which I did in both classes."
Journal Entry 3	"I didn't ask good enough leading questions to move the lesson ahead ... I didn't get through as many questions as I had hoped."

Journal Entry 4	“I was able to fix some of the issues that I had in the previous day’s lesson. But unfortunately that took time away from the spaghetti graphs project.”
Journal Entry 5	“I did not quite get as far as I had planned.”
Journal Entry 6	“I had originally planned on getting through the formal definition of the cosine function but instead I only got through the sine definition.”
Journal Entry 7	“I had planned to make it deeper into the investigation especially with 6 th hour that only got through Question #2.”
Journal Entry 8	Yes.
Journal Entry 9	“I would have liked to get farther into the Investigation 5.”
Journal Entry 10	“I was slightly behind 6 th hour and I didn’t get a chance to add the conversation about the cosine function.”
Journal Entry 11	“I wanted to get through Question 2 & 3 in the 1 st day but I had to spend time on finishing Question 3 on the 2 nd day of the lesson.”

I intended the third question in Table 62 (“Did your lesson deviate at all from your lesson plan? If so, how?”) to elicit David’s identification of and justification for *moments of instructional deviation*: instances in which David made spontaneous adjustments to his lesson plan in response to instructional constraints. David claimed in three of his eleven replies to this question that his lesson did not stray from his plan, suggesting that he often did not respond to unforeseen circumstances or events while teaching. In all eight responses in which David did claim to have deviated from his lesson plan, he described the deviation in terms of content coverage. For example, David explained, “I had planned on getting to at least Question #4 but I was only able to get through Question #1” (Journal Entry 1); “Because the class did not move through the questions as quickly as I had hoped I was not able to get as far as I had hoped” (Journal Entry 3); “[The lesson] did not move along as quickly as I had hoped” (Journal Entry 5); “It took longer than anticipated for the students to estimate the angle and corresponding vertical measures” (Journal Entry 6); “It just went a little longer than anticipated” (Journal Entry 11). These responses, and others, further reveal the extent to which David

appraised having insufficient time to progress through the curriculum as a constraint on his practice. Interestingly, in none of David's replies to the third question in Table 62 did he suggest that he deviated from his lesson plan as a result of having been responsive to students' thinking or their emerging ways of understanding.

In seven of his eleven responses to Question 4 in Table 62, David claimed he would not change anything if given the opportunity to teach a particular investigation again. As with several of his replies to other journal entry prompts, these responses suggest that either David was pleased with his spontaneous reactions to unanticipated circumstances or events that arose during the lesson, or that he did not recognize unanticipated circumstances or events to which he needed to respond. Also consistent with several of his answers to other questions, in three of the four replies to Question 4 in which David identified changes he would make to his instruction, he claimed that such changes were in the service of ensuring that he had sufficient time to cover the curriculum material. For example, David professed, "I wish that I had come up with better questions to move the conversation forward" (Journal Entry 3) and "I would have planned to take more time and maybe I needed to go through Question #1 with them" (Journal Entry 7). These suggested changes to his instruction reflect David's recognition that insufficient instructional time is a significant constraint on his practice. It is revealing that David did not justify the changes he suggested by appealing to the way these changes might improve the quality of the ways of understanding students have the opportunity to learn.

David claimed in four of his eleven responses to Question 5 in Table 62 ("Is there anything that would have made your teaching of the investigation more effective?") that

nothing would have improved the quality of his lesson. In his other responses to this question, David acknowledged that the following would have made his lesson more successful: better supplies (Journal Entry 2), asking better questions to support students' engagement (Journal Entry 3), teaching the previous lesson more effectively so as to ensure adequate pre-requisite knowledge (Journal Entry 4), communicating clearer expectations (Journal Entries 5 and 7), having the investigation more effectively organized (Journal Entry 10), and incorporating pre-requisite tasks to "smooth the transition" to the current investigation (Journal Entry 11). Several of David's responses revealed the following instructional goals: maintaining high student engagement (Journal Entry 3), supporting students' pre-requisite knowledge (Journal Entries 4 and 11), communicating clear directives and expectations (Journal Entries 5 and 7), and having sufficient time to cover the curriculum material (Journal Entry 10). That none of David's responses justified his proposals for improving his teaching by appealing to the ways of understanding students have the opportunity to construct demonstrates that supporting students in constructing particular ways of understanding was not among David's primary instructional objectives.

In six of his eleven responses to Question 6 in Table 62, David claimed that there was nothing problematic or unexpected about his lesson, thereby suggesting that he often did not appraise circumstances or events as imposing obstacles to the quality of his instruction. In his other responses, David identified the following as unexpected or problematic events from his teaching: an unanticipated comment from a student who claimed that degrees are a better unit of angle measure than radians (Journal Entry 3), students' incorrect responses to an investigation task or tasks (Journal Entries 5 and 10),

and general student struggles and difficulties (Journal Entries 7 and 11). It is telling that the events or circumstances David found problematic or unexpected had little if anything to do with how his students conceptualized mathematical ideas.

Question 7 of each teacher journal entry asked David to describe the degree to which his students understood the main ideas of particular investigations. I designed this question to reveal the extent to which David believed he had accomplished his instructional intentions, as well as to expose the circumstances David recognized as contributing to students not understanding foundational concepts. David's brief and underspecified responses did not provide much insight into these foci. David wrote, for example, "I don't think that they have a complete grasp of the topic but it is a good start" (Journal Entry 1) and "I think I was able to clear up many of the misconceptions that had arose from the previous lesson" (Journal Entry 4). Other responses were merely a faint summary of the understanding David believed students had constructed (e.g., "I feel very confident that the students' have a much better understanding of how the circle relates to the sine/cosine function" (Journal Entry 8)). It is noteworthy that none of David's responses characterized the understanding that students did or did not appear to have constructed. This further supports my claim from Chapter 7 that David was not consciously aware of the ways of understandings—in particular the mental actions and operations that comprise them—he intended his students develop.

Question 8 of each teacher journal entry asked David to identify any circumstances or events that inhibited his ability to accomplish his lesson objective. I phrased the question pointedly in an effort to elicit David's understanding of the instructional constraints that compromised the quality of his enacted mathematical

knowledge. In six of David's eleven responses to this question, he acknowledged that a deficit of instructional time hindered his ability to achieve his lesson objective. In four other responses David claimed that there were no constraints that affected his capacity to realize his instructional intentions. As with his responses to several other teacher journal prompts, David's replies to Question 8 identified circumstances that did not interfere with his achieving goals of student learning, but rather obstructed him from adhering to district and departmental pacing guidelines.

As with Question 8, I designed Question 9 in Table 62 to provide an occasion for David to identify constraints that compromised the quality of his instruction. David recognized a lack of instructional time as a constraint on his practice in four of his eleven responses. Two other responses revealed David's regret for not communicating less ambiguous directions and expectations during the lesson. Additionally, twice David indicated that nothing would have improved the quality of his lesson. In one other response David suggested that his lesson would have been more effective if he had used higher quality manipulatives (yarn instead of twine). Almost all of David's responses described what he could have done differently to make his lesson more effective. Importantly, in none of his responses to Question 9 did David justify his proposal for what he could have done differently by appealing to the quality of mathematical meanings his instruction afforded students the opportunity to construct.

Question 10 of each teacher journal entry asked, "If you had the power to change anything that could have improved the quality of how you taught this investigation, what would you change and why?" This is the kind of fanciful question that floats into the minds of frustrated and weary teachers during quiet planning periods and reflective drives

home from work. Casual conversations in teacher work areas and lunchrooms become impassioned as such conversations gravitate toward questions of this type. Perhaps this is to be expected from a population of professionals that are often denied the agency to design policy that directly affects their work. Bearing this in mind, in response to Question 10 David identified the following as things he would change if given the opportunity to do so (stated in abbreviated form):

- I wish I had managed my instructional time more efficiently (Journal Entries 3, 6, 7, 8, and 10).
- I wish I used yarn instead of twine in Investigation 2 (Journal Entry 2).
- I wish I would have provided clearer directions or made my expectations more explicit (Journal Entry 5).
- I wish I would have been more aware of my students' prior knowledge, which would have resulted in my having more realistic expectations regarding what we could cover in one class period (Journal Entry 7).
- Nothing (Journal Entries 1, 4, and 9).

I find it remarkable that David did not identify circumstances that many teachers identify as profoundly stifling their ability to support students' learning. Consistent with his responses to other journal entry questions, however, the constraints David recognized in response to Question 10 were not circumstances that impeded his ability to support students in constructing particular ways of understanding mathematical ideas. They were instead aspects of his own instruction that negatively influenced the rate at which he progressed through the curriculum, and which led to students' not having a clear image of what they were expected to do and how they were expected to do it.

Phase III Semi-Structured Clinical Interviews

In this section, I present my analysis of the Phase III semi-structured clinical interviews I conducted to reveal David's image of instructional constraints and to discern the role of these constraints on the mathematical knowledge David recruited to support students' learning. I begin by discussing my analysis of a series of preliminary questions I asked David prior to engaging him in conversations around the videos I selected to illustrate *moments of instructional deviation*, *moments of instructional incoherence*, and *moments of mathematical concession*. I decided to include these preliminary questions to test the following evidence-based hypotheses that emerged from my analysis of the videos of David's classroom teaching, pre-lesson interviews, and teacher journal entries:

- (1) David's unawareness of the mental activity comprising particular ways of conceptualizing mathematical ideas made him incapable of defining instructional goals and objectives in terms of the ways of understanding he intended students to construct; and
- (2) David's instructional goals, which had little or nothing to do with the meanings he wanted students to learn, conditioned the circumstances he appraised as constraints on his practice.

After my discussion of David's responses to these preliminary questions, I present my analysis of our conversation around pairs of audio or video excerpts that illustrate *moments of instructional deviation*, *moments of instructional incoherence*, and *moments of mathematical concession*.

Preliminary Questions

Table 64 contains the preliminary questions I asked David at the beginning of the first Phase III clinical interview.

Table 64

Phase III Clinical Interview Preliminary Questions

1. What are your goals as a teacher?
2. What things make it difficult for you to achieve these goals?
3. What factors inform your lesson design?
4. What do you primarily feel responsible for as a teacher?
5. What instructional constraints do you have to deal with?
6. What makes a highly effective mathematics teacher?
7. What are the characteristics of high-quality mathematics instruction?
8. When a student asks a question or makes a comment, what factors inform how you respond?
9. If a student teacher asked you, “What are the most important things I should know before walking into a class to teach a lesson?” how would you respond?
10. In your view, what is the role of a mathematics curriculum?
11. What is the process by which students learn mathematical ideas?

Unawareness of intended mental activity. Several of David’s responses to these questions support my claim from Chapter 7 that he was not consciously aware of the mental actions and operations that comprise his own ways of understanding, nor of those he intended students to construct. For example, in response to Question 3, David explained that his lesson design was informed by the outcome of previous lessons, the need to create opportunities to pursue students’ comments and questions, the amount of content he intended to cover, constraints on instructional time, and “classroom personality.” Interestingly, David did not claim that his lesson design was informed by his image of the ways of understanding he intended students to construct, nor did David’s response reveal his consideration of the mental activity in which he wanted students to engage.

David's response to Question 9 in Table 64 similarly suggests that his instructional design was likely uninformed by an awareness of the mathematical meanings he intended his instruction to support. In response to this question, David claimed that he would tell the student teacher that s/he should "know your students, know that you're not going to have all the answers and neither are they, ... [and] be flexible." David went on to explain that flexibility "is the important thing." David also stressed, "A student teacher I would expect to have content knowledge ... so I would be more worried about classroom management. So get to know your students, get to know the environment that you're walking into." Additionally, David explained that the student teacher must have an image of how the lesson will unfold, including anticipating how students will respond to specific problems. David concluded his response by claiming that his advise "would be more on the human relationship side ... because you need to concentrate on reading the kids and being comfortable with yourself and trust in your knowledge."

I asked David to describe why a student teacher must bear the aforementioned considerations in mind prior to teaching a lesson. David's response emphasized that it is important for the teacher to connect with his or her students, and he reiterated that he "wouldn't worry about content knowledge so much. ... They probably have the content knowledge." It is noteworthy that in his response to Question 9, David did not problematize the hypothetical student teacher's mathematical knowledge; there were clearly other competencies he considered more essential to being an effective teacher. It is further significant that nowhere in David's response did he convey the necessity of the

student teacher having a clear image of the meanings he or she would like the students to construct, preferring instead to emphasize affective considerations.

In an effort to discern the extent to which David's interaction with students was informed by his image of how he wanted them to understand mathematical ideas, I asked him to identify the factors that influence how he responds to students' questions or comments (see Question 8 in Table 64). David explained that he first attempts to clarify what the student is asking, and then determines whether he wants to pursue the student's question by assessing how "on topic" it is. David's response suggested that he perceives inquiries that are not "on topic" as unnecessary deviations and potential threats to the highly valued commodity of instructional time. David then explained that he pursues a student's question only if "it [is] getting me to where I want to be in the lesson" and if it is "moving me toward my goal." David summarized his response: "Really it comes down to processing what is it they're asking and where does this fall and then from there I decide how to deal with it." Interestingly, David's response did not convey the importance of attempting to discern what the student's question reveals about his or her way of conceptualizing a particular mathematical idea, nor did he explain that one of the factors that informs how he responds to students' questions or comments is his image of the way of understanding he expects his students to construct. As with his responses to Questions 7 and 9 in Table 64, David's remarks suggested that his instructional actions were uninformed by a conscious awareness of the mental actions and operations that comprise his own ways of understanding, or those he aspired to promote in his teaching.

Instructional goals and objectives. In an effort to discern the aptitudes David values in a teacher, I asked him to describe what makes a mathematics teacher highly effective (Question 6). I provide David's reply in Excerpt 83.

Excerpt 83

- 1 David: They have high content knowledge. Uh, they have to understand the way that, um, kids learn and process. And having that high content knowledge will help them because they'll understand, 'The kids are going to struggle at this topic. Is there another way I can understand it? Is there another way I can approach it? Is there a different spin on it or a connection I can make to it?' Um, they have to have that and that goes on to how kids learn and what the level of the kids are.

By "have high content knowledge" and "understand the way kids learn and process," David appeared to suggest that a highly effective teacher must have an awareness of the topics with which students typically struggle and must be prepared to offer alternative approaches to support students in overcoming these struggles. It is noteworthy that although David identified content knowledge as an important characteristic of a highly effective teacher, he appeared to understand the utility of such knowledge as allowing a teacher to make topics that are traditionally challenging for students more accessible by presenting alternative approaches. David's response did not suggest that a teacher's content knowledge allows him or her to engage students in experiences that support them in constructing particular ways of understanding.

I asked David to explain how he recruits his mathematical knowledge to support students' learning in an effort to determine whether he recognized that the primary, or perhaps only, benefit of strong content knowledge is that it enables a teacher to make topics that are traditionally challenging for students less complicated by presenting alternative approaches or explanations. David's response in Excerpt 84 is consistent with his remarks in the previous excerpt.

Excerpt 84

- 1 David: I understand where the hang-ups are going to be. Um, I understand where the kids are going to have issues and by having a big [knowledge] base, it allows me to say, 'Well let's take the approach that a geometry teacher would take.' ... So what would their geometry teacher have said? How would their geometry teacher have approached the idea that we're getting at? You know, we might be talking about proportionality, so how would the geometry teacher have approached it? So okay, then let's talk about similar triangles; let's talk about things we did last year with similar shapes and things like that. ... So I know this is where the calc teacher is going to want them to be. So what do I need to do to help make his transition easier? I know this is what the geometry teacher gave them. So what do I need to do to get them from where they were to where I want them to be?

David's response demonstrates that he considered his content knowledge useful because it allowed him to identify topics with which students generally struggle and to

present ideas in ways that make connections to what students have learned in the past. By attempting to connect what students are currently learning to their prior knowledge, and by presenting new ideas as a different instantiation on something they have already seen, David did not problematize the meanings he intended his students to construct. David's instruction, therefore, appeared to be informed by his answer to the question, "How is this topic similar to what my students have done in the past and how can I leverage their prior knowledge to get them to solve the problems I want them to solve?" Instructional actions informed by the answer to this question likely differ substantially from those driven by the following consideration: "What is the most advantageous way of understanding the central idea of this lesson and how can I design learning experiences that allow my students to engage in—and construct internalized representations of—the mental actions that characterize this understanding?" That David did not explain that his mathematical knowledge informs his characterization of the ways of understanding his instruction seeks to support, but rather that he uses his content knowledge to present alternative explanations of topics students typically find challenging, suggests that supporting students in constructing particular mathematical meanings was not among David's primary instructional goals.

David's responses to the first and fourth questions in Table 64 revealed with greater transparency his instructional goals and objectives. These questions respectively asked David to describe his instructional goals and to identify what he feels responsible for as a teacher. In response to Question 1, David disclosed that his goals are to "connect with students" and to "enhance their love of learning." David's response further revealed that he feels responsible for providing students access to high quality post-secondary

education by giving them “the skills so that they can be successful later on.” David also mentioned that one of his goals is to support students in appreciating the relevance of mathematics to their lives. It is telling that David’s response did not identify as an instructional goal supporting students in constructing particular ways of understanding mathematical ideas.

In response to my asking him to identify what he feels primarily responsible for as a teacher (Question 4 in Table 64), David explained that his foremost obligation is “creating the environment and the learning opportunities ... so creating a setup where [students] can actually learn.” I asked David to articulate the characteristics of the learning environment he aspires to create. In his reply, David explained that he attempts to cultivate a classroom environment wherein peer pressure does not discourage students from asking and answering questions or from participating in group or whole-class discussions. David further suggested that he works to create a learning environment in which students feel secure not knowing answers to questions and problems. David also acknowledged that he has a reputation for being a hard teacher, and expressed his desire for his students to feel comfortable interacting with him. David’s remarks clearly reveal his concern for constructing a learning environment wherein students feel safe and efficacious. Fostering such an environment, David suggested, is essential to supporting students’ learning. While he may have intended to imply it, David’s response did not explicitly recognize as an obligation the quality of mathematical meanings he afforded students the opportunity to construct. In particular, David did not appear to feel responsible for the mental imagery evoked when students assimilate mathematical symbols, interpret quantitative situations, or read novel problems. This suggests that

supporting students in constructing particular mental images and engaging in specific mental actions was not among David's primary instructional goals. It also substantiates my previous claim that David's instructional actions were not informed by an awareness of the mental activity in which he intended students to engage.

Instructional constraints. To provide an occasion for David to make explicit the circumstances he appraised as constraints on his practice, I asked him to describe the factors that make it difficult for him to achieve his previously mentioned goals (Question 2 in Table 64). In response, David explained that accomplishing his instructional aspirations becomes more difficult when students interpret his requests for them to explain or justify their answers as him being intimidating or coercive. David claimed that when this happens, students "shut down" and disengage, which compromises his capacity to create a learning environment wherein students feel safe to participate. David also recognized the cultural expectation that mathematics is a subject at which it is acceptable to be inept as a constraint on his practice because such a mindset inevitably contributes to low mathematical self-efficacy and perseverance, which further obstructs his ability to foster a classroom environment that encourages students' involvement.

In response to my directly asking David to identify the instructional constraints that hindered him from realizing of his instructional goals (Question 5 in Table 64), he cited a lack of instructional time and the unreceptiveness of the *Pathways* curriculum to supplementation as among the most significant. Regarding the former, David explained, "I have so many standards I have to cover, so many things I have to hit, and I have a finite number of days. I have a finite number of class periods." I asked David to describe how having insufficient instructional time affects his lesson planning and teaching.

David's response emphasized the process by which he and his colleagues accommodate for not having enough time to cover the required amount of material. Specifically, David explained that he and his colleagues determine when they need to be done with a module and together identify which investigations to skip and which topics they can cover more economically using other curriculum materials. David described that he and his professional learning community (PLC) design supplemental materials when they "feel the need to expand on a topic," want to "approach a topic a different way," or are accommodating for time constraints. David also claimed that some topics in *Pathways* "are overdone" and as a result cost he and the other teachers too much instructional time. David explained that in these situations he and his PLC design materials to simplify the content for the students. David then voiced his concern that the novelty of the *Pathways* curriculum does not easily allow teachers to incorporate worksheets and activities they have used in previous years—worksheets that permit teachers to cover the material at a quicker pace (see Excerpt 85).

Excerpt 85

- 1 David: If I was teaching out of a traditional book, I can just, you know, Google the topic and twenty worksheets pop up. But, um, some of the ways that the material is presented in this *Pathways* textbook ... if we want to create an additional worksheet—we don't think they got enough practice—creating a test, creating a quiz, it's not like we can go pull questions from another book. So materials do come in because sometimes we go, 'I really would like something else but there's nothing that can connect to this.'

It is interesting that David appraised the *Pathways* curriculum as an instructional constraint, not because it compromised the quality of mathematical meanings students had the opportunity to construct—the *Pathways* curriculum was designed for the purpose of supporting students’ conceptual mathematics learning—but rather because it obstructed his goal of having enough instructional time to cover a particular amount of material.

As with the instructional constraints David identified in the pre-lesson interviews and in his journal entry responses, David did not describe the constraints he cited in the first Phase III clinical interview, although some genuine and justifiable, as circumstances that impeded his ability to support students in constructing particular ways of understanding mathematical ideas.

Implicit theory of learning. In his response to Question 6 in Table 64, David explained that a highly effective teacher must “have high content knowledge” and must “understand the way kids learn and process.” To determine whether David’s latter statement suggested his recognition of the necessity of maintaining an explicit and disciplined way of thinking about what learning is and how one might engender it, I asked him to describe what he meant by his claim that a highly effective teacher must “understand the way kids learn and process.” In response to my question, David explained that teaching is an art and that he has observed colleagues who have pursued teaching as a second career who “know the mathematics as far as they can solve the equation, they can set up the equation, they can graph the equation, they can do everything you’d need them to do with the equation, but they have no idea how to connect that with the kids.” David continued by elaborating what these second-career

teachers cannot do: “They don’t understand pacing. They don’t understand ways to break it down.” I asked David to articulate specifically what these second-career teachers often do not know. I provide David’s response in Excerpt 86.

Excerpt 86

1 David: Usually people skills. That is the big thing that’s lacking. Uh, [they] don’t always understand adolescents. ... It’s a pitfall that a lot of teachers have is that you picked the subject because you fell in love with the subject. Um, and so then because you loved it, it probably was natural to you, it came easy to you so you don’t understand why they don’t understand it. ... I’m not sure I could teach somebody to do that ... to read kids and to understand what a kid’s need is. ... It can’t just be about knowledge base. It has to be about being able to connect with the students.”

David’s statement reveals that by “understand the way kids learn and process” he meant that a highly effective mathematics teacher must be able to comfortably interact with adolescents and must present ideas in a way that anticipates typical student errors. David’s instructional actions, therefore, did not appear to have been informed by an explicit theory of learning.

Moments of Instructional Deviation

There were several occasions in which during a pre-lesson interview David expressed his intention to support a way of understanding that differed from the understanding he conveyed in the context of instruction. I presented David with three such occasions in an effort to elicit his rationale for those instances in which he may have

intentionally deviated from his lesson plan in response to circumstances or events he appraised in the context of teaching as instructional constraints. Specifically, I presented David with three pairs of excerpts, each containing a selection of audio from a pre-lesson interview and a video clip from his classroom teaching, and asked David to characterize the meaning he conveyed in each excerpt and to compare and contrast these meanings. I then asked David to justify any discrepancies he recognized in the ways of understanding he conveyed in the pre-lesson interview and classroom instruction excerpts. In this subsection, I present my analysis of David's and my conversation around one of the three pairs of excerpts I selected to exemplify *moments of instructional deviation*. I discuss only one of these pairs of excerpts since the conclusions I draw from my analysis of it is consistent with the conclusions that resulted from my analysis of the other two.

The first excerpt I presented to David was from Pre-Lesson Interview 1. In this excerpt, David explained that he expected his students to conceptualize the task of assigning numerical values to the “openness” of an angle by “starting to think in terms of ratios.” David emphasized in this excerpt that the measure of an angle is a quantification of “the proportion of the circle” subtended by the angle, and that his instructional objective is to support students’ understanding of angle measure as “a part of the whole circle ... it’s a proportion of the total circumference.” The excerpt concluded with David reiterating that he would consider his lesson a success if students develop the understanding of “angle measures as part-whole, as the ratio of the part to the whole.” It is noteworthy that this way of understanding angle measure does not describe a quantity being measured, nor identify a unit with which to measure it.

I asked David to describe the meaning of angle measure he stressed in the pre-lesson interview excerpt. David claimed that he conveyed that openness “is hard to wrap your head around,” but that one might conceptualize an angle’s measure as the percentage or proportion of the circle the angle subtends. David explained, by way of example, that one degree subtends $1/360^{\text{th}}$ of the circle centered at the angle’s vertex, so even degrees convey a “part-whole relationship.” David’s interpretation of the meaning of angle measure he conveyed in this pre-lesson interview excerpt, while at times faint and circuitous, appeared consistent with my interpretation: Angle measure as a proportion of the circle centered at the vertex of the angle, or as the ratio of subtended arc length to circumference.

After David characterized the way of understanding angle measure he emphasized in the excerpt from Pre-Lesson Interview 1, I presented him with a video excerpt from the first lesson of Module 8. In this video, David endorsed, restated, and elaborated a students’ suggestion to compare the openness of two angles by examining the lengths of the arcs (of equal radii) the angles respectively subtend. In so doing, David supported students’ in recognizing subtended arc length as the quantity one measures when measuring angles. I again asked David to define the way of understanding he conveyed in the excerpt. David claimed that he described angle measure as “measuring along an arc.” David then explained that the student who made the initial suggestion was “not going to part-whole, ... not really doing the part-whole connection.” Again, David’s remarks gave me no reason to believe that his interpretation of the meaning he supported in this excerpt from Lesson 1 differed from my own interpretation.

After having listened to the audio excerpt from Pre-Lesson Interview 1 (in which David discussed angle measure as the fraction of the circle subtended by the angle), and after having watched the video excerpt from Lesson 1 (in which David described angle measure as the length of the arc the angle subtends), I asked David to compare the meaning of angle measure he conveyed/supported in each excerpt. (Recall that in Lesson 1 David discussed a way to compare the openness of two angles.) David confidently declared that the meanings he communicated in both excerpts were the same if one considers the radii of the circles respectively centered at the vertices of the angles from Lesson 1 to be one of some standard linear unit. David justified this condition by claiming that when it is satisfied, the ratio of subtended arc length to circumference is inconsequential because the same circumference divides both subtended arc lengths, so one can just compare the openness of the angles by comparing the lengths of their subtended arcs. That David interpreted the very different meanings (from my perspective) he conveyed in each excerpt as essentially equivalent since they produce the same result (an accurate comparison of the openness of two angles) reveals his inattention to the nuances of the ways of understanding he conveyed. Recognizing such nuances requires an awareness of the mental activity that characterize these ways of understanding, an awareness I have repeatedly demonstrated David did not possess.

Surprised that David did not recognize the meaning he described in the pre-lesson interview as differing from the meaning he supported in Lesson 1, I pointedly asked him to identify *any* discernable differences between the ways of understanding he conveyed in these excerpts. I provide David's reply in Excerpt 87.

Excerpt 87

1 David: I want [the student] to get to a proportion and he is still not thinking as a direct proportion. He's not making that connection yet so he's still unfortunately at this point stuck in physical measurements and using length units and that's what we want to get past but it's heading in the right direction. So he's not where I want him to be but he's moved me half way. At least we've gotten to arcs.

While David's response was not an answer to my question, he did explain that the student's way of understanding angle measure (and thus the understanding David endorsed) was unsatisfactory because the student did not describe angle measure as a proportion. David did, however, recognize the student's understanding as "heading in the right direction" since the student attended to one of the quantities that comprise the "proportion" of subtended arc length to circumference. That David interpreted the student's thinking as being en route to conceptualizing angle measure as a "direct proportion" is perhaps the reason he failed to recognize a significant distinction between the student's way of understanding angle measure and the way of understanding he professed the intention to support in the pre-lesson interview.

An objective interpretation of David's remark in Excerpt 87 would leave one with the impression that David had never previously demonstrated an understanding of angle measure as the length the arc an angle subtends measured in units that are proportionally related to the circumference of the circle containing the subtended arc. Of course, as I established in Chapter 6, David demonstrated this way of understanding on several occasions throughout the series of TBCIs. It is therefore revealing that David so

confidently interpreted the student's reasoning—which he himself supported and reiterated—as unproductive. That David questioned the utility of a meaning he had previously employed on several occasions further suggests that he had not achieved clarity relative to the ways of understanding he designed his instruction to promote.

I concluded our conversation of the excerpts from Pre-Lesson Interview 1 and Lesson 1 by asking David to explain why he endorsed the students' understanding of angle measure as the length of the arc an angle subtends, considering the way of understanding angle measure he professed the intention to support in his teaching was not entirely compatible with it. Excerpt 88 contains David's response.

Excerpt 88

- 1 David: It's because I know this was an early lesson so this is a positive step and it was a connection and I didn't just want to give him the answer. So, um, it was moving us in the right direction. ... I accepted it as in, 'This is where the students are now' and it's not necessarily where I want them to end up and it's something that I would note in the back of my mind, 'We need to fix this.' ... I know that future questions and future prompts are going to move us past that. ... This is a positive step forward to what my goal is. It's not there yet and either I thought I'd get to the goal later on or it was something that I thought I would address later."

David's reply reveals that the discrepancy I observed between the way of understanding angle measure he stated the intention to support during Pre-Lesson Interview 1 and the way of understanding angle measure he endorsed during the first

lesson was not the result of his responding to or accommodating for a particular event or circumstance he appraised as an instructional constraint.

Perhaps more revealing is that David rationalized his endorsement of the student's "angle measure as arc length" way of understanding because he saw it as a "positive step" toward the more advantageous understanding of "angle measure as a proportion." David contradicted his preference for this latter way of understanding at other moments during the first Phase III clinical interview. For example, David explained, "I think the length idea is more productive in the long run. . . . Part-whole is what they were used to. I want them to get to length." Therefore, on some occasions during the Phase III clinical interviews David claimed that understanding angle measure as a ratio of subtended arc length to circumference is more productive than understanding angle measure as the length of the subtended arc, whereas on other occasions he claimed to prefer the "arc length" way of understanding angle measure to the "proportion" way of understanding. David's inconsistent partiality for one of these ways of understanding angle measure over the other further supports my claim that his instructional actions were not informed by a clear and consistent image of the meanings he intended his students to construct.

Moments of Instructional Incoherence

In his teaching, David frequently supported dissimilar, often incompatible, ways of understanding particular mathematical ideas, what I refer to as *moments of instructional incoherence*. As with the previously discussed *moments of instructional deviation*, I used video excerpts of three such occasions as artifacts of classroom practice to discuss with David in the Phase III clinical interviews. My intention in doing so was to determine if the moments in which David conveyed discrepant meanings during

instruction resulted from his reacting to the circumstances and events he appraised as constraints on his practice, and if so, to ascertain the effect of David's construction of these constraints on the quality of his enacted mathematical knowledge. In this subsection, I discuss in detail my analysis of our conversation around one of the three pairs of video excerpts I selected from David's classroom teaching to exemplify a *moment of instructional incoherence*. I exclude a discussion of my analysis of David's and my conversation around the second and third pairs of video excerpts since the conclusions drawn therefrom are consistent with those presented.

The first video excerpt I showed David was from Lesson 4 and showed him facilitating a discussion around Task 10 on Investigation 2 (see Table 52 for a statement of the task and see Excerpt 60 for David's discussion of the task). This task displayed an angle with a measure of 0.45 radians with three circles centered at its vertex of radii 2 inches, 2.4 inches, and 2.9 inches respectively, and prompted students to determine the length of the three arcs subtended by the angle. The curriculum developers' intention was for students to interpret "0.45 radians" as suggesting that the length of all arcs subtended by the angle are 0.45 times as long as its radius. Therefore, one may determine the length of the three subtended arcs (in inches) by multiplying 0.45 by their respective radii.

Prior to discussing an approach to solving the problem, David reiterated that angle measures quantify "the portion of the circle," by which he meant the fraction of the circle's circumference subtended by the angle. Consistent with this emphasis, David supported his students in interpreting an angle with a measure of 0.45 radians as subtending $0.45/2\pi^{\text{th}}$ s of the circumference of the circle centered at its vertex (see Excerpt 60). David then asked, "How would I use that to figure out the arc length?" In doing so

David guided a student to propose the formula $S = \frac{0.45}{2\pi}(2\pi r)$, where S represents the length of the subtended arc (in inches) and r represents the radius of the subtended arc. As I discussed in my analysis of Excerpt 60, David's immediate inclination to support students in multiplicatively comparing the measure of the angle in radians with the number of radians "in a circle" to obtain the fraction of the circle's circumference subtended by the angle circumvented the curriculum designers' intention of providing an occasion for students to apply their understanding of radians as the length of the arc an angle subtends measured in units of the radius of the subtended arc.¹¹⁵

To determine whether David's interpretation of the meaning he conveyed in the first video excerpt was consistent with my interpretation, I asked him to describe the way of understanding angle measure he supported in his discussion of Task 10 on Investigation 2. David replied, "We were just looking for the proportion of a circle. ... So 0.45 is a radian measure so we can think of that as radius lengths. And out of the whole circle, which has 2π radius lengths in measure, so it's 0.45 divided by the 2π ." David's response reveals his recognition that he applied his "part-whole" understanding of angle measure while guiding a student to solve Task 10. David's and my interpretations of the first video excerpt, therefore, appeared compatible.

After David described the meaning of angle measure he conveyed in the first video excerpt, I prompted him to comment on the desirability of the student's suggestion to determine the length of the subtended arc by multiplying the circumference of the

¹¹⁵ Note that the discussion in Excerpt 60 occurred after David introduced students to the idea of radians.

circle containing the subtended arc by the fraction of the circumference of this circle the angle subtends (Excerpt 89).

Excerpt 89

- 1 Michael: Did the student approach the task in the way you were expecting or hoping he would?
- 2 David: ... The good thing about the way he approached it is that this is a way to get the solution. Uh, the good thing about this way is that from here we can see that the $2\pi s$ are going to cancel each other out and so we get where we want to go anyways, um, which is radian measure times radius length gives you arc length.

David's response in Line 2 of Excerpt 89 reveals his approval of the student's proposal to determine the length of the subtended arc using the formula $S = \frac{0.45}{2\pi}(2\pi r)$.

Prior to presenting David with the second video excerpt, I asked him to describe what he would change about his teaching of Task 10 on Investigation 2 if given the opportunity. David confidently replied, "I can't think of a way to present it better." Thus, David did not appear to recognize that in the first video excerpt he supported an unproductive way of understanding what it means to measure an angle in radians, or at least an understanding inconsistent with the curriculum designers' intentions.¹¹⁶

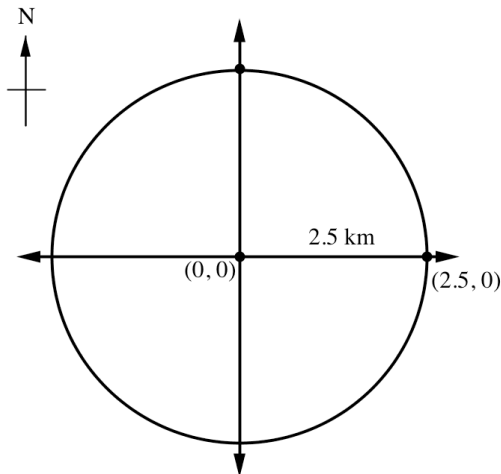
¹¹⁶ In the instructor version of Investigation 2, the *Pathways* curriculum contains the following note under Task 10: "Students should reason that 0.45 radians conveys that the arc length is 0.45 times as large as the length of the radius. If the radius length is 2 inches, then the subtended arc is $0.45(2) = 0.9$ inches." Therefore, David's endorsement of the student's proposal to determine the length of the subtended arc by multiplying the circumference of the circle by fraction of the circle's circumference subtended by the angle was inconsistent with the curriculum designers' intentions.

After our discussion the video depicting his teaching of Task 10 on Investigation 2, I presented David with a video excerpt from Lesson 9 that displayed him facilitating a conversation around Task 6(a) on Investigation 4 (see Table 65 for a statement of the task and see Excerpt 61 for David’s discussion of the task). A student, who for the sake of definiteness I refer to as Student 1, explained that her approach to solving Task 6(a) was to divide the length the skier traveled by the circumference of the circular trail ($\frac{2.75}{2.5(6.28)}$). David questioned the student’s approach by asking, “Why are we dividing by the whole circumference?” The student struggled to articulate her rationale so David asked another student (Student 2) to share his method for solving the problem. Student 2 determined the number of radius lengths the skier traveled around the trail by dividing the number of kilometers the skier traversed (2.75) by the radius of the circular trail (2.5 kilometers), an approach David validated.

Table 65

Investigation 4, Task 6(a) (Carlson, O’Bryan, & Joyner, 2013, p. 482)

For Exercises #6-8, use the following context: An arctic village maintains a circular cross-country ski trail that has a radius of 2.5 kilometers.



6. A skier started skiing from the position (2.5, 0) and skied counter-clockwise for 2.75 kilometers before stopping for a rest.

- a. How many *radius lengths* did the skier travel around the trail? Explain your reasoning.

I asked David to describe how Student 1 might have been thinking about the task. In response David explained, “The first student is trying to do part over whole. ... So they’re trying to figure out the percentage of the circle that they have. ... She’s changing this to the proportional thinking, so part over whole.” David then admitted, “So this (*Student 1’s reasoning*) is really similar to what we had in the last clip.” David obviously recognized the strategy Student 1 employed as analogous to the solution approach he supported in Lesson 4 while discussing Task 10 on Investigation 2. David later acknowledged, however, that Student 1 did not multiply the ratio she proposed ($\frac{2.75}{2.5(6.28)}$), which represented the fraction of the circular trail the skier traversed, by 2π to obtain the distance the skier traveled in units of radius lengths. David then remarked that even if Student 1 had done so, “there are units that are just going to cancel out and it makes it counterproductive because you’re going to circumference and there was no reason to actually make the trip all the way to circumference.” This criticism of Student 1’s suggestion is perplexing not only because David unnecessarily supported the same approach during his discussion of Task 10 on Investigation 2, but more so because he previously explained that he could not imagine presenting the solution to Task 10 in a more effective way.¹¹⁷

To my surprise, David claimed that Student 1 (who determined the number of radius lengths the skier traveled by dividing the number of kilometers the skier traversed

¹¹⁷ It is important to note that the *Pathways* curriculum supported the same way of understanding angle measure in radians at these two different points of instruction: *angle measure in radians as the length of the arc an angle subtends measured in units of the radius of the subtended arc.*

by the circumference of the circular trail in kilometers) and Student 2 (who determined the number of radius lengths the skier traveled by dividing the number of kilometers the skier traversed by the radius of the circular trail in kilometers) demonstrated “the same thinking” (Excerpt 90).

Excerpt 90

- 1 David: Both [students] have the same thinking. They’re setting up two proportions and they’re multiplying and doing operations to get the, um, the answer that they desire but they had to start with the proportion.
- 2 Michael: So what do you think the proportion was that this student (*Student 1*) started with?
- 3 David: So their proportion would have been (*writes “ $2.75/(2\pi \cdot 2.5) = \theta/(2\pi)$ ”*). ... So they have the angle is to 2π the same way that the arc length is to circumference. So that’s their proportional thinking. ... This (*Student 2’s solution*) is understanding that since we have 2π here and 2π here (*points to the 2π s on each side of the equality “ $2.75/(2\pi \cdot 2.5) = \theta/(2\pi)$ ”*) ... we can basically factor the 2π out and so we can just go with the simplified version.
- 4 Michael: ... How would the second student have known just to do this (*points to the fraction “ $2.75/2.5$ ”*)? ...
- 5 David: Basically it’s a unit change and we’re saying we have, uh, 2.75 kilometers and we want to divide it by the new unit of 2.5 kilometers in this new unit, which the new unit is radius lengths.

So he's almost probably thinking the unit conversion way (*writes* " $2.75 \text{ km} \cdot \frac{1 \text{ rad}}{2.5 \text{ km}}$ "). ... So the proportion he's thinking of is one radius length is to this (*points to where he wrote* "2.5 km") and then my final answer will be in radius lengths because of unit conversion.

- 6 Michael: How would this student (*Student 2*) have known to do this unit conversion?
- 7 David: Well this is the kind of thing they do in their science classes so a unit conversion like this would be pretty common.

Although David claimed that Student 1 and Student 2 demonstrated "the same thinking," his remarks in Excerpt 90 reveal that he did not interpret either student's solution strategy as suggesting a particular way of understanding, but instead was only able to discern the potential origin of the procedure each student enacted. To David, the similarity in the thinking both students demonstrated was a common characteristic of the supposed procedures from which these student's answers derived. That David described only hypothetical computations that resulted in the students' answers suggests his incapacity to characterize how his students conceptualize mathematical ideas, attending instead to the procedures they enact and the solutions they produce. I find it particularly remarkable that rather than recognizing that Student 2 likely interpreted the ratio $2.75/2.5$ as the measure of the distance the skier traveled in units of the radius of the circular trail (a quantitative operation), David claimed in Line 5 that the student probably used a unit conversion procedure regularly employed in science class. It seems reasonable to expect that David's inattention to the ways of understanding from which students' solution

strategies derived was a consequence of his unawareness of the mental activity that comprise the ways of understanding he intended his instruction to engender. After all, for a teacher to maintain a disposition to attend to students' thinking, the teacher must be aware of the fact that there is indeed thinking to which he or she may attend. David's remarks in Excerpt 90 illustrate the debilitating role that his unawareness of the mental activity comprising productive ways of understanding had on his capacity to construct models of students' thinking.

David later recognized that Student 2's solution may have resulted from him solving the equation $S = \theta \cdot r$ (where S represents the length of the subtended arc in some standard linear unit, r represents the length of the radius of the subtended arc in the same standard linear unit, and θ represents the measure of an angle in radians) for θ . David expressed preference for this approach "because it's simpler and it's to the direct variation relationship that we want to establish." I asked David to explain, then, why he led the student in the first video excerpt to determine the length of an arc an angle subtends by multiplying the circumference of the circle containing the subtended arc by the fraction of the circle's circumference subtended by the angle. I also asked David why he did not accept Student 1's approach of dividing a subtended arc length by the circumference, considering he supported this approach in the first video while discussing Task 10 on Investigation 2. In response, David explained that the purpose of Task 10 was to allow students to "discover that radian measure times radius length will give you arc

length.”¹¹⁸ Accordingly, David claimed that using the formula $S = \frac{\theta}{2\pi}(2\pi r)$ to determine the length of the subtended arc, S , allowed students to see that they can cancel the 2π s to obtain $S = \theta \cdot r$. After this formula was introduced in Investigation 2, David explained, he expected his students to use it to solve Task 6(a) on Investigation 4, which is why he privileged Student 2’s approach. Referring to Student 1’s method of dividing subtended arc length by circumference, David said, “this step was extra” and explained that “there is an easier formula to use.” What is noteworthy about David’s rationale is that it reveals that the discrepancy I observed between the video excerpts I presented to David did not result from his responding to or accommodating for particular circumstances or events he appraised as instructional constraints. The observed discrepancy was instead a consequence of David’s lacking mathematical knowledge, particularly his unawareness of the mental activity involved in constructing a productive and coherent way of understanding angle measure.

I explained at the beginning of this subsection that my intention in discussing with David videos of his teaching that exemplify *moments of instructional incoherence* was to determine if the moments in which David conveyed discrepant meanings during instruction resulted from his reacting to instructional constraints, and if so, to ascertain the effect of David’s construction of these constraints on the quality of his enacted mathematical knowledge. My analysis of David’s and my conversations around all three pairs of video excerpts revealed that the moments in which he conveyed discrepant, inconsistent, or incompatible ways of understanding in his instruction—which he often

¹¹⁸ That David interpreted this as the purpose of Task 10 on Investigation 2 further supports my claim that David was not consciously aware of the meanings he intended his instruction to support, nor of those promoted in the *Pathways* curriculum.

did not notice—were not occasioned by his responding to the circumstances and events he appraised as constraints on his practice.

Moments of Mathematical Concession

As I discussed in Chapter 7, on several occasions David demonstrated ways of understanding during the series of TBCIs that were inconsistent or incompatible with the ways of understanding his instruction supported. I selected three such *moments of mathematical concession* to discuss with David during the final Phase III clinical interview. Specifically, I presented David with three pairs of videos, each containing a video excerpt from the series of TBCIs and a video excerpt from his classroom teaching, for the purpose of determining if David willingly compromised the quality of his enacted mathematical knowledge in response to circumstances and events he interpreted as instructional constraints. In this subsection, I discuss in my analysis of our conversation around two of the three pairs of video excerpts I selected to illustrate *moments of mathematical concession*. I do not discuss my analysis of David's and my conversation around the third pair of video excerpts since the conclusions drawn therefrom are consistent with those I present below.

Moment of Mathematical Concession 1. I presented David with a video excerpt from Lesson 1 in which he explained that the measure of an angle is unit-less since the standard linear units one uses to measure the length of the subtended arc and the circumference of the circle containing the subtended arc “cancel” when one computes the ratio of these lengths (see Excerpt 56). As I previously explained in my analysis of this segment of instruction, David did not provide students with an opportunity to interpret the division of subtended arc length and circumference as the numerator measured in

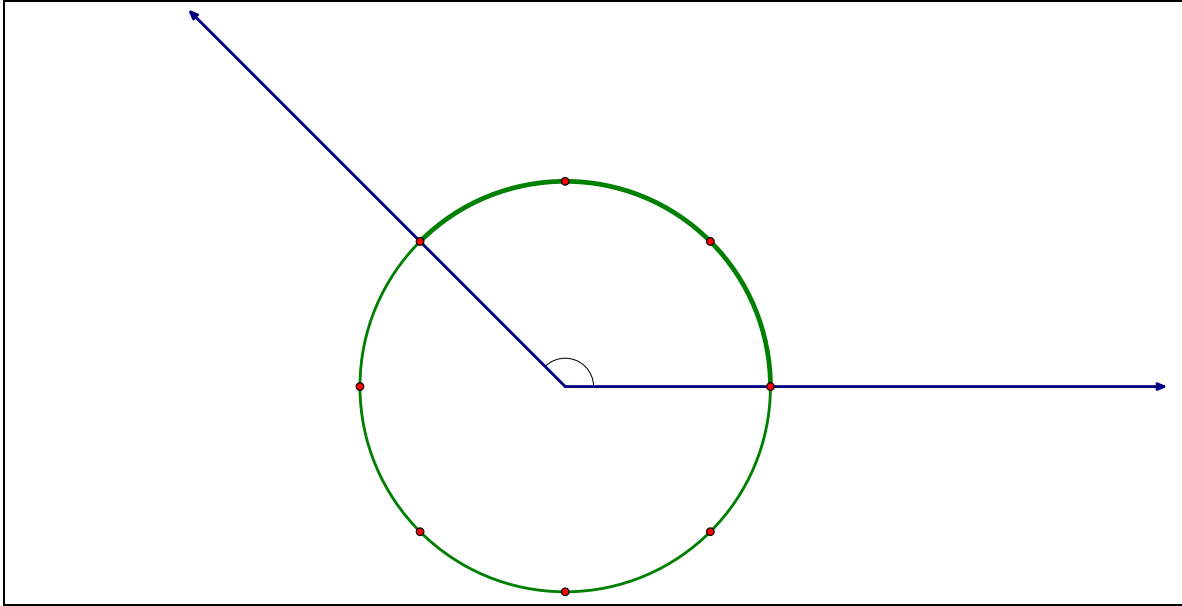
units of the denominator but rather as the ratio of two values. David therefore supported students in understanding the ratio of subtended arc length to circumference as an arithmetic operation as opposed to a quantitative operation. After having watched the video, I asked David to describe the meaning he conveyed. David's response, "Because we're comparing proportions of arc length to circumference, then we no longer care about the units of measure" led me to believe that he interpreted the video in the way I expected him to.

I then showed David the video excerpt from TBCI 3 in which he responded to the Nick-Meghan task (see Table 66 for a restatement of the task and see Excerpt 24 for David's response). Recall from my analysis of David's response to this task that he appeared to coordinate his two ways of understanding angle measure (*angle measure as the length of the subtended arc* and *angle measure as the fraction of the circle subtended by the angle*). In particular, David recognized that Nick measured the subtended arc length in units of circumference and Meghan measured the subtended arc length in units of $1/8^{\text{th}}$ of the circumference. David therefore assimilated Nick and Meghan's claims as two different instantiations of the same process: *measuring the subtended arc length in a particular unit*.

Table 66

Task 5 from TBCI 3

Nick claims that the measure of the angle shown is $3/8^{\text{th}}$ and Meghan claims that the measure of this angle is three. How is Nick thinking about measuring this angle? How is Meghan thinking about measuring the angle? Are they both correct?



After having watched the video clip in which he responded to the Nick-Meghan task, David reluctantly made the remark in Excerpt 91.

Excerpt 91

- 1 David: Now that I'm thinking about it and listening to what I said, I'm not even sure if I would call the full circle a unit. I think he's thinking more in terms of a proportion of the whole. I'm not really even sure if I want to call it a unit. ... Meghan is using a unit; she's using three quips or whatever. ... He's thinking of it as a proportion. Like I said I'm not sure if I would even want to call it a unit.

After having reflected for a few minutes, David recognized, "The only units that $3/8^{\text{th}}$ could go with is circumference. It's $3/8^{\text{th}}$ of circumference. That's the only unit, if we were to assign it a unit that's the only unit that I can think of that would be appropriate." David's hesitation and reluctance suggests that his interpretation of the way

of understanding he conveyed in his response to the Nick-Meghan task was probably not entirely consistent with my interpretation. David nonetheless acknowledged that the circumference is only unit to which Nick's measure of $3/8^{\text{th}}$ s could refer.

Once I assessed the extent to which David's interpretation of the video excerpts was consistent with mine, I asked him determine whether the ways of understanding he conveyed in the video clip from Lesson 1 (wherein he explained that angle measures are unit-less because units cancel when one computes the ratio of subtended arc length to circumference) was consistent with the way of understanding he conveyed in the video clip from TBCI 3 (wherein he recognized that both Nick and Meghan measured the length of the subtended arc, albeit in different *units*).

Excerpt 92

- 1 Michael: Is the way of understanding that you discuss in this clip (*TBCI 3*) consistent with the way of understanding you conveyed in this one here (*Lesson 1*)?
- 2 David: Um, yeah because we're trying to get to like Nick's way of thinking. So like Nick was just talking about the proportion and how he like didn't really kind of have units. ... So we were kind of leading to Nick's way of thinking about it so it does kind of match along with Nick's way of thinking.

David's response in Line 2 of Excerpt 92 plainly reveals that he did not recognize the discrepancy I noticed in the way of understanding angle measure he conveyed in both video excerpts. David interpreted $3/8^{\text{th}}$ s, the measure of the angle Nick proposed, as being a unit-less measure, which is consistent with the way of understanding he emphasized in

the video clip from Lesson 1. I subsequently asked David to describe what he would change about the instructional episode depicted in the video excerpt from Lesson 1. David responded, “I don’t know if I would actually change any of it because I do like the fact that when we get to radians that radians are kind of without units. So I do like the fact that we’re talking about the units kind of canceling.” David’s failure to recognize that he conveyed what I consider vastly different meanings of angle measure in the two video excerpts, as well as his assertion that he would not change anything about his instruction from the Lesson 1 video, even after having seen a video wherein he demonstrated a productive way of understanding, supports my oft-stated claim that David was not consciously aware of the mental actions that comprise the meanings he intended to promote in his instruction. Such conscious awareness would likely have equipped David with the cognitive schemes to recognize the discrepant and incompatible ways of understanding he conveyed in both video excerpts.

Moment of Mathematical Concession 2. I presented David with a video from TBCI 4 in which he used the Joe applet (see Table 67) to successfully approximate the values of $\sin(0.5)$ and $\cos(\frac{3}{4})$. David interpreted the task of approximating the value of $\sin(0.5)$ as, “Estimate how many radius lengths is Joe north of Abscissa Boulevard when the angle traced out by his path is 0.5 radians.” In particular, David interpreted the 0.5 as representing the number of radius lengths that Joe had traveled along Euclid Parkway and $\sin(0.5)$ as representing Joe’s distance north of Abscissa Boulevard in units of radius lengths. David similarly interpreted the task of approximating the value of $\cos(\frac{3}{4})$ in the following way: “Estimate how many radius lengths is Joe to the east of Ordinate Avenue when his path has traversed an arc that is $\frac{3}{4}$ ths times as long as the radius of Flatville.”

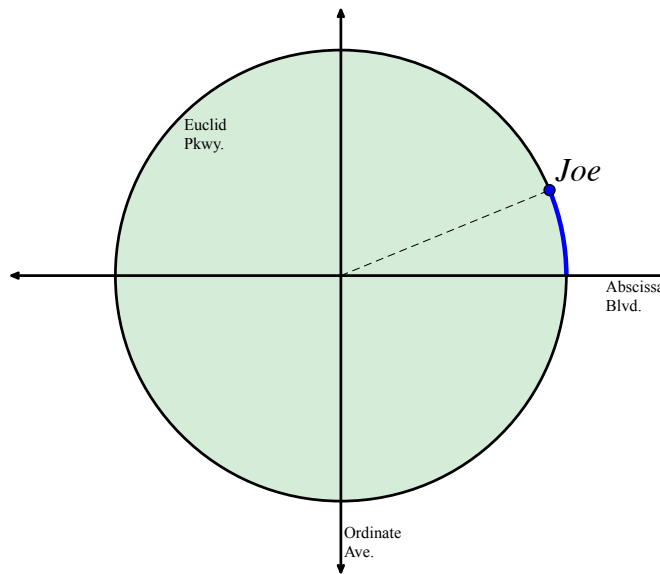
Upon having watched the video excerpt from TBCI 4, David described the way of understanding he demonstrated in a way that was compatible with my interpretation.

David also emphasized that the understanding he conveyed in the video is important for students to construct.

Table 67

Joe Applet

Suppose Joe is riding his bike on Euclid Parkway, a perfectly circular road that defines the city limits of Flatville. Ordinate Avenue is a road running vertically (north and south) through the center of Flatville and Abscissa Boulevard is a road running horizontally (east and west) through the center of Flatville. Assume Joe begins riding his bike at the east intersection of Euclid Parkway and Abscissa Boulevard in the counterclockwise direction.



I then presented David with a video excerpt from Lesson 7 in which he defined the outputs of sine and cosine relative to the following two cases: (1) when the radius of the circle centered at the vertex of an angle has a measure of one unit and (2) when this radius does not have a measure of one unit (see Excerpt 70). Specifically, in the video excerpt David claimed that if the radius of the circle has a measure of one unit, then the sine and cosine values of the angle's measure are respectively equal to the y - and x -

coordinate of the terminus of the subtended arc. David then explained that if the radius of the circle centered at the angle's vertex does not have a measure of one unit, then the values of sine and cosine are given by the respective ratios of the y - and x -coordinate of the terminal point to the length of the radius. After having watched the video excerpt from Lesson 7, David acknowledged that he was communicating the outputs of sine and cosine as ratios that "give the percentage of a radius length out."

I then asked David to determine if the way of understanding he supported in the video excerpt from Lesson 7 differed from the understanding he employed to approximate the value of $\sin(0.5)$ and $\cos(\frac{3}{4})$ in the video clip from TBCI 4 (Excerpt 93). Excerpt 93

- 1 Michael: Is there any way that the understanding of sine and cosine you convey in this clip (*Lesson 7*) is different from what you did here (*TBCI 4*)?
- 2 David: Only in the units of measure that we started with to obtain the ratio, but in the end we end up with an output that is a proportion of the entire radius. So in the end, no [they aren't different]. In the end they end up giving me the same thing.

As with so many previous occasions, David did not recognize the way of understanding he demonstrated in the first video excerpt as being fundamentally different from the way of understanding he conveyed in the second. David's response in Line 2 of Excerpt 93 focused primarily on the outcome of his application of two discrepant (from my perspective) ways of understanding instead of attending to the ways of understanding themselves. Like the several other occasions in which David demonstrated an incapacity

to attend to ways of understanding—either his own or his students’—his remarks in Excerpt 93 demonstrate that he had not achieved clarity relative to the mental activity involved in his own ways of understanding, nor of those he intended to support in his teaching. Had David done so, he would likely have been positioned to notice the discrepant meanings he conveyed in the videos I presented.

I presented David with three pairs of videos that exemplified *moments of mathematical concession* in an effort to determine if David willingly compromised the quality of his enacted mathematical knowledge in response to circumstances and events he interpreted as instructional constraints. My analysis of our conversations around all three pairs of video excerpts revealed that the moments in which David demonstrated ways of understanding that he did not support in his instruction—none of which David noticed—were not occasioned by his responding to the circumstances and events he appraised as instructional constraints.

Summary of Chapter

I conclude this chapter with a summary of the circumstances David appraised as constraints on his practice, and of the instructional goals and objectives such constraints obstructed David from achieving. I also describe the role of David’s mathematical knowledge in the construction of such instructional goals and objectives and propose an explanation for the rather insignificant role David’s image of instructional constraints had on the quality of his enacted subject matter knowledge.

David’s responses to select questions in the pre-lesson interviews revealed that he appraised the following circumstances as constraints on his practice: (1) having insufficient time to cover the required amount of curriculum material, (2) students’

lacking or disingenuous motivation and engagement, (3) his feeling responsible for preparing students to achieve procedural fluency in calculus, (4) students' insufficient prior knowledge, (5) students' inattention to instructions and instructional expectations, (6) inadequate time to assess students' learning and performance, and (7) the inconsistency with which students learn the facts and procedures he communicates. The instructional constraints David's responses in the pre-lesson interviews revealed suggest a number of his instructional goals: (1) progressing through the content at a pace consistent with departmental recommendations, (2) engineering a social content that encourages students' engagement and fosters their motivation, (3) equipping students with the factual and procedural knowledge that will enable them to fluently solve problems in calculus, and (4) creating a work environment that is free of time-wasting and energy-consuming annoyances and distractions.

The instructional goals that the constraints David identified in the pre-lesson interviews inhibited him from achieving were not goals defined in terms of the ways of understanding he intended his students to construct; that is, they were not goals of students' conceptual learning. That David did not recognize the aforementioned instructional constraints as obstructing the quality of mathematical meanings his students had the opportunity to construct is not altogether surprising considering that, as a result of his unawareness of the mental actions and operations comprising his own cognitive schemes, he was not positioned to define his instructional goals and objectives in terms of the ways of understanding he intended students learn (see Chapter 7). After all, one can only appraise circumstances as hindering students' ability to construct particular ways of

understanding if he or she maintains as a goal supporting students in constructing particular ways of understanding.

David's responses to the ten journal entry prompts in Table 62 transparently revealed the following as circumstances he appraised as constraints on his practice: (1) having insufficient time to progress through the curriculum materials, (2) students' lacking motivation and engagement, (3) students' deficient prior knowledge, (4) the inconsistency with which students learn facts and procedures, and (5) his tendency to communicate directions and expectations with insufficient clarity. As I emphasized in my discussion of David's journal entry responses, he did not appraise the aforesaid circumstances as obstructing his ability to support students in constructing particular ways of understanding mathematical ideas. This is plainly evidenced by the fact that David did not: describe deviations from his lesson plan as being in the service of supporting students' construction of particular mathematical meanings, rationalize suggested changes in his teaching by appealing to the way such changes might improve the quality of understandings his instruction supported, discuss the events or circumstances he found problematic or unexpected as having anything to do with how his students conceptualized mathematical ideas, describe unintended outcomes of particular lessons in terms of the understandings that students did or did not appear to have constructed, and discuss the consequences of the instructional constraints he identified as interfering with the ways of understanding he afforded students the opportunity to learn. Provided David did not define his instructional goals and objectives in terms of the ways of understanding he intended his students to construct, it is not surprising that the

instructional constraints he identified were not constraints on the quality of mathematical meanings his instruction supported.

David's responses to select preliminary questions during the first Phase III clinical interview revealed that his instructional goals consisted of the following: (1) connecting with students and enhancing their love of learning, (2) providing students access to high quality post-secondary education, (3) supporting students in appreciating the relevance of mathematics to their lives, (4) engineering a classroom environment wherein peer pressure does not discourage students from asking and answering questions or from participating in group or whole-class discussions, and (5) creating a learning context in which students feel secure not knowing answers to questions and problems. While David may have recognized that achieving these goals contributes to enhancing the quality of students' mathematics learning, it is noteworthy that David did not explicitly mention among these goals supporting students in constructing particular ways of understanding mathematical ideas. This is likely a byproduct of David having been unaware of the mental activity that comprises such ways of understanding.

David identified the following instructional constraints in his responses to several preliminary questions I asked at the outset of the Phase III clinical interviews: (1) a lack of instructional time, (2) the unreceptiveness of the *Pathways* curriculum to supplementation, (3) students interpreting his requests for them to explain or justify their answers as him being intimidating or coercive, and (4) the cultural expectation that mathematics is a subject at which it is acceptable to be inept. As with the instructional constraints David identified in the pre-lesson interviews and in his journal entry responses, David did not describe the constraints he cited in the first Phase III clinical

interview as circumstances that impeded his ability to support students in constructing particular ways of understanding. I again interpret this as likely a byproduct of David having been unaware of the mental activity that comprise the ways of understanding he intended to promote in his instruction.

My analysis of the preliminary questions I asked at the beginning of the first Phase III clinical interview therefore revealed that David's unawareness of the mental activity comprising particular ways of conceptualizing mathematical ideas made him incapable of defining instructional goals and objectives in terms of the ways of understanding he intended students to construct. My analysis of these preliminary questions further demonstrated that David's instructional goals, which had little or nothing to do with the meanings he wanted students to learn, conditioned the circumstances he appraised as constraints on his practice. David's responses to these questions also indicated that his instructional actions were uninformed by an explicit and disciplined way of thinking about what learning is and how a teacher might engender it.

During the Phase III clinical interviews, I presented David with audio and video excerpts from the series of TBCIs, pre-lesson interviews, and David's classroom instruction that exemplified *moments of instructional deviation*, *moments of instructional incoherence*, and *moments of mathematical concession*. My intention in doing so was to determine if the moments in which David conveyed/demonstrated discrepant ways of understanding resulted from his reacting to the circumstances and events he appraised as constraints on his practice, and if so, to establish the effect of David's construction of these constraints on the quality of his enacted mathematical knowledge.

My analysis of David's and my conversations around select pairs of audio and/or video excerpts that illustrated *moments of instructional deviation*, *moments of instructional incoherence*, and *moments of mathematical concession* revealed that David almost always failed to notice the discrepancy in the ways of understanding he conveyed/demonstrated in these excerpts. David's inability to recognize such discrepancies supports my claim that he was not consciously aware of the mental actions that comprise the meanings he intended to promote in his teaching, as such awareness would likely have equipped David with the cognitive schemes to recognize the discrepant and incompatible ways of understanding he conveyed in the excerpts we discussed. My analysis further revealed that the occasions in which David conveyed/demonstrated discrepant, inconsistent, or incompatible ways of understanding were not occasioned by his responding to or accommodating for the circumstances and events he appraised as constraints on his practice, but were rather a consequence of his unawareness of the mental activity involved in constructing particular ways of understanding mathematical ideas.

I argued in Chapter 7 that David was not consciously aware of the mental actions and operations that comprised the ways of understanding he intended his students to construct. In the current chapter, I have demonstrated that David identified circumstances as instructional constraints for reasons other than they hindered students' ability to construct particular ways of understanding. If David had been aware of the mental actions and operations that characterize productive mathematical ways of understanding, then it is at least possible that he would have appraised as constraints on his practice the circumstances that impeded students' ability to engage in or construct representations of

these mental actions. That he did not suggests that David's unawareness of the mental activity involved in particular ways of conceptualizing mathematical ideas conditioned how he defined his instructional goals and objectives, which further informed what he appraised as instructional constraints (see *Figure 76*). Therefore, rather than his enacted mathematical knowledge being compromised by his reactions to and accommodations for instructional constraints, David's mathematical knowledge appeared to condition the circumstances he appraised as constraints on his practice.

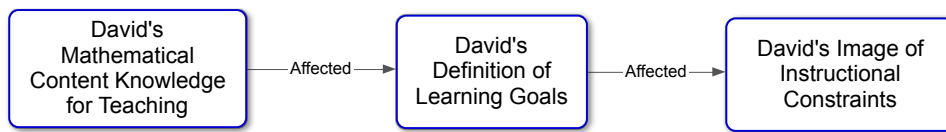


Figure 76. Factors affecting David's image of instructional constraints.

CHAPTER 9

DISCUSSION AND IMPLICATIONS

When my information changes, I alter my conclusions. What do you do, sir?

John Maynard Keynes

I begin this concluding chapter with a summary of the study's main findings and then proceed to discuss contributions and implications. I also identify the limitations of the study and elaborate future directions this work has enabled. I conclude the chapter with a reflective note about expectations.

Summary of Main Findings

Teachers must recognize the knowledge they possess as appropriate to employ in the process of achieving their goals and objectives in the context of practice. Such recognition is subject to a host of cognitive and affective processes that have thus far not been a central focus of research on teacher knowledge in mathematics education. To address this need, this dissertation study examined the role of a secondary mathematics teacher's image of instructional constraints on his enacted subject matter knowledge.

Understanding how the participating teacher's (i.e., David's) image of instructional constraints affected the mathematical knowledge he employed in the context of teaching involved: (1) constructing a model of David's mathematical knowledge independent of its instantiation in the context of classroom practice; (2) ascertaining how David perceived his environmental context while identifying the circumstances and events he appraised as instructional constraints and why; (3) constructing a model of the subject matter knowledge David utilized in the context of lesson planning and classroom practice so as to allow me to identify incongruities between his mathematical knowledge

and the mathematical knowledge he employed while teaching; and (4) apprehending the way and extent to which David's image of instructional constraints influenced the quality of his enacted subject matter knowledge. I collected data in three phases to accomplish these four objectives. First, I conducted a series of task-based clinical interviews (TBCIs) that allowed me to construct a model of David's mathematical knowledge of angle measure, the output quantities and graphical representations of the sine and cosine functions, and the period of the sine and cosine functions. In the second phase, I conducted pre-lesson interviews, collected journal entries, and examined David's instruction to characterize the mathematical knowledge he utilized in the context of designing and implementing lessons. Finally, in the third phase I conducted a series of semi-structured clinical interviews to identify the circumstances David considered constraints on his practice and to ascertain the role of these constraints on the quality of David's enacted subject matter knowledge. The procedures I employed to analyze the data derive from Strauss and Corbin's (1990) and Corbin and Strauss's (2008) grounded theory approach.

My analysis revealed that there were a number of occasions in which David demonstrated *particular* ways of understanding during the series of TBCIs that differed from *specific* meanings he supported in his instruction. It is important to note, however, that such discrepancies do not imply the nonexistence of consistencies between these two instantiations of David's mathematical knowledge. While there were few occasions in which David demonstrated a way of understanding in the series of TBCIs that he did not convey in his teaching, the several incongruities and contradictions I observed between specific ways of understanding David demonstrated in the series of TBCIs and particular

meanings he supported in his instruction were a reflection of the discrepant and incompatible meanings that characterized David's mathematical knowledge. In other words, while there were some true discrepancies and contradictions *between* David's subject matter knowledge and his enacted subject matter knowledge, it was far more often the case that David's enacted knowledge reflected the discrepancies and contradictions *within* his mathematical knowledge. This fact constitutes one of several lines of evidence in support of my claim that the incongruities I observed between David's subject matter knowledge and his enacted subject matter knowledge were not occasioned by his responding to the circumstances and events he appraised as constraints on the quality of his instruction. These observed inconsistencies were instead a consequence of David's mathematical knowledge consisting of disorganized and disconnected cognitive schemes. Assimilating external stimuli to such disconnected schemes resulted in David demonstrating observable products of his reasoning that were, from my perspective, incompatible, discrepant, and contradictory. This result supports Thompson, Carlson, and Silverman's (2007) observation:

If a teacher's conceptual structures comprise disconnected facts and procedures, their instruction is likely to focus on disconnected facts and procedures. In contrast, if a teacher's conceptual structures comprise a web of mathematical ideas and compatible ways of thinking, it will at least be possible that she attempts to develop these same conceptual structures in her students (p. 416-17).

That David's mathematical knowledge was not comprised of a coherent organization of cognitive schemes suggests that he had not sufficiently engaged in reflected abstraction—the mechanism of injecting coherence into systems of organized actions and operations constructed at the level of representation (i.e., schemes). It is through the process of reflected abstraction that one consolidates or connects disjoint

cognitive schemes, the result being a single coherent scheme or a network of related schemes to which one may assimilate a wider variety of stimuli. That David so often demonstrated discrepant and contradictory ways of understanding throughout the series of TBCIs and during his instruction reveals that he had not suitably engaged in reflected abstraction to construct such a coherent network of cognitive schemes.

There is a second consequence of not having sufficiently engaged in reflected abstraction that contributed to the incoherent ways of understanding David demonstrated in the series of TBCIs and conveyed in his instruction. A byproduct of engaging in reflected abstraction is that it allows a teacher to become consciously aware of the mental actions that comprise his or her way of understanding particular mathematical ideas. Such conscious awareness affects the pedagogical actions a teacher employs to support students' learning, and thus the coherence of the meanings the teacher conveys. That David's instruction supported discrepant and contradictory ways of understanding suggests that he was not consciously aware of the mental actions and operations that comprised his cognitive schemes. Equipped with such conscious awareness, it is at least possible that David would have engaged his students in learning experiences that promote their engagement in, and internalization of, the mental actions that comprise particular ways of conceptualizing mathematical ideas, and would therefore have contributed to more focused and coherent instruction.

Moreover, as a result of not having been consciously aware of the mental actions that comprised his own cognitive schemes, David was not positioned to define his instructional goals and objectives in terms of intended mental activity. Instead, David defined the intended outcomes of his instruction in terms of the observable behaviors he

wanted students to be able to demonstrate, the skills he hoped they would acquire, and the facts he expected them to know. If David had been consciously aware of the mental actions that comprise the schemes he wanted his students to construct, and if he had defined his instructional objectives to reflect this aspiration, he would have been positioned to employ pedagogical actions in the service of supporting students in engaging in the mental activity that characterize the ways of understanding he intended students to construct, which would likely have contributed to an instructional experience lacking the incoherence that was so prevalent in David's teaching. So, in the final analysis, David's unawareness of the mental actions that constitute the mathematical meanings he possessed imposed a limit on the quality of his enacted mathematical knowledge, far more so than did his reactions to, and accommodations for, the circumstances and events he appraised as constraints on his practice.

My analysis of the pre-lesson interviews, teacher journal entries, and Phase III semi-structured clinical interviews further support this conclusion. David's responses and remarks in these domains of data collection revealed that he appraised the following circumstances as constraints on his practice: (1) having insufficient time to cover the required amount of curriculum material, (2) students' lacking or disingenuous motivation and engagement, (3) his feeling responsible for preparing students to achieve procedural fluency in calculus, (4) students' insufficient prior knowledge, (5) students' inattention to instructions and instructional expectations, (6) inadequate time to assess students' learning and performance, (7) the inconsistency with which students learn the facts and procedures he communicates, (8) the unreceptiveness of the *Pathways* curriculum to supplementation, (9) students interpreting his requests for them to explain or justify their

answers as him being intimidating or coercive, and (10) the cultural expectation that mathematics is a subject at which it is acceptable to be inept. David did not appraise the aforesaid circumstances as obstructing his ability to support students in constructing particular ways of understanding mathematical ideas.

The instructional constraints David's responses in the pre-lesson interviews, teacher journal entries, and Phase III clinical interviews suggest a number of his instructional goals: (1) connecting with students and enhancing their love of learning, (2) providing students access to high quality post-secondary education, (3) supporting students in appreciating the relevance of mathematics to their lives, (4) progressing through the content at a pace consistent with departmental recommendations, (5) engineering a social content that encourages students' engagement and fosters their motivation, (6) equipping students with the factual and procedural knowledge that will enable them to fluently solve problems in calculus, and (7) creating a work environment that is free of time-wasting and energy-consuming annoyances and distractions.

These instructional goals do not reflect David's attention to the ways of understanding students have the opportunity to construct. This inattention was a consequence of David's unawareness of the mental activity comprising particular ways of conceptualizing mathematical ideas. That the circumstances and events David appraised as instructional constraints were not constraints on the quality of the ways of understanding his instruction supported reveals that he did not define his instructional goals and objectives in terms of the meanings he intended students to construct. This is testament to the fact that one can only appraise circumstances as obstructing students'

ability to construct particular mathematical meanings if he or she maintains as an instructional goal supporting students in constructing particular mathematical meanings.

During the Phase III semi-structured clinical interviews, I presented David with audio and video excerpts from the series of TBCIs, pre-lesson interviews, and his classroom instruction that illustrated *moments of instructional deviation*, *moments of instructional incoherence*, and *moments of mathematical concession*. Our discussion around these categories of instructional occasions revealed whether the moments in which David conveyed/demonstrated discrepant ways of understanding were a consequence of his reacting to the circumstances and events he appraised as instructional constraints, and if so, to ascertain the effect of David's construction of these constraints on the quality of his enacted subject matter knowledge.

My analysis of David's and my conversations around these select pairs of audio and/or video excerpts revealed that David consistently did not recognize the discrepancy in the ways of understanding he conveyed/demonstrated. David's incapacity to notice such discrepancies further supports my claim that he was not consciously aware of the mental actions that comprise the meanings he intended his instruction to support. Such awareness would likely have furnished David with a coherent organization of cognitive schemes that would have enabled him to recognize the discrepant and incompatible ways of understanding he conveyed in the excerpts we discussed. My analysis of the Phase III clinical interviews further supported my previous claim that the occasions in which David conveyed/demonstrated discrepant or contradictory ways of understanding were not occasioned by his responding to or accommodating for the circumstances and events he appraised as instructional constraints, but were rather a consequence of his unawareness

of the mental activity involved in constructing particular ways of understanding mathematical ideas.

My comparison of David's subject matter knowledge with his enacted subject matter knowledge revealed that he was not consciously aware of the mental actions and operations that comprised the ways of understanding he intended his students to construct. My analysis of the pre-lesson interviews, teacher journal entries, and Phase III semi-structured clinical interviews demonstrated that David's unawareness of the mental activity involved in particular ways of conceptualizing mathematical ideas conditioned how he defined his instructional goals and objectives, which further informed the circumstances and events he appraised as instructional constraints. The quality of David's enacted subject matter knowledge was therefore conditioned principally by a particular characteristic of this knowledge—his unawareness of the mental actions that comprise it—and not by his reactions to the circumstances and events he appraised as instructional constraints.

Contributions and Implications

The main findings of this dissertation study suggest that the mathematical knowledge required for effective teaching involves more than powerful understandings of mathematical ideas; it involves an awareness of the mental actions and operations that constitute these understandings. Therefore, what might be called *mathematical content knowledge for teaching* entails both strong subject matter knowledge as well as an awareness of the mental processes that characterize such knowledge. As I discussed in Chapter 3, reflecting abstraction is the process by which one constructs mathematical knowledge, or cognitive schemes, and reflected abstraction is the process whereby one

becomes consciously aware of the mental actions and operations that comprise these schemes.

A teacher's awareness of the mental actions and operations that constitute powerful ways of understanding mathematical ideas supports him or her in: (1) defining instructional goals and objectives in cognitive rather than behavioristic terms, (2) designing and/or selecting curriculum materials that seek to engender intended mental activity, (3) employing pedagogical actions that support students in engaging in—and thereby constructing internalized representations of—the mental actions that constitute productive mathematical meanings, (4) developing a disposition to attend to students' thinking, and (5) constructing models of students' epistemic ways of understanding. Additionally, this study has demonstrated that a secondary mathematics teacher's unawareness of the mental actions and operations that comprise particular ways of understanding mathematical ideas informed the circumstances and events he appraised (or did not appraise) as constraints on the quality of his instruction. These salient findings have contributed to my development of the theoretical framework for mathematical knowledge for teaching displayed in *Figure 77*.

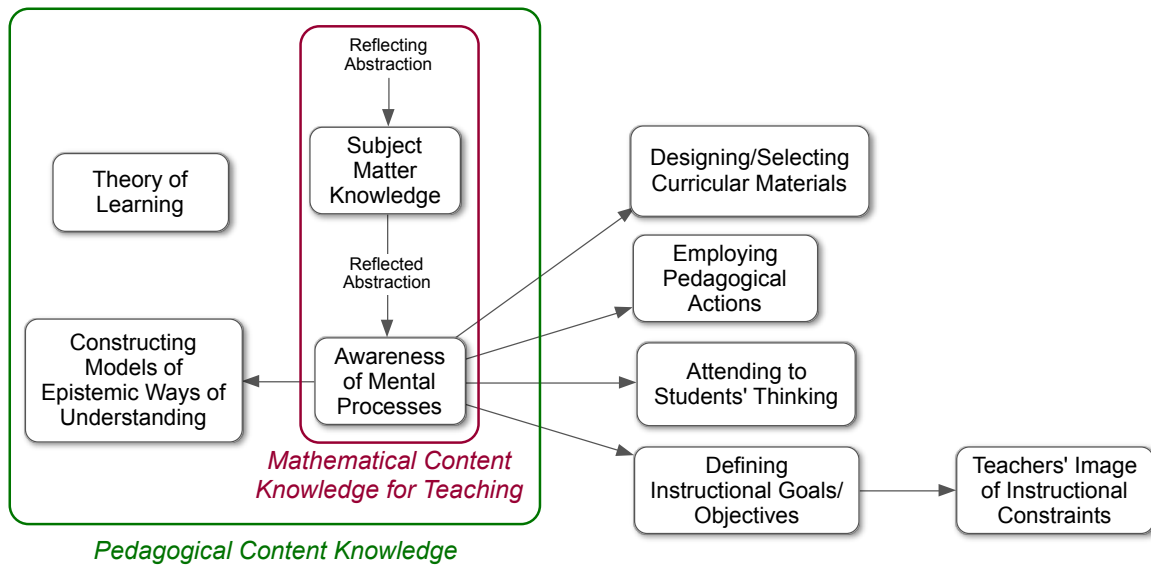


Figure 77. Theoretical framework for mathematical knowledge for teaching.

As this theoretical framework suggests, teachers’ awareness of the mental processes that constitute productive ways of understanding mathematical ideas supports them in reorganizing their mathematical knowledge into a form that is capable of informing effective instructional practices. This point supports Silverman and Thompson’s (2008) observation:

[W]e see a person’s [mathematical knowledge for teaching] as being grounded in a personally powerful understanding of particular mathematical concepts and as being created through the transformation of those concepts from an understanding having pedagogical potential to an understanding having pedagogical power.

In their theoretical framework for the development of mathematical knowledge for teaching, Silverman and Thompson’s (2008) propose that engaging in reflective abstraction is necessary for teachers to transform their mathematical knowledge from a state that maintains “pedagogical potential” into a form having “pedagogical power.” Since engaging in reflected abstraction is the means by which one brings into conscious awareness the mental actions and operations that comprise his or her cognitive schemes,

and since my investigation has revealed that such conscious awareness conditions the quality of the teacher's enacted subject matter knowledge, this dissertation study has provided empirical support for Silverman and Thompson's (2008) theoretical claim.

The theoretical framework displayed in *Figure 77*, as well as the central findings of this study on which it was based, exposes a common assumption underlying much of the literature in the area of mathematical knowledge for teaching, in addition to many mathematics teacher training and professional development programs: Mathematical knowledge for teaching is composed of a variety of distinct, but related, knowledge "types" that coalesce in the practice of teaching. Pre-service teacher preparation and in-service professional development programs based on this assumption focus on supporting teachers in developing these categories of knowledge without seriously attending to how knowledge in one domain informs or derives from knowledge in another. My results not only demonstrate that simply having powerful mathematical understandings is insufficient for supporting students' learning, but also that mathematical knowledge for teaching entails an awareness of the mental activity that constitute productive ways of understanding mathematical ideas. Moreover, in contrast to many of the existing theoretical frameworks for mathematical knowledge for teaching, several of which I discuss in Chapter 2, the theoretical framework presented in *Figure 77* does not consider a teacher's subject matter knowledge disjoint from, but somehow opaquely related to, his or her pedagogical content knowledge. Instead, the results of this study reveal that a teacher's subject matter knowledge is a fundamental component of his or her mathematical content knowledge for teaching but, importantly, does not constitute it. For this reason, the primary results of this dissertation study suggest that supporting pre- and

in-service teachers in transforming their mathematical knowledge into a form they can leverage to enhance students' learning involves researchers and professional development specialists in not only ensuring *that* teachers develop strong mathematical knowledge, but that teachers construct this knowledge in a way that makes them aware of the mental actions and operations that constitute desirable ways of understanding mathematical ideas.

Limitations

This dissertation study, as with any empirical investigation, is limited in a number of ways. First, the amount of data I collected, and the level of detail at which it needed to be analyzed to address the inquiry I proposed, necessitated my conducting a case study, which calls into question issues of generalizability. This dissertation study was largely a theory-building enterprise, and conducting a case study allowed me to accomplish this aim. Nonetheless, as a result of studying only one participant, my main findings cannot be said to apply to a population of secondary mathematics teachers at large.

Second, because of the amount of data I needed to analyze prior to designing and implementing the Phase III semi-structured clinical interviews, a full year elapsed between David concluding his instruction of Module 8 in the *Pathways Algebra II* curriculum and my administering these clinical interviews. David no doubt lost some degree of intimacy with the instructional choices for which I asked him to provide a rationale. A more immediate administration of these Phase III clinical interviews would certainly have enhanced the validity of this study's salient findings.

Third, I initially intended to conduct the series of TBCIs at least two weeks prior to David beginning Module 8. However, as a result of his department's decision to incorporate a brief unit on probability at the end of the semester to prepare students for a

district standardized assessment, David started teaching Module 8 two weeks earlier than he anticipated. I therefore had to conduct the TBCIs concurrently with David's teaching of Module 8. While I took care to administer each TBCI prior to David's teaching the concepts that were the focus of each respective interview, David very likely did not have sufficient time to solidify whatever understandings he constructed and/or refined during the series of TBCIs into a form that was conducive of informing his lesson design and instructional actions. This may explain some of the discrepancies and contradictions I observed between David's subject matter knowledge and his enacted subject matter knowledge.

Future Directions

Recall that in my opening chapter I identified the shortcomings of educational policy initiatives that derive from two polarized perspectives for the condition of education in the United States. Adherents to Perspective I (the "Deficit of Human Capital" perspective) assume that limitations in teacher aptitude lie at the heart of modest student achievement. Proponents of Perspective II (the "Perfect Storm of Constraints" perspective) contend that teachers operate under increasingly crippling circumstances that stifle their aptitude and make it nearly impossible for them to effectively do their work. I claimed that if what each perspective takes as problematic (teacher aptitude and environmental constraints respectively) were resolved, it is unlikely that teaching quality in the United States would drastically improve since the policies that originate from these two perspectives do not attend to the *interaction* between teachers' image of the constraints under which they work and the knowledge teachers bring to bear in the context of practice. I designed this dissertation study to attend to this interaction by

investigating how a secondary mathematics teacher's image of instructional constraints conditioned the subject matter knowledge he utilized while teaching. What I learned, however, was something about the effect of a secondary mathematics teacher's subject matter knowledge on his construction of instructional constraints.

In my opening chapter, I rationalized my intention to understand how teachers' image of instructional constraints affects their enacted subject matter knowledge by claiming that such an understanding may inform answers to important questions like, "How can teacher preparation and professional development programs be designed to support teachers in appraising what they commonly consider constraints on the quality of their teaching so as to minimize the effect of these perceived constraints on the quality of students' learning experiences?" and "How can teachers' work environments be engineered so as to minimize the potential that they will believe to be operating under constraints that have proven to condition the evocation of their subject matter knowledge in undesirable ways?" I argued in Chapter 4 that this dissertation study cannot contribute to clarifying the answers to such questions if the research participant maintains impoverished mathematical ways of understanding and unproductive mathematical ways of thinking. I claimed that it is of little consequence to examine how a teacher's weak mathematical knowledge is conditioned by his image of instructional constraints since students are not likely to substantially benefit from the teacher's uncompromised conveyance of such knowledge in the absence of these constraints. I explained that a teacher's appraisal of the constraints under which he works imposes an obstacle on the quality of his instruction if and only if the teacher's subject matter knowledge is profound enough to be conditioned in undesirable ways by his image of these constraints.

These claims transparently reveal my assumption that if a teacher possessed strong mathematical ways of understanding and productive ways of thinking, then any discrepancies between the teacher's subject matter knowledge and his or her enacted subject matter knowledge would necessarily result from the teacher willingly compromising the quality of his or her enacted mathematical knowledge in response to the circumstances and events he or she appraised as instructional constraints. This was an incorrect assumption. I did not entertain the possibility that a teacher's unawareness of the mental actions and operations that comprise his or her cognitive schemes could result in incongruities and contradictions between the teacher's mathematical knowledge and the mathematical knowledge he or she employs in the context of teaching. Therefore, while I did not learn what I intended to from this dissertation study, I did expose the following condition that must be satisfied prior to pursuing the inquiry I initially proposed: The participating teacher must become consciously aware of the mental actions and operations that comprise his or her mathematical schemes. My future research will therefore pursue the inquiry I initially proposed, but will do so after I taken care to ensure that the aforesaid condition is satisfied. Such research seeks to advance the field's understanding of the factors that condition the mathematical knowledge teachers employ in the context of teaching. More generally, the results of my future work aspire to reveal how mathematics educators may support teachers in favorably appraising their environmental circumstances, and managing their emotional responses to them, so as to minimize the effect of these circumstances on the quality of teachers' enacted knowledge.

Researchers have devoted considerable effort to characterizing the mathematical knowledge required for effective teaching and to understanding the experiences by which

teachers may construct such knowledge. However, teachers' knowledge is of interest to the field only because it has the potential to inform the mathematics students may learn. It is therefore imperative to examine the factors that condition the knowledge teachers utilize in the context of teaching. In order for current research on teacher knowledge in mathematics education to realize its intended effect of supporting teachers in engaging students in experiences whereby they may construct meaningful mathematical understandings, it is important to apprehend the factors that condition the knowledge teachers *do* possess in addition to characterizing the knowledge that teachers *should* possess. Ascertaining the factors that mediate the knowledge that resides in teachers' minds and the knowledge they bring to bear while teaching is indispensable to fashioning well-informed teacher preparation and professional development programs and educational policies that take seriously the effect of both teacher knowledge and those factors that compromise it. My future research aspires to contribute to this end.

A Concluding Note About Expectations

All but the most banal inquiries reveal the expectation, and often the aspiration, of the inquirer. This is particularly true when one seeks to ascertain the effect of something on something else. Proposing to examine the effect of *A* on *B* not only discloses the examiner's expectation that there is indeed an effect to examine, but also that the causal arrow extends *from A to B* and not the other way around. The investigation presented in this dissertation is no exception. Only the most unobservant reader would fail to infer from my opening chapter my expectation that the quality of a secondary mathematics teacher's enacted subject matter knowledge *would* be conditioned by his reaction to, and accommodation for, the circumstances and events he appraised as constraints on his

practice. This expectation derived, in part, from my experience as a secondary mathematics teacher, during which I regularly, and with complete awareness, supported my students in constructing understandings that were less powerful than I was capable of conveying. These conscious concessions to the quality of my enacted mathematical knowledge resulted from my attempts to compensate for, among other things, a pervasive culture of disinterest in mathematics as well as students' apathetic dispositions towards learning in general.

I confess that my expectation was also an aspiration. The inquiry I pursued was in no small part inspired by the contemptuous thought of a mathematics educator observing one of my lessons as a high school teacher, and coming to what often seems an inevitable conclusion: "This teacher does not know mathematics!" It is this type insensitivity to a teacher's inner dialogue that compelled me to provide what I considered a long overdue counter-narrative. I expected that David shared my experience of making conscious concessions to the quality of his enacted mathematical knowledge in response to instructional constraints, and as a result I aspired to tell my story through him.

Fortunately, even an inquiry that seeks to vindicate an expected causal relation, if conducted openly and honestly, may lead to unexpected insights and novel conclusions. I consider this dissertation an example of such an outcome. David's enacted mathematical knowledge was not conditioned by his image of instructional constraints. The causal arrow went the other way. Characteristics of David's mathematical knowledge conditioned how he defined his instructional goals and objectives, which further informed the events and circumstances David appraised as constraints on his practice. The

conclusions presented herein therefore fail to satisfy the aspiration I maintained at the outset of my inquiry.

Although this dissertation does not tell the story I expected it to, it nonetheless tells an important part of my story—a part with which I was unfamiliar prior to undertaking this investigation. I assumed that external circumstances imposed a limit on the quality of my instruction and took for granted the fact that if only these external circumstances were removed, I would finally be able to realize my potential as a mathematics teacher. Like David, however, I was only consciously aware of the observable actions I expected students to demonstrate and of the problems I intended them to solve. I did not feel responsible for the mental imagery evoked when my students assimilated mathematical symbols, interpreted quantitative situations, or read novel problems. Instead, I was concerned only with equipping my students with factual and procedural knowledge and with training them to recognize surface-level features of problem statements that might serve as cues for appropriately recalling these facts and enacting such procedures. I am convinced now that this misguided aspiration would certainly have survived the removal of any circumstances I appraised as constraints on the quality of my teaching. Therefore, like David, my unawareness of the mental actions that comprised my ways of understanding obstructed the quality of my instruction more profoundly than did my response to instructional constraints. For this reason, I am indebted to David for teaching me something about what it means to support pre- and in-service teachers in constructing mathematical content knowledge for teaching, as well as for teaching me something about myself.

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APPENDIX A
HUMAN SUBJECTS APPROVIAL LETTER



APPROVAL:CONTINUATION

Marilyn Carlson
Mathematics and Statistical Sciences, School of
480/965-6168
MARILYN.CARLSON@asu.edu

Dear Marilyn Carlson:

On 10/24/2014 the ASU IRB reviewed the following protocol:

Type of Review:	Continuing Review
Title:	Pathways to calculus:Disseminating and Scaling a Professional Development Model for Precalculus Teaching
Investigator:	Marilyn Carlson
IRB ID:	1108006730
Category of review:	(7)(b) Social science methods, (7)(a) Behavioral research

The IRB approved the protocol from 10/24/2014 to 8/29/2015 inclusive. Three weeks before 8/29/2015 you are to submit a completed "FORM: Continuing Review (HRP-212)" and required attachments to request continuing approval or closure.

If continuing review approval is not granted before the expiration date of 8/29/2015 approval of this protocol expires on that date. When consent is appropriate, you must use final, watermarked versions available under the "Documents" tab in ERA-IRB.

In conducting this protocol you are required to follow the requirements listed in the INVESTIGATOR MANUAL (HRP-103).

Sincerely,