

Operator-Valued Frames Associated with Measure Spaces

by

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## ABSTRACT

Since Duffin and Schaeffer's introduction of frames in 1952, the concept of a frame has received much attention in the mathematical community and has inspired several generalizations. The focus of this thesis is on the concept of an operator-valued frame (OVF) and a more general concept called herein an *operator-valued frame associated with a measure space* (MS-OVF), which is sometimes called a continuous g-frame. The first of two main topics explored in this thesis is the relationship between MS-OVFs and objects prominent in quantum information theory called positive operator-valued measures (POVMs). It has been observed that every MS-OVF gives rise to a POVM with invertible total variation in a natural way. The first main result of this thesis is a characterization of which POVMs arise in this way, a result obtained by extending certain existing Radon-Nikodym theorems for POVMs. The second main topic investigated in this thesis is the role of the theory of unitary representations of a Lie group  $G$  in the construction of OVFs for the  $L^2$ -space of a relatively compact subset of  $G$ . For  $G = \mathbb{R}$ , Duffin and Schaeffer have given general conditions that ensure a sequence of (one-dimensional) representations of  $G$ , restricted to  $(-1/2, 1/2)$ , forms a frame for  $L^2(-1/2, 1/2)$ , and similar conditions exist for  $G = \mathbb{R}^n$ . The second main result of this thesis expresses conditions related to Duffin and Schaeffer's for two more particular Lie groups: the Euclidean motion group on  $\mathbb{R}^2$  and the  $(2n + 1)$ -dimensional Heisenberg group. This proceeds in two steps. First, for a Lie group admitting a uniform lattice and an appropriate relatively compact subset  $E$  of  $G$ , the Selberg Trace Formula is used to obtain a Parseval OVF for  $L^2(E)$  that is expressed in terms of irreducible representations of  $G$ . Second, for the two particular Lie groups an appropriate set  $E$  is found, and it is shown that for each of these groups, with suitably parametrized unitary duals, the Parseval OVF remains an OVF when perturbations are made to the parameters of the included representations.

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## Chapter 1

### INTRODUCTION

#### 1.1 History

A frame for a complex, separable Hilbert space  $\mathcal{H}$  is a sequence  $\xi_1, \xi_2, \dots$  of members of  $\mathcal{H}$  that provides a stable way of recovering a vector  $\xi$  from its inner products with the sequence. The prototypical example is the case of an orthonormal basis, in which case

$$\xi = \sum_j \langle \xi, \xi_j \rangle \xi_j;$$

however, the elements of a frame need not be orthogonal, or even linearly independent. The study of frames has its origins in the work of Paley and Wiener [40] and others [23, 50] on bases of sines and cosines for  $L^2(-1/2, 1/2)$ , but the concept of a frame was not defined explicitly or studied systematically until the landmark paper of Duffin and Schaeffer entitled “A class of non-harmonic Fourier series” [25]. In this work, the authors established (1.) that a frame  $\{\xi_j\}$  provides, for every  $\xi \in \mathcal{H}$ , a basis-like expansion of the form  $\xi = \sum_j c_j \xi_j$  and (2.) a simple condition on the real numbers  $\{\dots, \lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \dots\}$  such that the system of exponentials  $\{e^{2\pi i \lambda_n \cdot} : n \in \mathbb{Z}\}$  is a frame. Despite the fact that their work would later have many applications, the study of frames lay mostly dormant for many years. The next major development was the 1986 paper of Daubechies, Grossman, and Meyer entitled “Painless nonorthogonal expansions” [21], which introduced wavelet frames and reignited interest in the subject of frames overall. Sometimes called *frame theory*, the study of frames now touches on areas as diverse as operator theory, pseudodifferential operators, multiple-access communication systems, tomography, compression,

signal transmission with erasures (*e.g.*, over the internet), phaseless reconstruction, sampling, analog-to-digital conversion, and other topics. For further background, standard references are [20, 33, 19, 31, 8]

Recently, several generalizations of frames have attracted the attention of those in the frame community. One of them, called a *g-frame* by Sun [48] and an *operator-valued frame* by Kaftal *et al.* [36], is a way to decompose  $\xi$  into a sequence of vectors, rather than scalars, in such a way that  $\xi$  can be recovered stably from them. In the former paper, the author establishes basic examples, facts, and terminology, and the latter paper adds to this body by proving some results about parameterization of OVF's. Since higher-dimensional information can be broken down into scalar components, every OVF arises, in general non-uniquely, as a sort of direct sum of a sequence of frames. However, it may be that no one of these direct-sum representations is more natural than any other. Two classes of examples of OVF's are the *fusion frames* of Casazza and Kutyniok [15] (see also [16]), which are operator-valued frames made up of orthogonal projections on  $\mathcal{H}$ , and the sets of “time-frequency localization operators” of Dörfler *et al.* [24], which are related to the windowed Fourier transform on  $\mathbb{R}^d$ . Although claims have been made that some of these OVF's can simplify the implementation of certain electronic systems, these claims have not to our knowledge been brought into practice and may be premature. However, given the vast success of the theory of frames, the subject of OVF's and related objects is still a potentially fruitful topic for research.

Another generalization, and one that has had some success, is that of a *frame associated with a measure space* (see [29]), or *continuous frame*, which can resolve  $\xi$  into a family of scalars indexed by a set more general than a discrete one—for example, a continuum. Sometimes called *coherent states* [3] in analogy with the physics literature, frames associated with measure spaces are known to simplify the

treatment of Calderon-Zygmund operators [28] and quasi-diagonalize certain classes of pseudodifferential operators [31].

Overarching these two generalizations is an object called a *continuous g-frame*, defined and studied independently by Abdollahpour and Faroughi [2] and Moran *et al.* [39]. We will use instead the term *operator-valued frame associated with a measure space*, or *MS-OVF*. Whereas an operator-valued frame decomposes  $\xi$  into a sequence of vectors and a frame associated with a measure space may decompose  $\xi$  into a continuum of scalars, an operator-valued frame associated with a measure space may decompose  $\xi$  into a continuum of vectors. As observed in [32] and [39], OVF's and MS-OVF's fit into the framework of a slight modification of a prominent object in quantum information theory, called a framed *positive operator-valued measure (POVM)*. In the latter document, the question of whether all framed POVMs correspond to some MS-OVF, or equivalently whether a POVM has a Radon-Nikodym derivative with respect to some  $\sigma$ -finite measure, arose. A positive answer to this question when the POVM's trace is finite and, respectively, when  $\mathcal{H}$  is finite-dimensional is obtained by Berezanskii and Kondratev in [9] and by Chiribella *et al.* in [18]; however, a full characterization of Radon-Nikodym differentiability appears to be lacking in the literature.

Two prominent examples of frames associated with measure spaces are the Fourier transform for  $\mathbb{R}$  restricted to  $L^2(-1/2, 1/2)$  and the Fourier transform for the circle applied to  $L^2(-1/2, 1/2)$ . The former can be used to represent a vector in  $L^2(-1/2, 1/2)$  as a continuous superposition of vectors from the family  $\{e^{2\pi i\lambda\cdot} : \lambda \in \mathbb{R}\}$ , and the latter can be used to represent members of  $L^2(-1/2, 1/2)$  as a discrete superposition of vectors from the family  $\{e^{2\pi in\cdot} : n \in \mathbb{Z}\}$ . One of several possible perspectives on the  $L^2$  convergence of the latter is that it is a consequence of the Poisson Summation Formula. As shown in Duffin and Schaeffer's paper [25], a family of the form



$\{e^{2\pi i\lambda_n \cdot} : n \in \mathbb{Z}\}$ , with more general real numbers  $\lambda_n$ , is a frame as well, assuming some mild conditions on the  $\lambda_n$ 's, so representing all vectors in  $L^2(-1/2, 1/2)$  as superpositions with respect to this family is also possible. This result was partially extended to the  $L^2$ -space of a ball in  $\mathbb{R}^d$  centered at the origin for  $d \geq 2$  by Beurling [10]. Both of these ideas extend easily to all connected, locally compact, Hausdorff abelian groups, each of which is just a product of  $\mathbb{R}^d$  and some compact abelian group  $K$ .

There is considerable literature devoted to whether the concepts of Fourier series and the Fourier transform for  $\mathbb{R}$  extend to general locally compact, Hausdorff groups  $G$ . If  $G$  is type I, second countable, and unimodular, then an analogue of the classical Plancherel theorem due to Segal [44, 45] and Mautner [38] holds. If  $G$  is a Lie group admitting a uniform lattice, then an analogue of the Poisson Summation Formula called the Selberg Trace Formula (see Selberg [46, 47] and Arthur [4]) holds. The latter is usually thought of as a pointwise formula relating linear forms for test functions on  $G$ . The question of when it provides a reproducing formula for members of the  $L^2$ -space of some relatively compact subset of  $G$ , analogous to a Fourier series for the  $L^2$ -space of  $(-1/2, 1/2)$  in  $\mathbb{R}$ , has not yet been addressed. This question is related to the more developed subject of finding Fourier series for compact Riemannian manifolds, but the two differ in the domain of functions considered. Another question not yet explored is whether the circle of results of Duffin and Schaeffer [25] and Beurling [10] extend to connected Lie groups beyond the abelian ones; *i.e.*, whether decompositions of functions in the  $L^2$ -space of a relatively compact subset  $E$  of  $G$  in terms of general irreducible unitary representations of  $G$  are possible.

## 1.2 Outline

In Chapter 2, we give notation, terminology, and preliminary results necessary to understand the rest of this thesis. An overview of the necessary concepts from functional analysis and harmonic analysis is given, as well as some basic background on frames and operator-valued frames. An example of an operator-valued frame for the  $L^2$ -space of a compact group is shown to follow from the Peter-Weyl theorem.

In Chapter 3, we give a study of MS-OVFs and POVMs. In Section 3.2, We first reproduce much of the basic theory of MS-OVFs introduced in Abdollahpour and Faroughi [2], including some basic facts about direct integrals of separable Hilbert spaces. We then fill the void of examples in their paper by providing an explicit example of an MS-OVF: namely, the example of the Fourier transform on a connected semisimple Lie group  $G$  restricted to the  $L^2$ -space of a relatively compact subset of  $G$ . In Section 3.3, the relationship between POVMs and MS-OVFs is described, and a characterization of MS-OVFs in terms of POVMs is obtained by extending the Radon-Nikodym theorems of Chiribella *et al.* [18] and Berezanskii and Kondratev [9]. Section 3.4 provides a conclusion and proposes some directions for future research.

In Chapter 4, the subject of extending the work of Duffin and Schaeffer [25] and Beurling [10] to a general connected Lie group  $G$  is explored. In Section 4.3, the idea of a Fourier series for  $L^2(-1/2, 1/2)$ , which can be thought of as a consequence of the Poisson Summation Formula, is extended, using the Selberg Trace Formula, to the  $L^2$ -space of certain relatively compact subsets of  $G$ , provided that  $G$  admits a so-called *uniform lattice*. In Section 4.4, this idea is applied to and extended for the Euclidean motion group for  $\mathbb{R}^2$  and for the Heisenberg group: for each of these examples, an appropriate relatively compact subset  $E$  of  $G$  is found, and the series coming from the trace formula is modified to give a more general class of decompositions of  $L^2(E)$ ,

which are similar to the decompositions of  $L^2(-1/2, 1/2)$  in Duffin and Schaeffer [25, Lemma III]. The decompositions obtained are expressed in terms of irreducible unitary representations of  $G$  and are examples of operator-valued frames, so they are given the name *OVFs of representations*. Section 4.5 provides a conclusion and proposes some directions for future research.

## Chapter 2

### PRELIMINARIES

#### 2.1 Introduction

This chapter establishes some preliminary notation, terminology, and results needed to understand the rest of the thesis. Important concepts include Hilbert spaces, measure spaces, topological groups, Fourier analysis, unitary representations, frames, and operator-valued frames. The focus of this thesis will be on separable Hilbert spaces,  $\sigma$ -finite measures, complex-valued functions, and on topological groups that are second countable, locally compact, and Hausdorff. These conditions should be assumed unless a statement is made to the contrary. In particular, these conditions on a topological group will be entailed when the terminology “locally compact group” is used. Throughout this thesis, the symbols  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{R}$ , and  $\mathbb{C}$  will refer to the natural numbers, the integers, the real numbers, and the complex numbers, with the natural numbers excluding 0, and the symbol  $\mathbb{T}$  will denote the multiplicative group of complex numbers of modulus one.

In Section 2.2, we give notation and terminology related to concepts from measure theory, functional analysis, and harmonic analysis. In Section 2.3, we give a review of frames and operator-valued frames and their basic properties. The concepts of analysis, synthesis, and frame operator are discussed, the frame algorithm for reconstruction is discussed, and several examples of frames and operator-valued frames are given.

## 2.2 Measure Theory, Functional Analysis, and Harmonic Analysis

For the rest of this document we will follow the notational conventions of [27, 26] unless an indication is made to the contrary.

As usual, a measurable space is denoted by an ordered pair, such as  $(X, \Sigma)$ , and a measure space is denoted by an ordered triple, such as  $(X, \Sigma, \mu)$ . Properties which are true of all  $x \in X$  except possibly on a set of  $\mu$ -measure zero are said to be true for  $\mu$ -almost every  $x$ , or  $\mu$ -a.e.  $x$ . The symbol  $\chi_E$  will denote the characteristic function of a set  $E$ . The symbol  $L^p(X, \mu)$ , or  $L^p(X, d\mu)$ , for  $p \geq 1$  will denote the Banach space of measurable functions  $f$  on  $X$  such that  $|f|^p$  is  $\mu$ -integrable. In particular, if  $\mu$  is the counting measure, then  $L^p(X, \mu)$  is denoted by  $\ell^p(X)$ , and if additionally  $X = \mathbb{N}$ , then  $\ell^p(X)$  is denoted by  $\ell^p$ .

Hilbert spaces will be denoted by calligraphic letters, such as  $\mathcal{H}$  and  $\mathcal{K}$ . If  $\mathcal{H}$  is a Hilbert space, the inner product and norm on  $\mathcal{H}$  will often be given a subscript: that is, if  $\xi, \eta \in \mathcal{H}$ , the inner product of  $\xi$  and  $\eta$  will be denoted by  $\langle \xi, \eta \rangle_{\mathcal{H}}$ , and the norm of  $\xi$  will be denoted by  $\|\xi\|_{\mathcal{H}}$ . When  $\mathcal{H}$  is understood, the subscripts will be dropped. Inner products will be conjugate linear with respect to the second variable. The Banach space of bounded operators between  $\mathcal{H}$  and  $\mathcal{K}$  with the usual operator norm is denoted  $\mathcal{L}(\mathcal{H}, \mathcal{K})$ , with  $\mathcal{L}(\mathcal{H}, \mathcal{H}) = \mathcal{L}(\mathcal{H})$ . The Hilbert-space adjoint of  $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$  will be denoted  $T^*$ . The set of positive operators from  $\mathcal{H}$  to  $\mathcal{H}$  will be denoted  $\mathcal{L}^+(\mathcal{H})$ , and the identity operator on  $\mathcal{H}$  will be denoted  $\mathbb{I}_{\mathcal{H}}$ .

For a Hilbert space  $\mathcal{H}$ , the ideal in  $\mathcal{L}(\mathcal{H})$  of trace-class operators on  $\mathcal{H}$  will be denoted  $L^1(\mathcal{H})$ , and if  $T \in L^1(\mathcal{H})$ , the trace of  $T$  is denoted  $\text{Tr}(T)$ . The Hilbert space of Hilbert-Schmidt class operators on  $\mathcal{H}$  will be denoted  $L^2(\mathcal{H})$ , with inner product  $\langle \cdot, \cdot \rangle_{\text{HS}}$  and norm  $\|\cdot\|_{\text{HS}}$ .

For a Hilbert space  $\mathcal{H}$  and a measurable space  $(X, \Sigma)$ , a map  $A : X \rightarrow \mathcal{L}(\mathcal{H})$

will be said to be *weakly measurable* if  $x \mapsto \langle A(x)\xi, \eta \rangle$  is measurable for all  $\xi, \eta \in \mathcal{H}$ . If  $\mu$  is a measure on  $(X, \Sigma)$ , if  $A : X \rightarrow \mathcal{L}(\mathcal{H})$  is weakly measurable, and if  $(\xi, \eta) \mapsto \int_X \langle A(x)\xi, \eta \rangle d\mu(x)$  is a bounded sesquilinear map, then we say that  $A$  is *weakly integrable* and we denote by  $\int_X A(x) d\mu(x)$  (the *weak integral* of  $A$  with respect to  $\mu$ ) the unique bounded operator  $S$  such that  $\langle S\xi, \eta \rangle = \int_X \langle A(x)\xi, \eta \rangle d\mu(x)$ . This notion of operator-valued integration is related to the commonly discussed concepts of Pettis integration and Bochner integration, but we will have no need to discuss these types of integration here.

If  $X$  is a locally compact, Hausdorff space,  $C_c(X)$  will denote the normed space of continuous, compactly-supported functions on  $X$ , with the uniform norm, denoted  $\|\cdot\|_\infty$ . If  $X$  is additionally a real  $C^k$  manifold, then  $C_c^k(X)$  is the set of  $k$ -times continuously differentiable, compactly-supported functions on  $X$ , where  $k \in \{1, 2, \dots, \infty\}$ . If  $E$  is an open subset of  $X$ , then  $C_E(X) := \{f \in C_c(X) : \text{supp} f \subset E\}$  and likewise for  $C_E^k(X) := \{f \in C_c^k(X) : \text{supp} f \subset E\}$ .

*Throughout this thesis, the letter  $G$  will be used to denote a locally compact group.* The identity of  $G$  will be denoted  $\mathbb{1}_G$ . Haar measure  $\mu$  for  $G$  will be left Haar measure unless a statement is made to the contrary. For  $\mu$ , the notation  $dx$  will sometimes be used in place of  $d\mu(x)$ , and we will denote by  $L^p(G)$  the space  $L^p(G, d\mu)$ ,  $p \geq 1$ . The convolution product on  $L^1(G)$  is denoted by  $(f * g)(x) = \int f(y)g(y^{-1}x) dy$ . The symbol  $\Delta_G$  will denote the *modular function* of  $G$ : *i.e.*, the unique function from  $G$  into  $(0, \infty)$  such that  $\mu(Ex) = \Delta_G(x)\mu(E)$  for all  $x \in G$  and all Borel  $E \subset G$ . We remind the reader that the modular function is identically 1 when  $G$  is discrete or abelian or compact, as well as in other cases.

A *unitary representation* (or simply, *representation*) of  $G$  is defined to be a homomorphism  $\pi$  from  $G$  into  $U(\mathcal{H}_\pi)$ , the group of unitary operators on some nonzero Hilbert space  $\mathcal{H}_\pi$ , that is continuous when  $U(\mathcal{H}_\pi)$  is given the strong-operator topol-

ogy. That is, a unitary representation is a map into  $U(\mathcal{H}_\pi)$  which satisfies  $\pi(xy) = \pi(x)\pi(y)$  and  $\pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*$ , and for which the map  $x \mapsto \pi(x)\xi$  is continuous for each  $\xi \in \mathcal{H}_\pi$ . The dimension of the space  $\mathcal{H}_\pi$  is called the *dimension* of the representation. Unitary representations are often referred to as ordered pairs  $(\pi, \mathcal{H}_\pi)$ , and two unitary representations  $(\pi_1, \mathcal{H}_1)$  and  $(\pi_2, \mathcal{H}_2)$  are said to be (*unitarily*) *equivalent* if there is a unitary  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  such that  $U\pi_1(x)U^* = \pi_2(x)$  for every  $x \in G$ . We will use the term “equivalent” in place of “unitarily equivalent” when speaking of representations. Any one-dimensional example is a continuous map into  $U(\mathbb{C}) \cong \mathbb{T}$ , and is called a (one-dimensional) *character* of  $G$ .

A prominent type of representation of  $G$  arises from the action of  $G$  on a locally compact, Hausdorff space  $S$ . If  $G$  acts continuously on  $S$  via  $(s, x) \in S \times G \mapsto s^x$ , if there is a  $G$ -invariant measure  $\mu$  on  $S$ , and if we define  $\mathcal{H} = L^2(S, d\mu)$ , a representation of  $G$  on  $\mathcal{H}$  is given by the following:

$$[\pi(x)f](s) = f(s^x)$$

for  $x \in G$ ,  $s \in S$ , and  $f \in \mathcal{H}$ . The operator  $\pi(x)$  is a unitary operator for each  $x \in G$  because  $\mu$  is  $G$ -invariant, the map  $\pi$  is clearly multiplicative, and the map  $\pi$  is continuous with respect to the strong-operator topology on  $\mathcal{H}$  by the argument that proves [26, Proposition 2.41]. If  $S = G$ ,  $s^x = sx$ , and  $\mu$  is right Haar measure,  $\pi$  is said to be the *right regular representation*. If  $S = G$ ,  $s^x = x^{-1}s$ , and  $\mu$  is left Haar measure,  $\pi$  is said to be the *left regular representation*. If  $S = H \backslash G$ ,  $s^x = sx$ , and there exists a right-invariant measure  $\mu$  on  $H \backslash G$  (conditions for this are described in [26, Theorem 2.49], with right cosets replacing left ones), then  $\pi$  is called the *right quasi-regular representation* for  $(G, H)$ . This representation is often denoted by  $R$  in the literature, and by definition

$$[R(y)\phi](x) = \phi(xy),$$

for  $x \in H \backslash G$ ,  $y \in G$ , and  $\phi \in L^2(H \backslash G, d\mu)$ . If  $S = G/H$ ,  $s^x = x^{-1}s$ , and there is a left-invariant measure  $\mu$  on  $G/H$ , then  $\pi$  is the *left quasi-regular representation* for  $(G, H)$ . In the absence of a specification of “left” or “right,” a quasi-regular representation will be assumed to be a right quasi-regular representation. Much of the focus in harmonic analysis has been on these two types representations and, more generally, on representations of  $G$  on  $L^2$ -spaces of locally compact Hausdorff spaces that admit a so-called *quasi-invariant measure*.

An important ingredient in the study of representations of  $G$  is their so-called *integrated form*. That is, if  $(\pi, \mathcal{H}_\pi)$  is a representation of  $G$  and  $f \in L^1(G)$ , then for a specified Haar measure  $dx$  we may define an operator on  $\mathcal{H}_\pi$  by

$$\pi(f) = \int_G f(x)\pi(x) dx,$$

interpreted as a weak integral. That this operator is well-defined for each such  $f$  and  $\pi$  can be seen as follows. Let  $((\xi, \eta))$  be defined for  $\xi, \eta \in \mathcal{H}_\pi$  as  $\int_G f(x) \langle \pi(x)\xi, \eta \rangle dx$ , which is finite since  $\int_G |f(x) \langle \pi(x)\xi, \eta \rangle| dx \leq \int |f(x)| dx \|\xi\| \|\eta\|$ . The map  $(\xi, \eta) \mapsto ((\xi, \eta))$  is clearly sesquilinear, and by the inequality just given it is a bounded form. Thus, there is a unique operator  $S \in \mathcal{L}(\mathcal{H}_\pi)$  such that  $((\xi, \eta)) = \langle S\xi, \eta \rangle$ . This is the operator we mean when we say  $\pi(f)$ . It is important to note that  $\pi$ , regarded as a map from  $L^1(G)$  into  $\mathcal{L}(\mathcal{H}_\pi)$ , is in fact a  $*$ -representation from the Banach  $*$ -algebra  $L^1(G)$  under convolution into  $\mathcal{L}(\mathcal{H}_\pi)$  [26, Theorem 3.9]. This idea is crucial in the proof of the Gelfand-Raikov theorem [26, Theorem 3.34], which states that there are enough *irreducible* representations (defined in the next paragraph) of  $G$  to separate points: *i.e.*, if  $x$  and  $y$  are distinct points of  $G$ , then there is an irreducible representation  $\pi$  of  $G$  such that  $\pi(x) \neq \pi(y)$ .

Given a representation  $\pi$  of  $G$  on  $\mathcal{H}$  we shall be interested in the case when there is a sequence of proper, nontrivial closed subspaces  $\{\mathcal{H}_j\}$ , each invariant under the



action of  $\pi$ , such that  $\mathcal{H} = \bigoplus \mathcal{H}_j$ . In this case we say  $\pi = \bigoplus \pi_j$ , where  $\pi_j$  is the representation on  $\mathcal{H}_j$  given by  $\pi_j(x) = \pi(x)|_{\mathcal{H}_j}$ . Whenever there is one such subspace, such a decomposition is possible, because, as is easily shown, if  $\mathcal{M} \subset \mathcal{H}$  is invariant under  $\pi$ , so is  $\mathcal{M}^\perp$ . In any case, if we have that  $\pi = \bigoplus \pi_j$ , we say that each  $\pi_j$  is a *subrepresentation* of  $\pi$ . Particularly interesting is when for each  $j$  the subrepresentation  $(\pi_j, \mathcal{H}_j)$  is an *irreducible* representation meaning that no proper, nontrivial, closed subspace of  $\mathcal{H}_j$  is invariant under  $\pi_j$ . Some representations, like the right regular representation on  $\mathbb{T}$  decompose into a direct sum of subrepresentations, and some, like the right regular representation on  $\mathbb{R}$ , do not.

If  $\pi$  can be decomposed as  $\bigoplus \pi_j$ , then although the  $\pi_j$ 's are not unique in and of themselves, the list of them is unique up to unitary equivalence. The number of times in  $\{0, 1, 2, \dots, \infty\}$  that  $\pi_j$  occurs, up to equivalence, in  $\bigoplus \pi_j$  is called its *multiplicity*. It will not be important to us here, but we note that similar decompositions exist in general if we introduce an object called a direct integral of representations, although these decompositions do not always possess the same uniqueness property. In any case, decomposing a general representation  $\pi$  of  $G$  explicitly into irreducible representations is a fundamental problem in harmonic analysis. Part of this problem is to describe the so-called unitary dual  $\hat{G}$  of  $G$ —the set of equivalence classes of irreducible representations of  $G$ . However, such descriptions and decompositions are only known for special classes of groups (*e.g.*, connected semi-simple Lie groups and connected nilpotent Lie groups, for two) and special representations (*e.g.*, regular, quasi-regular, actions on the  $L^2$ -spaces described above).

Finally, we make a note about the Fourier transform on a locally compact abelian group  $G$ . If  $\xi$  is a character on  $G$  and  $f \in L^1(G)$ , then we define

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_G f(x) \overline{\xi(x)} dx.$$

If  $\hat{G}$  is endowed with the operation of pointwise multiplication, then it, like  $G$ , is a locally compact abelian group. The Plancherel Theorem for  $G$ , a fundamental result, states that the map  $\mathcal{F}$  above, restricted to  $L^1(G) \cap L^2(G)$  extends uniquely to a unitary isomorphism from  $L^2(G)$  to  $L^2(\hat{G})$ , when Haar measure on  $\hat{G}$  is appropriately normalized. One special case of this of which we will make use is the case where  $G$  is compact. In this case,  $\hat{G}$  consists of a discrete set of characters  $\kappa_1, \kappa_2, \dots$ , and we will assume that Haar measure of  $G$  is normalized to 1. It follows from the Plancherel Theorem that these characters, when considered as members of  $L^2(G)$ , form an orthonormal basis for  $L^2(G)$ . Another important case is the case where  $G = \mathbb{R}^d$ . In this case, we identify a character  $\xi_\omega = e^{2\pi i \langle \omega, \cdot \rangle}$  with the real vector  $\omega$ , and the Fourier transform takes the form

$$\mathcal{F}f(\omega) = \hat{f}(\omega) = \int f(x) e^{-2\pi i \langle \omega, x \rangle} dx$$

for  $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ . The dot product  $\langle \omega, x \rangle$  of two real vectors will sometimes be written  $\omega \cdot x$ . If  $f$  is considered as a member of  $L^2(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \dots \times \mathbb{R}^{d_n})$ , then the symbol  $\mathcal{F}_j$  will denote the Fourier transform of  $f$  with respect to the  $j^{\text{th}}$  variable. For example,

$$\mathcal{F}_1 f(\omega_1, x_2, x_3, \dots, x_n) = \int_{\mathbb{R}^{d_1}} f(x_1, x_2, \dots, x_n) e^{-2\pi i \omega_1 \cdot x_1} dx_1.$$

This concludes this section.

### 2.3 Frame Theory

In this section, we review the concept of a frame and the concept of an operator-valued frame. In Section 2.3.1 and Section 2.3.2, basic properties of frames and operator-valued frames, respectively, are discussed, including the concepts of an analysis, synthesis, and frame operator. Methods of reconstruction using frames and

operator-valued frames are discussed. For each type of object, several examples are given, including the important example of a frame of exponentials (Example 2.3.3). Throughout this section, the symbol  $\mathcal{H}$  will denote a Hilbert space.

### 2.3.1 Frames

A frame for  $\mathcal{H}$  is a sequence  $\{\xi_j : j \in \mathbb{N}\} \subset \mathcal{H}$  such that there are constants  $B, A > 0$  for which

$$A \|\xi\|^2 \leq \sum_j |\langle \xi, \xi_j \rangle|^2 \leq B \|\xi\|^2 \quad (2.1)$$

for all  $\xi \in \mathcal{H}$ . If it is known only that there is a  $B \geq 0$  such that

$$\sum_j |\langle \xi, \xi_j \rangle|^2 \leq B \|\xi\|^2 \quad (2.2)$$

for all  $\xi \in \mathcal{H}$ , then we say  $\{\xi_j\}$  is a *Bessel sequence*. In this context, we may define a bounded operator  $T : \mathcal{H} \rightarrow \ell^2$  by  $\xi \mapsto \{\langle \xi, \xi_j \rangle\}$ . Such a map is called a *Bessel map* or *analysis operator* in the literature. The adjoint  $T^*$  is then called the *synthesis operator* for the sequence. Further, we will call  $R = T^*T$  the *resolvent* for the sequence. If  $\{\xi_j\}$  is a Bessel sequence,  $\{\xi_j\}$  is a frame if and only if the analysis operator  $T$  is invertible. In particular, this means that it is possible to approximately recover  $\xi$  from  $T\xi$  in the presence of limited noise added to the vector  $T\xi$ —in fact, even if limited noise is added to  $\xi$  before the application of  $T$ . Frames share this desirable property with orthonormal bases, and, as we will see in Example 2.3.6, non-orthogonal frames can sometimes be more convenient to work with than orthonormal bases. First let us discuss a simple example of a frame.

**Example 2.3.1.** Let  $\{\xi_j : j \in \mathbb{N}\}$  be an orthonormal system in  $\mathcal{H}$ . We will now check that  $\{\xi_j\}$  is a Bessel sequence, and for the particular case of an orthonormal basis, calculate its synthesis and resolvent operators. To check (2.2), simply cite Bessel's

inequality:

$$\sum_j |\langle \xi, \xi_j \rangle|^2 \leq \|\xi\|^2$$

for all  $\xi \in \mathcal{H}$ . If in addition  $\{\xi_j\}$  spans  $\mathcal{H}$ , then  $\{\xi_j\}$  is a frame with  $A = B = 1$ :

$$\sum_j |\langle \xi, \xi_j \rangle|^2 = \|\xi\|^2$$

for all  $\xi \in \mathcal{H}$ . For the resolvent, we first need an adjoint for  $T$ . We claim that  $T^*$  is given by the map  $S$  from  $\ell^2$  to  $\mathcal{H}$  defined by

$$\{c_j\} \mapsto \sum_j c_j \xi_j. \quad (2.3)$$

This sum converges in norm and is a bounded linear map since the  $|c_j|$ 's are square-summable. We will now check that  $T^* = S$ . Let  $\eta = \{c_j\} \in \ell^2$ . On one hand we have,

$$\begin{aligned} \langle \xi, S\eta \rangle &= \left\langle \xi, \sum_j c_j \xi_j \right\rangle \\ &= \sum_j \bar{c}_j \langle \xi, \xi_j \rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned} \langle T\xi, \eta \rangle_{\ell^2} &= \langle \{\langle \xi, \xi_j \rangle\}, \{c_j\} \rangle_{\ell^2} \\ &= \sum_j \bar{c}_j \langle \xi, \xi_j \rangle. \end{aligned}$$

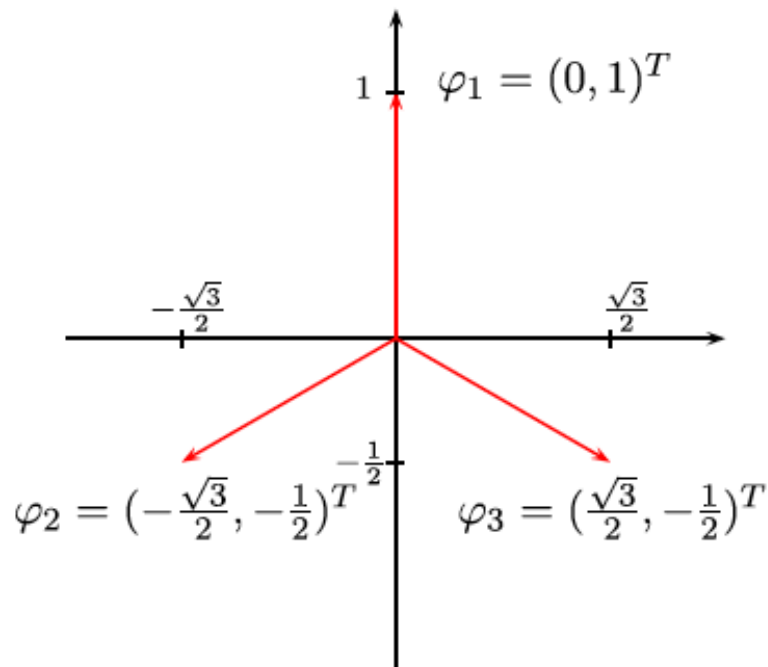
Thus  $T^*$  is given by (2.3). The resolvent  $T^*T$  is then

$$\xi \mapsto \sum_j \langle \xi, \xi_j \rangle \xi_j, \quad (2.4)$$

which is the identity operator on  $\mathcal{H}$ .

The above example implies that if  $\{\xi_j\}$  is an orthonormal *basis* for  $\mathcal{H}$ , then it is a frame with  $A = B = 1$ . Frames for which  $A = B$  are called *tight* frames. If in addition  $A = B = 1$ , the frame is called a Parseval frame. All orthonormal systems are Parseval frames for their closed span, but not all Parseval frames are orthonormal bases, as the following example indicates.

**Example 2.3.2.** (The “Mercedes” frame.) Consider the frame in Figure 2.1, made up of the three red vectors, for  $\mathcal{H} = \mathbb{C}^2$ . Each red vector lies in  $\mathbb{R}^2$ , but together they make a frame for  $\mathbb{C}^2$ . These vectors form a tight frame with  $A = B = 3/2$ . Scaling each vector by  $\sqrt{2/3}$  gives a Parseval frame whose members are nonorthogonal.



**Figure 2.1:** The “Mercedes” Frame

In the example above, the vectors  $\xi_1, \xi_2$ , and  $\xi_3$  all have length  $\sqrt{2/3}$ . For a general  $\mathcal{H}$ , if we were to require that  $\{\xi_1, \xi_2, \dots\}$  be a Parseval frame of *unit* vectors, then the only possibility is that  $\{\xi_j\}$  is an orthonormal basis. Indeed, the truth of

the equality

$$\sum_j |\langle \xi, \xi_j \rangle|^2 = \|\xi\|^2$$

for all  $\xi \in \mathcal{H}$  implies that for any  $k$  we have

$$\sum_j |\langle \xi_k, \xi_j \rangle|^2 = |\langle \xi_k, \xi_k \rangle|^2,$$

which implies that  $|\langle \xi_k, \xi_j \rangle| = 0$  for all  $k \neq j$ .

We provide in the following example an important class of infinite-dimensional frames that are in general non-orthogonal.

**Example 2.3.3.** (Frames of exponentials.) It is clear from the frame condition that any ordering of a frame is also a frame, so that we may consider sequences indexed by arbitrary discrete sets as frames. For  $x, \lambda \in \mathbb{R}^d$ , let  $e_\lambda(x) = e^{2\pi i \langle \lambda, x \rangle}$ . When  $\Lambda$  is a discrete subset of  $\mathbb{R}^d$  and  $E$  is a measurable subset of  $\mathbb{R}^d$ , several authors have investigated the question of when sequences of the form  $F(\Lambda) = \{e_\lambda : \lambda \in \Lambda\}$  are a frame for  $L^2(E)$ . The sequence  $F(\mathbb{Z}^d)$ , for example, is an orthonormal basis for  $L^2(E)$  with  $E = (-1/2, 1/2)^d$ , so it is of course also a Parseval frame for that space. In Duffin and Schaeffer's 1952 paper on non-harmonic Fourier series, the authors describe for  $d = 1$  a very general condition on  $\Lambda = \{\lambda_j : j \in \mathbb{Z}\}$  such that  $F(\Lambda)$  is a frame for  $L^2(E)$ , with  $E$  as above. Their conclusion: if there are  $M, \delta > 0$  such that  $|\lambda_j - j| < M$  for all  $j$  and  $|\lambda_i - \lambda_j| > \delta$  for all  $i \neq j$ , then  $F(\Lambda)$  is a frame for  $\mathcal{H}$ . It is often of interest to determine when a sequence  $\Psi = \{\psi_j\}$  that is a frame for  $\mathcal{H}$  is more strongly a *Riesz basis* for  $\mathcal{H}$ , which means  $\Psi$  is the image of an orthonormal basis  $\{f_j\}$  for  $\mathcal{H}$  under an invertible operator  $S : \mathcal{H} \rightarrow \mathcal{H}$ . Every Riesz basis is a frame since  $\sum_j |\langle \xi, S f_j \rangle|^2 = \|S^* \xi\|^2$ , which is bounded below by a positive constant times  $\|\xi\|^2$  since  $S^*$  is invertible. The famous "1/4-theorem," due to [35], states that  $F(\Lambda)$  is a Riesz basis for  $L^2(E)$  if we have that  $\sup_j |\lambda_j - j|$  is less than 1/4, and [7]

has extended this result further. Another result in this circle of ideas is one of [10], which describes a family of frames for the  $L^2$ -space of the unit ball  $B_d$  in  $\mathbb{R}^d$ . His result is that if  $\Lambda$  is a subset of  $\mathbb{R}^d$  such that  $\sup_{\zeta \in \mathbb{R}^d} \text{dist}(\zeta, \Lambda)$  is less than  $1/4$ , where  $\text{dist}(\zeta, \Lambda)$  is the Euclidean distance between the point  $\zeta$  and the set  $\Lambda$ , then  $F(\Lambda)$  is a frame for  $L^2(B_d)$ .

If  $\Phi = \{\phi_i\}$  and  $\Psi = \{\psi_j\}$  are frames for  $L^2(X, d\mu)$  and  $L^2(Y, d\nu)$ , respectively. Then, the set  $\Phi \otimes \Psi$  of all products of the form  $\phi_i \otimes \psi_j$  is a frame for  $L^2(X \times Y, \mu \times \nu)$ , where for  $\phi \in L^2(X)$  and  $\psi \in L^2(Y)$  the quantity  $\phi \otimes \psi$  is defined by  $(x, y) \in X \times Y \mapsto \phi(x)\psi(y)$ . Indeed, since  $L^2(X) \otimes L^2(Y)$  is norm-dense in  $L^2(X \times Y, \mu \times \nu)$ , it suffices to prove the frame inequalities hold on all  $f$  of the form  $g \otimes h$ , for  $g \in L^2(X)$  and  $h \in L^2(Y)$ . Suppose the frame bounds for  $\Phi$  are  $B_1, A_1 > 0$  and the frame bounds for  $\Psi$  are  $B_2, A_2 > 0$ . We have

$$\sum_{i,j} |\langle f, \phi_i \psi_j \rangle|^2 = \sum_i |\langle g, \phi_i \rangle|^2 \sum_j |\langle h, \psi_j \rangle|^2$$

and the right side is bounded below by  $A_1 A_2 \|g\|^2 \|h\|^2$  and above by  $B_1 B_2 \|g\|^2 \|h\|^2$ . Since  $\|f\|_{L^2(X \times Y)}^2 = \|g\|^2 \|h\|^2$ ,  $\Phi \otimes \Psi$  is a frame for  $L^2(X \times Y)$  with bounds  $A_1 A_2$  and  $B_1 B_2$ , as desired. This gives rise to an extension of the idea of frames of exponentials to a locally compact abelian group.

**Example 2.3.4.** (Frames of characters for connected locally compact abelian groups.) Suppose  $G$  is an abelian group. The Principle Structure Theorem for locally compact abelian groups (as found in [34]) ensures that  $G$  has an open subgroup isomorphic to  $\mathbb{R}^d \times K$  for some compact abelian group  $K$ . If we assume further that  $G$  is connected, then this subgroup must be equal to  $G$  since it is also closed. Thus, if  $G$  is connected, there is no loss in generality in assuming that  $G$  is *equal* to  $\mathbb{R}^d \times K$ . Suppose Haar measure of  $K$  is normalized to 1. Let  $e_\lambda$  for  $\lambda \in \mathbb{R}^d$  be the complex exponential

in Example 2.3.3, and let  $\Lambda = \{\lambda_1, \lambda_2, \dots\}$  and  $E_0$  be subsets of  $\mathbb{R}^d$  such that the sequence of exponentials  $F(\Lambda)$  is a frame for  $L^2(E_0)$ . Using [26, Proposition 4.6] all characters  $\pi$  of  $G$  are of the form  $\pi_{\lambda,l}(x, k) = e_\lambda(x)\kappa_l(k)$ , where  $x \in \mathbb{R}^d$ ,  $\lambda \in \mathbb{R}^d$ ,  $k \in K$ ,  $l \in \mathbb{N}$ , and  $\{\kappa_l : l \in \mathbb{N}\}$  is a list of all characters of  $K$ . Further,  $\{\kappa_l\}$  is an orthonormal basis for  $L^2(K)$ . By the remarks preceding this example,  $\{\pi_{\lambda_j,l}\} = \{e_{\lambda_j} \otimes \kappa_l\}$  is a frame for  $L^2(E_0 \times K)$ . We will refer to a frame obtained in this way as a *frame of characters* for  $L^2(E)$ .

For a general Bessel sequence  $\{\xi_j\}$ , calculations similar to those of Example 2.3.1 give the same results for  $T^*$  and  $R$ . Namely,

$$T^* : \ell^2 \rightarrow \mathcal{H} : \{c_j\} \mapsto \sum_j c_j \xi_j$$

and

$$R : \mathcal{H} \rightarrow \mathcal{H} : \xi \mapsto \sum_j \langle \xi, \xi_j \rangle \xi_j. \quad (2.5)$$

The proofs are given in [19, Lemma 3.2.1]. From this, we can make an alternative characterization of the upper and lower bounds in (2.1). To wit, they are equivalent to  $A \|\xi\|^2 \leq \|T\xi\|_{\ell^2}^2$  and  $\|T\xi\|_{\ell^2}^2 \leq B \|\xi\|^2$ , respectively, which are equivalent to  $A\mathbb{I}_{\mathcal{H}} \leq T^*T$  and  $T^*T \leq B\mathbb{I}_{\mathcal{H}}$ , respectively, in the positive-semidefinite partial ordering. (In this case, we call  $R$  the *frame operator*.) Now we see something important that distinguishes tight frames from other types of frames:  $\frac{1}{A}T^*T = \mathbb{I}_{\mathcal{H}}$ . This means we have the following simple reconstruction formula for obtaining  $\xi$  from  $T\xi$ :

$$\xi = \frac{1}{A} \sum_j \langle \xi, \xi_j \rangle \xi_j. \quad (2.6)$$

For a more general reproducing formula that works for non-tight frames, simply observe that by (2.5) we have

$$\xi = \sum_j \langle \xi, \xi_j \rangle R^{-1}\xi_j. \quad (2.7)$$



(It turns out that  $\mathcal{G} = \{R^{-1}\xi_j : j = 1, 2, \dots\}$  is a frame as well, called the *dual frame* of  $\{\xi_j\}$ , and has frame bounds  $\frac{1}{B}$  and  $\frac{1}{A}$ , and has the property that the dual frame of  $\mathcal{G}$  is again  $\{\xi_j\}$ .)

In light of the above, it is often desirable to invert  $R$ . For any positive operator  $S$  whose spectrum is bounded below by a positive number  $A$  and above by  $B$ , one can use the following recursive algorithm, found in [30], to compute approximants  $\xi^{(n)}$  to  $\xi$  given  $S\xi$ :

$$\xi^{(n)} = \xi^{(n-1)} + \frac{2}{A+B} S(\xi - \xi^{(n-1)}), \quad (n \geq 1), \quad (2.8)$$

Using this method we get exponential convergence:

$$\|\xi - \xi^{(n)}\| \leq \left(\frac{B-A}{B+A}\right)^n \|\xi\|. \quad (2.9)$$

In light of this convergence estimate, good bounds for the spectrum are essential to fast convergence. As explored in [30], it is sometimes computationally desirable to use the above algorithm when  $S$  is a frame operator, in which case the algorithm is called the *frame algorithm*.

**Remark 2.3.5.** Two problems that computational mathematicians are interested in are the efficiency of *storing* and *retrieving* information about a vector  $\xi$  using a frame. To achieve efficient storage, the sequence  $T\xi$  should be essentially zero except for a finite number of terms. To achieve efficient retrieval, the  $n^{\text{th}}$  partial sum of the series in Equation (2.7) should converge rapidly. This is of course only possible for some vectors  $\xi$ . The frames of Duffin and Schaeffer, described in Example 2.3.3, provide both efficient storage and efficient retrieval of  $k$ -times differentiable functions  $f$  supported on  $(-1/2, 1/2)$  in the following senses. For storage, the  $n^{\text{th}}$  term of sequence  $\{\langle f, \psi_j \rangle\}$  is equal to

$$\int_{-1/2}^{1/2} f(x) e^{-2\pi i \lambda_n x} dx.$$

Thus, integrating by parts  $k$  times gives a bound of  $\frac{C_f}{(n-M)^k}$  for  $n > M$  and for an appropriate constant  $C_f$  depending on  $f$ . For retrieval, the  $n^{\text{th}}$  partial sum in (2.7) then converges at a rate of roughly  $\frac{1}{k-1} \frac{C_f}{(n-M)^{k-1}}$ .

A fair question is what non-orthogonal frames do that orthonormal bases do not do? One example comes from the area of Gabor analysis.

**Example 2.3.6.** If  $g \in L^2(\mathbb{R})$  and  $a, b > 0$ , let  $g_{m,n}(x) = e^{2\pi imbx} g(x - na)$ . The set  $\{g_{m,n} : m, n \in \mathbb{Z}\}$  is called a *Gabor system*. Gabor systems are often used in applications to decompose members  $f$  of  $L^2(\mathbb{R})$  which are modeled as Schwarz-class signals. In light of the previous Remark, it is of interest from a computational standpoint to find Gabor systems that represent  $f$  efficiently. Given a Schwarz-class function  $g$  such that  $\{g_{m,n}\}$  is an orthonormal basis, the sequence  $\sum_{m,n} \langle f, g_{m,n} \rangle g_{m,n}$  will converge faster than any inverse power of  $(|m| + |n| + 1)$ , but in practice, finding such a  $g$  can be difficult or impossible. However, if the requirement of orthonormality is loosened to the requirement of being a frame, finding such a  $g$  is easy. In fact, as long as  $ab < 1$ , any  $g \in L^2(\mathbb{R})$  will give rise to not only a frame but a tight frame [21]. Such frames are called *Weyl-Heisenberg frames* and are a type of wavelet frame. In view of (2.6), the reconstruction formula corresponding to such a system is no more complicated than reconstruction using an orthonormal basis and exhibits the same property of rapid convergence just mentioned. Thus, non-orthogonal frames with good storage-and-retrieval properties can be easier to find than orthonormal bases with these properties.

### 2.3.2 Operator-Valued Frames

As suggested in the Section 1.1, frames are part of a larger family called *operator-valued frames*, which behave in largely the same way as frames but may be more

convenient for some purposes. Here we will define these objects and give two examples. Like a frame, an operator-valued frame has an analysis, synthesis, and frame operator. We describe the forms these operators take. Finally, we discuss the analogues of the reconstruction formulas in (2.6), (2.7), and (2.8).

Let  $\mathcal{K}_1, \mathcal{K}_2, \dots$  be a sequence of Hilbert spaces and  $\|\cdot\| = \|\cdot\|_{\mathcal{H}}$  and  $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathcal{H}}$ . If  $\{T_j : j \in \mathbb{N}\}$  is a sequence of bounded linear maps  $T_j : \mathcal{H} \rightarrow \mathcal{K}_j$  such that there exists  $B \geq 0$  such that

$$\sum_j \|T_j \xi\|_{\mathcal{K}_j}^2 \leq B \|\xi\|^2 \tag{2.10}$$

for all  $\xi \in \mathcal{H}$ , we will say that  $\{T_j\}$  is an *operator-valued Bessel sequence*. Intuitively, the difference between the Bessel sequences of the last section and those of this chapter is that those of this chapter resolve the vector  $\xi$  into a square-summable sequence of vectors  $\{T_j \xi\}$  rather than a square-summable sequence of scalars  $\{\langle \xi, \xi_j \rangle\}$ . More precisely,  $\{T_j\}$  resolves  $\xi$  into a sequence  $\{T_j \xi\} \in \Pi_j \mathcal{K}_j$  for which  $\sum_j \|T_j \xi\|_{\mathcal{K}_j}^2 < \infty$ . In what follows, we will freely pass back and forth between identifying  $\{T_j\}$  as a map ( $\xi \mapsto \{T_j \xi\}$ ) from  $\mathcal{H}$  into  $\bigoplus_j \mathcal{K}_j$  and identifying  $\{T_j\}$  as a sequence of operators. Also, for the rest of this section we will denote  $\bigoplus_j \mathcal{K}_j$  by  $\mathcal{K}$ , calling  $\mathcal{K}$  the *analysis space* of  $\{T_j\}$  and the  $\mathcal{K}_j$ 's the *component spaces* of  $\{T_j\}$ .

Given an operator-valued Bessel sequence  $T = \{T_j\}$ , the analysis, synthesis, and resolvent operators are defined as before, as  $T$ ,  $T^*$ , and  $T^*T$ , respectively. At this point it is pertinent to mention a difference between the Bessel sequences of the last chapter and those of this chapter. Those of the last chapter are not formally instances of those from this chapter, but they can be thought of as such by identifying a Bessel sequence  $\{\xi_j\}$  with an operator-valued Bessel sequence  $\{\langle \cdot, \xi_j \rangle\}$ . We do not lose anything by making this identification, as the purposes of Bessel sequences and operator-valued Bessel sequences are the same: to resolve a vector into constituent

parts.

An operator-valued Bessel sequence  $T = \{T_j\}$  is then an *operator-valued frame* for  $\mathcal{H}$  if, in addition to (2.10), there is  $A > 0$  such that

$$A \|\xi\|^2 \leq \sum_j \|T_j \xi\|_{\mathcal{K}_j}^2 \quad (2.11)$$

for all  $\xi \in \mathcal{H}$ . That is,  $T$  is an operator-valued frame for  $\mathcal{H}$  if and only if there are  $B, A > 0$  such that

$$A \|\xi\|^2 \leq \sum_j \|T_j \xi\|_{\mathcal{K}_j}^2 \leq B \|\xi\|^2$$

for all  $\xi \in \mathcal{H}$ . This concept is due to [48], although the terminology is due to [36]. If  $A = B$ , we say that  $T$  is a *tight OVF* for  $\mathcal{H}$ , and if  $A = B = 1$ , we say  $T$  is a *Parseval OVF*. We give now two examples: one which is investigated in the literature on frames and one which arises naturally from analysis on a compact group.

**Example 2.3.7.** (Time-frequency localization operators.) In this example we follow [24]. If  $f, g \in L^2(\mathbb{R}^d)$ , we define the windowed Fourier transform  $V_g$  to be

$$(V_g f)(t, \omega) = \int_{\mathbb{R}^d} f(x) \overline{g(x-t)} e^{-2\pi i \omega x} dx$$

We will use the notation  $\mathcal{S}_0(\mathbb{R}^d)$  to mean  $\{g \in L^2(\mathbb{R}^d) : V_g g \in L^1(\mathbb{R}^{2d})\}$ , the Feichtinger algebra. Let  $\phi$  be some function in  $S_0(\mathbb{R}^d)$ . Let  $\sigma$  be a bounded function on  $\mathbb{R}^{2d}$  with  $\sigma(x) \geq 0$  and compact support, and define the time-frequency localization operator  $H_\sigma$  corresponding to  $\sigma$  by  $H_\sigma f = V_\phi^* \sigma V_\phi f$ . Let  $\mathcal{K}_j = \overline{\text{range } H_{\sigma(\cdot - j)}} \subset \mathcal{H}$  for  $j \in \mathbb{Z}^{2d}$ . If  $\sigma \in \mathcal{S}_0(\mathbb{R}^{2d})$  and there are positive constants  $C_1, C_2$  such that

$$C_1 \leq \sum_{j \in \mathbb{Z}^{2d}} \sigma(\cdot - j) \leq C_2.$$

Then it is shown in [24] that  $\{H_{\sigma(\cdot - j)} : j \in \mathbb{Z}^{2d}\}$  is an operator-valued frame for  $L^2(\mathbb{R}^d)$ . That is, there are constants  $B, A > 0$  such that

$$A \|f\|_{L^2(\mathbb{R}^d)}^2 \leq \sum_{j \in \mathbb{Z}^{2d}} \|H_{\sigma(\cdot - j)} f\|_{\mathcal{K}_j}^2 \leq B \|f\|_{L^2(\mathbb{R}^d)}^2$$

for all  $f \in L^2(\mathbb{R}^d)$ .

**Example 2.3.8.** In this example we will use the treatment of the Peter-Weyl Theorem in [26, Chapter 5] to describe an OVF arising from analysis on a compact group  $G$ . By [26, Theorem 5.2], all irreducible representations of  $G$  are finite-dimensional. Fixing once and for all representatives  $(\pi_1, \mathcal{H}_{\pi_1}), (\pi_2, \mathcal{H}_{\pi_2}), \dots$  for the elements of  $\hat{G}$ , we may make the following definition (as made in *loc. cit.*):  $\hat{f}(\pi_j) = \int_G f(x)\pi_j(x^{-1}) dx$ . For each  $j$ , let  $d_j$  be the dimension of  $\pi_j$ . The above integral defines a member of  $\mathcal{L}(\mathcal{H}_{\pi_j})$ , which is in general not a one-dimensional vector space. If we define  $F_j f = \int_G f(x)\pi_j(x^{-1}) dx$ , we may view  $F_j$  as a map between Hilbert spaces by identifying each  $\mathcal{L}(\mathcal{H}_{\pi_j})$  with  $\mathcal{K}_j := L^2(\mathcal{H}_{\pi_j})$ . One interpretation of the Peter-Weyl theorem (as found in *loc. cit.*) tells us that

$$f(\cdot) = \sum_j \text{Tr}(\hat{f}(\pi_j)\pi_j(\cdot))d_j, \quad (2.12)$$

with convergence in  $L^2(G)$ . We will now put this into the context of operator-valued frames. Let  $T_j = \sqrt{d_j}F_j$ . We claim that  $F_j^* a = S_j a := \text{Tr}(a\pi_j(\cdot))$ , for  $a \in \mathcal{L}(\mathcal{H}_{\pi_j})$ . Indeed,

$$\begin{aligned} \langle F_j f, a \rangle_{\mathcal{K}_j} &= \int_G f(x) \langle \pi(x^{-1}), a \rangle_{\mathcal{K}_j} dx \\ &= \int_G f(x) \text{Tr}(a^* \pi(x^{-1})) dx \\ &= \int_G f(x) \text{Tr}(a^* \pi(x)^*) dx, \end{aligned}$$

whereas

$$\begin{aligned} \langle f, S_j a \rangle_{\mathcal{H}} &= \int_G f(x) \overline{\text{Tr}(a\pi_j(x))} dx \\ &= \int_G f(x) \text{Tr}(\pi_j(x)^* a^*) dx \\ &= \int_G f(x) \text{Tr}(a^* \pi_j(x)^*) dx. \end{aligned}$$

Thus,  $(T_j^* T_j f)(x) = \sqrt{d_j} \text{Tr}((T_j f) \pi_j(x)) = \text{Tr}(\hat{f}(\pi_j) \pi_j(x)) d_j$ . Thus, by Equation (2.12),

$$f = \sum_j T_j^* T_j f.$$

This means that  $T = \{T_j\} : \mathcal{H} \rightarrow \bigoplus_j \mathcal{K}_j$  is well-defined since

$$\begin{aligned} \sum_j \|T_j f\|_{\mathcal{K}_j}^2 &= \sum_j \langle T_j^* T_j f, f \rangle_{\mathcal{H}} \\ &= \left\langle \sum_j T_j^* T_j f, f \right\rangle_{\mathcal{H}} \\ &= \langle f, f \rangle_{\mathcal{H}} < \infty. \end{aligned}$$

The equality with  $\langle f, f \rangle_{\mathcal{H}}$  means that  $T$  is an OVF, as desired, with frame bounds in this case being  $A = B = 1$ . That is,  $T$  is a Parseval OVF. Moreover,  $T$  is orthogonal in the sense that  $T_k T_j^* = 0$  for  $k \neq j$ . Indeed, if  $a \in \mathcal{K}_j$ , then  $g = T_j^* a$  is defined by  $g(x) = \text{Tr}(a \pi_j(x))$  and thus is in the span of the matrix elements of the matrix-valued function  $x \mapsto \pi_j(x)$ . If  $k \neq j$ , then by the Schur Orthogonality Relations [26, 5.8],  $T_k$  applied to  $g$  is zero.

**Remark 2.3.9.** As noted in [36] an OVF is easily expanded into an ordinary frame. Indeed, if we are given an operator-valued frame  $\{T_j : j \in \mathbb{N}\}$ , with each  $T_j$  mapping into the Hilbert space  $\mathcal{K}_j$ , and an orthonormal basis  $\{e_{jk}\}_{k \geq 1}$  for each  $\mathcal{K}_j$ , it is easily seen that the set  $\{T_j^* e_{jk} : j, k \geq 1\}$  is an ordinary frame with the same frame bounds as  $\{T_j\}$ . So what is the point of working with OVFs rather than frames? One reason, to borrow a term from computer science, is that they provide some level of *procedural abstraction* over frames. That is, treating an OVF as an OVF and not a frame allows one to ignore how analysis is done on each of the  $\mathcal{K}_j$ 's and focus instead on the whole picture of how analysis is being done on  $\mathcal{H}$ . This makes it possible to avoid a choice of bases for the spaces  $\mathcal{K}_j$  when the operators  $T_j$  are already simply expressed, as in Example 2.3.7. Moreover, any such choice of bases may be somewhat arbitrary: for

example, in Example 2.3.8, the space  $\mathcal{K}_j$  corresponds to an irreducible representation  $\pi_j$  of  $G$ , so any proper decomposition of it must be a decomposition into subspaces of  $\mathcal{K}_j$  that are not  $\pi_j$ -invariant.

If  $T$  is an OVF, it is possible to iteratively reconstruct  $\xi$  from  $T\xi$  as in Section 2.3.1. This can be accomplished by simply setting  $S = R = T^*T$  in the algorithm (2.8), as before, and the same convergence rate applies.

Since reconstruction depends on  $R$ , it is natural to ask what form  $R$  takes, which depends on what form  $T^*$  takes. For both of these, the derivation is not appreciably different from the rank-one case, which as we have said is done in [19, Lemma 3.2.1], but for completeness we reproduce the details to encompass our more general situation.

**Proposition 2.3.10.** *Let  $T = \{T_j\}$  be an operator-valued Bessel sequence. Then if  $\eta = \{\eta_j\} \in \bigoplus_j \mathcal{K}_j$ , we have  $T^*\eta = \sum_j T_j^*\eta_j$ , with convergence in the weak topology on  $\mathcal{H}$ .*

*Proof.* Observe the following:

$$\begin{aligned} \langle T\xi, \eta \rangle_{\mathcal{K}} &= \langle \{T_j\xi\}, \{\eta_j\} \rangle_{\mathcal{K}} \\ &= \sum_j \langle T_j\xi, \eta_j \rangle_{\mathcal{K}_j} \\ &= \sum_j \langle \xi, T_j^*\eta_j \rangle_{\mathcal{H}}. \end{aligned}$$

Since also  $\langle T\xi, \eta \rangle_{\mathcal{K}} = \langle \xi, T^*\eta \rangle_{\mathcal{H}}$ , we have that  $\lim_{n \rightarrow \infty} \langle \xi, T^*\eta - \sum_{j=1}^n T_j^*\eta_j \rangle_{\mathcal{H}} = 0$ , which is precisely the statement that  $\sum_{j=1}^n T_j^*\eta_j$  tends weakly to  $T^*\eta$ .  $\square$

For  $R$ , then,  $T^*T\xi = T^*\{T_j\xi\} = \sum_j T_j^*T_j\xi$ , with convergence in the weak topology on  $\mathcal{H}$ . This means that for all  $\eta \in \mathcal{H}$ ,  $\langle T^*T\xi, \eta \rangle = \lim_{n \rightarrow \infty} \langle \sum_{j=1}^n T_j^*T_j\xi, \eta \rangle$ , which is precisely the same as saying that  $\sum_j T_j^*T_j$  converges in the weak-operator topology

to  $T^*T$ . But by the equivalence of WOT- and SOT-convergence for increasing nets of positive operators, we have that  $\sum_j T_j^*T_j$  converges to  $T^*T$  strongly. We have thus proved the following proposition.

**Proposition 2.3.11.** *Let  $T = \{T_j\}$  be an operator-valued Bessel sequence. Then  $\sum_j T_j^*T_j$  converges in the strong-operator topology to  $T^*T$ .*

Thus, we have an explicit way to calculate  $R\xi$ . In a similar observation to one made in Section 2.3.1, we may note that the frame bounds (2.11) and (2.10) are equivalent to  $A \|\xi\|^2 \leq \|T\xi\|_{\mathcal{K}}^2$  and  $\|T\xi\|_{\mathcal{K}}^2 \leq B \|\xi\|^2$ , respectively, which are equivalent to  $A\mathbb{I}_{\mathcal{H}} \leq T^*T$  and  $T^*T \leq B\mathbb{I}_{\mathcal{H}}$ , respectively. As before, we then have the following simple reconstruction formula when  $A = B$ :

$$\xi = \frac{1}{A}R\xi = \frac{1}{A} \sum_j T_j^*T_j\xi.$$

Further, in the case  $A \neq B$ , we have

$$\xi = R^{-1}R\xi = \sum_j R^{-1}T_j^*T_j\xi. \tag{2.13}$$



## Chapter 3

# OPERATOR-VALUED FRAMES ASSOCIATED WITH MEASURE SPACES AND POVMS

### 3.1 Introduction

In this chapter we describe two final levels of frame generalizations found in the literature and give the relationship between them. The first is the concept of an operator-valued frame associated with a measure space (MS-OVF), and the second is the concept of a positive operator valued measure (POVM). In Section 3.2, we develop MS-OVFs in much the same way as their introduction in Abdollahpour and Faroughi [2] does, with some elaboration. The extra details we provide are a brief summary of direct-integral theory and two examples of MS-OVFs, which are lacking in [2]. Then, in Section 3.3, we discuss the relationship between POVMs and MS-OVFs and give a characterization of MS-OVFs in terms of POVMs, extending the Radon-Nikodym theorems of Chiribella *et al.* [18] and Berezanskii and Kondratev [9] mentioned in Section 1.1. As in the last chapter,  $\mathcal{H}$  will denote a Hilbert space.

### 3.2 Operator-Valued Frames Associated with Measure Spaces

Many frames of interest arise from selecting a discrete subset of a “continuous frame,” or, frame associated with a measure space, in the terminology of [29]. That is, given some measure space  $(X, \Sigma)$  and family  $\{\psi_x : x \in X\} \subset \mathcal{H}$  with certain measurability requirements, a frame is obtained by selecting a discrete subset  $\{\psi_{x_1}, \psi_{x_2}, \dots\}$ . This is a process followed, for example, in wavelet and Gabor analysis [20]. Motivated by the relationship between frames and continuous frames, we consider in this section

an object which we call an *operator-valued frame associated with a measure space* or *MS-OVF*, which is the “continuous” analogue of an operator-valued frame. This concept was originally proposed by [2] under the term *continuous g-frame*, and we will largely follow their development, with some elaboration. As in the last section, we will indicate the form in which these objects arise in the literature and describe the analogues of the analysis, synthesis, and resolvent operators, and the analogue of the reconstruction formulas in (2.6), (2.7), and (2.8).

There are two equivalent common definitions of the direct integral of separable Hilbert spaces with respect to a measure  $\mu$ . For brevity we only present one. The other can be found for example in [12]. For our definition, we need the definition of a “measurable field of Hilbert spaces.”

**Definition 3.2.1.** [26, Chapter 7.4] Let  $(X, \Sigma)$  be a measurable space, let  $\{\mathcal{K}(x)\}_{x \in X}$  be separable Hilbert spaces, and let  $\tau_n \in \Pi_{x \in X} \mathcal{K}(x)$  ( $n = 1, 2, \dots$ ). We say that  $(\{\mathcal{K}(x)\}_{x \in X}, \{\tau_n\})$  (or  $\{\mathcal{K}(x)\}_{x \in X}$  for short) is a *measurable field of Hilbert spaces* if

1. for all  $x \in X$ ,  $\{\tau_n(x)\}_{n \in \mathbb{N}}$  is dense in  $\mathcal{K}(x)$ , and
2.  $x \mapsto \langle \tau_m(x), \tau_n(x) \rangle : X \rightarrow \mathbb{C}$  is measurable ( $m, n = 1, 2, \dots$ ).

Given a measurable field of Hilbert spaces  $\{\mathcal{K}(x)\}_{x \in X}$ , we say that an element  $\xi \in \Pi_{x \in X} \mathcal{K}(x)$  is a *measurable vector field* if  $x \mapsto \langle \xi(x), \tau_n(x) \rangle$  is measurable for all  $n$ . It is important to note that the map  $x \mapsto \langle \xi(x), \eta(x) \rangle$  is always measurable when  $\xi$  and  $\eta$  are measurable vector fields [26, Proposition 7.28]. Given a measure space  $(X, \Sigma, \mu)$ , the *direct integral* of the spaces  $\mathcal{K}(x)$  with respect to  $\mu$ , denoted  $\int_X^\oplus \mathcal{K}(x) d\mu(x) =: \mathcal{K}$ , is just the set of measurable vector fields  $\xi \in \Pi_{x \in X} \mathcal{K}(x)$  such that

$$\int_X \|\xi(x)\|_{\mathcal{K}(x)}^2 d\mu(x) < \infty,$$

equipped with the inner product

$$\langle \xi, \eta \rangle_{\mathcal{K}} = \int_X \langle \xi(x), \eta(x) \rangle_{\mathcal{K}(x)} d\mu(x),$$

modulo the null space of  $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ . As noted in [26],  $\mathcal{K}$  is actually complete with respect to  $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ , so that it is a Hilbert space.

Now we turn to defining an operator-valued Bessel field associated with a measure space (and, subsequently, an operator-valued frame associated with a measure space).

**Definition 3.2.2.** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(\{\mathcal{K}(x)\}_{x \in X}, \{\tau_n\})$  be a measurable field of Hilbert spaces, and let  $\mathcal{H}$  be a separable Hilbert space. Let  $T(x) : \mathcal{H} \rightarrow \mathcal{K}(x)$  be defined for  $\mu$ -a.e.  $x$ , and let  $T = \{T(x)\}_{x \in X}$ . We say that  $(X, \{\mathcal{K}(x)\}_{x \in X}, \{\tau_n\}, T, d\mu)$ —or simply  $(T, d\mu)$ , or  $T$ , if the other components are understood—is an *operator-valued Bessel field* if

1. for every  $\xi \in \mathcal{H}$ ,  $\{T(x)\xi\}_{x \in X} \in \Pi_{x \in X} \mathcal{K}(x)$  is a measurable vector field, and
2. for every  $\xi \in \mathcal{H}$ ,

$$\int_X \|T(x)\xi\|_{\mathcal{K}(x)}^2 d\mu(x) \leq B \|\xi\|^2. \quad (3.1)$$

Operator-valued Bessel sequences map  $\xi$  into a sequence of vectors whose norms are square-summable. The two items above say that operator-valued Bessel *fields* take  $\xi$  into a field of vectors whose norms are square-*integrable*. The measurability requirement is new because of the measure-theoretic nature of the situation at hand. The number  $B$  in the second condition is identical to the “upper frame bound” that we have seen twice before. Identifying the operator-valued Bessel field  $T$  with a linear map from  $\mathcal{H}$  into  $\Pi_{x \in X} \mathcal{K}(x)$  as before ( $\xi \mapsto \{T(x)\xi\}_{x \in X}$ ), these conditions are equivalent to requiring that  $T$  be a bounded linear map from  $\mathcal{H}$  into  $\int_X^{\oplus} \mathcal{K}(x) d\mu(x) =: \mathcal{K}$ . As before, we will often identify an operator-valued Bessel field  $T$  with this map.

If, in addition to 1. and 2., there is  $A > 0$  such that

$$A \|\xi\|^2 \leq \int_X \|T(x)\xi\|_{\mathcal{K}(x)}^2 d\mu(x), \quad (\xi \in \mathcal{H}) \quad (3.2)$$

we say that  $T$  is an *operator-valued frame associated with*  $(X, \mu)$ , or an *operator-valued frame associated with a measure space* if  $(X, \mu)$  is understood. For short, we will use the term *MS-OVF*. The following are two examples.

**Example 3.2.3.** (Fourier analysis on a connected semisimple Lie group.) Let  $\pi$  be a representation on a locally compact group  $G$  and  $f \in L^1(G)$ , then the weak integral  $\int_G f(x)\pi(x^{-1}) dx$  is easily seen to define a bounded sesquilinear form on  $\mathcal{H}_\pi \times \mathcal{H}_\pi$ , and we will denote this operator by  $\hat{f}(\pi)$ . We impose on  $\hat{G}$  the so-called Mackey-Borel measurable structure. Suppose that representatives  $\{(\pi_p, \mathcal{H}_p) : p \in \hat{G}\}$  of  $\hat{G}$  are chosen in such a way that  $\{\mathcal{H}_p\}$  is a measurable field of Hilbert spaces and for each measurable vector field  $p \mapsto \xi(p)$  in  $\Pi_p \mathcal{H}_p$  and each  $x \in G$ , the map  $p \mapsto \pi_p(x)\xi(p)$  is a measurable vector field. (This can be done by [26, Theorem 7.5] and [26, Lemma 7.39].) The Plancherel Theorem for  $G$  [26, Theorem 7.44] implies that if  $G$  is a unimodular, type I group, there is a measure  $\mu$  on  $\hat{G}$ , unique modulo positive scalars such that

1. the *Fourier transform*  $f \mapsto \hat{f}$  maps  $f \in L^1(G) \cap L^2(G)$  into  $\int^\oplus L^2(\mathcal{H}_p) d\mu(p)$ ,  
and

2. for  $f \in L^1(G) \cap L^2(G)$  one has the Parseval formula

$$\|f\|_{L^2(G)}^2 = \int \|\hat{f}(\pi_p)\|_{\text{HS}}^2 d\mu(p). \quad (3.3)$$

Let  $U$  be the map from  $L^2(G)$  to itself defined by  $f(x) \mapsto f(x^{-1})$ . Observing that  $U$  is unitary and replacing  $f$  with  $Uf$  in Equation 3.3 gives

$$\|f\|_{L^2(G)}^2 = \int \|\pi_p(f)\|_{\text{HS}}^2 d\mu(p).$$

In particular, this means that if  $E \subset G$  is open and relatively compact and  $L^2(E)$  is embedded naturally in  $L^2(G)$ , we have the same equality for all  $f \in L^2(E)$ . Thus, the family  $\{\pi_p : p \in \hat{G}\}$  is an MS-OVF associated with  $(\hat{G}, \mu)$  if we can show that  $\mu$ -almost every  $\pi_p$  is a bounded map from  $L^2(E)$  into a space of Hilbert-Schmidt class matrices. This is the case for connected semisimple Lie groups, which are known to be unimodular and type I. Suppose  $\pi \in \hat{G}$ . To prove that  $\pi$  is bounded, we use Harish-Chandra's regularity theorem, a reference for which is [6]. First, suppose that  $f \in C_E^\infty(G)$ . Using the notation  $f^*(x) = \overline{f(x^{-1})}$  and the fact that  $\pi$  is a \*-representation of  $L^1(G)$ , we have

$$\begin{aligned} \|\pi(f)\|_{\text{HS}}^2 &= \text{Tr}(\pi(f)^* \pi(f)) \\ &= \text{Tr}(\pi(f^* * f)). \end{aligned}$$

Harish-Chandra's regularity theorem states in particular that the map  $f \mapsto \text{Tr}(\pi(f))$  is a distribution and that it is given by  $\text{Tr}(\pi(f)) = \int f(y) \Theta_\pi(y) dy$  for some locally integrable function  $\Theta_\pi$ . Since  $f^* * f$  is in  $C_c^\infty(G)$  and supported on  $E^{-1}E$ , we have

$$\begin{aligned} \|\pi(f)\|_{\text{HS}}^2 &= \int_{E^{-1}E} (f^* * f)(y) \Theta_\pi(y) dy \\ &\leq \|f^* * f\|_{L^\infty(G)} \|\Theta_\pi\|_{L^1(E^{-1}E)} \\ &\leq \|f\|_{L^2(E)}^2 \|\Theta_\pi\|_{L^1(E^{-1}E)}. \end{aligned}$$

Thus,  $\pi$  is bounded on a dense subset of  $L^2(E)$ , and thus on all of  $L^2(E)$ .

**Example 3.2.4.** [19, Section 11.1] Let  $\mathcal{H} = L^2(\mathbb{R})$  and  $G$  be the “ $ax + b$ ” group:  $\mathbb{R}^+ \rtimes \mathbb{R}$ . Let  $X = G$  and  $\mu$  be the Haar measure on  $G$ :  $d\mu(a, b) = da db/a^2$ , where  $da$  and  $db$  denote Lebesgue measure. We say that  $\psi \in L^2(\mathbb{R})$  is *admissible* if

$$C_\psi := \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\gamma)|^2}{|\gamma|} d\gamma < \infty.$$

Let  $\psi$  be admissible. Finally, for each  $(a, b) \in X$ , let  $T(a, b) : \mathcal{H} \rightarrow \mathbb{C}$  be defined by

$$T(a, b)f = \int_{-\infty}^{\infty} f(y) \frac{1}{|a|^{1/2}} \overline{\psi\left(\frac{y-b}{a}\right)} dy,$$

where  $dy$  again denotes Lebesgue measure. It can be shown [19, Proposition 11.1.1], that for all  $f \in \mathcal{H}$ ,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |T(a, b)f|^2 d\mu(a, b) = C_\psi \|f\|^2.$$

Thus, if  $\mathcal{K}(x) = \mathbb{C}$  for each  $x$ , and if we enumerate the rationals by  $\{\rho_n\}$  and define  $\tau_n \in \Pi_{x \in X} \mathcal{K}(x)$  to be a constant function identically equal to  $\rho_n$ , we have that  $(X, \{\mathcal{K}(x)\}_{x \in X}, \{\tau_n\}, T, d\mu)$  is a tight (rank-one) operator-valued frame associated with  $(\mathbb{R}^2, d\mu)$ .

For the remainder of this chapter, the tuple  $(X, \{\mathcal{K}(x)\}_{x \in X}, \{\tau_n\}, T, d\mu)$  will denote an operator-valued Bessel field, and  $\mathcal{K}$  will denote  $\int^\oplus \mathcal{K}(x) d\mu(x)$ . As before, we define the *synthesis operator* of  $T$  to be  $T^*$  and the *resolvent* to be  $R = T^*T$ . If we wish to reconstruct  $\xi$  from knowledge of  $T$  and  $T\xi$ , we may again use the formula  $\xi = R^{-1}R\xi$ , which can again be calculated using the frame algorithm (2.8). Also, if  $A = B$ , then  $R = A\mathbb{I}_{\mathcal{H}}$ , so that  $\xi = \frac{1}{A}R\xi$ . Thus, we are interested again in  $R$  and  $T^*$ .

**Proposition 3.2.5.** *If  $\eta \in \mathcal{K}$ , then we have  $T^*\eta = \int_X T(x)^*\eta(x) d\mu(x)$  in the sense that for all  $\xi \in \mathcal{H}$ ,  $\langle \xi, T^*\eta \rangle_{\mathcal{H}} = \int_X \langle \xi, T(x)^*\eta(x) \rangle_{\mathcal{H}} d\mu(x)$ .*

*Proof.* Observe that since  $x \mapsto \langle T(x)\xi, \eta(x) \rangle_{\mathcal{K}(x)}$  is absolutely integrable, the same is true of  $x \mapsto \langle \xi, T(x)^*\eta(x) \rangle_{\mathcal{H}}$ . Thus,

$$\begin{aligned} \langle \xi, T^*\eta \rangle_{\mathcal{H}} &= \langle T\xi, \eta \rangle_{\mathcal{K}} \\ &= \int_X \langle T(x)\xi, \eta(x) \rangle_{\mathcal{K}(x)} d\mu(x) \\ &= \int_X \langle \xi, T(x)^*\eta(x) \rangle_{\mathcal{H}} d\mu(x), \end{aligned}$$

□

From this follows an easy corollary identifying  $R$ .

**Proposition 3.2.6.** *Let  $T$  be as above. Then we have*

$$R = T^*T = \int_X T(x)^*T(x) d\mu(x).$$

*Proof.* Let  $\xi_1, \xi_2 \in \mathcal{H}$ . Let  $\eta \in \mathcal{K}$  be defined by  $\eta(x) = T(x)\xi_2$ . Then, by this definition and the preceding proposition, we have

$$\begin{aligned} \langle \xi_1, T^*T\xi_2 \rangle_{\mathcal{H}} &= \langle \xi_1, T^*\eta \rangle_{\mathcal{H}} \\ &= \int_X \langle \xi_1, T(x)^*\eta(x) \rangle_{\mathcal{H}} d\mu(x) \\ &= \int_X \langle \xi_1, T(x)^*T(x)\xi_2 \rangle_{\mathcal{H}} d\mu(x). \end{aligned}$$

□

### 3.3 Positive Operator-Valued Measures

Given an MS-OVF  $(X, \{\mathcal{K}(x)\}_{x \in X}, \{\tau_n\}, T, d\mu)$ , it is often of interest to study *partial resolvents* of a vector  $\xi \in \mathcal{H}$ . By definition, we take these to be vectors of the form  $\int_E T(x)^*T(x)\xi d\mu(x)$  for  $E \in \Sigma$ . As shown in Example 2.3.3 and Example 2.3.6, these partial resolvents may converge quickly to  $\xi$  as the set  $E$  increases in a uniform way to  $X$ . In order to study these partial resolvents, it is of use to consider the partial resolvents of the frame operator itself: *i.e.*, operators of the form  $M_T(E) := \int_E T(x)^*T(x) d\mu(x)$  for  $E \in \Sigma$ . The set function  $E \in \Sigma \mapsto M_T(E) \in \mathcal{L}(\mathcal{H})$  has the special property that it is  $\sigma$ -additive with convergence in the weak operator topology. Indeed, for  $\xi \in \mathcal{H}$  and pairwise disjoint members of  $\Sigma$  called  $E_1, E_2, \dots$ , we have by monotone convergence

$$\langle M_T(\cup_{j=1}^{\infty} E_j) \xi, \xi \rangle = \sum_{j=1}^{\infty} \langle M_T(E_j) \xi, \xi \rangle.$$

(WOT convergence of  $\sum_j M_T(E_j)$  follows from polarization.) Since the operators  $M_T(E_j)$  are positive, sums of the form  $\sum_j M_T(E_j)$  are also SOT convergent, meaning that partial resolvents  $\sum_{j=1}^N M_T(E_j)\xi$  of a vector  $\xi$  converge in norm to  $M_T(X)\xi$ .

The map  $E \mapsto M_T(E)$  is an instance of an object with a special name in mathematical physics called a *positive operator-valued measure* (POVM). The general definition follows.

**Definition 3.3.1.** (As in [39].) Let  $(X, \Sigma)$  be a measurable space. If  $M : \Sigma \rightarrow \mathcal{L}^+(\mathcal{H})$ , then we will say that  $(X, \Sigma, M)$ , or simply  $M$ , is a *positive operator-valued measure* if

1.  $M(\emptyset) = 0$ , and
2. if  $E_1, E_2, \dots \in \Sigma$  are disjoint, then  $M(\cup_j E_j) = \sum_j M(E_j)$ , with the sum converging in the weak operator topology.

The case of most interest to us is the case where there is an  $A > 0$  such that  $A\mathbb{1}_{\mathcal{H}} \leq M(X)$ . In this case, we will say, as in [39], that  $M$  is a *framed POVM*, which is a general way of performing analysis on  $\mathcal{H}$  in the following sense. First, the convergence property of  $M$  implies norm convergence of  $\sum_j M(E_j)\xi$  for pairwise disjoint  $E_1, E_2, \dots \in \Sigma$  and  $\xi \in \mathcal{H}$ . Thus, any vector  $\xi$  may be expressed as the norm-convergent expansion  $M(X)^{-1}M(X)\xi = \sum_j M(X)^{-1}M(E_j)\xi$  for any pairwise disjoint  $E_j$ 's whose union is  $X$ . If  $M = M_T$  for some OVF  $T = \{T_j\}$ , then this formula is just (2.13), and we can think of  $\xi$  as being represented by the sequence  $\{M_T(\{j\})\xi : j = 1, 2, \dots\}$  instead of the sequence  $\{T_j\xi\}$ . As we will see in Remark 4.4.4 and Remark 4.4.8, avoiding the latter sequences in favor of the former sometimes offers an improvement in notational simplicity.

In the discrete case, given an OVF  $\{T_j\}$ , the POVM  $M_T$  is defined by  $E \in \mathcal{P}(\mathbb{N}) \mapsto \sum_{j \in E} T_j^* T_j$ . Further, every framed POVM on  $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$  arises in this way: given a



framed POVM  $M$ , just take  $T_j = \sqrt{M(\{j\})}$ . Thus, OVFs and framed POVMs on  $\mathbb{N}$  are in some sense equivalent. Given an arbitrary measurable space  $(X, \Sigma)$ , however, such an equivalence may not in general hold. Every MS-OVF  $T$  associated with with a  $\sigma$ -finite measure  $\mu$  will give rise to a framed POVM via  $T \mapsto M_T$ , as above, but it may be that not every framed POVM  $M$  arises from an MS-OVF associated with some  $\sigma$ -finite measure. In the remainder of this section, we give a necessary and sufficient condition for when  $M$  does.

The key question in this investigation is whether  $M$  is *decomposable*, meaning that there is an integral decomposition

$$M(E) = \int_E Q(x) d\mu(x)$$

for some  $\sigma$ -finite measure  $\mu$  and weakly  $\mu$ -measurable function  $Q : X \rightarrow \mathcal{L}^+(\mathcal{H})$ . Given such a function,  $M$  arises from the maps  $T(x) = \sqrt{Q(x)} : \mathcal{H} \rightarrow \overline{\text{range } Q(x)}$ . If  $\{\xi_n\}$  is an enumeration of the rational span of an orthonormal basis in  $\mathcal{H}$  and  $\mathcal{K}(x) = \overline{\text{range } Q(x)}$ , then the sequence  $\{\tau_n\} \subset \prod_{x \in X} \mathcal{K}(x)$  given by  $\tau_n(x) = \sqrt{Q(x)}\xi_n$  is as in Definition 3.2.1, so  $\{\mathcal{K}(x)\}_{x \in X}$  is a measurable field of Hilbert spaces. The family  $\{T(x)\}_{x \in X}$  is then an operator-valued Bessel sequence because for  $\xi \in \mathcal{H}$ ,

$$\begin{aligned} \langle T(x)\xi, \tau_n(x) \rangle &= \left\langle \sqrt{Q(x)}\xi, \tau_n(x) \right\rangle \\ &= \left\langle \sqrt{Q(x)}\xi, \tau_n(x) \right\rangle \\ &= \langle Q(x)\xi, \xi_n \rangle, \end{aligned}$$

a function which is measurable for all  $n$ . Another field of maps giving rise to  $M$  can be obtained by post-composing each  $T(x)$  with a unitary  $U(x)$  where  $\{U(x)\}_{x \in X}$  is a *measurable field of operators*. By definition, this means that for every measurable  $\eta \in \prod_{x \in X} \mathcal{K}(x)$ , the vector field  $\{U(x)\eta(x)\}_{x \in X}$  is also measurable. For the question of frame bounds for  $T$ , they are  $A$  and  $B$  if and only if  $A\mathbb{I}_{\mathcal{H}} \leq M(X) \leq B\mathbb{I}_{\mathcal{H}}$ .

The following is a simple criterion for decomposability.

**Theorem 3.3.2.** *Let  $(X, \Sigma, M)$  be a POVM and suppose there is a  $\sigma$ -finite measure  $\mu$  on  $(X, \Sigma)$  such that*

$$\|M(E)\| \leq \mu(E)$$

*for all  $E \in \Sigma$ . Then there exists a weakly measurable map  $Q : X \rightarrow \mathcal{L}^+(\mathcal{H})$ , defined on a set of full  $\mu$ -measure, with  $\langle Q(x)\xi, \xi \rangle \geq 0$  for every  $\xi \in \mathcal{H}$  and  $\mu$ -a.e.  $x$ , such that*

$$M(E) = \int_E Q(x) d\mu(x).$$

*Proof.* Let  $\mu_{\xi, \eta}(E) = \langle M(E)\xi, \eta \rangle$  for each  $\xi, \eta \in \mathcal{H}$ . Since  $|\mu_{\xi, \eta}(E)| \leq \mu(E) \|\xi\| \|\eta\|$ ,  $\mu_{\xi, \eta}$  is a complex measure for each  $\xi$  and  $\eta$ . By the Radon-Nikodym theorem, there therefore exists for each  $\xi, \eta \in \mathcal{H}$  a  $\mu$ -integrable function  $q(\cdot; \xi, \eta) : X \rightarrow \mathbb{C}$ , defined on a set of full  $\mu$ -measure, such that

$$\mu_{\xi, \eta}(E) = \int_E q(x; \xi, \eta) d\mu(x)$$

When  $\xi = \eta$ ,  $q(\cdot; \xi, \eta)$  is without loss of generality positive where defined.

Let  $\{e_j\}_{j=1}^{\infty}$  be an orthonormal basis of  $\mathcal{H}$ . Using sesquilinearity of  $(\xi, \eta) \in \mathcal{H} \times \mathcal{H} \mapsto \mu_{\xi, \eta}(E)$  and uniqueness of Radon-Nikodym derivatives, if  $a, b, c$ , and  $d$  are rational complex numbers and  $\xi, \xi', \eta$ , and  $\eta'$  are in the finite rational complex span  $\mathcal{M}$  of  $\{e_j\}$ . then there is a set

$$E_{a, b, c, d, \xi, \xi', \eta, \eta'}$$

of full  $\mu$ -measure such that for  $x$  in this set

$$\begin{aligned} q(x; a\xi + b\xi', c\eta + d\eta') &= a\bar{c}q(x; \xi, \eta) + b\bar{c}q(x; \xi', \eta) \\ &\quad + a\bar{d}q(x; \xi, \eta') + b\bar{d}q(x; \xi', \eta') \end{aligned}$$

Let  $X_0$  be the following intersection of such sets over all  $a, b, c, d \in \mathbb{Q} + i\mathbb{Q}$  and all  $\xi, \xi', \eta, \eta' \in \mathcal{M}$ :

$$X_0 = \cap E_{a,b,c,d,\xi,\xi',\eta,\eta'}$$

Since each of these sets has full  $\mu$ -measure, so does  $X_0$ .

For each  $x \in X_0$ ,  $(\xi, \eta) \in \mathcal{M} \times \mathcal{M} \mapsto q(x; \xi, \eta)$  defines a positive semidefinite sesquilinear form on  $\mathcal{M}$ . By Cauchy-Schwarz, then

$$|q(x; \xi, \eta)| \leq q(x; \xi, \xi)^{1/2} q(x; \eta, \eta)^{1/2}$$

for all  $\xi, \eta \in \mathcal{M}$ . We may now observe the following integral inequalities:

$$\begin{aligned} \int_E |q(x; \xi, \eta)| d\mu(x) &\leq \int_E q(x; \xi, \xi)^{1/2} q(x; \eta, \eta)^{1/2} d\mu(x) \\ &\leq \left( \int_E q(x; \xi, \xi) d\mu(x) \right)^{1/2} \left( \int_E q(x; \eta, \eta) d\mu(x) \right)^{1/2} \\ &= \langle M(E)\xi, \xi \rangle^{1/2} \langle M(E)\eta, \eta \rangle^{1/2} \\ &\leq \|M(E)\| \|\xi\| \|\eta\| \\ &\leq \|\xi\| \|\eta\| \mu(E) \end{aligned}$$

Thus, there is a set  $F_{\xi, \eta}$  of full  $\mu$ -measure such that  $|q(x; \xi, \eta)| \leq \|\xi\| \|\eta\|$  for  $x \in F_{\xi, \eta}$ .

Letting  $X_1$  be the intersection of the sets  $F_{\xi, \eta}$  as  $\xi, \eta$  range over  $\mathcal{M}$ ,  $X_1$  must have full  $\mu$ -measure. Thus, so does  $X_2 = X_1 \cap X_0$ .

For  $x \in X_2$ ,  $q(x; \cdot, \cdot)$  is a positive semidefinite bounded sesquilinear form. Henceforth, we will assume  $x$  belongs to this set. It is possible to extend the definition of  $q(x; \cdot, \cdot)$  to  $(\xi, \eta) \in \mathcal{H} \times \mathcal{H}$  using any sequences  $\{\xi_n\}, \{\eta_n\}$  in  $\mathcal{M}$  converging to  $\xi, \eta$ :

$$q(x; \xi, \eta) \equiv \lim_{n \rightarrow \infty} q(x; \xi_n, \eta_n)$$

This limit converges and is independent of the sequences  $\{\xi_n\}$  and  $\{\eta_n\}$  by boundedness on  $\mathcal{M} \times \mathcal{M}$ . The extended form  $q(x; \cdot, \cdot)$  is also positive semidefinite, bounded,

and sesquilinear because it has these properties on the dense subset  $\mathcal{M} \times \mathcal{M}$  of  $\mathcal{H} \times \mathcal{H}$ . Thus, for  $\mu$ -a.e.  $x$ , there is an operator  $Q(x) \in \mathcal{L}^+(\mathcal{H})$  with  $\langle Q(x)\xi, \eta \rangle = q(x; \xi, \eta)$ . Thus,

$$\langle M(E)\xi, \eta \rangle = \int_E \langle Q(x)\xi, \eta \rangle d\mu(x)$$

as desired. □

The above criterion is also necessary. To see this, suppose  $(X, \Sigma, M)$  is decomposable into  $Q : X \rightarrow \mathcal{L}(\mathcal{H})$  and a  $\sigma$ -finite measure  $\mu$  on  $(X, \Sigma)$ . Let  $E_n$  be a partition of  $X$  into  $\mu$ -finite sets and  $F_n = \{x \in X : n \leq \|Q(x)\| < n + 1\}$ . Then, the measure  $\nu$  defined by  $\nu(E) = \int_E \|Q(x)\| d\mu(x)$  is finite on the countable collection of sets  $E_i \cap F_j$ , and  $\|M(E)\| \leq \nu(E)$ .

One simple example of the above Theorem is when  $X = \mathbb{N}$ . It is clear that  $M$  is then decomposable, and a measure  $\mu$  which dominates  $\|M(\cdot)\|$  is given by  $\mu(\{j\}) = \|M(\{j\})\|$ . Another example of the Theorem is the case when there is a countable collection of subsets  $E_1, E_2, \dots$  of  $X$  such that  $X = \bigcup_{j=1}^{\infty} E_j$  and  $M$  is trace-class on each  $E_j$ . In this case, it is easily checked that the set function  $\mu$  given by  $\mu(E) = \text{Tr}(M(E))$  is  $\sigma$ -additive and dominates  $\|M(\cdot)\|$ . A proof of this for the special case that  $\text{Tr}(M(X)) < \infty$  that is similar to the proof of the above Theorem can be found in Berezanskii and Kondratev [9]. In another particular case, decomposability is satisfied if  $\mathcal{H}$  is finite-dimensional, as observed in Chiribella *et al.* [18].

We make here a couple of final remarks. First, as can be seen from the above arguments, for any decomposable POVM  $M$ , we may assume that the measure  $\mu$  satisfies  $\|M(E)\| \leq \mu(E)$  and the operator-valued function  $Q : X \rightarrow \mathcal{L}(\mathcal{H})$  satisfies  $\|Q(x)\| \leq 1$ . Second, in the case that  $M$  is decomposable, our decomposition implies a well-known one called the Naimark decomposition, which states that any POVM

can be represented as a map  $V$  from  $\mathcal{H}$  into another separable Hilbert space  $\mathcal{K}$ , a projection-valued measure  $S : \Sigma \rightarrow \mathcal{L}(\mathcal{K})$ , and the adjoint of  $V$ . That is, one can write  $M(E) = V^*S(E)V$  for every  $E \in \Sigma$ . Indeed, first take  $V = T : \mathcal{H} \rightarrow \mathcal{K}$ , as defined in Section 3.3. Since  $\mathcal{K} = \int_X^\oplus \mathcal{K}(x) d\mu(x)$ , we may define  $(S(E)\eta)(x)$  for  $\eta \in \mathcal{K}$  to be  $\chi_E(x)\eta(x)$ . From these definitions it follows easily that  $V^*S(E)V = \int_E Q(x) d\mu(x)$ .

### 3.4 Conclusion and Future Work

In this chapter, we have discussed MS-OVFs and some of their basic properties. Here, we have preferred to use the term “operator-valued frame associated with a measure space” in place of the term “continuous g-frame” found in some papers, to achieve consistency with the terms “operator-valued frame” and “frame associated with a measure space” that are used elsewhere.

In Section 3.2, we followed the discussion of Abdollahpour and Faroughi [2] in deriving the analysis, synthesis, and resolvent operators of an MS-OVF. We also filled in some details about direct integrals of Hilbert spaces, stated the applicability of the frame algorithm to MS-OVFs, and filled the void of examples of MS-OVFs in [2] with Example 3.2.3 and Example 3.2.4.

In Section 3.3, we discussed the relationship between MS-OVFs and framed POVMs. We found that every MS-OVF  $(T, d\mu)$  corresponds to a natural framed POVM  $M_T$ , and that every framed POVM  $M$  dominated by a measure  $\mu$  arises as  $M_T$  for some MS-OVF  $(T, d\mu)$ . For the latter result, we proved an operator-valued version of the Radon-Nikodym theorem, which is an extension of the results of Berezanskii and Kondratev [9] and Chiribella *et al.* [18] and is of interest in its own right. In particular, this theorem may have some application in the domain of quantum information theory, where the more specialized Radon-Nikodym theorem of [18], is evidently of some interest.

Given the close relationship between MS-OVFs and POVMs, it would be natural to try to find MS-OVFs with certain properties (symmetry, tightness, robustness against erasures, etc.) by drawing upon the rich set of examples of POVMs from the quantum-physics literature. Examples recently explored that correspond to MS-OVFs, with references, include *clean* POVMs [14], *symmetric, informationally-complete* POVMs [42], and *covariant* POVMs [17]. (For these and most other POVMs in the physics literature  $M(X) = \mathbb{I}_{\mathcal{H}}$ .) Although we have not discussed it here, a fruitful vein for future work might be to investigate whether these MS-OVFs or others from physics have any desirable properties with respect to the analysis of classical signals, which is the usual domain of frame theory.

Besides these examples, another part of the theory of POVMs that may have some impact on frame theory is its locally convex structure. The set  $\mathcal{M}(X, \mathcal{H}, 1)$  of all framed POVMs on  $X$  with  $A = B = 1$ , is convex and, given the appropriate topology, compact. By the Krein-Milman theorem, then, it is the closed convex hull of its set of extreme points  $\partial\mathcal{M}(X, \mathcal{H}, 1)$ . To be more explicit, the Choquet-Bishop-DeLeeuw theorem [11] expresses each  $M \in \mathcal{M}(X, \mathcal{H}, 1)$  as a weak superposition of extreme points using a probability measure  $\nu_M$  on  $\partial\mathcal{M}(X, \mathcal{H}, 1)$ :

$$\langle M(E)\xi, \eta \rangle = \int_{\partial\mathcal{M}(X, \mathcal{H}, 1)} \langle N(E)\xi, \eta \rangle d\nu_M(N)$$

for  $\xi, \eta \in \mathcal{H}$ . It is therefore of some interest to classify  $\partial\mathcal{M}(X, \mathcal{H}, 1)$ . For locally compact Hausdorff spaces  $X$  and finite-dimensional Hilbert spaces  $\mathcal{H}$ , a complete classification follows from a result of Arveson in [5] about completely positive maps. Noting that it is possible to remove the topological requirement on  $X$ , we now state this classification:

**Theorem 3.4.1.** *Suppose  $\mathcal{H}$  is a finite-dimensional Hilbert space. Then  $M$  is extremal in  $\mathcal{M}(X, \mathcal{H}, 1)$  if and only if there exist  $x_1, x_2, \dots, x_n \in X$  such that*

1.  $M(X \setminus \{x_1, x_2, \dots, x_n\}) = 0$ , and
2.  $M(\{x_1\})\mathcal{H}, M(\{x_2\})\mathcal{H}, \dots, M(\{x_n\})\mathcal{H}$  are weakly independent, meaning their tensor-squares in  $\mathcal{H} \otimes \mathcal{H}$  are linearly independent.

Since we are also interested in this chapter in POVMs that have frame bounds  $A$  and  $B$  not both equal to 1, we have obtained a preliminary result extending the one above to the set  $\mathcal{M}(X, \mathcal{H}, A, B)$  of all POVMs on  $X$  framed by constant numbers  $A$  and  $B$ , which is convex and compact in the same topology that was applied to  $\mathcal{M}(X, \mathcal{H}, 1)$ . The statement of the result is below, with  $\tilde{\mathcal{H}}$  being defined as the direct sum of the eigenspaces corresponding to eigenvalues of  $M(X)$  that are neither minimal nor maximal.

**Theorem 3.4.2.** *Suppose  $\mathcal{H}$  is a finite-dimensional Hilbert space. Then  $M$  is extremal in  $\mathcal{M}(X, \mathcal{H}, A, B)$  if and only if there exist  $x_1, x_2, \dots, x_n \in X$  such that*

1.  $M(X \setminus \{x_1, x_2, \dots, x_n\}) = 0$ , and
2.  $M(\{x_1\})\mathcal{H}, M(\{x_2\})\mathcal{H}, \dots, M(\{x_n\})\mathcal{H}$ , and  $\tilde{\mathcal{H}}$  are weakly independent.

The extreme points  $M$  in this extension vanish off a finite set of points in  $X$ , as do Arveson's, so the corresponding Choquet decompositions involve similar extreme points. We expect this result will have an impact on frame theory, provided Arveson's does, but we felt that it needed more development before being included in the main body of this thesis.

## Chapter 4

### OPERATOR-VALUED FRAMES OF REPRESENTATIONS

#### 4.1 Introduction

In this Chapter we explore the subject of extending the work of Duffin and Schaffer [25] and Beurling [10] to a general connected Lie group  $G$ . In analogy with the case of frames of exponentials for  $L^2(-1/2, 1/2)$ , our goal is to obtain OVF's involving representations of  $G$  for the  $L^2$ -space of a relatively compact subset of  $G$ . Because of their dependence on representations of  $G$ , these OVF's will be called *OVF's of representations*. Unlike the well-known continuous decompositions provided by the Plancherel Theorem, these decompositions will be discrete, and for the two groups we consider—the Euclidean motion group for  $\mathbb{R}^2$  and the Heisenberg group—these OVF's are apparently new.

For motivation, consider the case of frames of exponentials in  $\mathbb{R}^d$ . Letting  $Q_d = (-1/2, 1/2)^d$ , it is easily seen that  $F(\mathbb{Z}^d)$  is a Parseval frame for the  $L^2$ -space of any measurable set  $E \subset Q_d$ . Indeed, the inner products on  $L^2(E)$  and  $L^2(Q_d)$  coincide and  $F(\mathbb{Z}^d)$  is an orthonormal basis for the latter. If  $E$  is open, a rather more circuitous idea for proving this Parseval condition, but one that will nonetheless be useful later, is to first use the Poisson Summation Formula [26, Theorem 8.32] to prove it for all  $f \in C_E^\infty(G)$ . This formula states that

$$\sum_{k \in \mathbb{Z}^d} f(x + k) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{2\pi i n \cdot x},$$

with pointwise convergence for all  $x \in E$ . Since the support of  $f$  is contained in  $E$ ,



all of the terms corresponding to  $k \neq 0$  vanish:

$$f(x) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{2\pi i n \cdot x}.$$

Since the right-hand side converges in  $L^2(E)$ ,  $f$  satisfies Parseval's equality:

$$\|f\|_{L^2(E)}^2 = \sum_{n \in \mathbb{Z}^d} |\hat{f}(n)|^2.$$

Finally, since this equality holds for all  $f$  in a dense subspace of  $L^2(E)$ , it holds for all  $f \in L^2(E)$ .

If  $\Lambda$  is a non-degenerate lattice in  $\mathbb{R}^d$ , it follows more generally that the family  $F(\Lambda)$  is a tight frame for the  $L^2$ -space of some non-empty open subset of  $\mathbb{R}^d$ . This can be shown using a kind of generalized Poisson Summation Formula, but we will simply re-use the Parseval condition obtained above. Let  $\Lambda = T\mathbb{Z}^d$ , where  $T$  is an invertible operator from  $\mathbb{R}^d$  to  $\mathbb{R}^d$ , let  $E$  be an open set such that  $(T^\top)^{-1} E \subset Q_d$ , and let  $f \in L^2(E)$ . For  $n \in \mathbb{Z}^d$ , denote the element  $Tn$  of  $\Lambda$  by  $\lambda_n$ . We compute the inner product of  $f$  with an element of  $F(\Lambda)$ :

$$\begin{aligned} \langle f, e_{\lambda_n} \rangle_{L^2} &= \int f(x) e^{-2\pi i \langle \lambda_n, x \rangle} dx \\ &= \int f(x) e^{-2\pi i \langle Tn, x \rangle} dx \\ &= |\det T|^{-1} \int f((T^{-1})^\top x) e^{-2\pi i \langle Tn, (T^{-1})^\top x \rangle} dx \\ &= |\det T|^{-1} \int f((T^{-1})^\top x) e^{-2\pi i \langle n, x \rangle} dx. \end{aligned}$$

Thus, the sum over  $n$  of the terms  $|\langle f, e_{\lambda_n} \rangle|^2$ , is

$$|\det T|^{-2} \int |f((T^{-1})^\top x)|^2 dx,$$

which is just  $|\det T|^{-1} \|f\|_{L^2(E)}^2$ . This proves that  $F(\Lambda)$  is a tight frame for  $L^2(E)$  with frame bound  $|\det T|^{-1}$ .

It follows from the theory of frames of exponentials that  $F(\Lambda')$  can be a frame even if  $\Lambda'$  is not a lattice, as long as it is near  $\Lambda$ . Indeed, consider the family  $F(\Lambda + \epsilon)$ , where  $\epsilon$  is a sequence  $\{\epsilon_n : n \in \mathbb{Z}^d\}$  with each  $\epsilon_n \in \mathbb{R}^d$ . Although it is not strictly necessary, we restrict here to the case where  $(T^\top)^{-1}E$  is a subset of the unit ball  $B_d$ . In this case, if the elements of  $T^{-1}\epsilon$  are sufficiently small in norm, the result of Beurling [10] guarantees that  $F(\mathbb{Z}^d + T^{-1}\epsilon)$  is a frame for  $L^2(B_d)$ , and therefore for  $L^2((T^\top)^{-1}E)$ . By the above inner product computation, then,  $F(\Lambda + \epsilon)$  is, like  $F(\Lambda)$ , a frame for  $L^2(E)$ .

As we have mentioned, the focus of this chapter is on finding analogues of frames of exponentials for a connected Lie group  $G$ . For the abelian case, simply note that the above analysis does not change if  $G$  is a cartesian product of  $\mathbb{R}^d$  with a compact abelian group. By the analysis of Example 2.3.4,  $\{e_{\lambda_n + \epsilon_n} \otimes \kappa_l\}$  is a frame for  $L^2(E \times K)$  if  $\{e_{\lambda_n + \epsilon_n}\}$  is a frame for  $L^2(E)$  and  $\{\kappa_l\}$  is a list of all characters of  $K$ . Thus, in the abelian case, it is possible to find frames of characters with respect to any lattice or perturbed lattice in the Euclidean part of  $G$ .

In the remainder of this chapter, we will focus on the case when  $G$  is nonabelian. First, in Section 4.3, the idea of a frame of exponentials for which  $\Lambda$  is a lattice is extended, using a generalization of the Poisson Summation Formula called the Selberg Trace Formula, to groups  $G$  admitting a *uniform lattice* and a relatively compact *reproducing set*  $E \subset G$ . This construction results in a Parseval OVF of representations for  $L^2(E)$ . Then, in Section 4.4, the two examples of the Euclidean motion group for  $\mathbb{R}^2$  and the Heisenberg group are considered. Both are shown to admit Parseval OVFs of representations, and it is shown that under perturbations in the parameters of the included representations, these Parseval OVFs continue to behave as OVFs. These results resemble very closely Duffin and Schaeffer [25, Lemma III], a result which is generalized in Lemma 4.4.2 and used in the proofs. For each

OVF obtained in Section 4.4, it is also shown that decomposition with respect to the corresponding POVM takes a particularly simple form.

## 4.2 Quotients of $G$ and the Selberg Trace Formula

In this section we give some background on quotients of  $G$ , invariant measures, and the Selberg Trace Formula. A reference for invariant measures is [26], and two references for the latter are [22] and [4].

In analogy with the existence of Haar measure on  $G$ , it is sometimes possible to find an invariant measure on the quotient space of  $G$  by a closed subgroup  $H$ : *i.e.*, a nonzero measures  $\mu$  for which  $\mu(xyH) = \mu(yH)$ ,  $x, y \in G$ . If we denote by  $d\xi$  a left-invariant Haar measure on  $H$ , an important ingredient in the theory of such measures is the map  $P_H : C_c(G) \rightarrow C_c(G/H)$  given by  $f \mapsto \int_H f(x\xi) d\xi$ . That images under this map depend only on the coset in which  $x$  lives follows easily from left-invariance of  $d\xi$ , and continuity and compactness of support are also easily checked. The following theorem, states when measures of the above type exist.

**Theorem 4.2.1.** *(As stated in [26, Theorem 2.49].) Suppose  $H$  is a closed subgroup of  $G$ . There is a left  $G$ -invariant Radon measure  $\mu$  on  $G/H$  if and only if  $\Delta_G|_H = \Delta_H$ . In this case,  $\mu$  is unique up to a constant factor, and if this factor is suitably chosen we have*

$$\int_G f(x) dx = \int_{G/H} P_H f(xH) d\mu(xH) \quad (4.1)$$

for all  $f \in C_c(G)$ .

In this case,  $L^2(G/H)$  is defined to be  $L^2(G/H, d\mu)$ . Like the Haar measure,  $d\mu(xH)$  is sometimes denoted simply as  $dx$ .

**Remark 4.2.2.** If  $G/H$  admits such a measure, then as noted in [41, Chapter 8], the map  $P_H$  extends to a map from  $L^1(G)$  to  $L^1(G/H)$  and (4.1) holds for all  $f \in L^1(G)$ .

There is one more basic fact about quotients of  $G$  of which we make use:

**Proposition 4.2.3.** [26, Lemma 2.46] *If  $H$  is a closed subgroup of  $G$  and  $q : G \rightarrow G/H$  is the canonical quotient map and  $E$  is a compact subset of  $G/H$ , then there is a compact set  $K \subset G$  such that  $q(K) = E$ .*

*Proof.* Let  $V$  be an open, relatively compact neighborhood of  $\mathbb{1}_G$ . Then the sets of the form  $xV$  as  $x$  ranges over  $G$  form an open cover of  $G$ . Thus, the sets of the form  $q(xV)$  as  $x$  ranges over  $G$  form a cover of  $E$ . Since  $q$  is an open map, there must be finitely many points  $x_1, x_2, \dots, x_n \in G$  such that  $q(x_1V), q(x_2V), \dots, q(x_nV)$  cover  $E$ . We now set  $K = q^{-1}(E) \cap \cup_{j=1}^n x_j \bar{V}$ . Since  $E$  is closed and  $q$  is continuous,  $q^{-1}(E)$  is closed. Since  $\bar{V}$  is compact and group operations are continuous, the set  $\cup_{j=1}^n x_j \bar{V}$  is compact. Thus,  $K$  is compact. It now remains to show that  $q(K) = E$ . But this is clear because  $q(\cup_{j=1}^n x_j \bar{V}) \supset E$ .  $\square$

An important result that is not mentioned in most introductory harmonic analysis texts but will be useful here is the Selberg Trace Formula, or simply *trace formula*, which is a generalization of the Poisson Summation Formula. Fundamental to the setting of the trace formula is the concept of a *lattice*  $\Gamma$  in  $G$ : a discrete subgroup of  $G$  such that  $\Gamma \backslash G$  carries a right  $G$ -invariant measure  $\mu$  for which  $\mu(\Gamma \backslash G) < \infty$ . (Using right cosets is traditional in arithmetic geometry where the trace formula is often used.) An important fact about every lattice  $\Gamma$  in  $G$  is that it is closed. To see this, observe that since  $\Gamma$  is discrete, there is an open neighborhood  $U$  of  $\mathbb{1}_G$  in  $G$  such that  $U \cap \Gamma = \{\mathbb{1}_G\}$ , and let  $\{\gamma_i\}$  be a net in  $\Gamma$  converging to  $x \in G$ . Then, if  $V$  is an open neighborhood of  $\mathbb{1}_G$  such that  $VV \subset U$  and  $W = V \cap V^{-1}$ , then there is  $i_0$  such that  $i \geq i_0$  implies  $\gamma_i \in Wx$ . This implies that for  $i, j \geq i_0$  we have  $\gamma_i \gamma_j^{-1} \in WW^{-1} \subset U$ . Since also  $\gamma_i \gamma_j^{-1} \in \Gamma$ , we have  $\gamma_i = \gamma_j$  for all  $i, j \geq i_0$ , so that the net is eventually constant, which means that  $x \in \Gamma$ .

We say that a lattice  $\Gamma$  in  $G$  is *uniform* if it is *co-compact*: i.e., the quotient  $\Gamma \backslash G$  is compact. The trace formula applies only if  $G$  is unimodular, but in the case that  $G$  admits a uniform lattice, unimodularity is a straightforward consequence. This follows from [22, Proposition 9.1.2], which we state below without proof.

**Proposition 4.2.4.** *If  $G$  admits a unimodular closed co-compact subgroup, then  $G$  is unimodular.*

For the remainder of this chapter,  $\Gamma$  will be a uniform lattice,  $\mu$  will be a right-invariant measure with  $\mu(\Gamma \backslash G) = 1$ , Haar measure will be normalized according to (4.1) with the counting measure given to  $\Gamma$ ,  $q$  will be the canonical quotient map  $G \rightarrow \Gamma \backslash G$ , and  $R$  will be the right quasi-regular representation for  $(G, \Gamma)$ . The Hilbert space associated with  $R$  is  $\mathcal{H}_R = L^2(\Gamma \backslash G)$ . It is known [22, Lemma 9.2.7] that  $R$  decomposes discretely into irreducible subrepresentations with finite multiplicity. If  $\pi \in \hat{G}$ , let  $N_\Gamma(\pi)$  be the multiplicity of  $\pi$  in  $R$ , and let the set of elements of  $\hat{G}$  for which  $N_\Gamma(\pi) \neq 0$  be denoted by  $\hat{G}_\Gamma$ . Then the *trace formula* is the following

**Theorem 4.2.5.** [22, Chapter 9] *If  $f \in C_c^\infty(G)$ , then the operator  $R(f)$  is trace-class, and*

$$\mathrm{Tr} R(f) = \sum_{\pi \in \hat{G}_\Gamma} N_\Gamma(\pi) \mathrm{Tr} \pi(f) = \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} f(x^{-1}\gamma x) d\mu(\Gamma x)$$

The theorem comes from the fact that both sides of the equation are equal to  $\mathrm{Tr} R(f)$ . We note that the sum in the middle term can be replaced by a sum over any complete set of subrepresentations of  $R$ . We will only use theorem in the equivalent form that (1.)  $R$  decomposes discretely and (2.)  $\mathrm{Tr} R(f)$  equals the right side.

### 4.3 A Class of Parseval OVF's of Representations

As in the case of frames of exponentials, some of the simplest examples of OVF's of representations are ones that are Parseval. The purpose of this section is to give a

sufficient condition on  $G$  for when nontrivial Parseval OVF's of representations of  $G$  exist. First we define the term ‘‘OVF of representations.’’

Let  $dx = dm(x)$  be Haar measure on  $G$ , let  $E \subset G$  be a non-empty, open, relatively compact set, and let  $\{\pi_j : j \in \mathbb{N}\}$  be a set of representations of  $G$  on the Hilbert spaces  $\{\mathcal{H}_j : j \in \mathbb{N}\}$ . Further, assume that  $\{\pi_j\}$  has the property that, for each  $j$  and all  $f \in L^2(E)$ , the operator  $\pi_j(f)$  is a Hilbert-Schmidt class operator on  $\mathcal{H}_j$ . If  $\pi$  is such a representation, denote this map  $f \in L^2(E) \mapsto \pi(f) \in L^2(\mathcal{H}_j)$  as  $\check{\pi}$ . Then  $\{\pi_j\}$  will be called an *OVF of representations* for  $L^2(E)$  provided there exist  $B, A > 0$  such that

$$A \|f\|_{L^2(E)}^2 \leq \sum \|\pi_j(f)\|_{\text{HS}}^2 \leq B \|f\|_{L^2(E)}^2 \quad (4.2)$$

for all  $f \in L^2(E)$ . We will sometimes say in this case, abusing terminology slightly, that  $\{\pi_1, \pi_2, \dots\}$  is an OVF for  $L^2(E)$ , meaning that  $\{\check{\pi}_1, \check{\pi}_2, \dots\}$  is an OVF for  $L^2(E)$ . If  $A = B = 1$  we will say that  $\{\pi_j\}$  forms a *Parseval* OVF of representations. As we have suggested, in the case that  $G = \mathbb{R}$ , these are frames of exponentials.

By Proposition A.1, if  $G = \mathbb{R}^d$  and  $\Gamma = \mathbb{Z}^d$ , then  $R$  decomposes as  $\bigoplus_{j \in \mathbb{Z}^d} e_j$ . For a general lattice  $\Gamma$  in  $\mathbb{R}^d$ ,  $R$  similarly decomposes as  $\bigoplus_{\lambda \in \Lambda} e_\lambda$ , where  $\Lambda$  is the dual lattice of  $\Gamma$ . Given the motivation in the introduction to this chapter, it makes sense to say that the nonabelian analogue of a sequence of the form  $F(\Lambda)$ , for some lattice  $\Lambda$ , is a complete set of subrepresentations  $\{\pi_j\}$ , with multiplicity, of  $R$ . We call such a set a *harmonic* set of representations. Just as the sequence  $F(\Lambda)$  is a tight frame for some  $L^2$ -space in  $\mathbb{R}^d$  when  $\Lambda$  is a lattice, a harmonic set of representations will form a tight OVF for the  $L^2$ -space of an appropriately chosen subset  $E$  of  $G$ . When such an  $E$  exists and is nontrivial in measure, we will refer to the OVF  $\{\pi_j\}$  as a *harmonic OVF* for  $L^2(E)$ . With our choice of invariant measures, this OVF will actually be a Parseval OVF.

The appropriate choice of domain will turn out to be any  $(G, \Gamma)$  *reproducing set*:

a set  $E \subset G$  such that  $E$  is non-empty, open, and relatively compact, and such that  $EE^{-1}$  is disjoint from every conjugate of  $\Gamma - \{\mathbb{1}_G\}$ . Existence of  $E$  is equivalent to existence of an open, relatively compact, non-empty set  $U \subset G$  such that  $\cup_{g \in G} g^{-1}Ug$  intersects  $\Gamma$  only in the point  $\mathbb{1}_G$ . (For example, given  $U$ , take  $E$  to be a measurable set with  $EE^{-1} \subset U$ .) In the case  $G$  is abelian,  $U$  can be chosen to be any open subset of a lattice tile.

We now re-state the main result we wish to focus on.

**Theorem 4.3.1.** *Let  $G$  be a Lie group admitting a uniform lattice  $\Gamma$ , and let  $E$  be a  $(G, \Gamma)$  reproducing set. Then any decomposition of  $R$  into subrepresentations  $\{\pi_1, \pi_2, \dots\}$  (listed with multiplicities) gives that  $\{\pi_j\}$  forms a Parseval OVF of representations for  $L^2(E)$ .*

This begins with the preliminary result:

**Lemma 4.3.2.** *Let  $E \subset G$  be non-empty, open, and relatively compact, and let  $\mathcal{H} = L^2(E)$ . Then  $\check{R} : f \mapsto R(f)$  is a bounded linear map from  $\mathcal{H}$  into  $L^2(\mathcal{H}_R)$ .*

*Proof.* Let  $f \in \mathcal{H}$ . Let us apply  $R(f)$  to some  $\phi \in \mathcal{H}_R$ . For  $x \in G$ ,

$$\begin{aligned} (R(f)\phi)(\Gamma x) &= \left( \int_G f(y)R(y) dy \phi \right) (\Gamma x) \\ &= \int_G f(y) (R(y)\phi) (\Gamma x) dy \\ &= \int_G f(y)\phi(\Gamma xy) dy \\ &= \int_G f(x^{-1}y)\phi(\Gamma y) dy. \end{aligned}$$

By Remark 4.2.2, we may continue as follows:

$$\begin{aligned} &= \int_{\Gamma \backslash G} \sum_{\gamma} f(x^{-1}\gamma y)\phi(\Gamma \gamma y) d\mu(\Gamma y) \\ &= \int_{\Gamma \backslash G} \left( \sum_{\gamma} f(x^{-1}\gamma y) \right) \phi(\Gamma y) d\mu(\Gamma y). \end{aligned}$$

For each  $x, y \in G$ , the sum defining  $K(x, y) := \sum_{\gamma} f(x^{-1}\gamma y)$  is finitely supported, and thus convergent. Further, it depends only on the cosets of  $x$  and  $y$ . Treating  $K$  as a function on  $\Gamma \backslash G \times \Gamma \backslash G$ ,  $R(f)$  is then a kernel integral operator on  $\mathcal{H}_R$  with kernel  $K$ . As a result,  $R(f)$  has the following (possibly infinite) Hilbert-Schmidt norm:

$$\|R(f)\|_{\text{HS}}^2 = \int_{\Gamma \backslash G} \int_{\Gamma \backslash G} \left| \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y) \right|^2 d\mu(\Gamma x) d\mu(\Gamma y). \quad (4.3)$$

If  $q : G \rightarrow \Gamma \backslash G$  is the canonical quotient map and  $F$  is a subset of  $G$  of finite Haar measure, then by Remark 4.2.2 again,

$$\int_G \chi_F(x) dx = \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} \chi_F(\gamma x) d\mu(\Gamma x),$$

which is greater than or equal to  $\int_{\Gamma \backslash G} \chi_{q(F)} d\mu$ . By Proposition 4.2.3, there is a compact set  $K \subset G$  such that  $q(K) = \Gamma \backslash G$ . Thus, given  $S \subset \Gamma \backslash G$  and taking  $F = q^{-1}(S) \cap K$  in the above yields  $\int_K \chi_S \circ q(x) dx \geq \int_{\Gamma \backslash G} \chi_S d\mu$ . That is,  $\int_K g \circ q(x) dx \geq \int_{\Gamma \backslash G} g d\mu$  for all characteristic functions  $g$  on  $\Gamma \backslash G$ , and thus all non-negative measurable functions on  $\Gamma \backslash G$ . The right-hand side of (4.3) then becomes bounded by

$$\int_{\Gamma \backslash G} \int_K \left| \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y) \right|^2 dx d\mu(\Gamma y) \leq \int_K \int_K \left| \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y) \right|^2 dx dy.$$

The sum in the integrand vanishes off the set  $\Gamma_0 = \Gamma \cap KEK^{-1}$ , which is compact and discrete, hence finite. An application of the Cauchy-Schwarz inequality yields the following upper bound for  $\|R(f)\|_{\text{HS}}^2$ :

$$\begin{aligned} \|R(f)\|_{\text{HS}}^2 &\leq |\Gamma_0| \int_K \int_K \sum_{\gamma \in \Gamma_0} |f(x^{-1}\gamma y)|^2 dx dy \\ &\leq |\Gamma_0| \sum_{\gamma \in \Gamma_0} \int_K \int_G |f(x^{-1}\gamma y)|^2 dx dy \\ &\leq |\Gamma_0|^2 m(K) \|f\|_{L^2(E)}^2, \end{aligned}$$

as desired. □



Now, the decomposition  $R = \bigoplus \pi_j$  and the definition of  $R(f)$  yield a unitary  $V : \bigoplus \mathcal{H}_j \rightarrow \mathcal{H}_R$  for which, as an operator on  $\bigoplus \mathcal{H}_j$ ,

$$V^*R(f)V = \bigoplus \pi_j(f).$$

It follows that each  $\pi_j(f)$  is a Hilbert-Schmidt class operator on  $\mathcal{H}_j$  and that

$$\|R(f)\|_{\text{HS}}^2 = \sum \|\pi_j(f)\|_{\text{HS}}^2. \quad (4.4)$$

The condition for the operators  $\{\pi_j\}$ , which respectively map into the Hilbert spaces  $L^2(\mathcal{H}_j)$ , to form a Parseval OVF for  $L^2(E)$  is

$$\|f\|_{L^2(E)}^2 = \sum \|\pi_j(f)\|_{\text{HS}}^2. \quad (4.5)$$

In view of (4.4), this equality follows from  $\|f\|_{L^2(E)}^2 = \|R(f)\|_{\text{HS}}^2$ , a sufficient condition for which is that  $E$  is a  $(G, \Gamma)$  reproducing set. Verification of this sufficiency is achieved in the following lemmas.

**Lemma 4.3.3.** *Let  $M \in L^1(\mathcal{H}_R)$  and  $E$  be an non-empty, open, relatively compact subset of  $G$ . Then the function  $f^M : E \rightarrow \mathbb{C}$  defined by  $f^M(x) = \text{Tr}(R(x^{-1})M)$  is bounded, and  $\check{R}^*M = f^M$ .*

*Proof.* First it will be shown that  $f^M$  is well-defined. If  $M$  has eigenvalues  $\{\lambda_j : j \in \mathbb{N}\}$  and corresponding eigenbasis  $\{e_j\} \subset \mathcal{H}_R$ , and  $U$  is any unitary operator on  $\mathcal{H}_R$ , then

$$\begin{aligned} |\text{Tr}(UM)| &\leq \sum_j \left| \langle UM e_j, e_j \rangle_{\mathcal{H}_R} \right| \\ &\leq \sum_j \|UM e_j\|_{\mathcal{H}_R} \\ &= \sum_j \|M e_j\|_{\mathcal{H}_R} = \sum_j |\lambda_j|. \end{aligned}$$

Thus,  $f^M(x) = \sum_j \langle R(x^{-1}) M e_j, e_j \rangle$  converges absolutely to a bounded function on  $E$ .

It will now be shown that

$$\langle R(f), M \rangle_{L^2(\mathcal{H}_R)} = \langle f, f^M \rangle_{L^2(E)}.$$

The right-hand side is equal to

$$\int_E f(x) \operatorname{Tr}(M^* R(x)) dx.$$

As implied by the above estimates, the series  $\operatorname{Tr}(M^* R(x)) = \overline{\operatorname{Tr}(R(x^{-1}) M)}$ , expanded using  $\{e_j\}$ , converges absolutely to a bounded function. This means the integrand is dominated by a multiple of  $|f(x)|$  and, since  $f \in L^2(E) \subset L^1(E)$ , it follows from the dominated convergence theorem that

$$\langle f, f^M \rangle_{L^2(E)} = \operatorname{Tr} \left( \int_E f(x) M^* R(x) dx \right)$$

which is just

$$\operatorname{Tr} \left( M^* \int_E f(x) R(x) dx \right).$$

The latter is equal to  $\langle R(f), M \rangle_{L^2(\mathcal{H}_R)}$ , as desired.  $\square$

**Lemma 4.3.4.** *Let  $E$  be a  $(G, \Gamma)$  reproducing set and  $f \in L^2(E)$ . Then (4.5) holds.*

*Proof.* Suppose  $f \in C_E^\infty(G)$ . By Theorem 4.2.5,  $R(f)$  is trace-class. Thus, with the notation  $f_x(y) = f(yx)$ , Lemma 4.3.3 implies that the function  $\check{R}^* R(f)$  has the

following very specific form:

$$\begin{aligned}
(\check{R}^* R(f))(x) &= \text{Tr} (R (x^{-1}) R(f)) \\
&= \text{Tr} \left( R (x^{-1}) \int_G f(y) R(y) dy \right) \\
&= \text{Tr} \left( \int_G f(y) R(y) dy R (x^{-1}) \right) \\
&= \text{Tr} \left( \int_G f_x(y) R(y) dy \right) \\
&= \text{Tr} (R(f_x)) \\
&= \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} f_x(y^{-1} \gamma y) d\mu(\Gamma y) \\
&= f_x(\mathbb{1}_G) \mu(\Gamma \backslash G) + \int_{\Gamma \backslash G} \sum_{\mathbb{1}_G \neq \gamma} f_x(y^{-1} \gamma y) d\mu(\Gamma y),
\end{aligned} \tag{4.6}$$

where (4.6) follows from the Selberg Trace Formula (Theorem 4.2.5) applied to the function  $f_x$ . If  $x$  is such that  $\text{supp} f_x$  is disjoint from all conjugates of  $\Gamma - \{\mathbb{1}_G\}$ , then the integral term vanishes and the right-hand side becomes  $f_x(\mathbb{1}_G)$ , which is just  $f(x)$ . But this will happen if  $x \in E$ , since  $\text{supp} f_x \subset Ex^{-1} \subset EE^{-1}$ , which has the desired disjointness property.

Hence, for  $x \in E$  and  $f \in C_E^\infty(G)$ ,  $(\check{R}^* R(f))(x) = f(x)$ . Consequently,  $\|R(f)\|_{\text{HS}}^2 = \langle \check{R}^* R(f), f \rangle_{\mathcal{H}} = \|f\|_{\mathcal{H}}^2$  for all  $f$  in a dense subspace of  $\mathcal{H} = L^2(E)$ , and hence for all of  $\mathcal{H}$ . As noted above, the desired Parseval frame condition (4.5) follows from this equality.  $\square$

This section has established that if  $G$  is a Lie group that admits a uniform lattice  $\Gamma$  and a  $(G, \Gamma)$  reproducing set  $E$ , and if  $\{\pi_j : j \in \mathbb{N}\}$  is a list (with multiplicities) of the subrepresentations of the quasi-regular representation for  $(G, \Gamma)$ , then  $\{\pi_j\}$  is a Parseval OVF of representations for  $L^2(E)$ . In the next section we will see some examples of groups meeting the conditions above, and what the corresponding OVFs are. We will also see some examples of *non*-harmonic OVFs which are in a sense

nonabelian analogues of frames of exponentials associated with near-lattices.

#### 4.4 OVFs of Representations for Two Particular Groups

As we have indicated, all that is needed to specify a harmonic frame of representations for  $G$  is a uniform lattice  $\Gamma$  and a  $(G, \Gamma)$  reproducing set  $E$ . In this section we introduce some examples of OVFs of representations, both harmonic and not, for two specific groups—the real Heisenberg group  $H_n$  and the Euclidean motion group for  $\mathbb{R}^2$ .

##### 4.4.1 The Heisenberg Group

The real Heisenberg group  $G = H_n$ , or simply the *Heisenberg group*, is defined as  $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$  with the group law  $(x, \xi, t)(x', \xi', t') = (x+x', \xi+\xi', t+t' + \frac{1}{2}(x \cdot \xi' - x' \cdot \xi))$ . In this section we introduce some OVFs of representations of this group.

##### A harmonic OVF

Our first example of OVFs of representations of the Heisenberg group will be a harmonic one. In this section the discrete subgroup  $\Gamma$  consists of ordered triples in  $\mathbb{Z}^n \times \mathbb{Z}^n \times \frac{1}{2}\mathbb{Z}$  and the reproducing neighborhood  $E$  will be  $D \times (-1/4, 1/4)$ , where  $D = (-1/2, 1/2)^n \times (-1/2, 1/2)^n$ . For given  $x, \xi \in \mathbb{R}^n$ , the operation  $(x', \xi', t') \mapsto (x', \xi', t' + \frac{1}{2}(x \cdot \xi' - x' \cdot \xi))$  can be represented as a linear transformation on  $\mathbb{R}^{2n+1}$  with determinant 1, so Haar measure on  $H_n$  is just a product of Lebesgue measures.

It is necessary to verify that  $E$  really is a  $(H_n, \Gamma)$  reproducing set. To see this, first observe that  $\Gamma - \{0\}^{2n+1} = \Gamma_1 \cup \Gamma_2$  with  $\Gamma_1 = (\mathbb{Z}^{2n} - \{0\}^{2n}) \times \frac{1}{2}\mathbb{Z}$  and  $\Gamma_2 = \{0\}^{2n} \times (\frac{1}{2}\mathbb{Z} - \{0\})$ . Since the first  $2n$  scalar components of  $EE^{-1}$  lie in  $(-1, 1)$  and since the orbit of  $\Gamma_1$  under conjugation in  $G$  consists only of members of  $(\mathbb{Z}^{2n} - \{0\}^{2n}) \times \mathbb{R}$ ,  $EE^{-1}$  is disjoint from this orbit. On the other hand,  $\Gamma_2$  is in the center of  $H_n$ , so it is

equal to its orbit under conjugation. If  $(x, \xi, t) \in H_n$ , then  $(x, \xi, t)^{-1} = (-x, -\xi, -t)$ , so if  $(x, \xi, t), (x', \xi', t') \in E$  and if  $(x, \xi, t)(x', \xi', t')^{-1} \in \Gamma_2$ , then  $x = x', \xi = \xi'$ , and  $t - t' \in \frac{1}{2}\mathbb{Z} - \{0\}$ , which is impossible since  $t, t' \in (-1/4, 1/4)$ . Thus,  $EE^{-1}$  does not intersect  $\Gamma_2$ .

Let  $\{\pi_j\}$  be a list, with multiplicities, of irreducible subrepresentations of  $R$ . Then, according to Theorem 4.3.1,  $\{\pi_j\}$  is a harmonic OVF. Specifically,

$$\|f\|_{L^2(E)}^2 = \sum_j \|\pi_j(f)\|_{\text{HS}}^2 \quad (4.7)$$

for all  $f \in L^2(E)$ . We are interested in what form these  $\pi_j$ 's take, and fortunately, there are known formulas for all the representations of  $G$ . Up to equivalence, the infinite-dimensional representations of  $H_n$  have the form (see [49])  $\rho_\omega : H_n \times L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$

$$(\rho_\omega(x, \xi, t)\phi)(y) = e^{-2\pi i\omega(t+x\cdot y+\frac{1}{2}x\cdot\xi)}\phi(y+x)$$

with  $\omega \in \mathbb{R}^* = \mathbb{R} - \{0\}$  and  $\phi \in L^2(\mathbb{R}^n)$ . The others are the (one-dimensional) characters, given by  $\chi_{b,\beta}(x, \xi, t) = e^{-2\pi i(b\cdot x+\beta\cdot\xi)}$  for  $b, \beta \in \mathbb{R}^n$ . To decompose  $L^2(\Gamma \backslash H_n)$  into  $R$ -invariant subspaces, observe first that  $g \in L^2(\Gamma \backslash H_n)$  may be viewed as a function on  $H_n$  that is invariant under left translations in  $\Gamma$ . Such a function satisfies, in particular,  $g(x, \xi, t) = g(x, \xi, t + 1/2)$ . Thus,  $L^2(\Gamma \backslash H_n) = \bigoplus_{k \in \mathbb{Z}} \mathcal{K}_{2k}$ , where  $\mathcal{K}_{2k}$  is the  $R$ -invariant space  $\{h \in L^2(\Gamma \backslash H_n) : h(x, \xi, t) = e^{4\pi ikt}h(x, \xi, 0)\}$ . The action of  $R$  on  $\mathcal{K}_0$  factors through the action of the right regular representation of  $\mathbb{R}^{2n}$  on  $L^2(\mathbb{T}^{2n})$ , and can therefore be shown to decompose into the sum

$$\bigoplus_{a,\alpha \in \mathbb{Z}^n} \chi_{a,\alpha}$$

Further, it is shown in [49] that the action of  $R$  on  $\mathcal{K}_{2k}$ ,  $k \neq 0$ , splits into  $|2k|^n$  irreducible actions, each of which is equivalent by a Weil-Brezin-Zak transform to the

action of  $\rho_{2k}$  on  $L^2(\mathbb{R}^n)$ . Thus,

$$R \cong \bigoplus_{a, \alpha \in \mathbb{Z}^n} \chi_{a, \alpha} \oplus \bigoplus_{k \in \mathbb{Z}^*} |2k|^n \rho_{2k},$$

where  $\mathbb{Z}^* = \mathbb{Z} - \{0\}$ . From this it follows that  $\{\chi_{a, \alpha} : a, \alpha \in \mathbb{Z}^n\} \cup \{\rho_{2k} : k \in \mathbb{Z}^*\}$ , with  $\rho_{2k}$  repeated  $|2k|^n$  times for each  $k \neq 0$ , is a harmonic OVF. This may be summarized by the frame condition:

$$\|f\|_{L^2(E)}^2 = \sum_{a, \alpha \in \mathbb{Z}^n} |\chi_{a, \alpha}(f)|^2 + \sum_{k \neq 0} |2k|^n \|\rho_{2k}(f)\|_{\text{HS}}^2$$

for all  $f \in L^2(E)$ .

### Non-harmonic OVFs of representations

We will show in this section that some non-harmonic OVFs of representations of the Heisenberg group can be obtained by perturbing the parameters used to index the harmonic one above, just as a non-harmonic frame of exponentials can be obtained by replacing the parameters  $j$  indexing the exponentials  $\{e^{2\pi i j \cdot} : j \in \mathbb{Z}\}$  by nearby real numbers  $\{\lambda_j : j \in \mathbb{Z}\}$ . More precisely, we wish to prove the following result about replacing the equispaced parameters  $a$ ,  $\alpha$ , and  $2k$  with the real  $n$ -vectors  $\{b_a : a \in \mathbb{Z}^n\} \subset \mathbb{R}^n$  and  $\{\beta_\alpha : \alpha \in \mathbb{Z}^n\} \subset \mathbb{R}^n$  and real numbers  $\{\omega_k : k \in \mathbb{Z}^*\}$ .

**Theorem 4.4.1.** *Suppose  $\{b_a : a \in \mathbb{Z}^n\}$  and  $\{\beta_\alpha : \alpha \in \mathbb{Z}^n\}$  are sequences of real  $n$ -vectors and  $\{\omega_k : k \in \mathbb{Z}^*\}$  is a sequence of real numbers. Define*

$$M = \max \left\{ \sup_{a \in \mathbb{Z}^n} \|b_a - a\|_\infty, \sup_{\alpha \in \mathbb{Z}^n} \|\beta_\alpha - \alpha\|_\infty, \sup_{k \neq 0} |\omega_k - 2k| \right\}.$$

*If  $M > 0$  is sufficiently small, then there exist  $A = A(M) > 0$  and  $B = B(M)$  such that*

$$A \|f\|_{L^2(E)}^2 \leq \sum_{a, \alpha} |\chi_{b_a, \beta_\alpha}(f)|^2 + \sum_{k \neq 0} |2k|^n \|\rho_{\omega_k}(f)\|_{\text{HS}}^2 \leq B \|f\|_{L^2(E)}^2 \quad (4.8)$$

*holds for all  $f \in L^2(E)$ .*

For this we will need two lemmas, the first of which is a generalization of Duffin and Schaeffer [25, Lemma III] and provides explicit frame bounds for a perturbation of a frame of exponentials in  $\mathbb{R}^n$ .

**Lemma 4.4.2.** *Let  $E = (-1/2, 1/2)^n$  and identify  $L^2(E)$  with the set of functions in  $L^2(\mathbb{R}^n)$  vanishing off  $E$ . Given  $M > 0$  there is a number  $T(M) > 0$  such that if  $\{\mu_j\}$  and  $\{\lambda_j\}$  are sequences in  $\mathbb{R}^n$  such that  $\sup_j \|\mu_j - \lambda_j\|_\infty < M$  and*

$$A \|f\|_2^2 \leq \sum_j \left| \hat{f}(\lambda_j) \right|^2 \leq B \|f\|_2^2 \quad (4.9)$$

for all  $f \in L^2(E)$ , then

$$\sum_j \left| \hat{f}(\mu_j) - \hat{f}(\lambda_j) \right|^2 \leq T(M) \sum_j \left| \hat{f}(\lambda_j) \right|^2$$

for all  $f \in L^2(E)$ . Further,  $T(M)$  may be taken so that  $T(M) \rightarrow 0$  as  $M \rightarrow 0$ .

*Proof.* Suppose  $f \in L^2(E)$ ,  $h = \hat{f}$ , and  $\{\lambda_j\}$  and  $\{\mu_j\}$  are as above. The function  $h$  extends to a function on  $\mathbb{C}^n$  that is analytic:  $h(z) = \int_E f(y) e^{-2\pi iz \cdot y} dy$ . Thus, we may apply Taylor's theorem:

$$h(\mu_j) - h(\lambda_j) = \sum_{k \neq 0} \frac{h^{(k)}(\lambda_j)}{k!} (\mu_j - \lambda_j)^k,$$

where  $k \in \{0, 1, \dots\}^n$  is a multi-index. Denoting  $k_1 + k_2 + \dots + k_n$  by  $|k|$ , we next multiply and divide by  $\rho^{|k|}$  in each term, where  $\rho$  is some positive number, distribute a factor of  $(k!)^{1/2}$ , and apply Cauchy-Schwarz.

$$\begin{aligned} |h(\mu_j) - h(\lambda_j)|^2 &= \left| \sum_{k=1}^{\infty} \frac{h^{(k)}(\lambda_j)}{\rho^{|k|} (k!)^{1/2}} \frac{\rho^{|k|} (\mu_j - \lambda_j)^k}{(k!)^{1/2}} \right|^2 \\ &\leq \sum_{k=1}^{\infty} \frac{|h^{(k)}(\lambda_j)|^2}{\rho^{2|k|} k!} \sum_{k=1}^{\infty} \frac{(\rho M)^{2|k|}}{k!}. \end{aligned} \quad (4.10)$$

Consider the term

$$\sum_j |h^{(k)}(\lambda_j)|^2.$$

Since  $h^{(k)}(x) = \int_E f(y)(-2\pi iy)^k e^{-2\pi ix \cdot y} dy$ , the function  $h^{(k)}$  is the Fourier transform of  $g \in L^2(E)$ , where  $g(y) = f(y)(-2\pi iy)^k$ , and we have

$$A \|g\|_2^2 \leq \sum_j |h^{(k)}(\lambda_j)|^2 \leq B \|g\|_2^2.$$

Since  $\|g\|_2^2 \leq (2\pi)^{2|k|} \|f\|_2^2$  and since  $\|f\|_2^2 \leq \frac{1}{A} \sum_j |h(\lambda_j)|^2$  we get

$$\sum_j |h^{(k)}(\lambda_j)|^2 \leq (2\pi)^{2|k|} \frac{B}{A} \sum_j |h(\lambda_j)|^2.$$

Summing (4.10) over all  $j$  and using the above gives

$$\sum_j |h(\mu_j) - h(\lambda_j)|^2 \leq T(M, \rho) \sum_j |h(\lambda_j)|^2$$

with

$$T(M, \rho) = \frac{B}{A} \sum_{k \neq 0} \frac{(2\pi)^{2|k|}}{\rho^{2|k|} k!} \sum_{k=1}^{\infty} \frac{(\rho M)^{2|k|}}{k!}.$$

Letting  $T(M) = T(M, 1/\sqrt{M})$ , we see by dominated convergence that  $T(M) \rightarrow 0$  as  $M \rightarrow 0$ .

□

Next, we need a lemma that computes the quantity  $\|\rho_\omega(f)\|_{\text{HS}}^2$  for  $f \in L^1(H_n) \cap L^2(H_n)$  and  $\omega \neq 0$ .

**Lemma 4.4.3.** *Let  $\rho_\omega$  be an infinite-dimensional representation of the Heisenberg group  $H_n$ , and  $f \in L^1(H_n) \cap L^2(H_n)$ . Then*

$$\|\rho_\omega(f)\|_{\text{HS}}^2 = |\omega|^{-n} \int \int |\mathcal{F}_3 f(u, w, \omega)|^2 du dw.$$

*Proof.* For  $f \in L^1(H_n) \cap L^2(H_n)$  and  $b, \beta \in \mathbb{R}^n$  and  $\omega \in \mathbb{R}$  the (Euclidean) Fourier transform of  $f$  at  $(b, \beta, \omega)$  is

$$\hat{f}(b, \beta, \omega) = \int \int \int f(x, \xi, t) e^{-2\pi i(b \cdot x + \beta \cdot \xi + \omega t)} dx d\xi dt.$$



Let  $\mathcal{F}_1$ ,  $\mathcal{F}_2$ , and  $\mathcal{F}_3$  denote the corresponding Fourier transforms with respect to the first, second, and third variable, respectively. Applying  $\rho_\omega(f)$  to some  $\phi \in L^2(\mathbb{R}^n)$  gives

$$\begin{aligned} & \left( \int \int \int f(x, \xi, t) \rho_\omega(x, \xi, t) dx d\xi dt \phi \right) (y) \\ &= \int \int \int f(x, \xi, t) (\rho_\omega(x, \xi, t) \phi) (y) dx d\xi dt \\ &= \int \int \int f(x, \xi, t) e^{-2\pi i \omega(t + \xi \cdot x/2 - \xi \cdot y)} \phi(y - x) dx d\xi dt. \end{aligned}$$

Substituting  $x \leftarrow y - x$  and continuing, we get

$$\begin{aligned} & \int \int \int f(y - x, \xi, t) e^{-2\pi i \omega(t + \xi \cdot (y - x)/2 - \xi \cdot y)} \phi(x) dx d\xi dt \\ &= \int \int \int f(y - x, \xi, t) e^{-2\pi i \omega(t - \xi \cdot (x + y)/2)} d\xi dt \phi(x) dx. \end{aligned}$$

Thus,  $\rho_\omega(f)$  is an integral operator on  $L^2(\mathbb{R}^n)$  with kernel

$$\begin{aligned} K_\omega^f(y, x) &= \int \int f(y - x, \xi, t) e^{-2\pi i \omega(t - \xi \cdot (x + y)/2)} d\xi dt \\ &= \mathcal{F}_2 \mathcal{F}_3 f(y - x, -\omega(x + y)/2, \omega), \end{aligned}$$

where  $\mathcal{F}_2$  is the Fourier transform with respect to the second variable in the triple  $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ . The square of the Hilbert-Schmidt norm of this operator is as follows:

$$\|\rho_\omega(f)\|_{\text{HS}}^2 = \int \int |K_f(y, x)|^2 dx dy.$$

The substitution  $u = y - x$  and  $v = -\frac{1}{2}\omega(x + y)$  together with the Euclidean Parseval Formula for  $\mathbb{R}^n$  gives

$$\begin{aligned} \|\rho_\omega(f)\|_{\text{HS}}^2 &= \int \int |\mathcal{F}_2 \mathcal{F}_3 f(u, v, \omega)|^2 |\omega|^{-n} du dv \\ &= |\omega|^{-n} \int \int |\mathcal{F}_3 f(u, w, \omega)|^2 du dw, \end{aligned}$$

as desired. □

Now to the proof of the theorem.

*Proof.* Let  $f \in L^2(E)$ . The symbols  $p$ ,  $q$ , and  $r$  will denote the quadratic forms

$$q(f) = \sum_{a,\alpha} |\chi_{b_a, \beta_\alpha}(f)|^2$$

and

$$r(f) = \sum_{k \neq 0} |2k|^n \|\rho_{\omega_k}(f)\|_{\text{HS}}^2$$

and

$$p(f) = q(f) + r(f).$$

The result to be proven, in effect, is that for  $M > 0$  sufficiently small, the seminorm  $p^{1/2}$  is equivalent to  $\|\cdot\|_{L^2(E)}$ .

By Lemma 4.4.2 applied to  $f(\cdot, \cdot, \frac{1}{2}\cdot)$  in  $L^2((-1/2, 1/2)^{2n+1})$ , if  $\tilde{\cdot}$  is a map from  $J = \mathbb{Z}^n \times \mathbb{Z}^n \times 2\mathbb{Z}$  to  $\mathbb{R}^{2n+1}$  and the number

$$M' = \sup_{z \in J} \|\tilde{z} - z\|_\infty$$

is sufficiently small, there is  $T = T(M')$  such that

$$\sum_{z \in J} \left| \hat{f}(\tilde{z}) - \hat{f}(z) \right|^2 \leq T(M') \sum_{z \in J} \left| \hat{f}(z) \right|^2$$

for every  $f \in L^2(E)$ . By the triangle inequality, this means that the quantity

$$\sum_{z \in J} \left| \hat{f}(\tilde{z}) \right|^2 \tag{4.11}$$

is bounded above and below by the quantity

$$(1 \pm T(M')^{1/2})^2 \sum_{z \in J} \left| \hat{f}(z) \right|^2 = (1 \pm T(M')^{1/2})^2 \|f\|_{L^2(E)}^2.$$

Thus, it suffices to show that  $p(f)$  is bounded above and below by positive multiples of (4.11) for some  $\tilde{z}$ 's for which  $M' = M$ .

Let us now consider the terms of the form  $\|\rho_\omega(f)\|_{\text{HS}}^2$  for  $\omega \neq 0$ . By Lemma 4.4.3 such a term is equal to

$$\|\rho_\omega(f)\|_{\text{HS}}^2 = \frac{1}{|\omega|^n} \int \int |\mathcal{F}_3 f(u, v, \omega)|^2 du dv.$$

Further, the facts that  $g = \mathcal{F}_3 f(\cdot, \cdot, \omega)$  is supported on  $D$  and is square-integrable imply that  $\|\rho_\omega(f)\|_{\text{HS}}^2$  may be written using the  $2n$ -dimensional Fourier series expansion of  $g$  as

$$\|\rho_\omega(f)\|_{\text{HS}}^2 = \frac{1}{|\omega|^n} \sum_{a, \alpha \in \mathbb{Z}^n} |\mathcal{F}_1 \mathcal{F}_2 \mathcal{F}_3 f(a, \alpha, \omega)|^2 = \frac{1}{|\omega|^n} \sum_{a, \alpha} \left| \hat{f}(a, \alpha, \omega) \right|^2$$

for any  $\omega \neq 0$ .

Consider  $|r(f) - \phi(f)|$ , where

$$\begin{aligned} \phi(f) &= \sum_{k \neq 0} \sum_{a, \alpha} \left| \hat{f}(a, \alpha, \omega_k) \right|^2 \\ &= \sum_{k \neq 0} |\omega_k|^n \|\rho_{\omega_k}(f)\|_{\text{HS}}^2. \end{aligned} \tag{4.12}$$

The quantity has the following upper bound:

$$\begin{aligned} |r(f) - \phi(f)| &\leq \sum_{k \neq 0} \left| \left| \frac{2k}{\omega_k} \right|^n - 1 \right| |\omega_k|^n \|\rho_{\omega_k}(f)\|_{\text{HS}}^2 \\ &\leq \sup_{k \neq 0} \left| \left| \frac{2k}{\omega_k} \right|^n - 1 \right| \sum_{k \neq 0} |\omega_k|^n \|\rho_{\omega_k}(f)\|_{\text{HS}}^2 \\ &= \sup_{k \neq 0} \left| \left| \frac{2k}{\omega_k} \right|^n - 1 \right| \phi(f). \end{aligned}$$

For  $M \ll 1$ , a bound may be obtained by replacing  $|\omega_k|^n$  by  $|2k|^n - nM|2k|^{n-1}$ . A corresponding bound on the supremum terms is  $nM/(|2k| - nM)$ , which is decreasing in  $|k|$ . Thus, the supremum term is less than  $C(M) = nM/(2 - nM)$ , which goes to zero as  $M$  goes to zero. In other words,

$$(1 - C(M))\phi(f) \leq r(f) \leq (1 + C(M))\phi(f). \tag{4.13}$$

The inequality

$$(1 - C(M))(\phi(f) + q(f)) \leq p(f) \leq (1 + C(M))(\phi(f) + q(f)) \quad (4.14)$$

results from adding  $(1 - C(M))q(f) \leq q(f) \leq (1 + C(M))q(f)$  to (4.13). For each  $b, \beta \in \mathbb{R}^n$ , the quantity  $\chi_{b,\beta}(f)$  is equal to  $\hat{f}(b, \beta, 0)$ , so

$$q(f) = \sum_{a,\alpha \in \mathbb{Z}^n} \left| \hat{f}(b_a, \beta_\alpha, 0) \right|^2$$

for Haar measure as above. Thus, combining the above with (4.14) and (4.12) gives

$$(1 - C(M)) \sum_{z \in J} \left| \hat{f}(\tilde{z}) \right|^2 \leq p(f) \leq (1 + C(M)) \sum_{z \in J} \left| \hat{f}(\tilde{z}) \right|^2,$$

where, when  $k = 0$ ,  $(a, \alpha, 2k)^\sim = (b_a, \beta_\alpha, 0)$  and, when  $k \neq 0$ ,  $(a, \alpha, 2k)^\sim = (a, \alpha, \omega_k)$ .

For these values of  $\tilde{z}$ , the number  $M'$  is equal to  $M$ , and

$$\begin{aligned} & (1 - C(M))(1 - T(M)^{1/2})^2 \|f\|_{L^2(E)}^2 \\ & \leq p(f) \\ & \leq (1 + C(M))(1 + T(M)^{1/2})^2 \|f\|_{L^2(E)}^2, \end{aligned}$$

as desired. □

Observe that by making the perturbations small,  $A$  and  $B$  can be made as close to one as desired, resulting in a “nearly Parseval” OVF of representations. Thus, viewing the list of representations  $\{\chi_{b_a, \beta_\alpha}\} \cup \{\rho_{\omega_k}\}$  with the appropriate number of repetitions, the desired result about OVFs of representations on  $H_n$  is obtained: all that is needed to specify one is a sequence of numbers satisfying a Duffin-Schaeffer type stability condition.

**Remark 4.4.4.** (POVM formulation.) If  $T = \{\pi_j\}$  is an OVF of representations as above, each  $\pi_j$  is a map taking  $f \in L^2(E)$  to a kernel integral operator on  $L^2(\mathbb{R}^n)$ .

As claimed in Section 3.3, representing  $g \in \mathcal{H}$  by the sequence  $S = \{S_j g\}$ , for some OVF  $\{S_j\}$  for  $\mathcal{H}$ , is sometimes more complicated notationally than representing  $g$  by the sequence  $\{S_j^* S_j g\}$ , and the latter representation corresponds to doing analysis using the POVM  $M_S$ . In this Remark, we show that representing  $f$  by functions of the form  $\tilde{\pi}_j^* \tilde{\pi}_j f$  is particularly simple.

The terms in question are in fact similar to terms appearing in an exponential-frame expansion of  $L^2(E)$ . To see this, first note that if  $\chi_{b,\beta}$  is one of the Heisenberg group's characters and  $\check{\chi}_{b,\beta} : L^2(E) \rightarrow \mathbb{C}$  is the corresponding restriction to  $L^2(E)$ , then, just as in Fourier analysis of  $\mathbb{T}^{2n+1}$ ,

$$\begin{aligned} (\check{\chi}_{b,\beta}^* \check{\chi}_{b,\beta} f)(x, \xi, t) &= \int \int \int f(y, \eta, t) e^{-2\pi i(b \cdot y + \beta \cdot \eta)} dy d\eta dt e^{2\pi i(b \cdot x + \beta \cdot \xi)} \\ &= (\mathcal{F}_1 \mathcal{F}_2 \mathcal{F}_3 f)(b, \beta, 0) e^{2\pi i b \cdot x} e^{2\pi i \beta \cdot \xi}. \end{aligned}$$

Thus the character terms in the frame inequality (4.8) correspond to an exponential-frame expansion of  $(x, \xi) \mapsto \int_{\mathbb{R}} f(x, \xi, t) dt$ . Second, we consider the infinite dimensional irreducible representations  $\rho_\omega : H_n \rightarrow U(\mathcal{K})$ , with  $\mathcal{K} = L^2(\mathbb{R}^n)$ . Suppose  $\check{\rho}_\omega : L^2(E) \rightarrow L^2(\mathcal{K})$  is the corresponding restriction to  $L^2(E)$ ,  $f \in L^2(E)$ , and  $T_\omega$ ,  $\omega \neq 0$ , is the map  $T_\omega f(x, \xi, t) = |\omega|^{-n} \mathcal{F}_3 f(x, \xi, \omega) e^{2\pi i \omega t}$ . Then  $\check{\rho}_\omega^* \check{\rho}_\omega = T_\omega$ . To see this, observe first that by Lemma 4.4.3

$$\|\rho_\omega(f)\|_{\text{HS}}^2 = |\omega|^{-n} \int_{\mathbb{R}^{2n}} |\mathcal{F}_3 f(x, \xi, \omega)|^2 dx d\xi,$$

so that the right side is equal to  $\langle \check{\rho}_\omega^* \check{\rho}_\omega f, f \rangle_{L^2(E)}$ . But by Fubini's theorem, this quantity is also equal to  $\langle T_\omega f, f \rangle_{L^2(E)}$ , so that  $\check{\rho}_\omega^* \check{\rho}_\omega = T_\omega$ . We note that  $T_\omega$  is  $1/|\omega|^n$  times the orthogonal projection  $Q_\omega$  onto the space

$$\mathcal{K}_\omega = \{f \in L^2(E) : f(x, \xi, t) = f(x, \xi, 0) e^{2\pi i \omega t}\}.$$

Thus, letting  $e(b, \beta, \omega)$  be the function  $(x, \xi, t) \in E \mapsto e^{2\pi i(b \cdot x + \beta \cdot \xi + \omega t)}$ , the sum over all terms of the form  $\tilde{\pi}_j^* \tilde{\pi}_j$  then has the following expression in terms of orthogonal

projections:

$$\sum_{a,\alpha} \check{\chi}_{b_a,\beta_\alpha}^* \check{\chi}_{b_a,\beta_\alpha} + \sum_{k \neq 0} |k|^n \check{\rho}_{\omega_k}^* \check{\rho}_{\omega_k} = \sum_{a,\alpha} P_{e(b_a,\beta_\alpha,0)} + \sum_{k \neq 0} |k/\omega_k|^n Q_{\omega_k}.$$

**Remark 4.4.5.** It is a simple extension of Remark 2.3.5 to note that for functions  $f \in C_E^m(G)$ , the rank-one terms in partial sums of  $Sf$  converge rapidly, meaning on the order of  $1/(|a|^2 + |\alpha|^2)^{m/2}$ . Further, the norms of the terms of the form  $\check{\rho}_{\omega_k}^* \check{\rho}_{\omega_k} f$  will converge to zero at the same rate  $\|\rho_{\omega_k}(f)\|_{\text{HS}}$  does, by the Principle of Uniform Boundedness. For the latter, letting  $\partial_3$  be the partial derivative with respect to  $t$ ,

$$|\mathcal{F}_3 f(x, \xi, \omega)| \leq \frac{1}{|2\pi\omega|^m} |\mathcal{F}_3 \partial_3^m f(x, \xi, \omega)|,$$

so  $\|\rho_{\omega_k}(f)\|_{\text{HS}}$  goes to zero at a rate of  $1/|\omega_k|^m$ . This property may be important if these OVF's are to have any computational use.

#### 4.4.2 The Euclidean Motion Group for $\mathbb{R}^2$

In this section we will investigate OVF's of representations for the Euclidean motion group on  $\mathbb{R}^n$ , denoted  $E(n)$ , with  $n = 2$ . This group is defined to be  $H \rtimes K$  with  $H = \mathbb{R}^2$  and  $K$  being the matrix group  $\text{SO}(2)$  and  $h^k$  being the matrix product  $kh$ .

#### A harmonic OVF

By the uniqueness of Haar measure, Haar measure for  $G = E(2)$  is just a product of Lebesgue measure for  $\mathbb{R}^2$  and invariant measure on the circle  $K$ , which we normalize to 1. Let  $\Gamma$  be the lattice  $\mathbb{Z}^2 \times \{\mathbb{1}_K\}$  which is uniform and let  $E \subset G$  be  $\mathbb{D}/2 \times \text{SO}(2)$ , where  $\mathbb{D}$  is the open disc of radius 1 centered at the origin. To check that  $E$  is a  $(G, \Gamma)$  reproducing set, simply observe that  $gEg^{-1} = E$ , so that  $gEE^{-1}g^{-1}$  is disjoint from  $\Gamma - \{\mathbb{1}_G\}$  for all  $g \in G$  iff  $EE^{-1} = \mathbb{D} \times \text{SO}(2)$  disjoint from  $\Gamma - \{\mathbb{1}_G\}$ .

As before, if  $\{\pi_j\}$  is a complete list, with multiplicity, of subrepresentations of  $R$ , then  $\{\pi_j\}$  forms a harmonic OVF. It is shown in Appendix A that  $R = \bigoplus_{l \in \mathbb{Z}^2} \rho_l$ ,

where  $\rho_\lambda$  for the column vector  $\lambda \in \mathbb{R}^2$  is the representation of  $G$  on  $L^2(K)$  given by

$$(\rho_\lambda(h, k)\phi)(k_0) = e^{-2\pi i \lambda^{k_0} \cdot h} \phi(k^{-1}k_0)$$

for all  $\phi \in L^2(K)$ . (Two such  $\rho_\lambda$ 's are equivalent if and only if the parameters  $\lambda \in \mathbb{R}^2$  have the same length, so the given direct sum has repetitions in it.) Thus, we have the Parseval condition

$$\|f\|_{L^2(E)}^2 = \sum_{l \in \mathbb{Z}^2} \|\rho_l(f)\|_{\text{HS}}^2 \quad (4.15)$$

for all  $\phi \in L^2(K)$ . As stated in Appendix A, the representation  $\rho_\lambda$  is irreducible for all  $\lambda \neq 0$ .

### Non-harmonic OVF's of representations

The purpose of this section is to show existence of a class of non-harmonic OVF's of representations for the group  $E(2)$ . Specifically, we wish to find conditions on the parameters  $\lambda_l \in \mathbb{R}^2$  ( $l \in \mathbb{Z}^2$ ) such that  $\{\rho_{\lambda_l}\}$  is an OVF of representations:

$$A \|f\|_{L^2(E)}^2 \leq \sum_{l \in \mathbb{Z}^2} \|\rho_{\lambda_l}(f)\|_{\text{HS}}^2 \leq B \|f\|_{L^2(E)}^2 \quad (4.16)$$

for all  $\phi \in L^2(K)$ . For this we need the following lemma.

**Lemma 4.4.6.** *If  $f \in L^2(E)$ , then  $\|\rho_\lambda(f)\|_{\text{HS}}^2 = \int_K \int_K |\mathcal{F}_1 f(\lambda^k, k')|^2 dk dk'$ , where  $\mathcal{F}_1 f(\omega, k)$  denotes the Fourier transform of  $f(\cdot, k)$  at  $\omega \in \mathbb{R}^2$ .*

*Proof.* We first show that  $\rho_\lambda(f)$  is a kernel integral operator on  $L^2(K)$  by applying it to a function  $\phi \in L^2(K)$ .

$$\begin{aligned} \left( \int_K \int_H f(h, k) \rho_\lambda(h, k) dh dk \phi \right) (k_0) &= \int_K \int_H f(h, k) (\rho_\lambda(h, k)\phi)(k_0) dh dk \\ &= \int_K \int_H f(h, k) e^{-2\pi i \lambda^{k_0} \cdot h} \phi(k^{-1}k_0) dh dk \\ &= \int_K \int_H f(h, k_0 k^{-1}) e^{-2\pi i \lambda^{k_0} \cdot h} \phi(k) dh dk, \end{aligned}$$

where in the last step, we have used the fact that  $K$  is unimodular, so that  $d(k^{-1}) = dk$ . From the above we can see that  $\rho_\lambda(f)$  is an integral kernel operator on  $L^2(K)$ , as claimed, with kernel

$$\Phi(k, k_0) = \int_H f(h, k_0 k^{-1}) e^{-2\pi i \lambda^{k_0} \cdot h} dh.$$

The corresponding Hilbert-Schmidt norm is therefore

$$\|\rho_\lambda(f)\|_{\text{HS}}^2 = \int_K \int_K |\Phi(k, k_0)|^2 dk dk_0.$$

Making the substitution  $k \leftarrow k^{-1}k_0$  yields

$$\begin{aligned} \|\rho_\lambda(f)\|_{\text{HS}}^2 &= \int_K \int_K |\Phi(k^{-1}k_0, k_0)|^2 dk dk_0 \\ &= \int_K \int_K |\mathcal{F}_1 f(\lambda^{k_0}, k)|^2 dk dk_0. \end{aligned}$$

which is the desired result. □

We now prove (4.16) for appropriate  $\{\lambda_l\}_{l \in \mathbb{Z}^2}$ . In the following proof the norm  $\|\cdot\|$  applied to a vector in  $\mathbb{R}^2$  will be taken to be the Euclidean norm.

**Theorem 4.4.7.** *Suppose  $\{\lambda_l\}_{l \in \mathbb{Z}^2}$  is a subset of  $\mathbb{R}^2$ . Then, if  $M = \sup_{l \in \mathbb{Z}^2} \|\lambda_l\| - \|l\|$  is sufficiently small, there exist  $B, A > 0$  such that (4.16) holds for all  $f \in L^2(E)$ .*

*Proof.* Combining Equation (4.15) with Lemma 4.4.6, we get the following Parseval frame condition:

$$\|f\|_{L^2(E)}^2 = \sum_{l \in \mathbb{Z}^2} \int_K \int_K |\mathcal{F}_1 f(l^k, k')|^2 dk dk'.$$

We wish to prove that if  $M$  is sufficiently small, then the quantity

$$\sum_{l \in \mathbb{Z}^2} \int_K \int_K |\mathcal{F}_1 f(\lambda_l^k, k')|^2 dk dk'$$



is bounded above and below by positive multiples of  $\|f\|_{L^2(E)}^2$ . Let  $\kappa_l \in \mathbb{R}^2$  be defined for  $l = 0$  to be  $\lambda_0$  and for  $l \neq 0$  to be

$$\kappa_l = \frac{\|\lambda_l\|}{\|l\|} l.$$

Let  $k' \in K$ . By Lemma 4.4.2, there is  $T(M)$  depending only on

$$M = \sup_{l \in \mathbb{Z}^2} \|\kappa_l - l\|_\infty$$

such that

$$\sum_{l \in \mathbb{Z}^2} |\mathcal{F}_1 f(\kappa_l, k') - \mathcal{F}_1 f(l, k')|^2 \leq T(M) \sum_{l \in \mathbb{Z}^2} |\mathcal{F}_1 f(l, k')|^2.$$

Let  $k \in K$ . By the proof of the same Lemma, the same function  $T$  works in the inequality

$$\sum_{l \in \mathbb{Z}^2} |\mathcal{F}_1 f(\kappa_l^k, k') - \mathcal{F}_1 f(l^k, k')|^2 \leq T(M(k)) \sum_{l \in \mathbb{Z}^2} |\mathcal{F}_1 f(l^k, k')|^2, \quad (4.17)$$

where  $M(k) = \sup_{l \in \mathbb{Z}^2} \|\kappa_l^k - l^k\|_\infty$ . We make the definition

$$M' = \sup_{l \in \mathbb{Z}^2} \|\kappa_l^k - l^k\|.$$

Then

$$\begin{aligned} M' &= \left\| \frac{\|\lambda_l\|}{\|l\|} l^k - l^k \right\| \\ &= \left| \frac{\|\lambda_l\|}{\|l\|} - 1 \right| \|l^k\| \\ &= \left| \|\lambda_l\| - \|l\| \right|. \end{aligned}$$

Since  $M' \geq M(k)$  for all  $k$ , the number  $\sup_{k \in K} T(M(k))$  can be made smaller than some  $0 < C \ll 1$  by taking  $M'$  to be small. Thus, integrating (4.17) over  $k$  and  $k'$

and applying the triangle inequality gives

$$\begin{aligned}
& (1 - C^{1/2})^2 \sum_{l \in \mathbb{Z}^2} \int_K \int_K |\mathcal{F}_1 f(l^k, k')|^2 dk dk' \\
& \leq \sum_{l \in \mathbb{Z}^2} \int_K \int_K |\mathcal{F}_1 f(\kappa_l^k, k')|^2 dk dk' \\
& \leq (1 + C^{1/2})^2 \sum_{l \in \mathbb{Z}^2} \int_K \int_K |\mathcal{F}_1 f(l^k, k')|^2 dk dk'.
\end{aligned}$$

But this is the desired inequality because the first and last quantities are multiples of  $\|f\|_{L^2(E)}^2$ , and in the middle quantity  $\kappa_l$  can be replaced by  $\lambda_l$ .  $\square$

We have thus proved, in analogy with the corresponding result for the Heisenberg group, that for appropriate  $\lambda_l$ 's and  $E$ , the list  $\{\rho_{\lambda_l}\}$  is an OVF of representations for  $L^2(E)$ . Further, these representations are irreducible if no  $\lambda_l$  is equal to 0. We also make the note that the OVFs above are again “nearly Parseval” if the perturbations  $\|\lambda_l\| - \|l\|$  are chosen to be small.

**Remark 4.4.8.** (POVM formulation.) As in the Heisenberg case, if  $T = \{\rho_{\lambda_l}\}$  is an OVF of representations as described in this section, then representing  $f \in L^2(E)$  by  $\{\check{\rho}_{\lambda_l}^* \check{\rho}_{\lambda_l} f\}$  can be considerably simpler notationally than representing  $f$  by  $\{\rho_{\lambda_l}(f)\}$ . This again corresponds to doing analysis with the POVM  $M_T$ , as defined in Section 3.3, rather than by  $T$  directly.

Consider a term of the form  $\check{\rho}_{\lambda}^* \check{\rho}_{\lambda}$ . We will show that this product is the simple kernel integral operator  $S_{\lambda} : L^2(E) \rightarrow L^2(E)$  given by

$$(S_{\lambda} f)(h', k') = \int_H \left( \int_K e^{-2\pi i(h-h') \cdot \lambda^k} dk \right) f(h, k') dh.$$

Indeed, for  $f \in L^2(E)$ , we may apply Fubini's theorem in the following:

$$\begin{aligned}
\langle S_\lambda f, f \rangle_{L^2(E)} &= \int_{\mathbb{D}/2} \int_K (S_\lambda f)(h', k') \overline{f(h', k')} dh' dk' \\
&= \int_{\mathbb{D}/2} \int_K \int_{\mathbb{D}/2} \int_K e^{-2\pi i(h-h') \cdot \lambda^k} dk f(h, k') dh \overline{f(h', k')} dh' dk' \\
&= \int_K \int_K \int_{\mathbb{D}/2} f(h, k') e^{-2\pi i h \cdot \lambda^k} dh \int_{\mathbb{D}/2} \overline{f(h', k')} e^{2\pi i h' \cdot \lambda^k} dh' dk dk' \\
&= \int_K \int_K |\mathcal{F}_1 f(\lambda^k, k')|^2 dk dk' \\
&= \|\rho_\lambda(f)\|_{\text{HS}}^2 \\
&= \langle \check{\rho}_\lambda^* \check{\rho}_\lambda f, f \rangle_{L^2(E)}.
\end{aligned}$$

Thus,  $\check{\rho}_\lambda^* \check{\rho}_\lambda = S_\lambda$ , as claimed.

**Remark 4.4.9.** As in Remark 4.4.5, if the frame operator of  $\{\rho_{\lambda_l}\}_{l \in \mathbb{Z}^2}$  is applied to a function  $f$  with a certain smoothness property, then the terms in the resulting series expansion have a certain decay property. For completeness of discussion, we will now describe a result of this form with precision. Assume  $f \in C_E^{2m}(G)$ . In this case,  $f$  is  $2m$ -times differentiable with respect to its first argument ( $G = \mathbb{R}^2 \times \text{SO}(2)$ ). We will use the notation  $\Delta_1$  to denote the Laplacian with respect to  $\mathbb{R}^2$ . We claim that  $\|\check{\rho}_{\lambda_l}^* \rho_{\lambda_l}(f)\|_{L^2(E)}$  and  $\|\rho_{\lambda_l}(f)\|_{\text{HS}}$  go to zero on the order of  $|l|^{-2m}$ . By the Principle of Uniform Boundedness, it suffices to show that  $\|\rho_{\lambda_l}(f)\|_{\text{HS}}$  goes to zero on the order

of  $|l|^{-2m}$ . Given the formula in Lemma 4.4.6, we consider the integrand  $|\mathcal{F}_1 f(\lambda^k, k')|$ .

$$\begin{aligned}
|\mathcal{F}_1 f(\lambda^k, k')| &= \left| \int_{\mathbb{D}/2} e^{-2\pi i h \cdot \lambda^k} f(h, k') dh \right| \\
&= \frac{1}{\|\lambda\|_2^2} \left| \int_{\mathbb{D}/2} \left( \Delta_1 e^{-2\pi i h \cdot \lambda^k} \right) f(h, k') dh \right| \\
&= \frac{1}{\|\lambda\|_2^2} \left| \int_{\mathbb{D}/2} e^{-2\pi i h \cdot \lambda^k} \Delta_1 f(h, k') dh \right| \\
&= \dots \\
&= \frac{1}{\|\lambda\|_2^{2m}} \left| \int_{\mathbb{D}/2} e^{-2\pi i h \cdot \lambda^k} \Delta_1^m f(h, k') dh \right|
\end{aligned}$$

Thus, for each integer  $m > 0$ ,  $\|\rho_\lambda(f)\|_{\text{HS}}^2$  is bounded by

$$\frac{1}{\|\lambda\|_2^{4m}} \|\rho_\lambda(\Delta_1^m f)\|_{\text{HS}}^2$$

which goes to zero with order  $4m$  as  $\lambda \rightarrow \infty$ . Taking the square root gives the desired result.

## 4.5 Conclusion and Future Work

In this chapter we have constructed several types of OVF's of representations. In Section 4.3, we have constructed harmonic OVF's of representations of any Lie group  $G$  admitting a uniform lattice and a reproducing set  $E$ . In Section 4.4, we found a reproducing set  $E$  for the two examples of the Heisenberg group  $H_n$  and the motion group  $E(2)$ , and found that, for a natural parameterization of  $\hat{G}$ , the corresponding Parseval OVF's of representations remain OVF's of representations after perturbations of the representations' parameters.

Since an element  $f$  of a Hilbert space  $\mathcal{H}$  is uniquely specified by  $\{T_j f\}$  when  $\{T_j\}$  is an OVF for  $\mathcal{H}$ , one intriguing consequence of the latter result is a condition on  $\{\pi_j\}$  such that  $f \in L^2(E)$  is uniquely specified by  $\{\pi_j(f)\}$ . Another consequence of our results is that, as discussed in the subsections titled "POVM Formulation," it is

sometimes easier to represent  $f$  by  $\{\tilde{\pi}_j^* \pi_j(f)\}$  than it is to represent  $f$  by  $\{\pi_j(f)\}$ , providing a motivation for sometimes doing analysis using a POVM instead of the corresponding OVF. Finally, we note here that by making the perturbations in the representations' parameters small, the frame bounds  $A$  and  $B$  can be made as close to 1 as desired, resulting in a “nearly Parseval” OVF of representations. In view of (2.9), the frame algorithm for OVFs with  $A \approx B$  converges quickly, a property which would be desirable in any computational implementation of these OVFs.

In a thread related to the work undertaken in Section 4.4, one could consider, instead of the integer lattices of that section, more general lattices. In the case of the motion group, this is not particularly difficult, although doing it introduces a small degree of notational difficulty. In the case of the Heisenberg group, a modification of the results of [49] on the subrepresentations of  $R$  is needed. Following these ideas would provide a more satisfactory and complete theory than the one we have given.

Another interesting vein for future research may be the extension of our perturbative result to other Lie groups. One case in which this may be possible is the case when  $G$  is simply connected and nilpotent. In this case, let  $\mathfrak{g}$  be the corresponding Lie algebra, with real dual space  $\mathfrak{g}^*$ , and denote by  $\text{Ad}(x) : \mathfrak{g} \rightarrow \mathfrak{g}$  the action taking  $Y \in \mathfrak{g}$  to the tangent vector to the curve  $t \mapsto x[\exp tY]x^{-1}$  at  $t = 0$ . The *co-adjoint* action  $\text{Ad}^*$  of  $G$  on  $\mathfrak{g}^*$  is defined by  $\text{Ad}^*(x) = [\text{Ad}(x^{-1})]^*$ . By [37], there is a continuous bijection from the space of co-adjoint orbits  $\mathfrak{g}^*/G$  to  $\hat{G}$ , where the latter is given the so-called Fell topology. Further, by [13], this map is actually a homeomorphism. Thus, in this case, it may be possible to perturb some elements of  $\hat{G}$  by considering them as elements of  $\mathfrak{g}^*/G$ , which is a quotient space of a metric space.

Perhaps more interesting than the analysis of  $L^2(E)$  would be an analysis of spaces of the form  $L^2(E/K)$ , where  $K$  is a closed subgroup of  $G$ . The space  $G/K$  is an example of a *G-space*—*i.e.*, a locally compact, Hausdorff space acted on continuously

by the left action of  $G$ . In fact, as shown in [26, Proposition 2.44], if  $G$  is  $\sigma$ -compact, every  $G$ -space is homeomorphic to one of this form. Suppose  $K$  is compact,  $q : G \rightarrow G/K$  is the canonical quotient map,  $P_K : C_c(G) \rightarrow C_c(G/K)$  is defined as in Section 4.2, and  $\{\pi_j\}$  is an OVF of representations for  $L^2(E)$ . Define  $\pi'_j(f)$  for  $f \in C_{E/K}(G/K)$  to be  $\int_G f \circ q(x) \pi_j(x) dx$ . Since  $\{\pi_j\}$  is an OVF of representations and  $f \circ q$  is supported on the compact set  $EK$ , we have that

$$\sum_j \|\pi'_j(f)\|_{\text{HS}}^2$$

is bounded above and below by a nonzero multiple of

$$\int_G |f \circ q(x)|^2 dx.$$

Using Theorem 4.2.1, this bound becomes  $|K| \int_{G/K} |f(xK)|^2 d\mu(xK)$ . Since these bounds hold on a dense subset of  $L^2(E/K)$ , they hold on all of  $L^2(E/K)$ . As an example, let  $G = E(2)$ ,  $K = \{0\} \times \text{SO}(2)$ , and  $E = \mathbb{D}/2 \times \text{SO}(2)$ . In this case, the quotient  $E/K$  can be identified with  $\mathbb{D}/2$ , so  $\{\pi'_j\}$  forms an OVF for  $L^2(\mathbb{D}/2)$ . It would be interesting to see if this analysis extends to other quotient spaces.

Given an OVF of representations  $\{\pi_j\}$  for  $L^2(E)$ , a fruitful vein for future research may be the question of the existence of ‘‘Gabor systems’’ for  $L^2(G)$  derived from  $\{\pi_j\}$ . As described in Example 2.3.6, a Gabor system for  $L^2(\mathbb{R})$  is a system of vectors of the form  $g_{m,n}(x) = e^{2\pi i m x} g(x - nb)$  for some  $a, b > 0$  and some generator function  $g \in L^2(\mathbb{R})$ . Given  $ab < 1$ ,  $\{g_{m,n}\}$  is a (tight) frame for  $L^2(\mathbb{R})$ . One way to interpret this is that  $g_{m,\gamma}(x) = \chi_m(x) g(\gamma^{-1}x)$ , where  $\{\chi_m\}$  forms a frame of exponentials for the appropriate  $L^2$ -space, and  $\gamma$  is a member of some lattice  $\Gamma_1$ . Under this interpretation, the question is whether operators  $G_{j,\gamma}$  specified by  $f \in L^2(G) \mapsto \int_G f(x) \pi_j(x) \overline{g(\gamma^{-1}x)} dx$ , for some generator function  $g \in L^2(G)$ , form an OVF for  $L^2(G)$ . Such an analysis of  $L^2(G)$  would presumably be of interest as a possible discrete replacement for the (generally continuous) Fourier transform on  $L^2(G)$ .

Finally, we mention a possible research direction related to harmonic OVF's and sampling theory. By the Poisson Summation Formula, if  $f \in C_c^\infty(\mathbb{R})$  and  $\gamma > 0$ , then

$$\sum_k \hat{f}(\gamma k) e^{2\pi i \gamma k x} = f(x) + \sum_{j \neq 0} f(x + j/\gamma),$$

so that if  $f$  is supported on a set of measure larger than  $1/\gamma$ , the series on the left does not reconstruct  $f$  exactly, but rather up to some additional terms which are translates of  $f$ . The situation is similar for the Selberg Trace Formula for  $(G, \Gamma)$ , which states that for  $f \in C_c^\infty(G)$ ,

$$\sum_j \text{Tr} \left( \int_G f(y) \pi_j(y x^{-1}) dy \right) = f(x) + \int_{\Gamma \backslash G} \sum_{\mathbb{1}_G \neq \gamma} f(x^{-1} \gamma x) d\mu(\Gamma x).$$

If the support of  $f$  is contained in some reproducing set, then all of the terms on the right-hand side except the first one vanish. But if the support of  $f$  is not contained in such a set, some of the terms may not vanish. In the case of the Poisson Summation Formula, one is interested in how large the resulting reconstruction error inside  $L^2(E)$  is. In this case, this is easily done: the error is

$$\left\| \sum_{j \neq 0} f(\cdot + j/\gamma) \right\|_{L^2(-1/2\gamma, 1/2\gamma)}^2 = \int_{|x| \geq 1/2\gamma} |f(x)|^2 dx.$$

In the more general case, though it is not so easy to quantify this reconstruction error. Given that general members  $f$  of  $C_c^\infty(G)$  may be of interest, a fruitful question to pursue may be how large this error term is in terms of the support or other properties of  $f$ .

## REFERENCES

- [1] AMAL AAFIF, *Non-Commutative Harmonic Analysis on Certain Semi-Direct Product Groups*, PhD thesis, Drexel University, 2007.
- [2] M. R. ABDOLLAHPOUR AND M. H. FAROUGHI, *Continuous  $g$ -frames in Hilbert spaces*, Southeast Asian Bulletin of Mathematics, 32 (2008), pp. 1–19.
- [3] S. T. ALI, J. P. ANTOINE, AND J. P. GAZEAU, *Coherent States, Wavelets and Their Generalizations*, Springer Verlag, 2000.
- [4] JAMES ARTHUR, *An introduction to the trace formula*, Harmonic Analysis, the Trace Formula, and Shimura Varieties, 4 (2005), pp. 1–263.
- [5] W.B. ARVESON, *Subalgebras of  $C^*$ -algebras*, Acta Mathematica, 123 (1969), pp. 141–224.
- [6] M. ATIYAH, *Characters of semisimple Lie groups*, Collected Works, 4 (1976), pp. 487–556.
- [7] S. A. AVDONIN, *On the question of Riesz bases of exponential functions in  $L_2$* , Vestnik Leningrad Univ. Ser. Mat., 13 (1974), pp. 5–12.
- [8] JOHN J. BENEDETTO, *Wavelets: Mathematics and Applications*, vol. 13, CRC press, 1993.
- [9] I. U. M. BEREZANSKII AND I. U. G. KONDRATEV, *Spectral Methods in Infinite-Dimensional Analysis*, vol. 1, Springer, 1995.
- [10] ARNE BEURLING, *Local harmonic analysis with some applications to differential operators*, Some Recent Advances in the Basic Sciences, 1 (1966), pp. 1962–1964.
- [11] E. BISHOP AND K. DE LEEUW, *The representations of linear functionals by measures on sets of extreme points*, in Annales de l’institut Fourier, vol. 9, Institut Fourier, 1959, pp. 305–331.
- [12] O. BRATTELI AND D. W. ROBINSON, *Operator Algebras and Quantum Statistical Mechanics 1:  $C^*$ - and  $W^*$ -Algebras, Symmetry Groups, Decomposition of States*, vol. 1, Springer Verlag, 2nd ed., 1987.
- [13] IAN D. BROWN, *Dual topology of a nilpotent Lie group*, in Annales scientifiques de l’École Normale Supérieure, vol. 6, Société mathématique de France, 1973, pp. 407–411.
- [14] F. BUSCEMI, G. M. D’ARIANO, M. KEYL, P. PERINOTTI, AND R. WERNER, *Clean positive operator valued measures*, arXiv preprint quant-ph/0505095, (2005).
- [15] P. G. CASAZZA AND G. KUTYNIOK, *Frames of subspaces*, arXiv preprint math/0311384, (2003).



- [16] P. G. CASAZZA, G. KUTYNIOK, AND S. LI, *Fusion frames and distributed processing*, Applied and Computational Harmonic Analysis, 25 (2008), pp. 114–132.
- [17] G. CASSINELLI, E. DE VITO, AND A. TOIGO, *Positive operator valued measures covariant with respect to an irreducible representation*, arXiv preprint quant-ph/0302187, (2003).
- [18] G. CHIRIBELLA, G. M. D’ARIANO, AND D. SCHLINGEMANN, *Barycentric decomposition of quantum measurements in finite dimensions*, Journal of Mathematical Physics, 51 (2010), p. 022111.
- [19] O. CHRISTENSEN, *An Introduction to Frames and Riesz Bases*, Birkhäuser Boston, 2002.
- [20] INGRID DAUBECHIES ET AL., *Ten Lectures on Wavelets*, vol. 61, SIAM, 1992.
- [21] I. DAUBECHIES, A. GROSSMANN, AND Y. MEYER, *Painless nonorthogonal expansions*, Journal of Mathematical Physics, 27 (1986), p. 1271.
- [22] A. DIETMAR AND S. ECHTERHOFF, *Principles of Harmonic Analysis*, Springer, 2009.
- [23] ULISSE DINI, *Serie de Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale*, vol. 1, T. Nistri, 1880.
- [24] MONIKA DÖRFLER, HANS G. FEICHTINGER, AND KARLHEINZ GRÖCHENIG, *Time-frequency partitions for the Gelfand triple  $(S_0, L^2, S'_0)$* , Mathematica Scandinavica, 98 (2006), pp. 81–96.
- [25] R. J. DUFFIN AND A. C. SCHAEFFER, *A class of nonharmonic Fourier series*, Transactions of the American Mathematical Society, (1952), pp. 341–366.
- [26] G. B. FOLLAND, *A Course in Abstract Harmonic Analysis*, vol. 18, CRC Press LLC, 1995.
- [27] —, *Real Analysis: Modern Techniques and Their Applications*, vol. 361, Wiley New York, 1999.
- [28] MICHAEL FRAZIER, BJÖRN JAWERTH, AND GUIDO L. WEISS, *Littlewood-Paley Theory and the Study of Function Spaces*, vol. 79, American Mathematical Society, 1991.
- [29] JEAN-PIERRE GABARDO AND DEGUANG HAN, *Frames associated with measurable spaces*, Advances in Computational Mathematics, 18 (2003), pp. 127–147.
- [30] KARLHEINZ GRÖCHENIG, *Acceleration of the frame algorithm*, IEEE Transactions on Signal Processing, 41 (1993), pp. 3331–3340.
- [31] —, *Foundations of Time-Frequency Analysis*, Springer, 2001.

- [32] DEGUANG HAN, DAVID R. LARSON, BEI LIU, AND RUI LIU, *Operator-Valued Measures, Dilations, and the Theory of Frames*, American Mathematical Society, 2014.
- [33] C. E. HEIL AND D. F. WALNUT, *Continuous and discrete wavelet transforms*, SIAM Review, 31 (1989), pp. 628–666.
- [34] E. HEWITT AND K. A. ROSS, *Abstract Harmonic Analysis, vol. I*, Springer-Verlag, Berlin, 1963.
- [35] M. I. KADEC, *The exact value of the Paley-Wiener constant*, in Dokl. Akad. Nauk SSSR, vol. 155, 1964, pp. 1253–1254.
- [36] V. KAFTAL, D. R. LARSON, AND S. ZHANG, *Operator-valued frames*, Transactions of the American Mathematical Society, 361 (2009), pp. 6349–6385.
- [37] A. A. KIRILLOV, *Unitary representations of nilpotent Lie groups*, Russian Mathematical Surveys, 17 (1962), pp. 53–104.
- [38] F. I. MAUTNER, *Unitary representations of locally compact groups II*, Annals of Mathematics, (1950), pp. 528–556.
- [39] B. MORAN, S. HOWARD, AND D. COCHRAN, *Positive-operator-valued measures: A general setting for frames*, in Excursions in Harmonic Analysis, vol. 2, Springer, 2013, pp. 49–64.
- [40] RAYMOND PALEY AND NORBERT WIENER, *Fourier transforms in the complex domain*, vol. 19, American Mathematical Society, 1934.
- [41] HANS REITER AND JAN D. STEGEMAN, *Classical Harmonic Analysis and Locally Compact Groups*, Clarendon Press Oxford, 1968.
- [42] JOSEPH M. RENES, ROBIN BLUME-KOHOUT, ANDREW J. SCOTT, AND CARLTON M. CAVES, *Symmetric informationally complete quantum measurements*, arXiv preprint quant-ph/0310075, (2003).
- [43] BENJAMIN ROBINSON, WILLIAM MORAN, DOUGLAS COCHRAN, AND STEPHEN D. HOWARD, *Operator-valued frames for the Heisenberg group*, arXiv preprint arXiv:1409.7773, (2014).
- [44] I. E. SEGAL, *An extension of Plancherel’s formula to separable unimodular groups*, Annals of Mathematics, (1950), pp. 272–292.
- [45] —, *Decompositions of operator algebras, I*, Memoirs of the American Mathematical Society, 9 (1951), pp. 1–67.
- [46] ATLE SELBERG, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, Journal of the Indian Mathematical Society, 20 (1956), pp. 47–87.

- [47] —, *Discontinuous groups and harmonic analysis*, in Proceedings of the International Congress of Mathematicians, 1962, pp. 177–189.
- [48] WENCHANG SUN, *G-frames and g-Riesz bases*, Journal of Mathematical Analysis and Applications, 322 (2006), pp. 437–452.
- [49] SUNDARAM THANGAVELU, *Harmonic analysis on Heisenberg nilmanifolds*, Revista de la Unión Matemática Argentina, 50 (2009), pp. 75–93.
- [50] J. L. WALSH, *A generalization of the Fourier cosine series*, Transactions of the American Mathematical Society, 22 (1921), pp. 230–239.

## APPENDIX A

A DECOMPOSITION OF THE QUASI-REGULAR REPRESENTATION FOR  
 $(E(2), \mathbb{Z}^2)$

In this appendix we derive a decomposition of the quasi-regular representation for  $(E(2), \mathbb{Z}^2)$  into subrepresentations. To do this, we must make a brief digression into the topic of induced representations.

Let  $G$  be a locally compact group,  $H$  be a closed subgroup,  $q$  be the canonical projection of  $G$  onto  $G/H$ ,  $\sigma$  be a unitary representation of  $H$  on  $\mathcal{H}_\sigma$ , and the inner product and norm of  $\mathcal{H}_\sigma$  be  $\langle \cdot, \cdot \rangle_\sigma$  and  $\|\cdot\|_\sigma$ . We denote by  $C(G, \mathcal{H}_\sigma)$  the space of continuous functions from  $G$  to  $\mathcal{H}_\sigma$ . Let

$$\begin{aligned} \mathcal{F}_0 &= \{f \in C(G, \mathcal{H}_\sigma) : q(\text{supp} f) \text{ is compact and} \\ &\quad f(x\xi) = \sigma(\xi^{-1})[f(x)] \text{ for all } x \in G, \xi \in H\}. \end{aligned}$$

Suppose there exists a left invariant measure  $\mu$  on  $G/H$ . The representation  $\sigma \uparrow_H^G$  is defined to be the unique extension of the  $G$ -action of left translation on  $\mathcal{F}_0$  to the completion  $\mathcal{H}$  of  $\mathcal{F}_0$  with respect to the inner product

$$\langle f, g \rangle = \int_{G/H} \langle f(x), g(x) \rangle_\sigma d\mu(xH).$$

(Note that this inner product is well-defined since  $\langle f(x), g(x) \rangle_\sigma$  depends only on the coset of  $x$ .) That this process yields a unitary representation of  $G$  is checked in [26, Section 6.1].

Denoting by 1 the trivial representation of  $H$  on  $\mathbb{C}$ , then  $\mathcal{F}_0$  can be identified with  $C_c(G/H)$  and  $\mathcal{H}$  can be identified with  $L^2(G/H)$ . It is then clear that  $1 \uparrow_H^G$  is the left quasi-regular representation  $L$  for  $(G, H)$ .

Let  $q'$  be the canonical projection of  $G$  onto  $H \backslash G$  and  $\mathcal{F}'_0$  be defined by

$$\begin{aligned} \mathcal{F}'_0 &= \{f \in C(G, \mathcal{H}_\sigma) : q'(\text{supp} f) \text{ is compact and} \\ &\quad f(\xi x) = \sigma(\xi)[f(x)] \text{ for all } x \in G, \xi \in H\}. \end{aligned}$$

Then the same process, with right-translation in place of left-translation, yields a unitary representation of  $G$  that will denote  $\overset{G}{H} \uparrow \sigma$ . Suppose there exists a right-invariant measure  $\mu'$  on  $H \backslash G$ . If we again denote the trivial representation of  $H$  on  $\mathbb{C}$  by 1, then  $\overset{G}{H} \uparrow 1$  is the *right* quasi-regular representation  $R$  for  $(G, H)$ .

We recite here a property of  $\mathcal{F}'_0$  that will be useful: it follows from [26, Proposition 6.1] that for every  $h \in \mathcal{F}'_0$ , there is  $\alpha \in C_c(G)$  such that

$$h(x) = \int_H \alpha(\eta x) d\eta,$$

where  $d\eta$  denotes Haar measure on  $H$ .

We can now show the unitary equivalence of  $R$  and  $L$  when  $G$  and  $H$  are unimodular. In this case, both  $G/H$  and  $H \backslash G$  admit invariant measure  $\mu$  and  $\mu'$ , which we may normalize according to the normalization of  $dx$ , the normalization of  $d\eta$ , and Theorem 4.2.1. We first claim that the map  $U : \mathcal{F}_0 \rightarrow \mathcal{F}'_0$  given by  $f(x) \mapsto f(x^{-1})$  is

an isometry. First, we note that

$$\begin{aligned}
\langle Uf, Ug \rangle_{L^2(H \backslash G)} &= \int_{H \backslash G} f(x^{-1}) \overline{g(x^{-1})} d\mu'(Hx) \\
&= \int_{H \backslash G} \int_H \alpha(\eta x) d\eta d\mu'(Hx) \\
&= \int_G \alpha(x) dx,
\end{aligned}$$

where  $\alpha$  is chosen as above with  $h(x) = (f\bar{g})(x^{-1})$  and we have used Theorem 4.2.1 in the last step. On the other hand, with  $\alpha^\top(x) = \alpha(x^{-1})$ , and using unimodularity of  $G$  and  $H$ , we have

$$\begin{aligned}
\langle f, g \rangle_{L^2(G/H)} &= \int_{G/H} f(x) \overline{g(x)} d\mu(xH) \\
&= \int_{G/H} (f\bar{g})(x) d\mu(xH) \\
&= \int_{G/H} \int_H \alpha(\eta x^{-1}) d\eta d\mu(xH) \\
&= \int_{G/H} \int_H \alpha^\top(x\eta^{-1}) d\eta d\mu(xH) \\
&= \int_{G/H} \int_H \alpha^\top(x\eta) d\eta d\mu(xH) \\
&= \int_G \alpha^\top(x) dx \\
&= \int_G \alpha(x) dx.
\end{aligned}$$

Thus,  $U$  is an isometry. If  $\lambda$  is the action of left  $G$ -translation acting on  $\mathcal{F}_0$  and  $\rho$  is the action of right  $G$ -translation acting on  $\mathcal{F}'_0$ , then  $\rho U = U^{-1}\lambda$ . Thus, if  $\bar{U}$  is the unique unitary extension of  $U$  mapping  $L^2(G/H)$  onto  $L^2(H \backslash G)$ , then  $R\bar{U} = \bar{U}^{-1}L$ , as desired.

For the remainder of this appendix, we will focus on decomposing  $L$  for unimodular groups  $G$  and  $H$ , specializing at the end to  $G = E(2)$  and  $H = \mathbb{Z}^2 \times \{\mathbb{1}_K\}$ . For this, there are a few preliminary results that will be useful, two of which we mention without proof.

**Proposition.** [26, Proposition 6.9] *If  $\{\sigma_i\}$  is any family of representations of  $H$ , then  $(\bigoplus \sigma_i) \uparrow_H^G$  and  $\bigoplus \sigma_i \uparrow_H^G$  are unitarily equivalent.*

**Proposition.** [26, Theorem 6.14] *Suppose  $H$  is a closed subgroup of  $G$ ,  $K$  is a closed subgroup of  $H$ , and  $\sigma$  is a unitary representation of  $K$ . Then the representations  $\sigma \uparrow_K^G$  and  $\sigma \uparrow_K^H \uparrow_H^G$  are unitarily equivalent.*

**Proposition A.1.** *Let  $H \triangleleft G$  and  $G/H$  is compact and suppose  $\{(\sigma_i, \mathcal{H}_i)\}$  is a decomposition of the left regular representation  $\lambda$  on the group  $G/H$  into subrepresentations. Then  $1 \uparrow_H^G = \bigoplus \underline{\sigma}_i$ , where  $\underline{\sigma}_i(x) = \sigma_i(q(x))$ .*

*Proof.* We have chosen to assume  $G/H$  is compact so that we can restrict our attention to direct sums of representations. Notice that if we define  $\underline{\lambda}(x) = \lambda(q(x))$ , we have  $\underline{\lambda} = 1 \uparrow_H^G$ . Next, observe the following

$$\underline{\lambda}(x)|_{\mathcal{H}_i} = \lambda(q(x))|_{\mathcal{H}_i} = \sigma_i(q(x))|_{\mathcal{H}_i} = \underline{\sigma}_i(x)|_{\mathcal{H}_i}.$$

These equalities prove that  $\underline{\lambda}$  decomposes as  $\bigoplus \underline{\sigma}_i$ . We note that the representations  $\underline{\sigma}_i$  are also irreducible if the representations  $\sigma_i$  are.  $\square$

Suppose  $G = H \rtimes K$ , with  $H$  abelian,  $K$  compact, and  $\Gamma$  a co-compact subgroup of  $H$ . In what follows, we identify  $H \times \{\mathbb{1}_K\}$  with  $H$  and  $\Gamma \times \{\mathbb{1}_K\}$  with  $\Gamma$ . The decomposition of  $L$  is then given by

$$\begin{aligned} L &\cong 1 \uparrow_\Gamma^G \\ &\cong 1 \uparrow_\Gamma^H \uparrow_H^G \\ &\cong \left( \bigoplus \underline{\chi}_j \right) \uparrow_H^G, \end{aligned}$$

where  $\chi_j$  is a character of  $H/\Gamma$  and  $\underline{\chi}_j$  is the character on  $H$  given by  $\underline{\chi}_j(h) = \chi_j(h\Gamma)$ . We will denote  $\underline{\chi}_j$  by  $\nu_j$ . Continuing, the above is unitarily equivalent to

$$\bigoplus \nu_j \uparrow_H^G.$$

If  $G = E(n)$ , the index set for  $j$  is just  $\mathbb{Z}^n$ , and  $\nu_j$  is part of the larger family  $\nu_\lambda : \mathbb{R}^n \rightarrow \mathbb{C}$  given by  $\nu_\lambda(x) = e^{-2\pi i \lambda \cdot x}$  for  $\lambda \in \mathbb{R}^n$ . For  $n > 2$ , the representations  $\nu_\lambda \uparrow_H^G$  will not in general be irreducible. However, when  $n = 2$ , all of them are, except when  $\lambda = 0$ , as explained in [1, Section 6.1.2]. This author also proves, for  $n = 2$ , that  $\nu_\lambda \uparrow_H^G$  is equivalent to a representation  $\rho_\lambda$  acting on  $L^2(K)$  by

$$(\rho_\lambda(h, k)\phi)(k_0) = e^{-2\pi i \lambda^{k_0} \cdot h} \phi(k^{-1}k_0)$$

for  $h \in H$ ,  $k \in K$ , and  $\phi \in L^2(K)$ . Thus, for the quasi-regular representation for  $(E(2), \mathbb{Z}^2)$  we have the decomposition

$$R = \bigoplus_{j \in \mathbb{Z}^2} \rho_j, \tag{A.1}$$

as desired.

**Remark.** We note here that when  $j = 0$ , the representation  $\rho_0$  is reducible as the direct sum of characters of  $K \cong \mathbb{T}$ , but we choose not to decompose  $\rho_0$  so that *all* the representations occurring in the (A.1) can be “perturbed” in the sense of Theorem 4.4.7.