A Study on Constrained State Estimators

by

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ABSTRACT

This study focuses on state estimation of nonlinear discrete time systems with constraints. Physical processes have inherent in them, constraints on inputs, outputs, states and disturbances. These constraints can provide additional information to the estimator in estimating states from the measured output. Recursive filters such as Kalman Filters or Extended Kalman Filters are commonly used in state estimation; however, they do not allow inclusion of constraints in their formulation. On the other hand, computational complexity of full information estimation (using all measurements) grows with iteration and becomes intractable.

One way of formulating the recursive state estimation problem with constraints is the Moving Horizon Estimation (MHE) approximation. Estimates of states are calculated from the solution of a constrained optimization problem of fixed size. Detailed formulation of this strategy is studied and properties of this estimation algorithm are discussed in this work. The problem with the MHE formulation is solving an optimization problem in each iteration which is computationally intensive.

State estimation with constraints can be formulated as Extended Kalman Filter (EKF) with a projection applied to estimates. The states are estimated from the measurements using standard Extended Kalman Filter (EKF) algorithm and the estimated states are projected on to a constrained set. Detailed formulation of this estimation strategy is studied and the properties associated with this algorithm are discussed.

Both these state estimation strategies (MHE and EKF with projection) are tested with examples from the literature. The average estimation time and the sum of square estimation error are used to compare performance of these estimators. Results of the case studies are analyzed and trade-offs are discussed.

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Table of Contents

List of Tables

List of Figures

Figure Page

Figure Page

Chapter 1

INTRODUCTION

Many process control problems like pH neutralization, polymerization, temperature control and flow control exhibit nonlinear behavior. Linear state space modeling of these systems is not sufficient in controlling and forecasting them. However, these processes can be modeled accurately using nonlinear state space equations.

The states of the system summarize its past behavior and can be used to predict its future behavior. State estimation is crucial for control strategies like Model Predictive Control (MPC) and for monitoring process performance. In most cases, the states are not completely measurable and measurements of process variables can be used to estimate the states.

Constraints on the system states (for example concentrations cannot be negative), inputs (e.g., flows cannot be negative) and outputs (pH can take between value between 0 and 14) are inherent in process control models. Inclusion of constraints in the state estimation formulation helps in correcting modeling errors and other uncertainties associated with system operation.

Kalman Filter (KF) is a commonly used method in estimating states of a linear system. For linear dynamical systems, the Kalman Filter provides the optimal estimates of states from the measured input and output in the presence of state and output noise. Many people have investigated the state estimation problem for nonlinear linear systems. Some of the early works in nonlinear state estimation include formulation of state estimation as nonlinear state observer system (like Luenberger observer (Van Der Schaft, 1985; Tatiraju *et al.*, 1999)) and Extended Kalman filter

(EKF) (Maybeck, 1982; Ribeiro, 2004). The state observer formulation allows direct tuning of filter gain, which helps in obtaining sufficient conditions for asymptotic stability of the observer system. This method, however, does not allow the inclusion of information about measurement and state noise in its formulation. On the other hand, filter gain of EKF is obtained using Kalman update formula for the linearized system around previous estimates and covariance matrices of state and output noise. It is hard to obtain the required conditions for asymptotic stability of the estimator for EKF formulation. Because of its simplicity and low computational burden, EKF is widely used in the state estimation of nonlinear systems. Some of the applications of EKF are: State estimation and control of polymerization process (Kim and Choi, 1990; Crowley and Choi, 1996) and state estimation of systems described using deferential-algebraic equation (DAE) (Becerra *et al.*, 2001). Some of the shortcomings of EKF are addressed in a slightly different extension of the Kalman filter known as the Unscented Kalman Filter (UKF) (Julier and Uhlmann, 1997). None of these methods do not allow inclusion of constraints in their formulations.

State estimation can also be formulated as a solution of the optimization problem (minimization of weighted estimation error). A description of state estimation of linear systems without constraints as receding horizon estimator is provided by Thomas (Thomas, 1975), that involves solving fixed size optimization problem in each step to the estimate states of the system. Jang et al. formulated state estimation of nonlinear system without constraints as solution of fixed size optimization problem (Jang *et al.*, 1986). A constrained state estimator can be obtained by including constraints with in the optimization problem. Ideally, a state estimator has to minimize the weighted mean square estimation error by satisfying constraints for states and disturbances using all available outputs. The growth of size of the optimization problem with time makes this strategy impractical. Fixed size approximation of the full information optimization called Moving Horizon Estimation (MHE) is considered to make state estimation tractable. Derivation of MHE from full information estimator and asymptotic stability of the estimator for linear systems is described by Rao et al.(Rao *et al.*, 2001) and for nonlinear systems (Rao *et al.*, 2003; Rao and Rawlings, 2002). Robust MHE for linear systems is described by Alessandri et al. (Alessandri *et al.*, 2005). Inclusion of constraints makes MHE more robust, advantages of MHE over EKF is discussed by Haseltine and Rawlings (Haseltine and Rawlings, 2005).

MHE formulation is a practical strategy and can be used in online estimation of the states. Estimated states can be used for process monitoring and state feedback. Russo and Young utilized MHE in estimating states of industrial polymerization process (Russo and Young, 1999). MHE in state feedback is used for Model Predictive Control (MPC) with constraints (Rawlings, 2000; Sui *et al.*, 2008). Rao and Rawlings showed usage of MHE in process monitoring (Rao and Rawlings, 2002). Zavala and Biegler used MHE in state estimation in the operation of multi-zone low-density (LDPE) polyethylene tubular reactors (Zavala and Biegler, 2009). Application of MHE to an industrial gas phase polymerization reactor to improve estimates of current states and parameters was shown by Hedengren et al. (Hedengren *et al.*, 2007). Russo and Young discussed the usage of MHE to estimate states of industrial polymerization processes and issues encountered in implementation and choice of tuning parameters for MHE (Russo and Young, 1999). Application of alternate formulation of MHE(with integrators) in nonlinear model predictive control with non-zero mean disturbances have also been reported in the literature(Tenny and Rawlings, 2002; Tenny *et al.*, 2004).

Trying to formulate MHE for nonlinear systems can lead to non-convex optimizations. It is well known that solving non-convex optimization problem to find a global optimum is extremely time consuming and most of the existing solvers can end up trapped in a local optimum (Rao and Rawlings, 2002; Becerra *et al.*, 2001; Tenny *et al.*, 2004).

Many people have considered modified versions of Kalman Filter, EKF and UKF to include constraints. Different versions of Kalman Filter with constraints on states of the system for linear systems were proposed (Rengaswamy *et al.*, 2013; Simon and Chia, 2002; Yang and Blasch, 2006). However, these modifications fail to include constraints on disturbances. Modification of EKF algorithm to include constraints called Recursive Nonlinear Dynamic Data Reconciliation(RNDDR) was proposed by Vachhani et al. (Vachhani *et al.*, 2004, 2005). RNDDR is similar to MHE and has a huge computational burden. UKF with constraints called Unscented Recursive Nonlinear Dynamic Data Reconciliation (URNDDR) is discussed by Vachhani et al. (Vachhani *et al.*, 2006). URNDDR require more computational time than RNDDR. Less computationally intensive versions of EKF with constraints on the states were proposed by Rengaswamy et al. and Simon (Rengaswamy *et al.*, 2013; Simon, 2010). Still, constraints on disturbances were ignored by all the studies mentioned above. All these modifications of KF, EKF and UKF have similar or less computational burden as MHE. The methods with less computational burden do not include constraints on disturbances.

Inclusion of constraints in parameter estimation using projection is shown by Tsakalis (Tsakalis, 1998). Modification of EKF to include constraints using projection called EKF with projection is proposed in this study. This involves estimation of states of the system using EKF algorithm and estimated states are projected on to the constrained set to obtain constrained estimates.

This study focuses on constrained state estimation as MHE and EKF with projection. Summary of MHE derivation and properties are discussed in chapter 2.

Summary of EKF with projection algorithm and its properties are discussed in chapter 3. Examples from the papers (Rao *et al.*, 2003; Rao and Rawlings, 2002; Tenny *et al.*, 2004) are considered as case studies to evaluate performance of these algorithms. To maintain consistency both the algorithms are tuned with same parameters and same realizations of input/output are used as inputs for estimators. Computational advantage of EKF with projection over MHE is discussed.

Chapter 2

MOVING HORIZON ESTIMATION (MHE)

This chapter discusses the formulation of a constrained state estimation problem for the discrete time nonlinear system as a Moving Horizon Estimation(MHE). The theory discussed in this chapter is a summary of Rao *et al.* (2001) and Rao *et al.* (2003).

Consider following the discrete time nonlinear system

$$x_{k+1} = f_k(x_k, u_k) + w_k$$

$$y_k = h_k(x_k, u_k) + v_k \tag{2.1}$$

where states, inputs and disturbances satisfy the following constraints

$$x_k \in X_k \quad w_k \in W_k \quad v_k \in V_k \quad u_k \in U_k.$$

For all $k \geq 0$, it is assumed that functions f_k and h_k and sets $X_k \subseteq R^n$, $U_k \subseteq R^m$, $W_k \subseteq R^n$ and $V_k \subseteq R^p$ are closed with $0 \in W_k$ and $0 \in V_k$.

Let, $x(k; z, l, \{w_j\}, \{u_j\})$ is the solution of the system (2.1) at time instance k, when z is the state of the system at time l, $\{w_j\}_{j=l}^k$ and $\{u_j\}_{j=l}^k$ are input disturbances and input sequences respectively.

Let, $y(k; z, l, \{w_j\}, \{u_j\}) = h_k(x(k; z, l, \{w_j\}, \{u_j\}), u_k)$ denote the estimated output of the system (2.1) for the given state solution $x(k; z, l, \{w_j\}, \{u_j\})$. Let y_k denote the actual output of the system.

The full information state estimation problem for the system (2.1) can be formulated as the solution of the following optimization problem.

$$\Phi_T^* = \min_{x_0,\{w_j\}_{k=0}^{T-1}} \{\Phi_T(x_0,\{w_k\}) : (x_0,\{w_k\}) \in \Omega_\tau\} \tag{2.2}$$

$$\Omega_\tau = \left\{ (x_0,\{w_k\}) \begin{array}{cc} x(k,x_0,0,\{w_j\},\{u_j\}) \in X_k, & k=0,...T \\ w_k \in W_k, & k=0,...(T-1) \\ v_k = y_k - y(x(k,x_0,0,\{w_j\},\{u_j\})) \in V_k, & k=0,...(T-1) \end{array} \right\}.$$

The expression for $\Phi_T(x_0,\{w_k\})$ is given by the following equation

$$\Phi_T(x_0,\{w_k\}) = \left(\sum_{k=0}^{T-1}(w_k^T {Q_k}^{-1} w_k + v_k^T {R_k}^{-1} v_k)\right) + (x_0-\hat{x}_0)^T \Pi_0^{-1}(x_0-\hat{x}_0) \tag{2.3}$$

where, $\hat{x}_0$ is a-priori value of initial state and Q_k, R_k and Π_0 are positive definite matrices. The matrices Q_k and R_k are assumed to be covariance matrices of input and output noise respectively.

The solution of the above optimization problem yields to the optimal pair $\{\hat{x}_{0|T-1}, \{\hat{w}_{k|T-1}\}_{k=0}^{T-1}\}$. The estimate of state using this optimal pair is given by

$$\hat{x}_{k|T-1} = x(k;\hat{x}_{0|T-1},0,\{\hat{w}_{j|T-1}\},\{u_j\}). \tag{2.4}$$

This formulation of the state estimation problem is called full information state estimation as all available outputs y_k is used in estimating states. Obtaining the online solution of this optimization problem is impractical because of the increase in computational burden with time T. Forward dynamical programing can be used to make full information estimation tractable using approximation. But, the approximation should preserve stability and performance of full information estimation.

Using Markov property of the system (2.1), the objective function for full information estimation can be rearranged as follows

$$\Phi_T(x_0,\{w_k\}) = \left(\sum_{k=T-N}^{T-1}(w_k^T {Q_k}^{-1} w_k + v_k^T {R_k}^{-1} v_k)\right) + \left(\sum_{k=0}^{T-N-1}(w_k^T {Q_k}^{-1} w_k + v_k^T {R_k}^{-1} v_k)\right)$$

$$+(x_0-\hat{x}_0)^T\Pi_0^{-1}(x_0-\hat{x}_0)$$

$$=\left(\sum_{k=T-N}^{T-1}(w_k^TQ_k{}^{-1}w_k+v_k^TR_k{}^{-1}v_k)\right)+\Phi_{T-N}(x_0,\{w_k\}) \tag{2.5}$$

The reachable set R_τ of states at given time τ, for feasible x_0 and $\{w_k,u_k\}_{k=0}^{\tau}$ is defined as follows

$$R_\tau=\{x(\tau;x_0,0,\{w_j\},\{u_j\}):\{x_0,\{w_j\}\}\in\Omega_\tau\}. \tag{2.6}$$

And, the arrival cost at time τ, for $z\in R_\tau$ is defined as follows

$$Z_\tau(z)=\min_{z,\{w_k\}_{k=0}^{\tau-1}}\{\Phi_\tau(x_0,\{w_k\}):(x_0,\{w_k\})\in\Omega_\tau,x(\tau;x_0,0,\{w_j\},\{u_j\})=z\} \tag{2.7}$$

The arrival cost summarizes the effect of past input $\{u_k\}_{k=0}^{T-N-1}$ and measurements $\{y_k\}_{k=0}^{T-N-1}$ on state x_{T-N}.

Because, this term $\left(\sum_{k=T-N}^{T-1}w_k^TQ_k{}^{-1}w_k+v_k^TR_k{}^{-1}v_k\right)$ only depends on x_{T-N} and $\{w_k,v_k,u_k\}_{k=T-N}^{T-1}$, equivalence between fixed size estimation and full information problem can be established by reformulating full information estimation (2.5) as follows

$$\Phi_T(x_0,\{w_k\})=\left(\sum_{k=T-N}^{T-1}w_k^TQ_k{}^{-1}w_k+v_k^TR_k{}^{-1}v_k\right)+Z_{T-N}(z) \tag{2.8}$$

If the analytical expression for arrival cost exists, solution for the full information estimation problem can be obtained by solving the above fixed size optimization problem. Unfortunately, the majority of systems do not posses algebraic expression for the arrival cost. One of the exceptions is if the system is linear and unconstrained. The state estimate $\hat{x}_k$ is the same as the state estimate given by the Kalman Filter and the arrival cost is given by

$$Z_j(z)=(z-\hat{x}_j)^T\Pi_j^{-1}(z-\hat{x}_j)+\Phi_j^*$$

where $\hat{x}_j$ is the estimate of state at time instance j. Π_j can be calculated as the solution of the Kalman filtering Riccati equation

$$\Pi_{j+1}=Q_j+A_j\Pi_jA_j^T-A_j\Pi_jC_j^T(R_j+C_j\Pi_jC_j^T)^{-1}C_j\Pi_jA_j^T \tag{2.9}$$

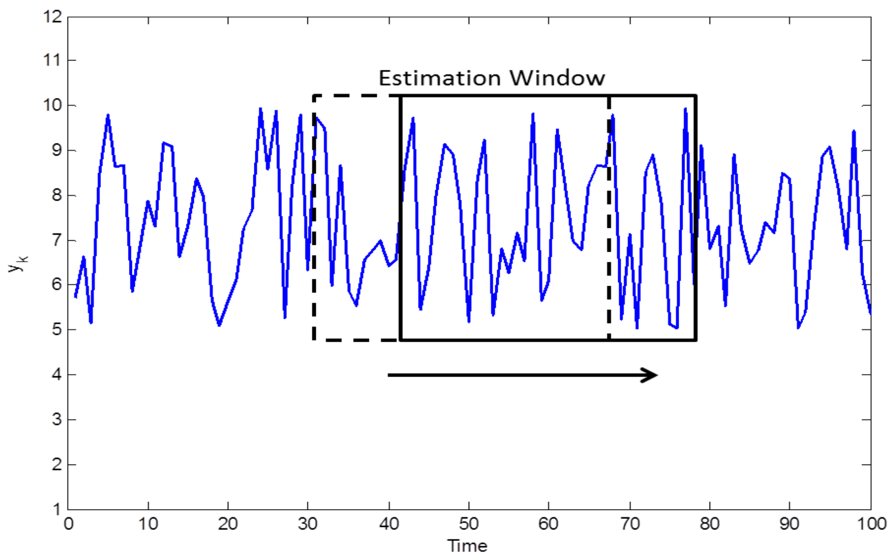

Figure 2.1: Visualization of Moving Horizon

The algebraic expression for arrival cost rarely exists if the system is nonlinear. In such cases, the state estimation (2.8) can be formulated as an MHE by considering approximation of arrival cost. The arrival cost approximation is used to account for the previous data outside the MHE window and it also provides a penalty for the deviation from past estimate. For chosen arrival cost approximation $\hat{Z}_i(.)$ the formulation of MHE is given below

$$\hat{\Phi}_T^* = \min_{z,\{w_j\}_{k=T-N}^{T-1}} \{\hat{\Phi}_T(z,\{w_k\}) : (z,\{w_k\}) \in \Omega_\tau^N\} \tag{2.10}$$

where $\hat{\Phi}_T(z,\{w_k\})$ is given by

$$\hat{\Phi}_T(z,\{w_k\}) = \left(\sum_{k=T-N}^{T-1} w_k^T {Q_k}^{-1} w_k + v_k^T {R_k}^{-1} v_k \right) + \hat{Z}_{T-N}(z)$$

$$\Omega_{\tau}^{N} = \left\{ (z, \{w_k\}) \begin{array}{rr} x(k, z, T-N, \{w_j\}, \{u_j\}) \in X_k, & k = (T-N), \\ & ...T \\ w_k \in W_k, & k = (T-N), \\ & ...(T-1) \\ v_k = y_k - y(x(k, z, T-N, \{w_j\}, \{u_j\})) \in V_k, & k = (T-N), \\ & ...(T-1). \end{array} \right\}$$

For the optimal pair $\{z^*, \{\hat{w}_{k|T-1}^{mh}\}_{k=T-N}^{T-1}\}$, moving horizon estimate of the state of the system $\{\hat{x}_{k|T-1}^{mh}\}_{k=T-N}^{T}$ is given by

$$\hat{x}_{k|T-1}^{mh} = x(k; z^*, T-N, \hat{w}_{j|T-1}^{mh}, u_j). \tag{2.11}$$

One strategy to approximate the arrival cost is by using the first order approximation of the Taylor series. Details of this approximation are given below

$$\hat{Z}_j(z) = (z - \hat{x}_j^{mh})^T \Pi_j^{-1} (z - \hat{x}_j^{mh}) + \Phi_T^*$$

As in Extended Kalman filter, Π_j is computed using Kalman filter covariance formula (2.9) with the linearized system matrices given below.

$$A_k := \frac{\partial f_k(x, 0, u_k)}{\partial x}|_{x=\hat{x}_k^{mh}}$$

$$G_k := \frac{\partial f_k(x_k^{mh}, w, u_k)}{\partial w}|_{w=0}$$

$$C_k := \frac{\partial h_k(x, u_k)}{\partial x}|_{x=\hat{x}_k^{mh}}.$$

MHE approximation gives fixed size approximation of the full information estimation problem. The sufficient conditions for stability of full information estimation for linear systems and necessary assumptions to guarantee stability of MHE approximation are discussed in are discussed in Tenny and Rawlings (2002). The sufficient conditions for asymptotic and bounded stability of full state estimator and MHE for nonlinear systems are discussed in Rao *et al.* (2003).

Some of the problems associated with MHE are stated below

- Non-convex nature of optimization problem may give local optima as solution.

- Solving optimization problems in each iteration has huge computational burden and this acts as a barrier for the online implementation.

- High computation time acts as a limitation on the closed-loop bandwidth, if MHE is used in state feedback control.

- Computational burden, also makes MHE undesirable for implementation on embedded real time boards.

- If the constraints are not chosen carefully, the estimator can provide spurious estimates.

Chapter 3

EXTENDED KALMAN FILTER(EKF) WITH PROJECTION

This chapter discusses the formulation of the standard Extended Kalman Filter (EKF) algorithm (Ribeiro (2004)) and formulation of a constrained state estimator as EKF with projection(Tsakalis (1998)).

Consider following discrete time nonlinear system

$$x_{k+1} = f_k(x_k, u_k) + w_k \tag{3.1}$$

$$y_k = h_k(x_k, u_k) + v_k \tag{3.2}$$

where,

$$u_k \in R^p$$

$$x_k \in R^n, f_k(x_k, u_k) : R^n \times R^p \to R^n$$

$$y_k \in R^r, h_k(x_k, u_k) : R^n \times R^p \to R^r$$

$$v_k \in R^r$$

$$w_k \in R^n$$

and $\{v_k\}$, $\{w_k\}$ are independent and identically distributed Gaussian random processes with zero mean and following covariance matrices

$$E[v_k v_k^T] = R_k$$

$$E[w_k w_k^T] = Q_k$$

.

The initial condition of the system x_0 is considered as a following Gaussian random vector given by,

$$x_0 \sim N(x_0, \Pi_0).$$

Let $\{y_i\}_k = \{y_1, y_2, ..., y_k\}$ be a set of system measurements. The goal of the estimator is to estimate states from the measurements $\{y_i\}_k$.

The estimator that minimizes the mean-square error evaluates the conditional mean of the PDF of x_k for given measurements $\{y_i\}_k$. Excluding special cases, it is necessary to have knowledge of the entire conditional PDF of x_k for given measurements $\{y_i\}_k$ to compute the conditional mean. One such exception is in the case of a linear system with Gaussian initial conditions, having state and process noise that are mutually independent, zero mean, white Gaussian processes. The conditional PDFs, $p(x_k|\{y_i\}_k)$, $p(x_{k+1}|\{y_i\}_k)$ and $p(x_{k+1}|\{y_i\}_{k+1})$, for this case are Gaussian and the Kalman Filter gives an iterative solution for state estimation.

If the system is nonlinear, the conditional PDFs, $p(x_k|\{y_i\}_k)$, $p(x_{k+1}|\{y_i\}_k)$ and $p(x_{k+1}|\{y_i\}_{k+1})$, are not Gaussian. The optimal estimator for nonlinear system has to propagate entire PDF in order to evaluate mean and variance of conditional PDFs, which results in heavy computational burden.

In order to make estimation less computational, approximation of estimation is considered. The Extended Kalman Filter (EKF) gives an approximation for the optimal estimate that minimizes estimation error for the linearized system. The nonlinear system is linearized around the last estimate and Kalman filter formulation is used to compute of the mean and covariance of the estimate.

The following consecutive steps are executed in each iteration of state estimation

1. Linearize non linear system dynamics $x_{k+1} = f_k(x_k, u_k) + w_k$ around last state estimate $\hat{x}_{k|k}$ and input u_k.

2. Prediction step of the Kalman Filter for the linearized system dynamics is used to compute $\hat{x}_{k+1|k}$ and $\Pi_{k+1|k}$.

3. Output equation of nonlinear system is linearized around $\hat{x}_{k+1|k}$ and u_{k+1}.

4. Filtering step of the Kalman Filter for linearized system dynamics is used to compute $\hat{x}_{k+1|k+1}$ and $\Pi_{k+1|k+1}$.

Following Matrices represent linearizaton

$$A_k := \frac{\partial f_k(x, u_k)}{\partial x}|_{x=\hat{x}_{k|k}}$$

$$C_{k+1} := \frac{\partial h_{k+1}(x, u_{k+1})}{\partial x}|_{x=\hat{x}_{k+1|k}}$$

Prediction and Filtering step of EKF are stated below

Prediction Step

$$\hat{x}_{k+1|k} = f_k(\hat{x}_{k|k}, u_k)$$

$$\Pi_{k+1|k} = A_k \Pi_{k|k} A_k^T + Q_k$$

Filtering Step

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + L_{k+1}[y_{k+1} - h_{k+1}(\hat{x}_{k+1|k}, u_{k+1})]$$

$$L_{k+1} = \Pi_{k+1|k} C_{k+1}^T [C_{k+1} \Pi_{k+1|k} C_{k+1}^T + R_{k+1}]^{-1}$$

$$\Pi_{k+1|k+1} = [I - L_{k+1} C_{k+1}] \Pi_{k+1|k}.$$

The Extended Kalman Filter (EKF) is not an optimal estimator and the matrices, $\Pi_{k+1|k}, \Pi_{k|k}$ and $\Pi_{k+1|k+1}$, do not represent covariances of the state estimate. Also, it is not possible to calculate the gain of this filter offline for steady-state implementation, since the linearized system matrices A_k and C_k are dependent on previous state estimates.

The stability of estimator is not guaranteed and is dependent on quality of approximation. That means if the approximations are not good, estimates of EKF may diverge from actual states.

Constraints on states and disturbance variables helps modeling uncertainties and process behaviors. The EKF algorithm can be made more robust by the inclusion of constraints. Formulation of a constrained state estimation as the Extended Kalman Filter with projection is given in the following section.

Extended Kalman Filter with projection

For the given vector $\grave{x}$, its projection of it on to set M with weight Π_T^{-1} can be formulated as the following minimization problem.

$$\Theta_T^* = \min_{\hat{x}} \{(\grave{x} - \hat{x})^T \Pi_T^{-1} (\grave{x} - \hat{x}) : \hat{x} \in M\}. \tag{3.3}$$

The vector $\hat{x}^*$ that minimizes above te objective function is called the weighted projection of $\grave{x}$ on to set M.

The Extended Kalman Filter algorithm is modified to include constraints by projecting the estimated state obtained from the standard Extended Kalman Filter on to the constrained set. The modified algorithm is called EKF with projection.

Details of the consecutive steps that need to be evaluated at every iteration are given below.

1. Linearize non linear system dynamics $x_{k+1} = f_k(x_k, u_k) + w_k$ around last state estimate $\hat{x}^*_{k|k}$ and input u_k.

2. Prediction step of the Kalman Filter for the linearized system dynamics is used to compute $\hat{x}_{k+1|k}$ and $\Pi_{k+1|k}$.

3. Output equation of nonlinear system is linearized around $\hat{x}_{k+1|k}$ and u_{k+1}.

4. Filtering step of the Kalman Filter for linearized system dynamics is used to compute $\hat{x}_{k+1|k+1}$ and $\Pi_{k+1|k+1}$.

5. Project $\hat{x}_{k+1|k+1}$ on to the constrained set M with $\Pi_{k+1|k+1}^{-1}$ as a weight to get the state estimate $\hat{x}^*_{k+1|k+1}$

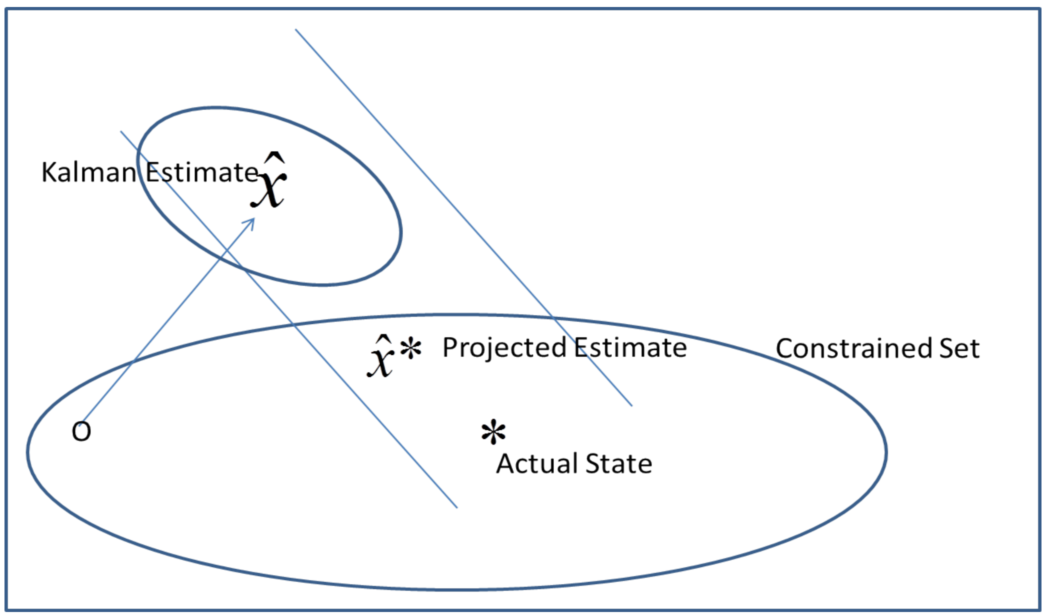

Figure 3.1: Visualization of the projection process.

The projection of states on to constrained set can be visualized as in the above figure. For any true state of the system in the constrained set, the constrained state estimate $\hat{x}^*_{k+1|k+1}$ is closer to the true state as compared to the unconstrained state estimate $\hat{x}_{k+1|k+1}$.

For linear systems the stability of the estimator is preserved and for the nonlinear systems the stability of the estimator is preserved locally and is dependent on the quality of approximation. In the case of large disturbances or unreasonable constraints, the true state can be outside constrained set. This may result in the state estimates getting stuck on the boundary of the constrained set.

Chapter 4

CASE STUDIES

Having studied formulation of MHE and EKF with projection, examples from the literature Rao and Rawlings (2002), Rao *et al.* (2003)and Tenny *et al.* (2004) are considered as case studies to test and compare the performance of these algorithms.

For these case studies the constrained state estimation problem is setup as EKF with projection, and MHE with horizon size $N = 1, 5$ and 10. To maintain consistency, values of matrices Q_k, R_k and Π_0 are chosen to be the same for MHE and EKF with projection and the same realizations of system input and system output are used as inputs for all estimators. The following figure 4.1 is a snap shot of the SIMULINK model setup used in the case studies.

All the case studies are simulated using MATLAB®2013a on a computer with 3rd Gen Intel®Core™i7-3770 processor (Quad Core, 3.40GHz, 8MB w/HD4000 Graphics) processor and 6GB, NON-ECC, 1600MHZ DDR3,2DIMM RAM.

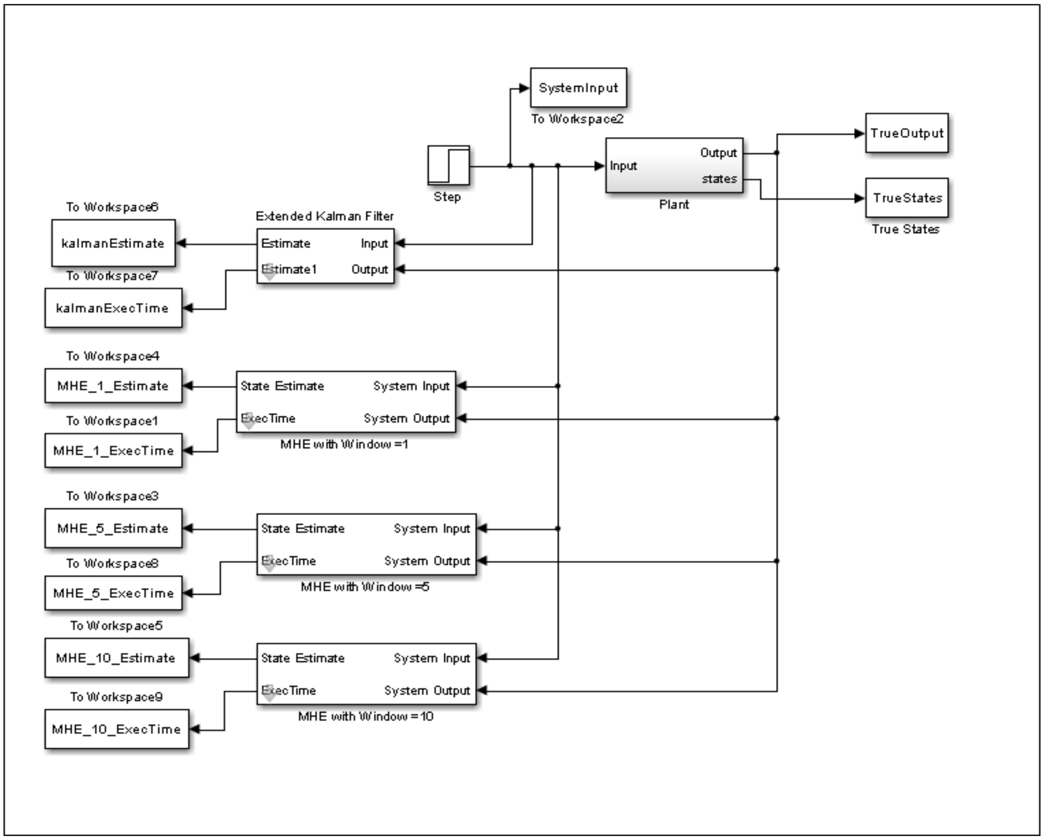

Figure 4.1: Snap shot of SIMULINK model file used in Case studies

the following metrics are used to compare the performances of the constrained state estimators, MHE and EKF with projection

- Sum of Square Estimation Error(SSEE)

$$\sum_{k=0}^{T}(x_k^j - \hat{x}_k^j)^2$$

  where x_k^j is j^{th} actual state value $\hat{x}_k^j$ is j^{th} state estimate.

- Average estimation time.

4.1 Case Study 1

4.1.1 Problem Description

Following linear discrete time system from Rao *et al.* (2003) is considered for this case study.

$$x_{k+1} = \begin{bmatrix} 0.99 & 0.2 \\ -0.1 & 0.3 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_k$$
$$y_k = \begin{bmatrix} 1 & -3 \end{bmatrix} x_k + v_k \tag{4.1}$$

It is assumed $\{v_k\}$ is a sequence of independent normally distributed random variables with zero mean and covariance of 0.01. And following scenarios are used to generate the sequence, w_k

1. $w_k = |z_k|$.

2. $w_k = min\{|z_k|\ , 2\}$.

where z_k is a sequence of normally distributed independent random variables with zero mean and covariance of identity. It is assumed that the initial state is normally distributed random variable with zero mean and covariance equal to the identity.

The constrained state estimation is formulated as MHE and Kalman filter with projection for this plant with $Q = 1$,$R = 0.01$, $\Pi_0 = 1$ and $\hat{x_0} = 0$. The matrix Π_k in MHE arrival cost is obtained from solving the discrete time matrix Riccati. $w_k \geq 0$ is chosen as a constraint for MHE and EKF with projection to capture the knowledge of the random variable w_k.

25 realizations of this state estimation problem are generated for the time length of 80 samples. The sum of square estimation error(SSEE) is computed for the average of 25 realizations. Results of the state estimation for this case study are shown in the following section.

Scenario 1 ($w_k = |z_k|$)

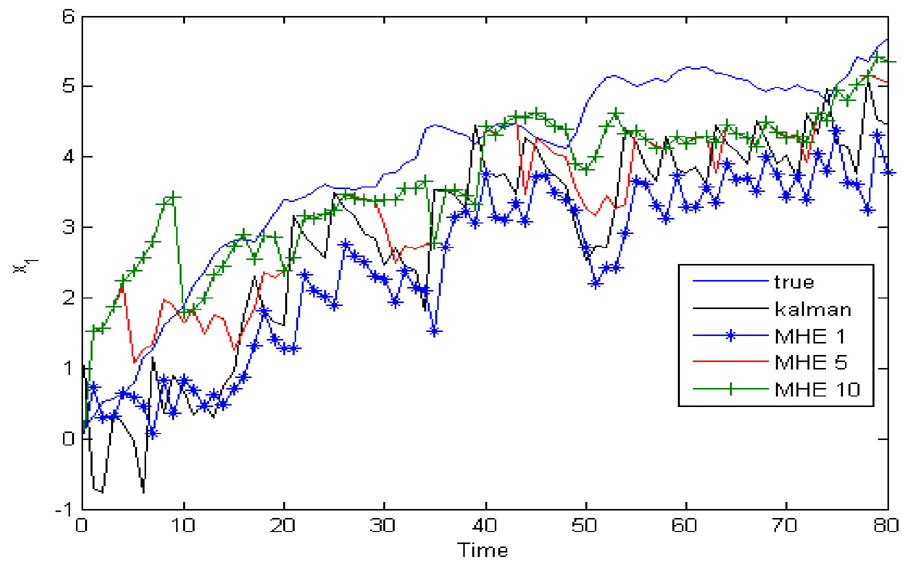

Figure 4.2: Comparison of estimators for x^1 for scenario 1 of case study 1

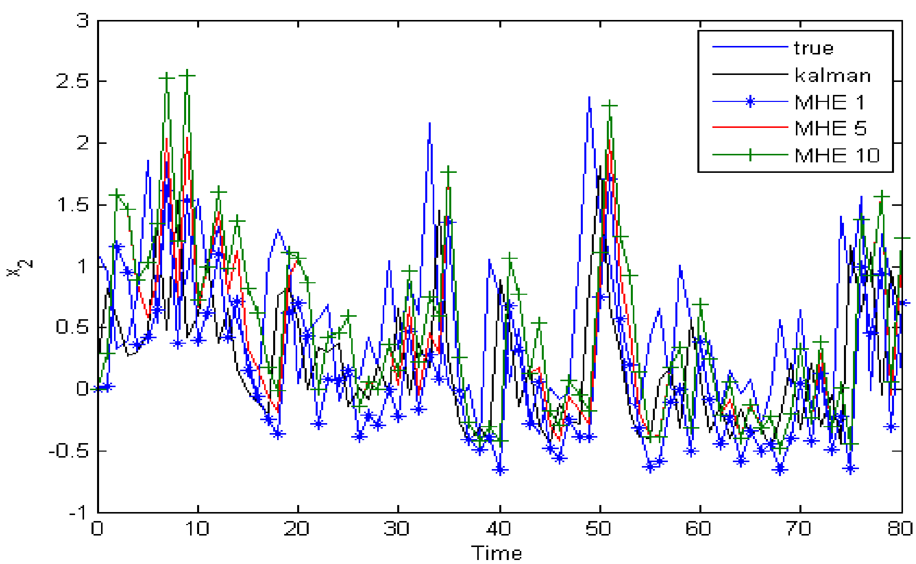

Figure 4.3: Comparison of estimators for x^2 for scenario 1 of case study 1

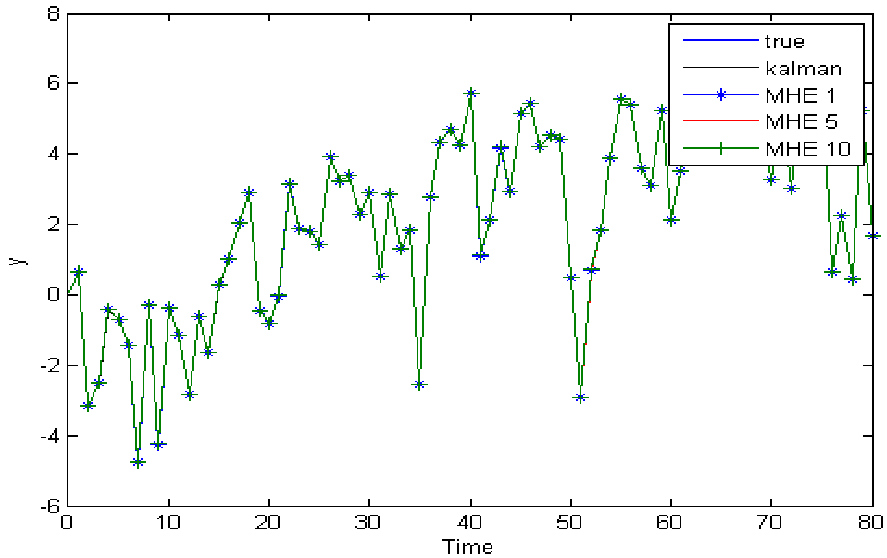

Figure 4.4: Comparison of estimators for y for scenario 1 of case study 1

	SSEE for x^1	SSEE for x^2	Average estimation time in sec
Kalman Filter with Projection	114.69	40.55	0.011
MHE $N = 1$	190.74	53.84	0.0195
MHE $N = 5$	62.97	48.35	0.1244
MHE $N = 10$	46.04	50.16	0.5428

Table 4.1: Performance metrics of the state estimators for scenario 1 of case study 1

Scenario 2 $(w_k = min\{|z_k|\ ,2\})$

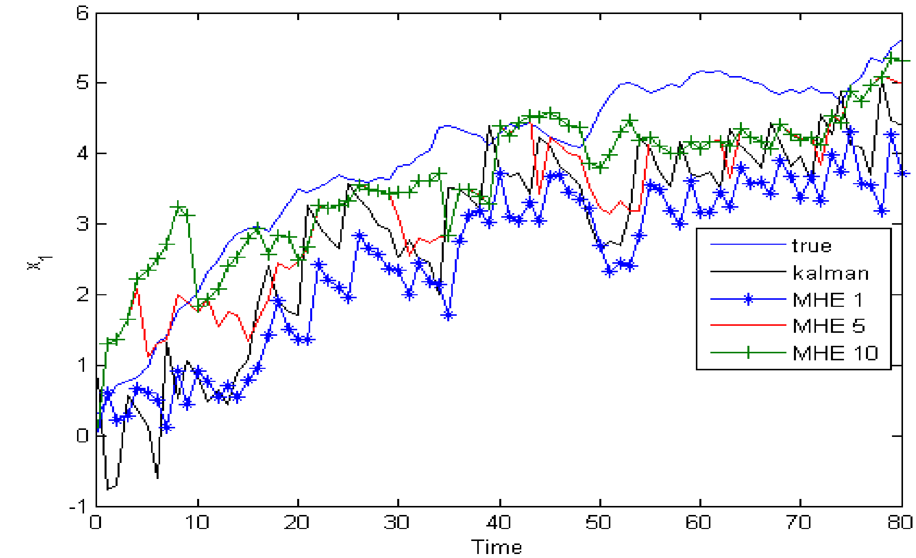

Figure 4.5: Comparison of estimators for x^1 for scenario 2 of case study 1

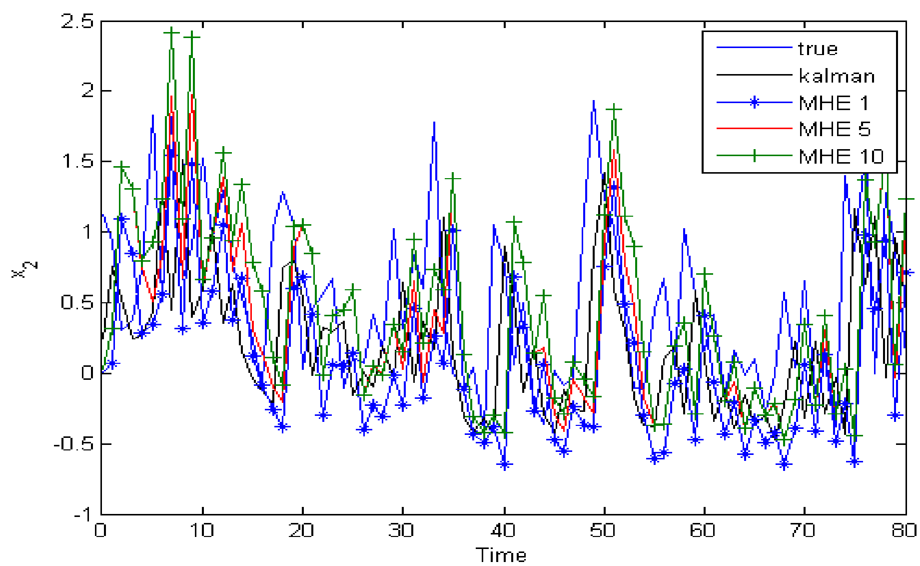

Figure 4.6: Comparison of estimators for x^2 for scenario 2 of case study 1

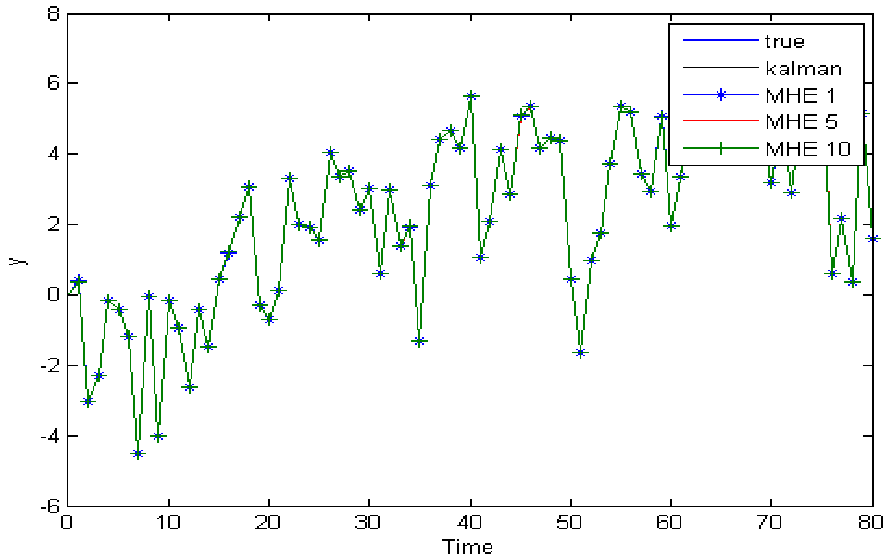

Figure 4.7: Comparison of estimators for y for scenario 2 of case study 1

	SSEE for x^1	SSEE for x^2	Average estimation time in sec
Kalman Filter with Projection	111.55	37.54	0.0116
MHE $N = 1$	187.58	49.01	0.0203
MHE $N = 5$	59.071	42.48	0.1293
MHE $N = 10$	38.73	43.8	0.5674

Table 4.2: Performance metrics of the state estimators for scenario 2 of case study 1

It can be seen from above tables 4.1 and 4.2 that for the state x^1, SSEE of Kalman Filter with projection is lower than SSEE of MHE with $N = 1$ and is higher than SSEE of MHE with $N = 5$ and $N = 10$. For the state x^2, SSEE of Kalman Filter with projection is lower than SSEE of MHE with $N = 1$, $N = 5$ and $N = 10$. The average estimation time of the Kalman Filter with projection is approximately half of the average estimation time for MHE with $N = 1$. The average estimation time for the MHE with $N = 10$ is approximately 50 times greater the average estimation time for Kalman Filter with Projection. Compared to the unconstrained Kalman filter in Rao *et al.* (2003) estimates of Kalman filter with projection follow the actual states and SSEE value is very low. Note: The mean of sequence w_k is not zero and w_k is not a Gaussian distribution. Inclusion of constraints allows nonzero mean non Gaussian disturbances.

4.2 Case Study 2

4.2.1 Problem Description

The following linear discrete time system from Rao and Rawlings (2002) is considered for this case study.

$$x_{k+1} = \begin{bmatrix} 0.9962 & 0.1949 \\ -0.1949 & 0.3815 \end{bmatrix} x_k + \begin{bmatrix} 0.03393 \\ 0.1949 \end{bmatrix} w_k$$

$$y_k = \begin{bmatrix} 1 & -3 \end{bmatrix} x_k + v_k \tag{4.2}$$

It is assumed $\{v_k\}$ is a sequence of independent normally distributed random variables with zero mean and covariance of 0.01. And, $w_k = |z_k|$, where z_k is a sequence of normally distributed independent random variables with zero mean and covariance of Identity. It is assumed that the initial state is normally distributed random variable with zero mean and covariance equal to the identity.

The constrained state estimation is formulated as MHE and Kalman filter with projection for this plant with $Q = 1$,$R = 0.01$, $\Pi_0 = 1$ and $\hat{x_0} = 0$. The matrix Π_k for MHE arrival cost is obtained from solving discrete time matrix Riccati. $w_k \geq 0$ is chosen as constraint for MHE and EKF with projection to capture knowledge of random variable w_k.

25 realizations of this state estimation problem are generated for the time length of 100 samples. The sum of square estimation error(SSEE) is computed for the average

of 25 realizations. Results of the state estimation for this case study are shown in the following section.

4.2.2 Results

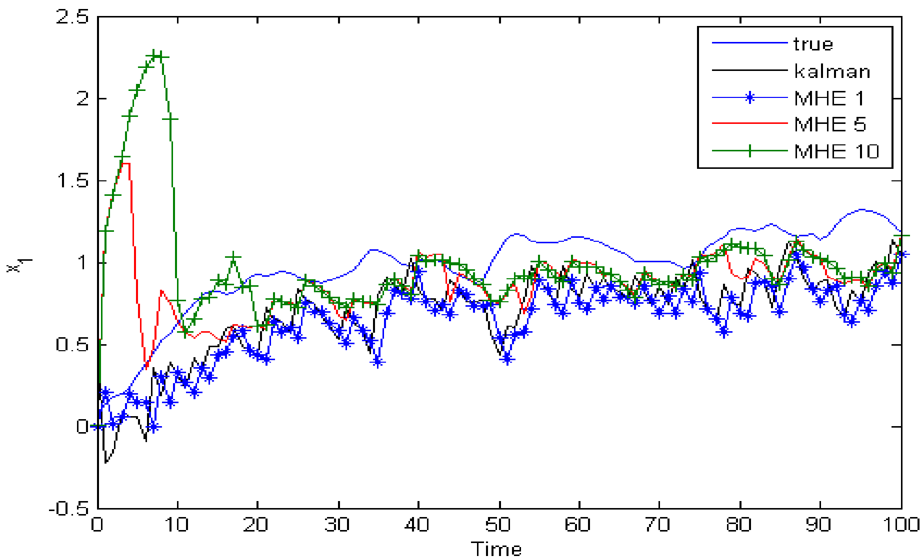

Figure 4.8: Comparison of estimators for x^1 for case study 2

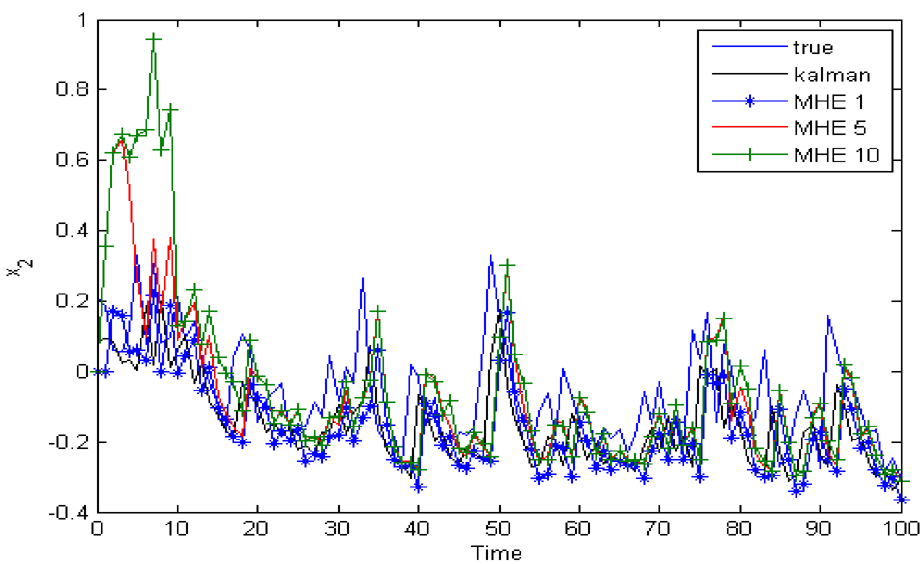

Figure 4.9: Comparison of estimators for x^2 for case study 2

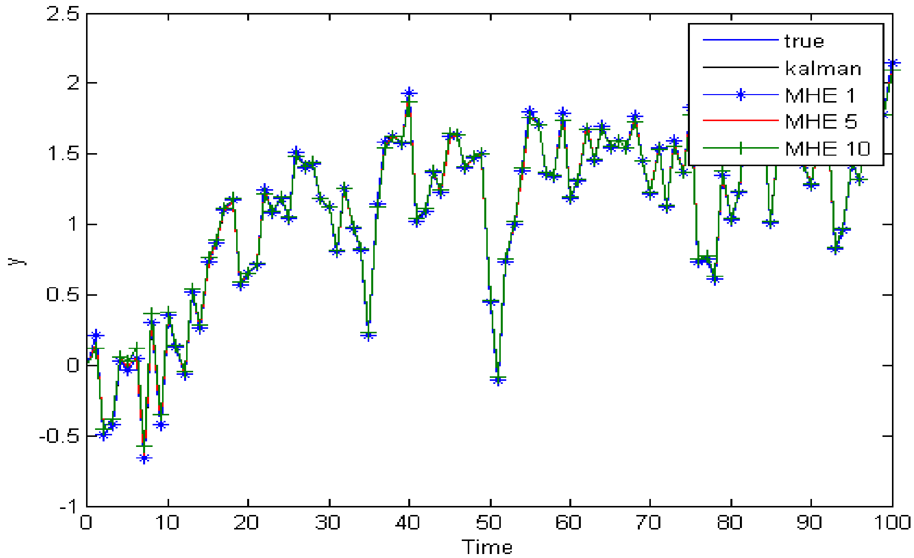

Figure 4.10: Comparison of estimators for y for case study 2

	SSEE for x^1	SSEE for x^2	Average estimation time in sec
Kalman Filter with Projection	10.09	2.05	0.0105
MHE $N = 1$	13.40	2.80	0.0190
MHE $N = 5$	11.27	3.23	0.1156
MHE $N = 10$	25.09	4.84	0.4895

Table 4.3: Performance metrics of the state estimators for case study 2

It can be seen from the table 4.2.2, the SSEE of Kalman Filter with projection is close to the SSEE of MHE with $N = 1, 5$. Because of high estimation error in initial window (from T= 1 to 10), the value of SSEE for MHE with $N = 10$ is high. It can be seen from the plots, the estimates of MHE with $N = 10$ are closer to actual states as compare to other estimators. The average estimation time of the Kalman Filter with projection is approximately half of the average estimation time for MHE with $N = 1$. The average estimation time for the MHE with $N = 10$ is approximately 45 times average estimation time for Kalman Filter with projection. Compared to the unconstrained Kalman filter in Rao and Rawlings (2002), SSEE for the Kalman filter with projection is very low.

4.3 Case Study 3

4.3.1 Problem Description

The following nonlinear discrete time system from Rao *et al.* (2003) is considered for this case study.

$$
\begin{aligned}
{x^1}_{k+1} &= 0.99{x^1}_k + 0.2{x^2}_k \\
{x^2}_{k+1} &= -0.1{x^1}_k + \frac{0.5{x^2}_k}{1 + ({x^2}_k)^2} + w_k \\
y_k &= {x^1}_k - 3{x^2}_k + v_k
\end{aligned}
\tag{4.3}
$$

It is assumed $\{v_k\}$ is sequence of independent normally distributed random variables with zero mean and covariance of 0.01. And, following scenarios are used to generate w_k sequence

1. $w_k = |z_k|$.

2. $w_k = min\{|z_k|\ , 2\}$

where, z_k is a sequence of normally distributed independent random variables with zero mean and covariance of identity. It is assumed that the initial state is normally distributed random variable with zero mean and covariance equal to the identity.

The constrained state estimation problem is formulated as MHE and EKF with projection for this plant with $Q = 1$,$R = 0.01$, $\Pi_0 = 1$ and $\hat{x_0} = 0$. The matrix Π_k for the MHE arrival cost is obtained from solving discrete time matrix Riccati. $w_k \geq 0$ is chosen as the constraint for MHE and EKF with projection to capture knowledge of the random variable w_k.

25 realizations of this state estimation problem are generated for the time length of 80 samples. The sum of square estimation error(SSEE) is computed for the average of 25 realizations. Results of the state estimation for this case study are shown in the following section.

4.3.2 Results

Scenario 1 ($w_k = |z_k|$)

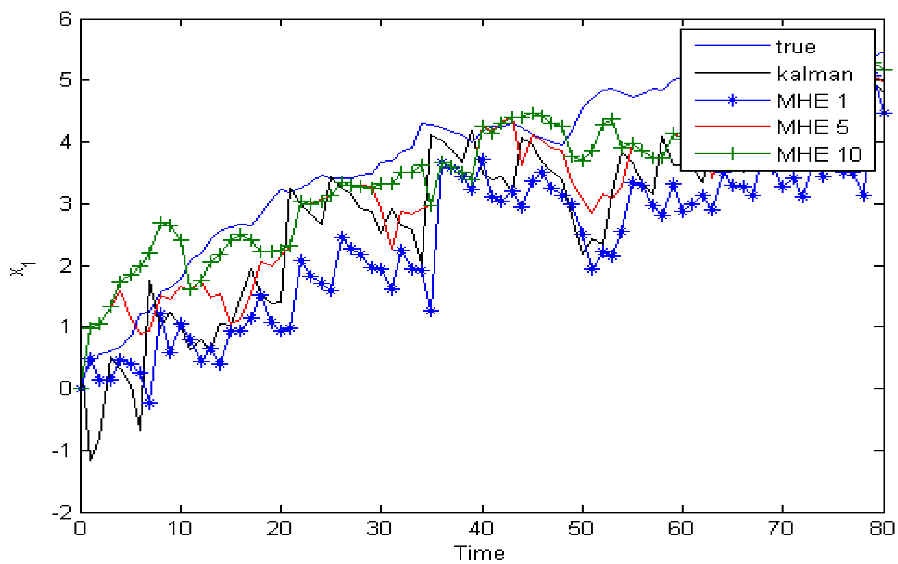

Figure 4.11: Comparison of estimators for x^1 for scenario 1 of case study 3

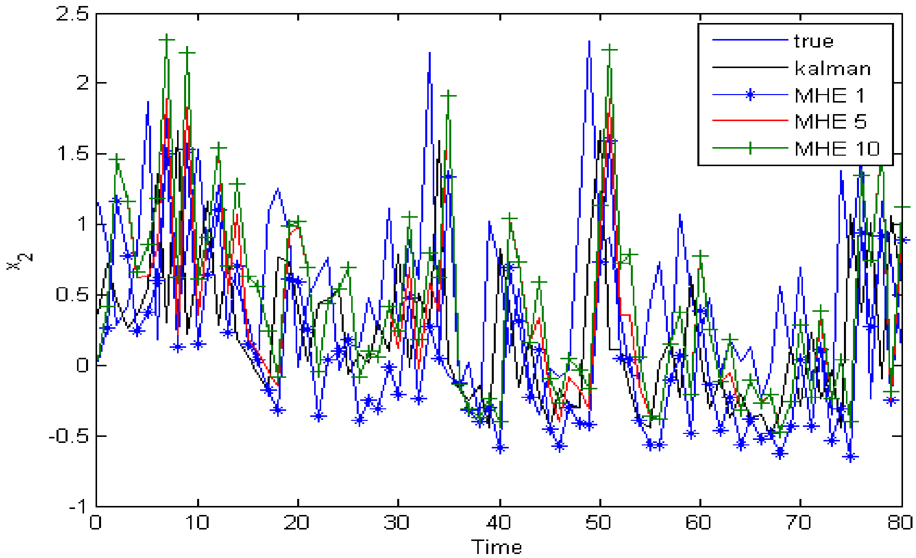

Figure 4.12: Comparison of estimators for x^2 for scenario 1 of case study 3

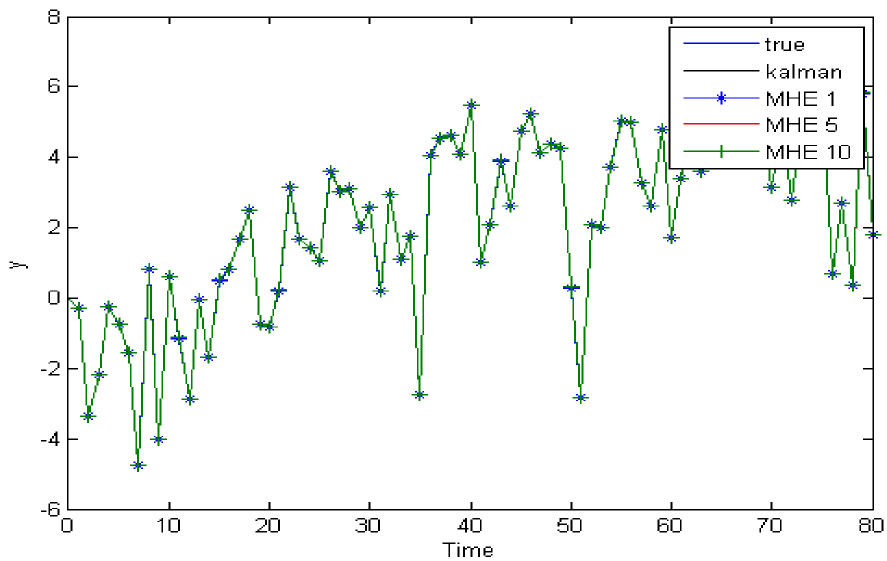

Figure 4.13: Comparison of estimators for y for scenario 1 of case study 3

	SSEE for x^1	SSEE for x^2	Average estimation time in sec
EKF with Projection	106.20	45.47	0.0126
MHE $N = 1$	196.34	54.27	0.0258
MHE $N = 5$	56.41	46.88	0.64
MHE $N = 10$	33.85	47.68	0.6700

Table 4.4: Performance metrics of state estimators for scenario 1 of case study 3

Case 2 ($w_k = min\{|z_k|\ ,2\}$)

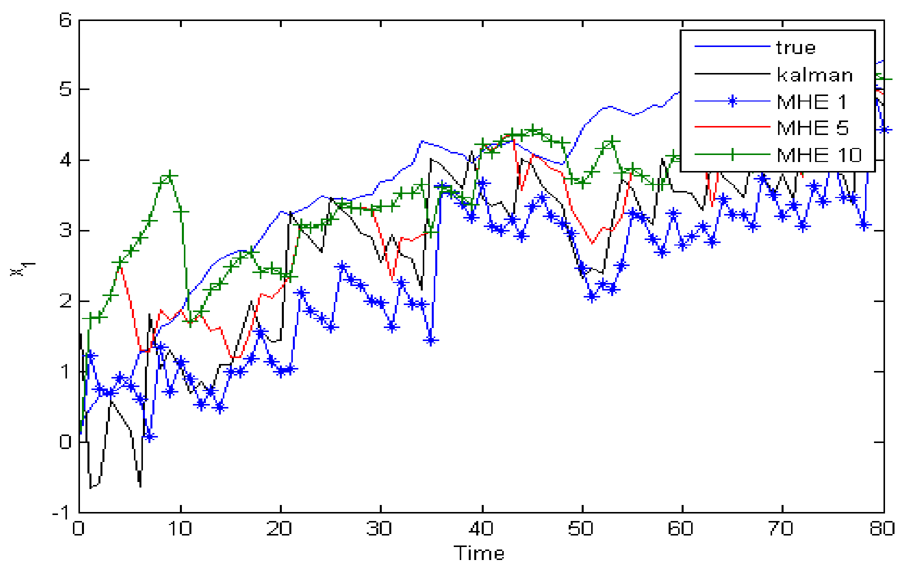

Figure 4.14: Comparison of estimators for x^1 for scenario 2 of case study 3

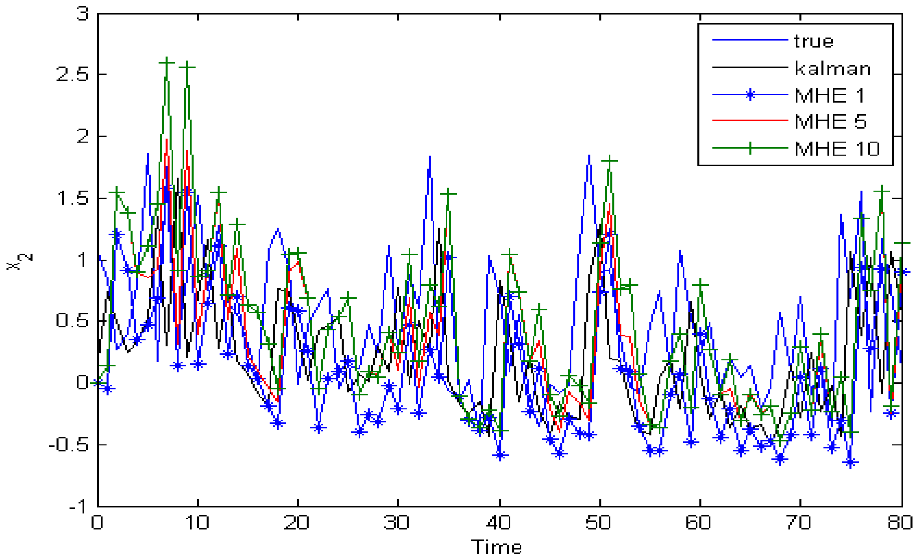

Figure 4.15: Comparison of estimators for x^2 for scenario 2 of case study 3

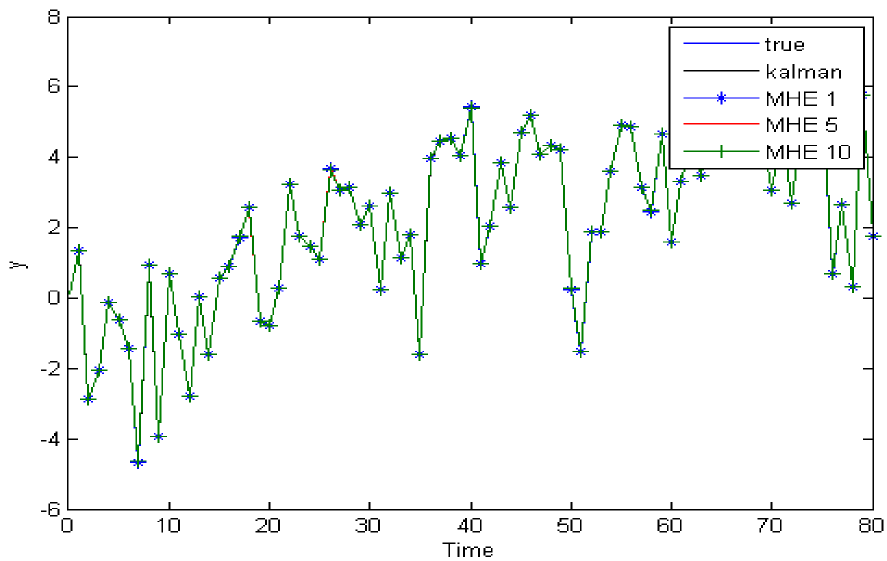

Figure 4.16: Comparison of estimators for y for scenario 2 of case study 3

	SSEE for x^1	SSEE for x^2	Average estimation time in sec
EKF with Projection	103.16	41.47	0.0102
MHE $N = 1$	190.31	49.42	0.0214
MHE $N = 5$	61.82	41.74	0.2352
MHE $N = 10$	52.97	44.25	0.5370

Table 4.5: Performance metrics of state estimators for scenario 2 of case study 3

It can be seen from above tables 4.4 and 4.5 that for the state x^1, SSEE of EKF with projection is lower than SSEE of MHE with $N = 1$ and is higher than SSEE of MHE with $N = 5$ and $N = 10$. For the state x^2, SSEE of the EKF with projection is lower than SSEE for MHE with $N = 1$, $N = 5$ and $N = 10$. The average estimation time of the Kalman Filter with projection is approximately half of the average estimation time for MHE with $N = 1$. The average estimation time for the MHE with $N = 10$ is approximately 50 times greater the average estimation time for Kalman Filter with Projection.

Compared to the unconstrained Extended Kalman filter in Rao *et al.* (2003), SSEE for the EKF with projection is very low and estimates converge with actual states.

4.4 Case Study 4

4.4.1 Problem Description

The process of Waste water treatment as discussed in Rao and Rawlings (2002) is considered in this case study. Constrained state estimation algorithms are used to detect location and quantity of leak in each tank. Block diagram of this process is shown below.

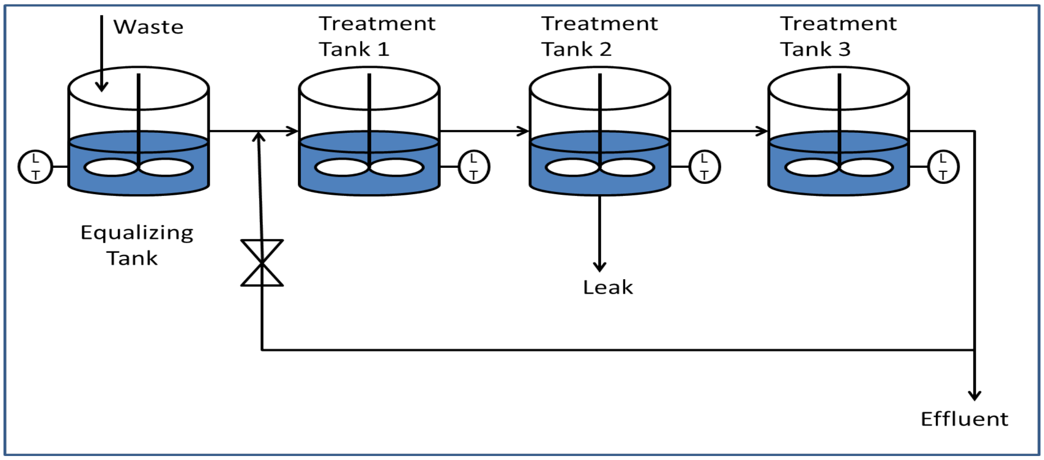

Figure 4.17: Waste water treatment process

This process is described by the following linear model.

$$x_{k+1} = \begin{bmatrix} 0.89168 & 0 & 0 & 0 & 1.0 \\ 0.10832 & 0.90518 & 0 & 0.04306 & 0 \\ 0 & 0.9482 & 0.89524 & 0 & 0 \\ 0 & 0 & 0.10476 & 0.89235 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} w_k$$

$$y_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & m \end{bmatrix} x_k + v_k \tag{4.4}$$

Meaning of the state variables is described in the table 4.6.

State	Description
x^1	Mass in equalizing tank
x^2	Mass in Tank 1
x^3	Mass in Tank 2
x^4	Mass in Tank 3
x^5	Mass of waste entering equalizing tank

Table 4.6: State Description for waste water treatment process

In this process, leak in the process is limited to Tank 2. The process is simulated with $w_k = |z_k|$ where z_k is a normally distributed random variable with the covariance given by following matrix.

$$Q_z = diag \begin{bmatrix} 0 & 0 & 5 & 0 & 15 \end{bmatrix}.$$

The value of m in the equation for y_k is chosen as 1 if the mass entering the equalizing tank is measured and chosen as 0 otherwise. It is assumed that the mass in the tank and mass flow rate entering the equalizing tank are measured and covariance of measurement error is

$$R = diag \begin{bmatrix} 8 & 8 & 8 & 8 & 4 \end{bmatrix}.$$

It is assumed that location of leak is unknown to the estimator and following matrix is chosen as covariance matrix of w_k for estimation

$$Q = diag \begin{bmatrix} 5 & 5 & 5 & 5 & 15 \end{bmatrix}.$$

Constrained state estimation is formulated as MHE and Kalman filter with projection for this plant with Q, R, $\Pi_0 = 1$ and $\hat{x_0} = 0$. $N = 1, 5$. To test the effectiveness of the state estimator in detection of leak, following scenarios are considered

1. Mass entering equalizing tank is measured and leak in tank 2.

2. Mass entering equalizing tank is measured and no leak in tank 2.

3. Mass entering equalizing tank is not measured and leak in tank 2.

4. Mass entering equalizing tank is not measured and no leak in tank 2.

Results of the state estimation and leak detection for this case study are shown in the following section.

4.4.2 Results

Scenario 1: Mass entering equalizing tank is measured and leak in tank 2

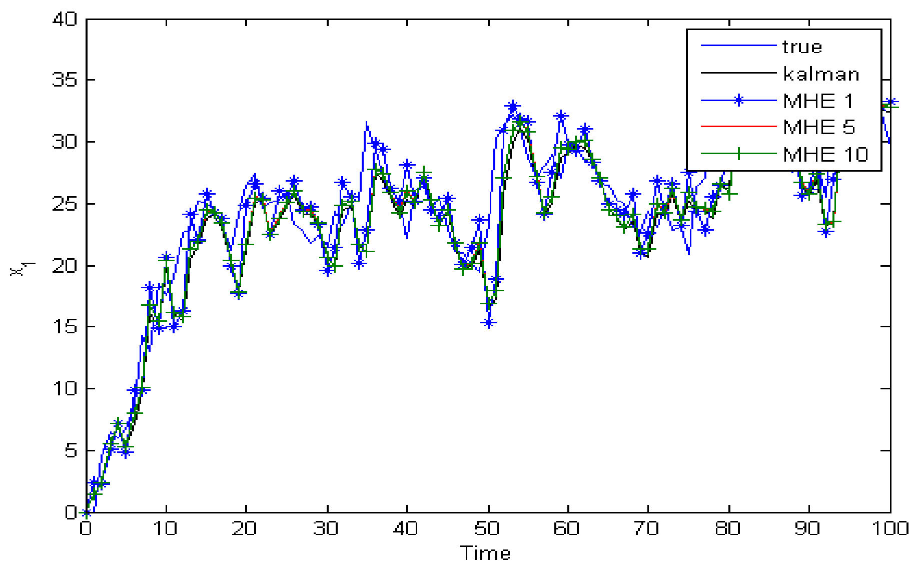

Figure 4.18: Comparison of estimators for x^1 for scenario 1 of case study 4

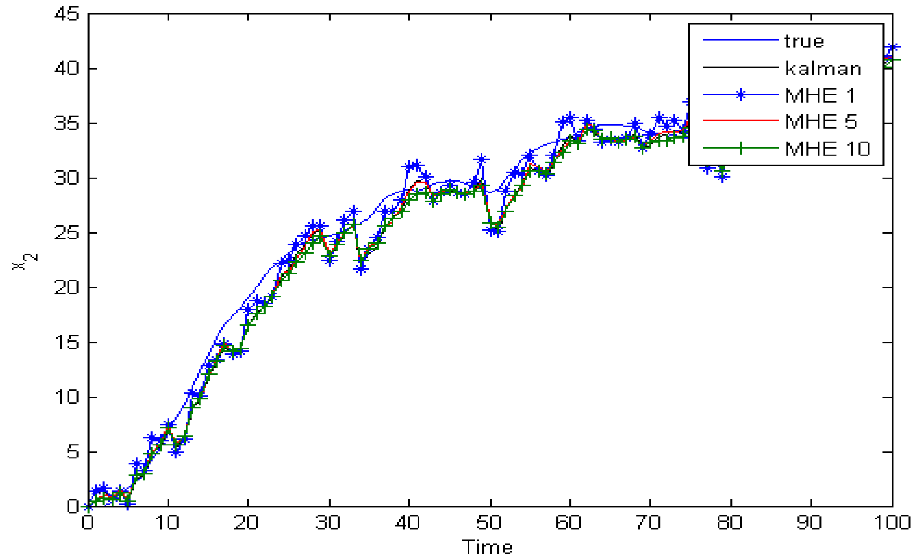

Figure 4.19: Comparison of estimators for x^2 for scenario 1 of case study 4

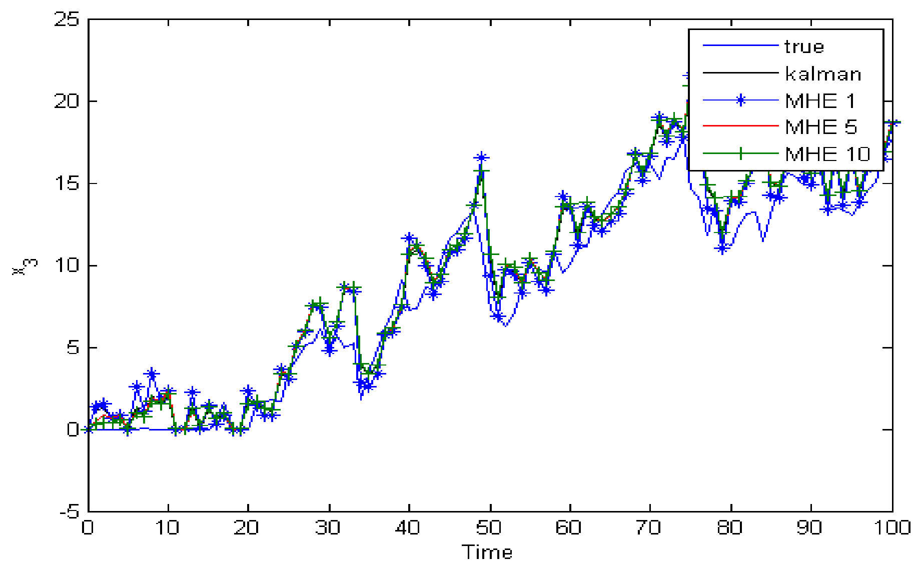

Figure 4.20: Comparison of estimators for x^3 for scenario 1 of case study 4

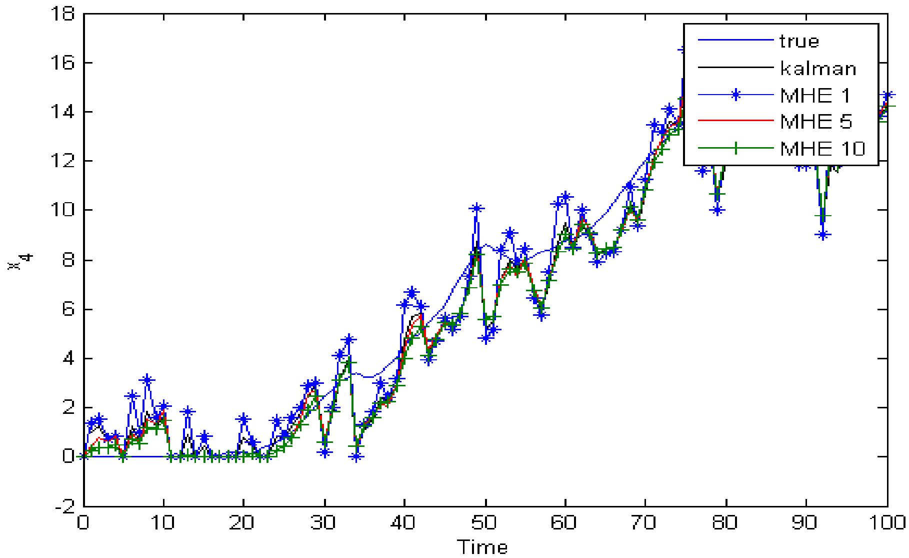

Figure 4.21: Comparison of estimators for x^4 for scenario 1 of case study 4

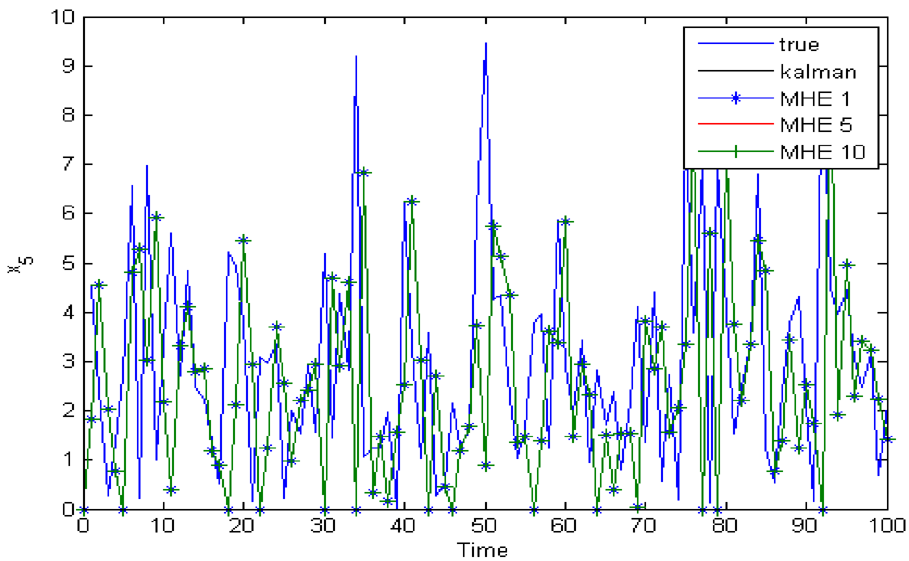

Figure 4.22: Comparison of estimators for x^5 for scenario 1 of case study 4

	SSEE for $x^1(\times 10^3)$	SSEE for $x^2(\times 10^3)$	SSEE for $x^3(\times 10^3)$	SSEE for $x^4(\times 10^3)$	SSEE for $x^5(\times 10^3)$	Average estimation time in sec
Kalman Filter with Projection	1.05	0.35	0.39	0.17	0.92	0.0232
MHE $N = 1$	1.56	0.47	0.59	0.25	0.80	0.0507
MHE $N = 5$	1.53	0.47	0.55	0.16	0.80	0.36
MHE $N = 10$	1.53	0.49	0.54	0.15	0.80	1.1406

Table 4.7: Performance metrics of state estimators for scenario 1 of case study 4

	Total losses	Mean losses in equalizing tank	Mean losses in tank 1	Mean losses in tank 2	Mean losses in tank 3
Actual	172.52	0	0	1.71	0
Kalman Filter with projection	300.95	0.6023	0.4552	1.428	0.4942
MHE N=1	136.0607	0.1986	0.2393	0.6717	0.2376
MHE N=5	172.3298	0.1974	0.2048	1.1114	0.1926
MHE N=10	168.5798	0.1675	0.1611	1.1926	0.1480

Table 4.8: Results of leak detection for scenario mass entering equalizing tank is measured and leak in tank 2

It can be seen from the table 4.7, the SSEE of Kalman filter with projection is smaller than the SSEE of MHE for states x^1, x^2, x^3, x^4 and is slightly higher than MHE for state x^5. The average estimation time of Kalman filter with projection is approximately 0.02 times the average estimation time of MHE with $N = 10$ and is approximately one half of the average estimation time of MHE with $N = 1$.

It can be seen from table 4.8 that the estimate of total leak in the system by Kalman filter with projection is higher than the actual value. But, Kalman filter with projection detects the location of the leak better than the MHE with $N = 1$. There is a bias in the estimates of leak for Kalman filter with projection and the value of bias is approximately equal to the mean of random variable $|z_k|$.

Compared to leak detection using unconstrained Kalman filter in Rao and Rawlings (2002), Kalman filter with projection estimates total leak and location of leak better.

Scenario 2: Mass entering equalizing tank is measured and no leak in tank 2

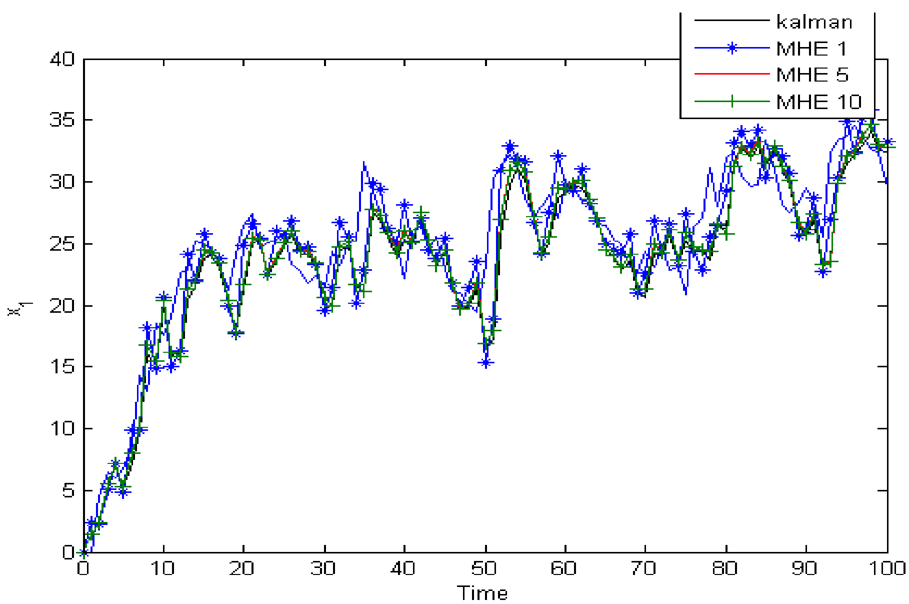

Figure 4.23: Comparison of estimators for x^1 for scenario 2 of case study 4

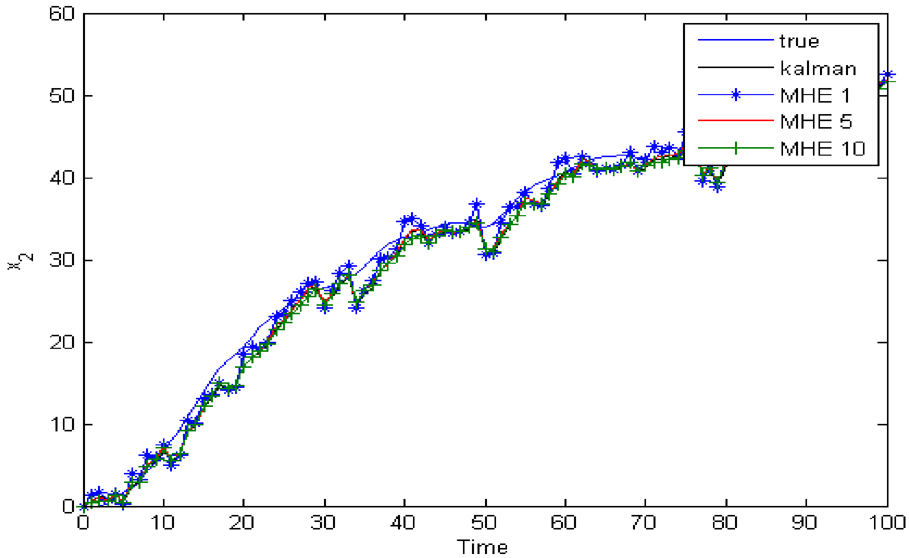

Figure 4.24: Comparison of estimators for x^2 for scenario 2 of case study 4

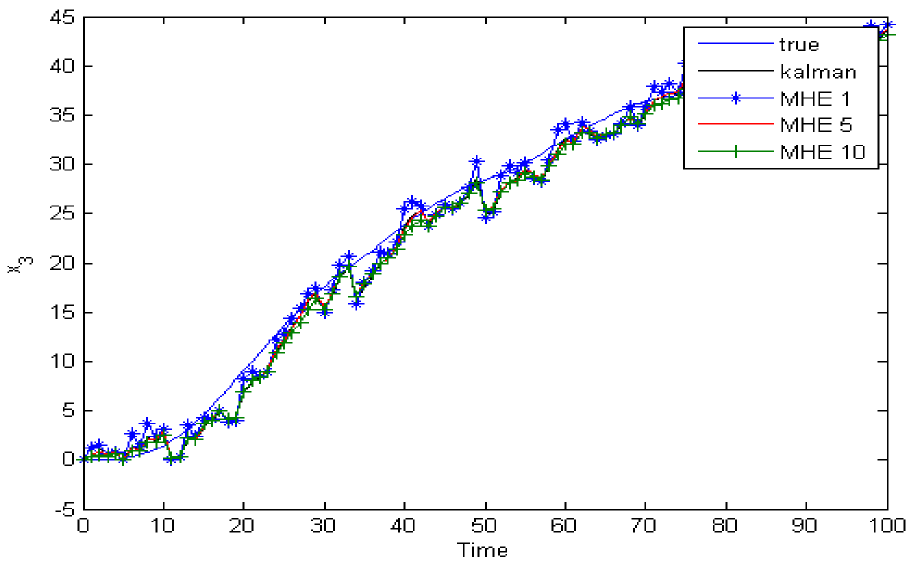

Figure 4.25: Comparison of estimators for x^3 for scenario 2 of case study 4

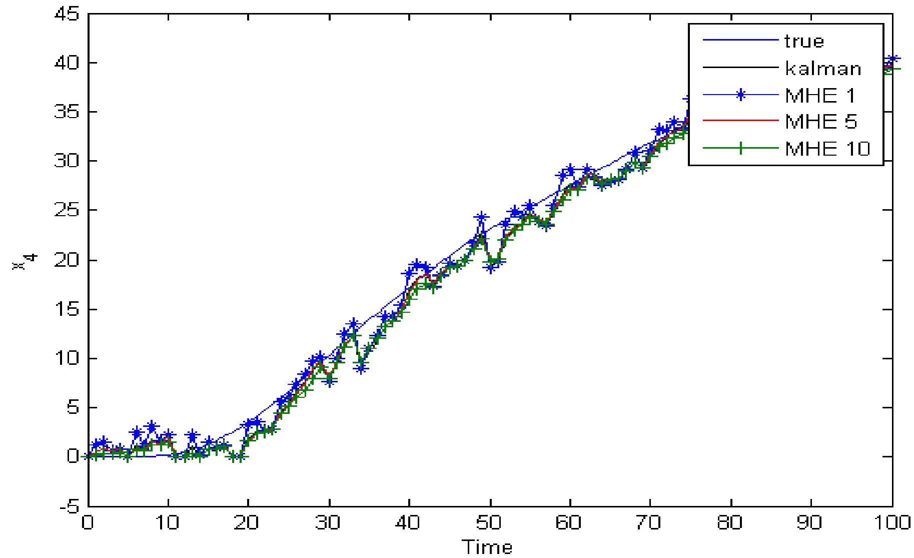

Figure 4.26: Comparison of estimators for x^4 for scenario 2 of case study 4

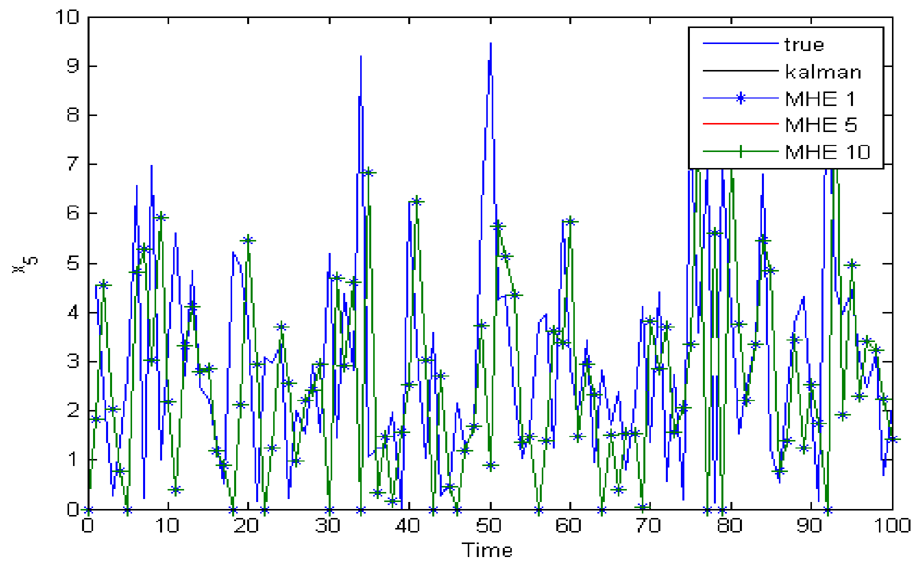

Figure 4.27: Comparison of estimators for x^5 for scenario 2 of case study 4

	SSEE for $x^1(\times 10^3)$	SSEE for $x^2(\times 10^3)$	SSEE for $x^3(\times 10^3)$	SSEE for $x^4(\times 10^3)$	SSEE for $x^5(\times 10^3)$	Average estimation time in sec
Kalman Filter with Projection	1.05	0.36	0.30	0.27	0.92	0.0240
MHE $N = 1$	1.56	0.49	0.40	0.36	0.80	0.0506
MHE $N = 5$	1.53	0.50	0.38	0.35	0.80	0.3167
MHE $N = 10$	1.53	0.53	0.41	0.38	0.80	1.0218

Table 4.9: Performance metrics of state estimators for scenario 2 of case study 4

	Total losses	Mean losses in equalizing tank	Mean losses in tank 1	Mean losses in tank 2	Mean losses in tank 3
Actual	0	0	0	0	0
Kalman Filter with projection	194.3080	0.6022	0.4527	0.4457	0.4232
MHE N=1	98.5818	0.1974	0.2640	0.2576	0.2571
MHE N=5	82.6727	0.1972	0.2178	0.2049	0.1986
MHE N=10	61.8128	0.1675	0.1583	0.1445	0.1417

Table 4.10: Results of leak detection for scenario mass entering equalizing tank is measured and no leak in tank 2

It can be seen from table 4.9, SSEE of Kalman filter with projection is smaller than SSEE of MHE for the states x^1, x^2, x^3, x^4 and is slightly higher than MHE for the state x^5. The average estimation time of Kalman filter with projection is approximately 0.02 times the average estimation time of MHE with $N = 10$ and is approximately one half of average estimation time of MHE with $N = 1$.

It can be seen from table 4.10 that the estimate of total leak in the system by Kalman filter with projection is higher than actual value. There is a bias in the estimates of leak for Kalman filter with projection and the value of bias is approximately equal to the mean of random variable $|z_k|$.

Compare to leak detection using unconstrained Kalman filter in Rao and Rawlings (2002), Kalman filter with projection estimates total leak and location of leak better.

Scenario 3: Mass entering equalizing tank is not measured and leak in tank 2

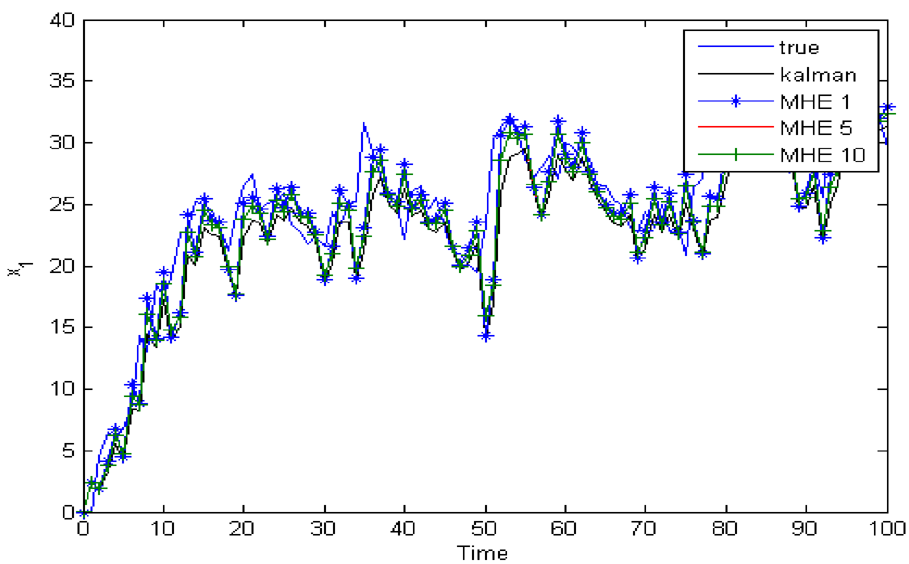

Figure 4.28: Comparison of estimators for x^1 for scenario 3 of case study 4

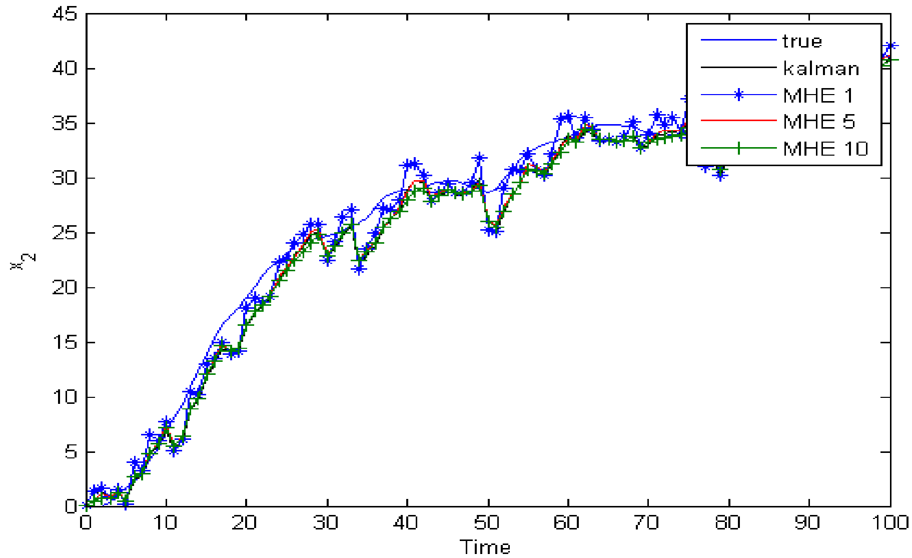

Figure 4.29: Comparison of estimators for x^2 for scenario 3 of case study 4

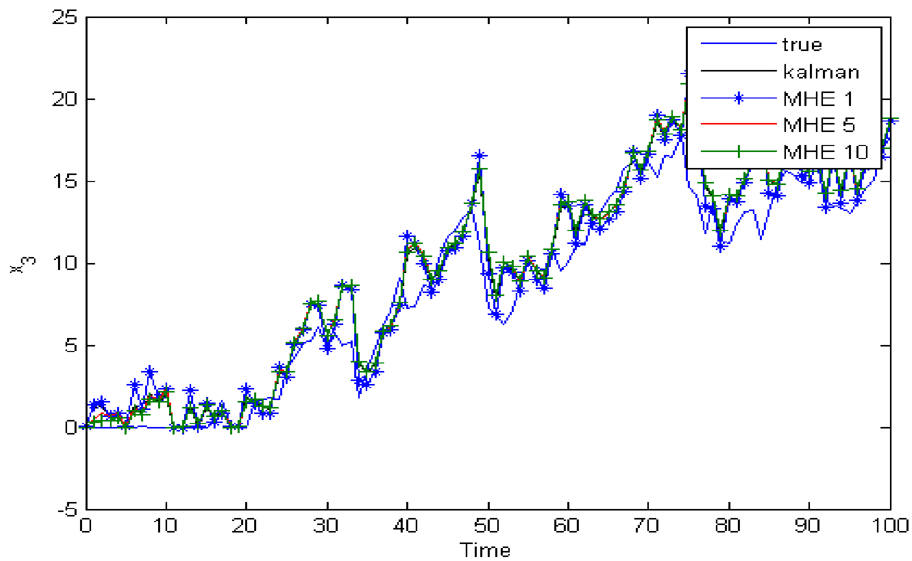

Figure 4.30: Comparison of estimators for x^3 for scenario 3 of case study 4

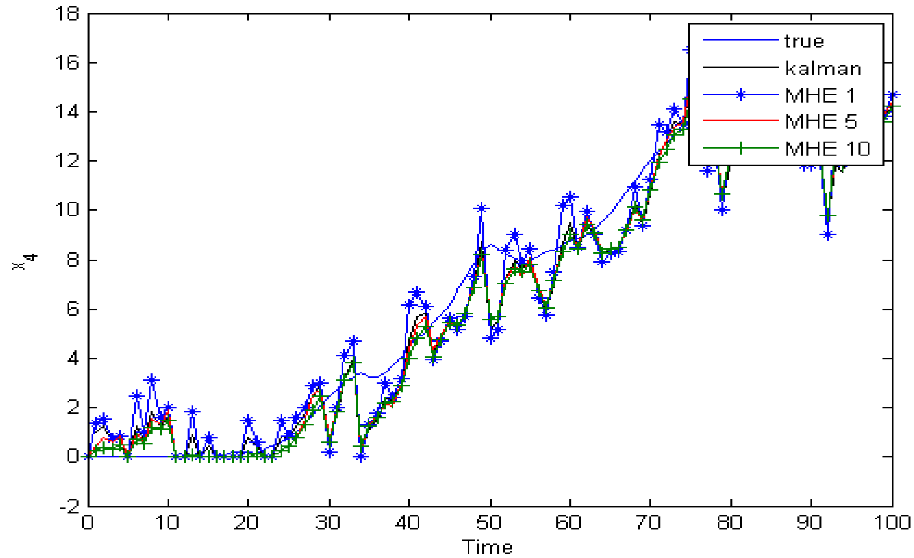

Figure 4.31: Comparison of estimators for x^4 for scenario 3 of case study 4

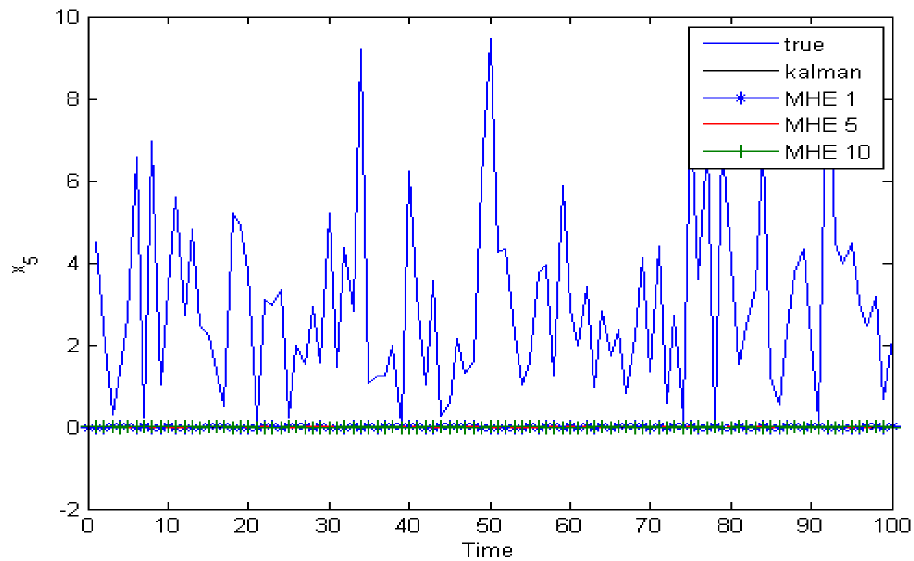

Figure 4.32: Comparison of estimators for x^5 for scenario 3 of case study 4

	SSEE for $x^1(\times 10^3)$	SSEE for $x^2(\times 10^3)$	SSEE for $x^3(\times 10^3)$	SSEE for $x^4(\times 10^3)$	SSEE for $x^5(\times 10^3)$	Average estimation time in sec
Kalman Filter with Projection	1.34	0.37	0.39	0.17	1.36	0.0301
MHE $N = 1$	1.66	0.46	0.59	0.25	1.36	0.0828
MHE $N = 5$	1.57	0.47	0.55	0.16	1.36	0.6067
MHE $N = 10$	1.58	0.49	0.54	0.15	1.36	1.8276

Table 4.11: Performance metrics of state estimators for scenario 3 of case study 4

	Total losses	Mean losses in equalizing tank	Mean losses in tank 1	Mean losses in tank 2	Mean losses in tank 3
Actual	172.5176	0	0	1.7081	0
Kalman Filter with projection	251.2802	0.1518	0.4180	1.4238	0.4944
MHE N=1	122.9867	0.0466	0.2631	0.6720	0.2360
MHE N=5	156.6337	0.0352	0.2115	1.1122	0.1919
MHE N=10	154.3659	0.0254	0.1621	1.1933	0.1476

Table 4.12: Results of leak detection for scenario mass entering equalizing tank is not measured and leak in tank 2

It can be seen from the table 4.11, SSEE of Kalman filter with projection is smaller than SSEE of MHE. The average estimation time of Kalman filter with projection is approximately 0.02 times the average estimation time of MHE with $N = 10$ and is approximately one half of the average estimation time of MHE with $N = 1$.

It can be seen from the table 4.12 the estimate of total leak in the system by

Kalman filter with projection is higher than the actual value. But, Kalman filter with projection detects the location of the leak better than the MHE with $N = 1$. There is a bias in the estimates of leak for Kalman filter with projection and the value of bias is approximately equal to the mean of random variable $|z_k|$.

Compare to leak detection using unconstrained Kalman filter in Rao and Rawlings (2002), Kalman filter with projection estimates total leak and location of leak better.

Scenario 4: Mass entering equalizing tank is not measured and no leak in tank 2

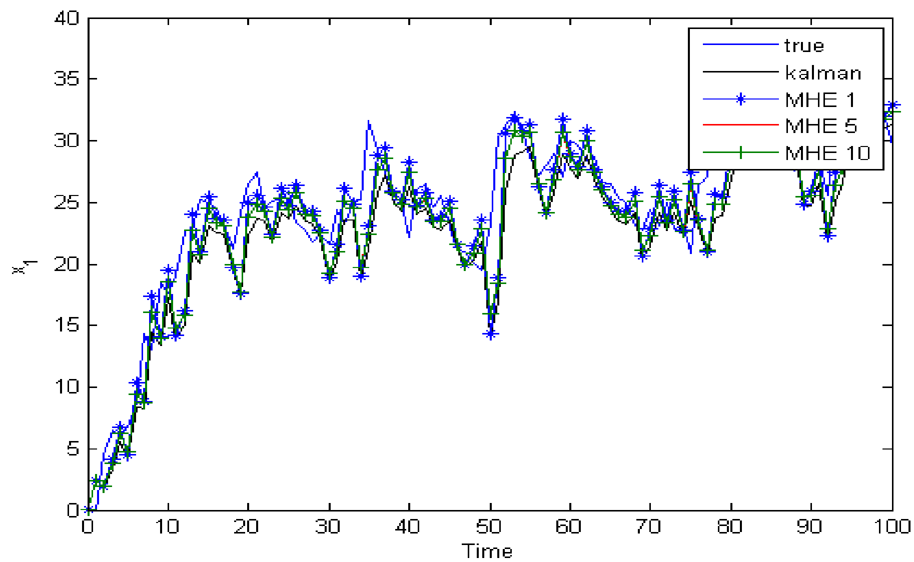

Figure 4.33: Comparison of estimators for x^1 for scenario 4 of case study 4

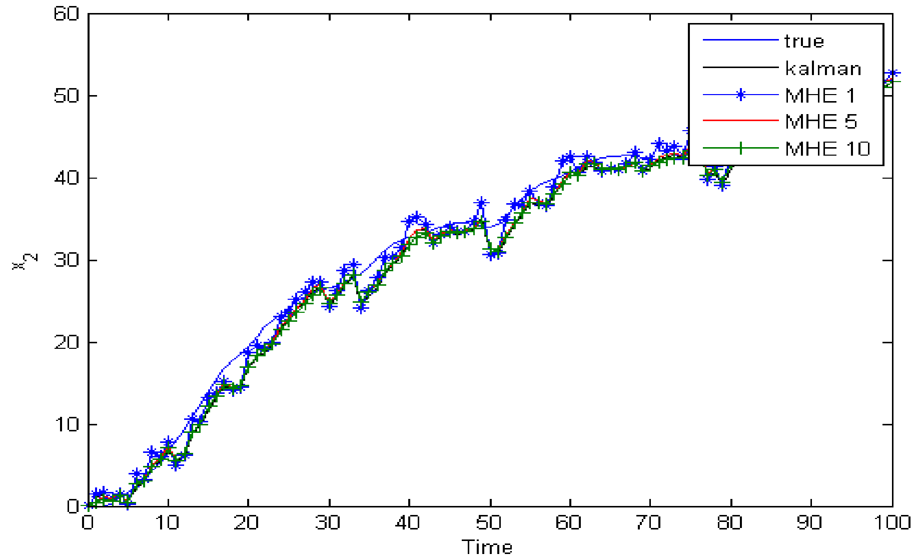

Figure 4.34: Comparison of estimators for x^2 for scenario 4 of case study 4

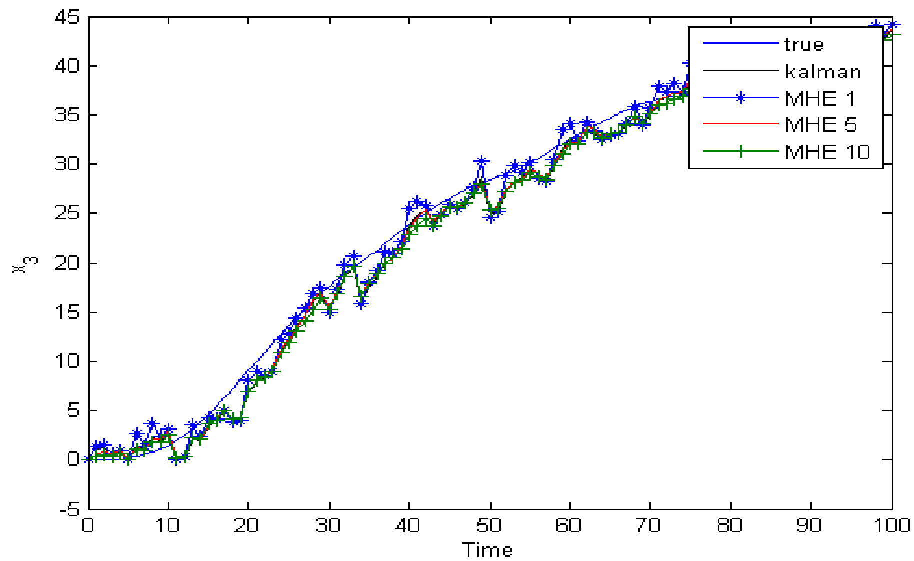

Figure 4.35: Comparison of estimators for x^3 for scenario 4 of case study 4

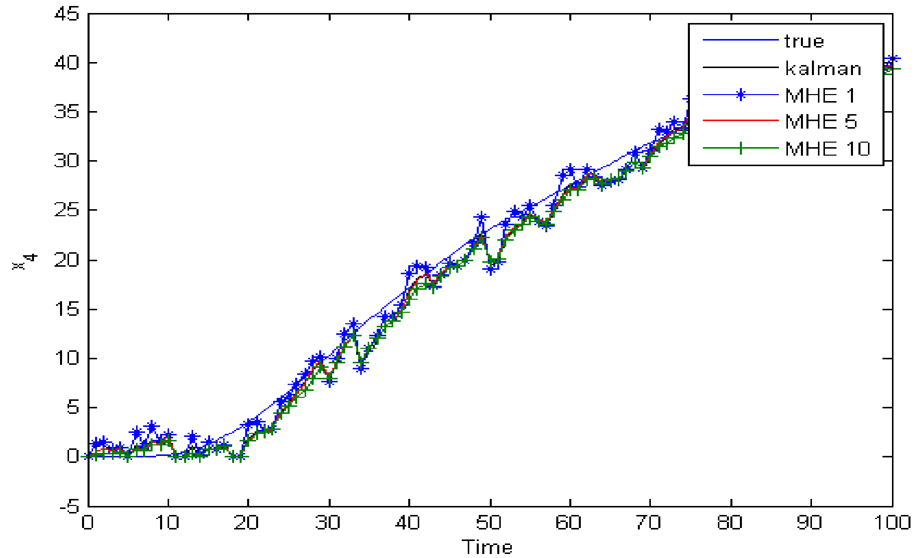

Figure 4.36: Comparison of estimators for x^4 for scenario 4 of case study 4

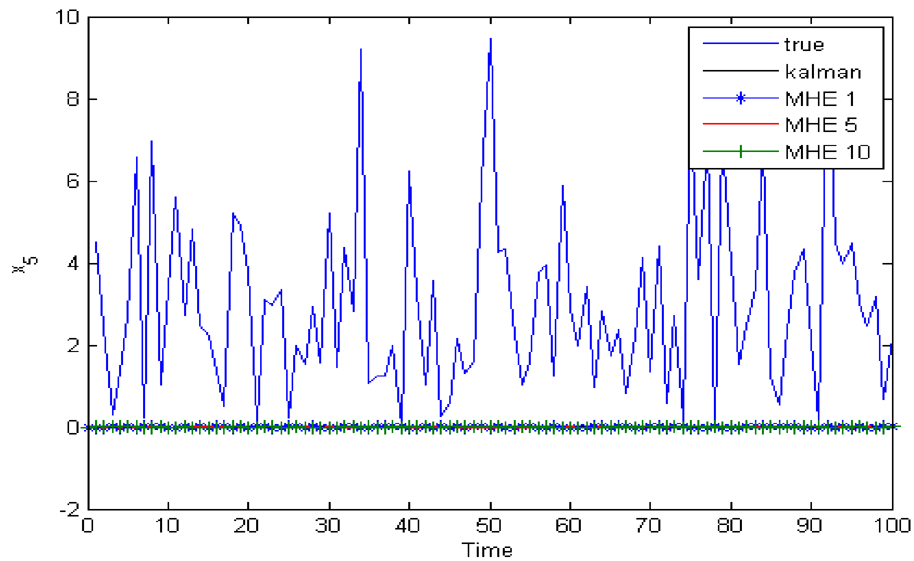

Figure 4.37: Comparison of estimators for x^5 for scenario 4 of case study 4

	SSEE for $x^1(\times 10^3)$	SSEE for $x^2(\times 10^3)$	SSEE for $x^3(\times 10^3)$	SSEE for $x^4(\times 10^3)$	SSEE for $x^5(\times 10^3)$	Average estimation time in sec
Kalman filter with Projection	1.34	0.38	0.29	0.27	1.36	0.0275
MHE $N = 1$	1.66	0.48	0.40	0.36	1.36	0.0817
MHE $N = 5$	1.57	0.50	0.38	0.35	1.36	0.5774
MHE $N = 10$	1.58	0.52	0.41	0.38	1.36	1.7871

Table 4.13: Performance metrics of state estimators for scenario 4 of case study 4

	Total losses	Mean losses in equalizing tank	Mean losses in tank 1	Mean losses in tank 2	Mean losses in tank 3
Actual	0	0	0	0	0
Kalman Filter with projection	144.7151	0.1518	0.4155	0.4423	0.4232
MHE N=1	85.8336	0.0457	0.2900	0.2583	0.2558
MHE N=5	67.1434	0.0354	0.2250	0.2062	0.1982
MHE N=10	47.6841	0.0255	0.1596	0.1455	0.1414

Table 4.14: Results of leak detection for scenario mass entering equalizing tank is not measured and no leak in tank 2

It can be seen from the table 4.13 SSEE of Kalman filter with projection is smaller than SSEE of MHE. The average estimation time of Kalman filter with projection is approximately 0.02 times the average estimation time of MHE with $N = 10$ and is approximately one third of the average estimation time of MHE with $N = 1$.

It can be seen from table 4.14 that the estimate of total leak in the system by Kalman filter with projection is higher than actual value. There is a bias in the estimates of leak for Kalman filter with projection and the value of bias is approximately equal to the mean of random variable $|z_k|$.

Compare to leak detection using unconstrained Kalman filter in Rao and Rawlings (2002), Kalman filter with projection estimates the total leak and the location of leak better.

4.5 Case Study 5

4.5.1 Problem Description

The model in example 3.2 of Tenny *et al.* (2004) is considered for this case study. Consider a continuously stirred tank reactor (CSTR) in which the isothermal irreversible reactions$A \longrightarrow B \longrightarrow C$ are taking place. Tank is fed with the stream that contains only chemical A . Maximum conversion of A to product B is desired. The concentration of the product B is measured and the process is regulated by adjusting the temperature of the reactor directly by a cascaded control system. The mass balance of these reactions are governed by the following equations

$$\dot{C}_A = \frac{F}{V}(C_{Af} - C_A) - k_1 C_A \exp\left(\frac{-E_1}{RT}\right) \tag{4.5}$$

$$\dot{C}_B = k_1 C_A \exp\left(\frac{-E_1}{RT}\right) - k_2 C_B \exp\left(\frac{-E_2}{RT}\right) - \frac{F}{V} C_B \tag{4.6}$$

where C_A is concentration of chemical A in the tank, C_B is concentration of chemical B in the tank, C_{Af} is concentration of chemical A in feed and T is temperature of reactor. C_A and C_B are considered as state variables and T is considered as manipulative variable. Nominal conditions of reactor are given in following table

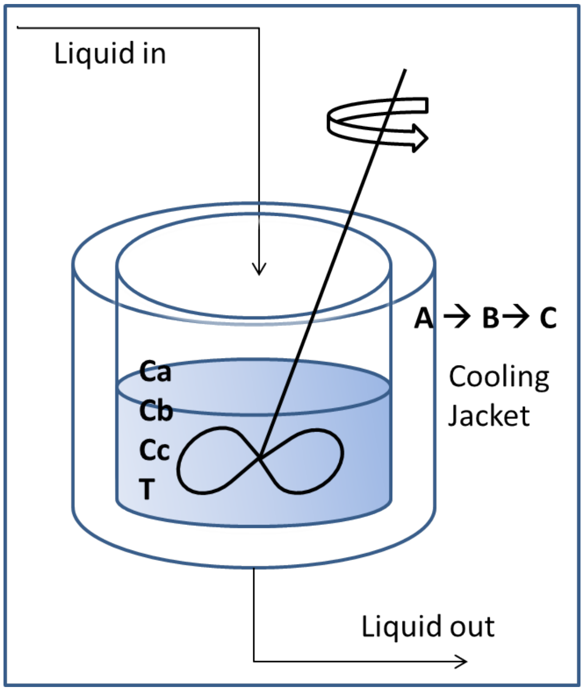

Figure 4.38: Block diagram of the CSTR

Variable	Value	Variable	Value
F	$100 \quad L/min$	C_{Af}	$1 \quad mol/L$
V	$100 \quad L$	$\frac{E_1}{R}$	$8750 \quad K$
k_1	$7.2 \times 10^{10} min^{-1}$	$\frac{E_2}{R}$	$9750 \quad K$
k_2	$5.2 \times 10^{10} min^{-1}$		

Table 4.15: Nominal operating conditions for CSTR

This system is discretized with sampling time of 0.01 min. It is assumed that the measurements of concentration of B have variance of 0.01. The following scenarios for concentration of chemical A in feed are considered.

1. $C_{Af} = |z_k|$, where z_k is a Gaussian random variable with mean 1 and variance 1.

2. $C_{Af} = |z_k|$, where z_k is a Gaussian random variable with mean 1 and variance 0.1.

Constrained state estimation is formulated as MHE and EKF with projection for this plant with $Q = 1, R = 0.01$, $\Pi_0 = 10$ and $\hat{C}_A, \hat{C}_B = 0$. The matrix Π_k for MHE arrival cost is obtained from solving discrete time matrix Riccati. $C_A \geq 0$ and $C_B \geq 0$ are chosen as constraint for MHE and EKF with projection. Results of the state estimation for this case study are shown in the following section

4.5.2 Results

Scenario 1

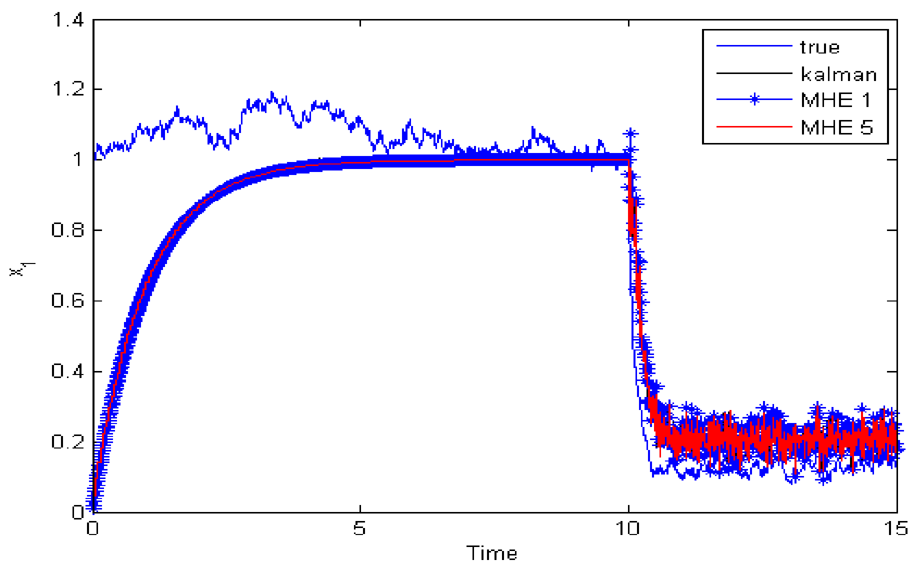

Figure 4.39: Comparison of estimators for C_A for scenario 1 of case study 5

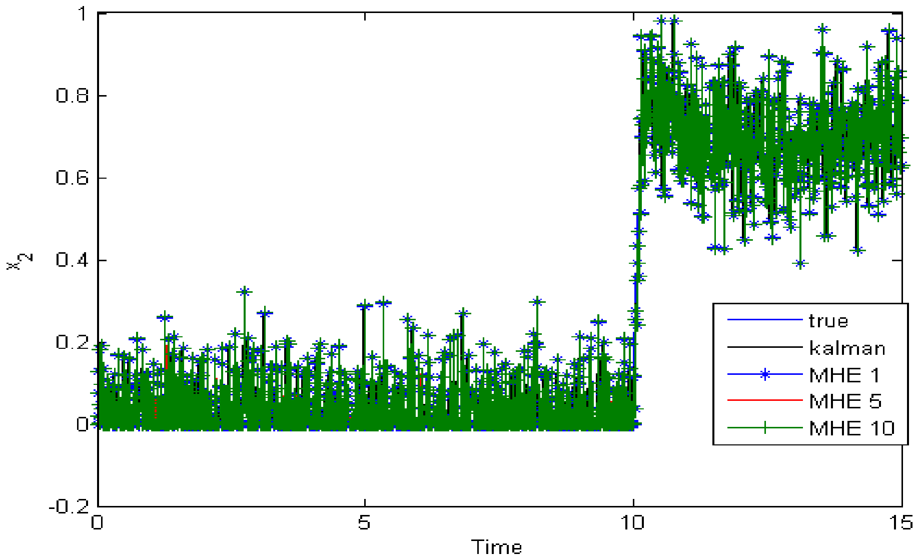

Figure 4.40: Comparison of estimators for C_B for scenario 1 of case study 5

	SSEE for C_A	SSEE for C_B	Average estimation time in sec
EKF with Projection	72.66	10.28	0.0086
MHE $N = 1$	75.94	10.38	0.2672
MHE $N = 5$	72.43	10.30	0.3325
MHE $N = 10$	69.47	10.47	1.2829

Table 4.16: Performance metrics of state estimators for scenario 1 of case study 5

Scenario 2

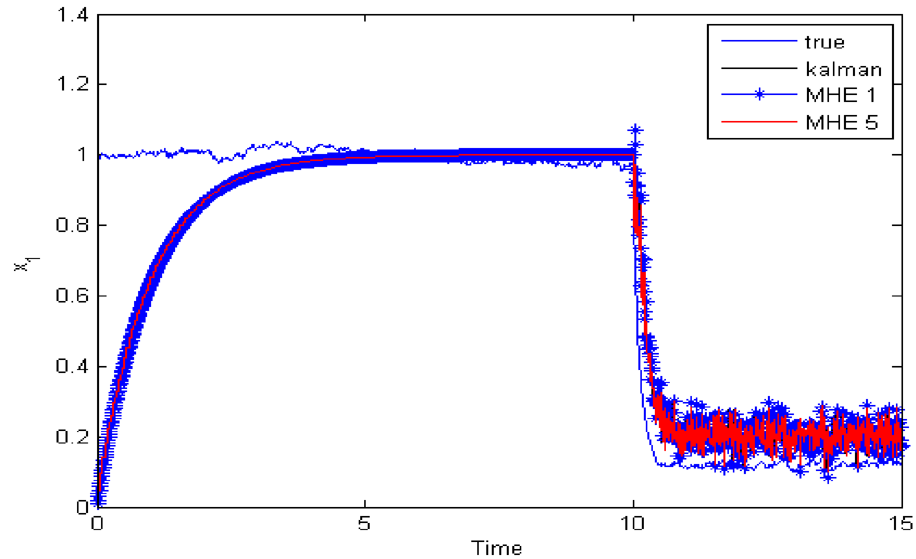

Figure 4.41: Comparison of estimators for C_A for scenario 2 of case study 5

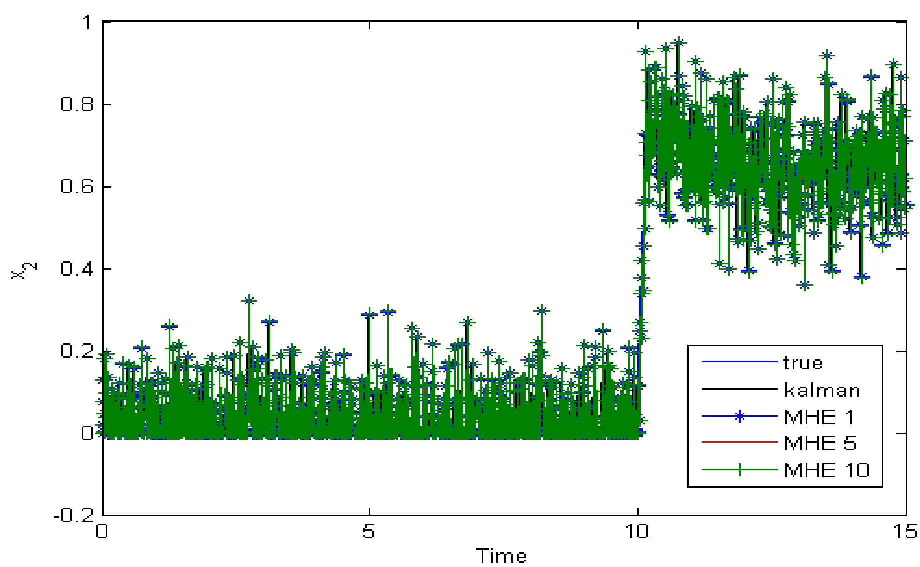

Figure 4.42: Comparison of estimators for C_B for scenario 2 of case study 5

	SSEE for C_A	SSEE for C_B	Average estimation time in sec
EKF with Projection	56.44	10.27	0.0092
MHE $N = 1$	58.13	10.40	0.29
MHE $N = 5$	54.65	10.320	0.36
MHE $N = 10$	52.25	10.50	1.3626

Table 4.17: Performance metrics of state estimators for scenario 2 of case study 5

It can be seen from above tables 4.1 and 4.2 that for the state C_A, SSEE of the Kalman Filter with projection is lower than SSEE of MHE with $N = 1$ and is higher than SSEE of MHE with $N = 5$ and $N = 10$. For the state C_B , SSEE of the Kalman Filter with projection is lower than SSEE of MHE with $N = 1$, $N = 5$ and $N = 10$. The average estimation time of the Kalman Filter with projection is approximately one third of the average estimation time for MHE with $N = 1$. The average estimation time for the MHE with $N = 10$ is approximately 50 times greater the average estimation time for Kalman Filter with Projection.

Chapter 5

CONCLUSION

State estimation for nonlinear discrete time systems in the presence of constraints on states, inputs and disturbances is investigated in this study. Inclusion of constraints in the state estimation formulation helps in correcting modeling errors and other uncertainties associated with system operation.

A brief derivation of constrained state estimation as Moving Horizon Estimation (MHE) from full information estimation was studied and practical problems associated with this algorithm was discussed in chapter 2. An alternative formulation of EKF known as Extended Kalman Filter (EKF) with projection algorithm was studied chapter 3.

The performance of these two estimators was tested with examples from literature. For these examples, constraints are active for huge amount of time and unconstrained Kalman filter/EKF produces poor results. Sum of square estimation error (SSEE) and average estimation time were used as metrics to compare their performances.

From the case studies 1, 2 and 3 it can be concluded that: compared to the unconstrained Kalman filter (or EKF), estimates of EKF with projection are close to the actual states. The SSEE of EKF with projection is less than SSEE of MHE with $N = 1$ and is greater than that with $N = 5$ and $N = 10$. The average estimation time for EKF with projection is approximately half as MHE with $N = 1$ and is significantly smaller than MHE with $N = 5$ and $N = 10$.

From case study 4 it can be concluded that: EKF with projection does not produce spurious results like unconstrained Kalman filter. The SSEE for EKF with projection are lower than MHE. Estimates of leaks for EKF with projection have a bias.

From case study 5 it can be concluded that: EKF with projection gives similar SSEE as MHE and it requires less computational time compared to MHE.

The following conclusions can be drawn from the results of case studies

- MHE
  - Pros
    - Low estimation error
    - Provision to increase horizon size for better estimation
  - Cons
    - High computational time
    - Increase in the average estimation time with increase in horizon size
    - Not suitable in state feedback control with high bandwidth
    - Computational burden makes it difficult to use in embedded systems.
- EKF with projection
  - Pros
    - Low computational time
    - Suitable in state feedback control with high bandwidth.
  - Cons
    - High estimation error compared to MHE with higher horizon size.

Therefore through this study it has been shown that EKF with projection offers viable alternative to MHE for the constrained state estimation problem.

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