Einstein Gravity and Beyond: Aspects of Higher–Curvature Gravity and Black

Holes

by

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ABSTRACT

This thesis explores the different aspects of higher curvature gravity. The "membrane paradigm" of black holes in Einstein gravity is extended to black holes in $f(R)$ gravity and it is shown that the higher curvature effects of $f(R)$ gravity causes the membrane fluid to become non-Newtonian. Next a modification of the null energy condition in gravity is provided. The purpose of the null energy condition is to filter out ill-behaved theories containing ghosts. Conformal transformations, which are simple redefinitions of the spacetime, introduces serious violations of the null energy condition. This violation is shown to be spurious and a prescription for obtaining a modified null energy condition, based on the universality of the second law of thermodynamics, is provided. The thermodynamic properties of the black holes are further explored using merger of extremal black holes whose horizon entropy has topological contributions coming from the higher curvature Gauss-Bonnet term. The analysis refutes the prevalent belief in the literature that the second law of black hole thermodynamics is violated in the presence of the Gauss-Bonnet term in four dimensions. Subsequently a specific class of higher derivative scalar field theories called the galileons are obtained from a Kaluza-Klein reduction of Gauss-Bonnet gravity. Galileons are null energy condition violating theories which lead to violations of the second law of thermodynamics of black holes. These higher derivative scalar field theories which are non-minimally coupled to gravity required the development of a generalized method for obtaining the equations of motion. Utilizing this generalized method, it is shown that the inclusion of the Gauss-Bonnet term made the theory of gravity to become higher derivative, which makes it difficult to make any statements about the connection between the violation of the second law of thermodynamics and the galileon fields.

*I dedicate this work to my family: my grandmother who has been my musical mentor for 15 years and to my parents.*

*Physical laws must possess mathematical beauty.*

*— P.A.M. Dirac*

# ACKNOWLEDGEMENTS

## Notations and conventions

We will be working with metrics in mostly plus signature $(-+++)$. Our convention follows that of Wald with the Riemann tensor defined as,

$$R^a_{bcd} = \partial_c \Gamma^a_{db} - \partial_d \Gamma^a_{cb} + \Gamma^a_{cm} \Gamma^m_{db} - \Gamma^a_{dm} \Gamma^m_{cb} \tag{1}$$

The Ricci tensor is defined as the contraction of the first and third index of the Riemann tensor: $R_{ab} = R^c_{acb}$ and the Ricci scalar is $R = R^a_a$.

The sign of the extrinsic curvature is such that its trace is positive for convex surfaces. It is defined as $K_{ab} = h^c_a \nabla_c n_b$ for surfaces with metric $h_{ab}$ and normal $n^a$.

We will use small Latin alphabets to express the indices in all quantities. This is different from the old convention of using the Greek alphabets as subscripts. We will write the metric as $g_{ab}$ instead of $g_{\mu\nu}$. $K_{AB}$ will be used to denote (D-1)–dimensional extrinsic curvature. Where a confusion might arise due to the presence of quantities of both higher and lower dimensions, the dimensionalities of the tensorial quantities will be made explicit by the use of projection operators.

Units to be used are the natural units with $\hbar = 1, c = 1, G = 1$ unless otherwise stated.

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Chapter 1

BLACK HOLES AND HIGHER CURVATURE GRAVITY

## 1.1   Black Holes

After Einstein discovered his theory of general relativity he attempted to find solutions that would explain the universe and its evolution. However the field equations in his theory were so complex and non-linear that he concluded that it was not possible to obtain a solution in a closed form. Yet, no more than a month later, an exact solution for a spherically symmetric spacetime was found by Schwarzschild. The solution was simple and elegant but confusing at the same time: it contained a singularity in the coordinate system at a fixed radius. This baffled mathematicians like Hilbert for years until 1924 when Eddington and Finkelstein realized that it must be a coordinate artifact, similar to the harmless breakdown of polar coordinates at the origin of the Euclidean plane, only this was at a nonzero radius. Moreover, once a particle crosses that particular radius (which marks the event horizon) it will not be able to come out. In the following years several solutions possessing these event horizons were obtained and these solutions collectively came to be known as *black holes*. The black holes are not just theoretical constructs. They are found to be strewn everywhere across the universe. In fact when a star whose mass exceeds four solar masses exhausts its fuel and collapses under its own gravity, it forms a black hole.

In this work, we are interested only in the theoretical aspects of the black holes. We wish to explore the properties of the horizon and the spacetime around it. Therefore we will need a formal definition of a black hole.

*Black holes are the complement of the past of future null infinity of a strongly asymptotically predictable spacetime. The boundary between the complement and the past of the future null infinity is called the event horizon.*

We do not need to go into the details of this definition. The essential point is that the event horizon is a kind of globally defined inner boundary of spacetime. An example of a black hole solution is a Schwarzschild solution.

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2sin^2(\theta)d\phi^2 \qquad (1.1)$$

The coordinates t describes time (loosely) outside the horizon while r is the radius and there is a coordinate singularity at $r = 2M$. This is the location of the *event horizon* or simply *horizon*. The horizon is a lightlike surface and if an observer wishes to visit $r = 2M + \epsilon$ and return to infinity from there, then he has to fire rockets which would propel him to almost light speed in order to escape the force of gravity. Otherwise he would fall into the black hole. Even photons, which are emitted near the horizon, have to work to overcome the tremendous gravitational potential of the black hole. The work done in climbing out of the potential will cause the photons to lose energy and get heavily redshifted. The redshift becomes more and more pronounced for photons emitted closer to the horizon and at the horizon the redshift becomes infinite. The photons can no longer escape and light gets trapped. This is why the horizon is called a *marginally trapped surface* and all the surfaces inside it are *trapped surfaces*. Since the horizon is a place of infinite redshift, a distant observer will be oblivious to its existence. All she will see are objects moving towards an imaginary surface but never reaching it. So she will conclude that the horizon is a boundary of the spacetime. And she will not be too far from the truth. The horizon is really an inner boundary of the spacetime and a major part of our work will be to explore this concept. The presence of a boundary naturally leads to the black hole membrane

paradigm and allows for a theory-independent definition of the black hole entropy called the *Wald entropy* (reviewed at the end of this chapter). We will have more to say about the ramification of the horizon being a boundary in a subsequent section.

### 1.1.1   Types of Black Holes

The black hole we saw before is a spherically symmetric black hole with mass as the only parameter. There was no charge or angular momentum. Most realistic black holes would have some angular momentum. The rotating black holes are called Kerr black holes. They are interesting in their own right being accurate descriptions of astrophysical black holes. However we do not need them for our purposes. The black holes we will be interested are the ones which have some electric charge. These are called Reissner-Nordström (RN) black holes. RN black holes are spherically symmetric and have two concentric horizons. They have an interesting property that they are traversable, meaning that observers can in principle come out of the black hole into another universe. When an observer first crosses the outer horizon, the radius r (which outside the black hole is a spatial coordinate) becomes a timelike coordinate and the observer has no choice but to move along the path of decreasing r. After crossing the inner horizon, the r coordinate becomes spatial again and the observer can choose to reverse her course to avoid falling to $r = 0$. When she reverses her course and moves outwards and crosses the inner horizon, the r coordinate is timelike again and motion of the observer is forced along the path of increasing r and continues till she comes out of the black hole to emerge in a parallel universe. In our subsequent chapters we will be working with extremal black holes. Therefore it would be instructive to review them.

**Extremal black holes:** The RN black holes have multiple horizons, one outer and one inner horizon, both of which are functions of the mass and charges. When the

mass equals the charge, the two horizons coincide and we get an extremal black hole. It is not possible to reduce the mass further than this without creating a "naked singularity" and violating the cosmic censorship conjecture. The Reissner-Nordström (RN) black hole metric is,

$$ds^2 = -\left(1 - 2M/r + Q/r^2\right)dt^2 + \frac{dr^2}{1 - 2M/r + Q/r^2} + r^2d\theta^2 + r^2 sin^2(\theta)d\phi^2$$

The horizons are at $r_\pm = M \pm \sqrt{M^2 - Q^2}$ and the black hole becomes extremal at $M = Q$. The metric of an extremal RN black hole is

$$ds^2 = -\left(1 - M/r\right)^2 dt^2 + \left(1 - M/r\right)^{-2} dr^2 + r^2 d\theta^2 + r^2 sin^2(\theta)d\phi^2$$

One interesting fact about these extremal black holes is that the gravitational attractive force is exactly balanced by electromagnetic repulsion. So two such black holes will exert no force on each other, a fact we will exploit in Chapter 5.

The Schwarzschild, Kerr, Reissner-Nordström and Kerr-Newman (having both electric charge and angular momentum) black holes are the only types of stationary black holes in four dimensions. In fact the *uniqueness theorem* [1–9] and *no-hair theorem* [10–13] together imply that the only parameters a stationary black hole may have are mass, charge, and angular momentum.

## 1.2   Black Hole Thermodynamics

During the early 70's Hawking discovered that, in the presence of quantum fields, black holes radiate thermally [14–16] and have a temperature that is proportional to the surface gravity of the black holes [1] . Earlier, Bekenstein [17–19] had argued that

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[1]**Surface gravity** is the work done per unit mass on a body to hold it at a fixed distance from the horizon of a black hole. It is the amount of acceleration that a fiducial observer (one who hovers over a black hole horizon) must have which will prevent her from falling into the black hole. Formally, it is the square root of the square of the divergence of the null generators of the horizon $\kappa^2 = -\frac{1}{2}\nabla_a l_b \nabla^a l^b$. For a Schwarzschild black hole of mass M, its value is $\kappa = \frac{1}{4M}$.

black holes could be attributed an entropy proportional to the area of the horizon. Thus a black hole behaves as a thermodynamic system with a Hawking temperature and a Bekenstein-Hawking entropy.

$$T = \frac{G\hbar}{c^3}\frac{\kappa}{2\pi} \qquad\qquad S_{BH} = \frac{k_B c^3}{\hbar}\frac{A}{4G} \qquad\qquad (1.2)$$

where $k_B$ is the Boltzmann constant. This is the last time we will use the universal constants explicitly in an equation. We would set $\hbar = 1, G = 1, c = 1, k_B = 1$ in the rest of the work. The presence of the $\hbar$ demonstrates the quantum mechanical origin of these quantities.

### 1.2.1 Raychaudhuri Equation

The proof of the second law of black hole thermodynamics is impossible without the existence of a geometrical relation known as the *Raychaudhuri equation*. This equation is a theory-independent statement about the behavior of a congruence of geodesics which are the generators of a null or timelike surface.

An observer on whom no force is acting moves along paths called *geodesics*. Evidently, this implies the geodesics should be defined as paths of zero acceleration. And since acceleration is defined as $A^b = v^a\nabla_a v^b$ for tangent vectors $v^a$ to a path, a path whose tangent vector satisfies the equation $v^a\nabla_a v^b = 0$ is a geodesic. A bundle of curves which are all geodesics is called a *geodesic congruence*. The proper time on a geodesic is called the affine time and any linear combination of it can be used as an "affine" parameter to label the geodesic. Consider the slice of a spacelike surface at a constant value of the affine parameter. The geodesics are proportional to the normal to this surface. If the surface is the horizon and the geodesics are null generators of this horizon then the horizon would be increasing in size if the geodesics are diverging (normals). And since the entropy is proportional to the area of a horizon, an

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increasing horizon would imply increasing entropy, thereby proving the second law.

To formally obtain the Raychaudhuri equation we require to understand how neighboring geodesics deviate from each other. This *geodesic deviation* is measured in terms of the derivative of the tangent vector to geodesics $B_{ab} = \nabla_a l_b$. This is a two tensor which means that it can be decomposed into a traceless symmetric part, an anti-symmetric part and a trace part.

$$B_{ab} = \frac{1}{D-2}\theta\gamma_{ab} + w_{ab} + \sigma_{ab} \tag{1.3}$$

This is the decomposition for a null geodesic congruence in D–dimensions. The trace part is the expansion scalar $\theta$ which measures how much the neighboring geodesics are diverging away from each other. The traceless symmetric part is the shear $\sigma_{ab}$. And the anti-symmetric part is the rotation $w_{ab}$. The Frobenius theorem states that if the geodesic congruence is hypersurface orthogonal [2], which is the case for the generators of the event horizon, then the rotation has to be zero [20]. The celebrated Raychaudhuri equation determines the derivative of the expansion scalar $\theta$ of the congruence with respect to the affine parameter $\lambda$

$$\frac{d\theta}{d\lambda} = -\frac{1}{D-2}\theta^2 - \sigma_{ab}\sigma^{ab} - R_{ab}l^a l^b \tag{1.4}$$

What this equation says is that the evolution of the expansion scalar $\theta$ is determined by the square of the geodesic deviation and by the Ricci curvature. If there is no curvature then the expansion is always slowing down. Gravity will generally enhance this effect (due to its attractive nature) provided the *Ricci convergence condition* $R_{ab}l^a l^b \geq 0$ is satisfied. Since $l^a$ are null vectors, Einstein's equations imply $R_{ab}l^a l^b = T_{ab}l^a l^b$. And since the Ricci convergence condition has no physical motivation it is

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[2]This means the tangent vectors to the geodesics generating the (D-1)–dimensional surface are also normal vectors to the (D-2)-dimensional spacelike cross-section of the horizon.

replaced by a more physical condition on matter called the *null energy condition* $T_{ab}l^a l^b \geq 0$. We will generalize this condition in Chapter 4.

### 1.2.2   Laws of Thermodynamics

The theory of general relativity is based on the simple principle of equivalence of gravity and curvature and that the laws of physics are invariant under arbitrary changes of the coordinate (diffeomorphism invariance). The simplest possible diffeomorphism invariant action we can write down is,

$$\int d^d x \sqrt{-g} R \qquad (1.5)$$

This is the Einstein-Hilbert action. We are intentionally working in d–dimensional spacetime to keep things as general as possible so that we can expand on this at a later stage. Unlike most Lagrangians, which are quadratic in first derivatives of the fields, the Einstein-Hilbert Lagrangian has second derivatives. Hence when varying this action, the Euler-Lagrange equations are not obtained unless the variation of the first derivatives at the boundary are taken care of. Hamilton's variational principle tells us that in presence of a boundary any variation of the action will generate boundary terms proportional to the momentum of the bulk fields [3] . Therefore, we need to cancel the unwanted term by adding another counter-term at the boundary which is proportional to the momentum of the bulk fields. This counter-term is called the Gibbons-Hawking term [21] and equals the trace of the extrinsic curvature for the case of the Einstein-Hilbert action.

$$\int_V d^d x \sqrt{-g} R + \int_{\partial V} d^{d-1} x \sqrt{h} \, K \qquad (1.6)$$

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[3] In this case the bulk field is the metric and the momentum is the extrinsic curvature $K_{ab}$ to the boundary.

Now, we have already established that the event horizon can be viewed as a boundary of spacetime. If we have a spacelike slice which runs from spacelike infinity to the black hole horizon we will have a bulk section with two boundaries – one outer and one inner boundary. On using the Gibbons-Hawking term from eqn. (1.6) and evaluating it over this slice it will lead to the ADM mass M at the outer boundary [4] and surface gravity times the area of the horizon at the inner boundary [22]. This ultimately leads to

$$dM = \frac{\kappa}{8\pi} dA \qquad (1.7)$$

and a slight redefinition of terms $S = A/4$ and $T = \kappa/2\pi$ yields the first law of black hole thermodynamics: $dM = TdS$. As noted in the beginning of the section, the surface gravity of the black hole is identified with the temperature of the black hole and the area over 4 with the entropy of the black hole. The existence of the first law is an encouragement to look for the second law of thermodynamics. We need to prove $dS > 0$ or rather $dA > 0$. The expansion scalar $\theta$ which measures the degree of divergence and convergence of a geodesic congruence is proportional to the rate of change of the area, $\theta = \frac{1}{A}\frac{dA}{d\lambda}$, $\lambda$ is the affine parameter parametrizing the geodesic congruence. When the geodesic congruence converges it forms a caustic. In physical spacetimes this corresponds to a naked singularity, something which is excluded by the cosmic censorship conjecture. Therefore a geodesic congruence should always diverge. This implies $\theta > 0 \Rightarrow dA > 0 \Rightarrow dS > 0$ proving the second law of thermodynamics. However a hidden assumption has been made about the matter content of the theory which we will revisit in Chapter 4.

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[4]The ADM mass of an asymptotically flat spacetime is defined as $M = \int_{S\to\infty}(K - K_0)$.

## 1.3  Higher Curvature Theories

Einstein had written down the simplest diffeomorphism invariant action possible. Nevertheless his action has the all-important property that not only is it diffeomorphism-invariant but it also gives second-order equations of motion. Therefore the theory does not contain any ghosts. However, we know that this theory is not renormalizable and loop corrections add counter-terms to the action which are higher order in the curvature scalars. Additionally, the Einstein-Hilbert action is obtained at the leading order in the $\alpha'$ expansion of string theory, where $\sqrt{\alpha'}$ is the string length. At higher order in $\alpha'$ expansion of string theory, higher curvature corrections appear. These expansions are motivation enough for us to extend gravity beyond Einstein theory and to explore the interesting black hole solutions in those theories. We would want to extend the theory while preserving the diffeomorphism invariance and at the same time preventing the appearance of ghosts.

### 1.3.1  $f(R)$ Theories

The simplest possible extension of the Einstein gravity appears to be adding the square of the Ricci scalar to the usual Einstein-Hilbert term.

$$\int R + \alpha R^2 \tag{1.8}$$

This action which is sometimes known as the Starobinsky model [23] is clearly diffeomorphism invariant. It is part of a much more general class of Lagrangians which are simple polynomial functions of the Ricci scalar. These theories are called $f(R)$ theories and are very good toy models (and sometimes realistic ones such as the one above) to test the laws of black hole thermodynamics. In fact they are equivalent to Einstein gravity minimally coupled to a scalar field. That is why they are also known as scalar-tensor theories. The scalar field turns out to be proportional to the first

derivative of the function $\phi = log(f'(R))$. Clearly, $f'(R)$ has to be positive, otherwise the map to scalar-tensor theory does not exist. This restricts the theories to only those which do not have contain ghosts. Therefore, even though the $f(R)$ theories have fourth-order equations of motion and by the Ostrogradski theorem [24] should contain ghosts, the scalar-tensor subset of the $f(R)$ theories do not have ghosts.

### 1.3.2  Lovelock Theories

The next class of theories which are interesting extensions of Einstein gravity are the ones which are explicitly constructed to have second order equations of motion. This is a unique class of theories called the Lovelock theories. They are well behaved theories without ghosts (by Ostrogradski) [25, 26]. The actions are topological invariants [5] in even dimensions [27]. A Lovelock term of order m is the Euler characteristic of dimension 2m. A general Lovelock term is written as,

$$L_m = \delta^{b_1 b_2 b_3 b_4 \cdots}_{a_1 a_2 a_3 a_4 \cdots} R_{a_1 a_2}{}^{b_1 b_2} R_{a_3 a_4}{}^{b_3 b_4} \cdots \tag{1.9}$$

where the $R_{a_1 a_2}{}^{b_1 b_2}$ are the Riemann tensors and the $\delta^{b_1 b_2 b_3 b_4 \cdots}_{a_1 a_2 a_3 a_4 \cdots}$ is the generalized Kronecker delta [6]. The first order Lovelock term is the Einstein-Hilbert term itself. The second one is called the Gauss-Bonnet term, $L_{GB} = R^2 - 4 R_{ab} R^{ab} + R_{abcd} R^{abcd}$. The third order and higher are not required for our purpose. The Gauss-Bonnet term is topological in four dimensions and the Einstein term is topological in two dimensions. However, they are topological over Riemannian manifolds which means the usual spacetimes used in general relativity need to be Euclideanized (Wick rotated)

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[5]A **topological invariant** is a quantity whose value depends only on the topology of a manifold. For example the genus is a topological invariant. It counts the number of handles in a manifold. The value of the genus is the same for any deformation of the manifold. The genus of a sphere is zero, that of a torus is 1, of a two-torus is two and so forth. There are many other topological invariants. The Euler character is one which is related to the genus of a manifold as $\chi = 2 - 2g$. The term topological is synonymous with non-dynamical in the present context as the action, being topological, has no infinitesimal variation.

[6]For example $\delta^{ab}_{cd} = \frac{1}{2!} \left( \delta^a_c \delta^b_d - \delta^a_d \delta^b_c \right)$.

before such statements can be made. This point is elaborated in Chapter 5. The Gauss-Bonnet term integrated over an Euclideanized manifold (without boundary) gives the Euler character of the manifold [28].

$$\chi_4 = \frac{1}{32\pi^2} \int d^4x \sqrt{(g)_E} \left( R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd} \right)_E \tag{1.10}$$

The fact that the Gauss-Bonnet term is topological makes it non-dynamical (does not affect the equations of motion) in four dimensions. A term in the action which does not affect the equations of motion behaves as a total derivative. In fact it can be demonstrated very easily, using differential forms, that the Gauss-Bonnet term in four dimensions is a total derivative (see Appendix A). However, this is true only in four dimensions. In lower dimensions it is identically zero. In higher dimensions it is dynamical. Therefore the equations of motion which exists in higher dimensions must become identically zero when the dimension is brought down to four. The reason for this phenomenon is that there exists an identity called the Bach-Lanczos identity [29],

$$C_{aklm}C_b^{klm} - \frac{1}{4}g_{ab}C_{ijkl}C^{ijkl} = 0 \tag{1.11}$$

where $C_{abcd}$ is the Weyl tensor. In four dimensions the variation of the Gauss-Bonnet Lagrangian is equal to the left-hand side of this identity.

The $\int R + L_{GB}$ is also the low energy effective action of heterotic string theory at sub-leading order in the $\alpha'$ expansion [30, 31]. Black hole solutions in Gauss-Bonnet gravity are known [32–51]. Black hole solutions were also found in other Lovelock theories [52–65].

### 1.3.3  Wald Entropy

For higher curvature theories of gravity, the temperature of the black holes is still proportional to the surface gravity, but the entropy is no longer proportional to the

area of the horizon. Wald [66] found the generalization of the Bekenstein-Hawking entropy for black holes of general theories of gravity. Given a gravitational Lagrangian of the form $L(g_{ab}, R_{abcd})$, the Wald entropy of a stationary black hole is [67, 68]

$$S = -2\pi \int P^{abcd} \epsilon_{ab} \epsilon_{cd} \tag{1.12}$$

where the $\epsilon_{ab} = k_a l_b - l_a k_b$ is the binormal to a bifurcate Killing horizon. For Einstein gravity, $P^{abcd} = \frac{\partial L}{\partial R_{abcd}} = \frac{1}{32\pi G} \left( g^{ac} g^{bd} - g^{ad} g^{bc} \right)$. Using this and the fact that the square of the binormals is $\epsilon_{ab} \epsilon^{ab} = -2$ we get back the usual area over four law for the entropy, $S = A/4$. The normalization of the quantity $P^{abcd}$ is due to convention used by Wald et al. for the Lagrangian, $L = \frac{1}{16\pi G} R$ (instead of just $L = R$).

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Chapter 2

EQUATIONS OF MOTION IN GENERALIZED GRAVITY

## 2.1   Introduction

A common action encountered in theories like Kaluza-Klein gravity, higher curvature gravity, and higher derivative scalar field theories is the one which has various forms of non-minimal scalar couplings to gravity. Some of them are even kinetically coupled like $G^{ab}\partial_a\phi\partial_b\phi$. A case by case evaluation of the equations of motion of these theories is a time consuming process. Existence of a generalized expression for the equations of motion, applicable to a wide variety of non-minimally coupled theories, will greatly simplify the process. Existing efforts [69] in that direction builds upon the vision of Wald et al. [70, 71] using a compact representation of the physical quantities (viz. entropy) in terms of generalized quantities. For example, when dealing with the entropy functional for non-Einsteinian theories of gravity an elegant representation of the generalized entropy can be made possible by defining an object $P_a{}^{bcd} = \partial L/\partial R^a{}_{bcd}$, which encodes the non-Einsteinian higher curvature effects. This quantity facilitates the writing of the Wald entropy [72] in a more compact manner and opens up the possibility for a proof of the second law of black hole thermodynamics for arbitrary theories of gravity. This approach is the motivation for obtaining a generalized expression for the equations of motion of higher curvature gravity.

In the following sections we will first derive a generalized equation of motion assuming the symmetries of the Riemann curvature tensor on the generalized quantity $P_{abcd}$. However our preliminary attempt will be met with failure, as several subtle features emerge. It appears to be obvious to assume that since the quantity $P_a{}^{bcd}$

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is obtained by differentiating the Lagrangian with respect to the Riemann tensor, it should inherit the symmetry properties of the Riemann tensor $R_{abcd}$ [69]. However the assumption about the symmetries of the quantity $P_{abcd}$ only works for theories which are homogeneous functions of the curvature scalars. For cases of non-minimal scalar kinetic couplings to gravity and in presence of the derivatives of the Riemann curvature in the Lagrangian the quantity $P_a{}^{bcd}$ does not inherit the symmetry properties of the Riemann curvature. Therefore the procedure needs to be revised, so as to accommodate the symmetries of the quantity $P_{abcd}$ (or lack thereof). For example, in galileon theories we encounter certain Lagrangians of the type $G^{ab}\partial_a\phi\partial_b\phi$ which causes this procedure to fail. Our task will be to extend the paradigm; in order to provide a more rigorous treatment of the derivation of the generalized equations of motion for all forms of non-minimally coupled Lagrangians (except the ones which contain derivatives of the Riemann curvature).

First we will perform a derivation of the generalized equations of motion using the assumption that the Lagrangian is a function of the curvature scalars only. Then in the subsequent section we will extend it to include all forms of non-minimally coupled Lagrangians (but not the ones which contain derivatives of the Riemann curvature). At the end we will verify our procedure by evaluating the equations of motion for a non-minimally kinetically coupled Lagrangian using brute force variational principle method and by comparing the resulting expression with the one obtained from our expression for the generalized equations of motion.

## 2.2 Derivation of Generalized Equations of Motion for Lagrangians without Kinetic Non-Minimal Couplings

In this section we will derive a generalized equation of motion for a Lagrangian which is a homogeneous function of curvature scalars. It can also include non-minimal

scalar couplings as long as the scalar couplings are not kinetic in nature. The procedure involves varying the Lagrangian with respect to the metric and the Riemann tensor separately and then reducing the varied expression further by expanding the variation of the Riemann tensor again. So it is not a Palatini variation even though we are varying using two independent quantities at first. We will assume that the Lagrangian is a scalar function of $g^{ab}$ and $R^a{}_{bcd}$ only, $L(g^{ab}, R^a{}_{bcd})$. The variation of the Lagrangian is [69]

$$
\begin{aligned}
\delta \left( L\sqrt{-g} \right) &= \left( \frac{\partial L\sqrt{-g}}{\partial g^{ab}} \right) \delta g^{ab} + \left( \frac{\partial L\sqrt{-g}}{\partial R^a{}_{bcd}} \right) \delta R^a{}_{bcd} \\
&= \left( \frac{\partial L\sqrt{-g}}{\partial g^{ab}} \right) \delta g^{ab} + \sqrt{-g} P_a^{bcd} \delta R^a{}_{bcd}
\end{aligned}
\tag{2.1}
$$

The variation of the determinant of the metric is $\delta log(g) = \delta g_{ab}\, g^{ab}$ which implies the following relation.

$$
\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{ab}\delta g^{ab}
\tag{2.2}
$$

To evaluate $\delta R^a{}_{bcd}$ we use the Palatini identity and use a gauge choice of the Riemann normal coordinates i.e. the Christoffel are all zero but their derivatives are not. This allows us to replace the partial derivatives with covariant derivatives. And since the variation of the Christoffel is a tensor quantity, the whole expression becomes gauge invariant and we no longer have to worry about whether we are in the Riemann normal coordinates or not.

$$
\begin{aligned}
\delta R^a{}_{bcd} &= \delta \left[ \partial_c \left( \Gamma^a_{db} \right) - \partial_d \left( \Gamma^a_{cb} \right) \right] = \nabla_c \left( \delta\Gamma^a_{db} \right) - \nabla_d \left( \delta\Gamma^a_{cb} \right) \\
&= \frac{1}{2}\nabla_c \left[ g^{ai} \left( -\nabla_i \delta g_{db} + \nabla_d \delta g_{bi} + \nabla_b \delta g_{di} \right) + \delta g^{ai}\Gamma_{idb} \right] - \{\text{term with } c \leftrightarrow d\} \\
&= \frac{g^{ai}}{2}\nabla_c \left[ -\nabla_i \delta g_{db} + \nabla_d \delta g_{bi} + \nabla_b \delta g_{di} \right] - \{\text{term with } c \leftrightarrow d\}
\end{aligned}
\tag{2.3}
$$

Using this expression the last term in eqn. (2.1) can be simplified as

$$
P_a^{bcd}\, \delta R^a{}_{bcd} = P^{ibcd}\nabla_c \left[ -\nabla_i \delta g_{db} + \nabla_d \delta g_{bi} + \nabla_b \delta g_{di} \right]
\tag{2.4}
$$

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where we have used the anti-symmetry of c and d indices. But $P^{ibcd}\nabla_c\nabla_d\delta g_{bi} = 0$ because of anti-symmetry in the indices $i$ and $b$ of $P^{ibcd}$. The rest of the terms $-\nabla_i\delta g_{db} + \nabla_b\delta g_{di}$ are anti-symmetric in $i$ and $b$. They contribute a total of $2\nabla_b\delta g_{di}$ and we obtain

$$
\begin{aligned}
P_a^{bcd}\delta R^a{}_{bcd} &= 2P^{abcd}\nabla_c\nabla_b(\delta g_{ad}) \\
&= 2\nabla_c\left[P^{abcd}\nabla_b\delta g_{ad}\right] - 2\nabla_b\left[\nabla_c P^{abcd}\delta g_{ad}\right] + 2\nabla_b\nabla_c P^{abcd}\delta g_{ad} \quad (2.5)
\end{aligned}
$$

Discarding the total derivatives, only the last term contributes to the equation of motion. However, it needs to be rearranged so that the variation is with respect to $\delta g^{ab}$. This is easily done with raising and lowering indices and keeping track of the indices. A useful identity which we will use often is the one involving contraction of the metric with both components of a two–tensor.

$$
A^{ab}\delta g_{ab} = A_{mn}(g^{am}g^{an}\delta g_{ab}) = A_{mn}(\delta g^{mn} - \delta g^{am}g^{bn}g_{ab} - g^{am}\delta g^{bn}g_{ab}) = -A_{mn}\delta g^{mn}
$$

$$(2.6)$$

This identity, when used with the above expression leads to, $2\nabla_b\nabla_c P^{abcd}\delta g_{ad} = 2\nabla^c\nabla^d P_{acbd}\delta g^{ab}$, where we have used both the above identity and the anti-symmetry property of $P_{abcd}$. Now going back to eqn. (2.1) and collecting terms proportional to $\delta g_{ab}$ we obtain a generalized equation of motion

$$\frac{\partial L}{\partial g^{ab}} - \frac{1}{2}g_{ab}L + 2\nabla^c\nabla^d P_{acbd} = 8\pi G T_{ab} \qquad (2.7)$$

Care needs to be taken while differentiating the Lagrangian with respect to the metric, because there may be contractions with $g_{ab}$. The $\delta g_{ab}$ term contributes a negative term to the equations of motion. As an example, let us consider $R_{abcd}R^{abcd}$. We need to rewrite this as $R^a{}_{bcd}R^p{}_{qrs}g_{ap}g^{bq}g^{cr}g^{ds}$ because the independent variable is the Riemann tensor with one up index and not the one with all index down. The variation of the three $g^{ab}$ and one $g_{ab}$ will give a total of $2\ R_{abcd}R_e{}^{bcd}$. The symmetry factor

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is the number of contravariant metric contractions minus the number of covariant metric contractions. As an application of this procedure, the equations of motion for the Gauss-Bonnet gravity is obtained in Appendix A using the generalized expression, eqn. (2.7).

<div align="center"><em>Failure of this method</em></div>

The method to derive a generalized equation of motion outlined in the previous section makes several assumptions. While the procedure still works for non-minimally coupled scalar fields as shown at the end of Appendix A, it fails for Lagrangians with a non-minimal kinetic scalar coupling, for example with $L = G^{ab} \partial_a \phi \partial_b \phi$. The failure is a result of the assumption that the quantity $P_{abcd}$ has the same symmetries as the Riemann curvature tensor, which is impossible to impose if the Lagrangian is not a homogeneous function of the curvature. This is because our variation was carried out using $R^a{}_{bcd}$ and not $R_{abcd}$. While the latter has all the symmetries of the Riemann curvature tensor the previous one does not have the anti-symmetry in the first two indices (see [73]). If the Lagrangian is not a homogeneous function of the curvature then $P^a{}_{bcd}$ will have some up index term which will not inherit the symmetries of the curvature tensor when lowered. To illustrate this with an example we will derive the equations of motion for the above Lagrangian, $L = G^{ab} \partial_a \phi \partial_b \phi$, by brute force variational method and then compare it with the one obtained from our generalized equations of motion. The calculation has been carried out in Appendix A. The equation of motion is

$$
(R_{ac} \partial_b \phi + R_{bc} \partial_a \phi) \partial^c \phi - \frac{1}{2} R \partial_a \phi \partial_b \phi - \frac{1}{2} (\partial \phi)^2 R_{ab} + \nabla_a \nabla^c \phi \nabla_b \nabla_c \phi - \Box \phi \nabla_a \nabla_b \phi
$$
$$
+ R_{acbd} \partial^c \phi \partial^d \phi + \frac{1}{2} g_{ab} \left( (\Box \phi)^2 - \nabla_c \nabla_d \phi \nabla^c \nabla^d \phi \right) - \frac{1}{2} g_{ab} \left( -\frac{1}{2} (\partial \phi)^2 R \right)
$$
$$
- g_{ab} R_{cd} \partial^c \phi \partial^d \phi = 8\pi G T_{ab} \tag{2.8}
$$

We shall not go through the steps but the reader can easily verify that this is not the same equation of motion which is obtained when eqn. (2.7) is used.

## 2.3 Derivation of Generalized Equations of Motion for Lagrangians with Kinetic Non-Minimal Couplings

We wish to find a generalized equation of motion from a generalized variational principle which applies to all kinds of Lagrangians, not just the ones which are homogeneous in the curvature scalars [1] . The Lagrangian which is a function of the metric and its derivatives $L(g, \partial g, \partial \partial g \ldots ; \phi, \partial \phi, \partial \partial \phi \ldots)$ can be written in a locally flat gauge $\partial g \sim 0$ (Riemann normal coordinates), which keeps the analysis covariant and the physics invariant. We also assume that the Lagrangian has derivatives of the metric (and all other fields) upto second order only. Any higher derivatives of the metric (or any other field) will introduce ghosts in the theory. The Lagrangian $L(g, \partial \partial g; \phi, \partial \phi, \partial \partial \phi; \ldots)$ can now be varied as $\delta L = \mathbf{H}\delta g + \mathbf{P}\delta(\partial \partial g)$. However, the double derivative of the metric only appears as the curvature scalars and their coupling to matter fields. So we can rewrite the variation as $\delta L = \mathbf{H}\delta g + \mathbf{P}\delta R$. Now the equation of motion depends on whether we vary with respect to the Riemann with all down indices or one up index. In the previous case the equation of motion is much more compact and $\mathbf{P}$ inherits all the symmetries of the Riemann while in the latter case the equation of motion is more complicated but no symmetries needs to be imposed on $\mathbf{P}$. We shall begin with the latter case first.

The variation procedure is similar to the one depicted in Section 2.2. All equations from eqn. (2.1) to eqn. (2.3) are still true, only we no longer assume or impose the symmetries of the Riemann curvature tensor on $P_{abcd}$. There is no eqn. (2.5) and the

---

[1]The Lovelock invariants are examples of homogeneous Lagrangians.

equation of motion is,

$$\frac{\partial L}{\partial g^{ab}} - \frac{1}{2}g_{ab}L + \frac{1}{4}\nabla^c\nabla^d\left[(P_{abcd} - P_{abdc}) + (P_{bacd} - P_{badc}) + (P_{cadb} - P_{cabd})\right.$$

$$\left. + (P_{cbda} - P_{cbad}) + (P_{bcad} - P_{bcda}) + (P_{acbd} - P_{acdb})\right] = 8\pi G T_{ab} \qquad (2.9)$$

So far we have varied the Riemann tensor and not the Ricci tensor. However, if the Lagrangian contains only the Ricci tensor and not the Riemann tensor, like the Einstein-Hilbert Lagrangian, then it makes sense to vary with respect to the Ricci tensor. The question is whether a generalized equation of motion derived from variation of the Ricci tensor would lead to a completely different equation of motion or whether our equation of motion can be reduced to the one obtained by variation of the Ricci tensor. To answer this let us find a generalized expression for the equation of motion using the variation of the Ricci tensor. We write the variation of the Lagrangian as,

$$
\begin{aligned}
\delta\left(L\sqrt{-g}\right) &= \left(\frac{\partial L\sqrt{-g}}{\partial g^{ab}}\right)\delta g^{ab} + \left(\frac{\partial L\sqrt{-g}}{\partial R_{ab}}\right)\delta R_{ab} \\
&= \left(\frac{\partial L\sqrt{-g}}{\partial g^{ab}}\right)\delta g^{ab} + \sqrt{-g}P^{ab}\,\delta R_{ab} \qquad (2.10)
\end{aligned}
$$

After discarding some total derivatives, the equation of motion is

$$\frac{\partial L}{\partial g^{ab}} - \frac{1}{2}g_{ab}L + \frac{1}{4}\left(\nabla^d\nabla_d(P_{ab} + P_{ba}) + 2g_{ab}\nabla^c\nabla^d P_{cd} - \nabla^d\nabla_b(P_{ad} + P_{da})\right.$$

$$\left. - \nabla^d\nabla_a(P_{bd} + P_{db})\right) = 8\pi G T_{ab} \qquad (2.11)$$

This is the equation the previous eqn. (2.9) should reduce to. The $P_{abcd}$ is actually related to $P_{ab}$ by $P_{abcd} = g_{ac}P_{bd}$. If we use this expression for $P_{abcd}$ in eqn. (2.9) we will obtain eqn. (2.11) thereby proving that the generalized equation of motion eqn. (2.9) works for all types of higher curvature Lagrangians (without scalar kinetic couplings).

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We can further simplify eqn. (2.9) to,

$$\frac{\partial L}{\partial g^{ab}} - \frac{1}{2}g_{ab}L + \frac{1}{2}\left(R_a^{kcd}P_{(kb)cd} + R_b^{kcd}P_{(ak)cd}\right) + \frac{1}{2}\nabla^c\nabla^d\left(P_{ca[db]} + P_{cb[da]}\right.$$

$$\left. + P_{bc[ad]} + P_{ac[bd]}\right) = 8\pi G T_{ab} \qquad (2.12)$$

The symmetrization symbols have symmetry factors in them $(ab) = \frac{1}{2}(ab+ba)$; $[ab] = \frac{1}{2}(ab-ba)$.

At the beginning of this section we had claimed that we will obtain the generalized equations of motion by varying with respect to the Riemann tensor with one up index, $R^a_{bcd}$ and then obtain it with by varying with respect to the Riemann tensor with all down indices, $R_{abcd}$. We did the first one and now we proceed to do the latter. We do a similar variation as eqn. (2.9) but with the Riemann tensor with all indices down. Here the anti-symmetry of the Riemann tensor in all of its indices are manifest. Hence when we write down the $P_{abcd}$ , it has to be correctly anti-symmetrized so that its indices have the same symmetries as the Riemann tensor. We can do this since all index lowered $P_{abcd}$ will inherit the symmetry properties of the Riemann tensor, as explained in the previous section. The variation of the Lagrangian is,

$$\delta\left(L\sqrt{-g}\right) = \left(\frac{\partial L\sqrt{-g}}{\partial g^{ab}}\right)\delta g^{ab} + \left(\frac{\partial L\sqrt{-g}}{\partial R_{abcd}}\right)\delta R_{abcd}$$

$$= \left(\frac{\partial L\sqrt{-g}}{\partial g^{ab}}\right)\delta g^{ab} + \sqrt{-g}P^{abcd}\delta R_{abcd}$$

$$= \sqrt{-g}\left(\left(\frac{\partial L}{\partial g^{ab}} - \frac{1}{2}g_{ab}L\right)\delta g^{ab} + P^{acde}R^b_{\ cde}\delta g_{ab} + P_a^{\ bcd}\delta R^a_{\ bcd}\right)(2.13)$$

There is a subtlelty here. Even though we have defined the generalized quantity $P_{abcd}$ as a variation of the Lagrangian with respect to $R_{abcd}$, the actual variation of the action still has to be with respect to the Riemann tensor with one up index, $R^a_{bcd}$. This has to do with the way the Riemann curvature tensor is defined. It is defined with respect to the Christoffel's with one index up. This adds an extra term $\mathbf{P} \cdot \mathbf{R}$ term to the variation which will be crucial to the analysis, as explained at the end.

We can now use the Palatini identity (eqn. (2.3)) to simplify the last term of this expression in a similar fashion to Section 2.2. Going back to eqn. (2.13) and collecting terms proportional to $\delta g^{ab}$ we have the generalized equations of motion,

$$\frac{\partial L}{\partial g^{ab}} - \frac{1}{2} g_{ab} L - \frac{1}{2} \left( P_a^{\ cde} R_{bcde} + P_b^{\ cde} R_{acde} \right) + \nabla^c \nabla^d \left( P_{acbd} + P_{bcad} \right) = 8\pi G T_{ab} \quad (2.14)$$

where the explicit symmetrization of the indices a & b is done as the equations of motion are expected to be symmetric in those indices. The rule regarding the derivative with respect to the metric, mentioned below eqn. (2.7), still applies and the symmetry factor is again the number of contravariant metrics minus the number of covariant metrics.

This equation of motion eqn. (2.14) differs from the one in eqn. (2.12) in many ways. In eqn. (2.12) the $P_{abcd}$ has no symmetries at all while in eqn. (2.14) it has the symmetries of the Riemann tensor. The appearance of the $\mathbf{P} \cdot \mathbf{R}$ terms in the equations of motion is very interesting. Even though it appears that the $\nabla\nabla\mathbf{P}$ term in the equations of motion accounts for the variation of the Ricci or the Riemann tensor, it is in fact not entirely true. A part of it is proportional to $\delta\mathbf{R}$ but there are extra terms which are cancelled by the $\mathbf{P} \cdot \mathbf{R}$ term. But there's more. The $\mathbf{P} \cdot \mathbf{R}$ will still have a remaining piece that cancels the extraneous terms which arises due to writing the Lagrangian in terms of the Riemann tensor when it can be expressed in terms of the Ricci tensor only. For example, if the Ricci scalar is written as $R_{abcd} \frac{1}{2} \left( g^{ac} g^{bd} - g^{ad} g^{bc} \right)$ instead of $R_{ab} g^{ab}$. This point becomes apparent while evaluating the terms in the generalized expression (see Appendix A and Appendix A)

For Lagrangians with a scalar field pre-factor, a further useful formula for the

equations of motion can be written down using $L = e^{\mu\phi}L'$ and $P_{abcd} = e^{\mu\phi}P'_{abcd}$,

$$e^{\mu\phi}\left(\frac{\partial L'}{\partial g^{ab}} - \frac{1}{2}g_{ab}L' - P^{cde'}_{(a}R_{b)cde} + \nabla^c\nabla^d\left(P'_{acbd} + P'_{bcad}\right)\right.$$

$$\left. + 2P'_{acbd}\left(\mu\nabla^c\nabla^d\phi + \mu^2\partial^c\phi\partial^d\phi\right)\right) = 8\pi GT_{ab} \qquad (2.15)$$

*Discussion*

We obtained a generalized equation of motion for higher curvature gravity. This expression works for most types of higher derivative Lagrangians expect for Lagrangians which contains derivatives of the curvature scalars, viz. $L = \nabla_c R_{ab}\nabla^c R^{ab}$. However, these Lagrangians are third order in the derivatives of the metric and will contain ghosts and therefore not realistic models of gravity. Our expression is the most generalized expression possible for any type of non-minimally coupled Lagrangian.

Chapter 3

THE BLACK HOLE MEMBRANE PARADIGM

## 3.1  Introduction

The membrane paradigm is the surprising idea that, to an outside observer, black hole horizons behave like fluid membranes. That is, when a black hole is perturbed by external fields, the equations of motion describing the response of the horizon are exactly what they would be if the fields were interacting instead with a bubble, or membrane, enveloping the horizon. The membrane is endowed with the sources for whatever external fields are present. In particular, to source the gravitational field, the membrane possesses the stress tensor of a viscous fluid. This external perspective of horizons as fluid membranes provides not only an intuitive way of understanding black hole interactions but also the original semiclassical realization of holography.

The membrane paradigm was first discovered [74, 75] by re-writing particular field equations of perturbed black hole horizons in terms of familiar nonrelativistic dissipative equations such as Ohm's law and the Navier-Stokes equation. A more systematic action-based derivation was obtained in [76], which in principle enabled membrane properties to be determined for arbitrary field theories. Nevertheless, many puzzles remain. For what gravitational theories does a black hole horizon behave as if it were a Newtonian fluid? What are the fluid transport coefficients in more general theories of gravity? Does the membrane always obey the Navier-Stokes equation? And more generally, for what gravitational theories does the membrane paradigm even exist? In this paper, we attempt to shed some light on these questions by considering the action formulation of the membrane paradigm for $f(R)$ theories

of gravity. These theories serve as a model for higher-derivative gravity; they are a simple extension of Einstein gravity in that they introduce exactly one extra degree of freedom.

## 3.2  Geometric Set-Up

Before entering into the details of the membrane action, it will help to specify precisely the geometric set-up. We will work in $D$ spacetime dimensions. Let the black hole event horizon, $H$, which is a $D-1$-dimensional null hypersurface, be generated by null geodesics $l^a$. We take these generators to have a nonaffine parameterization, $\tau$. That is, $l^a = (\partial/\partial\tau)^a$ and the geodesic equation is $l^a\nabla_a l^b = \kappa l^b$, rather than zero. Here $\kappa$ is a nonaffine coefficient; for a stationary spacetime, $l^a$ coincides with the null limit of the timelike Killing vector and $\kappa$ can then be interpreted as the surface gravity of the horizon.

Although the membrane paradigm can be formulated entirely on the event horizon, it proves convenient to introduce a timelike stretched horizon, $\Sigma$, positioned slightly outside $H$, the advantage being that a timelike surface has a nondegenerate metric which permits one to write down a conventional action. The precise choice of the timelike surface is somewhat arbitrary. We consider $\Sigma$ to be one among a foliation of timelike surfaces, each labeled by a parameter $\alpha$ such that in the limit $\alpha \to 0$, the stretched horizon approaches the true horizon. In the absence of horizon caustics, a one-to-one correspondence between points on $H$ and $\Sigma$ are always possible by, for example, using ingoing light rays that connect both the surfaces.

We can also regard the stretched horizon as the world-tube of a family of "fiducial" observers hovering just outside the black hole. We take these observers to have world lines $u^a$; then just as $H$ is generated by the null congruence $l^a$, the stretched horizon is generated by the timelike congruence $u^a$. The stretched horizon also has a spacelike,

outward pointing unit normal vector $n^a$.

In the limit $\alpha \to 0$, we require that $\alpha u^a \to l^a$ and $\alpha n^a \to l^a$ i.e. the stretched horizon tends to the true horizon in this limit, as we have already envisaged. This is nothing more than the statement that the null generator $l^a$ is both normal and tangential to the true horizon, which is the defining property of null surfaces. The metric $h_{ab}$ on the stretched horizon $\Sigma$ can be expressed in terms of the spacetime metric $g_{ab}$ and the normal vector $n^a$. Similarly we can define a metric $\gamma_{ab}$ on a $D-2$-dimensional spacelike cross-section of $\Sigma$, to which $u^a$ is normal:

$$h_{ab} = g_{ab} - n_a n_b \quad \text{and} \quad \gamma_{ab} = h_{ab} + U_a U_b. \tag{3.1}$$

We will choose the stretched horizon among all possible choices, such that the normal vector $n^a$ obeys an affine geodesic equation, $n^a \nabla_a n_b = 0$, and as a result, for any vector $v^a \in \Sigma$, we have $\nabla_a v^a = v^a_{|a}$ where $_{|a}$ is the covariant derivative with respect to the metric $h_{ab}$. Next, we denote $\{A, B, ...\}$ as the coordinates on the $D-2$-dimensional spacelike cross-section of $\mathcal{H}$ and $k^A_B = \gamma^d_B l^A_{||d}$ as the extrinsic curvature on the $D-2$-dimensional cross-section of the null surface, where $_{||A}$ is the covariant derivative with respect to $\gamma_{AB}$.

We define the extrinsic curvature of $\Sigma$ as $K^a_b = h^c_b \nabla_c n^a$. In the null limit $\alpha \to 0$, the various components of the extrinsic curvature become [74]

$$\text{As } \alpha \to 0 \quad : \quad K^U_U = K^a_b U_a U^b \to -\alpha^{-1} \kappa$$

$$K^U_A \to 0; K^A_B \to \alpha^{-1} k^A_B$$

$$K \to \alpha^{-1}(\theta + \kappa) \tag{3.2}$$

where $\theta$ is the expansion scalar of the geodesic congruence generating the horizon. Note that, for a $D-1$-dimensional timelike hypersurface, the extrinsic curvature of a $D-2$-dimensional spacelike section with respect to its timelike normal $u^a$ within the

hypersurface has nothing to do with the (projection of the) extrinsic curvature $K^A_B$ with respect to the spacelike normal $n^a$ off the hypersurface. However, in the null limit, both $u^a$ and $n^a$ map to the same null vector $l^a$. Hence we have $K^A_B \to \alpha^{-1} k^A_B$ where $k^A_B$ is the the extrinsic curvature of a $D-2$-dimensional spacelike section of the horizon. We can then decompose $k_{AB}$ into a traceless part and a trace as

$$k_{AB} = \sigma_{AB} + \frac{1}{D-2} \theta \gamma_{AB}$$

where $\sigma_{AB}$ is the shear of the null congruence. In the null limit, various components of the extrinsic curvature diverge and we need to renormalize them by multiplying by a factor of $\alpha$. The physical reason behind such infinities is that, as the stretched horizon approaches the true one, the fiducial observers experience more and more gravitational blue shift; on the true horizon, the amount of blue shift is infinite. This completes the description of our geometric set-up. Next, we review the derivation of the black hole membrane paradigm in standard Einstein gravity.

### 3.3 The Membrane Paradigm in Einstein Gravity

Since the region inside the event horizon cannot classically affect an outside observer, the classical equations of motion for such an observer must follow from the variation of the action restricted to the spacetime external to the black hole. However, the external action, $S^{\text{out}}$, is not stationary on its own because boundary conditions are not fixed at the horizon, and hence the boundary term in the derivation of the Euler-Lagrange equations does not vanish at the horizon as it does at infinity. In order to obtain the correct equations, we must add a surface term to the action whose variation cancels this residual boundary variation. We do this by splitting the action as

$$S = \left( S^{\text{in}} - S^{\text{surf}} \right) + \left( S^{\text{out}} + S^{\text{surf}} \right) , \tag{3.3}$$

where $S^{\text{surf}}$ is the requisite surface term, chosen so that $\delta S^{\text{out}} + \delta S^{\text{surf}} = 0$. The surface term corresponds to sources such as surface electric charges and currents for the Maxwell action, or a surface energy-momentum tensor for the gravitational action. These sources are fictitious in a traditional sense because an infalling observer passing through the horizon will not detect them. Nevertheless, to the external observer they are very much real and observable. An ontologically different stance, as advocated by the principle of observer complementarity, is that both the infalling and external viewpoints are equally valid, even though they seemingly contradict each other; indeed, the infalling observer is unable to detect Hawking radiation either but that is not usually regarded as implying that Hawking radiation is fictitious.

For Einstein gravity, the external action is given by

$$S^{\text{out}} = \frac{1}{16\pi G} \int d^D x \sqrt{-g}\, R + \frac{1}{8\pi G} \oint_{\infty} d^{D-1} x \sqrt{h}\, K \,, \tag{3.4}$$

where the second term is the Gibbons-Hawking boundary term required to cancel the normal derivatives of the variation of metric on the boundary at infinity. As before, extremizing this action does not yield the Einstein equations because of variational contributions at the (stretched) horizon. To cancel this contribution, we add a surface term $S^{\text{surf}}$ whose variation,

$$\delta S^{\text{surf}} = \frac{1}{2} \int d^{D-1} x \sqrt{h}\, t^{ab}\, \delta h_{ab} \,, \tag{3.5}$$

defines a surface energy-momentum tensor on the stretched horizon. This can be shown [76] to take the form

$$t^{ab} = \frac{1}{8\pi G} \left( K h^{ab} - K^{ab} \right) \,, \tag{3.6}$$

where $K_{ab}$ is the extrinsic curvature of the stretched horizon. By invoking the Gauss-Codazzi equations, the energy-momentum tensor can be shown to satisfy a conserva-

tion equation:

$$t^{ab}_{|b} = -h^a_c T^{cd} n_d ,\qquad(3.7)$$

where $T^{ab}$ is the energy-momentum tensor of real (bulk) matter outside the black hole. This is a continuity equation; it indicates that the divergence of the horizon energy-momentum tensor is equal to the flow of outside matter on to the horizon. The fact that the horizon energy-momentum tensor participates in a continuity equation with actual outside matter is crucial in sustaining the outside observer's belief that the surface energy-momentum tensor describes real matter.

In the limit that the stretched horizon approaches the true horizon, we can use 3.2 to express the regularized stress tensor in terms of the horizon expansion and shear. Remarkably, the stress tensor projected on a $D-2$-dimensional cross-section of the horizon then takes the form of the stress tensor of a viscous fluid [74, 76, 77]:

$$t^A_B = p\gamma^A_B - 2\eta\sigma^A_B - \zeta\theta\gamma^A_B ,\qquad(3.8)$$

where $p = \frac{\kappa}{8\pi G}$ is the pressure, $\eta = \frac{1}{16\pi G}$ the shear viscosity, and $\zeta = -\frac{1}{8\pi G}\frac{D-3}{D-2}$ the bulk viscosity of the membrane. The constancy of the transport coefficients means that the event horizon behaves as a $D-2$-dimensional Newtonian fluid. Note that, unlike ordinary fluids, the membrane has negative bulk viscosity. This would ordinarily indicate an instability against generic perturbations triggering expansion or contraction. It can be regarded as reflecting a null hypersurface's natural tendency to expand or contract.

Inserting the energy density $t^a_b U_a U^b = \Sigma = \alpha^{-1}\Sigma_{\mathcal{H}}$ into the conservation equation of the membrane stress tensor, we find that

$$\frac{d\Sigma_{\mathcal{H}}}{d\tau} + \Sigma_{\mathcal{H}}\theta = -p\theta + \zeta\theta^2 + 2\eta\sigma^2 + T^a_b l_a l^b .\qquad(3.9)$$

This is again the same as the energy conservation equation of a fluid with pressure $p$, shear viscosity $\eta$, and bulk viscosity $\zeta$ [78]. Next, inserting the $A$th-momentum

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density, $\pi_A = t_a^b \gamma_A^a U_b$, into the conservation equation of the membrane stress tensor, we arrive at the momentum conservation equation of the membrane:

$$\mathcal{L}_{l^a} \, \pi_A + \theta \, \pi_A \;\; = \;\; -\nabla_A p + 2\eta \sigma_{A||B}^B + \zeta \nabla_A \theta + T_A^l \, , \qquad (3.10)$$

where $\mathcal{L}_{l^a}$ is the Lie derivative along the null direction. Since the Lie derivative along a congruence plays the role of the convective derivative in ordinary fluid dynamics, we recognize this as the Navier-Stokes equation of a viscous fluid. This completes our short review of the membrane paradigm in Einstein gravity. Next, we turn to its extension to higher derivative gravity and treat $f(R)$ gravity as a special case.

### 3.4 The Membrane Paradigm in Higher Curvature Gravity

In our approach to derive a generalized expression for the boundary stress tensor we will need to assume that the generalized Lagrangian is homogenous. While membrane paradigm should work for most theories of gravity, it is impractical to analyze Lagrangians with non-homogeneous curvature couplings. This includes Lagrangians with derivatives of the curvature tensor. Because such Lagrangians have higher order equations of motion and by the Ostrogradski theorem they will contain ghosts. This analysis applies to all forms of higher curvature gravities. However whether all higher derivative theories admit black hole solutions is not known. Black hole solutions are known to exist for Gauss-Bonnet gravity [32], for higher order Lovelock gravity [52], for $f(R)$ gravity [79], for Weyl gravity [80], and Kaluza-Klein gravities [81]. We proceed by assuming that the higher derivative theory of gravity admits a black hole solution.

This general procedure also applies to Lagrangians with all forms of scalar couplings to the Riemann curvature tensor. But certain subtleties are involved in dealing with scalar couplings. Scalar fields add one extra degree of freedom to the theory

without adding any extra gauge degree of freedom. And we shall see later that the conservation of the membrane stress tensor is a result of the momentum constraint which in turn is a manifestation of the gauge degree of freedom of the theory. So the membrane stress tensor will no longer be conserved if one of the fields lacks gauge degree of freedom. This point is illustrated in Section 3.5 for the case of $f(R)$ theory which is a minimally coupled scalar-tensor theory.

For the purpose of generalized analysis consider a general diffeomorphism-invariant action with boundary terms

$$S = \frac{1}{16\pi G} \int d^D x \; L(g_{ab}, R_{abcd}) + S_\infty + S_{\text{matter}} \; , \tag{3.11}$$

where $S_\infty$ is the appropriate generalization of the Gibbons-Hawking term at infinity whose precise form we do not need. Its sole purpose is to cancel the variation of this action at infinity.

The surface term in generalized gravity is obtained by varying the Lagrangian and picking out the total derivative term, then decomposing the (D-1)–dimensional quantities on the (D-1)–dimensional surface into (D-2)–dimensional quantities and then taking the null limit. This will eliminate some terms as the event horizon is approached asymptotically. Therefore we can discard those terms in the (D-1)– dimensional stress tensor and define it as the membrane stress tensor. On-shell the variation of the action is just the surface term

$$\delta S_\Sigma = \frac{1}{8\pi G} \int d^{D-1}x \sqrt{-g} \, \nabla_a \left[ P^{abcd} \nabla_d \delta g_{bc} - \delta g_{bc} \nabla_d P^{abcd} \right] \tag{3.12}$$

$$= -\frac{1}{8\pi G} \int d^{D-1}x \sqrt{-h} \, n_a \left[ P^{abcd} \nabla_d \delta g_{bc} - \delta g_{bc} \nabla_d P^{abcd} \right] \tag{3.13}$$

We have used the fact that the stretched horizon is timelike (which gives a positive sign) and the normal $n_a$ is outward pointing (in comparison to the surface element, which gives a negative sign), $h_{ab}$ is the (D-1)–dimensional metric.

The action eqn. (3.13) still contains a D–dimensional total derivative

$$\delta S_\Sigma = -\frac{1}{8\pi G} \int d^{D-1}x \sqrt{-h} \left( \nabla_d(n_a P^{abcd}\delta g_{bc}) - (\nabla_d n_a)P^{abcd}\delta g_{bc} - 2\delta g_{bc}n_a\nabla_d P^{abcd} \right)$$

(3.14)

We cannot discard the total derivative because it is a D–dimensional derivative while the action is a (D-1)–dimensional boundary. However, as shown in Appendix B, that on asymptotically approaching the event horizon or equivalently the null limit of the stretched horizon this term becomes zero and the surface term becomes,

$$\delta S_\Sigma = \frac{1}{8\pi G} \int d^{D-1}x \sqrt{-h} \left[ (\nabla_d n_a)P^{abcd} + 2n_a\nabla_d P^{abcd} \right] \delta g_{bc}$$

(3.15)

However in this expression for the surface term, the integrand consists of D–dimensional quantities while the integration is over (D-1)–dimensions. We have to manipulate the integrand into quantities defined just on the stretched horizon. Most importantly the variation of the bulk metric $\delta g_{ab}$ at the boundary must be reduced to the variation of the boundary metric $\delta h_{ab}$. We can exploit the diffeomorphism degrees of freedom of the theory to set the variation of the normal vector to zero $\delta n_a = 0$. This would lead to $\delta g_{ab} \to \delta h_{ab}$. After some amount of algebra and using the properties of the stretched horizon (see Appendix B) it can be shown that eqn. (3.15) can be reduced to $P_{nklm}K^{mn}$ and an interesting extra term $2n^p\nabla^q P_{pklq}$ which is non-zero only for the theory of gravities which are not in the Lovelock-Lanczos form [1] .

$$\delta S_\Sigma = \frac{1}{8\pi G} \int d^{D-1}x \sqrt{-h} \left( P_{pklq}K^{pq}h^{kb}h^{lc} + 2n^p\nabla^q P_{pklq}h^{kb}h^{lc} \right) \delta h_{bc}$$

(3.16)

The resulting action reduces to the Einstein membrane action in the appropriate limit. The membrane stress-energy tensor on the (D-1)–dimensional hypersurface can be obtained directly from this action.

$$t^{ab} = -\frac{1}{4\pi G} \left( P_{nklm}K^{mn} + 2n^p\nabla^q P_{pklq} \right) h^{ka}h^{lb}$$

(3.17)

---

[1] This is because $\nabla^q P_{pklq} = 0$ for Lovelock-Lanczos theories, as a result of the Bianchi identity.

However, the transport coefficients cannot be read directly from this stress tensor. This stress tensor is a (D-1)–dimensional quantity and needs to be decomposed further and then evaluated on the spacelike section of the horizon. And this needs to be done case by case for every theory of gravity because $P_{abcd}$ contains powers of the Riemann tensor and its contractions, and the decomposition of $P_{abcd}$ is theory dependent.

The last term plays an important role for non-Lovelock theories. When we independently derive the $f(R)$ membrane stress tensor in the following section we will notice that the stress tensor could also be obtained directly from this generalized stress tensor.

## 3.5  Extension to $f(R)$ Gravity

In this section we derive the transport coefficients of the membrane fluid for black hole solutions in $f(R)$ gravity [79]. We specialize to the particular case for which the Lagrangian is a function $f(R)$ of the Ricci scalar only. The equations of motion for $f(R)$ gravity is

$$f'(R)R_{ab} - \nabla_a \nabla_b f'(R) + \left( \Box f'(R) - \frac{1}{2} f'(R) \right) g_{ab} = 8\pi G \, T_{ab} \,, \qquad (3.18)$$

where the prime denotes a derivative with respect to the argument; when $f(R) = R$, this reduces to Einstein's equation. In order to obtain this equation, we need to add a surface term to the action, with variation

$$\delta S_{\Sigma}^{\text{surf}} \;\; = \;\; -\frac{1}{8\pi G} \int d^{D-1}x \sqrt{-h} \, (F_1 + F_2) \,, \qquad (3.19)$$

where, using $P^{abcd} = \frac{1}{2}(g^{ac}g^{bd} - g^{ad}g^{bc})f'(R)$, the terms $F_1$ and $F_2$ are given by, [2]

$$F_1 = \frac{1}{2}h^{bc}\left(\nabla_c\left(f'(R)n^a\delta g_{ab}\right) - \nabla_a\left(f'(R)n^a\delta g_{bc}\right)\right) \tag{3.20}$$

$$F_2 = -\frac{1}{2}h^{bd}\left(\nabla_d n^c f'(R) + 2n^c\nabla_d f'(R)\right)\delta g_{bc} + h^{bc}\left(f'(R)\nabla_a n^a + 2n^a\nabla_a f'(R)\right)\delta g_{bc} \tag{3.21}$$

The contribution from the term $F_1$ can be shown to vanish in the limit in which the stretched horizon approaches the event horizon. We first note that any variations in the metric that are merely gauge transformations can be set to zero. Using a vector $v^a$ where $v^a$ vanishes on the stretched horizon, we can gauge away the variations in the normal direction so that $\delta g_{ab} \to \delta h_{ab}$. Next we notice that for any vector $v^a \in \Sigma$, we have $\nabla_a v^a = v^a_{|a}$ where $_{|a}$ is the covariant derivative with respect to $h_{ab}$; integration over any divergence term like $v^a_{|a}$ over the stretched horizon gives zero. We also use relations like $h_{ab}n^b = 0$ and $a^d = n^e\nabla_e n^d = 0$. Then the integral of the $F_1$ term is

$$\int d^3x\sqrt{-h}\, h^{bc}\left(\nabla_a\left(f'(R)n^a\delta h_{bc}\right) - \nabla_c\left(f'(R)n^a\delta h_{ab}\right)\right)$$

$$= \int d^3x\sqrt{-h}\left(\nabla_a\left(h^{bc}f'(R)n^a\delta h_{bc}\right) + \left(n^ca^b + n^ba^c\right)f'(R)\delta h_{bc} - \left(h^{bc}n^a f'(R)\delta h_{ab}\right)_{|c}\right.$$

$$\left. -f'(R)h^{bc}n^a\delta h_{ab}a_c - Kf'(R)n^bn^a\delta h_{ab} - a^bn^a f'(R)\delta h_{ab}\right)$$

$$= \int d^3x\sqrt{-h}\left(\nabla_a\left(h^{bc}f'(R)n^a\delta h_{bc}\right) - Kf'(R)n^bn^a\delta h_{ab}\right)$$

$$= \int d^3x\sqrt{-h}\left(\nabla_a\left(h^{bc}f'(R)n^a\delta h_{bc}\right) - Kf'(R)\left(\delta\left(n^bn^a h_{ab}\right) - n^a h_{ab}\delta n^b - n^b h_{ab}\delta n^a\right)\right)$$

$$= \int d^3x\sqrt{-h}\,\nabla_a\left(h^{bc}f'(R)n^a\delta h_{bc}\right) \tag{3.22}$$

Now let us take an auxiliary vector $k^a$ such that $n^a = u^a + \alpha k^a$. When $\alpha \to 0$, we have $n^a \to u^a \to \alpha^{-1}l^a$ on the true horizon. Then the term $F_1$ ultimately becomes

$$F_1 = \int d^3x\sqrt{-h}\,\nabla_a\left(h^{bc}f'(R)u^a\delta h_{bc}\right) + \alpha\int d^3x\sqrt{-h}\,\nabla_a\left(h^{bc}f'(R)k^a\delta h_{bc}\right) \tag{3.23}$$

---

[2]We have used the gauge choice $\delta n_a = 0$, which implies $g^{ab}\delta g_{bc} = h^{ab}\delta g_{bc} = g^{ab}\delta h_{bc} = h^{ab}\delta h_{bc}$.

The second term does not contribute in the null limit and the integrand of the first piece is of the form $\nabla_a v^a = v^a_{|a}$ where $v^a \in \Sigma$. Hence, this term also does not contribute anything. This completes our proof that $F_1$ term in (3.19) vanishes in the null limit.

After some straightforward manipulation we find that

$$
\begin{aligned}
\delta S_\Sigma^{\text{surf}} &= -\frac{1}{16\pi G} \int d^{D-1}x \sqrt{-h} \left( f'(R) \left( Kh^{ab} - K^{ab} \right) + 2n^d \nabla_d f'(R) h^{ab} \right) \delta h_{ab} \\
&\equiv -\frac{1}{2} \int d^{D-1}x \sqrt{-h} \, t^{ab} \, \delta h_{ab}
\end{aligned}
\tag{3.24}
$$

where $t^{ab}$ is the membrane stress tensor:

$$
t^{ab} = \frac{1}{8\pi G} \left( f'(R) \left( Kh^{ab} - K^{ab} \right) + 2n^d \nabla_d f'(R) h^{ab} \right) .
\tag{3.25}
$$

This is the stress tensor for the membrane in $f(R)$ gravity. However, taking its divergence does not give a conservation equation analogous to (3.7). This would seem to undermine the interpretation of $t^{ab}$ as real energy-momentum, which an observer will naturally require to be conserved. One way out is to note that the membrane stress tensor produces a discontinuity in the extrinsic curvature across the stretched horizon. The relationship between the discontinuity and the source term is given by the appropriate Israel junction condition [82] for $f(R)$ gravity, which is

$$
|f'(R) \left( Kh_{ab} - K_{ab} \right) + h_{ab} n^d \nabla_d f'(R)| = 8\pi G \, t_{ab} ,
\tag{3.26}
$$

where $|A| \equiv A_+ - A_-$ denotes the difference between the quantities evaluated on the stretched horizon between its embedding in the external universe and in the spacetime internal to the black hole. Comparing the junction condition (3.26) with the membrane stress tensor (3.25), we find

$$
f'(R) \left( Kh_{ab} - K_{ab} \right) + h_{ab} n^d \nabla_d f'(R)|_- + h_{ab} n^d \nabla_d f'(R)|_+ = 0 .
\tag{3.27}
$$

Now, junction conditions for a general $f(R)$ gravity theory have to be supplemented with an additional condition namely, the continuity of the Ricci scalar across the junction [82]. The reason behind this extra constraint is that, unlike for Einstein gravity, the equations of motion of a general $f(R)$ gravity is of fourth order. Unless the continuity of the Ricci scalar is imposed on the junction, the junction conditions do not reduce to the familiar Israel junction conditions as $f(R) \rightarrow R$. Another alternative but equivalent way to understand this extra constraint is that any $f(R)$ theory, other than Einstein gravity, can be cast into a scalar-tensor theory via a conformal transformation. Thus, apart from the tensor mode, a general $f(R)$ gravity also contains an extra scalar degree of freedom; it is the Ricci scalar that plays the role of this scalar field in the scalar-tensor picture. Hence, the continuity of the Ricci scalar actually ensures that the scalar degree of freedom is continuous across the junction. On the other hand, it is not possible to write down a conserved membrane source for scalar field theory due to the absence of a conserved current. By imposing this condition on our membrane, we are thereby effectively removing the scalar degree of freedom. The continuity of $R$ across the membrane leads to the continuity of the trace of extrinsic curvature $K$ [82]. Using this, (3.18) and (3.27), we find that

$$t^{ab}_{\phantom{ab}|b} = -h^a_c T^{cd} n_d \ . \tag{3.28}$$

This is once again a conservation law.

Although the use of the junction conditions unambiguously leads to the correct conservation law, there is still something dissatisfying about it. The whole idea of the membrane paradigm is that we should not have to consider conditions on the other side of the membrane; using junction conditions does not seem to fit that philosophy. It will be nice to find another motivation for (3.28). One intriguing possibility arises from observing that, for Einstein gravity, (3.6) is simply the momentum of gravity

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in a Hamiltonian picture where "time" runs in a spacelike direction off the stretched horizon. The existence of the conservation equation (3.7) can then be recast as the momentum constraint equation. That in turn arises because of gauge invariance. This viewpoint explains why scalar field theories do not have a realistic intepretation in terms of the membrane paradigm. Applied to $f(R)$ theory it suggests that, since the theory does possess diffeomorphism invariance, there must exist some kind of conservation equation for the membrane stress tensor. It could be very illuminating to make these ideas more precise. In particular, it suggests that Lovelock theories, which have the same number of degrees of freedom as Einstein gravity, might admit a very clean interpretation in terms of fluid membranes.

The projection of the membrane stress tensor, (3.25), to a spatial $(D$–$2)$-dimensional slice of the horizon gives

$$t^{AB} = \frac{1}{8\pi G}\left(\left(\kappa f'(R) + 2\frac{df'(R)}{d\tau}\right)\gamma^{AB} + \frac{(D-3)}{(D-2)}\theta f'(R)\gamma^{AB} - f'(R)\sigma^{AB}\right) \quad (3.29)$$

where $\tau$ is the nonaffine parameter of the null congruence and $l^a\nabla_a = d/d\tau$. The stress tensor resembles that of a viscous fluid and readily allows us to find the fluid transport coefficients:

$$
\begin{aligned}
\text{Pressure}: p &= \frac{1}{8\pi G}\left(\kappa f'(R) + 2\frac{df'(R)}{d\tau}\right) \\
\text{Shear viscosity}: \eta &= \frac{f'(R)}{16\pi G} \\
\text{Bulk Viscosity}: \zeta &= -\frac{(D-3)}{(D-2)}\frac{f'(R)}{8\pi G} \\
\text{Energy Density}: \Sigma_{\mathcal{H}} &= \frac{1}{8\pi G}\left(-\theta f'(R) - 2\frac{df'(R)}{d\tau}\right) \quad (3.30)
\end{aligned}
$$

As in Einstein gravity, then, the membrane stress tensor for any general $f(R)$ gravity can indeed be written as a fluid stress tensor. However, there are a few differences:

- Transport coefficients are not constants but depend on the flow, characteristic of a non-Newtonian fluid [83].

- For any $f(R)$ theory, the bulk viscosity coefficient is always negative provided $D > 3$. As in Einstein gravity, this is related to the teleological definition of the event horizon; it is independent of the theory of gravity.

- Since, the transport coefficients are not constants, the relevant Navier-Stokes equation, which involves the derivative of the stress tensor, will have extra terms proportional to the derivatives of these transport coefficients. For fluid with constant viscosity, such terms do not contribute and we obtain the regular Navier-Stokes equation [3].

Another important property of the membrane fluid is the saturation of the so-called KSS bound [84]. As in Einstein gravity, the ratio of the shear viscosity to entropy per unit area is $1/4\pi$. In fact, in the context of finite temperature AdS/CFT, it was suggested that, for any $f(R)$ gravity theory, the ratio $\eta/s$ always saturates the KSS bound [85]. Our result confirms this for the membrane fluid as well.

We can cast the conservation law into equations of fluid dynamics. Inserting the A-momentum density $\pi_A = t_a^b \gamma_A^a U_b$ into the conservation equation of the membrane stress tensor, we arrive at the momentum conservation equation of the membrane as:

$$\mathcal{L}_l \pi_A + \theta \pi_A \;\; = \;\; -\nabla_A p + 2\left(\eta \sigma_A^B\right)_{||B} + \zeta \nabla_A \theta + T_A^l + \theta \gamma_A^B \nabla_B \zeta \qquad (3.31)$$

This is identical to the Navier-Stokes equation of a viscous fluid provided we generalize the usual Navier-Stokes equation for the case of non-Newtonian fluids with nonconstant transport coefficients [78]. As a result, compared to Eq. (3.10), this equation has extra terms involving the change of transport coefficients along the flow. In Einstein gravity, all such terms vanish since the viscous transport coefficients in that case are constants.

---

[3] The derivation of the Navier-Stokes equation usually assumes the constancy of various transport coefficients. But it is straightforward to lift that assumption and derive a general Navier-Stokes equation for a fluid with nonconstant viscosity; see, e.g., section 15 in [78]

Let us summarize the broad picture as follows: we have derived the membrane stress tensor for a general $f(R)$ gravity theory and proved that the stress tensor behaves like that of a non-Newtonian viscous fluid provided we imposed the continuity of the scalar curvature across the membrane. This condition can be justified both by imposing the junction condition and also by the observation that in the scalar-tensor picture of $f(R)$ gravity, the scalar curvature plays the role of the scalar field, and the continuity of $R$ is merely a statement of the fact that there is no membrane paradigm for a scalar field.

We can also study the thermodynamics of the fluid membrane. We turn to that next.

### 3.6    Thermodynamics of the Membrane

In order to study the thermodynamics of the membrane, we will assume that the spacetime is stationary with a timelike Killing vector which is null on the horizon. In the semiclassical limit, where the dominant contribution comes from classical field configurations, the partition function is

$$Z \approx \exp\left(-(I_E^{\text{out}} + I_E^{\infty} + I_E^{\text{surf}})\right) = \exp\left(-\beta F\right), \tag{3.32}$$

where $I_E$ is the Euclidean action for the Euclideanized solution. Here $F$ is the free energy and $\beta$ is the periodicity of Euclidean time which is initially a free parameter but will ultimately be set to the inverse of Hawking temperature. The boundary term at infinity, $I_E^{\infty}$, is the appropriate generalization of the Gibbons-Hawking term, and is assumed to include any counter terms necessary to render the expression finite (such as a subtraction of the corresponding action for Minkowski space). Now, in any stationary spacetime, the last term in (3.25) vanishes in the null limit. The variation

of the membrane action therefore reduces to

$$\delta I_\Sigma^{\text{surf}} = -\frac{1}{8\pi G} \int d^{D-1}x \, \sqrt{-h} f'(R) \left( K h^{ab} - K^{ab} \right) \delta h_{ab} \, . \tag{3.33}$$

We will like to integrate this variation to obtain an action for the membrane. If we set the variation of the Ricci scalar to zero on the horizon, we can easily integrate the membrane action [76]. The result is

$$I^{\text{surf}} = -\frac{1}{8\pi G} \int d^{D-1}x \, \sqrt{-h} f'(R) K \, . \tag{3.34}$$

The condition $\delta R = 0$ on the horizon amounts to setting the variation of the scalar degree of freedom to zero in the scalar-tensor picture. In fact, the same condition is used on the external boundary to obtain the bulk equations of motion using a generalization of the Gibbons-Hawking surface term [25]. To evaluate the Euclidean action for the membrane, note that (3.2) gives $K = \alpha^{-1}\kappa$ in a stationary spacetime, while Euclidean time, $t_E$, runs from 0 to $\beta = 2\pi/\kappa$. Hence

$$I_E^{\text{surf}} = \frac{1}{4G} \int f'(R)\sqrt{\gamma} d^{D-2}x \, , \tag{3.35}$$

We can therefore calculate the entropy of the membrane:

$$S^{\text{surf}} = \beta^2 \frac{\partial F}{\partial \beta} = -\frac{1}{4G} \int f'(R)\sqrt{\gamma} d^{D-2}x \, . \tag{3.36}$$

The membrane entropy is *exactly* equal and opposite to the Wald entropy for $f(R)$ gravity [72, 66]. If the entropy of the external universe is the same as the Wald entropy, then the entropy of the total system "Membrane + External Universe" *vanishes*. In the membrane approach, this suggests the following interpretation. For an external observer, there is no black hole — only a membrane. The entropy of the external world is then simply the total entropy of everything outside, which is equal and opposite to the entropy of the membrane. This number decreases as matter leaves

the external system to fall through and be dissipated by the membrane. When all matter has fallen through the membrane, the outside is in a single state — vacuum — and has zero entropy.

To demonstrate this explicitly, consider the standard Schwarzschild spacetime of mass $M$ in four dimensions. This is still a solution of the vacuum $f(R)$ equations of motion. The stretched horizon can be taken to be simply a surface of constant Schwarzschild coordinate $r$. The bulk action vanishes on-shell and we find that

$$I_E^\infty = 4\pi G M^2 \ . \tag{3.37}$$

Hence, the entropy of the external universe is

$$S_{\text{ext}} = 4\pi G M^2 \ . \tag{3.38}$$

Now, for any polynomial $f(R)$ theory of the form $f(R) = R + ...$, we have $f'(R) = 1$ for this solution. In that case the entropy of the membrane is just

$$S_{\text{surf}} = -4\pi G M^2 \ . \tag{3.39}$$

The membrane entropy precisely cancels compensate the external entropy. As befits the generalized entropy, in a spacetime with no matter and no horizons, the total entropy is zero. That the membrane action reproduces (albeit with a properly interpreted minus sign) the black hole entropy is one of the advantages of the action formulation and one of the pleasing aspects of the membrane paradigm; it seems more satisfying that the horizon entropy can be traced to a term in the action that actually lives at the horizon, rather than at infinity.

## Summary

We have extended the membrane paradigm for black hole horizons to general $f(R)$ gravity theories. We have found that the membrane generically behaves like

a non-Newtonian fluid with curvature-dependent transport coefficients; the dynamical equations of the membrane are identical to the corresponding equations in fluid dynamics adapted to a fluid with inhomogeneous and velocity-gradient-dependent viscosity coefficients. We have also calculated the entropy of the membrane: it agrees with the suitable Wald entropy provided we set the variation of the Ricci scalar to zero on the horizon. Our calculations indicate that a membrane paradigm viewpoint may exist for general higher-derivative theories of gravity, but that there are subtleties, largely because there are additional physical degrees of freedom. It would be especially interesting to study the fluid properties of black hole horizons in Lovelock gravity, which has the same number of degrees of freedom as Einstein gravity.

Chapter 4

GENERALIZING THE ENERGY CONDITIONS

4.1    Introduction

By themselves, Einstein's equations impose virtually no restrictions on the kinds of spacetimes that are physically permissible. Any symmetric, suitably differentiable metric that satisfies the boundary conditions is a solution to the Einstein's equations since, formally, one can simply equate the energy-momentum tensor to the left-hand side of the Einstein's equations.

To restrict the solution space, one can impose some physical requirements. In particular, energy conditions are imposed on the types of matter considered. The various energy conditions — null (NEC), weak (WEC), dominant (DEC), and strong (SEC) — each express some seemingly reasonable expectation regarding matter, such as that the speed of energy flow be no greater than the speed of light. The energy conditions are inequalities that apply locally, and are asserted to hold everywhere in spacetime. They are generalizations of the notion that local energy density be non-negative and are implemented by requiring various linear combinations of the components of the matter energy-momentum tensor to be non-negative. The energy conditions are violated by potentially pathological forms of matter, such as certain instances of tachyons (WEC) and ghosts (NEC), and in turn eliminate many of the arbitrary metrics that would otherwise tautologically satisfy the Einstein's equations.

Furthermore, the energy conditions play a critical role in a variety of important theorems in general relativity. They are crucial, for example, to the singularity theorems which indicate that our universe began with a Big Bang singularity; almost

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all theoretical attempts to evade the Big Bang singularity require the violation of at least one or some of the energy conditions at some step. Energy conditions are also invoked in the topological censorship theorem, in positive energy theorems [86, 87], in prohibiting time machines [88], in the black hole no-hair theorem [10, 12, 13] and, in particular, in the area-law increase for black holes [17, 22, 89].

Given their importance, it is disturbing that the energy conditions are not derived from any fundamental principles. Indeed, the status of the energy conditions — why or even whether they hold — remains unclear [90]. For example it is unclear what happens to the energy conditions in higher derivative theories of gravity; should they be modified, or do the physical laws that depend on them (viz. the laws of black hole thermodynamics) require modification? In $f(R)$ gravity [91] and Brans-Dicke gravity [92] no modification is required. But the question is still open for other higher curvature theories of gravity. The null energy condition is routinely violated in higher derivative theories [93–96], in the presence of extra dimensions [97], in ghost condensate models [98], in string cosmology [99, 100] and in other cosmological models which evade the big-bang singularity with a cosmic bounce [101–103]. The presence of the quantum effects and backreaction also tend to introduce ambiguities in the energy conditions [104, 105]. Even if we restrict ourselves to only the lowest-curvature tree-level classical actions, ambiguities also appear when non-minimal couplings are introduced [106]. Indeed, even simple conformal frame transformations seem to violate the energy conditions [107]. The reason for this is easy to understand: the energy conditions are conditions on the stress tensor of matter but Weyl transformations (local re-scalings of the metric) mix matter and gravity, thereby altering the stress tensor non-trivially.

The role of the energy conditions in any theory of gravity is to select only the healthy theories from an infinite space of possible theories. However any deviations

(a) Null energy condition maps to a different set under conformal transformation

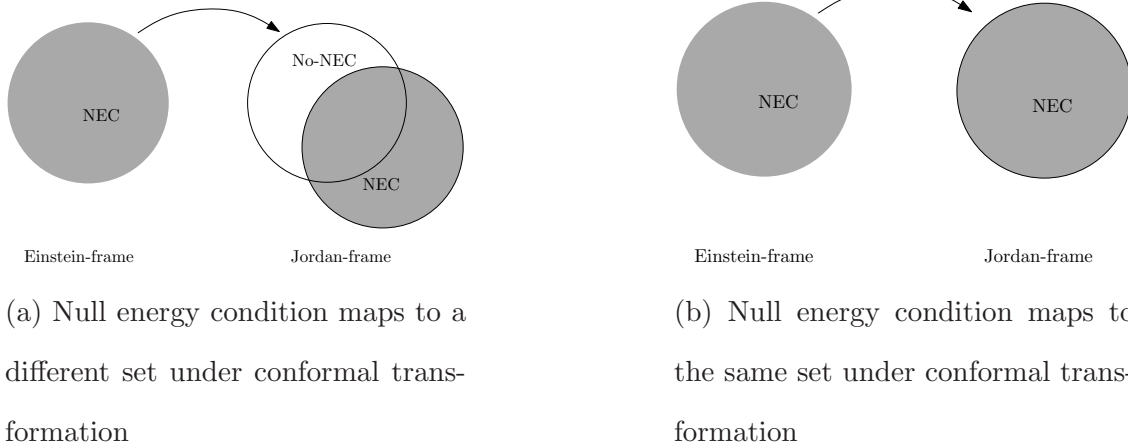(b) Null energy condition maps to the same set under conformal transformation

Figure 4.1: Null Energy Condition Before and After Conformal Transformation

from the minimally coupled Einstein gravity causes them to fail. Then there are higher derivative theories [93] which explicitly violate the energy conditions but still possess well behaved solutions. Therefore it is apparent that the energy conditions are not fulfilling their roles as good predictors of well behaved theories. Either the energy conditions are inadequate for the task or they require some form of modifications to remain a faithful indication of a well behaved theory. Discarding the energy conditions is not an option, therefore we will explore the latter option in considerable detail. To further illustrate this point let us look at Fig. 4.1. The figure on the left depicts what happens to the set of the null energy condition abiding theories under conformal transformations. They are mapped to another set, a part of which contains theories with ghosts. The theories did not change fundamentally under the conformal transformation and hence the illusion of becoming ghostly is undesirable. Our aim is to make sure that the null energy condition is also modified in such a way that the set of healthy theories maps to the set of healthy theories in the conformal frame.

To get this effect we will propose a new ad hoc form of the null energy condition, in Section 4.2.1, that is valid in all rescaled Weyl frames. Our new null energy

condition reduces to the usual form of the null energy condition in the Einstein frame and is consistent with the invariance of the second law of thermodynamics for black holes under conformal transformations. We use our modified null energy condition to supply a direct proof of the second law in the Jordan frame. The result motivates the following conjecture: the energy conditions, unlike the second law, does not appear to be fundamental. Therefore the second law should be taken as given and the energy conditions should be derived from them. Then the correct modification of the null energy condition in a given theory of gravity is that condition on matter that ensures that a classical black hole solution of the theory has an entropy that grows with time.

## 4.2   Energy Conditions in Jordan Frame

Consider a minimally coupled, Einstein-Hilbert scalar field action with canonically normalized scalar field $\phi$:

$$I = \int d^D x \sqrt{-g} \left( \frac{M_P^{D-2}}{2} R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right) \qquad (4.1)$$

Here $g_{ab}$ is the Einstein-frame metric that couples minimally to the scalar field. The Einstein-frame energy-momentum tensor of the scalar field, $T_{ab}^E$, manifestly satisfies the null energy condition:

$$T_{ab}^E k^a k^b = (k \cdot \partial\phi)^2 \geq 0 \qquad (4.2)$$

for any null vector $k^a$. Let us consider the modification of the Lagrangian due to a Weyl transformation, $\tilde{g}_{ab} = \Omega^2(x) g_{ab}$. The existence of a well-defined inverse of the metric requires a nowhere-vanishing conformal factor. In addition, we need $\Omega(x)$ to be related to the previously existing fields in Einstein frame; otherwise, we will have introduced a new degree of freedom. We therefore take $\Omega(x) = e^{\zeta\phi(x)/\mu}$ with $\mu = M_P^{\frac{D-2}{2}}$ where $\zeta$ is a dimensionless constant, and $\mu$ is a dimensionful constant with the same dimensions as $\phi$. We have chosen the conformal factor to be linear in

$\phi$ for simplicity; our results can trivially be extended to a general conformal factor of the form $\exp(f(\phi(x)/\mu))$.

It is important to emphasize here that we have merely redefined field variables from $(g_{ab}(x), \phi(x))$ to $(\tilde{g}_{ab}(x), \phi(x))$. Although the action is typically not invariant under such transformations (unless the metric transformation was induced by a diffeomorphism), the physics is exactly the same [108] provided only that the number of degrees of freedom is unchanged, which is the case here [1]. Field redefinitions between the matter and gravitational sectors are used routinely in both string theory and cosmology. In particular, if the energy conditions are to have physical significance, they too must continue to hold after field redefinition. Let us check whether this is the case.

Using the above choice of conformal factor the action reduces to

$$\int d^D x \sqrt{-\tilde{g}} \; e^{-\zeta(D-2)\phi/\mu} \left( \frac{M_P^{D-2}}{2} \tilde{R} - \frac{1}{2} \left( 1 - \zeta^2(D-2)(D-3) \right) (\partial_a \phi)^2 - e^{-2\zeta\phi/\mu} V(\phi) \right) \tag{4.3}$$

We will refer to this action as the Jordan-frame action because gravity is non-minimally coupled, via the term in which the Ricci scalar is directly coupled to matter. The various pieces of the Lagrangian no longer split cleanly into a gravity Lagrangian and a matter Lagrangian, and hence it is no longer sensible to define the Jordan-frame stress tensor, $T_{ab}^J$, naively, as the variation of the matter action:

$$T_{ab}^J \neq -\frac{2}{\sqrt{-\tilde{g}}} \frac{\delta I_{\text{matter}}}{\delta \tilde{g}^{ab}} \tag{4.4}$$

However, The equation of motion for $\tilde{g}_{ab}$ can be rewritten so as to get the Einstein tensor on the left-hand side. Then whatever is on the right-hand side is covariantly conserved as a result of the Bianchi identity. Thus we can simply define the

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[1] Indeed, this is true not only classically, but also quantum-mechanically. In the path integral, the fields are just integration variables. We can redefine these, just as we are free to redefine integration variables in ordinary calculus.

Jordan-frame stress tensor as the quantity proportional to the Einstein tensor in the gravitational equations:

$$\tilde{G}_{ab} \equiv \frac{1}{M_P^{D-2}} T_{ab}^J \tag{4.5}$$

Explicitly, the stress tensor in Jordan frame is

$$T_{ab}^J = (1 + \zeta^2(D-2))(\partial_a \phi)(\partial_b \phi) - \tilde{g}_{ab} \left( \frac{1}{2} \left(1 + \zeta^2(D-1)(D-2)\right)(\partial_a \phi)^2 \right.$$
$$\left. + e^{-2\zeta\phi/\mu} V(\phi) \right) + \zeta(D-2) M_P^{\frac{D-2}{2}} \left( \tilde{g}_{ab} \tilde{\nabla}^2 - \tilde{\nabla}_b \tilde{\nabla}_a \right) \phi \tag{4.6}$$

which is covariantly conserved:

$$\tilde{\nabla}^a T_{ab}^J = 0 \tag{4.7}$$

### 4.2.1 Null Energy Condition on the Stress Tensor in Jordan Frame

The covariantly-conserved stress tensor, eqn. (4.6), can still violate the null energy condition in its Einstein-frame form [107]. To see this, contract $T_{ab}^J$ with null vectors:

$$T_{ab}^J k^a k^b = \left(1 + \zeta^2(D-2)\right)(k^a \partial_a \phi)^2 - \zeta(D-2) M_P^{\frac{D-2}{2}} k^a k^b \tilde{\nabla}_a \tilde{\nabla}_b \phi \tag{4.8}$$

Notice that the positivity of the last term is ambiguous. If this were the correct form of the NEC, the set of physical solutions $(g_{ab}, \phi)$ in Einstein frame — namely, those that obey the Einstein frame null energy condition — will not map to the set of physical solutions $(\tilde{g}_{ab}, \phi)$ in Jordan frame — those that satisfy $T_{ab}^J k^a k^b \geq 0$. But the set of physical solutions should not change under field redefinitions. It must be, then, that this naive version of the null energy condition is incorrect in Jordan frame.

We propose that the correct null energy condition in Jordan frame is

$$\left[ T_{ab}^J + \zeta(D-2) M_P^{\frac{D-2}{2}} \tilde{\nabla}_a \tilde{\nabla}_b \phi \right] k^a k^b \geq 0 \tag{4.9}$$

Once this condition is imposed on the matter fields in Jordan frame the ambiguity is resolved and we have (solution set in Jordan frame) $\equiv$ (solution set in Einstein frame).

To see this explicitly, consider our energy-momentum tensor. Using the expression eqn. (4.6) in the relation eqn. (4.9), we find

$$\left(T^J_{ab} + \zeta(D-2)M_P^{\frac{D-2}{2}}\tilde{\nabla}_a\tilde{\nabla}_b\phi\right)k^a k^b = \left(1 + \zeta^2(D-2)\right)T^E_{ab}k^a k^b \qquad (4.10)$$

Hence, (modified null energy condition obeyed in Jordan frame) $\Leftrightarrow$ (usual null energy condition obeyed in Einstein frame). Our expression for the Jordan-frame null energy condition can be generalized to arbitrary non-minimal couplings of matter sources to the Ricci scalar. The generalized version is

$$\left[T^J_{ab} + (D-2)M_P^{D-2}\tilde{\nabla}_a\tilde{\nabla}_b\ln\Omega\right]k^a k^b \geq 0 \qquad (4.11)$$

This inequality reduces to the usual Einstein frame NEC, eqn. (4.2), in the case of minimal coupling ($\Omega = 1$) and to the modified NEC, eqn. (4.9), when $\Omega = \exp(\zeta\phi(x)/\mu)$.

To summarize, we propose a new form of the null energy condition, eqn. (4.11), that applies to non-minimally coupled scalar fields and whose solution set is the same as that of the usual Einstein frame null energy condition. This is as it should be, since the two frames are related by a field redefinition.

## 4.3   Entropy Increase in Jordan Frame

Our proposal for the modified null energy condition in Jordan frame was somewhat ad hoc. It happened to work for non-minimally coupled scalar fields in conformally transformed actions. We will now motivate the prescription by a robust physical principle. If we regard black hole entropy as counting the number of degrees of freedom via the dimensions of its Hilbert space, or alternatively, the number of possible initial configurations from which the black hole could be formed, then the entropy is clearly conformally invariant. If the entropy increases in one frame as a result of imposition of the null energy condition then it should also increase in the conformal frame without

any extra requirement, viz. the modified null energy condition of Jordan frame. In this section, we will prove that our modified null energy condition in Jordan frame indeed guarantees that, classically, black hole entropy never decreases.

Consider a stationary black hole solution to the Einstein equation in $D$ spacetime dimensions. The entropy of black hole solutions of the Einstein-Hilbert action is just proportional to its "area" [17], by which we mean a (D-2)–dimensional spacelike section of the horizon. But in dealing with spacetimes which are solutions to non-Einstein-Hilbert actions, such as the one in Jordan frame with non-minimal scalar coupling, a notion of entropy is absent. The Wald prescription of entropy [70] identifies a conserved charge with the entropy of the horizon for stationary solutions to these non-Einsteinian theories of gravity. The correct entropy to use for dynamical horizons is Jacobson-Myers entropy [71]. But since the metric has rescaled, we can obtain the entropy just by applying a field redefinition to the Bekentein-Hawking entropy [67]. The black hole entropy in Jordan frame is not simply proportional to the area. Rather it is

$$S = \frac{1}{4G_D} \int d^{D-2}x \sqrt{\tilde{\gamma}} \, \Omega^{-(D-2)} \tag{4.12}$$

The increase of entropy in Jordan frame has been studied before from the perspective of $f(R)$ theories [91] and the second law was proved in Jordan frame in [109] using the Einstein-frame NEC. We provide a proof directly in Jordan frame using our modified Jordan-frame NEC.

To see how the rate of change of black entropy in Jordan frame depends on the Jordan-frame null energy condition we first find the expression for the change in the black hole entropy.

$$\begin{aligned}
\frac{dS}{d\tilde{\lambda}} &= \frac{1}{4G_D} \int d^{D-2}x \sqrt{\tilde{\gamma}} \, \Omega^{-(D-2)} \left( \tilde{\theta} - (D-2)\frac{d\ln\Omega}{d\tilde{\lambda}} \right) \\
&\equiv \frac{1}{4G_D} \int d^{D-2}x \sqrt{\tilde{\gamma}} \, \Omega^{-(D-2)} \Theta
\end{aligned} \tag{4.13}$$

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where we have defined

$$\Theta(\tilde{\lambda}) = \tilde{\theta} - (D-2)\frac{d\ln\Omega}{d\tilde{\lambda}} \tag{4.14}$$

Here $\tilde{\theta}$ is the expansion scalar for the null generator $\tilde{k}^a$ of the horizon in Jordan frame:

$$\tilde{\theta} = \tilde{\nabla}_a\tilde{k}^a = \frac{d(\ln\sqrt{\tilde{\gamma}})}{d\tilde{\lambda}} \tag{4.15}$$

Black hole entropy would not decrease if $\Theta$ were non-negative, as evident from eqn. (4.13). We will now give a direct proof of the second law of thermodynamics for scalar-tensor theories in Jordan frame. As in the proof that the surface area of black holes in Einstein gravity always increases [22], we will show that, if $\Theta < 0$, then a caustic necessarily forms.

Notice that the vector $\tilde{k}^a = (d/d\tilde{\lambda})^a$ is related to $k^a = (d/d\lambda)^a$. This is easiest to see for a normalized timelike velocity vector, $u^a = (d/d\tau)^a$. Since $d\tau^2 = -g_{ab}dx^a dx^b$, rescaling the metric causes $\tau$ to scale: $\tilde{\tau} = \Omega\tau$. Then $\frac{d\tilde{\tau}}{d\tau} = \Omega$ and hence $\tilde{u}^a \equiv (d/d\tilde{\tau})^a = \frac{d\tau}{d\tilde{\tau}}(d/d\tau)^a = (1/\Omega)u^a$. Similarly, we have

$$\tilde{k}^a \equiv \left(\frac{d}{d\tilde{\lambda}}\right)^a = \frac{d\lambda}{d\tilde{\lambda}}\left(\frac{d}{d\lambda}\right)^a = \frac{1}{\Omega}k^a \tag{4.16}$$

Since $\Omega(x)$ is a function of space and time, $\tilde{k}^a$ is not in general affinely parameterized. Thus

$$\tilde{k}^b\nabla_b\tilde{k}^a = \kappa\tilde{k}^a \tag{4.17}$$

where

$$\kappa = \frac{d}{d\tilde{\lambda}}\ln\Omega \tag{4.18}$$

The corresponding Raychaudhuri equation for $\tilde{\theta}$ is

$$\frac{d\tilde{\theta}}{d\tilde{\lambda}} = \kappa\tilde{\theta} - \left(\frac{\tilde{\theta}^2}{D-2} + \tilde{\sigma}_{ab}\tilde{\sigma}^{ab} + \tilde{\omega}_{ab}\tilde{\omega}^{ab} + \tilde{R}_{ab}\tilde{k}^a\tilde{k}^b\right) \tag{4.19}$$

The presence of the $\kappa\tilde{\theta}$ term on the right is a sign that $\tilde{\lambda}$ is not an affine parameter. Hypersurface orthogonality of the null generators $\tilde{k}^a$ of the event horizon causes the

rotation ($\tilde{w} = 0$) to vanish by the Frobenius theorem. We can now use the equation of motion in Jordan frame and replace the Ricci tensor with the Jordan-frame stress tensor:

$$\tilde{R}_{ab}\tilde{k}^a\tilde{k}^b = \frac{1}{M_P^{D-2}}T_{ab}^J\tilde{k}^a\tilde{k}^b \tag{4.20}$$

Taking the derivative of eqn. (4.14) and substituting eqn. (4.19) then gives

$$
\begin{aligned}
\frac{d\Theta}{d\tilde{\lambda}} &= \frac{d\tilde{\theta}}{d\tilde{\lambda}} - (D-2)\frac{d^2}{d\tilde{\lambda}^2}\ln\Omega \\
&= \kappa\tilde{\theta} - \left(\frac{\tilde{\theta}^2}{D-2} + \tilde{\sigma}_{ab}^2 + \tilde{R}_{ab}\tilde{k}^a\tilde{k}^b\right) - (D-2)\frac{d^2}{d\tilde{\lambda}^2}\ln\Omega \\
&= -\frac{\Theta^2}{D-2} - \kappa\Theta - (D-2)\kappa^2 - \tilde{\sigma}_{ab}^2 - \frac{\tilde{k}^a\tilde{k}^b}{M_P^{D-2}}\left(T_{ab}^J + (D-2)M_P^{D-2}\tilde{\nabla}_a\tilde{\nabla}_b\ln\Omega\right)
\end{aligned}
\tag{4.21}
$$

The last line follows from the relation between $\kappa$ and $\Omega$ above eqn. (4.18). Note the presence of the terms in brackets in eqn. (4.21): this is proportional to precisely the expression that appears in our modified null energy condition in Jordan frame, eqn. (4.11). When that is obeyed we have

$$\frac{d\Theta}{d\tilde{\lambda}} \leq -\frac{\Theta^2}{D-2} - \kappa\Theta - (D-2)\kappa^2 \tag{4.22}$$

We will now analyze this equation carefully in order to prove that only solutions with $\Theta > 0$ do not have caustics. Cosmic censorship — which in this context means the prohibition of caustics — eliminates all solutions with $\Theta < 0$ and hence, by eqn. (4.13), the second law of black hole thermodynamics holds in Jordan frame.

Suppose, then, that at some parameter $\tilde{\lambda}_0$, a pencil of horizon null generators has $\Theta_0 < 0$. For a sufficiently thin pencil, the surface gravity is effectively constant over spacelike sections of the pencil. Therefore, we can regard $\kappa$ as a function of $\tilde{\lambda}$ only. First, consider $\kappa \leq 0$. But then every term on the right-hand side of eqn. (4.22) is nonpositive for $\Theta < 0$. Hence $\frac{d\Theta}{d\tilde{\lambda}} \leq 0$. In fact, $\frac{d\Theta}{d\tilde{\lambda}} \leq -\frac{\Theta^2}{D-2}$ whose solution is

$\Theta(\tilde{\lambda}) \leq \Theta_0 / (1 + \frac{\tilde{\lambda} - \tilde{\lambda}_0}{D-2} \Theta_0)$. For all negative values of $\Theta_0$, $\Theta(\tilde{\lambda})$ diverges at some finite $\tilde{\lambda}$, resulting in a caustic. Hence, for $\kappa \leq 0$, all solutions with $\Theta < 0$ lead to caustics.

Next, consider $\kappa > 0$. In this case, the term $-\kappa\Theta$ in eqn. (4.22) is positive for $\Theta < 0$. However, the three terms on the right-hand side together are always negative. To see this, consider the right-hand side of eqn. (4.22) as a quadratic polynomial in $\Theta$; this quadratic has no real roots for $\kappa > 0$. Hence again $\frac{d\Theta}{d\tilde{\lambda}} \leq 0$; $\Theta$ is a monotonically decreasing function of $\tilde{\lambda}$. However, to prove that this inevitably results in a caustic is more subtle because the positivity of the $-\kappa\Theta$ term does not permit us to write $\frac{d\Theta}{d\tilde{\lambda}} \leq -\frac{\Theta^2}{D-2}$.

A monotonically decreasing negative function $\Theta(\tilde{\lambda})$ can have three different asymptotic possibilities. Possibility 1 is that $\Theta$ asymptotically and monotonically approaches some finite negative value, $\Theta_{\min}$ i.e. $\lim_{\tilde{\lambda} \to \infty} \Theta(\tilde{\lambda}) = \Theta_{\min}$. Possibility 2 is that $\Theta$ is unbounded from below but reaches negative infinity only in the infinite future i.e. $\lim_{\tilde{\lambda} \to \infty} \Theta(\tilde{\lambda}) = -\infty$. This is not a caustic because $\Theta$ is finite at all finite values of $\tilde{\lambda}$. Possibility 3 is that $\Theta$ diverges at some finite $\tilde{\lambda}_c$: $\lim_{\tilde{\lambda} \to \tilde{\lambda}_c} \Theta(\tilde{\lambda}) = -\infty$. This corresponds to a caustic. These three possibilities are illustrated schematically by the curves in Fig. 4.2. We will now show that $\kappa > 0$ and $\Theta < 0$ always gives rise to possibility 3.

First we rule out possibility 1; $\Theta(\tilde{\lambda})$ does not asymptotically approach a finite value. Suppose that were true, then for large values of $\tilde{\lambda}$, we would have $\Theta \approx \Theta_{\min}$ and $\frac{d\Theta}{d\tilde{\lambda}} \approx 0$. Regarding the right-hand side of eqn. (4.22) as a quadratic in $\kappa$, we see that there are no real solutions for $\kappa$ when $\Theta = \Theta_{\min}$ and $\frac{d\Theta}{d\tilde{\lambda}} = 0$. Hence, possibility 1 is eliminated and $\Theta(\tilde{\lambda})$ is therefore unbounded from below.

Since $\Theta$ is unbounded from below, consider a very large (negative) value of $\Theta$. Focus on an infinitesimal interval of $\tilde{\lambda}$. In that interval, $\kappa(\tilde{\lambda})$ can be regarded effectively
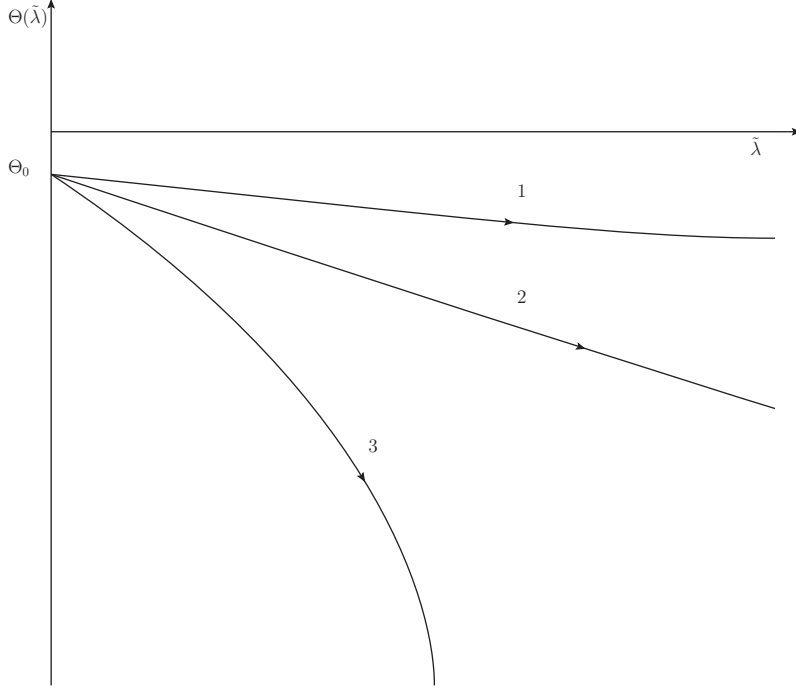
Figure 4.2: Possible Curves for a Negative Monotonically Decreasing Function $\Theta(\tilde{\lambda})$.

as a constant. Then we can integrate eqn. (4.22) to obtain

$$\Theta(\tilde{\lambda}) = \frac{\frac{\sqrt{3}}{2}\Theta_0 - \left(\Theta_0 + \frac{7}{4}(D-2)\kappa\right)\tan\left(\frac{\sqrt{3}}{2}\kappa(\tilde{\lambda} - \tilde{\lambda}_0)\right)}{\frac{\sqrt{3}}{2} + \left(\frac{\Theta_0}{\kappa(D-2)} + 1\right)\tan\left(\frac{\sqrt{3}}{2}\kappa(\tilde{\lambda} - \tilde{\lambda}_0)\right)} \tag{4.23}$$

Scrutiny of this reveals that the denominator vanishes for certain values of $\tilde{\lambda}$:

$$\tilde{\lambda} - \tilde{\lambda}_0 \approx -\frac{D-2}{\Theta_0} \tag{4.24}$$

Hence if $\Theta_0$ is large and negative, $\Theta$ becomes divergently negative in finite time: a caustic.

We have proven that, whether $\kappa$ is positive or negative, we always find a caustic in finite parameter $\tilde{\lambda}$ whenever $\Theta_0 < 0$. Cosmic censorship bans these solutions leaving only those with $\Theta \geq 0$. This in turn implies that black hole entropy must be non-decreasing in Jordan frame. Our proof relied crucially on eqn. (4.22), which follows from eqn. (4.21) only when our modification to the Jordan-frame null energy condition

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is satisfied. A quick check on our result comes from the observation that $\Theta = \theta_E/\Omega$, where $\theta_E$ is the expansion in Einstein frame. Then, when the usual Einstein-frame null energy condition is satisfied, $\theta_E$ is positive. This in turn means that $\Theta$ must be positive and that the Jordan-frame entropy also increases. Here we have proven that fact directly in Jordan frame without relying on a correspondence with Einstein frame.

## 4.4 Ambiguity in the Canonical Null Energy Condition in the presence of Multiple Fields

In the presence of multiple fields it becomes ambiguous whether to impose the null energy condition on each individual field or the entire field content as a whole. Even the physical principles will fail to provide any guidance since the physical laws (like the second law of thermodynamics) are insensitive to the microscopic details. This can be best illustrated with an example. We will show how the presence of ghosts can introduce ambiguities in the null energy condition in the Einstein frame and how our improved null energy condition salvages the situation.

### 4.4.1  A Ghost with a Scalar Field

Let us take a gravitational system minimally coupled to a free canonical scalar field and a free ghost field

$$\int d^D x \sqrt{-g} \left( R - \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\psi)^2 \right) \tag{4.25}$$

Here $\phi(x)$ is a regular canonical scalar field and $\psi(x)$ is a ghost field. Einstein frame null energy condition is

$$T^\phi_{ab}k^a k^b + T^\psi_{ab}k^a k^b = (k \cdot \partial\phi)^2 - (k \cdot \partial\psi)^2 \tag{4.26}$$

Depending on how strong the gradient of the ghost field is along the null vectors $k^a$ compared to that of the canonical scalar field, this quantity is sometimes positive and sometimes it is not. It is normal to assume that the null energy condition in this case should be applied to the individual pieces of the stress tensor, however this is too strong an assumption classically. From the point of view of quantum systems it makes perfect sense because the ghostlike degrees of freedom are undesirable and should be discarded. From classical point of view there is nothing wrong with the ghost fields in the presence of normal matter as long as they do not cause the total stress tensor to become unbounded from below.

Therefore, we propose a further conjecture that the matter fields should be integrated over all Jordan frames. This means that if any of the matter sources violate the null energy condition in any Jordan frame then the theory should be discarded altogether. To demonstrate our conjecture let us transform the above Einstein frame action to Jordan frame using the canonical scalar field $\phi(x)$ just like previous sections but this time keeping the coefficient of the $\phi(x)$ in the exponent arbitrary, i.e. $\Omega(x) = e^{\frac{n}{M_P}\phi(x)}$ [2] .

$$\int d^D x \sqrt{-\tilde{g}} e^{-\frac{n(D-2)}{M_P}\phi} \left( \frac{M_P^2}{2} \tilde{R} - \frac{1}{2} \left(1 - (D-2)(D-3)n^2\right) (\partial_a \phi)^2 + e^{-2n\phi/M_P} \frac{1}{2}(\partial_a \psi)^2 \right)$$

(4.27)

In the last equation we have assumed that the ghost field $\psi(x)$ does not transform under the conformal transformation. The stress tensor is a function of both the scalar

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[2]We can also transform using the ghost field. However, the final result will not be affected by our choice of the scalar field.

field and the ghost field.

$$T_{ab}^J = \left(1 + n^2(D-2)\right)\partial_a\phi\partial_b\phi - e^{-2n\phi/M_P}\partial_a\tilde{\psi}\partial_b\tilde{\psi}$$
$$- \tilde{g}_{ab}\left(\frac{1}{2}\left(1 + n^2(D-1)(D-2)\right)(\partial_a\phi)^2 - \frac{1}{2}e^{-2n\phi/M_P}(\partial_a\psi)^2\right)$$
$$+ n(D-2)M_P\left(\tilde{g}_{ab}\tilde{\nabla}^2 - \tilde{\nabla}_b\tilde{\nabla}_a\right)\phi \tag{4.28}$$

This stress tensor can be used in conjunction with the general form of the new improved null energy condition for Jordan frame (eqn. (4.11)).

$$T_{ab}^J k^a k^b + M_P^2\ k^a k^b\tilde{\nabla}_b\tilde{\nabla}_a\ ln\left(\frac{\partial L}{\partial R}\right) \geq 0$$

Instead of having the extra piece in terms of the scalar field $\phi(x)$ (as in eqn. (4.9)), we will write the expression in terms of the conformal factor $\Omega(x)$ so that the numerical pre-factor $n$ can be kept of arbitrary sign. The left hand side of the eqn. (4.11) in this case becomes,

$$T_{ab}k^a k^b + M_P^2\tilde{\nabla}_b\tilde{\nabla}_a ln\Omega = \left(1 + n^2(D-2)\right)(k\cdot\partial\phi)^2 - e^{-2n\phi/M_P}(k\cdot\partial\psi)^2 \tag{4.29}$$

It is apparent from this expression that in some Jordan frames the kinetic term of the ghost field will be enhanced and in some it will suppressed in comparison to the canonical scalar field. If in the Einstein frame it was suppressed then classically those theories are allowed because they are not forbidden by gravitational dynamics. However, if the same theories wreak havoc to thermodynamical laws in other conformal frames then they should be discarded. This principle of universal validity of theories, irrespective of frames considered, allows rejection of quantum mechanically sick theories in a classical manner. The prescription also removes any ambiguity in the case of multiple fields. We can conclude from this exercise that actions containing ghosts should not be considered even if they appear to naively satisfy the energy conditions in the Einstein frame.

**Example of Scalar and Ghost Field System in FRW Background**

We have conclusively proven that even the modified Jordan-frame null energy condition will be violated when matter fields contains ghost degrees of freedom. To complete the proof, we consider a concrete example of the action (eqn. (4.25)) in a FRW background. Assuming that the scalar and the ghost only have time dependence, this is an exactly solvable system with analytical solutions. To look for solutions we need to solve the equations of motion; which is the Friedman equations and the Klein-Gordon equations in curved spacetime (using $8\pi G = \frac{M_P^2}{2}$),

$$H^2 = \frac{2}{3M_P^2}\left(\rho + p\right) = \frac{2}{3M_P^2}\left(\dot{\phi}^2 - \dot{\psi}^2\right) \tag{4.30}$$

$$\ddot{\phi} + 3H\dot{\phi} = 0 \tag{4.31}$$

$$\ddot{\psi} + 3H\dot{\psi} = 0 \tag{4.32}$$

In eqn. (4.30), the pressure and the density are same since the potential is zero. This gives a stiff equation of state for matter $p = \omega\rho$ with $\omega = 1$. Equations 4.31 and 4.32 have the following solutions,

$$\dot{\phi} = \frac{A}{a^3}; \quad \dot{\psi} = \frac{B}{a^3} \tag{4.33}$$

Now, since the time dependence of the scale factor is $a(t) = t^{\frac{2}{3(1+\omega)}} \approx t^{1/3}$, the above equations can be solved for each field.

$$\phi = A\ ln(t) + C; \quad \psi = B\ ln(t) + C \tag{4.34}$$

The divergence of the scalar fields near the Big Bang arises due to the absence of a potential term. Normally the initial conditions are devised in a way that the fields go to zero at $t = 0$ and the constant is just the counter-term there. In this case the function is divergent at $t = 0$, choosing the constant to cancel this divergence is not the correct choice. It will make all future values of the solution become divergent.

The best thing to do, in this case, is to let the fields diverge near the Big Bang just the right amount so that $C = 0$

$$\phi = a \; M_P \; ln(t); \quad \psi = b \; M_P \; ln(t) \tag{4.35}$$

Here we have chosen a normalization of the fields which is the most obvious choice on dimensional grounds and $a$ and $b$ are just arbitrary constants whose values can only be decided from experiments performed. However, due to the Friedman equations eqn. (4.30), they must satisfy

$$a^2 - b^2 \geq 0 \tag{4.36}$$

Otherwise there is no gravitational dynamics. This inequality is nothing but the null energy condition of the Einstein frame [3]. So we see that the equation of motion implies the Einstein frame NEC. The Jordan-frame equations of motion have the same form in the Einstein frame but the null energy condition of Jordan frame is different. The Jordan-frame null energy condition can be potentially violated with an appropriately chosen conformal transformation. Putting the values of $\dot\phi$, $\dot\psi$ and $\phi$ in the modified null energy condition of Jordan frame (eqn. (4.29)) we have

$$\left(1 + n^2(D-2)\right)(k \cdot \partial\phi)^2 - e^{-2n\phi/M_P}(k \cdot \partial\psi)^2$$
$$\approx \frac{M_P^2}{a^6} \left( \left(1 + n^2(D-2)\right) a^2 - t^{-an}b^2 \right) \tag{4.37}$$

Since $t$ is always positive choosing $n = -\frac{2}{a}$ violates this null energy condition for all times later than $t = \left(1 + 4\frac{D-2}{a^2}\right)$. This demonstrates that even though the Einstein frame null energy condition may be satisfied, the modified Jordan-frame null energy condition will not fail to detect ghostly instabilities in the theory.

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[3] As the fields are only time dependent: $k \cdot \phi = \dot\phi = \frac{A}{a^3}$ and $k \cdot \psi = \dot\phi = \frac{B}{a^3}$; the null energy condition (eqn. (4.26)) produces this same equation.

*Discussion*

By using a field redefinition, we have seen that the form of the null energy condition is modified in Jordan frame. Similar rewritings lead to modified versions of other energy conditions, whose forms are not particularly illuminating. However, field redefinitions from the Einstein frame do not exist for generic theories, such as most higher-derivative theories. The question naturally arises as to what the appropriate generalization of the null energy condition is for such theories. A possible clue is to be found in eqn. (4.21). Given only that equation plus the requirement that the second law hold, one could in fact have inferred the modified null energy condition directly invoking neither a field redefinition nor even the original Einstein-frame null energy condition. This is because all terms on the right-hand side of eqn. (4.21) need to be negative in order to guarantee the validity of the second law. It is a very interesting observation since, rather than using the modified null energy condition to prove the second law, we will in this approach take the second law as a given and derive the appropriate condition for matter. The null energy condition – unlike the second law – does not seem to rest on any fundamental principles of physics. Therefore, in the same spirit as Jacobson's Einstein equation of state paper [110] (in which thermodynamic laws are taken as axioms rather than as statements to be proved), one should perhaps begin, not end, with the second law. This approach would generalize to other gravitational theories, provided one had the correct formulation of entropy, valid in non-stationary situations.

Chapter 5

THE SECOND LAW IN 4D EINSTEIN-GAUSS-BONNET GRAVITY

Bekenstein-Hawking entropy [17], which is proportional to the surface area of a black hole, always increases in time for classical processes [15, 22]. This is true even when the black hole is subject to large changes, as during black hole mergers [111]. However, the Bekenstein-Hawking entropy is the correct entropy only if the gravitational sector of the underlying theory is described by the Einstein-Hilbert action; when the action contains higher-order Riemann curvature terms a different expression for entropy is necessary. For example, Wald entropy [72] is constructed in order to explicitly satisfy the first law of thermodynamics for black holes in higher-curvature gravity. It remains an open question whether the entropy formulas for event horizons in these more general gravitational theories also obeys the second law of thermodynamics. Indeed, it has been argued in [112] and [113] (see also [71]) that the presence of a Gauss-Bonnet term in the four-dimensional gravitational action should — on general grounds that are reviewed below — lead to second law violations during black hole mergers. In the following sections, we examine this claim carefully and argue that no violations of the second law can occur in the regime where both Einstein-Gauss-Bonnet holds as an effective theory and black hole thermodynamics is valid. Our approach differs from Hawking's proof of the area theorem in three ways. First, we are including a Gauss-Bonnet term. Second, for black hole mergers, we are dealing with the micro-canonical ensemble (fixed total energy) whereas the area theorem applies to the canonical ensemble of fixed temperature. Third, a black hole merger is a topology-changing process rather than a small perturbation. Thus our demonstration of the validity of the second law is an addition to the pre-existing

proofs and does not automatically follow from them.

Consider the Einstein-Hilbert action with a Gauss-Bonnet term (disregarding surface terms [114]):

$$I = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(-2\Lambda + R + \alpha(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd})\right) \qquad (5.1)$$

Here $\alpha$ is a constant with dimensions of (length)$^2$. This combination of curvature-squared terms is non-dynamical in four dimensions. One quick way to see this is to Wick-rotate the action. Then the Euclideanized Gauss-Bonnet term integrated over a compact 4-surface is simply proportional to the Euler character of that surface:

$$\chi_4 = \frac{1}{32\pi^2} \int d^4x \sqrt{(g)_E} \left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right)_E \qquad (5.2)$$

Because it is topological, the Gauss-Bonnet term in four dimensions does not affect Einstein's equations, and therefore (when Wick-rotated back), this action has the same class of black hole solutions as pure Einstein gravity. Nevertheless, even though it does not contribute to the equations of motion, the extra Gauss-Bonnet contribution does have ramifications for semi-classical gravity (see, e.g. [112, 71, 115, 116]), because it alters the definition of entropy. In Lovelock gravity, the entropy is not given by the area but by the Jacobson-Myers entropy formula [71]; modification of the entropy formula is necessary for the validity of the first law. Jacobson-Myers entropy is suitable for dynamical black hole horizons similar to the boost-invariant form of the Wald entropy [70]. Since we are attempting to study black hole mergers, this entropy is preferable in form to the Wald entropy [72] which assumes the stationarity of horizons. The Jacobson-Myers entropy associated with arbitrary black holes of 4D Einstein-Gauss-Bonnet theory [71] is

$$S = \frac{1}{4G} \int dA \left(1 + \alpha R_{(2)}\right) \qquad (5.3)$$

Here $R_{(2)}$ is the Ricci scalar of a two–dimensional spacelike section of the event hori-zon. Integrating eqn. (5.3) is straightforward because the last part of the integral is simply the Euler character of the surface of the black hole:

$$\chi = \frac{1}{4\pi} \int dA \, R_{(2)} \tag{5.4}$$

(Since this integral is over a spacelike hypersurface it directly gives the Euler number without any Euclideanization; also, $\chi$ is the two-dimensional Euler character, which should not be confused with $\chi_4$ in eqn. (5.2).) The entropy is now the sum of two terms, the usual Bekenstein-Hawking term plus an additional term proportion to the Euler number:

$$S = \frac{A}{4G} + \pi \frac{\alpha}{G} \chi \tag{5.5}$$

At first glance, this formula opens the door to second law violations when $\alpha > 0$. To see this, consider the merger of two black holes with spherical topology ($\chi = 2$). For each black hole the Gauss-Bonnet term contributes $\frac{2\pi\alpha}{G}$ to the entropy, but after the merger only one hole exists. Thus, it might be that the increase in area entropy could be outweighed by the decrease in the topological contribution to the entropy.

However, we need to keep in mind two regimes of validity. Since gravity is not a renormalizable theory, the effective action consists of an infinite number of terms of ever-higher order in powers of the Riemann tensor. For example, the Einstein-Gauss-Bonnet action appears as only the leading terms in the low-energy effective action of heterotic string theory [26, 117]. In order to be able to neglect higher-order terms, a necessary condition is that $O(|\alpha|R) < 1$, where $R^2$ denotes some quadratic curvature scalar. Just on dimensional grounds, we see that the largest value the curvature scalar can have is of the order $R \sim 1/\ell_P^2 \sim M_P^2 \sim 1/G$, which implies that $|\alpha|/G < 1$ [1] . Moreover, in order for a semi-classical thermodynamic description to

---

[1] $\ell_P$ is the Planck length and $M_P$ is the Planck mass. In natural units, $\ell_P = 1/M_P$ and $\frac{1}{16\pi G} = \frac{M_P^2}{2}$.

be valid, at least one of the merging black holes must have a large entropy i.e. it must be macroscopic: $M \gg M_P$. Thus any semi-classical treatment of black holes in Einstein-Gauss-Bonnet gravity assumes the validity of two conditions:

$$\frac{|\alpha|}{G} < 1 \tag{5.6}$$

$$\frac{M}{M_P} \gg 1 \tag{5.7}$$

With these conditions in mind, let us attempt to force a violation of the second law in a merger. This can be attempted for black holes in asymptotically Minkowski, de Sitter, and anti-de Sitter space, depending on the value of any cosmological constant. We treat these three cases in turn.

## 5.1 Asymptotically Flat Black Hole Spacetimes

Since we are attempting to engineer a violation of the second law, let us first identify a scenario in which the increase in area is minimal, since any area increase contributes positively to the change in entropy. For a given mass, extremal black holes have the smallest area, $A = 4\pi G^2 M^2$ (compared with, say, $16\pi G^2 M^2$ for a Schwarzschild black hole). Let us therefore consider the merger of two extremal black holes (neglecting the loss due to gravitational waves emitted). Extremal black hole solutions with the same charge, known as Majumdar-Papapetrou black holes [118], have no mutual forces and hence solutions of single black holes can be superimposed to give exact multi-centered solutions. The entropy of the collection of black holes is then just the sum of the entropies of each individual black hole. The Bekenstein-Hawking entropy of a single extremal black hole in pure Einstein gravity is $S = \pi G M^2$. If two extremal black holes of masses $M$ and $M'$ merge, the net change in the area entropy is therefore $\Delta S = 2\pi G M M'$. For a macroscopic $M$, the smallest possible increase in area entropy occurs when the second black hole has a mass of $M_P$. Then

the change in area entropy is $\Delta S = 2\pi G M_P M$. Let us try to offset this by including the Gauss-Bonnet contribution. The initial entropy of the system is

$$S_i = \pi G(M^2 + M_P^2) + \frac{4\pi\alpha}{G} \tag{5.8}$$

The entropy after merger is

$$S_f = \pi G(M + M_P)^2 + \frac{2\pi\alpha}{G} \tag{5.9}$$

The change in entropy is then

$$\Delta S = 2\pi G M_P M - 2\pi\frac{\alpha}{G} \tag{5.10}$$

For a violation of the second law to occur, we require

$$\frac{\alpha}{G} > G M_P M = \left(\frac{M}{M_P}\right) \tag{5.11}$$

However, this requirement contradicts eqns. (5.6) and (5.7). Thus, in order for a second law violation to take place in Einstein-Gauss-Bonnet gravity, either we must have $M/M_P < 1$, in which case the "black hole" has no description in terms of classical geometry, or we must have $\alpha/G \gg 1$, invalidating Einstein-Gauss-Bonnet as an effective theory of gravity.

Although the coefficient $\alpha$ is positive in string theory, let us briefly consider the consequences of negative $\alpha$. When $\alpha < 0$, the merger process actually causes entropy to increase, by eqn. (5.10). However, now we have to check that the entropy of even one hole is positive. For the holes to have positive entropy,

$$\frac{|\alpha|}{G} < \frac{1}{2}\left(\frac{M}{M_P}\right)^2 \tag{5.12}$$

In our regime of validity, eqns. (5.6) and (5.7) are obeyed, this bound is automatically satisfied. Thus the negative $\alpha$ case presents no problems insofar as black hole thermodynamics is concerned.

For more general (e.g. Kerr) black holes, there are no exact stable two-black hole solutions. However, the area entropy of such black holes is greater than that of the Majumdar-Papapetrou black holes we considered. So one expects that in a merger of Kerr black holes, the entropy should increase even more.

## 5.2 Asymptotically de Sitter Black Hole Spacetimes

Consider next black hole mergers in asymptotically de Sitter space. If the two black holes are both much smaller than the de Sitter scale, the de Sitter curvature scale becomes irrelevant and our results for asymptotically flat black holes apply. Alternatively, if both black holes are large, we cannot merge them. This is because in de Sitter space, there is a maximum mass black hole, the Nariai solution, with $GM_{\mathrm{max}} = L/\sqrt{27}$, where $L$ is the de Sitter length. Hence we cannot merge two black holes whose combined mass exceeds the Nariai mass. Moreover, even if the total mass is less than $M_{\mathrm{max}}$, large black holes cannot be separated in a manner where we can reliably add their entropies. The only case left consists of one black hole has large mass and another with infinitesimal mass. Consider then a black hole with mass $M_{\mathrm{max}} - M_P$ and a black hole of mass $M_P$, so that the combined mass is the Nariai mass (for simplicity). The horizon of the large mass black hole has radius $r_1 = \frac{L}{\sqrt{3}} \left(1 - \epsilon - \frac{1}{6}\epsilon^2\right)$, where $\epsilon \equiv \sqrt{2M_P/3M} \ll 1$ [119], while the small mass black hole has radius $r_2 = 2GM_P$. The final configuration has only a Nariai black hole. Since the total mass is fixed, the cosmological horizon does not change during the merger and we can neglect the entropy contribution of the cosmological horizon. Considering only the black hole horizons, the Nariai black hole has entropy $\frac{\pi L^2}{3G} + \frac{2\pi\alpha}{G}$. On the other hand, the entropy of the large black hole and small black hole system is $\frac{\pi}{G}(r_1^2) + 4\pi GM_P^2 + \frac{4\pi\alpha}{G} \simeq \frac{\pi L^2}{3G}(1 - 2\epsilon) + 4\pi GM_P^2 + \frac{4\pi\alpha}{G}$. Note that the entropic contribution of the microscopic black hole and the Gauss-Bonnet terms are

both sub-leading in $\epsilon$. Hence, to leading order in $\epsilon$, the change in entropy is

$$\Delta S \simeq \pi G \sqrt{216 M_P M^3} \tag{5.13}$$

which is obviously positive. This result is consistent with earlier results [120].

## 5.3 Asymptotically Anti-de Sitter Black Hole Spacetimes

One of the features of asymptotically AdS spaces is that it allows black hole solutions with non-compact horizons. Even though the "uniqueness theorem" [1] dictates that the horizon topology of asymptotically flat 4D black holes must be spherical, no such restrictions apply for black holes in asymptotically AdS spaces; flat and hyperbolic horizon topologies are also allowed [121–126]. The generalization of the Schwarzschild solution is

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Sigma_2^2 \, , \tag{5.14}$$

where

$$f = k - \frac{2GM}{r} + \frac{r^2}{L^2} \tag{5.15}$$

Here $d\Sigma_2^2$ is the line element of the spacelike section of the horizon with constant curvature, $k = +1, 0, -1$, corresponding to a positive-, zero, or negative-curvature horizon respectively. The mass of the black hole is $M$, which is obtained by the Abbott-Deser-Tekin formalism for asymptotically AdS spaces [127, 128]. The $k = -1, 0$ black holes have infinite area but can be made compact by suitable identifications (e.g. [129]). In the hyperbolic case ($k = -1$), identification allows for horizons with different spatial topologies; spacelike sections with genus $g > 1$ are isomorphic to the quotient space of two-dimensional hyperbolic space under discrete isometries.

## Compact Black Holes in AdS

First we shall consider merger of black holes with compact horizons, one which is the solution eqn. (5.14) with $f(r) = 1 - 2GM/r + r^2/L^2$. However, black holes in AdS which have small masses are thermodynamically unstable by the Hawking-Page transition [130]. Therefore, we need to consider only large mass black holes. The horizon is at

$$r_h = L^2/3Q + Q; Q = \left( GML^2 + \sqrt{G^2M^2L^4 + L^6/27} \right)^{1/3} \tag{5.16}$$

In the large mass limit, $GM \gg L$. The horizon radius becomes

$$r_h \approx (2GML^2)^{1/3} \left( 1 + \frac{1}{3} \left( \frac{L}{2GM} \right)^{2/3} \right) \tag{5.17}$$

Including the Gauss-Bonnet term in the entropy and setting $\chi = 2$ for the compact horizon, we find

$$S = \frac{A}{4G} + \pi \frac{\alpha}{G} \simeq \frac{\pi}{G} \left( (2GML^2)^{2/3} + \frac{2}{3}L^2 + 2\alpha \right) \tag{5.18}$$

The entropy can potentially be rendered negative by the last Gauss-Bonnet term if the coupling constant $\alpha$ is negative. However, for effective field theory to be valid, the Gauss-Bonnet term in the action, eqn. (5.1), must be much smaller than the preceding terms. This means in particular that $\alpha R^2 \ll \Lambda \Rightarrow \alpha/L^4 \ll 1/L^2$. Therefore we have

$$\frac{|\alpha|}{L^2} \ll 1 \tag{5.19}$$

In view of this constraint, it is easy to see that the entropy is always positive.

Having established the positivity of the entropy, let us consider the merger, for simplicity, of two equal-mass AdS black holes. The change in the entropy due to this merger process would be

$$\Delta S \simeq \frac{\pi}{G} \left( (2^{2/3} - 2)(2GML^2)^{2/3} - \frac{2}{3} L^2 - 2\alpha \right) \tag{5.20}$$

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In the limit that we are working, $GM \gg L$, the first two terms always add up to a negative value and the last Gauss-Bonnet term just makes things worse when $\alpha > 0$. Thus it appears that, even in Einstein gravity, the merger of AdS black holes appears to violate the second law.

The resolution is as follows. Our approach has been to compare the final entropy of one black hole (given by an exact solution) to the initial entropies of two black holes. Except in the Majumdar-Papapetrou case, the two-black-hole geometries are not exact stable solutions of general relativity. To get the entropy, we have added the entropies of single black hole solutions. That is acceptable if the presence of each black hole is only a small perturbation on the geometry near the other black hole. In asymptotically flat space, we could consider two widely separated black holes. When $2GM/r \ll 1$, the metric near each black hole would be independent of the existence of the other black hole. In order to be able to add entropies, our initial configuration has to have a minimum separation of the holes. Now, in asymptotically AdS space, the minimum separation depends on the AdS scale too: $2GM/r \ll r^2/L^2$. This means that, in the background and coordinates of an AdS black hole of mass $M_1 = M$, the other black hole of mass $M_2$ must be located at least at $r_0 \gg (2GML^2)^{1/3} \simeq \eta(2GML^2)^{1/3}$ (where $\eta$ is a large number). However, in AdS, this requires climbing up a potential energy barrier. One can estimate how much energy is required to separate black holes to our minimum separation by considering a point particle in the geometry of the other. Let the energy of the black hole be $M_2 = -p_0$, where $p^\mu$ is the four-momentum of the black hole (we can treat it as a particle by ignoring its back reaction on the background). For a particle of mass $M$ at rest at a radius $r_0$ in an AdS space, $p_0 = -M\sqrt{1 + r_0^2/L^2}$, which follows from $-M^2 = p_0^2 g^{00}$. Hence when the second black hole is brought from $r = r_0$ to $r = 0$, the total energy of the combined black hole system is not just the sum of the two masses, but must also include this

potential energy:

$$M_{\text{tot}} = M_1 + M_2 = M + M\sqrt{1 + r_0^2/L^2} \simeq M\left(1 + \frac{r_0}{L}\right) \simeq \eta\left(\frac{2G}{L}\right)^{1/3} M^{4/3} \quad (5.21)$$

When we compare the entropy of the resultant black hole, we find that it exceeds the sum of the initial entropies:

$$\Delta S = S(M_{\text{tot}}) - 2S(M) \simeq \frac{\pi}{G}\left((2GL^2)^{2/3}\left(\left(\frac{2\eta^3 G}{L}\right)^{2/9} M^{8/9} - 2M^{2/3}\right) - \frac{2}{3}L^2\right) \quad (5.22)$$

The term in brackets is positive because $M > \frac{2^{7/2}L}{\eta^3 G}$ always, since $GM \gg L$ and $\eta \gg 1$. Also, it is of the order $M^{2/3}$ implying that $\Delta S \sim (GML^2)^{2/3} - L^2$ and since $GM$ is large compared to $L$, $\Delta S$ is positive.

Now we can add the Gauss-Bonnet contribution to the entropy. Taking minimum mass black holes so that the $M_{\text{tot}}^{2/3} - 2M^{2/3} \sim L^2$,

$$\Delta S \simeq \frac{\pi}{G}\left(L^2 - 2\alpha\right) \quad (5.23)$$

However, as argued earlier, the coupling constant $\alpha$ satisfies the constraint $|\alpha|/L^2 < 1$. Therefore, $\Delta S > 0$ even with the Gauss-Bonnet contribution to the entropy.

**Non-Compact Black Holes in AdS**

AdS also admits black holes with non-compact horizons. Consider a hyperbolic black hole with $f = -1 - 2GM/r + r^2/L^2$. The horizon is at

$$r_h = L^2/3Q + Q; Q = \left(GML^2 + \sqrt{G^2M^2L^4 - L^6/27}\right)^{1/3} \quad (5.24)$$

One important thing to note from this expression is that the mass of a hyperbolic black hole in AdS has a minimum value. It has to satisfy $GM \geq L/\sqrt{27}$. To obtain a finite entropy, we have to compactify the horizon by making discrete identifications. In the large mass limit, the total entropy of the compactified horizon of genus $g > 1$

is

$$
\begin{aligned}
S \quad &\simeq \quad \frac{A_0}{4G}\left((2GML^2)^{2/3} + \frac{2}{3}L^2\right) + \pi\frac{\alpha}{G}\chi \\
&\simeq \quad \frac{\pi|\chi|}{G}\left(\left(\frac{GML^2}{2}\right)^{2/3} + \frac{1}{3}L^2 - \alpha\right),
\end{aligned} \tag{5.25}
$$

where $A_0$ is the dimensionless area of the compact, orientable horizon of genus ($>$ 1) $= -2\pi\chi$ (by the Gauss-Bonnet theorem) which is positive since $\chi < 0$ for $g > 1$. Here we do not need to worry about the positivity of entropy as in eqn. (5.18) because the first two terms are order $\sim L^2$ and we have already seen that the coupling constant $\alpha$ satisfies the constraint $|\alpha|/L^2 \ll 1$.

After merging two such hyperbolic black holes of equal mass and ignoring the Gauss-Bonnet contribution for the moment, the change in entropy is

$$
\Delta S \simeq (\pi|\chi|/G)\left(2^{-1/3}(2^{2/3} - 2)\,(GML^2)^{2/3} - \frac{1}{3}\,L^2\right) \tag{5.26}
$$

which is negative. But we have again neglected the effect of the potential energy gained by the black holes while coming from a large distance in AdS. An analysis similar to that for compact AdS black holes leads to a change in entropy of the form

$$
\Delta S \simeq \frac{\pi|\chi|}{G}\left(\left(\frac{G^2L^4}{2}\right)^{1/3}\left(\left(\frac{2G\eta^3}{L}\right)^{2/9}M^{8/9} - 2M^{2/3}\right) - \frac{1}{3}\,L^2 + \alpha\right) \tag{5.27}
$$

Similar to the previous case of compact black holes, a black hole mass $M > M_P$ is enough for $\Delta S$ to be positive. The Gauss-Bonnet term again cause no trouble due to its smallness compared to the AdS scale.

For planar black holes, with $f = -2GM/r + r^2/L^2$, the analysis is similar. A finite entropy can be obtained by making a toroidal identification on the plane. Again, a proper accounting of the potential energy ensures that the entropy increases in a merger. The Gauss-Bonnet term has no effect here since the Euler character of a torus is zero.

*Conclusion*

We have investigated the validity of the second law of thermodynamics for black hole mergers in four-dimensional Einstein-Gauss-Bonnet effective theory. Contrary to previous claims in the literature, we see that the second law remains valid within the regime of validity of approximations, even though the presence of a topological term threatens to decrease the total entropy. Our calculations here are not at the level of a proof, and we did not consider the most general 4D black hole. But we have shown that the reasoning suggesting that the second law is violated does not apply and we see no reason to suspect second law violation for more general black holes. Nevertheless, it would be worth examining mergers of other types of black holes.

Of course, the second law is one of the most robust laws in physics: in any theory with a consistent underlying statistical mechanics, the coarse-grained entropy is expected to increase when two macroscopic systems merge. Had we found a violation of the second law for black holes in Einstein-Gauss-Bonnet gravity, it would have called into question not so much the second law as a principle of nature, but the semi-classical consistency of Einstein-Gauss-Bonnet theory. Our results show that the second law is indeed obeyed; we regard this as evidence that the thermodynamics of Einstein-Gauss-Bonnet gravity is consistent with some underlying statistical mechanics.

Chapter 6

GALILEONS FROM STRING THEORY

6.1  Introduction to Galileons

While attempting to explore the possibilities of modifying the long distance behavior of gravity by kinetically mixing a light scalar field, Nicolis et al. [93] came across a very specific type of higher-derivative scalar field theory that has the peculiar property that, in addition to the usual diffeomorphism degrees of freedom it also has an additional degree of freedom, called the galilean symmetry, reminiscent of the galilean transformation. This symmetry arose because the authors unmixed the scalar by a Weyl transformation but chose a gauge in which the theories are physically equivalent to each other; this means that they were dealing with a theory that is conformally coupled to gravity. The conformal invariance of the theory manifests itself in the galilean symmetry of the scalar field and generates a series of higher-derivative scalar field Lagrangians with the property that all of them have second-order equations of motion. The galileon (as obtained in [93]) propagates on a flat background. Covariantizing the galileons turns out to be straightforward [131] but with the side-effect that it introduces ghosts in the theory. These ghosts can only be removed by adding certain non-minimal couplings of gravity to the galileon action, which in turn breaks the galilean symmetry explicitly. Therefore, nowadays galileon is a generic term for higher-derivative scalar field theories with second-order equations of motion.

In fact galileons are not the most general higher-derivative scalar field actions with second-order equations of motion. They are actually a subset of a much more general class of theories with higher-derivative, non-minimally coupled scalar field actions

with second-order equations of motion called the Horndeski theories [132]. Let us demonstrate this connection by first writing down the Horndeski Lagrangians [132],

$$L_2 = \sqrt{-g}\left(P_2(\phi, X)R - 2\dot{P}_2(\phi, X)\left((\Box\phi)^2 - \nabla_c\nabla_d\phi\nabla^c\nabla^d\phi\right)\right) \qquad (6.1)$$

$$L_3 = \sqrt{-g}P_3(\phi, X)\left((\partial\phi)^2\Box\phi - \partial_c\phi\nabla^c\nabla^d\phi\partial_c\phi\right) \qquad (6.2)$$

where the $X = (\partial\phi)^2$ and the dot denotes differentiation with respect to $X$ i.e. $\dot{P}_2(\phi, X) = \partial P_2(\phi, X)/\partial X$. $P_i(\phi, X)$ are arbitrary functions of $\phi$ and $(\partial\phi)^2$. This is the most general Lagrangian possible which leads to second-order equations of motion when a scalar field is non-trivially (read kinetically) coupled to gravity. To demonstrate how the galileons end up being a special case of the Horndeski theories we write down the covariant galileons as obtained by [131],

$$L_4^{curved} = \sqrt{-g}(\partial\phi)^2\left(2(\Box\phi)^2 - 2\nabla_a\nabla_b\phi\nabla^a\nabla^b\phi - \frac{1}{2}(\partial\phi)^2 R\right) \qquad (6.3)$$

This Lagrangian contains a non-minimal coupling with the Ricci scalar, which is necessary to keep the equations of motion second-order when the background is curved. If we compare this Lagrangian with the Horndeski Lagrangian, eqn. (6.1), it is clear that this Lagrangian is a special case of the Horndeski Lagrangian $L_2$ with the choice of the function $P_2 = -\frac{1}{2}X^2 = -\frac{1}{2}(\partial\phi)^4$. Now we compare these Lagrangians with the original conformal galileons which were derived on a flat background [93].

$$L_4^{flat} = \left((\Box\phi)^2(\partial\phi)^2 - (\partial\phi)^2(\nabla_c\nabla_d\phi)(\nabla^c\nabla^d\phi) + 2\;\partial_a\phi(\nabla^a\nabla^b\phi)(\nabla_b\nabla^d\phi)\partial_d\phi\right.$$
$$\left. -2\;\Box\phi\;\partial_c\phi(\nabla^c\nabla^d\phi)\partial_d\phi\right) \qquad (6.4)$$

The eqn. (6.4) is the flat space cousin of the Lagrangian eqn. (6.3). To see this let us

evaluate the following quantity,

$$R_{ab}\partial^a\phi\partial^b\phi(\partial\phi)^2 = R_{ca}{}^c{}_b\partial^a\phi\partial^b\phi(\partial\phi)^2 = ((\nabla_c\nabla_a - \nabla_a\nabla_c)\partial^c\phi)\,\partial^a\phi(\partial\phi)^2$$

$$= (\Box\phi)^2(\partial\phi)^2 - (\partial\phi)^2(\nabla_c\nabla_d\phi)(\nabla^c\nabla^d\phi) - 2\partial_a\phi(\nabla^a\nabla^b\phi)(\nabla_b\nabla^d\phi)\partial_d\phi$$

$$+ 2\,\Box\phi\,\partial_c\phi(\nabla^c\nabla^d\phi)\partial_d\phi \tag{6.5}$$

where we have discarded some total derivatives at the end. It is clear that the two galileon Lagrangians are related as

$$L_4^{curved} = L_4^{flat} + G_{ab}\partial^a\phi\partial^b\phi(\partial\phi)^2 \tag{6.6}$$

This is the extra covariant coupling that needs to be added to the action in order to keep the equations of motion second-order. However the last term breaks the galilean invariance explicitly because the symmetry $\phi \to \phi + b_a x^a + c$ is broken in the presence of the first derivatives of the scalar field. Looking at the Lagrangians (6.1) – (6.3) we notice one important thing. These Lagrangians contain at most second-order derivatives of the scalar field and no more. This is in fact a result of a theorem in the Horndeski paper [132] which states that the most general second-order equations of motion can be obtained from Lagrangians which are at most second-order in the derivatives of the fields. We should note the similarity with the Lovelock Lagrangians. Even though the Lovelock theories are higher curvature theories of gravity they still possess second-order equations of motion and the action of these theories contain at most second-order in the derivatives of the metric. The connection is very evident and we are led to conjecture that the galileon field might not be an arbitrary scalar field but one which is related to a particular degree of freedom of the metric. In fact this insight turns out to be true. The galileons can be obtained as the volume modulus of extra dimensions [65] or as the position modulus of a brane embedded in a higher-dimensional bulk [133]. We will call the previous one conformal galileon and

the latter one DBI galileon (since the DBI action was used for the brane). It also has been shown recently [134] that the galileons can be obtained by Kaluza-Klein reduction of Lovelock theories over arbitrary cycles. This makes the galileons scalar field cousins of the Lovelock theories. They share the similar feature that both of them are high derivative field theories without ghosts which makes them part of a very attractive class of non-linear theories [1] . In this chapter, we will show how the galileons are recovered in the Kaluza-Klein reduction of the ten dimensional Heterotic string theory [2] [65, 138] (from five to four dimensions), how the resulting theory is related to the galilean genesis model of Creminelli et al. [139], and how this resolves a point of confusion in the literature regarding the connection between the null energy condition (NEC) and the stability of a non-linear theory.

### 6.1.1  Connection between Galileons, NEC Violation, and Stability

The galileons possess NEC (null energy condition) violating solutions with a stable, Poincaré invariant vacuum. They are known to allow stable NEC violating perturbations to propagate at subluminal speeds. There is of considerable interest in the literature regarding the NEC-stability connection and NEC-superluminality connection with the higher-derivative galileon theories being the test cases. Examples of theories which violate the NEC but have stable superluminal modes were found [140] and it was concluded that the correlation between NEC and stability is a weak one. However this superluminality is consistent with unitarity [141]. In fact the superluminal modes can be mapped onto subluminal modes when transforming from the Weyl to the DBI representation of the conformal galileons [142]. It was soon

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[1]The effects of non-linearity introduced by the galileons can provide explanation for the dark forces at work at large scales [135, 136].

[2]They were first discovered in [137] but the scalar terms were not recognized as galileons or part of the Horndeski theories.

realized [143] that these NEC violating modes lead to a dispersion relation of the form $\omega^2 = -ak^2 + bk^4$ which clearly has a gradient instability due to the wrong sign quadratic term coming from the kinetic part of the galileon field. However the instability is avoided at short wavelengths (large wavenumber) due to the presence of the higher-derivative term in the action. This severance of the connection between NEC and stability prompted others [144] to explore NEC $\leftrightarrow$ subluminality connection. The point was to find a NEC-violating background with a stable, Poincaré invariant vacuum so that the theory admits a NEC violating solution which is stable under generic perturbations of the background and which propagates subluminally. The trick is to parametrically distort the galileon genesis action [139] in a way that the NEC is violated. When the action is perturbed around a deSitter background, the gradient term has the right sign implying subluminality [3] . Similar NEC violating, subluminal solutions were found in [96] for DBI galileons [133].

The NEC violating subluminal perturbations violates the second law of black hole thermodynamics. The galileons are coupled to the Einstein-Hilbert term and therefore the galileons violating the NEC implies the violation of the Ricci convergence condition (by the Einstein's equation $R_{ab}k^a k^b = T_{ab}k^a k^b$) which in turn leads to the violation of the second law. We wish to demonstrate, however, that the connection between the null energy condition and the Ricci convergence condition is severed. This is because the galileons can be obtained at sub-leading order in the $\alpha'$ correction to the low energy effective action of Heterotic string theory by Kaluza-Klein reduction of the Gauss-Bonnet term. Therefore, it is not correct to just couple the bare galileons to the Einstein-Hilbert term alone but the Gauss-Bonnet term must also be included. This almost severs the NEC argument because the NEC violation would not reflect

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[3]The sound speed is proportional to the coefficient of the gradient term: $c_s^2 = \frac{2-\alpha}{2\alpha}$ where $\alpha$ is the coefficient of the $(\partial\phi)^4$ term in the galileon action.

directly on the Ricci tensor part alone; there will be other terms in the relation to absorb this apparent violation.

## 6.2   Galileans from Kaluza-Klein Reduction of String Theory

Let us consider an action of the form, $\int (R + L_{GB})$. When this action is Kaluza-Klein reduced over a circle, the third order galileon terms arise as volume modulus of the compact extra dimension. The lower dimensional action is the sum of the Gauss-Bonnet term and the third order galileon term (the other fields can be set to zero consistently). The equations of motion no longer leads to the equality of the Ricci scalar and the stress tensor.

In order to demonstrate our argument about the inclusion of the Gauss-Bonnet term, we begin by looking at the string effective action at sub-leading order in the $\alpha'$ expansion [116, 145] (put in this form by appropriate field redefinitions)

$$\frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g_s} e^{-2\phi} \left[ R + 4(\partial\phi)^2 + \alpha' \left( \frac{1}{8}(R_{abcd}^2 - 4R_{ab}^2 + R^2) - 2G^{ab}\nabla_a\phi\nabla_b\phi \right. \right.$$
$$\left. \left. + 2\Box\phi(\partial\phi)^2 - 2(\partial\phi)^4 \right) \right] \tag{6.7}$$

This action is in string-frame. We need to do a Weyl transformation to put it in the Einstein-frame. But if we look at the equations of motion, it is simple to convince ourselves that we can set the scalar field to zero consistently and the effective action reduces to,

$$\frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g_s} \left( R + \frac{\alpha'}{8}(R_{abcd}^2 - 4R_{ab}^2 + R^2) \right) \tag{6.8}$$

This is still in the string-frame as signified by the subscript "s". Now we want to Kaluza-Klein [146, 147] reduce this action. First, we have to reduce this ten–dimensional action to a five–dimensional action by compactifying the extra five dimensions over a five-torus, $T^5$. From there on, there are two ways to proceed. One

way is to remain in five–dimensions and remove the non-minimal coupling from the Gauss-Bonnet term and obtain galileons in five dimensions. The other way is to perform another Kaluza-Klein reduction over a $S^1$ and reduce this action to a four–dimensional action and then remove the non-minimal coupling from the Einstein part since the non-minimal coupling of curvature squared terms are non-removable in four–dimensions for dimensional reasons. The conformal transformation will still generate galileon terms but now there is a Gauss-Bonnet term non-minimally coupled to them.

Before proceeding it is interesting to note an interesting property specific to Lovelock terms which would simplify the analysis greatly. When Kaluza-Klein reducing the Lovelock terms (which includes the Einstein and the Gauss-Bonnet term) over a $S^1$, the kinetic terms of the scalar fields, which are generated in the action (while keeping the U(1) gauge fields fixed), turn out to be total derivatives.

$$
\frac{1}{2\kappa_{D+1}^2} \int d^{D+1}x \sqrt{-g_{D+1}} \left( R^{(D+1)} + \frac{\alpha'}{8} L_{GB}^{(D+1)} \right)
$$
$$
= \frac{1}{2\kappa_D^2} \int d^D x \sqrt{-g_D} e^\phi \left( R^{(D)} - 2((\partial\phi)^2 + \Box\phi) + \frac{\alpha'}{8} \left( L_{GB}^{(D)} + 8\tilde{G}^{ab}(\nabla_a\nabla_b\phi + \partial_a\phi\partial_b\phi) \right) \right)
$$
$$
= \frac{1}{2\kappa_D^2} \int d^D x \sqrt{-g_D} \ e^\phi \left( R^{(D)} + \frac{\alpha'}{8} L_{GB}^{(D)} \right) \tag{6.9}
$$

This action has only a non-minimal coupling of the scalar field. The equations of motion shows that this scalar field can also be set to zero consistently (provided the gravity part obeys the constraint of being Ricci flat) since there are no other matter fields present and what is left is a lower dimensional action with the same functional form as the higher dimensional action. And this procedure can be repeated multiple times for every dimension and the ten-dimensional action can be reduced to one in six–dimensions over a 4-dimensional torus of constant volume. The metric decomposition

looks like,

$$g_{\mu\nu} = \begin{pmatrix} g_{ab}^5 & & & & \\ & exp(2\phi) & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} \tag{6.10}$$

We can now perform a Kaluza-Klein reduction of the 6–dimensional action over a $S^1$ with the volume $e^\phi$. The resulting action in five dimensions is the five dimensional version of the last term in eqn. (6.9) with the right coefficients from eqn. (6.8) .

$$\frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g_5} \ e^\phi \left( R^{(5)} + \frac{\alpha'}{8} L_{GB}^{(5)} \right) \tag{6.11}$$

A conformal transformation needs to be performed to remove the non-minimal coupling of the scalar field. However, since the Einstein term and the quadratic term scale under different powers of the conformal factor the non-minimal couplings of both the terms cannot be removed simultaneously. We choose to remove the non-minimal coupling of the Gauss-Bonnet term. The resulting action is,

$$\frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g_5} \ \left[ e^{2\phi} \left( R^{(5)} - 2(\partial\phi)^2 - 6 \ \Box\phi \right) + \frac{\alpha'}{8} \left( L_{GB}^{(5)} + 8 G_{ab}^{(5)} \partial^a\phi\partial^b\phi \right. \right.$$
$$\left. \left. + 24\Box\phi(\partial\phi)^2 - 24(\partial\phi)^4 \right) \right] \tag{6.12}$$

The equations of motion for this action are still second-order. We can identify the galileon terms in this equation. However when we write down the equations of motion the Einstein tensor is no longer proportional to the matter part, which implies that the Ricci convergence condition is not obviously violated even when the NEC is violated by these galileon terms. Now we can go one dimension lower to four dimensions. We start with the five–dimensional action eqn. (6.12) and again set $\phi = 0$ consistently

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and reduce over a $S^1$ of size $e^\phi$. We get the four–dimensional version of the action eqn. (6.9).

However in four–dimensions, the non-minimal coupling of the curvature squared terms cannot be removed by any conformal transformation. Hence the only option is to remove the non-minimal coupling of the Einstein term giving the action.

$$\frac{1}{2\kappa_4^2} \int d^4x \sqrt{-g_4} \left[ R^{(4)} - 2(\partial\phi)^2 + \frac{\alpha'}{8} e^{-2\phi} \left( L_{GB}^{(4)} + 16 G_{ab}^{(4)} \partial^a \phi \partial^b \phi + 24 \Box \phi (\partial\phi)^2 \right) \right]$$

$$(6.13)$$

This matches the one found by the authors in [138]. However, this action do not contain all the terms from the galileon genesis action [139]. The $(\partial\phi)^4$ term is missing. The equations of motion are still second-order. This is because each term in eqn. (6.12) independently has second-order equation of motion. The scalar field $\phi$ is obtained as a Kaluza-Klein field and has mass dimension zero. It needs to be normalized properly to get a canonical scalar field action. Redefine $\phi \to \sqrt{2\pi G_4}\phi$. Since $\kappa_4^2 = 8\pi G_4$, the action eqn. (6.13) becomes

$$\frac{1}{16\pi G_4} \int d^4x \sqrt{-g_4} \left[ R^{(4)} + \frac{\alpha'}{8} e^{-\sqrt{8\pi G_4}\phi} L_{GB}^{(4)} \right] + \int d^4x \sqrt{-g_4} \left[ -\frac{1}{2}(\partial\phi)^2 \right.$$

$$\left. + \frac{\alpha'}{8} e^{-\sqrt{8\pi G_4}\phi} \left( 2 G_{ab}^{(4)} \partial^a \phi \partial^b \phi + 3\sqrt{2\pi G_4} \Box \phi (\partial\phi)^2 \right) \right] \qquad (6.14)$$

These actions have two different parameters and hence two different scales in them. However, the presence of non-perturbative objects like D-branes provides a relation between Newton's constant and the coupling constant $\alpha'$. The relation between them comes from the fact that the coupling of the supergravity action is obtained from Kaluza-Klein reduction of the M-theory action: $\kappa_{10}^2 = 8\pi G_{10} = \frac{1}{2}(2\pi)^7 g_s^2 \alpha'^4$. The string coupling constant can be assumed to be $g_s = 1$ and $G_{10} = G_4 V_6$ where the $G_d$ is the d–dimensional Newton's constant and $V_m$ is the volume of the m–dimensional internal space. The size of the internal compact space is of the order of the string scale. Hence its volume can be written as $V_6 = \eta'' l_s^6 = \eta' \alpha'^3$ with $l_s = \sqrt{2\pi\alpha'}$ being the

string length. Therefore $8\pi G_N = \frac{2^6\pi^7}{\eta'}\alpha' = \nu\alpha' \Rightarrow \alpha' = \mu G_N$ where we have chosen the numerical pre-factor for convenience and $G_N$ is the four–dimensional Newton's constant.

$$\int d^4x \left[ \frac{R}{16\pi G_N} - \frac{1}{2}(\partial\phi)^2 + \mu e^{-\sqrt{8\pi G_N}\phi} \left( \frac{L_{GB}^{(4)}}{128\pi} + \frac{G_N}{4}G_{ab}\partial^a\phi\partial^b\phi + \frac{3}{8}\sqrt{2\pi G_N^3}\Box\phi(\partial\phi)^2 \right) \right]$$

(6.15)

The equation of motion for this action can be easily obtained by using the generalized equations of motion derived in Chapter 2. The action contains a non-minimal scalar coupling to the Gauss-Bonnet term therefore we can use the eqn. (A.5).

$$G_{ab} - 8\pi G_N \left( \partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}(\partial\phi)^2 \right) + \frac{\mu}{4}e^{-\sqrt{8\pi G_N}\phi} \left( G_N\, H_{ab} - \sqrt{8\pi G_N^3}\, Q_{ab} \right.$$
$$\left. +8\pi G_N^2\, K_{ab} + \sqrt{128\pi^3 G_N^5}\, M_{ab} \right) = 0$$

(6.16)

with

$$H_{ab} = RR_{ab} - 2R_{ac}R_b^c - 2R^{cd}R_{acbd} + R_a^{cde}R_{bcde} - \frac{1}{4}L_{GB}$$

(6.17)

$$Q_{ab} = R_{acbd}\nabla^c\nabla^d\phi + R_{d(a}\nabla_{b)}\nabla^d\phi + \frac{R}{2}\left( g_{ab}\Box\phi - \nabla_a\nabla_b\phi \right) - R_{ab}\Box\phi - g_{ab}R_{cd}\nabla^c\nabla^d\phi$$

(6.18)

$$K_{ab} = \frac{3}{2}R\left( g_{ab}(\partial\phi)^2 - \partial_a\phi\partial_b\phi \right) - 2R_{ab}(\partial\phi)^2 - 4g_{ab}R_{cd}\partial^c\phi\partial^d\phi + 3R_{c(a}\partial_{b)}\phi\partial^c\phi$$
$$+ 3R_{acbd}\partial^c\phi\partial^d\phi + 2(\nabla_a\nabla^c\phi)(\nabla_b\nabla_c\phi) - 2\Box\phi\nabla_a\nabla_b\phi$$
$$+ g_{ab}\left( (\Box\phi)^2 - (\nabla_c\nabla_d\phi)(\nabla^c\nabla^d\phi) \right)$$

(6.19)

$$M_{ab} = (\nabla_a\nabla_b\phi)(\partial\phi)^2 + (\Box\phi)\partial_a\phi\partial_b\phi + 2\partial_{(a}\phi(\nabla_{b)}\nabla^c\phi)\partial_c\phi - 2g_{ab}\,\partial_c\phi(\nabla^c\nabla^d\phi)\partial_d\phi$$
$$- \frac{1}{2}g_{ab}(\Box\phi)(\partial\phi)^2$$

(6.20)

This equation of motion contains only one parameter, the four–dimensional Newton's constant $G_N$ and the terms are expanded in powers of $G_N$ which decides the mass dimensions of each order in the series. The equation is a modified form of the covariant

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galileons and is second-order. This equation of motion contracted with the null vectors will give the null energy condition for the galileon fields.

$$R_{ab}k^a k^b = 8\pi G_N (k \cdot \phi)^2 - \frac{\mu}{4} e^{-\sqrt{8\pi G_N}} \left( G_N \, H_{ab}k^a k^b - \sqrt{8\pi G_N^3} \, Q_{ab}k^a k^b \right.$$
$$\left. + 8\pi G_N^2 \, K_{ab}k^a k^b + \sqrt{128\pi^3 G_N^5} \, M_{ab}k^a k^b \right) \tag{6.21}$$

The violation of the NEC by the Galilean terms $M_{ab}$ does not imply the violation of the Ricci convergence condition since they are no longer linearly related by the equations of motion.

We have demonstrated that higher order, ghost free, kinetic terms of scalar fields like the galileons when coupled with gravity cannot appear with the Einstein term alone. The Gauss-Bonnet term is also present. Therefore the NEC violation of the galileon terms does not imply the violation of the Ricci convergence condition, $R_{ab}k^a k^b \geq 0$, since the equations of motion are no longer $G_{ab} = T_{ab}$ but $G_{ab} + \alpha/8 H_{ab} = T_{ab}$ where $H_{ab}$ is the Gauss-Bonnet contribution to the equations of motion. Since the black hole entropy for higher curvature gravity is no longer simply proportional to the area of the event horizon but proportional to a complex combination of curvature terms, the violation of the Ricci convergence condition is no longer related to the violation of the second law. Therefore the status of the second law of black hole thermodynamics is unclear. However, it is no longer clearly violated as before when the galileons were minimally coupled to the Einstein-Hilbert action.

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APPENDIX A

HIGHER CURVATURE GRAVITIES

## A.1   Equations of Motion for Gauss-Bonnet Gravity

The Lanczos-Lovelock Lagrangians are special in that they are higher curvature Lagrangians which have second order equation of motion. The first order Lovelock Lagrangian is the Einstein-Hilbert term which forms the basis of Einsteinian general relativity and the second order term is the Gauss-Bonnet Lagrangian. The Gauss-Bonnet Lagrangian is a topological term in four dimensions but has dynamical equations in higher dimensions. It is defined as,

$$L_m = \delta^{b_1 b_2 b_3 b_4}_{a_1 a_2 a_3 a_4} R_{a_1 a_2}{}^{b_1 b_2} R_{a_3 a_4}{}^{b_3 b_4} = R^2 - 4 R_{ab} R^{ab} + R_{abcd} R^{abcd} \tag{A.1}$$

The $P_{abcd}$ is straightforward to calculate. We just remove one Riemann $P_{ab}{}^{cd} = \delta^{cdb_3 b_4}_{ab a_3 a_4} R_{a_3 a_4}{}^{b_3 b_4}$. The generalized $\delta$ is the Kronecker delta. We will need to express everything in terms of $R^a_{bcd}$ in order to derive the equation of motion. Therefore, the Lagrangian could be rewritten as $L = P^{cd}_{ab} R^{ab}_{cd} = g_{pa} g^{qb} g^{rc} g^{sd} P^p_{qrs} R^a_{bcd}$. The one down and three up metrics contribute a factor of two. And $\nabla^a P_{abcd} = 0$ for all Lovelock theories. We can now use this in the master equation eqn. (2.7),

$$P^{pqr}_a R_{bpqr} - \frac{1}{4} g_{ab} L = 4\pi G T_{ab} \tag{A.2}$$

And using the specific form of $P_{abcd}$ for the Gauss-Bonnet

$$P_{abcd} = \frac{1}{2} \left( R(g_{ac} g_{bd} - g_{ad} g_{bc}) - 2(R_{ac} g_{bd} - R_{ad} g_{bc} - R_{bc} g_{ad} + R_{bd} g_{ac}) + 2 R_{abcd} \right) \tag{A.3}$$

we get the equation of motion for the Gauss-Bonnet Lagrangian. Explicitly,

$$R_a{}^{cde} R_{bcde} - 2 R_{ac} R^c_b - 2 R^{cd} R_{acbd} + R R_{ab} - \frac{1}{4} g_{ab} (R^2 - 4 R_{ab} R^{ab} + R_{abcd} R^{abcd}) = 4\pi G T_{ab} \tag{A.4}$$

This is a well known result. The real power of this general procedure is apparent when there are some non-minimal scalar couplings to the Lagrangian. For example, if we

want the equation of motion for a Lagrangian of the form $L = e^{\mu\phi}(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd})$ we repeat the procedure with $\nabla^c\nabla^d P_{acbd}$ with $P_{abcd} = e^{\mu\phi}P_{abcd}^{GB}$ where the $P_{abcd}^{GB}$ is the one from minimally coupled Gauss-Bonnet Lagrangian, eqn. (A.3). This $P_{abcd}$ no longer satisfies the Bianchi identity $\nabla^a P_{abcd} = 0$ because of the non-minimal coupling. The equation of motion can be easily calculated by evaluating

$$
e^{-\mu\phi}\nabla^c\nabla^d P_{acbd} = P_{acbd}^{GB}\left(\mu\nabla^c\nabla^d\phi + \mu^2\nabla^c\phi\nabla^d\phi\right)
$$

$$
= \mu\left(\frac{1}{2}R(g_{ab}\Box\phi - \nabla_a\nabla_b\phi) - R_{ab}\Box\phi - g_{ab}R_{cd}\nabla^c\nabla^d\phi + R_{ad}\nabla_b\nabla^d\phi + R_{bd}\nabla_a\nabla^d\phi\right.
$$

$$
\left. + R_{acbd}\nabla^c\nabla^d\phi\right)
$$

$$
+ \mu^2\left(\frac{1}{2}R(g_{ab}(\partial\phi)^2 - \partial_a\phi\partial_b\phi) - R_{ab}(\partial\phi)^2 - g_{ab}R_{cd}\partial^c\phi\partial^d\phi + R_{ad}\partial_b\phi\partial^d\phi + R_{bd}\partial_a\phi\partial^d\phi\right.
$$

$$
\left. + R_{acbd}\partial^c\phi\partial^d\phi\right) \tag{A.5}
$$

We have intentionally multiplied the derivative of $P_{abcd}$ with $e^{-\mu\phi}$ to get rid of that pre-factor. The equation of motion is now the same as eqn. (A.4) but with a factor $e^{-\mu\phi}$ on the stress tensor and with eqn. (A.5) included on the LHS of eqn. (A.4).

## A.2   Derivation of the Equations of Motion for $G_{ab}\partial^a\phi\partial^b\phi$ by Brute Force

Doing a straightforward variation of just $G_{ab}\partial^a\phi\partial^b$ (without the square root of the metric) leads to,

$$
\delta R_{ab}\partial^a\phi\partial^b\phi + R_{ac}\partial^c\phi\partial_b\phi\delta g^{ab} + R_{bc}\partial^c\phi\partial_a\phi\delta g^{ab} - \frac{1}{2}R\partial_a\phi\partial_b\phi\delta g^{ab} - \frac{1}{2}(\partial\phi)^2\delta R_{ab}g^{ab}
$$

$$
- \frac{1}{2}(\partial\phi)^2 R_{ab}\delta g^{ab} \tag{A.6}
$$

We can easily add the $-1/2g_{ab}L$ term to the equation of motion after evaluating this variation. Before proceeding with the variation to Ricci tensor terms we will need the form of their variation which is easily obtained by contracting eqn. (2.3),

$$
\delta R_{ab} = \frac{1}{2}\left(-\nabla^d\nabla_d\delta g_{ab} + \nabla^d\nabla_a\delta g_{bd} + \nabla^d\nabla_b\delta g_{ad} - \nabla_a\nabla_b(g^{cd}\delta g_{cd})\right) \tag{A.7}
$$

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A contraction with the metric gives,

$$\delta R_{ab} g^{ab} = \nabla^a \nabla^b \delta g_{ab} - \nabla^d \nabla_d (g^{cd} \delta g_{cd}) \tag{A.8}$$

Now we are equipped to handle the variation of the Ricci terms,

$$-\frac{1}{2}(\partial\phi)^2 (\delta R_{ab} g^{ab}) = \left( \nabla_b \nabla_a \nabla^c \phi \partial_c \phi + \nabla_a \nabla^c \phi \nabla_b \nabla_c \phi - g_{ab}(\nabla_d \nabla^d \nabla^c \phi \partial_c \phi \right.$$
$$\left. + \nabla_c \nabla_d \phi \nabla^c \nabla^d \phi) \right) \delta g^{ab} \tag{A.9}$$

where we have used eqn. (2.6) and discarded the total derivatives. The other term is longer. We will just quote the final result.

$$\delta R_{ab} \partial^a \phi \partial^b \phi = \frac{1}{2} \left( -2\Box\phi \nabla_a \nabla_b \phi + g_{ab}((\Box\phi)^2 + \nabla_c \nabla_d \phi \nabla^c \nabla^d \phi + \partial_d \phi \nabla^d \nabla_c \nabla^c \phi \right.$$
$$\left. + \partial_d \phi \nabla_c \nabla^d \nabla^c \phi) - 2\partial_d \phi \nabla^d \nabla_a \nabla_b \phi \right) \delta g^{ab} \tag{A.10}$$

Adding these two variations together,

$$-\frac{1}{2}(\partial\phi)^2 (\delta R_{ab} g^{ab}) + \delta R_{ab} \partial^a \phi \partial^b \phi$$
$$= \left( \nabla_a \nabla^c \phi \nabla_b \nabla_c \phi - \Box\phi \nabla_a \nabla_b \phi + R_{acbd} \partial^c \phi \partial^d \phi + \frac{1}{2} g_{ab} \left( (\Box\phi)^2 - \nabla_c \nabla_d \phi \nabla^c \nabla^d \phi \right) \right.$$
$$\left. - \frac{1}{2} g_{ab} R_{cd} \partial^c \phi \partial^d \phi \right) \delta g^{ab} \tag{A.11}$$

The full equation of motion for $G_{ab} \partial^a \phi \partial^b \phi$ is obtained by substituting this last expression in the eqn. (A.6) and also adding the $-1/2 g_{ab} L$ term from the square root of the metric,

$$(R_{ac} \partial_b \phi + R_{bc} \partial_a \phi) \partial^c \phi - \frac{1}{2} R \partial_a \phi \partial_b \phi - \frac{1}{2}(\partial\phi)^2 R_{ab} + \nabla_a \nabla^c \phi \nabla_b \nabla_c \phi - \Box\phi \nabla_a \nabla_b \phi$$
$$+ R_{acbd} \partial^c \phi \partial^d \phi + \frac{1}{2} g_{ab} \left( (\Box\phi)^2 - \nabla_c \nabla_d \phi \nabla^c \nabla^d \phi \right) - \frac{1}{2} g_{ab} \left( -\frac{1}{2}(\partial\phi)^2 R \right)$$
$$- g_{ab} R_{cd} \partial^c \phi \partial^d \phi = 8\pi G T_{ab} \tag{A.12}$$

Since all the work is already done we can use the brute-force method to obtain the equations of motion for the Lagrangian $L = R_{ab} \partial^a \phi \partial^b \phi$. This Lagrangian does

not possess the symmetry properties of the Lagrangian $L = G_{ab}\partial^a\phi\partial^b\phi$, which is a conserved quantity coupled with kinetic terms. The equation of motion for just $R_{ab}\partial^a\phi\partial^b\phi$ is,

$$(R_{ac}\partial_b\phi + R_{bc}\partial_a\phi)\partial^c\phi - \frac{1}{2}g_{ab}R_{cd}\partial^c\phi\partial^d\phi - \Box\phi\nabla_a\nabla_b\phi - \partial_d\phi\nabla^d\nabla_a\nabla_b\phi$$
$$+ \frac{1}{2}g_{ab}((\Box\phi)^2 + \nabla_c\nabla_d\phi\nabla^c\nabla^d\phi + \partial_d\phi\nabla^d\nabla_c\nabla^c\phi + \partial_d\phi\nabla_c\nabla^d\nabla^c\phi) = 8\pi G T_{ab} \quad \text{(A.13)}$$

## A.3 Equations of Motion for $R_{ab}\partial^a\phi\partial^b\phi$ using Generalized Equations of Motion

We are going to find the generalized equation of motion for the Lagrangian $L = R_{ab}\partial^a\phi\partial^b\phi$ using the formula eqn. (2.12) in Section 2.3. The $P_{abcd}$ with the right symmetries is

$$P_{acbd} = \frac{1}{4}\left(g_{ab}\partial_c\phi\partial_d\phi - g_{ad}\partial_b\phi\partial_c\phi + g_{cd}\partial_a\phi\partial_b\phi - g_{bc}\partial_a\phi\partial_d\phi\right) \quad \text{(A.14)}$$

and the derivative of the Lagrangian with the metric gives,

$$\frac{\partial L}{\partial g^{ab}} = (R_{bd}\partial_a\phi + R_{ad}\partial_b\phi)\partial^d\phi + R_{acbd}\partial^c\phi\partial^d\phi \quad \text{(A.15)}$$

We need the quantity $\mathbf{P}\cdot\mathbf{R}$ in the equation of motion

$$P_a^{cde}R_{bcde} = \frac{1}{2}R_{acbd}\partial^c\phi\partial^d\phi + \frac{1}{2}R_{bd}\partial_a\phi\partial^d\phi \quad \text{(A.16)}$$

and its $a \leftrightarrow b$ counterpart $P_b^{cde}R_{acde}$ together with the derivatives

$$\nabla^c\nabla^d P_{acbd} = \frac{1}{4}\left(-2\partial_d\phi\nabla^d\nabla_a\nabla_b\phi - 2\Box\phi\nabla_a\nabla_b\phi + g_{ab}((\Box\phi)^2 + \nabla_c\nabla_d\phi\nabla^c\nabla^d\phi\right.$$
$$\left.+\partial_d\phi\nabla^d\nabla_c\nabla^c\phi + \partial_d\phi\nabla_c\nabla^d\nabla^c\phi)\right) + \frac{1}{4}R_{bd}\partial_a\phi\partial^d\phi - \frac{1}{4}R_{acbd}\partial^c\phi\partial^d\phi \quad \text{(A.17)}$$

and $\nabla^c\nabla^d P_{bcad}$ all added together in eqn. (2.12) leads to

$$(R_{ac}\partial_b\phi + R_{bc}\partial_a\phi)\partial^c\phi - \frac{1}{2}g_{ab}R_{cd}\partial^c\phi\partial^d\phi - \Box\phi\nabla_a\nabla_b\phi - \partial_d\phi\nabla^d\nabla_a\nabla_b\phi$$
$$+ \frac{1}{2}g_{ab}((\Box\phi)^2 + \nabla_c\nabla_d\phi\nabla^c\nabla^d\phi + \partial_d\phi\nabla^d\nabla_c\nabla^c\phi + \partial_d\phi\nabla_c\nabla^d\nabla^c\phi) = 8\pi G T_{ab} \quad \text{(A.18)}$$

which is same as eqn. (A.13).

## A.4 Equations of Motion for $G_{ab}\partial^a\phi\partial^b\phi$ using Generalized Equations of Motion

In this section we see an explicit demonstration of the generalized method outlined in Section 2.3 for the Lagrangian $L = G_{ab}\partial^a\phi\partial^b\phi$. The $P_{abcd}$ with the right symmetries is

$$
P_{acbd} = \frac{1}{4}g_{ab}\left(\partial_c\phi\partial_d\phi - \frac{1}{2}(\partial\phi)^2 g_{cd}\right) - \frac{1}{4}g_{ad}\left(\partial_b\phi\partial_c\phi - \frac{1}{2}(\partial\phi)^2 g_{bc}\right)
$$
$$
+ \frac{1}{4}g_{cd}\left(\partial_a\phi\partial_b\phi - \frac{1}{2}(\partial\phi)^2 g_{ab}\right) - \frac{1}{4}g_{bc}\left(\partial_a\phi\partial_d\phi - \frac{1}{2}(\partial\phi)^2 g_{ad}\right) \quad \text{(A.19)}
$$

and the derivative of the Lagrangian with the metric gives,

$$
\frac{\partial L}{\partial g^{ab}} = (R_{bd}\partial_a\phi + R_{ad}\partial_b\phi)\partial^d\phi + R_{acbd}\partial^c\phi\partial^d\phi - \frac{1}{2}R\partial_a\phi\partial_b\phi - \frac{1}{2}(\partial\phi)^2 R_{ab} \quad \text{(A.20)}
$$

We are ignoring the term coming from the variation of the square root of the metric for the moment since it is trivially proportional to the Lagrangian itself.

$$
P_a^{cde}R_{bcde} = \frac{1}{2}R_{acbd}\partial^c\phi\partial^d\phi + \frac{1}{2}R_{bd}\partial_a\phi\partial^d\phi - \frac{1}{2}R_{ab}(\partial\phi)^2 \quad \text{(A.21)}
$$

The term left to calculate is the contribution by the variation of the Riemann,

$$
\nabla^c\nabla^d P_{acbd} = \frac{1}{4}g_{ab}\left((\Box\phi)^2 - \nabla_c\nabla_d\phi\nabla^c\nabla^d\phi\right) - \frac{1}{2}(\Box\phi\nabla_a\nabla_b\phi - \nabla_a\nabla^d\phi\nabla_b\nabla_d\phi)
$$
$$
- \frac{1}{4}g_{ab}R_{cd}\partial^c\phi\partial^d\phi + \frac{1}{4}\left(R_{acbd}\partial^c\phi\partial^d\phi + R_{bm}\partial^m\phi\partial_a\phi\right) \quad \text{(A.22)}
$$

Adding up all these expressions (A.20) (A.21) (A.22) with their $a \leftrightarrow b$ counterparts in eqn. (2.12) we get eqn. (A.12).

$$
(R_{ac}\partial_b\phi + R_{bc}\partial_a\phi)\partial^c\phi - \frac{1}{2}R\partial_a\phi\partial_b\phi - \frac{1}{2}(\partial\phi)^2 R_{ab} + \nabla_a\nabla^c\phi\nabla_b\nabla_c\phi - \Box\phi\nabla_a\nabla_b\phi
$$
$$
+ R_{acbd}\partial^c\phi\partial^d\phi + \frac{1}{2}g_{ab}\left((\Box\phi)^2 - \nabla_c\nabla_d\phi\nabla^c\nabla^d\phi\right) - \frac{1}{2}g_{ab}\left(-\frac{1}{2}(\partial\phi)^2 R\right)
$$
$$
- g_{ab}R_{cd}\partial^c\phi\partial^d\phi = 8\pi G T_{ab} \quad \text{(A.23)}
$$

## A.5  Demonstrating that the Gauss-Bonnet Term is a Total Derivative in Four Dimensions

The Gauss-Bonnet term can be written in form notation using the Riemann two forms (we are suppressing the tensorial indices),

$$G = R \wedge R \tag{A.24}$$

However the Riemann two form $R_b^a$ itself is defined as the exterior derivative of the connection one-forms (which is the exterior derivative over a fiber bundle with affine spin connection one-form $w_b^a$).

$$R = dw + w \wedge w \tag{A.25}$$

For an exposition to tensor valued forms see the appendix of [148]. Using this in the eqn. (A.24) and expanding

$$R \wedge R = dw \wedge dw + 2dw \wedge w \wedge w + w \wedge w \wedge w \wedge w \tag{A.26}$$

The first term can be converted to a total derivative and the last term is proportional to the volume form in four dimensions (only in four dimensions). The first term is,

$$dw \wedge dw = d(w \wedge dw) - w \wedge d^2w = d(w \wedge dw) \tag{A.27}$$

and for the second term we have the following identity,

$$dw \wedge w \wedge w = \frac{1}{3}d\left( w \wedge w \wedge w \right) \tag{A.28}$$

which is obvious and collectively they lead to,

$$R \wedge R = d\left( w \wedge dw + \frac{2}{3}w \wedge w \wedge w \right) + \epsilon \tag{A.29}$$

For a vector valued (D-1)–form $\alpha_{n_1...n_{D-1}} = \epsilon_{an_1...n_{D-1}}v^a$, the exterior derivative is a total derivative times the volume form, $d\alpha = \nabla_a v^a \epsilon$. Therefore the Gauss-Bonnet

term in eqn. (A.29) is clearly a total derivative plus a constant. A proof for the general Lovelock theories in d–dimensions is a straightforward extension to this proof [149].

# APPENDIX B

# MEMBRANE PARADIGM CALCULATIONS

## B.1 Derivation of the Surface Term for Generalized Gravity

The surface term in generalized gravity could be obtained by variation of the Lagrangian $L\sqrt{-g}$ . The variation has terms proportional to second derivatives of the variation of the metric, the first derivatives of the variation of the metric and just the variation of the metric. The term proportional to $\delta g^{ad}$ contributes to the equation of motion so we drop that term. The surface term is then,

$$\delta S_\Sigma = \frac{1}{16\pi G} \int d^{D-1}x\sqrt{-g} \left[ \nabla_c \left( 2P^{abcd}\nabla_b\delta g_{ad} \right) - 2\nabla_b \left( \nabla_c P^{abcd}\delta g_{ad} \right) \right]$$
$$= \frac{1}{8\pi G} \int d^{D-1}x\sqrt{-g} \, \nabla_c \left[ P^{abcd}\nabla_b\delta g_{ad} - \delta g_{ad}\nabla_b P^{acbd} \right] \tag{B.1}$$

Rearranging the indices we get,

$$\delta S_\Sigma = \frac{1}{8\pi G} \int d^{D-1}x \sqrt{-h} \, \nabla_a \left[ P^{abcd}\nabla_d\delta g_{bc} - \delta g_{bc}\nabla_d P^{abcd} \right] \tag{B.2}$$

## B.2 Decomposition of the D-Dimensional Derivative

D-dimensional derivative could be decomposed into the (D-1)-dimensional derivative along the (D-1)-dimensional surface and the components along the normal to the (D-1)-dimensional surface.

$$\nabla_i \mathbf{A} = A^j_{|i}\mathbf{e}_j + K_{ij}A^j \frac{\mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \tag{B.3}$$

Since $\mathbf{n}$ is spacelike and the extrinsic curvature $K_{ij} = h^c_i\nabla_c n_j$ we have

$$\nabla_i A^k = A^j_{|i}e^k_j + (h^c_i\nabla_c n_j)A^j n^k \tag{B.4}$$

Taking trace of this equation,

$$\nabla_i A^i = A^i_{|i} + (h^c_i\nabla_c n_j)A^j n^i = A^i_{|i} \tag{B.5}$$

## B.3 To show that the Total Derivative Term does not contribute

We begin by taking the general boundary term and decomposing it into its normal and tangential components. $h_b^a$ are projection operators and $n^a$ are the normals to the surface. Following Wald's [148] notation, the indices are not replaced from Roman to Greek as was previously done in the literature [74, 76]. Instead the 3-quantities are understood by the presence of the projection operators $h_b^a$. For example,

$$
\begin{aligned}
n_a P^{abcd} \nabla_d \delta g_{bc} &= n_a P^{aijk} h_i^b h_j^c h_k^d \nabla_d \delta g_{bc} + n_a P^{aijk} h_i^b h_j^c n^d n_k \nabla_d \delta g_{bc} \\
&\quad + n_a P^{aijk} h_i^b n^c n_j h_k^d \nabla_d \delta g_{bc}
\end{aligned}
\tag{B.6}
$$

All others terms are zero due to symmetry of $P^{abcd}$ (which is same as the Riemann tensor). The projection operators needs to be present due to our convention. For example, the index "a" can be used to denote any dimension but the projection operator acting on this index will decide the dimension it will represent.

We need to show that the D–dimensional total derivative term in eqn. (3.14) vanishes in the null limit. First we will need to decompose the total derivative into (D-1)–dimensional quantities and then show that each term becomes zero when the stretched horizon approaches the event horizon asymptotically. To achieve this, we decompose both the quantity $P^{abcd}$ and the total derivative using the relation: $g_f^d = h_f^d + n^d n_f$.

$$
\begin{aligned}
\nabla_d(n_a P^{abcd} \delta g_{bc}) &= \nabla_d(n_a P^{abcd} \delta h_{bc}) \\
&= h_d^e \nabla_e(n_a P^{abcf} h_f^d \delta h_{bc}) + n^e n_d \nabla_e(n_a P^{abcf} h_f^d \delta h_{bc}) \\
&\quad + h_d^e \nabla_e(n_a P^{abcf} n^d n_f \delta h_{bc}) + n^e n_d \nabla_e(n_a P^{abcf} n^d n_f \delta h_{bc}) \\
&= A + B + C + D
\end{aligned}
\tag{B.7}
$$

We now treat each term in the above equation separately and obtain the form they

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would reduce to, as the null limit of the stretched horizon is approached asymptotically.

**A:** $h^e_d \nabla_e(n_a P^{abcf} h^d_f \delta h_{bc})$

We use the relation $n^a - u^a = \alpha k^a$ where $k^a$ is the auxiliary null vector. In the limit the stretched horizon approaches the event horizon, $\alpha \to 0$ and the normal vector $n^a$ can be replaced by the vector $u^a$, which is tangent to the horizon. Now since the entire quantity $h^e_d \nabla_e(u_a P^{abcf} h^d_f \delta h_{bc})$ is projected onto the (D-1)–dimensional surface, the D–dimensional derivative can be replaced by the (D-1)–dimensional derivative.

$$h^e_d \nabla_e(n_a P^{abcf} h^d_f \delta h_{bc}) \overset{\alpha \to 0}{=} (u_a P^{abcf} h^d_f \delta h_{bc})_{|e} h^e_d \tag{B.8}$$

**B:** $n^e n_d \nabla_e(n_a P^{abcf} h^d_f \delta h_{bc})$

$$n^e n_d \nabla_e(n_a P^{abcf} h^d_f \delta h_{bc})$$
$$= n^e n_d (\nabla_e n_a P^{abcf} h^d_f \delta h_{bc} + n_a \nabla_e P^{abcf} h^d_f \delta h_{bc} + n_a P^{abcf} \nabla_e h^d_f \delta h_{bc}$$
$$+ n_a P^{abcf} h^d_f \nabla_e \delta h_{bc}) \tag{B.9}$$

Now the (D-1)–dimensional metric $h^d_f$ contracted with the normal $n_d$ (which is orthogonal to it) gives zero and also the (D-1)–dimensional metric does not change in the normal direction,

$$n^e \nabla_e h^d_f = n^e \nabla_e(g^d_f - n^d n_f) = 0 \tag{B.10}$$

So all the terms on the right hand side of eqn. (B.9) vanishes.

$$n^e n_d \nabla_e(n_a P^{abcf} h^d_f \delta h_{bc}) = 0 \tag{B.11}$$

**C:** $h^e_d \nabla_e(n_a P^{abcf} n^d n_f \delta h_{bc})$

We use the relation $n^a - u^a = \alpha k^a$ again where $k^a$ is the auxiliary null vector and in the null limit we just replace all $n^a$ by $u^a$ which converts the D-dimensional derivative

into a (D-1)–dimensional derivative.

$$h_d^e \nabla_e (n_a P^{abcf} n^d n_f \delta h_{bc}) \overset{\alpha \to 0}{=} (u_a P^{abcf} u^d u_f \delta h_{bc})_{|e} h_d^e \tag{B.12}$$

**D:** $n^e n_d \nabla_e (n_a P^{abcf} n^d n_f \delta h_{bc})$

We use the relation $n^a - u^a = \alpha k^a$, $k^a$ is the auxiliary null vector.

$$n^e n_d \nabla_e (n_a P^{abcf} n^d n_f \delta h_{bc})$$

$$= n^e n_d \nabla_e (n_a P^{abcf} (u^d + \alpha k^d) n_f \delta h_{bc})$$

$$= n^e n_d \nabla_e (n_a P^{abcf} u^d n_f \delta h_{bc}) + \alpha n^e n_d \nabla_e (n_a P^{abcf} k^d n_f \delta h_{bc})$$

$$= (n_a P^{abcf} n_f \delta h_{bc}) n_d n^e \nabla_e u^d + \alpha n^e n_d \nabla_e (n_a P^{abcf} k^d n_f \delta h_{bc}) \tag{B.13}$$

$$\overset{\alpha \to 0}{=} 0 \tag{B.14}$$

where in the eqn. (B.13) we have used the fact that the vector $u^d$ is orthogonal to the normal vector $n_d$ and in the last line we have used the fact that the acceleration of the vector $n^d$ is zero: $n_d n^e \nabla_e u^d = -u^d n^e \nabla_e n_d = 0$.

Therefore, the terms B and D are zero in the null limit of the stretched horizon. And in that limit the terms A and C add up to

$$(u_a P^{abcf} h_f^d \delta h_{bc})_{|e} h_d^e + (u_a P^{abcf} u^d u_f \delta h_{bc})_{|e} h_d^e$$

$$= (u_a P^{abcf} h_f^d \delta h_{bc})_{|e} h_d^e + (u_a P^{abcf} n^d n_f \delta h_{bc})_{|e} h_d^e + O(\alpha) \overset{\alpha \to 0}{=} (u_a P^{abcd} \delta h_{bc})_{|d} \tag{B.15}$$

Therefore the D–dimensional total derivative on the stretched horizon becomes the (D-1)–dimensional total derivative in the null limit.

$$-\frac{1}{8\pi G} \int d^{D-1} x \sqrt{-h} \left( \nabla_d \left( n_a P^{abcd} \delta g_{bc} \right) \right)$$

$$\overset{\alpha \to 0}{=} -\frac{1}{8\pi G} \int d^{D-1} x \sqrt{-h} \left( u_a P^{abcd} \delta h_{bc} \right)_{|d}$$

$$= -\frac{1}{8\pi G} \left( \int_{t=end} d^{D-2} x \sqrt{\gamma} \left( u_d u_a P^{abcd} \delta h_{bc} \right) - \int_{t=begin} d^{D-2} x \sqrt{\gamma} \left( u_d u_a P^{abcd} \delta h_{bc} \right) \right)$$

$$= 0 \tag{B.16}$$

$u_d$ is normal to $n_a$ and to the (D-2)–dimensional spacelike section of the horizon. Eqn. B.16 is zero because the variation of the (D-1) metric goes to zero on the two boundaries of stretched horizon which are located at timelike infinity.

## B.4   The Black Hole Membrane Stress Tensor

The first term in eqn. (3.15) can be simplified by using the properties of the stretched horizon: eqns. B.18-B.21. We can use the diffeomorphism degrees of freedom to set the variation of the normal vector to zero, $\delta n_a = 0$. This leads to $\delta g_{ab} \to \delta h_{ab}$. We also use the fact that $n^a$ is normal to the (D-1)–dimensional metric $h_{ab}$: $h^{ab}n_a = 0$.

$$
\begin{aligned}
P^{abcd}(\nabla_d n_a)\delta g_{bc} &= P^{abc}{}_m g^{md}(\nabla_d n_a)\delta g_{bc} = P^{abc}{}_m(h^{md} + n^m n^d)(\nabla_d n_a)\delta g_{bc} \\
&= P^{abc}{}_m K_a^m \delta g_{bc} & \text{[Using B.18]} \\
&= P^{abc}{}_m K_a^m \delta h_{bc} \\
&= P^a{}_{klm} K_a^m g^{kb} g^{lc} \delta h_{bc} = P^a{}_{klm} K_a^m (h^{kb} + n^k n^b)(h^{lc} + n^l n^c)\delta h_{bc} \\
&= P_{aklm} K^{am} h^{kb} h^{lc} \delta h_{bc} & \text{[Using B.19 \& B.21]}
\end{aligned}
$$

$$(\text{B.}17)$$

where we have used the relations :

$$a_a = n^d \nabla_d n_a = 0 \tag{B.18}$$

$$n^c h^{kb}\delta h_{bc} = n^c(-\delta h^{kb} h_{bc}) = 0 \tag{B.19}$$

$$n^b n^c \delta h_{bc} = \delta(n^b n^c h_{bc}) - h_{bc}\delta n^b n^c - h_{bc} n^b \delta n^c = 0 \tag{B.20}$$

$$h^{ab}n_a = 0 \tag{B.21}$$

For the second term in the eqn. (3.15) we use the projection operators on the indices (bc) of $P^{abcd}$ and hence effectively project these indices onto the (D-1)–dimensional

surface.

$$\delta S_\Sigma = \frac{1}{8\pi G} \int d^{D-1}x \ \sqrt{-h} \ \left[ (\nabla_d n_a) P^{abcd} + 2n_a \nabla_d P^{abcd} \right] \delta g_{bc}$$

$$= \frac{1}{8\pi G} \int d^{D-1}x \ \sqrt{-h} \ \left[ P_{pklq} K^{pq} h^{kb} h^{lc} + 2n^p \nabla^q P_{pklq} h^{kb} h^{lc} \right] \delta h_{bc} \qquad (\text{B.22})$$

$$= -\frac{1}{2} \int d^{D-1}x \ \sqrt{-h} \ t^{ab} \ \delta h_{ab} \qquad (\text{B.23})$$

The stress-energy tensor on the (D-1)–hypersurface is,

$$t^{ab} = -\frac{1}{4\pi G} \left( P_{nklm} K^{mn} + 2n^p \nabla^q P_{pklq} \right) h^{ka} h^{lb} \qquad (\text{B.24})$$

APPENDIX C

JORDAN FRAME

## C.1 Conformal Transformation of the Ricci Scalar

Given the conformal transformation of the metric $g_{ab}$ with a conformal factor $\Omega(x)$,

$$\tilde{g}_{ab} = \Omega(x)^2 g_{ab} \tag{C.1}$$

the Ricci scalar transforms as [148] (we will suppress the coordinate dependence of the conformal factor from now on),

$$\tilde{R} = \Omega^{-2} \left( R - (D-2)(D-1)(\partial ln\Omega)^2 - 2(D-2)\partial^2 ln\Omega \right)$$
$$R = \Omega^2 \left( \tilde{R} - (D-2)(D-1)(\partial ln\Omega)^2 + 2(D-2)\partial^2 ln\Omega \right) \tag{C.2}$$

The Einstein-Hilbert term which is the product $\sqrt{-g}R$, transforms as

$$\sqrt{-\tilde{g}} \, \Omega^{-D} \left( \Omega^2 \tilde{R} - (D-2)(D-1)(\partial \Omega)^2 + 2(D-2)\Omega^2 \partial^2 ln\Omega \right)$$
$$= \sqrt{-\tilde{g}} \left( \Omega^{-(D-2)} \tilde{R} + (-(D-2)(D-1) + 2(D-2)^2)\Omega^{-D}(\partial\Omega)^2 \right.$$
$$\left. + 2(D-2)\partial(\Omega^{-(D-2)}\partial\Omega) \right)$$
$$= \sqrt{-\tilde{g}} \left( \Omega^{-(D-2)} \tilde{R} + (D-2)(D-3)\Omega^{-D}(\partial\Omega)^2 + \text{total derivative} \right) \tag{C.3}$$

## C.2 Equations of Motion in Jordan Frame

A variation of the action eqn. (4.3) with respect to the conformally transformed metric $\tilde{g}_{ab}$ would give the equations of motion,

$$\frac{M_P^2}{2} \left( \tilde{R}_{ab} - \frac{1}{2}\tilde{g}_{ab}\tilde{R} \right) = \frac{1}{2}f(\partial_a\phi)(\partial_b\phi) - \frac{1}{2}\tilde{g}_{ab} \left( \frac{1}{2}f(\partial_a\phi)^2 + V_2(\phi) \right)$$
$$- \frac{M_P^2}{2}e^{2\phi/M_P} \left( \tilde{g}_{ab}\tilde{\nabla}^2 - \tilde{\nabla}_b\tilde{\nabla}_a \right) e^{-2\phi/M_P}$$

$$\text{where,} \quad f = 1 - 4\left( \frac{D-3}{D-2} \right) \quad \text{and} \quad V_2(\phi) = e^{\frac{-4}{D-2}\phi}V(\phi) \tag{C.4}$$

In the equation of motion the last term can be expanded,

$$e^{2\phi/M_P} \tilde{\nabla}_b \tilde{\nabla}_a e^{-2\phi/M_P} = \frac{2}{M_P^2} \left( 2\tilde{\nabla}_a \phi \tilde{\nabla}_b \phi - \tilde{\nabla}_b \tilde{\nabla}_a \phi \right)$$

$$e^{2\phi/M_P} \tilde{\nabla}^2 e^{-2\phi/M_P} = \frac{2}{M_P^2} \left( 2(\tilde{\nabla}_a \phi)^2 - \tilde{\nabla}^2 \phi \right) \tag{C.5}$$

and the equations of motion be rewritten in a manner which will have the Einstein tensor on the left hand side of the equations of motion so that the terms on the right hand side of the equation is covariantly conserved as a result of the Bianchi identity. And it is customary to identify the quantity on the right hand side of the Einstein's equations as the stress tensor.

$$\tilde{R}_{ab} - \frac{1}{2} \tilde{g}_{ab} \tilde{R} = \frac{1}{M_P^2} T_{ab}^J \tag{C.6}$$

where the modified stress tensor in Jordan frame is,

$$T_{ab}^J = f_1 (\partial_a \phi)(\partial_b \phi) - \tilde{g}_{ab} \left( \frac{1}{2} f_2 (\partial_a \phi)^2 + V_2(\phi) \right) + 2M_P \left( \tilde{g}_{ab} \tilde{\nabla}^2 - \tilde{\nabla}_b \tilde{\nabla}_a \right) \phi$$

$$f_1 = 1 + \left( \frac{4}{D-2} \right); \quad f_2 = 1 + 4 \left( \frac{D-1}{D-2} \right); \quad V_2(\phi) = e^{\frac{-4}{D-2}\phi} V(\phi) \tag{C.7}$$

This stress tensor is conserved (by the Bianchi identity),

$$\tilde{\nabla}^a T_{ab}^J = 0 \tag{C.8}$$