Spiral laser beams in inhomogeneous media

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Explicit solutions of the inhomogeneous paraxial wave equation in a linear and quadratic approximation are applied to wave fields with invariant features, such as oscillating laser beams in a parabolic waveguide and spiral light beams in varying media. A similar effect of superfocusing of particle beams in a thin monocrystal film, harmonic oscillations of cold trapped atoms, and motion in magnetic field are also mentioned.

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Green function and generalized Fresnel integrals. In the context of quantum mechanics, a one-dimensional (1D) linear Schrödinger equation for generalized driven harmonic oscillators,

$$i\psi_t = -a(t)\psi_{xx} + b(t)x^2\psi - ic(t)x\psi_x - id(t)\psi - f(t)x\psi + ig(t)\psi_x, \quad (1)$$

(a, b, c, d, f, and g are suitable real-valued functions of time t only), can be solved by the integral superposition principle:

$$\psi(x,t) = \int_{-\infty}^{\infty} G(x,y,t)\psi(y,0)\mathrm{d}y, \quad (2)$$

where

$$G(x,y,t) = [2\pi\mu_0(t)]^{-1/2} \times \exp[i(\alpha_0(t)x^2 + \beta_0(t)xy + \gamma_0(t)y^2 + \delta_0(t)x + \varepsilon_0(t)y + \kappa_0(t))], \quad (3)$$

for certain initial data $\psi(x,0) = \varphi(x)$ (see [1–4] and the references therein for more details).

The intrinsic connection between Hamiltonian mechanics and the process of wave propagation is anything but a new idea [5,6]. Yet, in paraxial optics, when the time variable t represents the coordinate, say s, in the direction of wave propagation, Eqs. (2) and (3) can be thought of as a generalization of the Fresnel integral [7–10].

In the paraxial approximation, a 2D coherent light field in a parabolic inhomogeneous medium with coordinates $(\mathbf{r}, s) = (x, y, s)$ is described by the following equation for the complex field amplitude:

$$iA_s = -a(A_{xx} + A_{yy}) + b(x^2 + y^2)A - ic(xA_x + yA_y) - 2idA - (xf_1 + yf_2)A + i(g_1A_x + g_2A_y), \quad (4)$$

where a, b, c, d, $f_{1,2}$, and $g_{1,2}$ are real-valued functions of the coordinate in the direction of wave propagation s. The latter equation can be reduced to the standard form

$$-i\chi_\tau + \chi_{\xi\xi} + \chi_{\eta\eta} = c_0(\xi^2 + \eta^2)\chi, \quad (5)$$

($c_0 = 0, 1$) by the following Ansatz:

$$A = \mu^{-1} e^{i(\alpha(x^2+y^2)+\delta_1 x+\delta_2 y+\kappa_1+\kappa_2)}\chi(\xi,\eta,\tau),$$

where $\xi = \beta(s)x + \varepsilon_1(s)$, $\eta = \beta(s)y + \varepsilon_2(s)$, and $\tau = \gamma(s)$ (see Lemma 1 of [10] for a detailed statement).

The corresponding 2D Fresnel integral for inhomogeneous media in the linear and quadratic approximation is obtained in [10] (which may include intensity fluctuations from a random phase modulation). The Gaussian–Hermitian beams are given by separation of the variables

$$A_{nm}(\mathbf{r},s) = \frac{e^{i(\kappa_1+\kappa_2)+2i(n+m+1)\gamma}}{\sqrt{2^{n+m}n!m!\pi}}\beta \times e^{i(\alpha(x^2+y^2)+\delta_1 x+\delta_2 y)-(\beta x+\varepsilon_1)^2/2-(\beta y+\varepsilon_2)^2/2} \times H_n(\beta x+\varepsilon_1)H_m(\beta y+\varepsilon_2), \quad (6)$$

in terms of solutions of certain Ermakov-type systems, which are known in quadratures [2] [see Eqs. (9)–(14) below for an important explicit special case].

Oscillating and breathing laser beams. For a 1D paraxial wave equation with quadratic refractive index,

$$2iA_s + A_{xx} - x^2A = 0, \quad (7)$$

an important class of Gaussian–Hermitian modes can be presented as follows:

$$A_n(x,s) = e^{i(\alpha x^2+\delta x+\kappa)+i(2n+1)\gamma}\sqrt{\frac{\beta}{2^n n!\sqrt{\pi}}} \times e^{-(\beta x+\varepsilon)^2/2}H_n(\beta x+\varepsilon), \quad (8)$$

where $H_n(x)$ are the Hermite polynomials [11] and

$$\alpha(s) = \frac{\alpha_0 \cos 2s + \sin 2s(\beta_0^4 + 4\alpha_0^2 - 1)/4}{\beta_0^4 \sin^2 s + (2\alpha_0 \sin s + \cos s)^2}, \quad (9)$$

$$\beta(s) = \frac{\beta_0}{\sqrt{\beta_0^4 \sin^2 s + (2\alpha_0 \sin s + \cos s)^2}}, \quad (10)$$

$$\gamma(s) = -\frac{1}{2}\arctan\frac{\beta_0^2 \tan s}{1 + 2\alpha_0 \tan s}, \qquad (11)$$

$$\delta(s) = \frac{\delta_0(2\alpha_0 \sin s + \cos s) + \varepsilon_0\beta_0^3 \sin s}{\beta_0^4 \sin^2 s + (2\alpha_0 \sin s + \cos s)^2}, \qquad (12)$$

$$\varepsilon(s) = \frac{\varepsilon_0(2\alpha_0 \sin s + \cos s) - \beta_0\delta_0 \sin s}{\sqrt{\beta_0^4 \sin^2 s + (2\alpha_0 \sin s + \cos s)^2}}, \qquad (13)$$

$$\kappa(s) = \sin^2 s\,\frac{\varepsilon_0\beta_0^2(\alpha_0\varepsilon_0 - \beta_0\delta_0) - \alpha_0\delta_0^2}{\beta_0^4 \sin^2 s + (2\alpha_0 \sin s + \cos s)^2} + \frac{1}{4}\sin 2s\,\frac{\varepsilon_0^2\beta_0^2 - \delta_0^2}{\beta_0^4 \sin^2 s + (2\alpha_0 \sin s + \cos s)^2}. \qquad (14)$$

The real- or complex-valued parameters α_0, $\beta_0 \neq 0$, $\gamma_0 = 0$, δ_0, ε_0, and $\kappa_0 = 0$ are initial data of the corresponding Ermakov-type system [2,12,13]. A direct Mathematica verification can be found in Media 1. (Harmonic motion of cold trapped atoms is experimentally realized [14].)

These explicit solutions that are omitted in all textbooks on quantum mechanics (see [13,15]) provide a new multiparameter family of oscillating Gaussian–Hermitian beams in parabolic (self-focusing fiber) waveguides, which deserve an experimental observation; special cases were theoretically studied earlier in [8,16]. Examples are shown on Figs. 1 and 2. (Particular solutions in terms of Airy functions can be obtained in analogy with [5,17–19].)

Spreading solutions. The homogeneous paraxial wave equation,

$$2iB_s + B_{xx} = 0, \qquad (15)$$

can be transformed by the substitution

$$B(x,s) = \frac{1}{(s^2+1)^{1/4}} \exp\left(\frac{isx^2}{2(s^2+1)}\right) A\left(\frac{x}{\sqrt{s^2+1}}, \arctan s\right), \qquad (16)$$

into the inhomogeneous form in Eq. (7) (see [12] and the references therein). Composition of Eqs. (8) and (16) results in multiparameter solutions to parabolic Eq. (15):

$$B_n(x,s) = [((2\alpha_0 s+1)^2 + \beta_0^4 s^2)]^{-1/4} \times \sqrt{\frac{\beta_0}{2^n n!\sqrt{\pi}}} \exp\left(\frac{ix^2((4\alpha_0^2+\beta_0^4)s + 2\alpha_0)}{2((2\alpha_0 s+1)^2+\beta_0^4 s^2)}\right) \times \exp\left(ix\frac{(2\alpha_0 s+1)\delta_0 + s\beta_0^3\varepsilon_0}{(2\alpha_0 s+1)^2+\beta_0^4 s^2}\right) \times \exp\left(is\frac{(2\alpha_0 s+1)(\beta_0^2\varepsilon_0^2 - \delta_0^2) - 2s\beta_0^3\delta_0\varepsilon_0}{2((2\alpha_0 s+1)^2+\beta_0^4 s^2)}\right) \times \exp\left(-i\left(n+\frac{1}{2}\right)\arctan\left(\frac{\beta_0^2 s}{2\alpha_0 s+1}\right)\right) \times \exp\left(-\frac{(\beta_0(x-\delta_0 s) + \varepsilon_0(2\alpha_0 s+1))^2}{2((2\alpha_0 s+1)^2+\beta_0^4 s^2)}\right) \times H_n\left(\frac{\beta_0(x-\delta_0 s) + \varepsilon_0(2\alpha_0 s+1)}{\sqrt{(2\alpha_0 s+1)^2+\beta_0^4 s^2}}\right). \qquad (17)$$

Their direct Mathematica verification is also provided in Media 1 (see also [8]).

Breathing spiral laser beams. By the Ansatz $\Psi(X,Y,T) = \chi(\xi,\eta,\tau)$, $T = -\tau$, and

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\omega\tau & -\sin\omega\tau \\ \sin\omega\tau & \cos\omega\tau \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \qquad (18)$$

($\omega =$ constant), Eq. (5) with $c_0 = 1$ can be transformed to the equation of motion for the isotropic planar harmonic oscillator in a perpendicular uniform magnetic field (in the rotating frame of reference):

$$i\Psi_T + \Psi_{XX} + \Psi_{YY} = (X^2+Y^2)\Psi + i\omega(X\Psi_Y - Y\Psi_X). \qquad (19)$$

The latter equation was solved in the early days of quantum mechanics by Fock [20,21] in polar coordinates, $X = R\cos\Theta$ and $Y = R\sin\Theta$:

$$\Psi(R,\Theta,T) = \sqrt{\frac{n!}{\pi(n+|m|)!}}e^{-iET} \times e^{im\Theta}R^{|m|}e^{-R^2/2}L_n^{|m|}(R^2), \quad E = 4n + 2(|m|+1) - m\omega, \qquad (20)$$

($m = \pm 0, \pm 1, \ldots, n = 0, 1, \ldots$) in terms of Laguerre polynomials [11]. This wave function coincides, up to

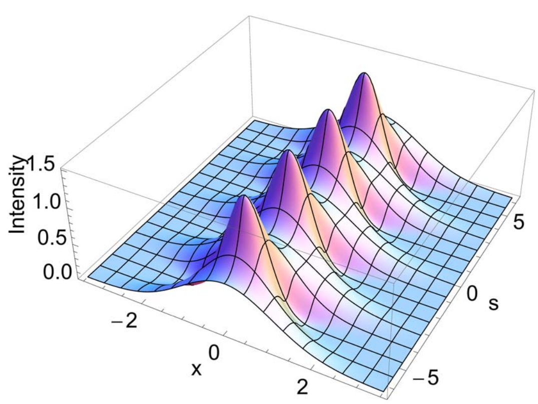

Fig. 1. Breathing Gaussian mode.

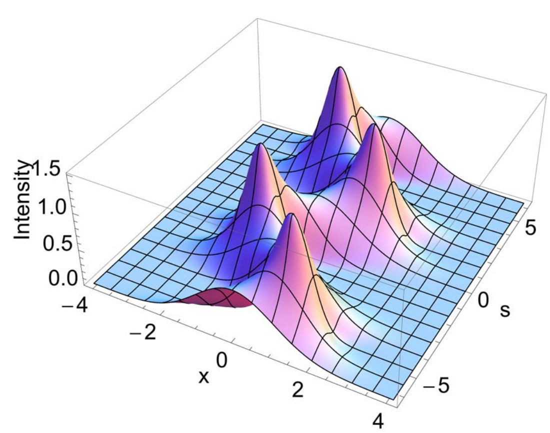

Fig. 2. Bending and breathing Gaussian mode.

a simple factor, with the one for a flat isotropic oscillator without magnetic field. Therefore, its development in terms of Eq. (6) for standard harmonics is a 2D special case of the multidimensional expansions from [11] (see also [22,23] and the references therein).

As a result, by back substitution one arrives at a general family of spiral solutions in inhomogeneous media. For example, the 2D paraxial wave equation ($\omega = 0$),

$$2iA_s + A_{xx} + A_{yy} = (x^2 + y^2)A, \tag{21}$$

possesses the following Gaussian–Laguerre modes:

$$\begin{aligned} A_n^m(x,y,s) &= \beta\sqrt{\frac{n!}{\pi(n+m)!}} \\ &\times e^{i(\alpha(x^2+y^2)+\delta_1 x+\delta_2 y+\kappa_1+\kappa_2)}e^{i(2n+m+1)\gamma} \\ &\times (\beta(x\pm iy)+\varepsilon_1 \pm i\varepsilon_2)^m e^{-(\beta x+\varepsilon_1)^2/2-(\beta y+\varepsilon_2)^2/2} \\ &\times L_n^m((\beta x+\varepsilon_1)^2+(\beta y+\varepsilon_2)^2), \qquad m\ge 0, \end{aligned} \tag{22}$$

by the explicit action of Schrödinger's group (see [10,12] and the references therein for classical accounts). Here, Eqs. (9) through (14) are utilized for real or complex parameters α_0, $\beta_0 \neq 0$, $\delta_0^{(1,2)}$, and $\varepsilon_0^{(1,2)}$ (the last two sets may be different). Examples are shown in Figs. 3 and 4.

Spreading and rotating solutions. The homogeneous parabolic equation,

$$2iB_s + B_{xx} + B_{yy} = 0, \tag{23}$$

and Eq. (21) are related by the transformation

$$\begin{aligned} B(x,y,s) &= \frac{1}{(s^2+1)^{1/2}}\exp\left(\frac{is(x^2+y^2)}{2(s^2+1)}\right) \\ &\times A\left(\frac{x}{\sqrt{s^2+1}}, \frac{y}{\sqrt{s^2+1}}, \arctan s\right). \end{aligned} \tag{24}$$

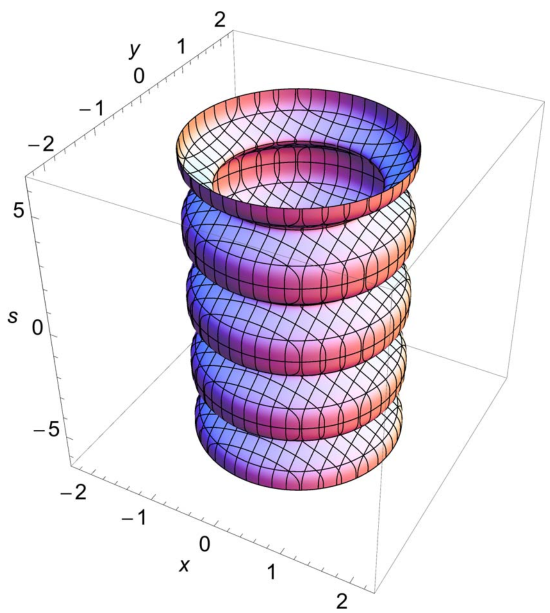

Fig. 3. Breathing Gaussian mode: surface where the intensity $|A|^2$ changes by the factor e.

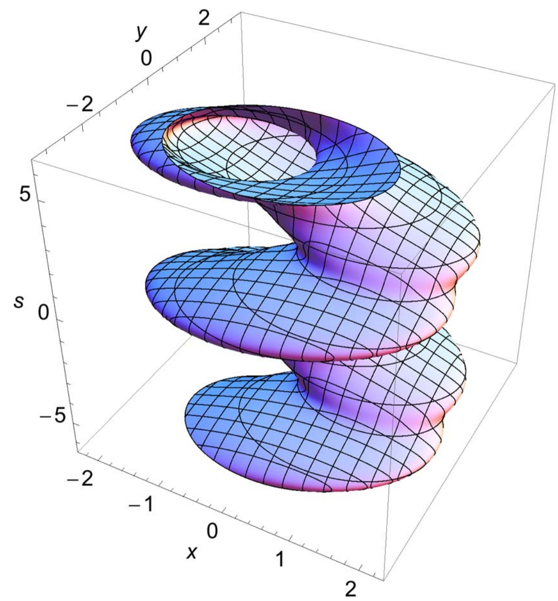

Fig. 4. Breathing and rotating Gaussian mode: surface where the intensity $|A|^2$ changes by the factor e.

Examples of spiral laser beams in a uniform medium are discussed in [24–26] (see also [8,16]).

A multiparameter solution is given by

$$\begin{aligned} B_n^m(x,y,s) &= \frac{e^{is(\delta_0^{(1)^2}+\delta_0^{(2)^2})/(2(1+2\alpha s))}}{\sqrt{(2\alpha_0 s+1)^2+\beta_0^4 s^2}} \\ &\times \exp\left(-i(1+m+2n)\arctan\left(\frac{s\beta_0^2}{1+2\alpha_0 s}\right)\right) \\ &\times \exp\left(i\,\frac{\alpha_0(x^2+y^2)+x\delta_0^{(1)}+y\delta_0^{(2)}}{2\alpha_0 s+1}\right) \\ &\times \exp\left[-\frac{(\beta_0(x-\delta_0^{(1)}s)+\varepsilon_0^{(1)}(2\alpha_0 s+1))^2}{2(2\alpha_0 s+1+i\beta_0^2 s)(1+2\alpha_0 s)}\right] \\ &\times \exp\left[-\frac{(\beta_0(y-\delta_0^{(2)}s)+\varepsilon_0^{(2)}(2\alpha_0 s+1))^2}{2(2\alpha_0 s+1+i\beta_0^2 s)(1+2\alpha_0 s)}\right] \\ &\times \left[\frac{\beta_0(x+iy)-(\delta_0^{(1)}+i\delta_0^{(2)})s}{\sqrt{(2\alpha_0 s+1)^2+\beta_0^4 s^2}}\right. \\ &\left.+\frac{(\varepsilon_0^{(1)}+i\varepsilon_0^{(2)})(2\alpha_0 s+1)}{\sqrt{(2\alpha_0 s+1)^2+\beta_0^4 s^2}}\right]^m \\ &\times L_n^m\left[\frac{(\beta_0(x-\delta_0^{(1)}s)+\varepsilon_0^{(1)}(2\alpha_0 s+1))^2}{(2\alpha_0 s+1)^2+\beta_0^4 s^2}\right. \\ &\left.+\frac{(\beta_0(y-\delta_0^{(2)}s)+\varepsilon_0^{(2)}(2\alpha_0 s+1))^2}{(2\alpha_0 s+1)^2+\beta_0^4 s^2}\right]. \end{aligned} \tag{25}$$

A similar effect of superfocusing of proton beams in a thin monocrystal film was discussed in [27] (validity of the 2D harmonic crystal model had been confirmed by Monte Carlo computer experiments). Among other quantum mechanical analogs, the minimum-uncertainty squeezed states for atoms and photons in a cavity are reviewed in [28]. (See also [6,10,19,29] and the references therein for extensions to nonlinear geometrical optics; an optoacoustic experiment is proposed in [30].)

In summary, we present multiparameter solutions to homogeneous and inhomogeneous paraxial wave equations which may be of interest in adaptive optics of (partially) coherent beams propagating through an atmospheric turbulence [17,31–33] and deserve an experimental observation.

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